Kindergarten

Volume II: Real Analysis

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Preface

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How to Read?

Chapter 0

Preliminaries

0.1 Ordering

Chapter 1

User Guides for General Topology

1.1 Basic terminologies related to the topology

In Volume 1, we explored basic terminologies and their properties of metric spaces. In this chapter, we will extend those by introducing the concept called "topology".

Definition 1.1.1
Let X be a set. A *topology* is a collection τ of some subsets of X satisfies the following conditions.

(i) $\varnothing, X \in \tau$;

(ii) If $\mathscr{U} \subset \tau$, then $\bigcup \mathscr{U} \in \tau$;

(iii) If $\mathscr{F} \subset \tau$ be a finite subcollection, then $\bigcap \mathscr{F} \in \tau$.

Every element of a topology is called an *open set*, and complement of open sets are called *closed sets*.

Everywhere in this book, **topological spaces are not empty**. By the definition of closed sets, we can clearly obtain the following.

Proposition 1.1.2

(i) If $\mathscr{U}$ is a collection of some closed sets, then $\bigcap \mathscr{U}$ is also closed.

(ii) If $\mathscr{F}$ is a finite collection of some closed sets, then $\bigcup \mathscr{F}$ is closed.

If X is a topological space, we should define closures and interiors differently. However, if one studied well about closures and interiors in metric spaces, below definition makes sense.

Definition 1.1.3 (Closures and interiors)
Let X be a topological space and A a subset of X. Then a *closure* of A is an intersection of all closed sets each of which contains A, and an *interior* of A is a union of all open sets each of which is contained in A.
We write a closure of A by $\mathrm{cl}A$, and an interior of A by $\mathrm{int}A$.

Remark 1.1.4
Notation $\overline{A}$ is also used for denoting the closure of A, and A° for writting the interior of A.

Now we define neighborhoods of a point in topological spaces. Although we defined this to be open in metric spaces, for some purpose, we define neighborhoods *need not to be* *open*.

Definition 1.1.5 (Neighborhoods)
Let X be a topological space and $x \in X$. Then a subset $N \subset X$ is called a *neighborhood of x* if its interior contains x. That is, $x \in \text{int}N$.

1.2 Filters and ultrafilters

In the basis analysis, one is taught that if X is a metric space and A is a subset of X then $x \in \overline{A}$ if and only if there is a sequence $(x_n)_n$ in A that converges to x. However, such fact does not hold for general topological spaces. Actually, several properties of sequences in metric spaces break when we generalize those in the arbitrary topological spaces. This makes a reason introducing an object which acts like a sequence more general sense. There are two things that generalize sequences: filters, and nets. In here, we are going to study about filters.

First, we need to see what are filters. Note that filters are defined even if X has no topology.

Definition 1.2.1 (Definition of filters)
Let X be a nonempty set. A *filter* $\mathcal{F}$ is a nonempty collection of some subsets of X satisfies the following conditions.

(i) $\varnothing \notin \mathcal{F}$;

(ii) If $A \in \mathcal{F}$, then $B \in \mathcal{F}$ for every $B \supset A$;

(iii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Clearly $\{X\}$ is a filter in X, so there exists a filter for any X. In addition, $\{X\}$ is the smallest filter we can ever make. Then, is there exists a maximal filter in X with respect to the set theoretical inclusion? Below proposition shows that indeed such filter exists, so this proposition justifies that Definition 1.2.3 is well-defined.

Proposition 1.2.2
Let X be a nonempty set and $\mathcal{F}$ be a filter in X. Then there exists a maximal filter $\mathcal{U} \supset \mathcal{F}$ in X meaning that there is no filter $\mathcal{G}$ in X which contains $\mathcal{U}$.

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proof. Fix a filter $\mathcal{F}$ in X and define an ordered set

$$\mathscr{C} := \{\mathcal{B} \supset \mathcal{F} : \mathcal{B} \text{ is a filter in } X\}$$

with an inclusion order. Then by the Hausdorff maximal theorem, there exists a maximal total ordered subset $\mathscr{D}$ of $\mathscr{C}$. Claim that $\bigcup \mathscr{D}$ is a filter and $\bigcup \mathscr{D} \supset \mathcal{F}$. Notice that the second one is trivial, so we are going to focus the first assertion.
Clearly $\varnothing \notin \bigcup \mathscr{D}$ and if $E \in \bigcup \mathscr{D}$ and $F \supset E$, then $F \in \bigcup \mathscr{D}$. If $E_1, E_2 \in \bigcup \mathscr{D}$, we can two filters $\mathcal{F}_1, \mathcal{F}_2 \in \mathscr{D}$ such that $E_1 \in \mathcal{F}_1$ and $E_2 \in \mathcal{F}_2$, respectively. Since $\mathscr{D}$ is a chain, it suffices to assume that $\mathcal{F}_1 \subset \mathcal{F}_2$. In this case, we have $E_1 \cap E_2 \in \mathcal{F}_2 \subset \bigcup \mathscr{D}$, proving that $\bigcup \mathscr{D}$ is a filter.
Notice that $\bigcup \mathscr{D} \in \mathscr{C}$ so $\bigcup \mathscr{D}$ is the desired one, otherwise we can take a filter $\mathcal{G} \supset \bigcup \mathscr{D}$ but this makes a contradiction of the maximality of $\mathscr{D}$. $\square$

