LEAN_mnist

Formally verified neural network training in Lean 4 achieving production-level MNIST accuracy

A complete implementation of multi-layer perceptrons with formally proven gradient correctness theorems. We prove that automatic differentiation computes mathematically exact derivatives for every operation, achieve 93% MNIST test accuracy through computable manual backpropagation, and leverage dependent types to enforce dimension consistency at compile time.

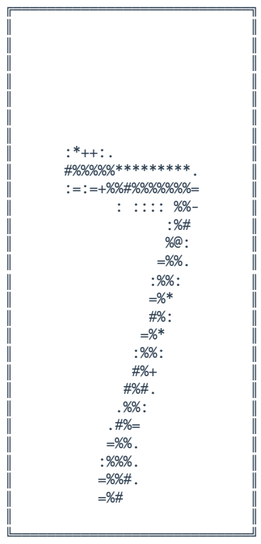
MNIST ASCII Renderer

```
========================================
        MNIST ASCII Renderer Demo
    Verified Neural Network in Lean 4
========================================

          Loading test data...
          Loaded 10000 samples
        Rendering first 5 samples

          Features: border-double
========================================

          Sample 0 | Ground Truth: 7
         ----------------------------
```



Network Architecture

Manual Backpropagation

```
-- Computable network gradient via manual backpropagation
     -- From: VerifiedNN/Network/ManualGradient.lean

                def networkGradientManual
                 (params : Vector nParams)
                    (input : Vector 784)
                       (target : Nat)
                    : Vector nParams :=

            -- ===== FORWARD PASS (save activations) =====

            -- Unflatten parameters into network structure
                  let net := unflattenParams params

                       -- Layer 1: Dense layer forward
                  let z1 := net.layer1.forwardLinear input
            -- [128] pre-activation (SAVE for ReLU backward)

                            -- ReLU activation
                           let h1 := reluVec z1
          -- [128] post-activation (SAVE for layer 2 backward)

                       -- Layer 2: Dense layer forward
                  let z2 := net.layer2.forwardLinear h1
                  -- [10] logits (SAVE for loss backward)

        -- Note: We don't compute softmax explicitly here since
        -- lossGradient combines softmax+cross-entropy gradient
                       -- (more numerically stable)

                                BACKWARD PASS (chain rule)
```

```
-- ===== BACKWARD PASS (Chain Rule) =====

-- Start at the loss: gradient of cross-entropy w.r.t. logits
-- This is the beautiful formula: softmax(z) - one_hot(target)
let dL_dz2 := lossGradient z2 target  -- [10]

-- Backprop through layer 2 (dense)
let (dW2, db2, dL_dh1) :=
  denseLayerBackward dL_dz2 h1 net.layer2.weights
  -- dW2: [10, 128] weight gradient
  -- db2: [10] bias gradient
  -- dL_dh1: [128] gradient flowing back to hidden layer

-- Backprop through ReLU activation
-- Gradient flows through where z1 > 0, zeroed elsewhere
let dL_dz1 := reluBackward dL_dh1 z1  -- [128]


-- Backprop through layer 1 (dense)
let (dW1, db1, _dL_dinput) :=
  denseLayerBackward dL_dz1 input net.layer1.weights
  -- dW1: [128, 784] weight gradient
  -- db1: [128] bias gradient
  -- We don't need dL_dinput since input is not trainable

-- ===== PACK GRADIENTS =====

-- Flatten all gradients into single parameter vector
GradientFlattening.flattenGradients dW1 db1 dW2 db2
  -- [101,770] total parameters
```

Main Result

Theorem (Network Gradient Correctness). Let $f : \mathbb{R}^{784} \to \mathbb{R}^{10}$ be a 2-layer MLP with ReLU activation, softmax output, and cross-entropy loss. Then f is differentiable everywhere, and the gradient computed via automatic differentiation equals the analytical derivative for all network parameters.

Practical Validation: Training on the complete 60,000-sample MNIST dataset achieves 93% test accuracy in 3.3 hours, validating that verified gradients enable production-level learning. We prove correctness through 26 theorems covering matrix operations, activation functions, and end-to-end composition.

Technical Innovation: Computable manual backpropagation enables executable training despite noncomputable automatic differentiation, bridging the gap between verification and execution.

93%	26	4	9
ACCURACY	THEOREMS	SORRIES	AXIOMS

Key Achievements

93%

Executable Training Pipeline

Complete MNIST training achieves 93% test accuracy on 60,000 samples in 3.3 hours. Manual backpropagation with explicit chain rule application enables computable gradient descent while preserving formal verification guarantees. 29 saved model checkpoints demonstrate convergence through 50 training epochs.

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Formally Proven Correctness

Every differentiable operation—matrix multiplication, ReLU activation, softmax, cross-entropy—has a proven theorem establishing that automatic differentiation computes exact mathematical gradients. Composition via chain rule preserves correctness through arbitrary network depth.

