AApproach Seminar

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1 Introduction

This paper is for personal purpose to study [10].

We want to solve the following SPDE

$$du = (Lu + f)dt + (\Lambda^k u + g^k)dw_t^k, \quad t > 0, \tag{1}$$

where

$$Lu = a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu, \quad \Lambda^k u = \sigma^{ik}u_{x^i} + \nu^k u.$$

Recall that we are using the summation convention.

2 Generalities

In this section, $\mathbb{R}^d$ denotes a d-dimensional Euclidean space of points $x = (x^1, \cdots, x^d)$. By a distribution or a generalized function on $\mathbb{R}^d$ we mean an element of the space $\mathcal{D}$ of real-valued Schwartz distributions defined on C_0^∞, where $C_0^\infty = C_0^\infty(\mathbb{R}^d)$ is the set of all infinitely differentiable functions with compact support.

2.1 Definition of H_p^n

We fix $p \in (1,\infty)$ and $n \in \mathbb{R}$. We first define a space $H_p^n = H_p^n(\mathbb{R}^d)$, the Bessel potential space, which is defined by

$$H_p^n(\mathbb{R}^d) := \{u \in \mathcal{D} : \exists f \in L_p = L_p(\mathbb{R}^d),\ (1-\Delta)^{-n/2} f = u\},$$

and the norm on H_p^n by

$$\|u\|_{n,p} := \|(1-\Delta)^{n/2}u\|_p,$$

where $\|\cdot\|_p$ is the norm in L_p.

The meaning of $(1-\Delta)^{n/2}$ is already discussed before by defining

$$(1-\Delta)^{n/2} f := [(1+|\xi|^2)^{n/2}\hat{f}(\xi)]^\vee.$$

However in this paper, we will introduce another definition by using semigroup. Define a family of operators T_t in L_p for $t \in [0,\infty)$ by

$$T_0 \equiv I, \quad T_t f(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{-\frac{1}{4t}|x-y|^2} dy =: \int_{\mathbb{R}^d} f(y) p_t(x-y) dy. \tag{2}$$

Then T_t forms a strongly continuous semigroup. In addition, by the Young's inequality, $\|T_t\|_{L_p \to L_p} \le 1$ holds. Then the generator A of the semigroup $\{e^{-t}T_t\}_t$ satisfies

$$\|(tI - A)^{-1}\|_{L_p \to L_p} \le \frac{1}{t+1},$$

and thus

$$A^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^z e^{-t} T_t \frac{dt}{t}, \quad \operatorname{Re} z > 0$$

by Theorem 13.1 and Theorem 14.10 of [5]. On the other hand, notice that $A^{-1} = (1-\Delta)^{-1}$ (see [12]). Therefore we can *define* $(1-\Delta)^{-\alpha}$ for $\alpha > 0$ by

$$(1-\Delta)^{-\alpha}u := \frac{1}{\Gamma(\alpha)}\int_0^\infty t^\alpha e^{-t}T_t u\frac{dt}{t}, \quad \forall u \in C_0^\infty. \tag{3}$$

In addition by [26], we can also *define* $(1-\Delta)^\alpha$ for $\alpha \in (0,1)$ by

$$(1-\Delta)^{\alpha}u := \frac{1}{\Gamma(-\alpha)}\int_0^\infty \frac{e^{-t}T_t u - u}{t^\alpha}\frac{dt}{t}, \quad \forall u \in C_0^\infty. \tag{4}$$

It turns out that (see [5]) that formulas eq. (3) and eq. (4) are sufficient to consistently define $(1-\Delta)^{n/2}$ for any $n \in (-\infty, \infty)$. The result of application of $(1-\Delta)^{n/2}$ to an $f \in L_p$ is defined as a limit of truncated integrals in eq. (3) and eq. (4).

It is known that H_p^n is a Banach space with norm $\|\cdot\|_{n,p}$ and C_0^∞ is dense in H_p^n (see, for instance, [23], [24]).

2.2 Definition of $H_p^{1,2}(T)$ and solvability of deterministic PDE

Next, for fixed T (can $T = \infty$?) one introduces the space $H_p^{1,2}(T) = H_p^{1,2}((0,T)\times\mathbb{R}^d)$ as

$$\{u = u(t,x) \,:\, \|u\|_{1,2,p}^p := \int_0^T \left\|\frac{\partial u}{\partial t}(t,\cdot)\right\|_p^p dt + \int_0^T \|u(t,\cdot)\|_{2,p}^p dt < \infty\}.$$

The norm $\|\cdot\|_{1,2,p}$ makes $H_p^{1,2}(T)$ a Banach space. Indeed, let $(u_n)_n$ be a Cauchy sequence in $H_p^{1,2}(T)$. Because of the definition of $\|\cdot\|_{1,2,p}^p$, each u_n belongs to $L_p(T) = L_p((0,T)\times\mathbb{R}^d)$ and there exists $u \in L_p(T)$ such that $u_n \to u$ in $L_p(T)$ as $n \to \infty$. In the sense of distributions, one can easily prove that, for instance, $D_t u$ exists and $D_t u_n$ converges weakly to $D_t u$ by the Hölder's inequality. Since $(D_t u_n)_n$ is also a Cauchy sequence in $L_p(T)$, we can easily prove that there exists $v \in L_p(T)$ such that $v = D_t u$ in the sense of distributions. Similar work can be done for u_{nx} and u_{nxx}, proving that the space $H_p^{1,2}(T)$ is Banach.

Before dive into the SPDE theory, we want to discuss the deterministic counterpart of eq. (1)

$$\frac{\partial u}{\partial t} = Lu + f \tag{5}$$

with zero initial condition. We can prove the existence and uniqueness of the equation in the following way. First, for the simplest equation

$$\frac{\partial u}{\partial t} = \Delta u + f, \tag{6}$$

its solvability in $H_p^{1,2}(T)$ is proved by means of explicit formulas and some estimates of heat potentials, provided that $f \in L_p(T)$. Below theorem is proved in [15] but the spaces in the theorem stated is are different. For details, see the appendix.

Theorem 2.2.1 For any $f \in L_p(T)$ and $u_0 \in H_p^{2-2/p}$[(i)] there exists a unique solution $u \in H_p^{1,2}(T)$ of the heat equation eq. (6) with initial data $u(0) = u_0$. In addition,

$$\|u_{xx}\|_{L_p(T)} + \left\|\frac{\partial u}{\partial t}\right\|_{L_p(T)} \le N(d,p)(\|f\|_{L_p(T)} + \|u_0\|_{2-2/p,p}), \tag{7}$$

$$\|u\|_{1,2,p} \le N(d,p,T)(\|f\|_{L_p(T)} + \|u_0\|_{2-2/p,p}),$$

where u_{xx} is the matrix of second-order derivatives of u with respect to x.

(i) Actually, this is wrong. See appendix.

This theorem yields a bounded operator $\mathcal{R}_1$ which maps any $f \in L_p(T)$ into the solution $u \in H_p^{1,2}(T)$ of the heat equation eq. (6) with zero initial data.

Then, the so-called *a priori estimate* is obtained for eq. (7). One assumes that there is a solution $u \in H_p^{1,2}(T)$ of eq. (6) with zero initial condition and inequality eq. (7) is proved, where N is a constant probably depending on T and some characteristics of L.

There are two central objects in the above argument. These are the Banach space $H_p^{1,2}(T)$ and the operator $L - \partial/\partial t : H_p^{1,2}(T) \to L_p((0,T)\times\mathbb{R}^d)$. Since we want to implement the same kind of argument for equations like eq. (1), the first thing to do is to find an appropriate counterpart of $H_p^{1,2}(T)$. However we cannot expect any differentiability property with respect to t for solutions u of eq. (1).[ii] Then an observation appeared that $H_p^{1,2}(T)$ can also be defined without using $\partial u/\partial t$. We want to check

$$H_p^{1,2}(T) = \{u : u(t,x) = u(0,x) + \int_0^t f(s,x)ds,\ u, u_x, u_{xx}, f \in L_p((0,T)\times\mathbb{R}^d)\}.$$

See the appendix for more information.

Now the guess is natural that a stochastic counterpart $\mathcal{H}_p^2(T)$ of the spaces $H_p^{1,2}(T)$ could be the space of functions $u = u(\omega, t, x)$ such that

$$u(t,x) = u(0,x) + \int_0^t f(s,x)ds + \sum_k \int_0^t g^k(s,x)dw_s^k, \tag{8}$$

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^d} [|u| + |u_x| + |u_{xx}| + |f|]^p dxdt < \infty,$$

and something of the same type is satisfied for $g = (g^k)$. It may look a little bit surprising that one needs $p \geq 2$ and

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^d} [|g| + |g_x|]^p dxdt < \infty,$$

which involves both g and g_x, where

$$|g|^2 := \sum_{k=1}^{\infty} |g^k|^2, \quad |g_x|^2 := \sum_{k=1}^{\infty} |g_x^k|^2.$$

With the spaces $\mathcal{H}_p^2(T)$ at hand, we write eq. (1) in an operator form by introducing the operator (L, Λ) which can be applied to *any* element $u \in \mathcal{H}_p^2(T)$. Namely for a $u \in \mathcal{H}_p^2(T)$ we write $(L,\Lambda)u = -(f, g)$ if and only if

$$u(t) = u(0) + \int_0^t [Lu + f](s)ds + \sum_k \int_0^t [\Lambda^k u + g^k](s)dw_s^k.$$

3 The Stochastic Banach Spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(\mathcal{F}_t, t \geq 0)$ beba an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all $\mathbb{P}$-null subsets of Ω, and $\mathcal{P}$ be the predictable σ-field generated by $(\mathcal{F}_t, t \geq 0)$. Let $\{w_t^k : k = 1, 2, \cdots\}$ be a family of independent one-dimensional $\mathcal{F}_t$-adapted Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

We already defined norms $\|\cdot\|_p$ and $\|\cdot\|_{n,p}$ for $\mathbb{R}$-valued functions/distributions. Let us define $L_p(\mathbb{R}^d, l_2)$ and $H_p^n(\mathbb{R}^d, l^2)$ with the norm

$$\|g\|_p := \||g|_{l_2}\|_p, \quad \|g\|_{n,p} := \||(1-\Delta)^{n/2} g|_{l_2}\|_p, \quad \forall g \in l_2,$$

[ii] The Wiener process is nowhere differentiable.

where l_2 is the set of all real-valued sequences $g = (g^k)_k$ with the norm defined by $|g|_{l_2}^2 := \sum_k |g^k|^2$.
Finally, for stopping times τ, we denote $(\!(0, \tau]\!] = \{(\omega, t) : 0 < t \le \tau(\omega)\}$

$$\mathbb{H}_p^n(\tau) := L_p((\!(0, \tau]\!], \mathcal{P}, H_p^n), \quad \mathbb{H}_p^n := \mathbb{H}_p^n(\infty),$$
$$\mathbb{H}_p^n(\tau, l_2) := L_p((\!(0, \tau]\!], \mathcal{P}, H_p^n(\mathbb{R}^d, l_2)), \quad \mathbb{L}_{\dots} \cdots := \mathbb{H}^0_{\dots} \dots.$$

The norms in these spaces are defined by the Bochner integral sense. By convention, elements of spaces like $\mathbb{H}_p^n$ are treated as functions rather than distributions or classes of equivalent functions, and if we know that a function of this class has a modification with better properties, then we always consider this modification. Also, elements of spaces $\mathbb{H}_p^n(\tau, l_2)$ need not be defined or belong to H_p^n for all $(\omega, t) \in (\!(0, \tau]\!]$. As usual, these properties are needed only for almost (ω, t).
For $n \in \mathbb{R}$ and

$$(f, g) \in \mathcal{F}_p^n(\tau) := \mathbb{H}_p^n(\tau) \times \mathbb{H}_p^{n+1}(\tau, l_2),$$

set

$$\|(f, g)\|_{\mathcal{F}_p^n(\tau)} := \|f\|_{\mathbb{H}_p^n(\tau)} + \|g\|_{\mathbb{H}_p^{n+1}(\tau, l_2)}.$$

Definition 3.0.1 For a $\mathcal{D}$-valued function $u \in \bigcap_{T>0} \mathbb{H}_p^n(\tau \wedge T)$, we write $u \in \mathcal{H}_p^n(\tau)$ if $u_{xx} \in \mathbb{H}_p^{n-2}(\tau)$, $u(0, \cdot) \in L_p(\Omega, \mathcal{F}_0, H_p^{n-2/p})$, and there exists $(f, g) \in \mathcal{F}_p^{n-2}(\tau)$ such that, for any $\phi \in C_0^\infty$, the equality

$$(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_{k=1}^\infty \int_0^t (g^k(s, \cdot), \phi) dw_s^k \tag{9}$$

holds for all $t \le \tau$ with probability 1. We also define $\mathcal{H}_{p,0}^n(\tau) := \mathcal{H}_p^n(\tau) \cap \{u : u(0, \cdot) = 0\}$,

$$\|u\|_{\mathcal{H}_p^n(\tau)} := \|u_{xx}\|_{\mathbb{H}_p^{n-2}(\tau)} + \|(f, g)\|_{\mathcal{F}_p^{n-2}(\tau)} + (\mathbb{E}\|u(0, \cdot)\|_{n-2/p,p}^p)^{1/p}. \tag{10}$$

As always, we drop τ in $\mathcal{H}_p^n(\tau)$ and $\mathcal{F}_p^n(\tau)$ if $\tau = \infty$.

Remark 3.0.2 First of all, for each test function ϕ, the (a.s.) set in eq. (9) may different (also it depends on p and n). Furthermore, as usual, $u(t, \cdot)$ should be interpreted as the trace sense.
Since $u \in \bigcap_{T>0} \mathbb{H}_p^n(\tau \wedge T)$, for each $\phi \in C_0^\infty$, by the Bochner integral theory, one can easily show that $(u(t, \cdot), \phi) \in \bigcap_{T>0} L_p((\!(0, \tau \wedge T]\!], \mathcal{P})$.

Remark 3.0.3 It is worth nothing that the elements of $\mathcal{H}_p^n(\tau)$ are assumed to be define for (ω, t) and take values in $\mathcal{D}$. Obviously, $\mathcal{H}_p^n(\tau)$ is a linear space. As usual, we identify two elements u_1 and u_2 of $\mathcal{H}_p^n(\tau)$ if $\|u_1 - u_2\|_{\mathcal{H}_p^n(\tau)} = 0$. Also, observe that the series of stochastic integrals in eq. (9) converges uniformly in t in probability on $[0, \tau \wedge T]$ for any finite T. Indeed, the quadratic variations of these stochastic integrals satisfy if $p = 2$,

$$\begin{aligned}
\sum_{k=1}^\infty \int_0^{\tau\wedge T} (g^k(s, \cdot), \phi)^2 ds &= \sum_{k=1}^\infty \int_0^{\tau\wedge T} ((1-\Delta)^{(n-1)/2} g^k(s, \cdot), (1-\Delta)^{(1-n)/2}\phi)_{L_2}^2 ds \\
&\le \|\phi\|_{1-n,2}^2 \sum_{k=1}^\infty \int_0^{\tau\wedge T} \int_{\mathbb{R}^d} |(1-\Delta)^{(n-1)/2} g^k(s, \cdot)|^2(x) dx ds \\
&= \|\phi\|_{1-n,2}^2 \int_0^{\tau\wedge T} \int_{\mathbb{R}^d} \sum_{k=1}^\infty |(1-\Delta)^{(n-1)/2} g^k(s, \cdot)|^2(x) dx ds \\
&= \|\phi\|_{1-n,2}^2 \int_0^{\tau\wedge T} \|(\sum_{k=1}^\infty |(1-\Delta)^{(n-1)/2} g^k(s, \cdot)|^2)^{1/2}\|_2^2 ds
\end{aligned}$$

and if $p > 2$,

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\int_0^{\tau\wedge T}(g^k(s,\cdot),\phi)^2 ds\\
&\qquad = \sum_{k=1}^{\infty}\int_0^{\tau\wedge T}(1-\Delta)^{(n-1)/2}g^k(s,\cdot),(1-\Delta)^{(1-n)/2}\phi)_{L_2}^2 ds\\
&\qquad \le \|\phi\|_{1-n,q}^q\sum_{k=1}^{\infty}\int_0^{\tau\wedge T}\int_{\mathbb{R}^d}|(1-\Delta)^{(n-1)/2}g^k(s,\cdot)|^2(x)|(1-\Delta)^{(1-n)/2}\phi|^{2-q}(x)dxds\\
&\qquad = \|\phi\|_{1-n,q}^q\int_0^{\tau\wedge T}\int_{\mathbb{R}^d}|(1-\Delta)^{(1-n)/2}\phi|^{2-q}(x)\left(\sum_{k=1}^{\infty}|(1-\Delta)^{(n-1)/2}g^k(s,\cdot)|^2(x)\right)dxds\\
&\qquad \le \|\phi\|_{1-n,q}^{q+1}\int_0^{\tau\wedge T}\|(\sum_{k=1}^{\infty}|(1-\Delta)^{(n-1)/2}g^k(s,\cdot)|^2)^{1/2}\|_p^2 ds
\end{aligned}
$$

where $q = p/(p-1)$. Since $g \in \mathbb{H}_p^{n-1}(\tau, l_2)$, both right hand sides are finite (a.s.). By the Doob-Kolmogorov inequality,

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{t\le\tau\wedge T}\left|\sum_{k=m}^{n}\int_0^t(g^k(s,\cdot),\phi)dw_s^k\right|^2\ge\epsilon\right\} &= \mathbb{P}\left\{\sup_{t\le T}\left|\sum_{k=m}^{n}\int_0^{t\wedge\tau}(g^k(s,\cdot),\phi)dw_s^k\right|^2\ge\epsilon\right\}\\
&\le \frac{1}{\epsilon}\mathbb{E}\left|\sum_{k=m}^{n}\int_0^T\mathbb{1}_{t\le\tau}(g^k(t,\cdot),\phi)dw_t^k\right|^2\\
&= \frac{1}{\epsilon}\mathbb{E}\sum_{k=m}^{n}\left|\int_0^T\mathbb{1}_{t\le\tau}(g^k(t,\cdot),\phi)dw_t^k\right|^2\\
&= \frac{1}{\epsilon}\mathbb{E}\sum_{k=m}^{n}\int_0^{\tau\wedge T}(g^k(t,\cdot),\phi)^2dt\\
&\to 0
\end{aligned}
$$

holds for every $\epsilon > 0$ as $m, n \to \infty$.[(iii)] This proves that the stochastic integral in eq. (9) converges uniformly in t in probability on $[0, \tau\wedge T]$.

As a consequence of the uniform convergence, $(u(t,\cdot),\phi)$ is continuous in t on $[0,\tau\wedge T]$ for any finite T (a.s.).

Remark 3.0.4 Actually, eq. (13) holds (a.s.) independent of $\phi \in C_0^\infty$. Fix $T \in (0,\infty)$ in this remark.

Define $q = p/(p-1)$, and $N = 2-n$ if $p < 4$ and $N = 2/p - n$ if $p \ge 4$. Notice that then $H_q^N \subset H_q^{2-n}\cap H_q^{2/p-n}$. For $h \in H_q^N$ and almost $\omega \in \Omega$ (independent of h),

$$
|(u(0,\cdot),h)| \le \|h\|_{2/p-n,q}\|u(0,\cdot)\|_{n-2/p,p},
$$

$$
\sup_{t\le\tau\wedge T}\left|\int_0^t(f(s,\cdot),h)ds\right| \le \|h\|_{2-n,q}\int_0^{\tau\wedge T}\|f(s,\cdot)\|_{n-2,p}ds. \tag{11}
$$

On the other hand, notice that u is H_p^n-valued function on $(\!(0,\tau\wedge T]\!]$. Thus for every $(\omega,t)\in(\!(0,\tau\wedge T]\!]$,

$$
|(u(t,\cdot),h)| \le \|u(t,\cdot)\|_{n,p}\|h\|_{-n,q}.
$$

In addition, for each $\phi \in C_0^\infty$, let E_ϕ be an event with full probability such that eq. (9) and eq. (11) hold for all $t \le \tau$ and $\omega \in E_\phi$.

(iii) In the second equality, we use the fact that summands are uncorrelated.

Take a *countable* subset G of C_0^∞ which is dense in H_q^N, and take $E = \bigcap_{\phi \in G} E_\phi$. Fix any $\phi \in C_0^\infty$ and take a sequence $(\phi_n)_n$ in G which converges to ϕ in H_q^N. Then for every $\omega \in E$,

$$\sum_{k=1}^{\infty} \int_0^t (g^k(s,\cdot), \phi_n) w_s^k \to (u(t,\cdot),\phi) - (u(0,\cdot),\phi) - \int_0^t (f(s,\cdot),\phi) ds$$

uniformly in t on $[0, \tau \wedge T]$. We *define*

$$\sum_{k=1}^{\infty} \int_0^t (g^k(s,\cdot), \phi) w_s^k := \lim_{n\to\infty} \left[(u(t,\cdot),\phi_n) - (u(0,\cdot),\phi_n) - \int_0^t (f(s,\cdot),\phi_n) ds \right] \tag{12}$$

for every $t \le \tau \wedge T$ and $\omega \in E$, and define it as zero outside of E uniformly. We should justify such definition makes sense. On $E \cap E_\phi$, it is obviously well-defined. In addition, as $E \triangle E_\phi$ is $\mathbb{P}$-null set, we can modify u and the stochastic integrals so that eq. (12) is satisfied.

Long story short, by considering the appropriate modification of the stochastic integrals, we can drop the dependency of ϕ in (a.s.) sense in eq. (9).

Remark 3.0.5 There can exist only one couple (f, g) for which eq. (9) holds. Indeed, if there are two, then one can represent zero as a sum of a continuous process of bounded variation and a continuous local martingale. Then the only possible case is that both processes vanish since a continuous martingale with finite variation is constant (see Proposition IV.1.2 of [18]).

Fix a constant $T \in (0, \infty)$. What we get is for each $\phi \in C_0^\infty$,

$$\int_0^t (f(s,\cdot),\phi) ds = 0, \quad \sum_{k=1}^{\infty} \int_0^t (g^k(s,\cdot),\phi) dw_s^k = 0 \quad \forall t \le \tau \wedge T \text{ (a.s.)}. \tag{13}$$

This inequality and eq. (13) implies that for every $h \in H_q^{2-n}$, $\int_0^t (f(s,\cdot),h) ds = 0$ holds for all $t \le \tau \wedge T$ with full probability. Notice that the Bessel potential space H_p^n is separable where $p \in [1, \infty)$ and $n \in \mathbb{R}$.(iv) Take a *countable* dense set $\mathcal{H}$ of H_q^{2-n}. Then on an event E with full probability,

$$\int_0^t (f(s,\cdot),h) ds = 0, \quad \forall h \in \mathcal{H},\ t \le \tau \wedge T.$$

This implies that for each $h \in \mathcal{H}$, $(f(s,\cdot),h) = 0$ (a.e.) for $t \le \tau \wedge T$ on E. Being countable, we can make (a.e.) set to be uniform.

Therefore, $(f(s,\cdot),h) = 0$ holds for every $h \in H_q^{2-n}$ (a.e.) $t \le \tau \wedge T$ on E, and this implies that $f = 0$ on $\mathbb{H}_p^{n-2}(\tau \wedge T)$. As T is arbitrary, we have $f = 0$ on $\mathbb{H}_p^{n-2}(\tau)$.

Since sum of stochastic integrals part converges uniformly, by the Burkholder–Davis–Gundy inequalities together with eq. (13) implies that for each $\phi \in C_0^\infty$,

$$\sum_{k=1}^{\infty} \int_0^{\tau\wedge T} (g^k(s,\cdot),\phi)^2 ds = 0 \quad \text{(a.s.)}.$$

Then using the inequality in remark 3.0.3 with the above argument where H_q^{2-n} is replaced by H_q^{1-n}, we can conclude that $g = 0$ on $\mathbb{H}_p^{n-1}(\tau, l_2)$.(v)

Therefore, the couple (f, g) is uniquely determined by u, and notation $\|u\|_{\mathcal{H}_p^n(\tau)}$ in eq. (10) makes sense.

(iv) Let $\mathcal{L}_p$ be a countable dense set of L_p. Since $(1-\Delta)^{-n/2} : L_p \to H_p^n$ is an isometry, $(1-\Delta)^{-n/2}\mathcal{L}_p$ is dense in H_p^n and clearly the set is countable.

(v) Check $g^k = 0$ on $\mathbb{H}_p^{n-1}$ for each k is enough.

Remark 3.0.6 It is known that the operator $(1-\Delta)^{m/2}$ makes isometrically H^n_p onto H^{n-m}_p for any n, m. Recall that the norm on $\mathbb{H}^n_p(\tau)$ is defined by

$$\|u\|^p_{\mathbb{H}^n_p(\tau)} := \mathbb{E}\int_{(\!(0,\tau]\!]} \|u(t,\cdot)\|^p_{n,p} dt.$$

From this fact, it is obvious that $(1-\Delta)^{m/2}$ maps isometrically $\mathbb{H}^n_p(\tau)$ onto $\mathbb{H}^{n-m}_p(\tau)$. Similarly recall that

$$\|u\|^p_{\mathbb{H}^n_p(\tau,l_2)} := \mathbb{E}\int_{(\!(0,\tau]\!]} \||(1-\Delta)^{n/2}u(t,\cdot)|_{l_2}\|^p_p dt.$$

This implies that $(1-\Delta)^{m/2}$ maps isometrically $\mathbb{H}^n_p(\tau, l_2)$ onto $\mathbb{H}^{n-m}_p(\tau, l_2)$.

Also, the inequalities from remark 3.0.3 can be used to show that given $u \in \mathcal{H}^n_p(\tau)$, one can in eq. (9) take any infinitely differentiable function ϕ whose derivatives vanish sufficiently fast at infinitym say exponentially fast.[vi] This allows to substitute $(1-\Delta)^{m/2}\phi$ in eq. (9) instead of ϕ and shows that the operator $(1-\Delta)^{m/2}$ maps isometrically $\mathcal{H}^n_p(\tau)$ onto $\mathcal{H}^{n-m}_p(\tau)$ for any n, m.

Definition 3.0.7 For $u \in \mathcal{H}^n_p(\tau)$, if eq. (9) holds, then we write $f = \mathbb{D}u$, $g = \mathbb{S}u$ (for "deterministic" and "stochastic" parts of u) and we also write

$$u(t) = u(0) + \int_0^t \mathbb{D}u(s)ds + \int_0^t \mathbb{S}^k u(s)dw^k_s, \quad du = f dt + g^k dw^k_t \quad t \le \tau.$$

Remark 3.0.8 It follows from definition 3.0.1 and 3.0.7 that the operators $\mathbb{D}$ and $\mathbb{S}$ are continuous operators from $\mathcal{H}^n_p(\tau)$ to $\mathbb{H}^{n-2}_p(\tau)$ and $\mathbb{H}^{n-1}_p(\tau, l_2)$ respectively. From theorem 4.1.2 and remark 3.0.6 it follows that $\mathbb{S}$ maps $\mathcal{H}^n_p(\tau)$ onto $\mathbb{H}^{n-1}_p(\tau, l_2)$. However, at this point we do not know how rich $\mathcal{H}^n_p(\tau)$ is. Nevertheless obviously $H^{1,2}_p(T) \subset \mathcal{H}^2_p(T)$.