Definition 1.2.3 (Definition of ultrafilters)
Let X be a set. An *ultrafilter* $\mathcal{U}$ in X is a maximal filter.

As we know that such definition is quite difficult to draw some image what ultrafilters look like. There is a equivalent statement that describes ultrafilters.

Proposition 1.2.4
Let X be a nonempty set and $\mathcal{U}$ be a filter in X. Then $\mathcal{U}$ is an ultrafilter if and only if for any $A \subset X$, either A or $X \setminus A$ belongs to $\mathcal{U}$.

- -

proof. To prove the necessity, assume that there exists a subset $A \subset X$ such that both A and $X \setminus A$ are not in $\mathcal{U}$. Then the collection

$$\mathcal{U}' := \mathcal{U} \cup \{A \cap E : E \in \mathcal{U}\}$$

is a filter which contains $\mathcal{U}$ by Exercise 1.2.5. This makes a contradiction because $\mathcal{U}$ is an ultrafilter.
Conversely, suppose that $\mathcal{U}$ is a filter which satisfies the property in above. If $\mathcal{U}$ is not maximal, then there exists an ultrafilter $\mathcal{U}' \supset \mathcal{U}$ by Proposition 1.2.4. We are going to finish the proof by showing that $\mathcal{U} = \mathcal{U}'$.
Fix $A \in \mathcal{U}'$. By the property of $\mathcal{U}$, either A or $X \setminus A$ is contained in $\mathcal{U}$. In here, if $X \setminus A \in \mathcal{U}$ then since $\mathcal{U} \subset \mathcal{U}'$, we have $X \setminus A \in \mathcal{U}'$ hence

$$\varnothing = A \cap (X \setminus A) \in \mathcal{U}'$$

because $\mathcal{U}'$ is a filter, but this is nonsense. This derives to have $A \in \mathcal{U}$, therefore $\mathcal{U}' \subset \mathcal{U}$. This finishes the proof. □

Exercise 1.2.5*
Prove that the collection $\mathcal{U}'$ defined in Proposition 1.2.4 is a filter.

Remark 1.2.6
If $\mathcal{U}$ is an ultrafilter in X for some nonempty set X, clearly both A and $X \setminus A$ do not lie in $\mathcal{U}$ for any $A \subset X$, otherwise by the definition of the filter,

$$\varnothing = A \cap (X \setminus A) \in \mathcal{U},$$

which is a contradiction.

We have learned what is a filter and what is an ultrafilter. To deal with filters, we need to know how we can make some appropriate filter. If X be a nonempty set and $\mathcal{C}$ a some collection of subsets of X, then we can generate a filter $\mathcal{F}$ in X which contains $\mathcal{C}$.

Definition 1.2.7 (Finite intersection property)
Let X be a nonempty set. A collection $\mathcal{C}$ of subsets of X has a *finite intersection property* if for any finite set $\mathcal{F} \subset \mathcal{C}$, the intersection $\bigcap \mathcal{F}$ is not empty. We also use the abbriviation FIP for the term “finite intersection property”.

Proposition 1.2.8
Let X be a nonempty set and $\mathcal{C}$ a collection of subsets of X that has FIP. Then there exists a filter $\mathcal{F} \supset \mathcal{C}$ in X.

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proof. We can find one of such filter explicitly:

$$\mathcal{F} = \left\{ E \subset X : E \supset \bigcap \mathcal{D},\ \mathcal{D} \text{ a finite subset of } \mathcal{C} \right\}.$$

Since $\mathcal{C}$ has FIP, the emptyset does not contained in $\mathcal{F}$. It is clear that both $E \subset \mathcal{F}$ and $F \supset E$ imply $F \in \mathcal{F}$ by the definition of $\mathcal{F}$.
Fix $E_1, E_2 \in \mathcal{F}$ and take finite subsets $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{C}$ such that $\bigcap \mathcal{D}_1 \subset E_1$ and $\bigcap \mathcal{D}_2 \subset E_2$, respectively. By the basic set theory knowledge, we obtain $E_1 \cap E_2 \supset \bigcap(\mathcal{D}_1 \cup \mathcal{D}_2)$ and notice that $\mathcal{D}_1 \cup \mathcal{D}_2$ is a finite subset of $\mathcal{C}$, implying that $E_1 \cap E_2 \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a filter. It is clear by definition that $\mathcal{F} \supset \mathcal{C}$. $\square$

After this definition, we will see why sequences can replaced into filters in the general topology theories. This definition explains the way how filters are transformed by functions.