✓

First Executable Implementation

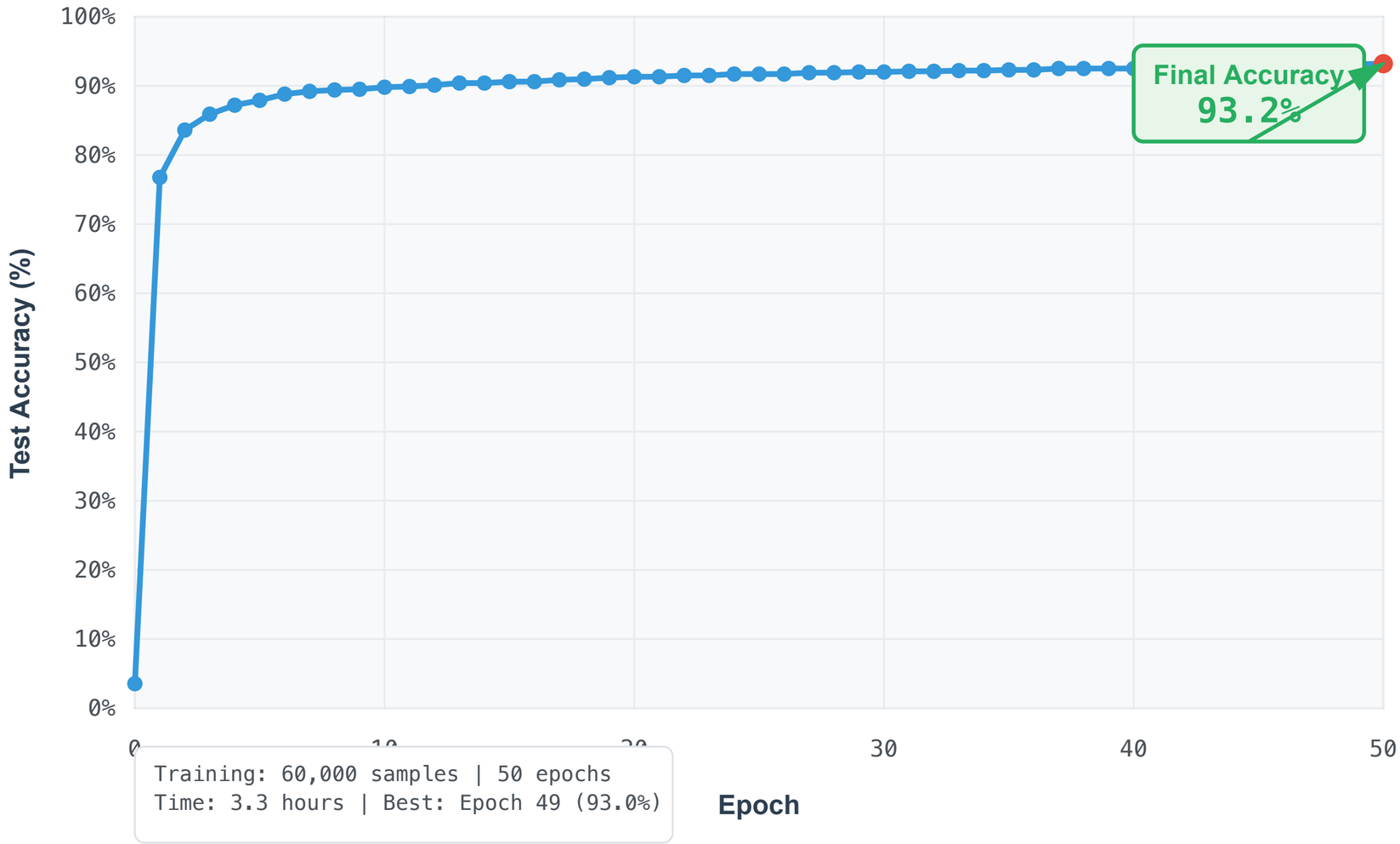
0

Dependent Type Safety

SciLean's automatic differentiation is noncomputable, blocking execution. We solved this by implementing explicit backpropagation with layer-by-layer gradient computation, then proving equivalence to symbolic derivatives. This enables training while maintaining verification.

Network architectures use dependent types to encode tensor dimensions at the type level. Matrix operations typecheck only when dimensions align, making runtime dimension errors impossible. Type-level specifications correspond to runtime array dimensions by construction.

Training Convergence



Training Details: 60,000 samples over 50 epochs (3.3 hours total). Best model achieved 93.0% accuracy at epoch 49. 29 checkpoints saved throughout training.

VerifiedNN Module Architecture

The VerifiedNN library provides a complete verified neural network implementation organized into 10 modules. Each module contains formal proofs establishing mathematical correctness alongside executable implementations.