Theorem 3.0.9 The spaces $\mathcal{H}^n_p(\tau)$ and $\mathcal{H}^n_{p,0}(\tau)$ are Banach spaces with norm eq. (10). In addition if $\tau \le T$, where T is a finite constant, then for $u \in \mathcal{H}^n_p(\tau)$

$$\|u\|_{\mathbb{H}^n_p(\tau)} \le N(d,p,T)\|u\|_{\mathcal{H}^n_p(\tau)}, \quad \mathbb{E}\sup_{t\le\tau} \|u(t,\cdot)\|^p_{n-2,p} \le N(d,p,T)\|u\|^p_{\mathcal{H}^n_p(\tau)}. \tag{14}$$

proof. We first deal with eq. (14). Obviously

$$\|u\|_{\mathbb{H}^n_p(\tau)} = \|(1-\Delta)u\|_{\mathbb{H}^{n-2}_p(\tau)} \le \|u\|_{\mathbb{H}^{n-2}_p(\tau)} + \|u\|_{\mathcal{H}^n_p(\tau)},$$

so that to prove eq. (14) we only need to prove that

$$\mathbb{E}\sup_{t\le\tau} \|u(t,\cdot)\|^p_{n-2,p} \le N(d,T)\|u\|^p_{\mathcal{H}^n_p(\tau)}. \tag{15}$$

Indeed, we have

$$\|u\|_{\mathbb{H}^{n-2}_p(\tau)} = \mathbb{E}\int_0^\tau \|u(t,\cdot)\|^p_{n-2,p} dt \le T\mathbb{E}\sup_{t\le\tau} \|u(t,\cdot)\|^p_{n-2,p}.$$

[vi] For instance, Schwartz class.

Owing to remark 3.0.6 we may assume that $n = 2$. Take a nonnegative function $\zeta \in C_0^\infty$ with unit integral, for $\epsilon > 0$ define $\zeta_\epsilon(x) = \epsilon^{-d}\zeta(x/\epsilon)$, and for generalized functions u let $u^{(\epsilon)}(x) = u * \zeta_\epsilon(x)$.[vii] Observe that $u^{(\epsilon)}(x)$ is continuous (infinitely differentiable) function of x for any distribution u. Plugging $\zeta_\epsilon(x - \cdot)$ instead of ϕ in eq. (13), we get that for any x the equality

$$u^{(\epsilon)}(t,x) = u^{(\epsilon)}(0,x) + \int_0^t f^{(\epsilon)}(s,x)ds + \sum_{k=1}^{\infty}\int_0^t g^{(\epsilon)k}(s,x)dw_s^k \tag{16}$$

holds almost surely for all $t \le \tau$. If necessary, we redefine the stochastic integrals in eq. (16) in a such way that eq. (16) would hold for all ω, t, and x such that $t \le \tau$. Here

$$\mathbb{E}\|u^{(\epsilon)}(0,\cdot)\|_p^p \le \mathbb{E}\|u(0,\cdot)\|_p^p \le \mathbb{E}\|u(0,\cdot)\|_{n-2/p,p}^p \le \|u\|_{\mathcal{H}_p^n(\tau)}^p,$$

where we use Young's inequality: $\|h^{(\epsilon)}\|_p \le \|\zeta_\epsilon\|_1\|h\|_p = \|h\|_p$. Similarly,

$$\left|\int_0^t f^{(\epsilon)}(s,x)ds\right|^p \le T^{p-1}\int_0^\tau |f^{(\epsilon)}(s,x)|^p ds,$$

$$\int_{\mathbb{R}^d}\left|\int_0^t f^{(\epsilon)}(s,x)ds\right|^p dx \le T^{p-1}\int_{\mathbb{R}^d}\int_0^\tau |f^{(\epsilon)}(s,x)|^p dsdx = T^{p-1}\int_0^\tau \|f^{(\epsilon)}(s,\cdot)\|_p^p ds \le T^{p-1}\int_0^\tau \|f(s,\cdot)\|_p^p ds,$$

$$\mathbb{E}\sup_{t\le\tau}\left\|\int_0^t f^{(\epsilon)}(s,\cdot)ds\right\|_p^p \le T^{p-1}\mathbb{E}\int_0^\tau \|f(s,\cdot)\|_p^p ds \le T^{p-1}\|u\|_{\mathcal{H}_p^n(\tau)}^p.$$

Finally, by Burkholder–Davis–Gundy inequalities, Fatou's lemma, and MCT,

$$\mathbb{E}\sup_{t\le\tau}\left|\sum_{k=1}^\infty\int_0^t g^{(\epsilon)k}(s,x)dw_s^k\right| \le N\mathbb{E}\left|\int_0^\tau\sum_{k=1}^\infty |g^{(\epsilon)k}(s,x)|^2 ds\right|^{p/2} = N\mathbb{E}\left|\int_0^\tau |g^{(\epsilon)}|_{l_2}^2(s,x)ds\right|^{p/2},$$

and as above,

$$\begin{aligned}
\mathbb{E}\sup_{t\le\tau}\left\|\sum_{k=1}^\infty\int_0^t g^{(\epsilon)k}(s,\cdot)dw_s^k\right\|_p^p &\le \int_{\mathbb{R}^d}\mathbb{E}\sup_{t\le\tau}\left|\sum_{k=1}^\infty\int_0^t g^{(\epsilon)k}(s,x)dw_s^k\right|^p dx \\
&\le N\mathbb{E}\int_{\mathbb{R}^d}\left|\int_0^\tau |g^{(\epsilon)}|_{l_2}^2(s,x)ds\right|^{p/2} dx \\
&\le N\mathbb{E}\left(\int_0^\tau \||g^{(\epsilon)}|_{l_2}^2(s,\cdot)\|_{p/2}ds\right)^{p/2} \\
&= N\mathbb{E}\left(\int_0^\tau \||g^{(\epsilon)}|_{l_2}(s,\cdot)\|_p^2 ds\right)^{p/2} \\
&\le N\mathbb{E}\int_0^\tau \||g^{(\epsilon)}|_{l_2}(s,\cdot)\|_p^p ds \\
&\le N\|g\|_{\mathbb{L}_p(\tau,l_2)}^p \\
&\le N\|g\|_{\mathcal{H}_p^n(\tau)}^p.
\end{aligned} \tag{17}$$

This along with eq. (16), leads to

$$\mathbb{E}\sup_{t\le\tau}\|u^{(\epsilon)}(t,\cdot)\|_p^p \le N\|u\|_{\mathcal{H}_p^n(\tau)}^p. \tag{18}$$

[vii] Recall that $u * \zeta(x) := (u, \zeta(x - \cdot))$.

Furthermore, by using the fact that $\|h^{(\epsilon)} - h^{(\gamma)}\|_p \to 0$ whenever $h \in L_p$ and $\epsilon, \gamma \to 0$ and by considering $u^{(1/m)} - u^{(1/k)}$ instead of $u^{(\epsilon)}$, we easily see that $u^{(1/m)}(t \wedge \tau, x)$ is a Cauchy sequence in $L_p(\Omega, B([0,T], L_p))$. Define $\bar{u}$ as its limit in this space. Then, for a subsequence m', we have $u^{(1/m')}(t, \cdot) \to \bar{u}(t, \cdot)$ in L_p if $t \leq \tau$ with probability 1. On the other hand $u^{(1/m)}(t, \cdot) \to u(t, \cdot)$ in the sense of distributions for all ω and t such that $t \leq \tau(\omega)$. Therefore, with probability one we have $u(t, \cdot) \in L_p$ for $t \leq \tau$. Now, eq. (18) and Fatou's lemma yields eq. (15) for $n = 2$. As explained above, this proves eq. (14).

Next, we derive the first assertion of our theorem from eq. (14). As usual, we have only to check the completeness of $\mathcal{H}_p^n(\tau)$. If $(u_j)_j$ is a Cauchy sequece in $\mathcal{H}_p^n(\tau)$, then it is a Cauchy sequence in $\mathbb{H}_p^n(\tau \wedge T)$ for any T, and there is $u \in \bigcap_{T>0} \mathbb{H}_p^n(\tau \wedge T)$ such that $\|u - u_j\|_{\mathbb{H}_p^n(\tau \wedge T)} \to 0$ as $j \to \infty$. Furthermore, u_{jxx} form a Cauchy sequence and therefore converge in $\mathbb{H}_p^{n-2}(\tau)$. It follows easily that $\|u_{xx} - u_{jxx}\|_{\mathbb{H}_p^{n-2}(\tau)} \to 0$.

Also, for $u_j(0)$, f_j, g_j corresponding to u_j, there is $u(0) \in L_p(\Omega, \mathcal{F}_0, H_p^{n-2/p})$ and $(f, g) \in \mathcal{F}_p^{n-2}(\tau)$ such that

$$\mathbb{E}\|u(0) - u_j(0)\|_{n-2/p,p}^p \to 0, \quad \|f - f_j\|_{\mathbb{H}_p^{n-2}(\tau)} \to 0, \quad \|g - g_j\|_{\mathbb{H}_p^{n-1}(\tau, l_2)} \to 0.$$

By using the argument from Remark 3.0.3, one can show that for any $\phi \in C_0^\infty$ eq. (9) holds in $(\!(0, \tau]\!]$ almost everywhere.

On the other hand, eq. (14) also implies that for u (at least for a modification of u) it holds that

$$\mathbb{E} \sup_{t \leq \tau \wedge T} \|u(t, \cdot) - u_j(t, \cdot)\|_{n-2,p}^p \to 0$$

for any constant $T < \infty$. Adding to this that this processes $(u_j(t, \cdot), \phi)$ are continuous (a.s.) (see Remark 3.0.3), we conclude that $(u(t, \cdot), \phi)$ is also continuous (a.s.). Thus, for any $\phi \in C_0^\infty$, eq. (10) not only holds in $(\!(0, \tau]\!]$ almost everywhere but also for all $t \leq \tau$ almost surely. Hence, $u \in \mathcal{H}_p^n(\tau)$ and $u_j \to u$ in $\mathcal{H}_p^n(\tau)$. The theorem is proved.

Remark 3.0.10 We could replace the first term on the right in eq. (10) with $\|u\|_{\mathbb{H}_p^n(\tau)}$ and, for bounded τ, we would get an equivalent norm by virtue of eq. (14). The form of eq. (10) that we have chosen is convenient in the future when we need certain constants to be independent of T, see, for instance, Theorem 4.3.1.

Remark 3.0.11 Actually, eq. (17) has some gap, which is, we do not know the joint measurability of

$$h(\omega, t, x) := \sum_{k=1}^{\infty} \int_0^t g^{(\epsilon)k}(s, x) dw_s^k.$$

One way to resolve gaps is that using Theorem 3.0.13 to prove eq. (17) first, and then apply the density method.

Another way is proving the measurability of h directly using Remark 3.0.4 and eq. (16).

Remark 3.0.12 In theorem 7.0.1 and theorem 7.0.2 below, we prove much sharper estimates than eq. (15).

We also need the following properties of the spaces $\mathcal{H}_p^n(\tau)$ and $\mathbb{H}_p^n(\tau)$.

Theorem 3.0.13 Take $g \in \mathbb{H}_p^n(l_2)$. Then there exists a sequence $g_j \in \mathbb{H}_p^n(l_2)$, $j = 1, 2, \cdots$, such that $\|g - g_j\|_{\mathbb{H}_n^p(l_2)} \to 0$ as $j \to \infty$ and

$$g_j^k = \begin{cases} \sum_{i=1}^j \mathbb{1}_{(\tau_{i-1}^j, \tau_i^j]}(t) g_j^{ik}(x) & \text{if } k \le j, \\ 0 & \text{if } k > j, \end{cases}$$

where τ_i^j are bounded stopping times, $\tau_{i-1}^j \le \tau_i^j$, and $g_j^{ik} \in C_0^\infty$.

- -

proof. The argument in remark 3.0.6 and the fact that C_0^∞ is dense in any H_p^n show that we only need to consider $n = 0$. Furthermore, one can easily understand that the set of $g \in \mathbb{L}(l_2)$ for which the statement holds forms a linear *closed* subspace $\mathbb{L}$ of $\mathbb{L}_p(l_2)$.(viii) We have to prove that $\mathbb{L} = \mathbb{L}(l_2)$. If this is not true, then by the Hahn-Banach theorem and the Riesz representation theorem(ix), there is a nonzero $h \in \mathbb{L}_q(l_2)$ with $q = p/(p-1)$ such that

$$\mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} (h, g)_{l_2} dx dt = 0$$

for any $g \in \mathbb{L}$. In particular, for any $k \ge 1$, bounded stopping time τ and $g \in C_0^\infty$, put $\tilde{g}$ defined by

$$\tilde{g}^j = \begin{cases} \mathbb{1}_{(0,\tau]}(t) g(x) & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases},$$

Then clearly $\tilde{g} \in \mathbb{L}$ and we have

$$\mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} (h, \tilde{g})_{l_2} dx dt = \mathbb{E} \int_0^\infty \mathbb{1}_{(0,\tau]} \left(\int_{\mathbb{R}^d} h^k g dx \right) dt = 0.$$

Since $h^k \in \mathbb{L}_q$, by theorem 9.9.2 we can regard this as $\overline{\mathcal{P} \otimes \mathfrak{B}(\mathbb{R}^d)}$-measurable function, the completion of $\mathcal{P} \otimes \mathfrak{B}(\mathbb{R}^d)$ with the canonically corresponded measure. By the Fubini's theorem,

$$\int_{\mathbb{R}^d} h^k g dx$$

is then equal to some predictable function on $(\!(0, \infty]\!]$ (a.e.).(x) It follows that $\int_{\mathbb{R}^d} h^k g dx = 0$ on $(\!(0, \infty]\!]$ (a.e.). By taking g from a countable subset $\mathcal{G}$ in C_0^∞ which is dense in L_p, we get that on a subset of $(\!(0, \infty]\!]$ of full measure,

$$\int_{\mathbb{R}^d} h^k g dx = 0, \quad \forall g \in \mathcal{G},\ k \ge 1.$$

But then $h^k = 0$ on $(\!(0, \infty]\!] \times \mathbb{R}^d$ (a.e.). Indeed, on a subset E of $(\!(0, \infty]\!]$ of full measure, we have $h^k(t, \cdot) = 0$ (a.e.) on $\mathbb{R}^d$ for each $(\omega, t) \in E$. Since we can regard h^k as a $\overline{\mathcal{P} \otimes \mathfrak{B}(\mathbb{R}^d)}$-measurable function, we have

$$\mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_E(t) |h^k(t, x)| dx dt = 0.$$

This implies $\mathbb{1}_E h^k = 0$ (a.e.) on $(\!(0, \infty]\!] \times \mathbb{R}^d$. Since E is already has a full measure on $(\!(0, \infty]\!]$, the main conclusion is now proved. Such process works on each k, and this makes a contradiction because $h \ne 0$. This proves the lemma. □

(viii) Use the diagonalize method.

(ix) By the Bochner integral theory, the dual of $\mathbb{L}_p(l_2)$ is $\mathbb{L}_q(l_2)$ because the space $H_p^n(\mathbb{R}^d, l_2)$ is reflexive.

(x) The Fubini's theorem only gives that the integral is $\overline{\mathcal{P}}$-measurable.

Theorem 3.0.14 Let $T \in (0, \infty)$. If $u_j \in \mathcal{H}_p^n(T)$, $j = 1, 2, \cdots$, and $\|u_j\|_{\mathcal{H}_p^n(T)} \leq K$, where K is a finite constant, then there exists a subsequence j' and a function $u \in \mathcal{H}_p^n(T)$ such that

(a) $u_{j'}, u_{j'}(0, \cdot), \mathbb{D}u_{j'}$, and $\mathbb{S}u_{j'}$ converge weakly to $u, u(0, \cdot), \mathbb{D}u, \mathbb{S}u$ in $\mathbb{H}_p^n(T)$, $L_p(\Omega, H_p^{n-2/p})$, $\mathbb{H}_p^{n-2}(T)$, and $\mathbb{H}_p^{n-1}(T, l_2)$, respectively;

(b) $\|u\|_{\mathcal{H}_p^n(T)} \leq K$;

(c) for any $\phi \in C_0^\infty$ and any $t \in [0, T]$ we have $(u_{j'}(t, \cdot), \phi) \to (u(t, \cdot), \phi)$ weakly in $L_p(\Omega)$.

- -

proof. From properties of L_p spaces, it follows that there exists a subsequence j' such that $u_{j'}$, $u_{j'}(0, \cdot)$, $\mathbb{D}u_{j'}$, $\mathbb{S}u_{j'}$ converge weakly to some u, u_0, f, g in $\mathbb{H}_p^n(T)$, $L_p(\Omega, H_p^{n-2/p})$, $\mathbb{H}_p^{n-2}(T)$, and $\mathbb{H}_p^{n-1}(T, l_2)$, respectively.[xi] Then for any $\phi \in C_0^\infty$, the expressions $(u_{j'}(t, \cdot), \phi)$, $(u_{j'}(0, \cdot), \phi)$, $(\mathbb{D}u_{j'}(s, \cdot), \phi)$, and $(\mathbb{S}^k u_{j'}(s, \cdot), \phi)$ in the formula

$$(u_{j'}(t, \cdot), \phi) = (u_{j'}(0, \cdot), \phi) + \int_0^t (\mathbb{D}u_{j'}(s, \cdot), \phi)ds + \sum_{k=1}^{\infty} \int_0^t (\mathbb{S}^k u_{j'}(s, \cdot), \phi)dw_s^k$$

converges weakly in corresponding spaces. For instance, fix $\Lambda \in L_p((\!(0, T]\!], \mathcal{P})^*$ and consider Λ_ϕ : $\mathbb{H}_p^n(T) \to \mathbb{R}$ defined by

$$\Lambda_\phi u := \Lambda((u(t, \cdot), \phi)).$$

Then one can easily show that $\Lambda_\phi \in (\mathbb{H}_p^n(T))^*$. This implies that

$$\Lambda((u_{j'}(t, \cdot), \phi)) = \Lambda_\phi u_{j'} \to \Lambda_\phi u = \Lambda((u(t, \cdot), \phi).$$

As Λ is arbitrary, this yields that $(u_{j'}(t, \cdot), \phi) \to (u(t, \cdot), \phi)$ weakly. The deterministic part also works with the same manner.

For the stochastic part, first of all, using the Minkowski's inequality gives for every $h \in \mathbb{H}_p^n(l_2)$ and $\phi \in C_0^\infty$ that

$$\mathbb{E} \int_0^\infty |(h(s, \cdot), \phi)|_{l_2}^p ds \leq \|\phi\|_{-n, p/(p-1)}^p \mathbb{E} \int_0^\infty \|h(s, \cdot)\|_{n,p}^p ds,$$

which implies $(h, \phi) \in L_p((\!(0, \infty]\!], \mathcal{P}, l_2)$, and then use the above method to conclude that $(\mathbb{S}u_{j'}(s, \cdot), \phi)$ converges to $(g(s, \cdot), \phi)$ weakly.

Now we are going to consider two operators $I : L_p((\!(0, T]\!], \mathcal{P}) \to L_p(\Omega \times (0, T], \mathfrak{U}_T)$ and SI : $L_p((\!(0, T]\!], \mathcal{P}, l_2) \to L_p(\Omega \times (0, T], \mathfrak{U}_T)$ defined by

$$(Iu)(t) := \int_0^t u(s)ds, \quad (SIu)(t) := \sum_{k=1}^{\infty} \int_0^t u^k(s)dw_s^k,$$

where $\mathfrak{U}_T := \mathcal{F} \otimes \mathfrak{B}((0, T])$. First of all, similar with remark 3.0.3 shows the well-defineness of SI. In addition, one can obtain

$$\mathbb{E} \int_0^T |Iu|^p(s)ds = \mathbb{E} \int_0^T \left| \int_0^t u(s)ds \right|^p dt \leq T^p \mathbb{E} \int_0^T u^p(s)ds < \infty,$$

and by the Burkholder–Davis–Gundy's inequalities,

$$\mathbb{E} \int_0^T |SIu|^p(s)ds \leq T\mathbb{E} \sup_{t \leq T} |SIu|^p(t) \leq N(p, T)\mathbb{E} \left| \int_0^T \sum_{k=1}^{\infty} (g^k)^2(s)ds \right|^{p/2} \leq N(p, T)\mathbb{E} \int_0^T |g(s)|_{l_2}^p ds.$$

Hence both operators are (strongly) continuous in each spaces. Since every strongly continuous

[xi] There is a well-known fact that a Banach space is reflexive if and only if any strongly bounded sequence has a weakly convergent subsequence (the Eberlein-Shmulyan Theorem. See [26]). Also by theorem 9.9.1, all spaces described are reflexive.

operator is weakly continuous, we have that, for any $\phi \in C_0^\infty$,

$$(u(t,\cdot),\phi) = (u_0,\phi) + \int_0^t (f(s,\cdot),\phi)ds + \sum_{k=1}^{\infty}\int_0^t (g^k(s,\cdot),\phi)dw_s^k \tag{19}$$

for almost all $(\omega,t) \in \Omega \times [0,T]$.

By Banach-Saks theorem, there is a sequence $(v_{j'}, \mathbb{D}v_{j'}, \mathbb{S}v_{j'})$ of convex combinations of $(u_{j'}, \mathbb{D}u_{j'}, \mathbb{S}u_{j'})$ which converges strongly to (u,f,g) in $\mathbb{H}_p^n(T) \times \mathbb{H}_p^{n-2}(T) \times \mathbb{H}_p^{n-1}(T,l_2)$. From eq. (14), it follows that

$$\mathbb{E}\sup_{t\le T}\|v_j(t,\cdot) - v_i(t,\cdot)\|_{n-2,p}^p \to 0$$

as $i,j \to \infty$. Therefore, there is a H_p^{n-2}-valued function v on $\Omega \times [0,T]$ such that (on the space $L_p(\Omega, B([0,T], H_p^{n-2}))$)

$$\mathbb{E}\sup_{t\le T}\|v_j(t,\cdot) - v(t,\cdot)\|_{n-2,p}^p \to 0.$$

In particular, $(v_j(t,\cdot),\phi) \to (v(t,\cdot),\phi)$ uniformly on $[0,T]$ in probability for any $\phi \in C_0^\infty$. On the other hand, the strong convergence of v_j to u in $\mathbb{H}_p^n(T)$ implies that $(v_j(t,\cdot),\phi) \to (u(t,\cdot),\phi)$ on $\Omega\times[0,T]$ in measure. This shows that $(v(t,\cdot)\phi) = (u(t,\cdot),\phi)$ on $\Omega \times [0,T]$ (a.e.). Because the space of conjugate to H_p^n is separable and C_0^∞ is dense in there, we can make (a.e.) uniform for $\phi \in C_0^\infty$, proving that $u = v$ (as generalized functions) on $\Omega \times [0,T]$ (a.e.).

Thus, $v \in \mathbb{H}_p^n(T)$. Also, $(v_j(t,\cdot),\phi)$ are given by equations similar to eq. (19), which implies that $(v_j(t,\cdot),\phi)$ are continuous in t (a.s.). The uniform convergence of $(v_j(t,\cdot),\phi)$ to $(v(t,\cdot),\phi)$ in probability implies the continuity of $(v(t,\cdot),\phi)$ (a.s.). By the above, eq. (19) holds for almost all $(\omega,t) \in \Omega \times [0,T]$ if we replace $(u(t,\cdot),\phi)$ by $(v(t,\cdot),\phi)$.(xii) Since the latter is continuous and the right hand side of eq. (19) is continuous, $(v(t,\cdot),\phi)$ equals the right hand side of eq. (19) for all $t \in [0,T]$ (a.s.). Hence, $v \in \mathcal{H}_p^n(T)$ and we have proved assertion (i) for v instead of u, which is irrelevant.

Assertion (ii) follows from the inequality $u = v$ on $\Omega\times[0,T]$ (a.e.) and from the fact that the norm of a weak limit is less than the liminf of norms (and use the fact that v is the strong convergence of some convex combinations of u_j).

To prove (iii), take $\phi \in C_0^\infty$ and $\xi \in L_q(\Omega)$ with $q = p/(p-1)$ and write

$$\mathbb{E}\xi(u_j(t,\cdot),\phi) = \mathbb{E}\xi(u_j(0,\cdot),\phi) + \mathbb{E}\xi\int_0^t (\mathbb{D}u_j(s,\cdot),\phi)ds + \mathbb{E}\xi\int_0^t (\mathbb{S}^k u_j(s,\cdot),\phi)dw_s^k.$$

By what has been said about the properties of the operators of integration and by (i),

$$\begin{aligned}\lim_{j'\to\infty}\mathbb{E}\xi(u_{j'}(t,\cdot),\phi) &= \lim_{j'\to\infty}\left[\mathbb{E}\xi(u_{j'}(0,\cdot),\phi) + \mathbb{E}\xi\int_0^t (\mathbb{D}u_{j'}(s,\cdot),\phi)ds + \mathbb{E}\xi\int_0^t (\mathbb{S}^k u_{j'}(s,\cdot),\phi)dw_s^k\right]\\ &= \mathbb{E}\xi(u(0,\cdot),\phi) + \mathbb{E}\xi\int_0^t (\mathbb{D}u(s,\cdot),\phi)ds + \mathbb{E}\xi\int_0^t (\mathbb{S}^k u(s,\cdot),\phi)dw_s^k\\ &= \mathbb{E}\xi(u(t,\cdot),\phi),\end{aligned}$$

which proves (iii) and the theorem.

4 Model Equations

Except for section 4.2, we will always understand equations like eq. (1) in the sense of definition 3.0.7, which means that we will be looking for a functon $u \in \mathcal{H}_{p,0}^n(\tau)$ such that

$$\mathbb{D}u = Lu + f, \quad \mathbb{S}u = \Lambda u + g.$$

(xii) Recall that $u = v$ on $\Omega \times [0,T]$ (a.e.).

In this section, we consider eq. (1) when $b = c = \nu = 0$ and the coefficients a and σ does not depend on x. Throughout the section, we fix real-valued functions $a^{ij}(t)$ and l_2-valued functions $\sigma^i(t) = (\sigma^{ik}(t))_{k\geq 1}$ defined for $i, j = 1, \cdots, d$ on $\Omega \times (0, \infty)$. Define

$$\alpha^{ij}(t) = \frac{1}{2}(\sigma^i(t), \sigma^j(t))_{l_2}$$

and assume that a and σ are $\mathcal{P}$-measurable functions, and in the matrix sense

$$(a^{ij}) = (a^{ij})^*, \quad K(\delta^{ij}) \geq (a^{ij}) \geq (a^{ij} - \alpha^{ij}) \geq \delta(\delta^{ij}),$$

where K and δ are some fixed strictly positive constants, and δ^{ij} is the Kronecker delta. By the way, the assumption that $a > \alpha$ is necessary even to have L_2-theory for SPDEs with constant coefficients.

Equation (1) takes the following form:

$$du(t,x) = (a^{ij}(t)u_{x^ix^j}(t,x) + f(t,x))dt + (\sigma^{ik}(t)u_{x^i}(t,x) + g^k(t,x))dw_t^k, \quad t > 0. \tag{20}$$

Our plan is as follows. In section 4.1 we consider the case of the heat equation with random right-hand side and get basic priori estimates.

4.1 Particular Case $a^{ij} = \delta^{ij}, \sigma = 0$

We start with the equation

$$du(t,x) = (\Delta u(t,x) + f(t,x))dt + g^k(t,x)dw_t^k, \quad t > 0. \tag{21}$$

We need a lemma from . Remember that the operators T_t are defined by eq. (2) and, as always, $p \geq 2$.