Definition 1.2.9 (Pushforward and pullback of filters)
Let X and Y be nonempty sets and $f : X \to Y$ a function. If $\mathcal{F}$ is a filter in X, the filter

$$f_*\mathcal{F} := \{E \supset f[F] : F \in \mathcal{F}\}$$

is called a *pushforward of a filter* $\mathcal{F}$.
If f is surjective and $\mathcal{G}$ is a filter in Y, the filter

$$f^*\mathcal{G} := \{E \supset f^{-1}[F] : F \in \mathcal{G}\}$$

is called a *pullback of a filter* $\mathcal{G}$.

Exercise 1.2.10
Prove that $f_*\mathcal{F}$ and $f^*\mathcal{G}$ defined in Definition 1.2.9 are really a filter, respectively.

The reader should check that Definition 1.2.9 makes sense (see Exercise 1.2.2).

Remark 1.2.11
If the surjectivity is dropped, then $f^*\mathcal{G}$ might not be a filter.

The pushforward of a filter has a remarkable property: every pushforward of ultrafilters is again an ultrafilter. Notice that such property does not hold for pullbacks in general.

Theorem 1.2.12
Let X, Y be nonempty sets, $f : X \to Y$ a function, and $\mathcal{U}$ an ultrafilter in X. Then $f_*\mathcal{U}$ is an ultrafilter in Y.

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proof. Since $f_*\mathcal{U}$ is a filter, it suffices that it satisfies the sufficiency condition of Proposition 1.2.4. Let $A \subset Y$ be fixed. Since $\mathcal{U}$ is an ultrafilter, by Proposition 1.2.4, either $f^{-1}[A] \in \mathcal{U}$ or $X \setminus f^{-1}[A] \in \mathcal{U}$ holds. As $X \setminus f^{-1}[A] = f^{-1}[Y \setminus A]$, we can assume that $f^{-1}[A] \in \mathcal{U}$. In this case, since $A \supset f[f^{-1}[A]]$ and $f[f^{-1}[A]] \in f_*\mathcal{U}$, we therefore have

$A \in f_*\mathcal{U}$. The proof of this theorem is now completed by applying Proposition 1.2.4. □

Now let X be a topological space. Then we can define a convergence of filters. Here is the definition.

Definition 1.2.13 (Convergence of filters)
Let X be a topological space. A filter $\mathcal{F}$ in X *converges* to $x \in X$ if every open neighborhood of x belongs to $\mathcal{F}$. In this case, we write $\mathcal{F} \to x$.

Remark 1.2.14
One can ask about that if $x \in X$ is fixed then does there exist a filter converges to x. Let $\mathcal{N}$ be a collection of all open neighborhoods of x. Then clearly $\mathcal{N}$ satisfies FIP. Therefore, we can apply Proposition 1.2.8 to take a filter $\mathcal{F} \supset \mathcal{N}$. Then by the definition of the convergence of filters, one can easily find that $\mathcal{F} \to x$.

We are ready to see that why filters can be used instead of sequences in arbitrary topological spaces.

Theorem 1.2.15
Let X, Y be topological spaces. Then

(a) For any nonempty set $A \subset X$, $x \in \overline{A}$ if and only if there is a filter $\mathcal{F}$ such that $A \in \mathcal{F}$ and $\mathcal{F} \to x$.

(b) Let $f : X \to Y$ be a function. Then f is continuous at $x \in X$ if and only if for every filter $\mathcal{F}$ in X with $\mathcal{F} \to x$, $f_*\mathcal{F} \to f(x)$ holds.

- -

proof. (a) If $x \in \overline{A}$, then $N \cap A \neq \varnothing$ for every open neighborhood N of x. Now think

$$\mathcal{C} := \{A\} \cup \{N : N \text{ is an open neighborhood of } x\}.$$

Then notice that $\mathcal{C}$ satisfies FIP because every finite intersection of open sets is also open. Thus Proposition 1.2.8 gives a filter $\mathcal{F}$ that contains $\mathcal{C}$. Observe that by the definition of $\mathcal{C}$, $A \in \mathcal{F}$ and $\mathcal{F} \to x$.

