Crystallographic Restriction Theorem

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Chapter 1

Introduction

The *Crystallographic Restriction Theorem* characterizes exactly which orders can be achieved by integer matrices of a given dimension. For an $N \times N$ matrix M with integer entries, we say M has *finite order* m if $M^m = I$ and m is the smallest such positive integer. The theorem provides a complete answer to the question: which values of m can occur as orders of $N \times N$ integer matrices?

Main Theorem. *An $N \times N$ integer matrix can have finite order m if and only if $\psi(m) \leq N$, where ψ is defined below.*

The function ψ measures the "arithmetic complexity" of m in terms of Euler's totient function φ. For a prime power p^k, we define the *prime power contribution*:

$$\psi_{\mathrm{pp}}(p, k) = \begin{cases} 0 & \text{if } p = 2 \text{ and } k = 1, \\ \varphi(p^k) & \text{otherwise.} \end{cases}$$

The special case $\psi_{\mathrm{pp}}(2, 1) = 0$ reflects the fact that order 2 can be achieved by the 0×0 empty matrix (or equivalently, $-I$ achieves order 2 without increasing dimension). For a general positive integer m with prime factorization $m = \prod_i p_i^{k_i}$, we define:

$$\psi(m) = \sum_i \psi_{\mathrm{pp}}(p_i, k_i).$$

Proof Strategy. The proof splits naturally into two directions.

Forward direction ($m \in \mathrm{Ord}_N \Rightarrow \psi(m) \leq N$): If M is an $N \times N$ integer matrix with order m, then M satisfies $M^m = I$, so its minimal polynomial divides $X^m - 1$. Since M has exact order m, the m-th cyclotomic polynomial Φ_m divides the minimal polynomial, and hence divides the characteristic polynomial of M. By the degree constraint, $\deg(\chi_M) = N \geq \deg(\Phi_m) = \varphi(m) \geq \psi(m)$.

Backward direction ($\psi(m) \leq N \Rightarrow m \in \mathrm{Ord}_N$): We construct an integer matrix of order m with dimension exactly $\psi(m)$, which can then be embedded into any larger dimension. The construction uses companion matrices of cyclotomic polynomials. For a prime power p^k with $\psi_{\mathrm{pp}}(p, k) > 0$, the companion matrix of Φ_{p^k} is a $\varphi(p^k) \times \varphi(p^k)$ integer matrix with order p^k. For composite m, we combine companion matrices for each prime power factor in a block diagonal arrangement. The coprimality of orders ensures the combined matrix has order lcm of the individual orders, which equals m.

This formalization follows the approach of Bamberg, Cairns, and Kilminster [1], who characterized achievable orders in arbitrary dimension using the ψ function.

Bibliography

[1] J. Bamberg, G. Cairns, and D. Kilminster. The crystallographic restriction, permutations, and Goldbach's conjecture. *Amer. Math. Monthly*, 110(3):202–209, 2003.

Chapter 2

The Psi Function

The ψ function measures the "arithmetic complexity" of a positive integer m. For a prime power p^k, we define:

- $\psi_{\mathrm{pp}}(p, k) = \varphi(p^k)$ for p odd or $k \geq 2$
- $\psi_{\mathrm{pp}}(2, 1) = 0$

For a general integer m with prime factorization $m = \prod_i p_i^{k_i}$, we have:

$$\psi(m) = \sum_i \psi_{\mathrm{pp}}(p_i, k_i)$$

Chapter 3

Integer Matrix Orders

We define the set Ord_N of achievable orders for $N \times N$ integer matrices.

Chapter 4

Companion Matrices

Companion matrices provide a key construction for achieving orders via cyclotomic polynomials.

Chapter 5

The Crystallographic Restriction Theorem

The proof splits into two directions: showing $\psi(m) \leq N$ is necessary (forward) and sufficient (backward) for $m \in \mathrm{Ord}_N$.

Appendix A

Appendix

This appendix collects technical lemmas used throughout the proof. These are general-purpose results about finite sets, coprime products, Euler's totient function, and matrix orders that support the main arguments.