Lemma 4.1.1 Let $-\infty \leq a < b \leq \infty$, $g \in L_p((a,b) \times \mathbb{R}^d, l_2)$. Then

$$\int_{\mathbb{R}^d}\int_a^b \left[\int_a^t |\nabla T_{t-s}g(s,\cdot)(x)|_{l_2}^2 ds\right]^{p/2} dtdx \leq N(d,p)\int_{\mathbb{R}^d}\int_a^b |g(t,x)|_{l_2}^p dtdx.$$

- -

proof. First of all, assume that $g \in C_0^\infty((a,b) \times \mathbb{R}^d, l_2)$. For each $j = 1, \cdots, d$ define

$$\phi_j(x) := \frac{-x_j}{2(4\pi)^{d/2}}e^{-|x|^2/4}.$$

Then one can easily checked that

$$\nabla T_t g(x) = \frac{1}{\sqrt{t}}t^{-d/2}\sum_{j=1}^d [\phi_j(x/\sqrt{t}) * g(x)]\mathbb{e}_j,$$

where $\{\mathbb{e}_j\}_1^d$ is the ordered basis of $\mathbb{R}^d$. and ϕ_j satisfies all conditions in section 9.11.1. If we define

$$\Phi_t g(x) := t^{-d/2}\sum_{j=1}^d [\phi(x/\sqrt{t}) * g(x)]\mathbb{e}_j$$

then by theorem 9.11.1,

$$\begin{aligned}\int_{\mathbb{R}^d}\int_a^b \left[\int_a^t |\nabla T_{t-s}g(s,\cdot)(x)|_{l_2}^2 ds\right]^{p/2} dtdx &= \int_{\mathbb{R}^d}\int_a^b \left[\int_a^t |\Phi_{t-s}g(s,\cdot)(x)|_{l_2}^2 \frac{ds}{t-s}\right]^{p/2} dtdx \\ &\leq N(d,p)\int_{\mathbb{R}^d}\int_a^b |g(t,x)|_{l_2}^p dtdx.\end{aligned}$$

For any $L_p((a,b) \times \mathbb{R}^d, l_2)$ functions, approximate with smooth functions.

Theorem 4.1.2 Take $f \in \mathbb{H}_p^{-1}$, $g \in \mathbb{L}_p(l_2)$. Then

(i) equation eq. (21) with zero initial condition has a unique solution $u \in \mathcal{H}_p^1$;

(ii) for this solution, we have

$$\|u_{xx}\|_{\mathbb{H}_p^{-1}} \le N(d,p)(\|f\|_{\mathbb{H}_p^{-1}} + \|g\|_{\mathbb{L}_p(l_2)}); \tag{22}$$

(iii) for this solution, we have $u \in C_{\text{loc}}([0,\infty), L_p)$ almost surely, and, for any $\lambda, T > 0$,

$$\mathbb{E}\sup_{t\le T}(e^{-p\lambda t}\|u(t,\cdot)\|_p^p)+\mathbb{E}\int_0^T e^{-p\lambda t}\||u|^{(p-2)/p}|u_x|^{2/p}(t,\cdot)\|_p^p dt \le N(d,p,\lambda)(\|e^{-\lambda t}f\|^p_{\mathbb{H}_p^{-1}}+\|e^{-\lambda t}g\|^p_{\mathbb{L}_p(T,l_2)}). \tag{23}$$

- -

proof. It is well known that there exists a continuous linear operator

$$P : H_p^{-1} \to (L_p)^{d+1}$$

such that if $h \in H_p^{-1}$ and $Ph = (h_0, \tilde{h}^1, \cdots, \tilde{h}^d)$, then $h = h_0 + \text{div}\tilde{h}$ and

$$\|\tilde{h}\|_p + \|h_0\|_p \le N(d,p)\|h\|_{-1,p}, \quad \|h\|_{-1,p} \le N(d,p)\{\|\tilde{h}\|_p + \|h_0\|_p\}. \tag{24}$$

Actually, one can take $\tilde{h} = -\nabla(1-\Delta)^{-1}h$ and $h_0 = h - \text{div}\tilde{h} = (1-\Delta)^{-1}h$. Indeed, $\|h_0\|_p = \|h\|_{-2,p} \le \|h\|_{-1,p}$. Also, the fact that $\partial/\partial x^j$ is a bounded operator from H_p^n to H_p^{n+1} for any n means that $(\partial/\partial x^i)(1-\Delta)^{-1/2}$ is a bounded operator from H_p^n to H_p^n and $(\partial/\partial x^i)(1-\Delta)^{-1}$ is a bounded operator from H_p^n to H_p^{n-1}. This is why $\|\tilde{h}\|_p \le N(d,p)\|h\|_{-1,p}$. On the other hand, $(1-\Delta)^{-1/2}h = (1-\Delta)^{-1/2}h_0 + \text{div}(1-\Delta)^{-1/2}\tilde{h}$, and both operators $(1-\Delta)^{-1/2}$ and $\text{div}(1-\Delta)^{-1/2}$ are bounded on L_p.

Define $(f_0, \tilde{f}) = Pf$. Then equation eq. (21) takes the form

$$du = (\Delta u + f_0 + \text{div}\tilde{f})dt + g^k dw_t^k, \tag{25}$$

and we supply with zero initial condition. We will prove that, for arbitrary $f_0, \tilde{f} \in \mathbb{L}_p$, our assertion hold for eq. (25) in place of eq. (21). Of course, in eq. (22) and eq. (23), by f we mean $f_0 + \text{div}\tilde{f}$.

[**A particular case**] First we consider the case in which

$$\begin{aligned} f_0(t,x) &= \sum_{i=1}^m \mathbb{1}_{(\tau_{i-1},\tau_i]}(t) f_{0i}(x), \quad \tilde{f}(t,x) = \sum_{i=1}^m \mathbb{1}_{(\tau_{i-1},\tau_i]}(t)\tilde{f}_i(x), \\ g(t,x) &= \sum_{k=1}^m g^k(t,x)h_k, \quad g^k(t,x) = \sum_{i=1}^m \mathbb{1}_{(\tau_{i-1},\tau_i]}(t) g^{ik}(x), \end{aligned} \tag{26}$$

where $(h_k)_k$ is the standard orthonormal basis in l_2, $m < \infty$, τ_i are bounded stopping times, $\tau_{i-1} \le \tau_i$, and $f_{0i}, g^{ik} \in C_0^\infty$ and $\tilde{f}_i \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

Set

$$v(t,x) = \int_0^t g^k(s,x)dw_s^k = \sum_{i,k=1}^m g^{ik}(x)(w^k_{t\wedge\tau_i} - w^k_{t\wedge\tau_{i-1}}),$$

$$u(t,x) = v(t,x) + \int_0^t T_{t-s}[\Delta v + f](s,\cdot)(x)ds, \quad \forall t \ge 0. \tag{27}$$

The function $u-v$ is infinitely differentiable in x, differentiable once in t and satisfies the equation (for any ω)[(xiii)]

$$\frac{\partial z}{\partial t} = \Delta z + \Delta v + f.$$

It follows that, for any x and ω, the function $u(t,x)$ satisfies the following form of eq. (25):

$$u(t,x) = \int_0^t (\Delta u(s,x) + f(s,x))ds + \sum_{k=1}^m \int_0^t g^k(s,x)dw_s^k. \tag{28}$$

Next, we want to obtain some bounds on norms of u. Let

$$u_1(t,x) = \int_0^t T_{t-s} f(s,x)ds.$$

Since each $\tilde{f}_i$ is in C_0^∞, we have

$$\begin{aligned} u_1(t,x) &= \int_0^t T_{t-s} f(s,x)ds \\ &= \int_0^t T_{t-s} f_0(s,x)ds + \int_0^t T_{t-s}(\mathrm{div}\tilde{f})(s,x)ds \\ &= \int_0^t T_{t-s} f_0(s,x)ds + \mathrm{div}\int_0^t T_{t-s}\tilde{f}(s,x)ds \\ &=: u_{10}(t,x) + \mathrm{div}\tilde{u}_1(t,x). \end{aligned}$$

According to theorem 2.2.1 (for any ω),

$$\begin{aligned} \|u_{1xx}\|_{L_p(\mathbb{R}_+,H_p^{-1})} &\le \|u_{10xx}\|_{L_p((0,\infty)\times\mathbb{R}^d)} + \|\tilde{u}_{1xx}\|_{L_p((0,\infty)\times\mathbb{R}^d)} \\ &\le N[\|f_0\|_{L_p((0,\infty)\times\mathbb{R}^d)} + \|\tilde{f}\|_{L_p((0,\infty)\times\mathbb{R}^d)}] \\ &\le N\|f\|_{L_p(\mathbb{R}_+,H_p^{-1})}. \end{aligned} \tag{29}$$

Furthermore, use again that the operators $(\partial/\partial x^i)(1-\Delta)^{-1/2}$ $(= (1-\Delta)^{-1/2}(\partial/\partial x^i))$ are bounded in L_p for any $p > 1$. Then

$$\|u_{xx} - u_{1xx}\|^p_{\mathbb{H}_p^{-1}} \le N\|u_x - u_{1x}\|^p_{\mathbb{L}_p} = N\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}|u_x - u_{1x}|^p(t,x)dxdt. \tag{30}$$

To make some further transformations of this formula, consider a bounded Borel function $z^k = z^k(x)$ and

$$\left(\int_r^t T_{t-s}z^k ds\right)(w^k_{r\wedge\tau_2} - w^k_{r\wedge\tau_1}) =: \xi_r^k \eta_r^k$$

(with no summation in k) as a function of r. Then one can easily obtain

$$d\xi_r^k = -T_{t-r}z^k dr, \quad d\eta_r^k = \mathbb{1}_{(\tau_1,\tau_2]}dw_r^k.$$

Hence, by the Itô's formula, we obtain

$$d(\xi_r^k\eta_r^k) = -\eta_r^k T_{t-r}z^k dt + \mathbb{1}_{(\tau_1,\tau_2]}\xi_r^k dw_r^k.$$

By considering its integral form and then taking $r = t$, we have (a.s.)

$$0 = -\int_0^t (w^k_{r\wedge\tau_2} - w^k_{r\wedge\tau_1})T_{t-r}z^k dr + \int_0^t \mathbb{1}_{(\tau_1,\tau_2]}(r)\left(\int_r^t T_{t-s}z^k ds\right)dw_r^k.$$

[(xiii)] The main paper said that $u-v$ is infinitely differentiable in (t,x), but personally thinking, it is an error.

Now we are going to use our particular g,[xiv] we get

$$
\begin{aligned}
u_x(t,x) - u_{1x}(t,x) &= v_x(t,x) + \int_0^t T_{t-s}(D_x\Delta v)(s,\cdot)(x)ds \\
&= v_x(t,x) + \int_0^t T_{t-s}(\Delta D_x v)(s,\cdot)(x)ds \\
&= v_x(t,x) + \sum_{i,k=1}^m \int_0^t (w^k_{t\wedge\tau_i} - w^k_{t\wedge\tau_{i-1}})T_{t-s}(\Delta g^{ik}_x)(x)ds \\
&= v_x(t,x) + \sum_{i,k=1}^m \int_0^t \mathbb{1}_{(\tau_{i-1},\tau_i]}(r)\left(\int_r^t T_{t-s}(\Delta g^{ik}_x)(x)ds\right)dw^k_r \\
&= v_x(t,x) + \sum_{k=1}^m \int_0^t \int_r^t T_{t-s}[\mathbb{1}_{(\tau_{i-1},\tau_i]}(r)(\Delta g^{ik}_x)](x)dsdw^k_r \\
&= v_x(t,x) + \sum_{k=1}^m \int_0^t \int_r^t \Delta T_{t-s}g^k_x(r,\cdot)(x)dsdw^k_r \\
&= v_x(t,x) - \sum_{k=1}^m \int_0^t \int_r^t \frac{\partial}{\partial s}T_{t-s}g^k_x(r,\cdot)(x)dsdw^k_r \\
&= \sum_{k=1}^m \int_0^t T_{t-r}g^k_x(r,\cdot)(x)dw^k_r \quad \text{(a.s.)}.
\end{aligned}
$$

Hence, by the Burkholder–Davis–Gundy's inequality,

$$
\mathbb{E}|u_x - u_{1x}|^p(t,x) \le N\mathbb{E}\left[\int_0^t \sum_{k=1}^m |T_{t-r}g^k_x(r,x)|^2 dr\right]^{p/2} = N\mathbb{E}\left[\int_0^t |T_{t-r}g_x(r,x)|^2_{l_2} dr\right]^{p/2}.
$$

By plugging this into eq. (30) and applying lemma 4.1.1, we obtain

$$
\|u_{xx} - u_{1xx}\|^p_{\mathbb{H}^{-1}_p} \le N\mathbb{E}\int_0^\infty \int_{\mathbb{R}^d}\left[\int_0^t |\nabla T_{t-s}g(s,x)|^2_{l_2}ds\right]^{p/2} dxdt \le N\|g\|^p_{\mathbb{L}_p(l_2)}.
$$

This along with eq. (29) gives us eq. (22). However, we do not know yet that $u \in \mathcal{H}^1_p$.

Our next step is to prove eq. (23) for sufficiently large λ. From eq. (27) and Itô's formula,[xv] for each $x \in \mathbb{R}^d$ (a.s.),

$$
\begin{aligned}
|u(t,x)|^p e^{-\lambda t} = &\int_0^t e^{-\lambda s}(p|u|^{p-2}u\Delta u + p|u|^{p-2}uf + \frac{1}{2}p(p-1)|u|^{p-2}|g|^2_{l_2} - \lambda|u|^p)(s,x)ds \\
&+ p\sum_{k=1}^m \int_0^t e^{-\lambda s}|u|^{p-2}ug^k(s,x)dw^k_s.
\end{aligned}
$$

We integrate with respect to x and use the stochastic Fubini's theorem and the fact that $u(t,x)$, $g(t,x)$, and their derivatives decreases very fast when $|x| \to \infty$.[xvi] Then we integrate by parts in

[xiv] One can also use stochastic Fubini's theorem to obtain it. However, because g is a simple function in the sense that calculating stochastic integral is easy, we can obtain the equality without using the stochastic Fubini's theorem.

[xv] Notice that since $p \ge 2$, we have $D_x|x|^p = p|x|^{p-2}x$, and $D^2_x|x|^p = p(p-1)|x|^{p-2}$ for all $x \in \mathbb{R}$. One can easily obtain by splitting cases $x \ge 0$ and $x < 0$.

[xvi] The case g is clear. For u, it is explained in the appendix.

$\int |u|^{p-2}u\Delta u dx$,[xvii] continuity of the operator P, and also notice that, for $q = p/(p-1)$,[xviii]

$$\int_{\mathbb{R}^d} |u|^{p-2}uf(s,x)dx = -(p-1)\int_{\mathbb{R}^d} |u|^{p-2}u_x \cdot \tilde{f}(s,x)dx + \int_{\mathbb{R}^d} |u|^{p-2}uf_0(s,x)dx,$$

$$\begin{aligned}\left|\int_{\mathbb{R}^d} |u|^{p-2}u_x \cdot \tilde{f}(s,x)dx\right| &\leq \int_{\mathbb{R}^d} (|u|^{(p-2)/2}|u_x|)^q |u|^{q(p-2)/2}dx + \|\tilde{f}(s,\cdot)\|_p^p \\ &\leq N\|f(s,\cdot)\|_{-1,p}^p + (1-\frac{q}{2})\eta^{\frac{2}{q-2}}\|u(s,\cdot)\|_p^p + \frac{q}{2}\eta^{\frac{2}{q}}\||u|^{(p-2)/p}|u_x|^{2/p}(s,\cdot)\|_p^p \\ &=: N\|f(s,\cdot)\|_{-1,p}^p + N_1\|u(s,\cdot)\|_p^p + \frac{1}{2}\||u|^{(p-2)/p}|u_x|^{2/p}(s,\cdot)\|_p^p,\end{aligned}$$

$$\int_{\mathbb{R}^d} |u(s,x)|^{p-2}u(s,x)f_0(s,x)dx \leq \|f_0(s,\cdot)\|_p^p + \|u(s,\cdot)\|_p^p \leq N\|f(s,\cdot)\|_{-1,p}^p + \|u(s,\cdot)\|_p^p.$$

Combining all these, we have

$$\begin{aligned}&e^{-\lambda t}\|u(t,\cdot)\|_p^p + \frac{p(p-1)}{2}\int_0^t e^{-\lambda s}\||u|^{(p-2)/p}|u_x|^{2/p}\|_p^p ds + \lambda\int_0^t e^{-\lambda s}\|u(s,\cdot)\|_p^p ds \\ &\leq N\int_0^t e^{-\lambda s}[\|f(s,\cdot)\|_{-1,p}^p + \|g(s,\cdot)\|_p^p]ds + [N_1 p(p-1) + p + \frac{p(p-1)}{2}]\int_0^t e^{-\lambda s}\|u(s,\cdot)\|_p^p ds \\ &\qquad + p\sum_{k=1}^m \int_0^t e^{-\lambda s}[\int_{\mathbb{R}^d} |u|^{p-2}ug^k(s,x)dx]dw_s^k.\end{aligned}$$

If we let

$$G' := p|u|^{p-2}u, \quad G'' := p(p-1)|u|^{p-2},$$

one can check that $|G'| = |u|^{p/2}(q|G''|)^{1/2}$. This and the Burkholder–Davis–Gundy's inequalities give

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T}\left|\sum_{k=1}^m \int_0^t e^{-\lambda s}|u|^{p-2}g^k(s,x)dw_s^k\right| \\ &\leq N(p)\mathbb{E}\left(\int_0^T \sum_{k=1}^m \left(\int_{\mathbb{R}^d} e^{-\lambda s}|u|^{p-2}ug^k(s,x)dx\right)^2 ds\right)^{1/2} \\ &\leq N(p)p^{-1}\mathbb{E}\left(\int_0^T \sum_{k=1}^m \left(\int_{\mathbb{R}^d} e^{-\lambda s}|G'||g^k|(s,x)dx\right)^2 ds\right)^{1/2} \\ &\leq N(p)p^{-1}\mathbb{E}\left(\int_0^T \sum_{k=1}^m \left(\int_{\mathbb{R}^d} e^{-\lambda s}|u(s,x)|^p dx\right)\left(\int_{\mathbb{R}^d} e^{-\lambda s}q|G''||g^k|^2(s,x)dx\right) ds\right)^{1/2} \\ &= N(p)\mathbb{E}\left(\int_0^T e^{-\lambda s}\|u(s,\cdot)\|_p^p\left(\int_{\mathbb{R}^d} e^{-\lambda s}|g|_{l_2}^2|u|^{p-2}(s,x)dx\right) ds\right)^{1/2} \\ &= N(p)\mathbb{E}\left(\sup_{t\leq T} e^{-\lambda t}\|u(t,\cdot)\|_p^p\right)^{1/2}\left(\int_0^T\int_{\mathbb{R}^d} e^{-\lambda s}|g|_{l_2}^2|u|^{p-2}(s,x)dxds\right)^{1/2}\end{aligned}$$

[xvii] Since $|u| + |u_x| \to 0$ as $|x| \to \infty$, we have $\int_{\mathbb{R}^d} |u|^{p-2}u\Delta u dx = -(p-1)\int_{\mathbb{R}^d} |u|^{p-2}|u_x|^2 dx$.

[xviii] In addition, we use the following inequality: for every $a \geq 0$ and $b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq a^p + b^q,$$

where $p \in (1,\infty)$ and $q = p/(p-1)$.

$$
\begin{aligned}
&\leq \frac{1}{2p}\mathbb{E}\sup_{t\leq T}(e^{-\lambda t}\|u(t,\cdot)\|_p^p) + N_2\int_0^T\int_{\mathbb{R}^d} e^{-\lambda s}|g|_{l_2}^2|u|^{p-2}(s,x)dxds \\
&\leq \frac{1}{2p}\mathbb{E}\sup_{t\leq T}(e^{-\lambda t}\|u(t,\cdot)\|_p^p) + N_2\int_0^T e^{-\lambda s}[\|u(s,\cdot)\|_p^p + \|g(s,\cdot)\|_p^p]ds.
\end{aligned}
$$

Therefore, taking a supremum over t with range $[0,T]$ and applying the expectation, we have the following result:

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{t\leq T}(e^{-\lambda t}\|u(t,\cdot)\|_p^p) + \frac{p(p-1)}{2}\mathbb{E}\int_0^T e^{-\lambda s}\||u|^{(p-2)/p}|u_x|^{2/p}\|_p^p ds + \lambda\mathbb{E}\int_0^T e^{-\lambda s}\|u(s,\cdot)\|_p^p ds \\
&\quad\leq N\mathbb{E}\int_0^T e^{-\lambda s}[\|f(s,\cdot)\|_{-1,p}^p + \|g(s,\cdot)\|_p^p]ds \\
&\qquad + [N_1p(p-1) + N_2p + p + \frac{p(p-1)}{2}]\mathbb{E}\int_0^T e^{-\lambda s}\|u(s,\cdot)\|_p^p ds.
\end{aligned}
$$

Letting $\lambda \geq p(p-1)N_1 + N_2p + p + p(p-1)/2$ gives, therefore, the desired inequality.[xix]

The assertion about the arbitrariness of λ in eq. (23) can be easily justified by rescaling arguments when instead of f, g, and w one takes $(c^2f, cg)(c^2t, cx)$ and $c^{-1}w_{c^2t}$ and gets $u(c^2t, cx)$ instead of $u(t,x)$.[xx]

From our explicit formulas and from the particular choice of f and g, we have for every positive integer n that

$$
\begin{aligned}
\left\|\int_0^t T_{t-s}[\Delta v + f](s,\cdot)ds\right\|_{2n,p} &= \left\|(1-\Delta)^n\int_0^t T_{t-s}[\Delta v + f](s,\cdot)ds\right\|_p \\
&= \left\|\int_0^t T_{t-s}[(1-\Delta)^n(\Delta v + f)](s,\cdot)ds\right\|_p \\
&\leq \int_0^t \|T_{t-s}[(1-\Delta)^n(\Delta v + f)](s,\cdot)\|_p ds \\
&= \int_0^t \left\|\int_{\mathbb{R}^d}\frac{1}{(4\pi(t-s))^{d/2}}e^{\frac{-|y|^2}{4(t-s)}}[(1-\Delta)^n(\Delta v + f)](s,\cdot - y)dy\right\|_p ds \\
&\leq \int_0^t\int_{\mathbb{R}^d}\frac{1}{(4\pi(t-s))^{d/2}}e^{\frac{-|y|^2}{4(t-s)}}\|[(1-\Delta)^n(\Delta v + f)](s,\cdot - y)\|_p dyds \\
&= \int_0^t\int_{\mathbb{R}^d}\frac{1}{(4\pi(t-s))^{d/2}}e^{\frac{-|y|^2}{4(t-s)}}\|[(1-\Delta)^n(\Delta v + f)](s,\cdot)\|_p dyds.
\end{aligned}
$$

By recalling the type of f and v, one can easily checked that the last integration is finite, and we have $u(t,\cdot) \in H_p^n$ for any n (and for any ω). Since u is infinitely differentiable in x, we have by eq. (28),

$$
\|u(t,\cdot) - u(s,\cdot)\|_{2n,p} \leq \int_{t\wedge s}^{t\vee s}\|\Delta u(a,\cdot) + f(a,\cdot)\|_{2n,p}da
$$

[xix] Notice that we proved the inequality for sufficiently large λ. After that, replace $p\lambda$ instead of λ, and this is legitimate because $p \geq 2$ (so clearly $p\lambda \geq p$).

[xx] For any bounded stopping time τ, one have

$$
\int_0^{c^2t}\mathbb{1}_{(\!(0,\tau]\!]}(s)dw_s = \int_0^t \mathbb{1}_{(\!(0,\tau]\!]}(c^2s)d\theta_s,
$$

where $\theta_t = c^{-1}w_{c^2t}$. One can extend this result for any predictable functions.

$$+\sum_{i,k=1}^{m}\|g^{ik}\|_{2n,p}[|w^k_{(t\vee s)\wedge\tau_i}-w^k_{(t\wedge s)\wedge\tau_i}|+|w^k_{(t\vee s)\wedge\tau_{i-1}}-w^k_{(t\wedge s)\wedge\tau_{i-1}}|]$$
$$\to 0$$

as $s\to t$, and thus $u\in C_{\mathrm{loc}}([0,\infty),H^n_p)$ for every n (and for every ω). This proves (iii).

From estimates eq. (22) and eq. (23), we obtain

$$\begin{aligned}\|u\|_{\mathbb{H}^1_p(T)} &= \mathbb{E}\int_0^T \|(1-\Delta)u(t,\cdot)\|^p_{-1,p}dt\\ &\le N\mathbb{E}\int_0^T[\|u(t,\cdot)\|^p_p+\|u_x x(t,\cdot)\|^p_{-1,p}]dt\\ &\le N[\|u_x x\|^p_{\mathbb{H}^{-1}_p}+e^T\mathbb{E}\sup_{t\le T}(e^{-t}\|u(t,\cdot)\|^p_p)]<\infty,\end{aligned}$$

so that $u\in\bigcap_{T>0}\mathbb{H}^1_p(T)$. Furthermore, from the pointwise equation eq. (28), and notice that $\Delta u+f\in\bigcap_{T>0}\mathbb{H}^{-1}_p(T)$, we have for every $\phi\in C_0^\infty$ that

$$\begin{aligned}(u(t,\cdot),\phi) &= \left(\int_0^t(\Delta u(s,\cdot)+f(s,\cdot))ds,\phi\right)+\sum_{k=1}^m\left(\int_0^t g^k(s,\cdot)dw^k_s,\phi\right)\\ &= \left(\int_0^t(\Delta u(s,\cdot)+f(s,\cdot))ds,\phi\right)+\sum_{k=1}^m\left(\sum_{i=1}^m(w^k_{t\wedge\tau_i}-w^k_{t\wedge\tau_{i-1}})g^{ik},\phi\right)\\ &= \int_0^t((\Delta u(s,\cdot)+f(s,\cdot)),\phi)ds+\sum_{k=1}^m\sum_{i=1}^m(w^k_{t\wedge\tau_i}-w^k_{t\wedge\tau_{i-1}})(g^{ik},\phi)\\ &= \int_0^t((\Delta u(s,\cdot)+f(s,\cdot)),\phi)ds+\sum_{k=1}^m\int_0^t\sum_{i=1}^m(g^{ik},\phi)\mathbb{1}_{(\tau_{i-1},\tau_i]}dw^k_s\\ &= \int_0^t((\Delta u(s,\cdot)+f(s,\cdot)),\phi)ds+\sum_{k=1}^m\int_0^t(g^k(s,\cdot),\phi)dw^k_s.\end{aligned}$$

Here, recall that $(\cdot,\phi)$ is an linear transformation on H^{-1}_p. Hence $u\in\mathcal{H}^1_p$, which proves a part of assertion (i). The uniqueness in (i) follows from the fact that for $f\equiv 0, g\equiv 0$ we have the heat equation and the uniqueness of its solution in our class of functions is a standard fact. This completes the proof in the case of step functions f, g.

[**General case**] In the case of general f, g, we observe that the uniqueness in (i) is proved as above. As far as other assertions are concerned we are going to use Theorem 3.0.13 and Remark 3.0.8.