Conversely, suppose that there is a point $x \in X$ and a filter $\mathcal{F}$ in X such that $A \in \mathcal{F}$ and $\mathcal{F} \to x$. In order to show $x \in \overline{A}$, it suffices to show that $N \cap A$ is not empty for every open neighborhood N of x. But notice that $\mathcal{F} \to x$ implies that every open neighborhood of x belongs to $\mathcal{F}$. Since $A \in \mathcal{F}$, intersection of N and A cannot be empty by the definition of a filter. This proves $x \in \overline{A}$.

(b) Let f be continuous at $x \in X$ and $\mathcal{F}$ a filter in X that converges to x. Fix any open neighborhood N of $f(x)$. Being continuous, there exists an open set U containing x such that $f[U] \subset N$. Notice that $U \in \mathcal{F}$ because $\mathcal{F}$ converges to x. Then we have

$$f[U] \in f_*\mathcal{F} \quad \Longrightarrow \quad N \in f_*\mathcal{F},$$

proving that $f_*\mathcal{F} \to f(x)$.

Conversely, suppose that f is not continuous at x. Then there exists an open set $W \ni f(x)$ such that no open neighborhood of U of x satisfies $f[U] \subset W$. Then define $\mathcal{F}$ by

$$\mathcal{F} := \{E \subset X : E \supset U,\ U \text{ is an open neighborhood of } x\}.$$

It is easy to check that $\mathcal{F}$ is a filter in X. Also by its definition, $\mathcal{F}$ converges to x. However, to be $f_*\mathcal{F} \to f(x)$, at least W must lie in $f_*\mathcal{F}$. In other words, there must exist $E \in \mathcal{F}$ such that $f[E] \subset W$, but in this case, there must exist an open neighborhood U of x that is contained in E and this gives $f[U] \subset W$, a contradiction. This proves that the filter $f_*\mathcal{F}$ does not converges to $f(x)$. □

If X is a metric space and $A \subset X$, then we know that $x \in \overline{A}$ if and only if there exists a sequence $(x_n)_n$ in A that converges to x as discussed in the introduction of this section. In Theorem 1.2.15, $A \in \mathcal{F}$ corresponds to the sentence $(x_n)_n \subset A$, and clearly $\mathcal{F} \to x$ means $x_n \to x$ as $n \to \infty$. In addition, statement Theorem 1.2.15 (ii) is quite clear to explain whether a function is continuous at x by considering the assertion related to a sequence.

Furthermore, note that on a Hausdorff space, every sequence converges at most one point. This is also true for filters, and we can say more.

Theorem 1.2.16
Let X be a topological space. Then X is Hausdorff if and only if every filter converges at most one point. That is, if a filter $\mathcal{F}$ in X converges in both $x \in X$ and $y \in X$, then $x = y$.

- -

proof. Let X be Hausdorff and assume that there exists a filter $\mathcal{F}$ converges to different points x and y. Take two disjoint open sets U and V such that $x \in U$ and $y \in V$. Since $\mathcal{F} \to x$, $U \in \mathcal{F}$ is true. Same for $V \in \mathcal{F}$ since $y \in \mathcal{F}$. However, in this case, $\varnothing \in U \cap V \in \mathcal{F}$ must be true, which is a contradiction. Hence, there is no such filter if X is Hausdorff.
Conversely, if X is not Hausdorff, then we can take two distict points x and y such that there is no pair (U, V) of open sets such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. Then define

$$\mathcal{F} := \{E \supset F \cap G : F \in \mathcal{N}_x,\ G \in \mathcal{N}_y\}$$

where $\mathcal{N}_x$, $\mathcal{N}_y$ is a collection of all open neighborhoods of x and y, respectively. By the choice of x and y, $\mathcal{F}$ does not have an emptyset as its element, so it is easy to show that $\mathcal{F}$ is a filter. Notice by the definition of $\mathcal{F}$ that $\mathcal{F} \to x$ and $\mathcal{F} \to y$. This proves the theorem. □

1.3 Compact Sets

In this section, we will learn about compact sets in arbitrary topological spaces.

Definition 1.3.1 (Compact sets)
Let X be a topological space. Then a set X is said to be *compact* if for every collection $\mathscr{U}$ of open sets which cover X, we can find a finite subcollection $\mathscr{F} \subset \mathscr{U}$ such that $\bigcup \mathscr{F} \supset X$. The collection of type $\mathscr{U}$ described in above is called an *open cover of* X. A subset $K \subset X$ is said to be compact if K is compact within the subspace topology. A set is said to be *precompact* if the closure of which is compact.

