Crystallographic Restriction Theorem

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Chapter 1

Introduction

The *Crystallographic Restriction Theorem* characterizes exactly which rotation orders are achievable by integer matrices. Specifically, an $N \times N$ integer matrix can have finite order m if and only if $\psi(m) \leq N$, where ψ is a function based on Euler's totient function.

Chapter 2

Supporting Lemmas

Chapter 3

The Psi Function

The ψ function measures the "arithmetic complexity" of a positive integer m. For a prime power p^k, we define:

- $\psi_{pp}(p, k) = \varphi(p^k)$ for p odd or $k \geq 2$
- $\psi_{pp}(2, 1) = 0$

For a general integer m with prime factorization $m = \prod_i p_i^{k_i}$, we have:

$$\psi(m) = \sum_i \psi_{pp}(p_i, k_i)$$

Chapter 4

Integer Matrix Orders

We define the set Ord_N of achievable orders for $N \times N$ integer matrices.

Chapter 5

Companion Matrices

Companion matrices provide a key construction for achieving orders via cyclotomic polynomials.

Chapter 6

The Crystallographic Restriction

The proof splits into two directions: showing $\psi(m) \leq N$ is necessary (forward) and sufficient (backward) for $m \in \mathrm{Ord}_N$.

Chapter 7

Main Theorem