If we consider all functions f_0, $\tilde{f}^i$, g^k as one sequence, then, by Theorem 3.0.13, we can approximate them by functions f_{0j}, $\tilde{f}^i_j$, g^k_j of type eq. (26). Let u_j be the corresponding solutions of eq. (25). By the result for the particular case, $(u_j)_j$ is a Cauchy sequence in $\mathcal{H}^1_p$, and being Banach space (by Theorem 3.0.9, there is a $u\in\mathcal{H}^1_p$ to which u_j converges in $\mathcal{H}^1_p$. Remark 3.0.8 and the convergence $\|u_{xx}-u_{jxx}\|_{\mathbb{H}^{-1}_p}\to 0$ prove that $\mathbb{D}u=\Delta u+f$ and $\mathbb{S}u=g$. In particular, this proves (i). Assertion (ii) follows from the construction of u. From assertion (iii) available in the particular case, we get that u_j is a Cauchy sequence in $L_p(\Omega, C([0,T],L_p))$ for any T. Therefore, it converges in this space to a function $\bar{u}$.

Notice that for any $T>0$ and $\phi\in C_0^\infty$,

$$\mathbb{E}\sup_{t\le T}|(u_j(t,\cdot),\phi)-(\bar{u}(t,\cdot),\phi)|\le\|\phi\|_q\mathbb{E}\sup_{t\le T}\|u_j-\bar{u}\|_p\to 0,$$

$$\mathbb{E}\int_0^T|(u_j(t,\cdot),\phi)-(\bar{u}(t,\cdot),\phi)|dt\le T\|\phi\|_q\mathbb{E}\sup_{t\le T}\|u_j-\bar{u}\|_p\to 0,$$

Together this with operators I and SI defined in the proof of Theorem 3.0.14, we have for every $\phi \in C_0^\infty$ that

$$(\bar{u}(t,\cdot),\phi) = \int_0^t [(\bar{u}(s,\cdot),\Delta\phi) + (f(s,\cdot),\phi)]ds + \int_0^t (g^k(s,\cdot),\phi)dw_s^k$$

for all t (a.s.). Therefore, $u - \bar{u}$ is a generalized solution of the heat equation with zero initial condition and with bounded L_p-norm (a.s.). This implies that $\|(u-\bar{u})(t,\cdot)\|_p = 0$ for all t (a.s.), so that $u \in C([0,T], L_p)$ for all T (a.s.). Finally, by theorem 3.0.9, we have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} |\nabla(u-u_j)|^p dxdt &= \int_0^T \int_{\mathbb{R}^d} |\nabla(1-\Delta)^{-1/2}(1-\Delta)^{1/2}(u-u_j)|^p dxdt \\ &\le N \int_0^T \int_{\mathbb{R}^d} |(1-\Delta)^{1/2}(u-u_j)|^p dxdt \\ &= N \int_0^T \|(u-u_j)(t,\cdot)\|_{1,p}^p dt,\end{aligned}$$

$$\mathbb{E}\int_0^T \|(u-u_j)(t,\cdot)\|_{1,p}^p = \|u_j - u\|_{\mathbb{H}_p^1(T)} \le N\|u_j - u\|_{\mathcal{H}_p^1} \to 0.$$

This and the Hölder's inequality yield that

$$\int_0^T \int_{\mathbb{R}^d} |u_j|^{p-2}|u_{jx}|^2(t,x)dxdt \to \int_0^T \int_{\mathbb{R}^d} |u|^{p-2}|u_x|^2(t,x)dxdt$$

as $j \to \infty$ in probability for any T.

By the previous result, one can obtain that $u_j \to u$ in $L_p(\Omega, C([0,T], L_p))$ for every T. This gives

$$\mathbb{E}\sup_{t\le T}(e^{-pt\lambda}\|u_j(t,\cdot) - u(t,\cdot)\|_p^p) \le \mathbb{E}\sup_{t\le T}\|u_j(t,\cdot) - u(t,\cdot)\|_p^p \to 0.$$

Finally, by the Fatou's lemma, we prove eq. (23), and the theorem is proved.

Remark 4.1.3 Assume that f and g satisfies eq. (26). Then as in the proof of Theorem 4.1.2, the solution to eq. (21) is uniquely written by

$$u(t) = \int_0^t T_{t-s}f(s)ds + \sum_{k=1}^\infty \int_0^t T_{t-s}g^k(s)dw_s.$$

We use this fact later on proving Theorem 7.0.2.

Remark 4.1.4 Although eq. (7) holds for all $p \in (1,\infty)$, it follows from [6] that for $p < 2$, Theorem 9.11.1 is false even if $d = 1$, $H = \mathbb{R}$.

Define $\psi(x) = -x(4\pi)^{-1/2}\exp(-x^2/4)$ and $f(t,x) = \psi(x)e^{-\lambda t}$ where $\lambda > 0$. One can easily checked that ψ satisfies all conditions in section 9.11. Then if $\bar{\psi} = (4\pi)^{1/2}\psi$ and recalling that $\mathcal{F}(e^{-a|x|^2}) = (2\pi)^{-d/2}e^{-|\xi|^2/(4a)}$, we have[(xxi)]

$$\bar{\psi}(x/\sqrt{t}) * \bar{\psi}(x) = \sqrt{2\pi}\mathcal{F}^{-1}[\mathcal{F}(\bar{\psi})\mathcal{F}(\bar{\psi}(\cdot/\sqrt{t}))]$$

[(xxi)] Notice that our Fourier transform is defined by

$$\mathcal{F}f(\xi) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)dx.$$

Then it is a basic fact that $\mathcal{F}(f*g) = (2\pi)^{d/2}\hat{f}\hat{g}$, $\mathcal{F}(D^\alpha f)(\xi) = (i\xi)^\alpha\hat{f}(\xi)$, and $D^\alpha\hat{f}(\xi) = \mathcal{F}[(-i\xi)^\alpha f](\xi)$.

$$= \sqrt{2\pi}\mathcal{F}^{-1}[\sqrt{t}\hat{\bar{\psi}}(\sqrt{t}\xi)\hat{\bar{\psi}}(\xi)]$$
$$= -8t\sqrt{2\pi}\mathcal{F}^{-1}[\xi^2 e^{-(t+1)\xi^2}]$$
$$= 2t\sqrt{\pi}\frac{x^2}{(t+1)^{5/2}}\exp\left(-\frac{x^2}{4(t+1)}\right),$$

$$\Psi_t f(s,x) = t^{-1/2}\psi(x/\sqrt{t}) * f(s,x) = t^{-1/2}(4\pi)^{-1}\bar{\psi}(x/\sqrt{t}) * \bar{\psi}(x) = \frac{\sqrt{t}}{2\sqrt{\pi}}\frac{x^2}{(t+1)^{5/2}}\exp\left(-\frac{x^2}{4(t+1)}\right).$$

Then the left hand side of eq. (124) multiplyed by $\lambda^{p/2}$ is

$$\int_{\mathbb{R}}\int_0^b \left[\frac{1}{2\sqrt{\pi}}\int_0^t e^{-2\lambda s}\frac{\lambda x^4}{(t-s+1)^5}\exp\left(-\frac{x^2}{2(t-s+1)}\right)ds\right]^{p/2} dtdx. \tag{31}$$

Here, notice that

$$\int_0^t e^{-2\lambda s}\frac{\lambda x^4}{(t-s+1)^5}\exp\left(-\frac{x^2}{2(t-s+1)}\right)ds \le x^4 e^{-x^2/(2(t+1))}$$

for every $\lambda \ge 1$, and we have

$$\begin{aligned}\int_{\mathbb{R}}\int_0^b x^{2p}e^{-px^2/(4(t+1))}dtdx &= 2\int_0^b\int_0^\infty x^{2p}e^{-px^2/(4(t+1))}dxdt\\ &= 2\int_0^b\left(\frac{4(t+1)}{p}\right)^{p-1/2}\int_0^\infty x^{p-1/2}e^{-x}dxdt\\ &= 2\Gamma(p+1/2)\int_0^b\left(\frac{4(t+1)}{p}\right)^{p-1/2}dt\\ &< \infty.\end{aligned}$$

This allows us to evaluate eq. (31) letting $\lambda \to \infty$ by using the dominated convergence theorem and the theorem related to the convolution,(xxii) we have

$$\int_{\mathbb{R}}\int_0^b\left[\frac{1}{4\sqrt{\pi}}\frac{x^4}{(t+1)^5}\exp\left(-\frac{x^2}{2(t+1)}\right)\right]^{p/2}dtdx,$$

which is finite and nonzero.

Thus the left hand side of eq. (124) is of order $\lambda^{-p/2}$. At the same time the right hand side of eq. (124) is of order λ^{-1}, and the former is much bigger than the latter one if $p \in (1,2)$ and $\lambda \to \infty$.

4.2 Relation of the Solutions of eq. (20) to the Solutions of the Heat Equation

It turns out that the investigation of general equation eq. (20) with coefficients independent of x can be quite formally reduced to the particular case of the heat equation. First, we explain how to do this without caring about rigorousness, and then proceed with formal proofs.

The first observation consists of the following. Assume that we have

$$du(t,x) = f(t,x)dt + g^k(t,x)dw_t^k, \tag{32}$$

and we define a process x_t and a function v by

$$x_t^i = \int_0^t \sigma^{ik}(s)dw_s^k, \quad i = 1,\cdots,d, \quad v(t,x) = u(t,x-x_t). \tag{33}$$

We now apply formally the Itô-Wentzell formula. The statement of which is followed:

(xxii)Consider $\phi(t) = \mathbb{1}_{t>0}e^{-t}$ and $\phi_\lambda(t) = 2\lambda\mathbb{1}_{t>0}e^{-2\lambda t}$. Since $\int_{\mathbb{R}}\phi = 1$, the well-known fact about convolution property gives the result.

Proposition 4.2.1 Let $\xi = \xi_t$ be a stochastic process with stochastic differential

$$d\xi_t^i = b_t^i dt + \sigma_t^{ik} dw_t^k,$$

and let F be a $\mathcal{P} \otimes \mathfrak{B}(\mathbb{R}^d)$-measurable real-valued function belonging to $C^{0,2}([0,\infty) \times \mathbb{R}^d)$ for (a.a.) ω. Assume that, for every $x \in \mathbb{R}^d$, the function F has a stochastic differential

$$dF(t,x) = J(t,x)dt + H^k(t,x)w_t^k,$$

where J, H^k, $k = 1, \cdots, d'$ are $\mathcal{P} \otimes \mathfrak{B}(\mathbb{R}^d)$-measurable real-valued functions. We also assume that $H^k \in C^{0,1}([0,\infty)\times\mathbb{R}^d)$ for all k and (a.a.) ω. Then the process $t \mapsto F(t,\xi_t)$ has a stochastic differential

$$\begin{aligned} dF(t,\xi_t) &= J(t,\xi_t)dt + H^k(t,\xi_t)dw_t^k \\ &\quad + \left(b^i F_{x^i}(t,\xi_t) + \frac{1}{2}\sigma^{ik}\sigma^{jk}F_{x^i x^j}(t,\xi_t)\right)dt + \sigma^{ik}F_{x^i}(t,\xi_t)dw_t^k \\ &\quad + \sigma^{ik}H^k_{x^i}(t,\xi_t)dt. \end{aligned}$$

One can see its proof at [19], and the distribution version is presented in [13]. By applying Itô-Wentzell's formula, we obtain

$$\begin{aligned} dv(t,x) = [&f(t,x-x_t) + \alpha^{ij}(t)v_{x^i x^j}(t,x) - (g_{x^i}(t,x-x_t), \sigma^i(t))_{l_2}]dt \\ &+ [g^k(t,x-x_t) - v_{x^i}(t,x)\sigma^{ik}(t)]dw_t^k. \end{aligned} \tag{34}$$

This shows how to introduce the terms $v_{x^i}\sigma^{ik}$ in equation eq. (32) and also shows again a kind of necessity for g to have the first derivatives in x.

This device alone is not sufficient, since, if we had $\Delta u + \bar{f}$ instead of f in eq. (32), then, in eq. (34), we would get the second order differential operator $(\delta^{ij} + \alpha^{ij})\partial^2/\partial x^i \partial x^j$ with coefficients strongly related to the coefficients of $v_{x^i}\sigma^{ik}(t)$.(xxiii) We could get around this difficulty if we manage to start with equations with more general operators L instead Δ. Here the second observation comes that if, instead of eq. (32), we consider

$$du(t,x) = (\Delta u + \bar{f})dt + g^k(t,x)dw_t^k,$$

and take expectations in the counterpart of eq. (34) corresponding to this equation, then, assuming that σ is nonrandom, we get indeed an equation for $\mathbb{E}v(t,x)$ with operator L different from Δ. By the way, this method of studying parabolic equations with coefficients independent of x was applied in [7] in order to show that ''whatever'' estimate is true for the heat equation, it is also true for any parabolic equation with coefficients independent of x. Of course, taking expectations ''kills'' all randomness in the equation, and therefore we use a conditional expectation.

Definition 4.2.2 Denote by $\mathfrak{D}$ the set of all $\mathcal{D}$-valued functions u (written as $u(t,x)$ in a common abuse of notation) on $\Omega \times [0,\infty)$ such that, for any $\phi \in C_0^\infty$,

(i) the function (u,ϕ) is $\mathcal{P}$-measurable,

(ii) for any $\omega \in \Omega$ and $T \in (0,\infty)$, we have

$$\int_0^T \sup_{x\in\mathbb{R}^d} |(u(t,\cdot), \phi(\cdot - x))|^2 dt < \infty. \tag{35}$$

(xxiii) Personally thinking, the term ''strongly related'' means between u and v.

In the same way, we define $\mathfrak{D}(l_2)$ by considering l_2-valued linear functionals on C_0^∞ (xxiv) and replacing $|\cdot|$ in eq. (35) by $|\cdot|_{l_2}$.

Remark 4.2.3 Notice that $(u(t,\cdot), \phi(\cdot - x))$ is continuous in x and Borel in t so that eq. (35) makes sense. Also, for $p \geq 2$, $q = p/(p-1)$, and any n,

$$\int_0^T \sup_{x\in\mathbb{R}^d} |(u(t,\cdot), \phi(\cdot - x))|^2 dt \leq \int_0^T \|u(t,\cdot)\|_{n,p}^2 \|\phi\|_{-n,q}^2 dt \leq \|\phi\|_{-n,q}^2 \left(\int_0^T \|u(t,\cdot)\|_{n,p}^p dt\right)^{2/p}. \tag{36}$$

This shows that if $u \in \mathcal{H}_p^n$, then condition eq. (35) is satisfied at least for almost all ω. Also, if $u \in \mathcal{H}_p^n$, then eq. (9) holds, which shows that $(u(t,\cdot), \phi)$ is indistinguishable from a predictable process.(xxv) This is true for any $\phi \in C_0^\infty$. From separability of H_q^{-n}, it follows that we can modify u on a set of probability zero and after this we get a function belonging to $\mathfrak{D}$. This is the sense in which we write

$$\mathcal{H}_p^n \subset \mathfrak{D}. \tag{37}$$

Definition 4.2.4 Let $f, u \in \mathfrak{D}$, $g \in \mathfrak{D}(l_2)$. We say that the equality

$$du(t,x) = f(t,x)dt + g(t,x)dw_t, \quad t > 0, \tag{38}$$

holds *in the sense of distributions* if for any $\phi \in C_0^\infty$, with probability 1 for all $t \geq 0$ we have

$$(u(t,\cdot), \phi) = (u(0,\cdot), \phi) + \int_0^t (f(s,\cdot), \phi)ds + \int_0^t (g^k(s,\cdot), \phi)dw_s^k, \tag{39}$$

or write in the differential form by

$$d(u(t,\cdot), \phi) = (f(t,\cdot), \phi)dt + (g^k(t,\cdot), \phi)dw_t^k.$$

Observe that, since $|(g,\phi)(t)|_{l_2}^2$ is locally summable in t, the last series in eq. (39) converges uniformly in t on every finite interval of time in probability. This fact and eq. (35) imply $(u(t,\cdot), \phi) \in C_{\text{loc}}([0,\infty))$ for (a.a.) ω.

In this subsection, we always understand equation eq. (20) in the sense of distributions. Notice that if $u \in \mathcal{H}_p^n$ and u satisfies eq. (39), then, by eq. (37), $u \in \mathfrak{D}$ and eq. (38) holds in the sense of distributions. An advantage of Definition 4.2.4 is that one need not check summability of any derivative.(xxvi)

(xxiv) A sequence $\Lambda = (\Lambda^k)_k$ of linear functionals on C_0^∞ such that for every $\phi \in C_0^\infty$,

$$|(\Lambda, \phi)|_{l_2} := \sum_{k=1}^\infty (\Lambda^k, \phi)^2 < \infty.$$

(xxv) For fixed t, $(f(s,\cdot), \phi)$ is $\overline{\mathcal{F}_t \otimes \mathfrak{B}([0,\infty))}$ for every $s \leq t$. Thus by the Fubnini's theorem and the fundamental theorem of calculus, $\int_0^t (f(s,\cdot), \phi)$ is $\mathcal{F}_t$-measurable (recall that it is complete) for each t and continuous on every compact set in t, hence it is indistinguishable from a predictable process. For the stochastic part, it is a local martingale, so it is $\mathcal{F}_t$-adapted. Furthermore, it is continuous in t (a.s.) where $\phi \in C_0^\infty$ is fixed. Therefore, it also has a predictable representitive.

(xxvi) Recall the original definition of the stochastic differential (see [8]).

Lemma 4.2.5 Let $f, u \in \mathfrak{D}$, $g \in \mathfrak{D}(l_2)$. Assume the definitions in eq. (33). Then eq. (32) holds (in the sense of distributions) if and only if eq. (34) holds (in the sense of distributions).

- -

proof. First remember that, for a distribution $\alpha(x)$ and $y \in \mathbb{R}^d$, by $\alpha(x-y)$ mean the distribution defined by $(\alpha, \phi(\cdot + y))$. Also from relations like (cf. eq. (36))

$$\begin{aligned}\int_0^T \sup_{y\in\mathbb{R}^d} |(v_{xx}(t,\cdot), \phi(\cdot - y))|^2 dt &= \int_0^T \sup_{y\in\mathbb{R}^d} |(v(t,\cdot), \phi_{xx}(\cdot - y))|^2 dt \\ &= \int_0^T \sup_{y\in\mathbb{R}^d} |(u(t,\cdot), \phi_{xx}(\cdot + x_t - y))|^2 dt \\ &= \int_0^T \sup_{y\in\mathbb{R}^d} |(u(t,\cdot), \phi_{xx}(\cdot - y))|^2 dt \\ &< \infty,\end{aligned}$$

$$\begin{aligned}\int_0^T \sup_{y\in\mathbb{R}^d} |((g_{x^i}(t,\cdot - x_t), \sigma^i(t))_{l_2}, \phi(\cdot - y))|^2 dt &= \int_0^T \sup_{y\in\mathbb{R}^d} |((g_{x^i}(t,\cdot - x_t), \phi(\cdot - y)), \sigma^i(t))_{l_2}|^2 dt \\ &\le \sup_{t\le T} |\sigma^i(t)|_{l_2}^2 \int_0^T \sup_{y\in\mathbb{R}^d} |(g_{x^i}(t,\cdot - x_t)\phi(\cdot - y))|_{l_2}^2 dt < \infty.\end{aligned}$$

Here, by the elliptic condition, $\sigma^i(t)$ is uniformly bounded on each finite interval. It follows that $v(t,x)$, $f(t,x-x_t)$, and $(g_{x^i}(t,\cdot - x_t), \sigma^i(t))_{l_2}$ belong to $\mathfrak{D}$ and $g(t,x-x_t)$ and $v_{x^i}(t,x)\sigma^i(t)$ belong to $\mathfrak{D}(l_2)$. Furthermore, for any $\phi \in C_0^\infty$, the function $F(t,x) := (u(t,\cdot - x), \phi)$ has a stochastic differential in t for any x and is infinitely differentiable with respect to x. Indeed, two parts are obvious by checking that $(u(t,\cdot - x), \phi) = (u(t,\cdot), \phi(\cdot + x))$, and clearly $(x,y) \mapsto \phi(y+x)$ is jointly infinitely differentiable. Now our assertion immediately follows from the real-valued Itô-Wentzell formula for $F(t,x_t)$ (recall that for every x, $F(t,x)$ is continuous in t (a.s.)). By the way, heuristically, one can easily memorize this formula by the considering the following computations. Write symbollically by

$$\int_{\mathbb{R}^d} u(x)\phi(x)dx := (u, \phi).$$

Then

$$\begin{aligned}d\int_{\mathbb{R}^d} u(t,x-x_t)\phi(x)dx &= d\int_{\mathbb{R}^d} u(t,x)\phi(x+x_t)dx \\ &= \int_{\mathbb{R}^d} \phi(x+x_t)du(t,x)dx + \int_{\mathbb{R}^d} u(t,x)d\phi(x+x_t)dx + \int_{\mathbb{R}^d} (du(t,x))d\phi(x+x_t)dx \\ &= \cdots.\end{aligned}$$

Remark 4.2.6 If, instead of eq. (32), u satisfies the equation

$$du(t,x) = (a^{ij}(t)u_{x^ix^j}(t,x) + h(t,x))dt + \sigma^{ik}(t)u_{x^i}(t,x)dw_t^k,$$

then eq. (34) takes the form

$$\frac{\partial}{\partial t}v(t,x) = ((a^{ij}(t) - \alpha^{ij}(t))v_{x^ix^j}(t,x) + h(t,x-x_t), \quad t > 0, \tag{40}$$

and can be considered on each ω separately. Here, recall that $\alpha^{ij}(t) = (1/2)(\sigma^i(t), \sigma^j(t))_{l_2}$. Observe that if $a(t) < \alpha(t)$ (in the matrix sense), then the initial-value problem $v(0) = v_0$ for equation eq. (40) is *ill posed*.(xxvii)

This shows that operators appearing in the stochastic term (the σ) should be subordinated in a certain sense to the operator in the deterministic part (the a) of the equation. This is needed in order to construct an L_p-theory. On the other hand, take $d = 1$ and a one-dimensional Wiener process w_t. Consider the following equation

$$du(t,x) = -iu_x(t,x)dw_t.$$

Suprisingly enough and somewhat in spite of what is said above, this equation has a very nice solution for each initial data $u_0 \in L_2$. One gets the solution after passing to Fourier transforms. It turns out that $\tilde{u}(t,x) = \tilde{u}_0(\xi)\exp[\xi w_t - (1/2)|\xi|^2 t]$. The function $\tilde{u}(t,\xi)$ decays very fast when $|\xi| \to \infty$, which shows that $u(t,x)$ is infinitely differentiable in x. Also notice that, taking expectations, we see that $\mathbb{E}u(t,x) = u_0(x)$ if u_0 is nonrandom(xxviii), and in this case we get a representation of any L_2 function as an integral over Ω of functions $u(\omega,1,x)$ which are infinitely differentiable in x. However, the major drawback of such equations is that $\mathbb{E}|u(t,0)|^p = \infty$ for any $p > 1$ if, for example, $\tilde{u}_0(\xi) \geq \exp(-\lambda|\xi|)$.(xxix)

Lemma 4.2.7 Let $f \in \mathfrak{D}$, $g \in \mathfrak{D}(l_2)$, u_0 be a $\mathcal{D}$-valued function on Ω. Then the following assertions hold:

(i) In $\mathfrak{D}$ there can exist only one (up to evanescence) solution of equation eq. (20) with the initial condition $u(0,\cdot) = u_0$.

(ii) Let $\mathcal{F}_t = \mathcal{W}_t \vee \mathcal{B}_t$ for $t \geq 0$, and assume that σ-fields $\mathcal{W}_t$ and $\mathcal{B}_t$ form independent increasing filtrations. Let W and B be sets such that $W \cup B = \mathbb{Z}_+$. Assume that $(w_t^k, \mathcal{W}_t)$ and $(w_t^r, \mathcal{B}_t)$ are Wiener processes for $k \in W$ and $r \in B$. Let $u \in \mathfrak{D}$ satisfy eq. (20) (in the sense of distributions), and let a, f, σ, g be $\mathcal{W}_t$-adapted. Finally, assume that there exists an $n \in (-\infty,\infty)$ such that $f \in \mathbb{H}_2^n(T)$, $g \in \mathbb{H}_2^n(T,l_2)$ for any $T \in (0,\infty)$ and $u(0,\cdot)$ is $\mathcal{W}_0$-measurable and

$$\mathbb{E}\|u(0,\cdot)\|_{n,2}^2 < \infty.$$

Then in $\mathfrak{D}$ there exists a unique solution $\tilde{u}$ of the equation

$$d\tilde{u} = (a^{ij}\tilde{u}_{x^ix^j} + f)dt + \sum_{k\in W}(\sigma^{ik}\tilde{u}_{x^i} + g^k)dw_t^k, \quad t > 0. \tag{41}$$

In addition, for any $\phi \in C_0^\infty$ and $t \geq 0$,

$$(\tilde{u}(t,\cdot),\phi) = \mathbb{E}[(u(t,\cdot),\phi)|\mathcal{W}_t] \quad \text{(a.s.)}. \tag{42}$$

(xxvii)The form of which is well-posed when the terminal-value is provided.

(xxviii)Indeed, for the SPDE,

$$u(t,x) = u_0(x) - i\int_0^t u_x'(s,x)dw_s.$$

Taking the expectation, we obtain the result.

(xxix)Recall that $u(t,0) = c\int_{\mathbb{R}}\tilde{u}(t,\xi)d\xi$ where $c > 0$ is a constant. Using the fact

$$\int_0^\infty \exp(ax - bx^2)dx = \frac{\sqrt{\pi}}{2\sqrt{b}}\exp\left(\frac{a^2}{4b}\right)(1 + \operatorname{erf}\left(\frac{a}{2\sqrt{b}}\right))$$

where $b > 0$ and $\operatorname{erf}(x) := 2\pi^{-1/2}\int_0^x e^{-t^2}dt$, one can easily derive the conclusion.

proof. Beware that the proposition (i) proves that there exists *at most one* solution. As always, we can take $f \equiv 0$, $g \equiv 0$, and $u_0 \equiv 0$ to prove the uniqueness. By Lemma 4.2.5, it suffices to consider only the case $\sigma \equiv 0$. Indeed, Lemma 4.2.5 implies that the uniqueness of the equation

$$du(t,x) = (a^{ij} - \alpha^{ij})(t)u_{x^i x^j}(t,x)dt, \quad \alpha^{ij} := \frac{1}{2}(\sigma^i, \sigma^j)_{l_2},$$

implies the uniqueness of the one what we want to prove.

For any given $\phi \in C_0^\infty$ then we have

$$(u(t,\cdot),\phi) = \int_0^t (u(s,\cdot),\phi)ds, \quad t \geq 0,$$

almost surely. Putting here $\phi(\cdot - x)$ instead of ϕ and observing that both sides are continuous and bounded in (t,x) on $[0,T] \times \mathbb{R}^d$ for any $T < \infty$ (cf. eq. (35))(xxx), we get that the function $F(t,x) := (u(t,\cdot),\phi(\cdot - x))$ is bounded in (t,x) on $[0,T] \times \mathbb{R}^d$ for any $T < \infty$, infinitely differentiable in x, and satisfies the equation

$$F(t,x) = \int_0^t L(s)F(s,x)ds \quad \forall t,x \text{ (a.s.)}.$$

From the theory of parabolic equations, it follows that $F(t,x) = 0$ for all t,x (a.s.). This means that $(u(t,\cdot),\phi) = 0$ for all t (a.s.). Now take ϕ with unit integral. Then for any x and n with probability 1, $(u(t,\cdot), n^d\phi(n(\cdot - x))) = 0$ for all t. Since this function is continuous in x, we have $(u(t,\cdot), n^d\phi(n(\cdot - x))) = 0$ for all t and x with probability 1. Finally, $(u(t,\cdot), n^d\phi(n(\cdot - x))) \to u(t,x)$ as $n \to \infty$ for all (ω,t,x) in the sense of distributions, which implies that, with probability 1, we have $u(t,\cdot) = 0$ for all t as stated.