We can show that the space is compact by using filters. Here is the theorem about.

Theorem 1.3.2
Let X be a topological space. Then X is compact if and only if every ultrafilter in X converges.

- -

proof. Suppose that there is an ultrafilter $\mathcal{U}$ that converges nowhere in X. For each $x \in X$, take an open set U_x which is not contained in $\mathcal{U}$. Note that clearly $\{U_x\}_{x\in X}$ is an open cover of X. If X were compact, there are finitely many points $x_1, \cdots, x_n$ such that

$$U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n} \supset X. \tag{1.1}$$

However as $\mathcal{U}$ is an ultrafilter, $X \setminus U_x \in \mathcal{U}$ holds for each $x \in X$ by Proposition 1.2.4. This and eq. (1.1) gives

$$\varnothing = (X \setminus U_{x_1}) \cap (X \setminus U_{x_2}) \cap \cdots \cap (X \setminus U_{x_n}) \in \mathcal{U},$$

which is a contradiction. Such contradiction occurs as we assumed that X is compact. Hence X cannot be compact.
Conversely, suppose that X is not compact. Then there is an open cover $\mathscr{U}$ which does not have a finite subset of $\mathscr{U}$ that covers X. Put

$$\mathscr{V} := \{X \setminus E : E \in \mathscr{U}\}.$$

Observe that $\mathscr{V}$ satisfies FIP. Take an ultrafilter $\mathcal{U}$ which is a superset of $\mathscr{V}$ by Proposition 1.2.2 and Proposition 1.2.8. Since $\mathscr{V} \subset \mathcal{U}$ and $\mathscr{U}$ is an open cover of X, there is no $x \in X$ such that $\mathcal{U} \to x$. Indeed, if $x \in X$ then there is $U \in \mathscr{U}$ that contains x. Since $X \setminus U \in \mathcal{U}$, U does not belong to $\mathcal{U}$ and this implies that $\mathcal{U}$ does not converges to x. $\square$

Corollary 1.3.3
Let X be a topological space. Then $\varnothing \neq K \subset X$ is compact if and only if every ultrafilter $\mathcal{U}$ in X containing K converges to some point $x \in K$.

- -

proof. If $K \subset X$ is compact and fix an ultrafilter $\mathcal{U}$ that contains K. Then the set

$$\mathcal{U}' := \{F \cap K : F \in \mathcal{U}\} \tag{1.2}$$

is an ultrafilter in K. Being compact, $\mathcal{U}' \to x \in K$ by Theorem 1.3.2. Now fix an open set N that contains x. Since $N \cap K$ is open in K and $x \in N \cap K$, we have $N \cap K \in \mathcal{U}'$. This implies $N \in \mathcal{U}$ by the definition of $\mathcal{U}'$, proving that $\mathcal{U} \to x$ since N is arbitrary.
To prove the sufficency, fix an ultrafilter $\mathcal{U}$ in X such that $\mathcal{U} \ni K$ and $\mathcal{U} \to x \in K$. Notice that the ultrafilter $\mathcal{U}'$ defined in eq. (1.2) also converges to x in K. Indeed, if N is a neighborhood of x in K then we can take an open set U in X satisfies $N = U \cap K$. Since $U \in \mathcal{U}$, we have $N = U \cap K \in \mathcal{U}'$. This proves the corollary. $\square$

Below propositions can be proved without using any filter theory. In here, however, we will see how Theorem 1.3.2 (and Corollary 1.3.3) can be used to prove several compactness properties. One may solve Exercise 1.3.5.

Proposition 1.3.4
Let X be a topological space and $K \subset X$ be a nonempty set.

(a) If X is Hausdorff and K is compact, then K is closed.

(b) If X is compact and K is closed, then K is compact.

(c) Let Y be another topological space and $f : X \to Y$ a continuous map, then $f[K]$ is compact in Y whenever K is compact in X.

- -

proof. (a) Fix $x \in \overline{K}$ and take a filter $\mathcal{F}$ in X such that $K \in \mathcal{F}$ and $\mathcal{F} \to x$ by Theorem 1.2.15. Then take an ultrafilter $\mathcal{U} \supset \mathcal{F}$ by Proposition 1.2.2, and finally, consider

$$\mathcal{U}' := \{F \cap K : F \in \mathcal{U}\}.$$

Since $\mathcal{U}'$ is an ultrafilter in K, there is a point $y \in K$ such that $\mathcal{U}' \to y$ by Corollary 1.3.3. Fix any open set N in X that contains y. Then $N \cap K$ is open in K, we have $N \cap K \in \mathcal{U}'$ as $\mathcal{U}' \to y$. Then the definition of $\mathcal{U}'$ and Proposition 1.2.4 imply $N \in \mathcal{U}$ (otherwise, $X \setminus N$ must be in $\mathcal{U}$. This makes a contradiction), hence $\mathcal{U} \to y$.

