

# Crystallographic Restriction Theorem

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# Chapter 1

## Introduction

The *Crystallographic Restriction Theorem* characterizes exactly which orders can be achieved by integer matrices of a given dimension. For an  $N \times N$  matrix  $M$  with integer entries, we say  $M$  has *finite order*  $m$  if  $M^m = I$  and  $m$  is the smallest such positive integer. The theorem provides a complete answer to the question: which values of  $m$  can occur as orders of  $N \times N$  integer matrices?

**Main Theorem.** *An  $N \times N$  integer matrix can have finite order  $m$  if and only if  $\psi(m) \leq N$ , where  $\psi$  is defined below.*

The function  $\psi$  measures the “arithmetic complexity” of  $m$  in terms of Euler’s totient function  $\varphi$ . For a prime power  $p^k$ , we define the *prime power contribution*:

$$\psi_{\text{pp}}(p, k) = \begin{cases} 0 & \text{if } p = 2 \text{ and } k = 1, \\ \varphi(p^k) & \text{otherwise.} \end{cases}$$

The special case  $\psi_{\text{pp}}(2, 1) = 0$  reflects the fact that order 2 can be achieved by the  $0 \times 0$  empty matrix (or equivalently,  $-I$  achieves order 2 without increasing dimension). For a general positive integer  $m$  with prime factorization  $m = \prod_i p_i^{k_i}$ , we define:

$$\psi(m) = \sum_i \psi_{\text{pp}}(p_i, k_i).$$

**Proof Strategy.** The proof splits naturally into two directions.

*Forward direction* ( $m \in \text{Ord}_N \Rightarrow \psi(m) \leq N$ ): If  $M$  is an  $N \times N$  integer matrix with order  $m$ , then  $M$  satisfies  $M^m = I$ , so its minimal polynomial divides  $X^m - 1$ . Since  $M$  has exact order  $m$ , the  $m$ -th cyclotomic polynomial  $\Phi_m$  divides the minimal polynomial, and hence divides the characteristic polynomial of  $M$ . By the degree constraint,  $\deg(\chi_M) = N \geq \deg(\Phi_m) = \varphi(m) \geq \psi(m)$ .

*Backward direction* ( $\psi(m) \leq N \Rightarrow m \in \text{Ord}_N$ ): We construct an integer matrix of order  $m$  with dimension exactly  $\psi(m)$ , which can then be embedded into any larger dimension. The construction uses companion matrices of cyclotomic polynomials. For a prime power  $p^k$  with  $\psi_{\text{pp}}(p, k) > 0$ , the companion matrix of  $\Phi_{p^k}$  is a  $\varphi(p^k) \times \varphi(p^k)$  integer matrix with order  $p^k$ . For composite  $m$ , we combine companion matrices for each prime power factor in a block diagonal arrangement. The coprimality of orders ensures the combined matrix has order lcm of the individual orders, which equals  $m$ .

This formalization follows the approach of Bamberg, Cairns, and Kilminster [1], who characterized achievable orders in arbitrary dimension using the  $\psi$  function.

# Bibliography

- [1] J. Bamberg, G. Cairns, and D. Kilminster. The crystallographic restriction, permutations, and Goldbach's conjecture. *Amer. Math. Monthly*, 110(3):202–209, 2003.

## Chapter 2

# The Psi Function

The  $\psi$  function measures the “arithmetic complexity” of a positive integer  $m$ . For a prime power  $p^k$ , we define:

- $\psi_{\text{pp}}(p, k) = \varphi(p^k)$  for  $p$  odd or  $k \geq 2$
- $\psi_{\text{pp}}(2, 1) = 0$

For a general integer  $m$  with prime factorization  $m = \prod_i p_i^{k_i}$ , we have:

$$\psi(m) = \sum_i \psi_{\text{pp}}(p_i, k_i)$$

## Chapter 3

# Integer Matrix Orders

We define the set  $\text{Ord}_N$  of achievable orders for  $N \times N$  integer matrices.

## Chapter 4

# Companion Matrices

Companion matrices provide a key construction for achieving orders via cyclotomic polynomials.

## Chapter 5

# The Crystallographic Restriction Theorem

The proof splits into two directions: showing  $\psi(m) \leq N$  is necessary (forward) and sufficient (backward) for  $m \in \text{Ord}_N$ .

# Appendix A

## Auxiliary Lemmas

This appendix collects technical lemmas used throughout the proof. These are general-purpose results about finite sets, coprime products, Euler's totient function, and matrix orders that support the main arguments.