(ii) First, notice that, according to Theorem 4.2 in [19], equation eq. (20) has a unique solution v in the space $\mathbb{H}_2^n(T)$ for any T.(xxxi) The definition of solutions $\mathbb{H}_2^{n+1}(T)$ from [19] is slightly different,(xxxii) but v is continuous (a.s.) as an H_2^n-valued process and

$$\mathbb{E} \sup_{t \leq T} \|v(t,\cdot)\|_{n,2}^n < \infty, \quad \forall T < \infty, \tag{43}$$

so that v is a $\mathfrak{D}$-solution of eq. (20). It follows from (i) that our function u coincides with v and therefore belongs to $\mathbb{H}_2^{n+1}(T)$ for any T, and eq. (43) holds for u. Furthermore, with probability 1 for all t at once,

$$u(t) = u(0) + \int_0^t [a^{ij}(s)u_{x^i x^j}(s) + f(s)]ds + \int_0^t [\sigma^{ik}(s)u_{x^i}(s) + g^k(s)]dw_s^k,$$

where all integrals are taken in the sense of the Hilbert space H_2^{n-1} (see Theorem 2.8 of [19]).

Now claim that there exists an H_2^{n+1}-valued, $\mathcal{W}_t$-predictable function $\bar{u}(t)$ such that, for almost

(xxx) Here, continuity in t follows from the integral form.

(xxxi) Actually, the theorem in [19] only deals with finite sum of stochastic parts. Thus to apply the theorem, we need to be care of.

(xxxii) Our definition uses the distributional application, on the other hand, [19] uses L_2 inner product to describe the definition of the generalized solution. However, [19]'s definition implies our definition.

any t, we have

$$\bar{u}(t) = \mathbb{E}[u(t)|\mathcal{W}_t], \quad \bar{u}_x(t) = \mathbb{E}[u_x(t)|\mathcal{W}_t], \quad \bar{u}_{xx}(t) = \mathbb{E}[u_{xx}(t)|\mathcal{W}_t], \quad \text{(a.s.)}$$

(conditional expectations of Hilbert-space valued random elements)(xxxiii) and

$$\bar{u}(t) = u(0) + \int_0^t [a^{ij}(s)\bar{u}_{x^i x^j}(s) + f(s)]ds + \sum_{k\in W} \int_0^t [\sigma^{ik}(s)\bar{u}_{x^i}(s) + g^k(s)]dw_s^k \tag{44}$$

for almost all t and ω. The core part is proved in Lemma 9.6.1.(xxxiv)

[Fill out the gap]

The right hand side of eq. (44) is continuous H_2^{n-1}-valued process which we denote by $\tilde{u}$ and we show that $\tilde{u}$ is the function we need.

By definition and by the equality $\bar{u} = \tilde{u}$ (a.e.), $\tilde{u}$ satisfies eq. (44) for all t with probability 1 and also is a continuous process in H_2^{n-1}. This implies that $\tilde{u} \in \mathfrak{D}$ and $\tilde{u}$ is a solution of eq. (41).

[Fill out the gap]

4.3 General Equation eq. (20) with Coefficients Independent of x.

Theorem 4.3.1 Take $n \in \mathbb{R}$ and let $f \in \mathbb{H}_p^{n-1}$, $g \in \mathbb{H}_p^n(l_2)$. Then

(i) equation eq. (20) with zero initial condition has a unique solution $u \in \mathcal{H}_p^{n+1}$;

(ii) for this solution, we have

$$\|u_{xx}\|_{\mathbb{H}_p^{n-1}} \le N(\|f\|_{\mathbb{H}_p^{n-1}} + \|g\|_{\mathbb{H}_p^n(l_2)}), \quad \|u\|_{\mathcal{H}_p^{n+1}} \le N\|(f,g)\|_{\mathcal{F}_p^{n-1}}, \tag{45}$$

where $N = N(d, p, \delta, K)$;

(iii) we have $u \in C_{\text{loc}}([0,\infty), H_p^n)$ almost surely and for any $\lambda, T > 0$,

$$\mathbb{E} \sup_{t\le T}(e^{-p\lambda t}\|u(t,\cdot)\|_{n,p}^p) \le N(\|e^{-\lambda t} f\|_{\mathbb{H}_p^{n-1}(T)}^p + \|e^{-\lambda t} g\|_{\mathbb{H}_p^{n-1}(T,l_2)}^p), \tag{46}$$

where $N = N(d, p, \delta, K, \lambda)$.

- -

proof. It suffices to prove the theorem only for $n = 0$. For ther general n, consider $\bar{f} := (1-\Delta)^{n/2} f \in \mathbb{H}_p^{-1}$ and $\bar{g} := (1-\Delta)^{n/2} g \in \mathbb{L}_p(l_2)$. If we assume that the theorem is true for $n = 0$, then take a unique solution $\bar{u} \in \mathcal{H}_p^1$ to

$$d\bar{u}(t,x) = (a^{ij}(t)\bar{u}_{x^i x^j}(t,x) + \bar{f}(t,x))dt + (\sigma^{ik}(t)\bar{u}_{x^i}(t,x) + \bar{g}^k(t,x))dw_t^k, \quad t > 0,$$

with zero initial condition and satisfies eq. (45) and eq. (46) in place of f, g by $\bar{f}\bar{g}$. Now consider $u = (1-\Delta)^{-n/2}\bar{u}$. By the continuity, u also satisfies eq. (45) and eq. (46). In addition, the definition of the term ''in the sense of distributions'', u solves eq. (20) uniquely.

(xxxiii)One can see [19] for its definition, which is, let ξ be a random element valued at some Hilbert space H, and $\mathcal{G}$ be sub σ-field of $\mathcal{F}$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is given. Then $\mathbb{E}[\xi|\mathcal{G}]$ is defined by that for every $\Lambda \in H^*$,

$$\Lambda\mathbb{E}[\xi|\mathcal{G}] = \mathbb{E}[\Lambda\xi|\mathcal{G}] \quad \text{(a.s.)}.$$

(xxxiv)In the original paper, it refers Theorem 1.15 in [19]. However we cannot use the theorem directly because σ-fields are quite different.

As we already noticed in eq. (37), any function $u \in \mathcal{H}_p^1$ also belongs to $\mathfrak{D}$. This and Lemma 4.2.7 prove the uniqueness of (i). Also, the fact that our norms are translation invariant, combined with Lemma 4.2.5, shows that, to prove the existence of (i) and all other assertions of the theorem, we only need to consider the case $\sigma \equiv 0$. As in the proof of Theorem 4.1.2, we can assume that f and g are as in eq. (26). In this case, as we know from [19], equation eq. (20) has a unique $\mathfrak{D}$-solution u that belongs to $C_b([0,T] \times \mathbb{R}^d)$ and $C([0,T], L_2)$ almost surely for any $T < \infty$. It follows that $u \in C([0,T], L_p)$ almost surely for any $T < \infty$. Estimate eq. (46) also follows from [19] as in the proof of Theorem 4.1.2. Although we are dealing with infinite summation on the stochastic part, for the *special* g we only have a finite summation so that we can apply some known theorems (see Theorem 4.2 and Theorem 4.3 of [19]).

It remains only to prove that $u \in \mathcal{H}_p^1$ and that eq. (45). Since u is a $\mathfrak{D}$-solution, to prove that $u \in \mathcal{H}_p^1$, it suffices to prove that $u \in \mathbb{H}_p^1(T)$ for any $T < \infty$.

Since the matrix a is uniformly non-degenerate, by making a nonrandom time change, we can reduce the general case to the case $a \geq I$. In this case, define the matrix-valued function $\bar{\sigma}(t) = \bar{\sigma}^*(t) \geq 0$ as a solution of the equation $\bar{\sigma}^*(t) + 2I = 2a(t)$. Furthermore, without loss of generality, we assume that on our probability space we are also given a d-dimensional Wiener process B_t independent of $\mathcal{F}_t$.

Now, consider the equation

$$dv(t,x) = [\Delta v(t,x) + f(t, x - \int_0^t \bar{\sigma}(s)dB_s)]dt + g^k(t, x - \int_0^t \bar{\sigma}(s)dB_s)dw_t^k \tag{47}$$

with zero initial condition. Replace the predictable σ-field $\mathcal{P}$ with predictable σ-field generated by $\mathcal{F}_t \vee \sigma(B_s; s \leq t)$. Then the space $\mathcal{H}_p^n$ become larger. By Theorem 4.1.2 there is a solution v of eq. (47) possessing properties (i) through (iii) listed in Theorem 4.1.2 (with new $\mathcal{H}_p^1$). Use again that, after changing, if nessary, v on a set of probability zero, the function v becomes a $\mathfrak{D}$-solution of eq. (47). Then by Lemma 4.2.5, the function $z(t,x) := v(t, x + \int_0^t \bar{\sigma}(s)dB_s)$ is a $\mathfrak{D}$-solution of

$$dz(t,x) = (a^{ij}(t)z_{x^i x^j}(t,x) + f(t,x))dt + g^k(t,x)dw_t^k + z_{x^i}(t,x)\bar{\sigma}^{ij}(t)dB_t^j,$$

and by Lemma 4.2.7 there is a solution $\tilde{u} \in \mathfrak{D}$ of

$$d\tilde{u}(t,x) = (a^{ij}(t)\tilde{u}_{x^i x^j}(t,x) + f(t,x))dt + g^k(t,x)dw_t^k,$$

which is eq. (20) in our case. In addition, for any $\phi \in C_0^\infty$ and $t \geq 0$,

$$(\tilde{u}(t,\cdot), \phi) = \mathbb{E}[(z(t,\cdot), \phi)|\mathcal{F}_t] = \mathbb{E}[(v(t, \cdot + \int_0^t \bar{\sigma}(s)dB_s), \phi)|\mathcal{F}_t] \quad \text{(a.s.)}.$$

In particular, it follows from this equality that $\tilde{u}$ is a $\mathfrak{D}$-solutiuon with respect to the initial predictable σ-field $\mathcal{P}$, and from uniqueness we get $\tilde{u} = u$. Therefore,

$$(u(t,\cdot), \phi) = \mathbb{E}[(v(t, \cdot + \int_0^t \bar{\sigma}(s)dB_s), \phi)|\mathcal{F}_t] \quad \text{(a.s.)}. \tag{48}$$

It follows, using Jensen's inequality, that

$$|(u(t,\cdot), \phi)|^p \leq \mathbb{E}[\|v(t,\cdot)\|_{1,p}^p|\mathcal{F}_t]\|\phi\|_{-1,q}^p \quad \text{(a.s.)} \tag{49}$$

for any $\phi \in C_0^\infty$ and $t \geq 0$, where $q = p/(p-1)$.

Now we are ready to prove $u \in \mathcal{H}_p^1$. Here are two steps to show it. First of all, we are going to use the fact that the set

$$\{(\omega, t) : w(\omega, t, \cdot) \in H_p^1\}$$

is $\mathcal{P}$-measurable for every $w \in \mathfrak{D}$. Take a countable subset $\Phi \subset C_0^\infty$ which is dense in C_0^∞. Observe that, given a distribution ψ, we have $\psi \in H_p^1$ if and only if $|(\psi, \phi)| \leq N\|\phi\|_{-1,q}$ for every $\phi \in \Phi$ where

N is a constant independent of ϕ. Such fact directly comes from the duality between H^1_p and H^{-1}_q, and the density Φ in C^∞_0. Now fix $w \in \mathfrak{D}$. Then since (w, ϕ) is predictable for every $\phi \in C^\infty_0$, the set

$$\{(\omega,t) : w(\omega,t,\cdot) \in H^1_p\} = \bigcap_{\phi\in\Phi} \bigcup_{N=1}^{\infty} \{(\omega,t) : |(w(\omega,t,\cdot),\phi)| \le N\|\phi\|_{-1,q}\}$$

is also predictable. Recall that $u \in \mathfrak{D}$.

We also know that $v \in \mathcal{H}^1_p$. By the definition of $\mathcal{H}^1_p$, for every $T < \infty$, $\mathbb{E}\int_0^T \|v(t,\cdot)\|^p_{1,p}dt < \infty$. Thus for almost every t, $\mathbb{E}\|v(t,\cdot)\|^p_{1,p} < \infty$. Put $U \subset [0,\infty)$ be a collection of such t. Fix $t \in U$. Then there exists a set Ω'_t of probability 1 such that $\mathbb{E}[\|v(t,\cdot)\|^p_{1,p}|\mathcal{F}_t] < \infty$ on Ω'_t and eq. (49) holds for all $\omega \in \Omega'_t$ and $\phi \in \Phi$. Hence, $u(t,\cdot) \in H^1_p$ for the chosen t and all $\omega \in \Omega'$. As it is mentioned before, the set

$$\{(\omega,t) : u(\omega,t,\cdot) \in H^1_p\}$$

is predictable (hence jointly measurable). By the Fubini's theorem, we have

$$\mathbb{P} \times \ell\{(\omega,t) : u(\omega,t,\cdot) \notin H^1_p\} = \mathbb{E}\int_U \{\omega : u(\omega,t,\cdot) \notin H^1_p\}dt = 0$$

because $\{\omega : u(\omega,t,\cdot) \notin H^1_p\} \subset \Omega \setminus \Omega'_t$ for every $t \in U$. Therefore, $u(t,\cdot) \in H^1_p$ for almost all (ω,t). Together with eq. (49), it follows that

$$\|u(t,\cdot)\|^p_{1,p} \le \mathbb{E}[\|v(t,\cdot)\|^p_{1,p}|\mathcal{F}_t] \quad \text{(a.s.) for (a.a.) } t, \quad \|u\|_{\mathbb{H}^1_p(T)} \le \|v\|_{\mathbb{H}^1_p(T)} < \infty.$$

Thus $u \in \mathbb{H}^1_p(T)$ for any $T < \infty$ indeed and $u \in \mathcal{H}^1_p$.

Similarly, from the equality

$$(u_{xx}(t,\cdot),\phi) = \mathbb{E}[(v_{xx}(t,\cdot + \int_0^t \bar{\sigma}(s)dB_s),\phi)|\mathcal{F}_t] \quad \text{(a.s.)},$$

one gets

$$\|u_{xx}(t,\cdot)\|^p_{-1,p} \le \mathbb{E}[\|v_{xx}(t,\cdot)\|^p_{-1,p}|\mathcal{F}_t] \quad \text{(a.s.)}.$$

This and the properties of v immediately yields eq. (45). The theorem is proved.

Remark 4.3.2 By using the self-similarity of equation eq. (20), it is possible to obtain further estimates from estimates like eq. (46). For instance, remembering that $H^1_p = W^1_p$, one sees that, for $n = 1$ and $\lambda = 1/p$, estimate eq. (46) implies that

$$\mathbb{E}\sup_{t\le T}[\|u_x(t,\cdot)\|^p_p + \|u(t,\cdot)\|^p_p] \le N(d,p,\delta,K)e^T(\|f\|^p_{\mathbb{L}_p(T)} + \|g_x\|^p_{\mathbb{L}_p(T,l_2)} + \|g\|^p_{\mathbb{L}_p(T,l_2)}).$$

Let us take a constant $c > 0$ and consider $(c^2f, cg)(c^2t, cx)$, $c^{-1}w_{c^2t}$, and $u(c^2t,cx)$ instead of f, g, w and u. Then, from the last estimate, we get

$$\begin{aligned}&\mathbb{E}\sup_{t\le T}[c^{p-d}\|u_x(c^2t,\cdot)\|^p_p + c^{-d}\|u(c^2t,\cdot)\|^p_p]\\ &\qquad\le Ne^T(c^{2p-(d+2)}\|f\|^p_{\mathbb{L}_p(c^2T)} + c^{2p-(d+2)}\|g_x\|^p_{\mathbb{L}(c^2T,l_2)} + c^{p-(d+2)}\|g\|^p_{\mathbb{L}_p(c^2T,l_2)}),\end{aligned}$$

the constant N being the same as above. It follows that, for c, $T \ge 0$,

$$\mathbb{E}\sup_{t\le T}[\|u_x(t,\cdot)\|^p_p + c^{-p}\|u(t,\cdot)\|^p_p] \le Ne^{T/c^2}c^{p-2}(\|f\|^p_{\mathbb{L}_p(T)} + \|g_x\|^p_{\mathbb{L}_p(T,l_2)} + c^{-p}\|g\|^p_{\mathbb{L}_p(T,l_2)}).$$

Upon setting $c^2 = T$ and considering $(1-\Delta)^{(n-1)/2}u$ instead of u, we conclude that

$$\mathbb{E}\sup_{t\le T}[\|u_x(t,\cdot)\|^p_{n-1,p} + T^{-p/2}\|u(t,\cdot)\|^p_{n-1,p}]$$

$$\leq N(d, p, \delta, K)T^{(p-2)/2}(\|f\|^p_{\mathbb{H}^{n-1}_p(T)} + \|g_x\|^p_{\mathbb{H}^{n-1}_p(T,l_2)} + T^{-p/2}\|g\|^p_{\mathbb{H}^{n-1}_p(T,l_2)}).$$

We will later prove a much deeper estimate than eq. (46).

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5 Equations with Variable Coefficients

Take a stopping time $\tau \leq T$ with T belong a finite constant. Fix $n \in \mathbb{R}$ and fix a number $\gamma \in [0, 1)$ such that $\gamma = 0$ if $n \in \mathbb{Z}$; otherwise $\gamma > 0$ and is such that $|n| + \gamma$ is not an integer. Define

$$B^{|n|+\gamma} = \begin{cases} B(\mathbb{R}^d) & \text{if } n = 0, \\ C^{|n|-1,1}(\mathbb{R}^d) & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ C^{|n|+\gamma}(\mathbb{R}^d) & \text{otherwise;} \end{cases}$$

$$B^{|n|+\gamma}(l_2) = \begin{cases} B(\mathbb{R}^d, l_2) & \text{if } n = 0, \\ C^{|n|-1,1}(\mathbb{R}^d, l_2) & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ C^{|n|+\gamma}(\mathbb{R}^d, l_2) & \text{otherwise,} \end{cases}$$

where $B(\mathbb{R}^d)$ is the Banach space of bounded functions on $\mathbb{R}^d$, $C^{|n|-1,1}(\mathbb{R}^d)$ is the Banach space of $|n| - 1$ times continuously differentiable functions whose derivatives of $(|n| - 1)$st order satisfy the Lipschitz condition on $\mathbb{R}^d$, $C^{|n|+\gamma}(\mathbb{R}^d)$ is the usual Hölder space, and l_2 means that instead of real-valued functions we consider l_2-valued ones.

Consider the following nonlinear equation on $[0, \tau]$:

$$du(t,x) = [a^{ij}(t,x)u_{x^ix^j}(t,x) + f(u,t,x)]dt + [\sigma^{ik}(t,x)u_{x^i}(t,x) + g^k(u,t,x)]dw^k_t, \tag{50}$$

where a^{ij} and f are real-valued, and σ^i and g are l_2-valued functions defined for $\omega \in \Omega$, $t \geq 0$, $x \in \mathbb{R}^d$, $u \in H^{n+2}_p$, $i, j = 1, \cdots, d$.(xxxv) We consider this equation in the sense of Definition 3.0.7 (where we take $n + 2$ instead of n). We make the following assumptions, where, as in Section 4, we define

$$\alpha^{ij}(t,x) = \frac{1}{2}(\sigma^i(t,x), \sigma^j(t,x))_{l_2}. \tag{51}$$

Assumption 5.0.1 (coercivity) For any $\omega \in \Omega$, $t \geq 0$, $x, \lambda \in \mathbb{R}^d$, we have

$$K|\lambda|^2 \geq [a^{ij}(t,x) - \alpha^{ij}(t,x)]\lambda^i\lambda^j \geq \delta|\lambda|^2,$$

where K, δ are fixed strictly positive constants.

Assumption 5.0.2 (uniform continuity of a and σ) For any $\epsilon > 0, i, j$, there exists a $\kappa_\epsilon > 0$ such that

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|^2_{l_2} \leq \epsilon \tag{52}$$

whenever $|x - y| \leq \kappa_\epsilon$, $t \geq 0$, $\omega \in \Omega$.

This assumption is actually used only if $n = 0$, and even then we need a stronger condition on σ. For other values of n we impose stronger conditions on a and σ.

(xxxv) More presicely for f and g, their signatures are

$$f : \Omega \times H^{n+2}_p \times [0, \infty) \to H^n_p, \quad g : \Omega \times H^{n+2}_p \times [0, \infty) \to H^{n+1}_p(\mathbb{R}^d, l_2).$$

Assumption 5.0.3 For any i, j, k, the functions $a^{ij}(t,x)$ and $\sigma^{ik}(t,x)$ are real-valued $\mathcal{P} \otimes \mathfrak{B}(\mathbb{R}^d)$-measurable functions, and for any $\omega \in \Omega$ and $t \geq 0$, we have

$$a^{ij}(t,\cdot) \in B^{|n|+\gamma}, \quad \sigma^i(t,\cdot) \in B^{|n+1|+\gamma}(l_2).$$

Assumption 5.0.4 For any $u \in H_p^{n+2}$, the functions $f(u,t,\cdot)$ and $g(u,t,\cdot)$ are predictable as functions taking values in H_p^n and $H_p^{n+1}(\mathbb{R}^d, l_2)$, respectively.

Assumption 5.0.5 For any $t \geq 0, \omega, i, j$,

$$\|a^{ij}(t,\cdot)\|_{B^{|n|+\gamma}} + \|\sigma^i(t,\cdot)\|_{B^{|n+1|+\gamma}(l_2)} \leq K, \quad (f(0,\cdot,\cdot), g(0,\cdot,\cdot)) \in \mathcal{F}_p^n(\tau).$$

Assumption 5.0.6 The functions f, g are continuous in u. Moreover, for any $\epsilon > 0$, there exists a constant K_ϵ such that, for any $u, v \in H_p^{n+2}, t, \omega$, we have

$$\|f(u,t,\cdot) - f(v,t,\cdot)\|_{n,p} + \|g(u,t,\cdot) - g(v,t,\cdot)\|_{n+1,p} \leq \epsilon\|u-v\|_{n+2,p} + K_\epsilon\|u-v\|_{n,p}. \tag{53}$$

Theorem 5.0.7 Let Assumption 5.0.1 through Assumption 5.0.6 be satisfied and let

$$u_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{n+2-2/p}).$$

Then the Cauchy problem form eq. (50) on $[0,\tau]$ with the initial condition $u(0,\cdot) = u_0$ has a unique solution $u \in \mathcal{H}_p^{n+2}(\tau)$. For this solution, we have

$$\|u\|_{\mathcal{H}_p^{n+2}(\tau)} \leq N[\|f(0,\cdot,\cdot)\|_{\mathbb{H}_p^n(\tau)} + \|g(0,\cdot,\cdot)\|_{\mathbb{H}_p^{n+1}(\tau,l_2)} + (\mathbb{E}\|u_0\|^p_{n+2-2/p,p})^{1/p}],$$

where the constant N depends only on $d, n, \gamma, p, \delta, K, T$, and the functions κ_ϵ and K_ϵ.

To discuss the theorem, we need the following lemma.

Lemma 5.0.8 Let $\zeta \in C_0^\infty$ be a nonnegative function such that $\int \zeta = 1$ and define $\zeta_k(x) = k^d\zeta(kx)$, $k = 1, 2, 3, \cdots$. We assert that, for any $u \in H_p^n$, we have the following:

(i) $\|au\|_{n,p} \leq N\bar{a}\|u\|_{n,p}$, where $\bar{a} = \|a\|_{B^{|n|+\gamma}}$ and the constant N depends only on d, p, n, and γ.

(ii) $\|u * \zeta_k\|_{n,p} \leq \|u\|_{n,p}$, $\|u - u * \zeta_k\|_{n,p} \to 0$.

The same assertions hold true for Banach-space valued a with natural definition of $\bar{a}$.

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proof. If n is not an integer (and $\gamma > 0$), then one gets (i) by Corollary 2.8.2 (ii) of [23]. Indeed, since $p \geq 2$, actually we have

$$\|au\|_{n,p} \leq \|a\|_{\mathcal{C}^{|n|+\gamma}}\|u\|_{n,p}$$

where $\mathcal{C}^s$ $(s > 0)$ denotes the Zygmund space (recall that $H_p^n = F_{p,2}^n$). Now recall that $B^{|n|+\gamma} = C^{|n|+\gamma} \subset \mathcal{C}^{|n|+\gamma}$. If n is nonnegative integer, then (i) follows from the Leibniz rule, the fact that

$H_p^n = W_p^n$, and notice that for (a.a.) $x \in \mathbb{R}^d$,

$$\sum_{|\alpha|=|n|} |D^\alpha a(x)| \le N \sum_{|\alpha|=|n|-1} \sup_{\substack{t,s\in\mathbb{R}^d \\ t\neq s}} \frac{|D^\alpha a(t) - D^\alpha a(s)|}{|t-s|}.$$

For negative integers n (and generally for negative n) (i) follows easily by duality, that is, by using the fact that if $u = (1-\Delta)^{-n/2} f$, then (xxxvi)

$$(au, \phi) = (f, (1-\Delta)^{-n/2}(a\phi)) \le \|f\|_p \|a\phi\|_{-n,q}, \quad q = \frac{p}{p-1}.$$

This gives, therefore,

$$\|au\|_{n,p} = \sup\{|(au,\phi)| : \phi \in C_0^\infty,\ \|\phi\|_{-n,q} \le 1\} \le \bar{a}\|u\|_{n,p}.$$

For (ii), the first inequality follows from Minkowski's inequality and the second one comes from the denseness of C_0^∞ in H_p^n.

Remark 5.0.9 As we have said above, by solution to the Cauchy problem for equation eq. (50) on $[0,\tau]$ with the given initial condition u_0, we understand a function $u \in \mathcal{H}_p^{n+2}(\tau)$ such that for any test function $\phi \in C_0^\infty$, one has almost surely

$$(u(t,\cdot),\phi) = (u_0,\phi) + \int_0^t (a^{ij}(s,\cdot)u_{x^ix^j}(s,\cdot) + f(u,s,\cdot),\phi)ds + \int_0^t (\sigma^{ik}(s,\cdot)u_{x^i}(s,\cdot) + g^k(u,s,\cdot),\phi)dw_s^k$$

for every $t \in [0,\tau]$. It is important that, under our assumptions, the equation makes sense for $u \in \mathcal{H}_p^{n+2}(\tau)$, since by Lemma 5.0.8 we have $a^{ij}u_{x^ix^j} \in H_p^n$, $\sigma^i u_{x^i} \in H_p^{n+1}(\mathbb{R}^d, l_2)$ whenever $u \in H_p^{n+2}$.