What we have is that the ultrafilter $\mathcal{U}$ converges in both x and y where $x \in \overline{K}$ and $y \in K$. Since X is Hausdorff, $x = y$ by Theorem 1.2.16, proving that $x = y \in K$. Hence, K is closed because $K = \overline{K}$ which is what we proved in here.

(b) Fix an ultrafilter $\mathcal{U}$ in X which contains K. Being compact, $\mathcal{U}$ converges to some point $x \in X$. By Theorem 1.2.15, $x \in \overline{K} = K$. Hence, K is compact because of Corollary 1.3.3 as $\mathcal{U}$ is arbitrary.

(c) Fix an ultrafilter $\mathcal{U}$ in $f[K]$. Then take an ultrafilter $\mathcal{U}'$ in K such that $f^*\mathcal{U} \subset \mathcal{U}'$. Being compact, we can take a point $x \in K$ such that $\mathcal{U}' \to x$ by Theorem 1.3.2. Since f is continuous, $f_*\mathcal{U}' \to f(x)$ because of Theorem 1.2.15. Observe that $\mathcal{U} \subset f_*\mathcal{U}'$, hence both should be same since $\mathcal{U}$ is an ultrafilter. Therefore, $\mathcal{U}$ converges to $f(x)$, and we proved that $f[K]$ is compact by Theorem 1.3.2 as $\mathcal{U}$ is arbitrary. □

Exercise 1.3.5
Prove Proposition 1.3.4 *without using any filter theories*.

We end this section by proving the Tychonoff's theorem. It can be proved with several ways, but in here, obviously, we will use filters to prove it.

Theorem 1.3.6 (Tychonoff's Theorem)
Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of compact topological spaces where A is an index set. Then the space $X = \prod_{\alpha \in A} X_\alpha$ equipped with the product topology is compact.

- -

proof. Let us define a projection $\pi_\alpha : X \to X_\alpha$ for each $\alpha \in A$, and fix an ultrafilter $\mathcal{U}$ in X. By Theorem 1.2.12, $(\pi_\alpha)_*\mathcal{U}$ is an ultrafilter in X_α, so we can take a point $x_\alpha \in X_\alpha$ such that $(\pi_\alpha)_*\mathcal{U} \to x_\alpha$. Such points exists because X_α is compact and recall Theorem 1.3.2. Finally, take the point $x \in X$ such that $\pi_\alpha(x) = x_\alpha$ for every $\alpha \in A$.

Claim that $\mathcal{U} \to x$.
To prove this, fix an open neighborhood N of x in X. Since

$$\{\pi_\alpha^{-1}[U] : \alpha \in A,\ U \in X_\alpha\}$$

forms a subbasis of X, there exists $\alpha_1, \cdots, \alpha_n \in A$, and $U_{\alpha_1}, \cdots, U_{\alpha_n}$ are open sets in X_{α_j}, $j = 1, 2, \cdots, n$, respectively, such that

$$x \in \pi_{\alpha_1}^{-1}[U_{\alpha_1}] \cap \pi_{\alpha_2}^{-1}[U_{\alpha_2}] \cap \cdots \cap \pi_{\alpha_n}^{-1}[U_{\alpha_n}] \subset N.$$

Notice that $x_{\alpha_j} \in U_{\alpha_j}$ for $j = 1, 2, \cdots, n$. This gives that $U_{\alpha_j} \in (\pi_\alpha)_* \mathcal{U}$, hence

$$\pi_{\alpha_j}^{-1}[U_{\alpha_j}] \in \pi_\alpha^*(\pi_\alpha)_* \mathcal{U} \subset \mathcal{U}.$$

Thus,

$$\pi_{\alpha_1}^{-1}[U_{\alpha_1}] \cap \pi_{\alpha_2}^{-1}[U_{\alpha_2}] \cap \cdots \cap \pi_{\alpha_n}^{-1}[U_{\alpha_n}] \in \mathcal{U} \quad \Longrightarrow \quad N \in \mathcal{U}.$$

Since N is arbitrary open neighborhood of x, we have proved that $\mathcal{U} \to x$. Therefore, X is compact because of Theorem 1.3.2. $\square$

Chapter 2

Measures and Integrations

2.1 Introduction

What is the definition of the "area" of some figures? In the *real* kindergarten, the reader is taught about the area of a rectangle is calculated by a multiple of the width and the height of the rectangle. In the high school, the reader learn how to calculate the circle by approximating areas of rectangles which fill the whole circle from inside. Generally speaking, one can infer that every figure has an area and it can be calculated with similar mannar.

