

Crystallographic Restriction Theorem

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Chapter 1

Introduction

The *Crystallographic Restriction Theorem* characterizes exactly which orders can be achieved by integer matrices of a given dimension. For an $N \times N$ matrix M with integer entries, we say M has *finite order* m if $M^m = I$ and m is the smallest such positive integer. The theorem provides a complete answer to the question: which values of m can occur as orders of $N \times N$ integer matrices?

Main Theorem. *An $N \times N$ integer matrix can have finite order m if and only if $\psi(m) \leq N$, where ψ is defined below.*

The function ψ measures the “arithmetic complexity” of m in terms of Euler’s totient function φ . For a prime power p^k , we define the *prime power contribution*:

$$\psi_{\text{pp}}(p, k) = \begin{cases} 0 & \text{if } p = 2 \text{ and } k = 1, \\ \varphi(p^k) & \text{otherwise.} \end{cases}$$

The special case $\psi_{\text{pp}}(2, 1) = 0$ reflects the fact that order 2 can be achieved by the 0×0 empty matrix (or equivalently, $-I$ achieves order 2 without increasing dimension). For a general positive integer m with prime factorization $m = \prod_i p_i^{k_i}$, we define:

$$\psi(m) = \sum_i \psi_{\text{pp}}(p_i, k_i).$$

Proof Strategy. The proof splits naturally into two directions.

Forward direction ($m \in \text{Ord}_N \Rightarrow \psi(m) \leq N$): If M is an $N \times N$ integer matrix with order m , then M satisfies $M^m = I$, so its minimal polynomial divides $X^m - 1$. Since M has exact order m , the m -th cyclotomic polynomial Φ_m divides the minimal polynomial, and hence divides the characteristic polynomial of M . By the degree constraint, $\deg(\chi_M) = N \geq \deg(\Phi_m) = \varphi(m) \geq \psi(m)$.

Backward direction ($\psi(m) \leq N \Rightarrow m \in \text{Ord}_N$): We construct an integer matrix of order m with dimension exactly $\psi(m)$, which can then be embedded into any larger dimension. The construction uses companion matrices of cyclotomic polynomials. For a prime power p^k with $\psi_{\text{pp}}(p, k) > 0$, the companion matrix of Φ_{p^k} is a $\varphi(p^k) \times \varphi(p^k)$ integer matrix with order p^k . For composite m , we combine companion matrices for each prime power factor in a block diagonal arrangement. The coprimality of orders ensures the combined matrix has order lcm of the individual orders, which equals m .

This formalization follows the approach of Bamberg, Cairns, and Kilminster [1], who characterized achievable orders in arbitrary dimension using the ψ function.

Bibliography

- [1] J. Bamberg, G. Cairns, and D. Kilminster. The crystallographic restriction, permutations, and Goldbach's conjecture. *Amer. Math. Monthly*, 110(3):202–209, 2003.

Chapter 2

The Psi Function

The ψ function measures the “arithmetic complexity” of a positive integer m . For a prime power p^k , we define:

- $\psi_{\text{pp}}(p, k) = \varphi(p^k)$ for p odd or $k \geq 2$
- $\psi_{\text{pp}}(2, 1) = 0$

For a general integer m with prime factorization $m = \prod_i p_i^{k_i}$, we have:

$$\psi(m) = \sum_i \psi_{\text{pp}}(p_i, k_i)$$

Chapter 3

Integer Matrix Orders

We define the set Ord_N of achievable orders for $N \times N$ integer matrices.

Chapter 4

Companion Matrices

Companion matrices provide a key construction for achieving orders via cyclotomic polynomials.

Chapter 5

The Crystallographic Restriction Theorem

The proof splits into two directions: showing $\psi(m) \leq N$ is necessary (forward) and sufficient (backward) for $m \in \text{Ord}_N$.

Appendix A

Auxiliary Lemmas

This appendix collects technical lemmas used throughout the proof. These are general-purpose results about finite sets, coprime products, Euler's totient function, and matrix orders that support the main arguments.