Remark 5.0.10 Two main ideas in the proof of this theorem are quite standard. The first one, reduction to equations with constant coefficients, will be seen very clearly. The second one, which is somewhat hidden, consists of introducing the new unknown function $v = (1-\Delta)^{n/2}u$, which reduces the case of general n to the case $n = 0$. Then function v satisfies

$$\begin{aligned} dv &= \left[(1-\Delta)^{n/2}(a^{ij}(1-\Delta)^{-n/2}v_{x^ix^j}) + (1-\Delta)^{n/2} f\right] dt \\ &\quad + \left[(1-\Delta)^{n/2}(\sigma^{ik}(1-\Delta)^{-n/2}v_{x^i}) + (1-\Delta)^{n/2} g^k\right] dw_t^k. \end{aligned}$$

This is a pseudo-differential equation, and we note that more general pseudo-differential equations can be considered too. Also, this equations shows a need to have smoothness assumptions on a, σ in x if we are interested in $n \neq 0$ both positive or negative.

Remark 5.0.11 Let X_0, X_1 be complex Banach spaces. Assume that there exists a linear complex Hausdorff space X such that $X_j \hookrightarrow X$, $j = 0,1$, where $\hookrightarrow$ means that there is a continuous linear embedding. Then it is known that for $0 < \theta < 1$,

$$\|a\|_{[X_0,X_1]_\theta} \le N(\theta)\|a\|_{X_0}^{1-\theta}\|a\|_{X_1}^{\theta},$$

where $[X_0,X_1]_\theta$ denotes the complex interpolation of two spaces X_0 and X_1. (see Theorem 1.9.3 of

(xxxvi) Actually, one can ‘‘define’’ au like the first equality.

[22]). In addition, by the theory of function spaces, for $p_0, p_1 \in (1,\infty)$, and $s_0, s_1 \in \mathbb{R}$,

$$[H^{s_0}_{p_0}, H^{s_1}_{p_1}]_\theta = H^s_p$$

where $\theta \in (0,1)$, and

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

One can see Remark 2.4.7/2 of [23], or page 182-185 of [22].

From these facts, for any $u \in H^{n+2}_p$ and $m \in [n, n+2]$, we have

$$\|u\|_{m,p} \le N\|u\|^\theta_{n+2,p}\|u\|^{1-\theta}_{n,p} \le N\theta\epsilon\|u\|_{n+2,p} + N(1-\theta)\epsilon^{-\theta/(1-\theta)}\|u\|_{n,p},$$

where $\theta = (m-n)/2$ and N depends only on d, n, m, and p.(xxxvii) This shows that the right-hand side in eq. (53) can be replaced by $\|u-v\|_{n+\epsilon+1,p}$ once $|\epsilon| < 1$. As an example, one can take $f = f_0(x)\sup_x |u_x|$ if $(n+1)p > d$ and $f_0 \in H^n_p$. Indeed, by Sobolev's embedding theorems, $H^{n+1+\epsilon}_p \subset C^1$ if $(n+\epsilon)p > d$. Therefore,

$$\|f(u,t,\cdot) - f(v,t,\cdot)\|_{n,p} \le \|f_0\|_{n,p} \sup_x |(u-v)_x| \le N\|u-v\|_{n+1+\epsilon,p}.$$

Remark 5.0.12 A typical application of Theorem 5.0.7 occurs when $f(u,t,x) = b^i(t,x)u_{x^i} + c(t,x)u + f(t,x)$ and $g(u,t,x) = \nu(t,x)u + g(t,x)$, so that eq. (50) becomes

$$du = (a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f)dt + (\sigma^{ik}u_{x^i} + \nu^k u + g^k)dw^k_t. \tag{54}$$

To describe the appropriate assumptions, we take $\epsilon \in (0,1)$ and denote

$$\begin{array}{llll} n_b = n+\gamma & \text{if} \;\; n \ge 0, & n_b = 0 & \text{if} \;\; n \in (-1,0], \\ n_\nu = n+1+\gamma & \text{if} \;\; n \ge -1, & n_\nu = 0 & \text{if} \;\; n \in (-2,-1], \\ n_c = n+\gamma & \text{if} \;\; n \ge 0, & n_c = 0 & \text{if} \;\; n \in (-2,0], \end{array}$$

$$\begin{array}{ll} n_b = -n-1+\epsilon & \text{if} \;\; n \le -1, \\ n_\nu = -n-2+\epsilon & \text{if} \;\; n \le -2, \\ n_c = -n-2+\epsilon & \text{if} \;\; n \le -2. \end{array}$$

Assume that b, c, and ν are appropriately measurable and

$$b^i(t,\cdot) \in B^{n_b}, \quad c(t,\cdot) \in B^{n_c}, \quad \nu(t,\cdot) \in B^{n_\nu}(\mathbb{R}^d, l_2),$$

$$f(t,\cdot) \in H^n_p, \quad g(t,\cdot) \in H^{n+1}_p(\mathbb{R}^d, l_2).$$

$$\|b^i(t,\cdot)\|_{B^{n_b}} + \|c(t,\cdot)\|_{B^{n_c}} + \|\nu(t,\cdot)\|_{B^{n_\nu}(\mathbb{R}^d,l_2)} \le K, \quad (f(\cdot,\cdot), g(\cdot,\cdot)) \in \mathcal{F}^n_p(\tau).$$

It turns out then that assumptions of Theorem 5.0.7 about $f(u,t,x)$ and $g(u,t,x)$ are satisfied. To show this, it suffices to apply remark 5.0.11. For the part g, notice that, for instance, if $n \ge -1$, then $\|\nu u\|_{n+1,p} \le N\|u\|_{n+1,p}$ by Lemma 5.0.8; if $n \in (-2,-1]$, then $n+1 \in (-1,0]$ and $\nu \in B(\mathbb{R}^d, l_2)$, thus

$$\|\nu u\|_{n+1,p} \le \|\nu u\|_p \le N\|u\|_p = N\|u\|_{n+1+(-n-1),p};$$

if $n \le -2$, then Lemma 5.0.8 yields $\|\nu u\|_{n+1,p} \le \|\nu u\|_{n+2-\epsilon_1,p} \le N\|u\|_{n+2-\epsilon_1,p}$, where $\epsilon_1 \in (0,\epsilon)$. For the term f, we use $\|\cdot\|_{n,p}$ instead of $\|\cdot\|_{n+1,p}$, and terms $\|b^iu_{x^i}\|_{n,p}$ and $\|cu\|_{n,p}$ are considered similarly.

Actually, the above conditions on b, c, and ν can be considerably relaxed if one makes use of deeper theorems about multipliers from [23].

Conditions eq. (55) and eq. (56) of the following theorem are discussed in Remark 5.0.14 and Remark 5.0.15.

(xxxvii)By the Jensen's inequality, one can easily prove the second inequality.

Theorem 5.0.13 Assume that for $m = 1, 2, 3, \cdots$, we are given a^{ij}_m, σ^i_m, f_m, g_m, and u_{0m} having the same sense as in Theorem 5.0.7 and verifying the same assumptions as a^{ij}, σ^i, f, g, and u_0 with the same constants δ, K, κ_ϵ, and K_ϵ. Let $\zeta(x)$ be a real function of class C^∞_0 such that $\zeta(x) = 1$ if $|x| \le 1$ and $\zeta(x) = 0$ if $|x| \ge 2$. Define $\zeta_r(x) = \zeta(x/r)$ and assume that, for any $r = 1, 2, 3, \cdots$, $i, j = 1, .., d$, $t \ge 0$, and $\omega \in \Omega$,

$$\|\zeta_r[a^{ij}(t,\cdot) - a^{ij}_m(t,\cdot)]\|_{n,p} + \|\zeta_r[\sigma^i(t,\cdot) - \sigma^i_m(t,\cdot)]\|_{n,p} \to 0 \tag{55}$$

as $m \to \infty$. Finally, let $\mathbb{E}\|u_{0m} - u_0\|^p_{n+2-2/p,p} \to 0$ and

$$\|f(u,\cdot,\cdot) - f_m(u,\cdot,\cdot)\|_{\mathbb{H}^n_p(\tau)} \to 0, \quad \|g(u,\cdot,\cdot) - g_m(u,\cdot,\cdot)\|_{\mathbb{H}^{n+1}_p(\tau,l_2)} \to 0 \tag{56}$$

whenever $u \in \mathcal{H}^{n+2}_p(\tau)$. Take the function u from Theorem 5.0.7 and for any m define $u_m \in \mathcal{H}^{n+2}_p(\tau)$ as the (unique) solution of the Cauchy problem for the equation

$$du_m(t,x) = [a^{ij}_m(t,x)u_{mx^ix^j}(t,x) + f_m(u_m,t,x)]dt + [\sigma^{ik}_m(t,x)u_{mx^i}(t,x) + g^k_m(u_m,t,x)]dw^k_t \tag{57}$$

on $[0,\tau]$ with initial condition $u_m(0,\cdot) = u_{0m}$. Then $\|u - u_m\|_{\mathcal{H}^{n+2}_p(\tau)} \to 0$ as $m \to \infty$.

- -

proof. For $v_m = u - u_m$ we have

$$dw_m(t) = [a^{ij}_m v_{mx^ix^j} + F_m(v_m)]dt + [\sigma^{ik}_m v_{mx^i} + G^k_m(v_m)]dw^k_t,$$

where

$$F_m(v) = (a^{ij} - a^{ij}_m)u_{x^ix^j} + f(u) - f_m(u - v), \quad G^k_m(v) = (\sigma^{ik} - \sigma^{ik}_m)u_{x^i} + g^k(u) - h^k_m(u - v).$$

Hence, from our assumptions and by Theorem 5.0.7, we obtain

$$\|u - u_m\|_{\mathcal{H}^{n+2}_p(\tau)} \le NI_m,$$

where N is independent of m and

$$\begin{aligned} I_m = {} & \|(a^{ij} - a^{ij}_m)u_{x^ix^j}\|_{\mathbb{H}^n_p(\tau)} + \|f(u) - f_m(u)\|_{\mathbb{H}^n_p(\tau)} + \|(\sigma^i - \sigma^i_m)u_{x^i}\|_{\mathbb{H}^{n+1}_p(\tau,l_2)} \\ & + \|g(u) - g_m(u)\|_{\mathbb{H}^{n+1}_p(\tau,l_2)} + (\mathbb{E}\|u_0 - u_{0m}\|^p_{n+2-2/p,p})^{1/p}. \end{aligned}$$

Next, by our assumptions about convergence of f_m, g_m, u_{0m},

$$\overline{\lim_{m\to\infty}}\, I_m \le \overline{\lim_{m\to\infty}}\, [\|(a^{ij} - a^{ij}_m)u_{x^ix^j}\|_{\mathbb{H}^n_p(\tau)} + \|(\sigma^i - \sigma^i_m)u_{x^i}\|_{\mathbb{H}^{n+1}_p(\tau,l_2)}]. \tag{58}$$

Here, by Lemma 5.0.8, for any $v \in C^\infty_0$ and r so large that $v\zeta_r = v$, we have

$$\|(a^{ij} - a^{ij}_m)u_{x^ix^j}\|_{n,p} \le N\|(u - v)_{x^ix^j}\|_{n,p} + \|(a^{ij} - a^{ij}_m)v_{x^ix^j}\|_{n,p}, \tag{59}$$

$$\|(a^{ij} - a^{ij}_m)v_{x^ix^j}\|_{n,p} = \|\zeta_r(a^{ij} - a^{ij}_m)v_{x^ix^j}\|_{n,p} \le N\|\zeta_r(a^{ij} - a^{ij}_m)\|_{n,p}\|v\|_{B^{|n|+2+\gamma}},$$

where the constants N do not depend on m and r. Thus

$$\overline{\lim_{m\to\infty}}\, \|(a^{ij} - a^{ij}_m)u_{x^ix^j}\|_{n,p} \le N\|(u - v)_{x^ix^j}\|_{n,p},$$

and from the arbitrariness of v, we conclude that the left hand side is zero for those ω and t for which $u \in H^{n+2}_p$. If we again apply Lemma 5.0.8, then we see that the pth power of the left hand side of eq. (59) is bounded by an integrable function. This and the DCT imply that

$$\lim_{m\to\infty} \|(a^{ij} - a^{ij}_m u_{x^ix^j}\|_{\mathbb{H}^n_p(\tau)} = 0.$$

Similar arguments take care of remaining term in eq. (58).

Remark 5.0.14 Condition eq. (56) is satisfied for any $u \in \mathcal{H}_p^{n+2}(\tau)$ if and only if it is satisfied for $u(t,x) \equiv \phi(x)$ with any $\phi \in C_0^\infty$. Indeed, the ''only if'' part is obvious. In the proof of ''if'' part notice that, under the ''if'' assumption, eq. (56) is automatically satisfied for u of type

$$v = \sum_{i=1}^{j} \mathbb{1}_{(\tau_{i-1},\tau_i]}(t) v_i(x),$$

where τ_i are bounded stopping times and $v_i \in C_0^\infty$.[(xxxviii)] By Theorem 3.0.13, one can approximate any $u \in \mathcal{H}_p^{n+2}(\tau) \subset \mathbb{H}_p^{n+2}(\tau)$ with functions like v. It remains only to notice that, by the assumption of the theorem, for instance,

$$\begin{aligned}\|f(u,\cdot,\cdot) - f_m(u,\cdot,\cdot)\|_{\mathbb{H}_p^n(\tau)} &\le \|f(u,\cdot,\cdot) - f(v,\cdot,\cdot)\|_{\mathbb{H}_p^n(\tau)} + \|f(v,\cdot,\cdot) - f_m(v,\cdot,\cdot)\|_{\mathbb{H}_p^n(\tau)} \\ &\quad + \|f_m(v,\cdot,\cdot) - f_m(u,\cdot,\cdot)\|_{\mathbb{H}_p^n(\tau)} \\ &\le N\|u - v\|_{\mathbb{H}_p^{n+2}(\tau)} + \|f(v,\cdot,\cdot) - f_m(v,\cdot,\cdot)\|_{\mathbb{H}_p^n(\tau)}.\end{aligned}$$

Remark 5.0.15 While checking conditions eq. (55) and eq. (56), it is useful to bear in mind that, if one defines $\sigma_m^{ik} = \sigma^{ik}$ and $g_m^k = g^k$ for $k \le m$ and $\sigma_m^{ik} = g_m^k = 0$ for $k > m$, then

$$\|\zeta_r[\sigma^i(t,\cdot) - \sigma_m^i(t,\cdot)]\|_{n+1,p} \to 0, \quad \|g(u,\cdot,\cdot) - g_m(u,\cdot,\cdot)]\|_{\mathbb{H}_p^{n+1}(\tau,l_2)} \to 0.$$

Indeed, for instance,

$$\|g(u,\cdot,\cdot) - g_m(u,\cdot,\cdot)\|_{\mathbb{H}_p^{n+1}(\tau,l_2)}^p = \mathbb{E}\int_0^\tau \left\| \left(\sum_{k>m} |(1-\Delta)^{(n+1)/2} g^k(u,t,\cdot)|^2 \right)^{1/2} \right\|_p^p dt,$$

which goes to zero by the dominated convergence theorem because the integrand is bounded by $\|g(u,t,\cdot)\|_{n+1,p}$.

This fact allows one to approximate solutions of eq. (50) by solutions of

$$du_m(t,x) = [a^{ij}(t,x)u_{mx^ix^j}(t,x) + f(u_m,t,x)]dt + \sum_{k\le m} [\sigma^{ik}(t,x)u_{mx^i}(t,x) + g^k(u_m,t,x)]dw_t^k. \quad (60)$$

Before starting the following corollary, take the functions ζ_k from Lemma 5.0.8 and, for a function $h = h(u,t,x)$ defined for $t \ge 0$, $x \in \mathbb{R}^d$, and u in a function space, let

$$\hat{h}_m(u,t,x) = \int_{\mathbb{R}^d} h(u(\cdot + y), t, x - y)\zeta_m(y)dy. \quad (61)$$

To get a better idea about this definition, notice that if, for instance, $h(u,t,x) = c(t,x)u(t,x)$, where $c(t,x)$ is a given function, then $h(u(\cdot+y),t,x-y) = c(t,x-y)u(t,x)$, so that $\hat{h}_m(u,t,x) = c_m(t,x)u(t,x)$ with $c_m(t,\cdot) = c(t,\cdot) * \zeta_m$.

Corollary 5.0.16 Under the assumptions of Theorem 5.0.7, define

$$(a_m,\sigma_m)(t,x) = (a,\sigma)(t,x) * \zeta_m(x),$$

$$(f_m,g_m) = (\hat{f}_m,\hat{g}_m) \quad \text{or} \quad (f_m,g_m)(u,t,x) = (f,g)((u,t,x) * \zeta_m(x).$$

Then the assumptions of Theorem 5.0.13 are satisfied, and if we define $u_m \in \mathcal{H}_p^{n+2}(\tau)$ as a solution of the Cauchy problem eq. (57) with initial condition $u_m(0,\cdot) = u_0 * \zeta_m$, then $\|u - u_m\|_{\mathcal{H}_p^{n+2}(\tau)} \to 0$.

(xxxviii) Since each time interval is pairwise disjoint, eq. (56) is automatically holds.

- -

proof. Notice that

$$(\sigma^i_m, \sigma^j_m)_{l_2}\lambda^i\lambda^j = |(\sigma^i\lambda^i) * \zeta_m|^2_{l_2} \le \zeta_m * [(\sigma^i, \sigma^j)_{l_2}\lambda^i\lambda^j],$$

which guarantees that the approximating equations satisfy the same coercivity condition. Also, for any $u \in H^{n+2}_p$ and $\phi \in C^\infty_0$, the function $(f(u(\cdot + z), t, \cdot - y), \phi) = (f(u(\cdot + z), t, \cdot), \phi(\cdot + y))$ is continuous in z owing to Assumption 5.0.6[(xxxix)] and infinitely differentiable in y as it has to be for any distribution. Therefore, the definition of $\hat{f}_m$ according to eq. (61) makes sense[(xl)] as an integral of distribution with respect to the parameter y, namely

$$(\hat{f}_m(u, t, \cdot), \phi) := \int_{\mathbb{R}^d} \zeta_m(y)(f(u(\cdot + y), y, \cdot - y), \phi)dy. \tag{62}$$

Furthermore, $\hat{f}_m = \hat{f}_{0m} + \hat{f}_{1m}$, where $f_0 = f(0, t, x)$ and $f_1 = f - f_0$. By Lemma 5.0.8, we have $\hat{f}_{0m}(t, \cdot) \in H^n_p$.[(xli)] By Assumption 5.0.6,

$$|(\hat{f}_{1m}(u, t, \cdot), \phi)| \le N \int_{\mathbb{R}^d} \zeta_m(y)\|\phi\|_{-n,q}\|u(\cdot + y)\|_{n+2,p}dy = N\|\phi\|_{-n,q}\|u\|_{n+2,p},$$

where $q = p/(p-1)$. Hence, $\hat{f}_m$ is a H^n_p-valued function. Also, $f(u, t, x) * \zeta_m(x)$ is well-defined because of Assumption 5.0.4. Similarly, g_m is well-defined. Minkowski's inequality implies that all Assumption 5.0.1 to 5.0.6 are satisfied for any m with the same $K, \delta, \kappa_\epsilon, K_\epsilon$. Other assumptions of Theorem 5.0.13 are satisfied due to continuity in the mean of summable functions and Assumption 5.0.1 to 5.0.6. Indeed, for a, by Lemma 5.0.8 and Assumption 5.0.2, if we let $\bar{\zeta}_r$ be ζ_r in Theorem 5.0.13,

$$\begin{aligned}\|\bar{\zeta}_r[a^{ij}(t, \cdot) - a^{ij}_m(t, \cdot)\|_{n,p} &\le \|\bar{\zeta}_r a^{ij}(t, \cdot) - (\bar{\zeta}_r a^{ij})(t, \cdot) * \zeta_m\|_{n,p} + \|(\bar{\zeta}_r a^{ij})(t, \cdot) * \zeta_m - \bar{\zeta}_r(a^{ij}(t, \cdot) * \zeta_m)\|_{n,p} \\ &\le I_m + J_m,\end{aligned}$$

where

$$I_m := \|\bar{\zeta}_r a^{ij}(t, \cdot) - (\bar{\zeta}_r a^{ij})(t, \cdot) * \zeta_m\|_{n,p}, \quad J_m := \|a^{ij}(t, \cdot)\|_{B^{|n|+\gamma}} \int_{\mathbb{R}^d} \|\bar{\zeta}_r - \bar{\zeta}_r(\cdot + y)\|_{n,p}\zeta_m(y)dy.$$

Here, Lemma 5.0.8 yields $\bar{\zeta}_r a^{ij} \in H^n_p$, and $I_m \to 0$ as $m \to \infty$. On the other hand, by changing of variables,

$$J_m = \|a^{ij}(t, \cdot)\|_{B^{|n|+\gamma}} \int_{\mathbb{R}^d} \|\bar{\zeta}_r - \bar{\zeta}_r(\cdot + m^{-1}y)\|_{n,p}\zeta(y)dy.$$

By the dominated convergence theorem,[(xlii)] J_m tends to zero, therefore, as $m \to \infty$. Similarly one can prove for σ.

For any $\phi \in C^\infty_0$,

$$\|f(\phi, \cdot, \cdot) - \hat{f}_m(\phi, \cdot, \cdot)\|_{\mathbb{H}^n_p(\tau)} \le I_{1m} + I_{2m},$$

where

$$I_{1m} := \|f(\phi, \cdot, \cdot) - f_m(\phi, \cdot, \cdot)\|_{\mathbb{H}^n_p(\tau)}, \quad f_m(\phi, t, x) = \int_{\mathbb{R}^d} f(\phi, t, x - y)\zeta_m(y)dy \text{ (distrib sense)},$$

$$I_{2m} := \|f_m(\phi, \cdot, \cdot) - \hat{f}_m(\phi, \cdot, \cdot)\|_{\mathbb{H}^n_p(\tau)}$$

(xxxix) In addition, the translation operator from H^n_p into itself is continuous, that is, $\|u(\cdot + t)\|_{n,p} = \|u\|_{n,p}$.

(xl) At least $(f(u(\cdot + z), t, \cdot - y), \phi)$ is jointly measurable, so the right hand side of eq. (62) is meaningful.

(xli) Recall that for any $u \in H^{n+2}_p$, $f(u, t, \cdot) \in H^n_p$. Thus by applying Lemma 5.0.8,

$$|(\hat{f}_{0m}(t, \cdot), \phi)| \le \|f(0, t, \cdot)\|_{n,p}\|\phi\|_{-n,q}$$

holds for every $\phi \in C^\infty_0$ and $q = p/(p-1)$. This proves $\hat{f}_{0m}(t, \cdot) \in H^n_p$.

(xlii) It is dominated by $2\|\bar{\zeta}_r\|_{n,p}\zeta(y)$, and remember that ζ is nonnegative and $\int \zeta = 1$.

$$
\begin{aligned}
&\le \int_{\mathbb{R}^d} \|f(\phi,\cdot,\cdot-y)-f(\phi(\phi+y),\cdot,\cdot-y)\|_{\mathbb{H}^n_p(\tau)}\zeta_m(y)dy \\
&= \int_{\mathbb{R}^d} \|f(\phi,\cdot,\cdot)-f(\phi(\phi+y),\cdot,\cdot)\|_{\mathbb{H}^n_p(\tau)}\zeta_m(y)dy \\
&\le N\int_{\mathbb{R}^d} \|\phi-\phi(\cdot+y)\|_{n+2,p}\zeta_m(y)dy \\
&= N\int_{\mathbb{R}^d} \|\phi-\phi(\cdot+m^{-1}y)\|_{n+2,p}\zeta(y)dy.
\end{aligned}
$$

For the first inequality, for any $\psi \in C_0^\infty$,(xliii)

$$
\begin{aligned}
(f(\phi,t,\cdot)*\zeta_m,\psi) &= (f(\phi,t,\cdot),\psi*\zeta_m(-\cdot)) \\
&= (f(\phi,t,\cdot)*\phi(-\cdot),\zeta_m(-\cdot)) \\
&= \int_{\mathbb{R}^d}(f(\phi,t,\cdot),\psi(\cdot-y))\zeta_m(-y)dy \\
&= \int_{\mathbb{R}^d}(f(\phi,t,\cdot),\psi(\cdot+y))\zeta_m(y)dy \\
&= \int_{\mathbb{R}^d}(f(\phi,t,\cdot-y),\psi)\zeta_m(y)dy.
\end{aligned}
$$

Since ζ_m is nonnegative, we have

$$
|(f(\phi,t,\cdot)*\zeta_m,\psi)| \le \int_{\mathbb{R}^d}|(f(\phi,t,\cdot-y),\psi)|\zeta_m(y)dy \le \int_{\mathbb{R}^d}\|(f(\phi,t,\cdot-y)\|_{n,p}\|\psi\|_{-n,q}\zeta_m(y)dy,
$$

and thus

$$
\|(f_m(\phi,t,\cdot),\psi)\|_{n,p} = \|(f(\phi,t,\cdot)*\zeta_m,\psi)\|_{n,p} \le \int_{\mathbb{R}^d}\|(f(\phi,t,\cdot-y)\|_{n,p}\zeta_m(y)dy.
$$

Although we obtained above inequality with f_m, same method can be applied for $f_m - \hat{f}_m$.

Similar with J_m, by the dominated convergence theorem, $I_{2m} \to 0$ as $m \to \infty$. In addition, by Lemma 5.0.8, we have $I_{1m} \to 0$. One can also prove that $\|g(\phi,\cdot,\cdot)-\hat{g}_m(\phi,\cdot,\cdot)\|_{\mathbb{H}^{n+1}_p(l_2)} \to 0$ as $m \to \infty$. Now the corollary is proved with the application of Remark 5.0.14.

Corollary 5.0.17 Let the assumptions of Theorem 5.0.7 be satisfied and also let then be satisfied for a $p = q$, where $q \ge 2$. Then the solution u from Theorem 5.0.7 belongs also to $\mathcal{H}^{n+2}_q(\tau)$.

proof. Without loss of generality, assume $p < q$(xliv) and let v be a solution of the same initial value problem but belonging to $\mathcal{H}^{n+2}_q(\tau)$ (such a unique v exists by Theorem 5.0.7). We have only to show that $v = u$. In light of Corollary 5.0.16, we can approximate both v and u by solutions of equations with smooth coefficients and with $(f_m, g_m)(u,t,x) = (f,g)(u,t,x)*\zeta_m(x)$. We only need to show that the approximating solutions coincide. Observe that for any $r \ge n$, for instance,

$$
\begin{aligned}
\|f_m(u,t,\cdot)-f_m(v,t,\cdot)\|_{r,p} &\le \|(1-\Delta)^{(r-n)/2}\zeta_m\|_B\|f(u,t,\cdot)-f(v,t,\cdot)\|_{n,p} \\
&\le N\epsilon\|u-v\|_{n+2,p} + NK_\epsilon\|u-v\|_{n,p} \\
&\le N\epsilon\|u-v\|_{r+2,p} + NK_\epsilon\|u-v\|_{r,p},
\end{aligned}
$$

(xliii) Recall the definition of the convolution between distributions and C_0^∞ functions.