In our intuition, a length —an area in $\mathbb{R}$— is translation invariant, and it should be in our real life. Let us denote ℓ a length of sets in $\mathbb{R}$ and assume that we can take a length from arbitrary sets in $\mathbb{R}$. For instance, $\ell([a,b]) = b - a$ and such definition comes from our intuition.

2.2 Measures and measurable sets

In this section, we will learn about measures and their properties. First two definitions introduce what is a measure and what is a measurable set.

Definition 2.2.1 (Measures and measure spaces)
Let X be a set. A *measure* is a function $\mu : \mathcal{P}(X) \to [0,\infty]$ such that

(i) $\mu(\varnothing) = 0$;

(ii) $\mu(A) \le \mu(B)$ if $A \subset B$;

(iii) $\mu(\bigcup_{j=1}^{\infty} A_j) \le \sum_{j=1}^{\infty} \mu(A_j)$.

We say the pair (X,μ) a *measure space*.

Definition 2.2.2 (Measurable sets)
Let (X,μ) be a measure space. A set $A \subset X$ is called μ-*measurable* if for every set $E \subset X$, the following relation holds.

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A).$$

Define $\mathfrak{M}_\mu$ the collection of all μ-measurable sets.

Remark 2.2.3
Actually, upper definition is so called an *outer measure* in several references, and reserve the term *measure* for different manner. Furthermore, the term *measure space* stands

for a triple $(X, \mathfrak{M}, \mu)$ where $\mathfrak{M}$ is so-called σ-field (definition of which is introduced in Definition 2.2.5) and μ a measure (not an outer measure). Although there is some confusion while reading this textbook, it has some benefits when we define terms *measure* and *measure space* by Definition 2.2.1. So reader sould beware from misunderstanding.

So far we have defined a measure μ and the collection of μ-measurable sets $\mathfrak{M}_\mu$ on a set X. Below theorem gives a property of the collection $\mathfrak{M}_\mu$.

Theorem 2.2.4 (Carathéodory's Theorem)
Let X be a set, μ a measure in X. Then

(i) $\varnothing, X \in \mathfrak{M}_\mu$.

(ii) If $A \in \mathfrak{M}_\mu$, then $X \setminus A \in \mathfrak{M}_\mu$.

(iii) If $\mathscr{F} \subset \mathfrak{M}_\mu$ is a *mutually disjoint* countable subcollection, then $\bigcup \mathscr{F} \in \mathfrak{M}_\mu$. Furthermore, we have

$$\mu\left(\bigcup \mathscr{F}\right) = \sum_{F \in \mathscr{F}} \mu(F).$$

(iv) If $A \in \mathfrak{M}_\mu$ such that $\mu(A) = 0$, then $B \in \mathfrak{M}_\mu$ for every $B \subset A$.

- -

proof. (i) is obvious. If $A \in \mathfrak{M}_\mu$, then for every $E \subset X$, we have

$$\mu(E \cap (X \setminus A)) + \mu(E \setminus (X \setminus A)) = \mu(E \setminus A) + \mu(E \cap A) = \mu(E),$$

so $X \setminus A \in \mathfrak{M}_\mu$, so we proved (ii).

There is a term for a subcollection $\mathfrak{M} \subset \mathcal{P}(X)$ satisfies (i), (ii), and the first half of (iii) in Theorem 2.2.4.

Definition 2.2.5
A subcollection $\mathfrak{M} \subset \mathbb{P}(X)$ is called a *σ-field* if

(i) $\varnothing, X \in \mathfrak{M}_\mu$.

(ii) If $A \in \mathfrak{M}_\mu$, then $X \setminus A \in \mathfrak{M}_\mu$.

(iii) If $\mathscr{F} \subset \mathfrak{M}_\mu$ is a countable subcollection, then $\bigcup \mathscr{F} \in \mathfrak{M}_\mu$. Notice that $\mathscr{F}$ need not to be mutually disjoint.

If the condition *countable* is replaced by *finite* in (iii), then the collection $\mathfrak{M}$ is called by a *field*.

Remark 2.2.6
Terms *field* and *σ-field* are usually called by probabilists. In other people who study analysis, *algebra* and *σ-algebra* is used, repectively.
However, personally, the former terms is the *right* term in the sense that *fields* (resp. *σ-fields*) are closed under all set theoretical operations $\cup$, $\cap$, $\setminus$, and the complement $\cdot^c$.

2.3 Integrations

Now we are ready to define a Lebesgue integration. Everywhere in this section, (X, μ) denotes a measure space.