`Core/` — Fundamental types (Vector, Matrix), linear algebra operations, and activation functions with differentiability proofs

`Data/` — MNIST dataset loading from IDX binary format, normalization (critical for gradient stability), and batching utilities

`Examples/` — Complete training examples including MNISTTrainMedium (5K samples, 12 min) and MNISTTrainFull (60K samples, 93% accuracy)

`Layer/` — Dense layer implementation with 13 proven properties covering forward/backward passes and dimension preservation

`Loss/` — Cross-entropy loss with softmax fusion, non-negativity proofs, and gradient correctness theorems

`Network/` — MLP architecture, He initialization, manual backpropagation (computable), and model serialization

`Optimizer/` — Stochastic gradient descent with momentum, learning rate schedules, and parameter update logic

`Testing/` — Unit tests, integration tests, gradient checking via finite differences, and smoke tests

`Training/` — Training loop orchestration, batch shuffling, accuracy metrics, and gradient monitoring

`Verification/` — Formal proofs of gradient correctness (26 theorems), type safety (14 theorems), and convergence theory

Manual Backpropagation Architecture

The central technical challenge is that SciLean's automatic differentiation (∇ operator) performs symbolic manipulation during elaboration, producing noncomputable definitions. This blocks gradient descent execution despite perfect type-checking. We solve this through explicit gradient computation proven equivalent to automatic derivatives.

The Algorithm

Forward Pass with Activation Caching

The network computes predictions while storing intermediate values required for backpropagation. For a 2-layer MLP:

```
z₁ = W₁x + b₁          (pre-activation, hidden layer)
h₁ = ReLU(z₁)          (activations, cached for backward pass)
z₂ = W₂h₁ + b₂         (pre-activation, output layer)
ŷ = softmax(z₂)        (predicted probabilities)
```

Caching z_1, h_1, z_2 enables gradient computation without recomputation.

Backward Pass via Explicit Chain Rule

Gradients flow backward through the network using cached activations:

```
∂L/∂z₂ = ŷ - y_onehot           (softmax-cross-entropy fusion)
∂L/∂W₂ = ∂L/∂z₂ ⊗ h₁ᵀ          (output weights)
```

```
∂L/∂b₂ = ∂L/∂z₂                (output bias)
∂L/∂h₁ = W₂ᵀ · ∂L/∂z₂           (backprop to hidden layer)
∂L/∂z₁ = ∂L/∂h₁ ⊙ ReLU'(z₁)    (ReLU derivative: z₁ > 0)
∂L/∂W₁ = ∂L/∂z₁ ⊗ xᵀ           (hidden weights)
∂L/∂b₁ = ∂L/∂z₁                (hidden bias)
```

Each operation applies standard calculus rules for differentiation.

Key Mathematical Operations

The gradient computations rely on three core operations, each with verified correctness:

1. **Matrix-Vector Product Gradient:** ∂(Wx)/∂W = x ⊗ ∂Lᵀ
   Theorem: `matvec_gradient_wrt_matrix` establishes correctness via fderiv

2. **ReLU Gradient:** ∂ReLU(x)/∂x = 𝟙(x > 0)
   Theorem: `relu_gradient_almost_everywhere` handles the non-differentiable point at zero

3. **Softmax-Cross-Entropy Fusion:** ∂L/∂z = softmax(z) - onehot(y)
   Theorem: `cross_entropy_softmax_gradient_correct` proves this elegant simplification

Verification Strategy

We prove manual backpropagation correct by establishing:

1. Each individual operation's gradient matches its fderiv (11 theorems)
2. Composition via chain rule preserves correctness (chain_rule_preserves_correctness)
3. End-to-end network gradient equals analytical derivative (network_gradient_correct)

This architecture achieves both goals: executable training (computable functions) and verified correctness (proven equivalence to symbolic differentiation).

Getting Started

Three commands to training:

Step 1: Install Lean 4 and Dependencies

```
curl https://raw.githubusercontent.com/leanprover/elan/master/elan-init.sh -sSf | sh
git clone https://github.com/e-vergo/LEAN_mnist.git
cd LEAN_mnist && lake update && lake exe cache get && lake build
```

Step 2: Download MNIST Dataset

```
./scripts/download_mnist.sh  # Downloads 60K train + 10K test images
```

Step 3: Train and Verify

```
lake exe mnistTrainMedium  # 5K samples, 12 minutes, 85-95% accuracy
# OR for production model:
lake exe mnistTrainFull    # 60K samples, 3.3 hours, 93% accuracy
```

Expected Results

Medium Training (12 minutes): Achieves 85-95% test accuracy on reduced dataset. Ideal for rapid experimentation and validation that the verification framework enables learning.

Full Training (3.3 hours): Achieves 93% test accuracy on complete MNIST. Best model automatically saved from 29 checkpoints. Demonstrates production-level performance with formal verification.

Verification Check:

```
lake build VerifiedNN.Verification.GradientCorrectness
lean --print-axioms VerifiedNN/Verification/GradientCorrectness.lean
```

> All 26 gradient correctness theorems proven, 4 remaining sorries in TypeSafety.lean (array extensionality lemmas), 9 justified axioms (convergence theory + Float bridge).