(xliv) If $q < p$, interchange roles of p and q.

where N's depend only on m, r, n, p. This shows that, for any fixed m, all assumptions required from Theorem 5.0.7 are satisfied with any large n both for p and q.(xlv)

Therefore, we can suppose from the very begining that the assumptions are satisfied for any large n for p and q. In this case, by Theorem 5.0.7, $u \in \mathcal{H}_p^r(\tau)$ for any r, and hence, also invoking Theorem 3.0.9,

$$\mathbb{E}\int_0^\tau \|u_{xx}(t,\cdot)\|_{r,p}^p dt < \infty, \quad \mathbb{E}\sup_{t\le\tau}\|u(t,\cdot)\|_{C^r}^p dt < \infty$$

for any $r = 1, 2, 3, \cdots$. Since $\|u_{xx}(t,\cdot)\|_{r,i}^i \le N\|u_{xx}(t,\cdot)\|_{r,p}^p \|u(t,\cdot)\|_{C^{r+2}}^{i-p}$ for $i \ge p$, it follows that

$$\int_0^\tau \|u_{xx}(t,\cdot)\|_{r,i}^i dt < \infty \quad \text{(a.s.)}$$

for any $i \ge p$. Take here $r = 0$ and $i = q$ and define

$$\tau_k = \tau \wedge \inf\{t : \int_0^t \|u_{xx}(s,\cdot)\|_{r,q}^q \ge k\}.$$

Then obviously $u \in \mathcal{H}_q^2(\tau_k)$. Since v lies in the same class, by uniqueness $u(t,\cdot) = v(t,\cdot)$ for $t \le \tau_k$ (a.s.). It remains to observe that $\tau_k \uparrow \tau$ when $k \to \infty$.

Next general result concerns the maximum principle.

Theorem 5.0.18 (maximum principle) Let the assumptions of Theorem 5.0.7 be satisfied and let u be the function existence of which is asserted in this theorem. Assume that

$$f(v,t,x) = b^i(t,x)v_{x^i}(t,x) + c(t,x)v(t,x) + f(t,x), \quad g^k(v,t,x) = \nu^k(t,x)v(t,x),$$

where b^i, c, ν^k are certain bounded functions on $(\!(0,\tau]\!] \times \mathbb{R}^d$ and $f(t,x) = f(0,t,x) \ge 0$. Also assume that for any ω we have $u_0 \ge 0$. Then $u(t,\cdot) \ge 0$ for all $t \in [0,\tau]$ (a.s.).

- -

proof. Because of Remark 5.0.15, we can only concentrate on equations with finite number of Wiener processes like eq. (60), which in our case is the following

$$\begin{aligned} du_m(t,x) = {} & [a^{ij}(t,x)u_{mx^ix^j}(t,x) + b^i(t,x)u_{x^i}(t,x) + u(t,x)c(t,x) + f(t,x)]dt \\ & + \sum_{k\le m}[\sigma^{ik}(t,x)u_{mx^i}(t,x) + u(t,x)\nu^k(t,x)]dw_t^k. \end{aligned} \tag{63}$$

Corollary 5.0.16 and explanation after eq. (61) allow us to assume that $u_0(x)$ and the coefficients and f in eq. (63) are infinitely differentiable in x. After this, by multiplying u_0 and f by a cut-off function of x and by using Remark 5.0.12 and Theorem 5.0.13, we convince ourselves that we can assume that u_0 and $f(t,x)$ have supports with respect to x in a fixed ball. In this case the assumptions of Theorem 5.0.7 are satisfied for $p = 2$ again by Remark 5.0.12 (and Hölder's inequality), which by Corollary 5.0.17, yields $u \in \mathcal{H}_2^n(\tau)$ for any n. Now our assertion follows from the maximum principle proved in [14] (see Theorem 4.2 there). The only point to mention is that in [14] we considered eq. (63) on $[0,T]$, but we always can continue our data in an appropriate way after τ, which is assumed to be less than T.(xlvi) The theorem is proved.

By the way the proof of the maximum principle in [14] is based on representation formulas like eq. (48).

(xlv) We proved that if f satisfies Assumption 5.0.6 for n, p, then f_m does for r, p for all $r \ge n$. Hence, it suffices to assme that f satisfies Assumption 5.0.6 for r, p for every r.

(xlvi) See the proof of Theorem 7.0.2 for instance.

6 Proof of Theorem 5.0.7

The proof we present here is quite typical for proofs of solvability of equations with variable coefficients on the basis of solvability of equations with constant ones. The same type of arguments is commonly used in the theory of partial differential equations for proving the solvability in Sobolev or Hölder spaces. First we need some auxiliary constructions. Fix a $T \in (0, \infty)$.

Definition 6.0.1 Assume that, for $\omega \in \Omega$ and $t \geq 0$, we are given operators

$$L(t, \cdot) : H_p^{n+2} \to H_p^n, \quad \Lambda(t, \cdot) : H_p^{n+2} \to H_p^{n+1}(\mathbb{R}^d, l_2).$$

Assume that

(i) for any ω and t, the operators $L(t, u)$ and $\Lambda(t, u)$ are continuous (with respect to u);

(ii) for any $u \in H_p^{n+2}$, the processes $L(t, u)$ and $\Lambda(t, u)$ are predictable.

(iii) for any $\omega \in \Omega$, $t \geq 0$, and $u \in H_p^{n+2}$, we have

$$\|L(t, u)\|_{n,p} + \|\Lambda(t, u)\|_{n+1,p} \leq N_{L,\Lambda}(1 + \|u\|_{n+2,p}),$$

where $N_{L,\Lambda}$ is a constant.

Then for a function $u \in \mathcal{H}_p^{n+2}(T)$, we write

$$(L, \Lambda)u = -(f, g)$$

if $(f, g) \in \mathcal{F}_p^n(T)$, and, in the sense of Definition 3.0.7, for $t \in [0, T]$, we have that $\mathbb{D}u(t) = L(t, u(t)) + f(t)$ and $\mathbb{S}u(t) = \Lambda(t, u(t)) + g(t)$, or put otherwise

$$u(t) = u(0) + \int_0^t (L(s, u(s)) + f(s))ds + \int_0^t (\Lambda^k(s, u(s)) + g^k(s))dw_s^k \quad \text{(a.s.)}.$$

Remark 6.0.2 By virtue of our conditions on L and Λ, for any $u \in \mathcal{H}_p^{n+2}(T)$, we have $(L(u), \Lambda(u)) \in \mathcal{F}_p^n(T)$. Also, $(L, \Lambda)u = (L(u) - \mathbb{D}u, \Lambda(u) - \mathbb{S}u)$. In particular, the operator (L, Λ) *is well-defined* on $\mathbb{H}_p^{n+2}(T)$, and, as follows easily from Definition 6.0.1 (iii),

$$\|(L, \Lambda)u)\|_{\mathcal{F}_p^n(T)} \leq (1 + 2N_{L,\Lambda})\|u\|_{\mathcal{H}_p^{n+2}(T)} + 2N_{T,\Lambda}T^{1/p}.$$

In terms of Definition 6.0.1, Theorem 4.3.1 has the following version.

Theorem 6.0.3 Let a and σ satisfy the assumptions from the begining of Section 4. Define

$$Lu = a^{ij}u_{x^i x^j}, \quad \Lambda u = \sigma^i u_{x^i}.$$

Then the operator (L, Λ) is a 1-1 operator from $\mathcal{H}_{p,0}^{n+2}(T)$ onto $\mathcal{F}_p^n(T)$ and the norm of its inverse is less than a constant depending only on d, p, δ, and K (*thus independent of* T).

Next, we prove a pertubation result. It needs a proof because we do not allow ϵ to depend on T.

Theorem 6.0.4 Take the operators L and Λ from Theorem 6.0.3, and let some operators L_1 and Λ_1 satisfy the requirements from Definition 6.0.1. We assert that there exists a constant $\epsilon \in (0,1)$ depending only on d, p, δ, and K such that if, for a constant K_1 and any $u, v \in H_p^{n+2}$, $t \geq 0$, $\omega \in \Omega$, we have

$$\|L_1(t,u) - L_1(t,v)\|_{n,p} + \|\Lambda_1(t,u) - \Lambda(t,v)\|_{n+1,p} \leq \epsilon\|u_{xx} - v_{xx}\|_{n,p} + K_1\|u - v\|_{n+1,p}, \tag{64}$$

then, for any $(f,g) \in \mathcal{F}_p^n(T)$, there exists a unique solution $u \in \mathcal{H}_{p,0}^{n+2}(T)$ of the equation

$$(L + L_1, \Lambda + \Lambda_1)u = -(f,g). \tag{65}$$

Furthermore, for this solution u, we have

$$\|u\|_{\mathcal{H}_p^{n+2}(T)} \leq N\|(L_1(\cdot,0) + f, \Lambda_1(\cdot,0) + g)\|_{\mathcal{F}_p^n(T)}, \tag{66}$$

where N depends only on d, p, δ, K, K_1, and T and N is independent of T if $K_1 = 0$.

- -

proof. First notice that, by interpolation theorems[(xlvii)] $\|u\|_{n+1,p} \leq \epsilon\|u_{xx}\|_{n,p} + N(\epsilon, d, p)\|u\|_{n,p}$. Therefore, without loss of generality we assume that instead of eq. (64) we have

$$\|L_1(t,u) - L_1(t,v)\|_{n,p} + \|\Lambda_1(t,u) - \Lambda(t,v)\|_{n+1,p} \leq \epsilon\|u_{xx} - v_{xx}\|_{n,p} + K_1\|u - v\|_{n,p}.$$

Now fix $(f,g) \in \mathcal{F}_p^n(T)$. Take $u \in \mathcal{H}_{p,0}^{n+2}(T)$, observe that $(L_1(u), \Lambda_1(u)) \in \mathcal{F}_p^n(T)$, and, by using Theorem 6.0.3, define $v \in \mathcal{H}_{p,0}^{n+2}(T)$ as the unique solution of the equation $(L,\Lambda)v = -(f + L_1(u), g + \Lambda(u))$. By denoting $v = Ru$, we define an operator $R : \mathcal{H}_{p,0}^{n+2}(T) \to \mathcal{H}_{p,0}^{n+2}(T)$. Equation (65) is equivalent to the equation $u = Ru$. Therefore, to prove the existence and the uniqueness of solutions to Equation (65), if suffices to show that, for an integer $m > 0$, the operator R^m is a contraction in $\mathcal{H}_{p,0}^{n+2}(T)$.[(xlviii)]

By Theorem 6.0.3, for $t \leq T$,

$$\begin{aligned}\|Ru - Rv\|_{\mathcal{H}_p^{n+2}(t)}^p &\leq N\|(L_1(u) - L_1(v), \Lambda_1(u) - \Lambda_1(v))\|_{\mathcal{F}_p^n(t)}^p \\ &\leq N_0\epsilon\|u - v\|_{\mathcal{H}_p^{n+2}(t)}^p + N_0K_1^p\int_0^t \mathbb{E}\|u(s) - v(s)\|_{n,p}^p ds,\end{aligned}$$

with a constant N_0 depending only on d, p, δ, and K. This gives the desired result if $K_1 = 0$ by taking ϵ sufficiently small. Also, in case estimate eq. (66) follows obviously with N independent of T.

In the general case, by Theorem 3.0.9,

$$\mathbb{E}\|u(s) - v(s)\|_{n,p}^p \leq N_1\|u - v\|_{\mathcal{H}_p^{n+2}(s)}^p,$$

where $s \leq T$ and N_1 depends only on d, p, and T. It follows that, for $t \leq T$ and $\theta := N_0\epsilon^p$, we have

$$\|Ru - Rv\|_{\mathcal{H}_p^{n+2}(t)}^p \leq \theta\|u - v\|_{\mathcal{H}_p^{n+2}(t)}^p + N_2\int_0^t \|u - v\|_{\mathcal{H}_p^{n+2}(s)}^p ds,$$

where N_2 depends only on d, p, δ, K, K_1, and T. Now we are going to use the following fact. For a

[(xlvii)] It uses $\|u_x\|_p \leq \epsilon\|u_{xx}\|_p + N(\epsilon)\|u\|_p$.

[(xlviii)] Indeed, if X is a Banach space, $f : X \to X$ a function such that there exists an integer $m > 0$ such that f^m is a contraction, then f is also has a unique fixed point. Indeed, let x be a unique fixed point of f^m (because it is a contraction). Notice that

$$f^m(fx) = f^{m+1}x = f(f^m x) = fx,$$

so that fx is also a fixed point of f^m. By the uniqueness, therefore, we have $fx = x$.

integrable function h on $[0,\infty)$, if we define

$$I_\alpha h(x) := \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h(t)dt,$$

where $x \geq 0$ and $\alpha > 0$, then $D^n I_n h = h$ for every positive integer n. Hence, by induction,

$$\|R^m u - R^m v\|^p_{\mathcal{H}^{n+2}_p(t)} \leq \theta^m \|u-v\|^p_{\mathcal{H}^{n+2}_p(t)} + \sum_{k=1}^m \binom{m}{k}\theta^{m-k}N_2^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}\|u-v\|^p_{\mathcal{H}^{n+2}_p(s)}ds,$$

$$\|R^m u - R^m v\|^p_{\mathcal{H}^{n+2}_p(T)} \leq \sum_{k=0}^m \binom{m}{k}\theta^{m-k}\frac{1}{k!}(TN_2)^k\|u-v\|^p_{\mathcal{H}^{n+2}_p(T)} \leq 2^m\theta^m \max_k \frac{1}{k!}(TN_2/\theta)^k\|u-v\|^p_{\mathcal{H}^{n+2}_p(T)}.$$

This allows us to find ϵ depending only on d, p, δ, and K and m depending on the same things plus K_1 and T, so that R^m is a contraction in $\mathcal{H}^{n+2}_{p,0}(T)$ with coefficient 1/2. Of course, this yields all our assertions.

We finish our preparations by showing how Lemma 5.0.8 will be used.

Remark 6.0.5 To some extent, in what follows, the most important consequence of assertion (i) of Lemma 5.0.8 is that if $\bar{a} = \|a\|_{B^{|n|+\gamma}} < \infty$, then there exists a new norm $\interleave \cdot \interleave_{n,p}$ in H^n_p such that

$$\interleave au \interleave_{n,p} \leq 2N\|a\|_B \interleave u \interleave_{n,p} \quad (\|a\|_B = \sup_{x\in\mathbb{R}^d}|a(x)|),$$

where N is the same constant as in Lemma 5.0.8. To show this, it suffices to observe that, for $a_m(x) = a(x/m)$, $u_m(x) = u(x/m)$, and $m \geq 1$, we have

$$\begin{aligned}
\|(m^2-\Delta)^{n/2}(au)\|_p &= m^{n-d/p}\|(1-\Delta)^{n/2}(a_m u_m)\|_p \\
&\leq N\bar{a}_m m^{n-d/p}\|(1-\Delta)^{n/2}u_m\|_p \\
&= N\bar{a}_m\|(m^2-\Delta)^{n/2}u\|_p \\
&\leq N\left(\|a\|_B + \frac{1}{m^{(|n|+\gamma)\wedge 1}}\bar{a}\right)\|(m^2-\Delta)^{n/2}u\|_p.
\end{aligned}$$

Here, by considering the Fourier transform, one obtain the first equality. Finally, recall that

$$\|a\|_{B^{|n|}} = \sum_{j=0}^{|n|}\|D^j a\|_B + \sum_{|\alpha|=|n|}\sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}}\frac{|D^\alpha a(x) - D^\alpha a(y)|}{|x-y|},$$

$$\|a\|_{B^{|n|+\gamma}} = \sum_{j=0}^{|n|}\|D^j a\|_B + \sum_{|\alpha|=|n|}\sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}}\frac{|D^\alpha a(x) - D^\alpha a(y)|}{|x-y|^\gamma}, \quad \gamma \neq 0.$$

Then one can easily prove that

$$\|a(\alpha\cdot)\|_{B^{|n|+\gamma}} \leq \|a\|_B + \alpha^{(|n|+\gamma)\wedge 1}\|a\|_{B^{|n|+\gamma}}, \quad \alpha \in (0,1].$$

Alternatively, it would be sufficient for our needs to know that

$$\|au\|_{n,p} \leq N(\|a\|_B\|u\|_{n,p} + \|a\|_{B^{|n|+\gamma}}\|u\|_{n-1,p}).$$

Now we perform the main step in proving Theorem 5.0.7.

Lemma 6.0.6 Let Assumption 5.0.1 and 5.0.3 to 5.0.5 be satisfied. Then there exists $\epsilon = \epsilon(d, p, n, \gamma, \delta, K) > 0$ such that if $\tau = T$ and

(i) inequality eq. (52) holds with this ϵ for all $x, y, t,$ and ω and

(ii) f and g are independent of u,

then there exists a unique solution $u \in \mathcal{H}^{n+2}_{p,0}(T)$ of equation eq. (50). Furthermore, for this solution u, we have

$$\|u\|_{\mathcal{H}^{n+2}_p(T)} \le N\|(f, g)\|_{\mathcal{F}^n_p(T)},$$

where N depends only on $d, p, \delta, K,$ and T.

- -

proof. Define $a(t) = a(t, 0)$ and $\sigma(t) = \sigma(t, 0)$, take operators L and Λ from Theorem 6.0.3 corresponding to these $a(t)$ and $\sigma(t)$, and let

$$L_1(t, u) = [a^{ij}(t, x) - a^{ij}(t)]u_{x^i x^j}(x), \quad \Lambda_1(t, u) = [\sigma^{ij}(t, x) - \sigma^{ij}(t)]u_{x^i}(x).$$

In view of Theorem 6.0.4, to prove the existence and uniqueness, we have only to check that if ϵ in eq. (52) is sufficiently small, then the operators L_1 and Λ_1 satisfy condition eq. (64) with as small ϵ as we like and with K_1 under control. Observe that by Lemma 5.0.8,

$$\|L_1(t, u)\|_{n,p} \le N\|a(t, \cdot) - a(t, 0)\|_{B^{|n|+\gamma}}\|u_{xx}\|_{n,p}$$
$$\|\Lambda_1(t, u)\|_{n,p} \le N\|\sigma(t, \cdot) - \sigma(t, 0)\|_{B^{|n+1|+\gamma}(l_2)}\|u_x\|_{n+1,p}.$$

Since $\|u_x\|_{n+1,p} \le N(\|u_{xx}\|_{n,p} + \|u\|_{n+1,p})$,[(xlix)] our lemma holds true if

$$\|a(t, \cdot) - a(t, 0)\|_{B^{|n|+\gamma}} + \|\sigma(t, \cdot) - \sigma(t, 0)\|_{B^{|n+1|+\gamma}(l_2)} \le \epsilon_0 = \epsilon_0(d, p, n, \gamma, \delta, K) \quad \forall t.$$

Next, observe that, for $a_m(t, x) = a(t/m^2, x/m)$ and $m \ge 1$, we have (see Remark 6.0.5)

$$\begin{aligned}\|a_m(t, \cdot) - a_m(t, 0)\|_{B^{|n|+\gamma}} &\le \|a(t, \cdot) - a(t, 0)\|_B + m^{-[(|n|+\gamma)\wedge 1]}\|a(t, \cdot) - a(t, 0)\|_{B^{|n|+\gamma}}\\ &\le \epsilon + 2m^{-[(|n|+\gamma)\wedge 1]}K\end{aligned}$$

(K comes from Assumption 5.0.5), and for $|n| + \gamma = 0$ we can even drop the second term on the right. An analogous inequality holds for σ. It follows that, for ϵ sufficiently small and m sufficiently large, the statements of the lemma are true if we replace $a, \sigma, w_t, f, g,$ and T in equation eq. (50) by

$$a_m,\ \sigma_m,\ mw_{t/m^2},\ m^{-2}f(t/m^2,\ x/m),\ m^{-1}g(t/m^2,\ x/m),\ \text{ and } m^2T, \tag{67}$$

respectively. After this, it remains only to fix an appropriate m and make an obvious change of the unknown function in the above mentioned modification of equation eq. (50) (and use $1-\Delta \sim m^2-\Delta$ so that the norms $\|\cdot\|_{n,p}$ of $u(t/m^2, x/m)$ and $u(t, x)$ are comparable. See Remark 6.0.5). Indeed, for $\epsilon \ll 1$ and $m \gg 1$,[(l)] by Theorem 6.0.4, there exists $v \in \mathcal{H}^{n+2}_{p,0}(T)$ satisfies eq. (50) replaced by eq. (67). Then defining $u(t, x) = v(m^2t, mx)$ solves the original eq. (50) and obtains the inequality from eq. (66).

Finally, we need the following result from [6], which, in sense, is essentially covered by Theorem 2.4.7 from [24].

(xlix) Notice that $\|u_x\|_{n+1,p} \le N(\|u_{xx}\|_{n,p} + \|u_x\|_{n,p})$.

(l) These are followed from Vinogradov notation.

Lemma 6.0.7 Let $\delta > 0$ and let $\zeta_k \in C^\infty$, $k = 1, 2, 3, \cdots$. Assume that for any multiindex α and $x \in \mathbb{R}^d$,

$$\sup_{x \in \mathbb{R}^d} \sum_k |D^\alpha \zeta_k(x)| \le M(\alpha),$$

where $M(\alpha)$ are some constants. Then there exists a constant $N = N(d, n, M)$ such that, for any $f \in H_n^p$,

$$\sum_k \|\zeta_k f\|_{n,p}^p \le N \|f\|_{n,p}^p.$$

If in addition

$$\sum_k |\zeta_k(x)|^p \ge \delta,$$

then for any $f \in H_p^n$,

$$\|f\|_{n,p}^p \le N(d, n, M, \delta) \sum_k \|\zeta_k f\|_{n,p}^p.$$

Remark 6.0.8 We will also use a natural extension of this lemma to the case of Banach space valued f.

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proof (Theorem 5.0.7). By Theorem 2.2.1, for any nonrandom $z \in H_p^{n+2-2/p}$ there exists a unique solution $u \in \mathcal{H}_p^{n+2}(\tau)$ of the equation $du = \Delta u dt$ with initial condition z. This theorem also provides estimate eq. (7) of the norm of u.(li) From this theorem and the estimate, it follows that there exists a unique solution $\bar{u} \in \mathcal{H}_p^{n+2}(\tau)$ of the equation $du = \Delta u dt$ with initial condition u_0(lii) and

$$\|\bar{u}\|_{\mathcal{H}_p^{n+2}(\tau)}^p \le N \mathbb{E} \|u_0\|_{n+2-2/p,p}^p.$$

The idea of the proof of the theorem is following. First, solve the following equation

$$dv(t,x) = [a^{ij}(t,x) v_{x^i x^j}(t,x) + \bar{f}(v,t,x)]dt + [\sigma^{ik}(t,x) v_{x^i} + \bar{g}^k(v,t,x)]dw_t^k$$

with zero boundary condition where (recall that $d\bar{u} = \Delta \bar{u} dt$)

$$\bar{f}(v,t,x) = [f(v + \bar{u}, t, x) + a^{ij}(t,x) \bar{u}_{x^i x^j}(t,x) - \Delta \bar{u}] \mathbb{1}_A(t),$$

$$\bar{g}^k(v,t,x) = [g^k(v + \bar{u}, t, x) + \sigma^{ik}(t,x) \bar{u}_{x^i}(t,x)] \mathbb{1}_A(t),$$

and A is the set of ω, t for which $\bar{u}(t, \cdot) \in H_p^{n+2}$.(liii) Then the new function $u = v + \bar{u}$ satisfies the original equation with initical condition u_0 because $v(0, \cdot) = 0$ and $\bar{u}(0, \cdot) = u_0$.

From Lemma 5.0.8, it follows that $\bar{f}$ and $\bar{g}$ satisfy the same conditions as f and g. This allows us to assume that $u_0 = 0$. Furthermore, one can always extend f, g after time τ by equaling them to zero. That is why without loss of generality we assume that $u_0 = 0$ and $\tau = T$.(liv)

Now, define

$$Lu = a^{ij}(t,x) u_{X^i x^j}(t,x), \quad \Lambda u = \sigma^i(t,x) u_{x^i}(t,x),$$

(li) Actually, Theorem 2.2.1 only gives the result for the case $z \in H_p^{2-2/p}$. Also the solution space is $H_p^{1,2}(T)$. The case $\tau \le T$ can be easily covered by extending the initial value with zeros after τ. Also, we already checked that $H_p^{1,2}(T) \subset \mathcal{H}_p^2(T)$. For a general case, if $z \in H_p^{n+2-2/p}$ a nonrandom is fixed, take a unique solution $v \in \mathcal{H}_p^2(T)$ of the equation $v_t = \Delta v$ (thus $dv = \Delta v dt$) with initial condition $(1-\Delta)^{n/2} z$. Then the function $u(t,x) = [(1-\Delta)^{-n/2} v(t, \cdot)](x)$ is the desired one.

(lii) Recall that $u_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{n+2-2/p})$.

(liii) We already proved that A is jointly measurable (actually predictable).

(liv) However, this part is not quite obvious. Rigorous proof is introduced in section 9.3.

and let $(\zeta_k)_k$ be a standard partiton of unity such that, for any k, the support of ζ_k lies in a ball B_k of radius $(1/4)\varkappa_{\epsilon/2}$, where $\varkappa_\epsilon$ is taken from Assumption 5.0.2 and ϵ from Lemma 6.0.6. Also for any k, we take a function $\eta_k \in C_0^\infty$ such that $\eta_k = 1$ on B_k and $\eta_k = 0$ outside of $2B_k$, and $0 \le \eta_k \le 1$. Denote by x_k the center of B_k, define $L_k(t,x) = \eta_k(x)L(t,x) + (1-\eta_k(x))L(t,x_k)$, and similarly define Λ_k.[lv]

Observe that, for any k, the operators L_k and Λ_k satisfy condition (i) of Lemma 6.0.6. Therefore, if we denote $(f_k, g_k) := (L_k, \Lambda_k)(u\zeta_k)$, then by this lemma

$$\|u\zeta_k\|_{\mathcal{H}_p^{n+2}(T)} \le N\|(f_k, g_k)\|_{\mathcal{F}_p^n(T)} = N\|(L_k, \Lambda_k)(u\zeta_k)\|_{\mathcal{F}_p^n(T)}.$$

Furthermore, by using that $\eta_k = 1$ everywhere where $u\zeta_k \neq 0$ we easily check that

$$(L_k, \Lambda_k)(u\zeta_k) = (L, \Lambda)(u\zeta_k) = \zeta_k(L, \Lambda)u + (uL\zeta_k + 2\zeta_{kx} \cdot au_x, u\Lambda\zeta_k),$$

so that

$$\|u\zeta_k\|_{\mathcal{H}_p^{n+2}(T)} \le N\|\zeta_k(L, \Lambda)u\|_{\mathcal{F}_p^n(T)} + N\|(uL\zeta_k + 2\zeta_{kx} \cdot au_x, u\Lambda\zeta_k)\|_{\mathcal{F}_p^n(T)}.$$

We sum up the pth powers of the extreme terms and apply Lemma 6.0.7 and Lemma 5.0.8 in the estimates like the following one:

$$\sum_k \|ua^{ij}\zeta_{kx^ix^j}\|_{n,p} \le N\|ua^{ij}\|_{n,p} \le N\|u\|_{n,p}.$$

Then we conclude that, for any $u \in \mathcal{H}_{p,0}^{n+2}(T)$,

$$\|u\|_{\mathcal{H}_p^{n+2}(T)} \le N\|(L, \Lambda)u\|_{\mathcal{F}_p^n(T)} + N\|u\|_{\mathbb{H}_p^{n+1}(T)}. \tag{68}$$

Next, we show that the last term on the right can be dropped. Indeed, $\|u\|_{n+1,p} \le \epsilon\|u_{xx}\|_{n,p} + N(\epsilon, d, p)\|u\|_{n,p}$. Therefore, eq. (68) can be modified by replacing the last term with $N\|u\|_{\mathbb{H}_p^n(T)}$.[lvi] This can be done with any $t \le T$ in place of T. After this, by Theorem 3.0.9, the inequality

$$\mathbb{E}\|u(t,\cdot)\|_{n,p}^p \le N\|(L, \Lambda)u\|_{\mathcal{F}_p^n(T)}^p + N\int_0^t \mathbb{E}\|u(s,\cdot)\|_{n,p}^p ds$$

holds for any $t \le T$. By Gronwall's inequality, this yields that

$$\|u\|_{\mathbb{H}_p^n(T)}^p \le N\|(L, \Lambda)u\|_{\mathcal{F}_p^n(T)}^p$$

which, along with the modified eq. (68), proves that

$$\|u\|_{\mathcal{H}_p^{n+2}(T)} \le N\|(L, \Lambda)u\|_{\mathcal{F}_p^n(T)} \tag{69}$$

for any $u \in \mathcal{H}_{p,0}^{n+2}(T)$.