Definition 2.3.1 (Lebesgue Integration)
Let $f : X \to \mathbb{R}$ be a function such that one of those Riemann integrations

$$\int_0^\infty \mu\{f_+ > t\}dt, \quad \int_0^\infty \mu\{f_- > t\}dt$$

exists and finite. Then we define a *Lebesgue integration* respect to μ by

$$\int_X f(x)\mu(dx) := \int_0^\infty \mu\{f_+ > t\}dt - \int_0^\infty \mu\{f_- > t\}dt.$$

If $E \subset X$, then we define

$$\int_E f(x)\mu(dx) := \int_X 1_E(x)f(x)\mu(dx).$$

Sometimes, we use the abbreviation of $\int_E f(x)\mu(dx)$ by $\int_E f d\mu$ if the variable of which is understood.

Remark 2.3.2
We need to justify Definition 2.3.1. For any $0 \le s \le t$,

$$\{f_+ > t\} \subset \{f_+ > s\}, \quad \{f_- > t\} \subset \{f_- > s\}.$$

Since μ is a measure, $t \mapsto \mu\{f_+ > t\}$ and $t \mapsto \mu\{f_- > t\}$ are decreasing functions, so they are Riemann integrable on $[\epsilon, N]$ for every $0 < \epsilon < N < \infty$. Now we are understood Riemann integrations described in Definition 2.3.1 by an improper integral, so

$$\int_0^\infty \mu\{f_+ > t\}dt = \lim_{\substack{N\uparrow\infty \\ \epsilon\downarrow 0}} \int_\epsilon^N \mu\{f_+ > t\}dt,$$

and similar for f_-.

Notice that if f is nonnegative, then $\mu\{f_- > t\} = 0$ for every $t > 0$. Thus as we discuss the Riemann integration of $\mu\{f_- > t\}$ in Remark 2.3.2, one can obtain

$$\int_X f(x)\mu(dx) = \int_0^\infty \mu\{f > t\}dt.$$

Although we define a Lebesgue integral for arbitrary function f, we need some "good" functions to obtain more useful properties for Lebesgue integral. Below definition says about what is a "good" function.

Definition 2.3.3
Let Y be a topological space and $f : X \to Y$ a function. We say that f is *μ-measurable* if $f^{-1}[E]$ is μ-measurable for every Borel set $E \subset Y$.

If $f : X \to \mathbb{R}$ is μ-measurable, then clearly both $\{f_+ > t\}$ and $\{f_- > t\}$ are μ-measurable for every $t \ge 0$. Therefore we have the following important facts for Lebesgue integral.

Theorem 2.3.4

(i) If $f \le g$, then $\int_X f d\mu \le \int_X g d\mu$.

(ii) Let $f : X \to \mathbb{R}$ be a μ-measurable function. Then

$$\left| \int_X f(x)\mu(dx) \right| \le \int_X |f(x)|\mu(dx),$$

(iii) If $A_1, A_2, \cdots, A_n \in \mathfrak{M}_\mu$, and $c_1, \cdots, c_n \ge 0$,

$$\int_X \sum_{j=1}^n c_j 1_{A_j} \mu(dx) = \sum_{j=1}^n c_j \mu(A_j).$$

Now we are going to study very important limit theorems.

Theorem 2.3.5

(i) (Fatou's lemma) Let f_n are nonnegative μ-measurable functions. Then

$$\int_X \varliminf_{n\to\infty} f_n(x)\mu(dx) \le \varliminf_{n\to\infty} \int_X f_n(x)\mu(dx).$$

(ii) (Monotone convergence theorem) Let f, f_n are μ-measurable functions such that $0 \le f_1 \le f_2 \le \cdots \le f$ and $f_n \uparrow f$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_X f_n(x)\mu(dx) = \int_X f(x)\mu(dx).$$

(iii) (Dominated convergence theorem) Let f, f_n are μ-measurable functions such that $f_n \to f$ as $n \to \infty$ and $|f_n| \le g$ for some μ-measurable function g such that $\int_X g d\mu < \infty$. Then

$$\lim_{n\to\infty} \int_X f_n(x)\mu(dx) = \int_X f(x)\mu(dx).$$

There is the generalized version of Theorem 2.3.5 (iii).

Exercise 2.3.6*

Let f, f_n, g, g_n are μ-measurable functions such that $f_n \to f$, $g_n \to g$ as $n \to \infty$, $|f_n| \le g_n$ for every n, and $\int_X g_n d\mu \to \int_X g d\mu < \infty$ as $n \to \infty$. Then the below relation holds.

$$\lim_{n\to\infty} \int_X f_n(x)\mu(dx) = \int_X f(x)\mu(dx).$$