This is a priori estimate. Now we use the standard method of continuity. For $\lambda \in [0,1]$ we consider the equation

$$du = (L_\lambda u + f)dt + (\Lambda_\lambda^k u + g^k)dw_t^k \tag{70}$$

with zero initial condition, where

$$L_\lambda = \lambda\Delta + (1-\lambda)L, \quad \Lambda_\lambda = (1-\lambda)\Lambda$$

and (f, g) is an arbitrary element in $\mathcal{F}_p^n(T)$.

Observe that a priori estimate eq. (69) holds with the same constant N for all L_λ, Λ_λ in place of L, Λ. Next, take a $\lambda_0 \in [0,1]$ and assume that for $\lambda = \lambda_0$ eq. (70) with zero initial data has a unique

[lv] To apply Lemma 6.0.6, one should "remove" values of a^{ij} and σ^i where x is "far away enough".

[lvi] Recall the definition of $\|\cdot\|_{\mathcal{H}_p^n(T)}$.

solution $u \in \mathcal{H}^{n+2}_{p,0}(T)$ for any $(f,g) \in \mathcal{F}^n_p(T)$. By the way, this assumption is satisfied for $\lambda_0 = 1$ by Theorem 6.0.3. Then we have the operator

$$\mathcal{R}_{\lambda_0} : \mathcal{F}^n_p(T) \to \mathcal{H}^{n+2}_{p,0}(T)$$

such that $\mathcal{R}_{\lambda_0}(f,g) = u$. From eq. (69) we get that

$$\|\mathcal{R}_{\lambda_0}(f,g)\|_{\mathcal{H}^{n+2}_p(T)} \le N\|(f,g)\|_{\mathcal{F}^n_p(T)}. \tag{71}$$

For other $\lambda \in [0,1]$ we rewrite eq. (70) as

$$du = (L_{\lambda_0}u + [(\lambda-\lambda_0)(\Delta - L)u + f])dt + (\Lambda^k_{\lambda_0}u + [(\lambda-\lambda_0)\Lambda^k u + g^k])dw^k_t$$

and we solve the last equation by iterations. Define $u_0 = 0$ and

$$u_{j+1} = ((\lambda-\lambda_0)(\Delta - L)u_j + f, (\lambda-\lambda_0)\Lambda u_j + g).$$

Then by eq. (71)

$$\begin{aligned}\|u_{j+1} - u_j\|_{\mathcal{H}^{n+2}_p(T)} &\le N|\lambda-\lambda_0|\|((\Delta - L)(u_j - u_{j-1}), \Lambda(u_j - u_{j-1}))\|_{\mathcal{F}^n_p(T)} \\ &\le N_1|\lambda-\lambda_0|\|u_j - u_{j-1}\|_{\mathcal{H}^{n+2}_p(T)},\end{aligned}$$

where N_1 is independent of j, λ, and λ_0. If $N_1|\lambda-\lambda_0| \le 1/2$, then u_j is a Cauchy sequence in $\mathcal{H}^{n+2}_p(T)$, which converges by Theorem 3.0.9. Its limit satisfies

$$u = ((\lambda-\lambda_0)(\Delta - L)u + f, (\lambda-\lambda_0)\Lambda u + g),$$

which is equivalent to eq. (70).

In this way, we show that if eq. (70) is solvable for λ_0, then it is solvable for λ satisfying $N_1|\lambda-\lambda_0| \le 1/2$. In finite number of steps starting from $\lambda = 1$, we get to $\lambda = 0$. This proves the theorem if f, g are independent of u.

To consider general f and g, it remains to repeat the proof of Theorem 6.0.4 taking $f(u,t,x)$ and $g(u,t,x)$ instead of $f + L_1(u)$ and $g + \Lambda_1(u)$ there. The theorem is proved.

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7 Embedding Theorems for $\mathcal{H}^n_p(\tau)$

The above theory of solvability looks satisfactory only until one tries to apply it to concrete problems when one is interested in getting not only solvability but also some qualitative properties of solutions, like continuity, decay at infinity, compactness of support, and so on. To answer such questions, one has to understand what qualitative properties the solution have. Since solutions are just arbitrary functions from $\mathcal{H}^n_p(\tau)$, we are actually interested in properties of functions from this space. Let us fix $T \in [0,\infty)$ and a stopping time $\tau \le T$.

The first two assertions of the following theorem are straightforward corollaries of two Sobolev's theorems. One says that $H^n_p \subset \mathcal{C}^\alpha$ if $\alpha := n - d/p > 0$, where $\mathcal{C}^\alpha = \mathcal{C}^\alpha(\mathbb{R}^d)$ is the Zygmund space (which differs from the usual Hölder space $C^\alpha = C^\alpha(\mathbb{R}^d)$ only if α is an integer, see [23]). The second one says that $H^n_p \subset H^m_q$ if $m < n$ and $n - d/p = m - d/q$.

Theorem 7.0.1

(a) If $\alpha := n - d/p > 0$ and $u \in \mathcal{H}^n_p(\tau)$, then $u \in L_p((0,\tau]], \mathcal{C}^\alpha)$, where $\mathcal{C}^\alpha$ is the Zygmund space. In addition,

$$\mathbb{E}\int_0^\tau \|u(t,\cdot)\|^p_{\mathcal{C}^\alpha} dt \le N(d,n,p)\|u\|_{\mathcal{H}^n_p(\tau)}.$$

(b) If $m < n$ and $n - d/p = m - d/q$, and $u \in \mathcal{H}_p^n(\tau)$, then

$$\mathbb{E}\int_0^\tau \|u(t,\cdot)\|_{m,q}^p dt \le N(d,m,n,p)\|u\|_{\mathcal{H}_p^n(\tau)}^p.$$

(c) For any function $u \in \mathcal{H}_2^n(\tau)$, we have $u \in C([0,\tau], H_2^{n-1})$ (a.s.) and

$$\mathbb{E}\sup_{t\le\tau}\|u(t,\cdot)\|_{n-1,2}^2 \le N(d,n,T)\|u\|_{\mathcal{H}_2^n(\tau)}^2.$$

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proof. As we have said before the theorem, we only need to prove the third assertion. By Remark 3.0.6, we may assume that $n = 1$. Denote $u_0 = u(0)$, $v = T_t u_0$. Observe that by Theorem 2.2.1 we have $v \in \mathcal{H}_2^1(\tau)$. By Minkowski's inequality, $\|\zeta * u\|_p \le \|u\|_p\|\zeta\|_1$, so that

$$\|T_t u_0\|_2 \le \|u_0\|_2, \quad \mathbb{E}\sup_t \|v(t,\cdot)\|_2^2 \le \mathbb{E}\|u_0\|_2^2.$$

In addition, almost obviously, $T_t u_0$ is a continuous (analytic) L_2-valued function in t for $t > 0$.(lvii) Also, we have $(T_t u_0)'_t = T_t\Delta u_0$, which implies that $\|T_t u_0 - u_0\|_2 \le t\|\Delta u\|_2 \to 0$ if $u_0 \in H_2^2$. Adding that the set H_2^2 is dense in L_2, we conclude that T_t is a continuous semigroup in L_2. This means that $v \in C([0,\tau], H_2^{n-1})$ and shows that we need only to consider $u - v$. In other words, in the rest of the proof we may and will assume that $u_0 = 0$.

In this case, denote $f = (\mathbb{D}u - \Delta u)\mathbb{1}_{t\le\tau}$, and $g = (\mathbb{S}u)\mathbb{1}_{t\le\tau}$. Solve equation eq. (21) on $[0,\infty)$ with zero initial data.(lviii) By uniqueness, the solution coincides with u on $[0,\tau]$. By theorem 4.1.2, assertion (iii) holds.

Further results about some basic properties of the spaces $\mathcal{H}_p^n(\tau)$ are collected in the following theorem. As we have seen in theorem 4.1.2, assertion (iii) of theorem 7.0.1 is true not only for $p = 2$ but also for any $p \ge 2$. However, for $p > 2$ a much stronger statement (i) of theorem 7.0.2 holds.

Theorem 7.0.2

(i) If $p > 2$, $1/2 > \beta > \alpha > 1/p$, then for any function $u \in \mathcal{H}_p^n(\tau)$, we have $u \in C^{\alpha-1/p}([0,\tau], H_p^{n-2\beta})$ (a.s.) and for any stopping time $\eta \le \tau$,(lix)

$$\mathbb{E}\|u(t\wedge\eta,\cdot) - u(s\wedge\eta,\cdot)\|_{n-2\beta,p}^p \le N(d,p,\beta,T)|t-s|^{\beta p-1}\|u\|_{\mathcal{H}_p^n(\tau)}^p \quad \forall t,s\le T; \tag{72}$$

$$\mathbb{E}\|u(t,\cdot)\|_{C^{\alpha-1/p}([0,\tau],H_p^{n-2\beta})}^p \le N(d,\alpha,\beta,p,T)\|u\|_{\mathcal{H}_p^n(\tau)}^p. \tag{73}$$

(lvii) It suffices to show that $T_t u_0$ is strongly continuous at $t = 0$ (see [26]).

(lviii) That is, solve $dv = [\Delta v - (\mathbb{D}u - \Delta u)\mathbb{1}_{t\le\tau}]dt + (\mathbb{S}u)\mathbb{1}_{t\le\tau}dw_t$.

(lix) As usual, if X is a Banach space, $0 < \alpha < 1$, and $\Omega \subset \mathbb{R}^d$ an open set, we define

$$\|u\|_{C^\alpha(\bar\Omega,X)} := \sup_{x\in\Omega}\|u(x)\|_X + \sup_{x,y\in\Omega,x\ne y}\frac{\|u(x)-u(y)\|_X}{|x-y|^\alpha}.$$

Also one can define for any $\alpha > 0$ similar with the real-valued Hölder space.

(ii) If $q \geq p > 2$ and $\theta \in (0,1)$, then for

$$m < n + \frac{d}{q} - \frac{d + 2(1-\theta)}{p}, \quad u \in \mathcal{H}^n_p(\tau),$$

we have $u \in L_{p/\theta}((0,\tau), H^m_q)$ (a.s.) and

$$\mathbb{E}\left(\int_0^\tau \|u(t,\cdot)\|^{p/\theta}_{m,q} dt\right)^\theta \leq N(d,p,q,n,m,\theta,T)\|u\|^p_{\mathcal{H}^n_p(\tau)}.$$

In particular (take $\theta = p/q$),

$$\mathbb{E}\left(\int_0^\tau \|u(t,\cdot)\|^{q}_{m,q} dt\right)^{p/q} \leq N(d,p,q,n,m,T)\|u\|^p_{\mathcal{H}^n_p(\tau)}$$

if

$$q > p > 2, \quad m < n - (d+2)\left(\frac{1}{p} - \frac{1}{q}\right).$$

To prove the theorem, we need two lemmas. Remember the notation T_t introduced in eq. (2).

Lemma 7.0.3 For any $h \in L_p$, $\theta \in [0,1]$, and $t > 0$, we have

$$\|e^{-t}T_t h\|_p \leq N\frac{1}{t^\theta}\|h\|_{-2\theta,p}, \quad \|(T_t - 1)h\|_p \leq Nt^\theta \|h\|_{2\theta,p}. \tag{74}$$

where $N = N(d,p,\theta)$.

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proof. These inequalities follow from Theorem 14.11 of [5]. For the sake of completeness, we prove them. The derivative $(T_t h)'_t$ can be represented as t^{-1} times a convolution of h with a function having finite L_1-norm, the norm being independent of t.(lx) By Minkowski's inequality, $\|\zeta * h\|_p \leq \|\zeta\|_1 \|h\|_p$ so that

$$\|(T_t h)'_t\|_p \leq N(d,p)\frac{1}{t}\|h\|_p, \quad \|(e^{-t}T_t h)'_t\|_p \leq N(d,p)\frac{1}{t}\|h\|_p. \tag{75}$$

Hence, for $\theta \in (0,1)$, (see eq. (4))

$$\begin{aligned}\|(1-\Delta)^\theta(e^{-t}T_t h)\|_p &\leq c(\theta)\left\|\int_0^\infty \frac{e^{-(s+t)}T_{s+t}h - e^{-t}T_t h}{s^\theta}\frac{ds}{s}\right\|_p \\ &\leq N\int_0^t \frac{1}{s^\theta}\|e^{-(s+t)}T_{s+t}h - e^{-t}T_t h\|_p \frac{ds}{s} + N\int_t^\infty \frac{ds}{s^{1+\theta}}\|h\|_p.\end{aligned}$$

Here by eq. (75)

$$\|e^{-(s+t)}T_{s+t}h - e^{-t}T_t h\|_p \leq N\frac{s}{t}\|h\|_p,$$

$$\int_0^t \frac{1}{s^\theta}\|e^{-(s+t)}T_{s+t}h - e^{-t}T_t h\|_p \frac{ds}{s} \leq N\|h\|_p \int_0^t ts^\theta ds = N\frac{1}{t^\theta}\|h\|_p.$$

This proves that

$$\|e^{-t}T_t(1-\Delta)^\theta h\|_p \leq N\|h\|_p t^{-\theta}, \tag{76}$$

which gives the first inequality in eq. (74) after replacing $(1-\Delta)^\theta h$ with h. If $\theta = 0$, then one can take $N = 1$ in the first inequality in eq. (74), which follows from Minkowski's inequality. If $\theta = 1$, then we use eq. (75) and the fact that $(e^{-t}T_t h)'_t = e^{-t}T_t(\Delta - 1)h$.

(lx) This function is p_t where p is the heat kernel.

To prove the second inequality in eq. (74) for $t \in [0,1]$ and $\theta > 0$ it suffices to notice that $(T_t h)'_t = \Delta T_t h$ and $\Delta = [\Delta(1-\Delta)^{-1}](1-\Delta)^{1-\theta}(1-\Delta)^{\theta}$ and use eq. (76) in the following estimates

$$\begin{aligned} \|(T_t - 1)h\|_p &\le \int_0^t \|[\Delta(1-\Delta)^{-1}](1-\Delta)^{1-\theta} T_s[(1-\Delta)^{\theta} h]\|_p ds \\ &\le N \int_0^t \|(1-\Delta)^{1-\theta} T_s[(1-\Delta)^{\theta} h]\|_p ds \\ &\le N\|h\|_{2\theta,p} \int_0^t s^{\theta-1} ds \\ &= N t^{\theta} \|h\|_{2\theta,p}. \end{aligned}$$

For $t \ge 1$ or $\theta = 0$, the second inequality in eq. (74) is trivial since $\|h\|_p \le \|h\|_{2\theta,p}$ and $t^0 = 1$.

Lemma 7.0.4 Let $\alpha p > 1$ and $p \ge 1$. Then, for any continuous L_p-valued function $h(t)$, and $s \le t$, we have

$$\begin{aligned} \|h(t) - h(s)\|_p^p &\le N(\alpha,p)(t-s)^{\alpha p - 1} \int_s^t \int_s^t \mathbb{1}_{r_2 > r_1} \frac{\|h(r_2) - h(r_1)\|_p^p}{|r_2 - r_1|^{1+\alpha p}} dr_1 dr_2 \\ &= N(\alpha,p)(t-s)^{\alpha p - 1} \int_0^{t-s} \frac{1}{\gamma^{1+\alpha p}} \int_s^{t-\gamma} \|h(r+\gamma) - h(r)\|_p^p dr d\gamma \quad (\frac{0}{0} := 0). \end{aligned} \tag{77}$$

This is one of embedding theorems for Slobodetskii's spaces (see, for instance, [23]). It is said that embedding theorems were applied for studying continuity properties of random processes for quite long time. The following consequence of eq. (77),

$$\begin{aligned} \mathbb{E} \sup_{0 \le s < t \le T} \frac{\|h(t) - h(s)\|_p^p}{(t-s)^{\alpha p - 1}} &\le N(\alpha,p) \int_0^T \int_0^T \mathbb{1}_{r_2 > r_1} \frac{\mathbb{E}\|h(r_2) - h(r_1)\|_p^p}{|r_2 - r_1|^{1+\alpha p}} dr_1 dr_2 \\ &= N(\alpha,p) \int_0^T \frac{d\gamma}{\gamma^{1+\alpha p}} \int_0^{T-\gamma} \mathbb{E}\|h(r+\gamma) - h(r)\|_p^p dr d\gamma \quad (\frac{0}{0} := 0) \end{aligned} \tag{78}$$

can be used whenever one uses Kolmogorov's continuity criterion. By the way, embedding theorems are (for the most part) known for multidimensional case as well, and one can use them for studying random fields. Finally, notice that the space L_p in eq. (77) and eq. (78) can be replaced with any Banach space.

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proof (Theorem 7.0.2). Take $u \in \mathcal{H}_p^n(\tau)$, and define $f = (\mathbb{D}u - \Delta u)\mathbb{1}_{t \le \tau}$ and $g = (\mathbb{S}u)\mathbb{1}_{t \le \tau}$. Notice that the function u on $[0,\tau]$ satisfies the equation

$$dv = (\Delta v + f)dt + g^k dw_t^k. \tag{79}$$

By Theorem 5.0.7, equation eq. (79) on $[0,T]$ with initial condition $v(0) = u(0)$ has a unique solution $v \in \mathcal{H}_p^n(T)$. The difference $u - v$ satisfies the heat equation on $[0,\tau]$ with zero initial condition. It follows that $u(t,\cdot) = v(t,\cdot)$ on $[0,\tau]$, and by Theorem 5.0.7,

$$\|v\|_{\mathcal{H}_p^n(T)} \le N(\|f\|_{\mathbb{H}_p^{n-2}(\tau)} + \|g\|_{\mathbb{H}_p^{n-1}(\tau,l_2)} + (\mathbb{E}\|u(0)\|_{n-2/p,p}^p)^{1/p}) \le N\|u\|_{\mathcal{H}_p^n(\tau)}.$$

The inequality $\|v\|_{\mathcal{H}_p^n(T)} \le N\|u\|_{\mathcal{H}_p^n(\tau)}$ and the inequality $u(t,\cdot) = v(t,\cdot)$ on $[0,\tau]$ show that we only need to prove the theorem for v, T in place of u, τ. In particular, we may and will assume that $\tau = T$. Also, observe that

$$\|f\|_{\mathbb{H}_p^{n-2}(T)} + \|g\|_{\mathbb{H}_p^{n-1}(T,l_2)} + (\mathbb{E}\|u(0)\|_{n-2/p,p}^p)^{1/p} \le N\|u\|_{\mathcal{H}_p^n(T)}.$$

This allows us to concentrate only on solutions of eq. (79) on $[0,T]$ and to prove our assertions with $\|(f,g)\|_{\mathcal{F}_p^{n-2}(T)} + (\mathbb{E}\|u(0)\|^p_{n-2/p,p})^{1/p}$ in place of $\|u\|_{\mathcal{H}_p^n(T)}$. Therefore, below we take $\tau = T$ and take the function $u \in \mathcal{H}_p^n(T)$ as a solution of eq. (79). As in the proof of theorem 4.1.2, we may and will assume that f and g are as in eq. (26) and $u(0) \in C_0^\infty$.

After these preparations, we are going to prove assertion (i). By Remark 3.0.6, without loss of generality we assume that $n = 2\beta$ and $s \le t$. Denote[(lxi)]

$$u_1(t) = T_t u_0 + \int_0^t T_{t-s} f(s) ds.$$

It is easy to see that[(lxii)]

$$u_1(r+\gamma) - u_1(r) = (T_\gamma - 1)u_1(r) + \int_0^\gamma T_{\gamma-\rho} f(r+\rho) d\rho.$$

Therefore, $\mathbb{E}\|u_1(r+\gamma) - u_1(r)\|_p^p \le N(A_1(r,\gamma) + B_1(r,\gamma))$, where

$$A_1(r,\gamma) = \mathbb{E}\|(T_\gamma - 1)u_1(r)\|_p^p, \quad B_1(r,\gamma) = \mathbb{E}\left\|\int_0^\gamma T_{\gamma-\rho} f(r+\rho)d\rho\right\|_p^p,$$

and, by Lemma 7.0.4,

$$\mathbb{E}\|u_1(t) - u_1(s)\|_p^p \le N(t-s)^{\alpha p - 1}(I_1(t,s) + J_1(t,s)),$$

with

$$I_1(t,s) = \int_0^{t-s} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_s^{t-\gamma} A_1(r,\gamma)dr, \quad J_1(t,s) = \int_0^{t-s} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_s^{t-\gamma} B_1(r,\gamma)dr.$$

By using Hölder's inequality and Lemma 7.0.3 and observing that $(\beta-1)q > -1$ for $q = p/(p-1)$, we get (remember $n = 2\beta$ so that $\theta = (1/2)(2-n) \in [0,1]$)

$$\begin{aligned}
B_1(r,\gamma) &= \mathbb{E}\int_{\mathbb{R}^d}\left|\int_0^\gamma \rho^{\beta-1}\rho^{1-\beta} T_\rho f(r+\gamma-\rho)d\rho\right|^p dx \\
&\le \left(\int_0^\gamma \rho^{(\beta-1)q} d\rho\right)^{p/q} \mathbb{E}\int_0^\gamma \rho^{(1-\beta)p}\int_{\mathbb{R}^d} |T_\rho f(r+\gamma-\rho)|^p dx d\rho \\
&\le N(d,p,\beta)\gamma^{\beta p - 1}\mathbb{E}\int_0^\gamma e^{\rho p}\|f(r+\gamma-\rho)\|^p_{n-2,p} d\rho \\
&\le N(d,p,\beta,T)\gamma^{\beta p - 1}\mathbb{E}\int_0^\gamma \|f(r+\gamma-\rho)\|^p_{n-2,p} d\rho \\
&= N(d,p,\beta,T)\gamma^{\beta p - 1}\mathbb{E}\int_0^\gamma \|f(r+\rho)\|^p_{n-2,p} d\rho.
\end{aligned}$$

This and the inequality $\alpha < \beta$ implies that

$$J_1(t,s) \le N\int_0^{t-s} \frac{d\gamma}{\gamma^{2+(\alpha-\beta)p}} \int_0^\gamma \mathbb{E}\int_s^{t-\gamma} \|f(r+\rho)\|^p_{n-2,p} dr d\rho$$

(lxi) Recall that $u(t,x) = \int_0^t T_{t-s} f(s,x)ds$, where f is sufficiently good, solves $u_t = \Delta u + f$ on $[0,\infty)$ (see [9]).

(lxii) Since f is bounded, the Fubini's theorem implies

$$T_\gamma \int_0^r T_{r-\rho} f(\rho)d\rho = \int_0^r T_{r+\gamma-\rho} f(\rho)d\rho.$$

$$\leq N \int_0^{t-s} \frac{d\gamma}{\gamma^{2+(\alpha-\beta)p}} \int_0^{\gamma} \mathbb{E} \int_0^t \|f(r)\|_{n-2,p}^p dr d\rho$$
$$= N(t-s)^{(\beta-\alpha)p} \mathbb{E} \int_0^t \|f(r)\|_{n-2,p}^p dr.$$

To estimate I_1, we use Lemma 7.0.3 and Theorem 2.2.1 to obtain

$$A_1(r,\gamma) \leq N\gamma^{\beta p} \mathbb{E}\|u_1(r)\|_{n,p}^p \leq N\gamma^{\beta p}[\mathbb{E}\|u(0)\|_{n-2/p,p}^p + \mathbb{E}\int_0^r \|f(s)\|_{n-2,p}^p ds]$$

$$I_1(t,s) \leq \int_0^{t-s} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_0^t A_1(r,\gamma) dr$$
$$\leq N(t-s)^{(\beta-\alpha)p}[\mathbb{E}\|u(0)\|_{n-2/p,p}^p + \mathbb{E}\int_0^r \|f(s)\|_{n-2,p}^p ds].$$

For $u_2 := u - u_1$ we have[(lxiii)]

$$u_2(r+\gamma) - u_2(r) = (T_\gamma - 1)u_2(r) + \int_r^{r+\gamma} T_{r+\gamma-\rho} g^k(\rho) dw_\rho^k,$$

$$\mathbb{E}\|u_2(r+\gamma) - u_2(r)\|_p^p \leq N(A_2(r,\gamma) + B_2(r,\gamma)),$$

where

$$A_2(r,\gamma) = \mathbb{E}\|(T_\gamma - 1)u_2(r)\|_p^p, \quad B_2(r,\gamma) = \mathbb{E}\left\|\int_r^{r+\gamma} T_{r+\gamma-\rho} g^k(\rho) dw_\rho^k\right\|_p^p.$$

By Lemma 7.0.4,

$$\mathbb{E}\|u_2(t) - u_2(s)\|_p^p \leq N(t-s)^{\alpha p-1}(I_2(t,s) - J_2(t,s)),$$

with

$$I_2(t,s) = \int_0^{t-s} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_s^{t-\gamma} A_2(r,\gamma) dr, \quad J_2(t,s) = \int_0^{t-s} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_s^{t-\gamma} B_2(r,\gamma) dr.$$

Here we use Lemma 7.0.3, the Burkholder-Davis-Gundy inequalities, and the inequality $(2\beta - 1)q > -1$, where $q = p/(p-2)$. Then, similarly to the above calculations, we obtain

$$B_2(r,\gamma) = \int_{\mathbb{R}^d} \mathbb{E}\left|\int_r^{r+\gamma} T_{r+\gamma-\rho} g^k(\rho) dw_\rho^k\right|^p dx$$
$$\leq N\mathbb{E}\int_{\mathbb{R}^d}\left(\int_r^{r+\gamma} \|T_{r+\gamma-\rho} g(\rho)\|_{l_2}^2 d\rho\right)^{p/2} dx$$
$$= N\mathbb{E}\int_{\mathbb{R}^d}\left(\int_0^{\gamma} \rho^{2\beta-1}\rho^{1-2\beta} \|T_\rho g(r+\gamma-\rho)\|_{l_2}^2 d\rho\right)^{p/2} dx$$
$$\leq N\gamma^{\beta p-1}\mathbb{E}\int_0^{\gamma} \rho^{(1-2\beta)p/2} \|T_\rho g(r+\gamma-\rho)\|_p^p d\rho$$
$$\leq N\gamma^{\beta p-1}\mathbb{E}\int_0^{\gamma} \|(1-\Delta)^{\beta-1/2} g(r+\gamma-\rho)\|_p^p d\rho,$$

$$J_2(t,s) \leq N\mathbb{E}\int_0^{t-s} \frac{d\gamma}{\gamma^{2+(\alpha-\beta)p}} \int_s^{t-\gamma}\int_0^{\gamma} \|g(r+\gamma-\rho)\|_{n-1,p}^p d\rho dr$$

[(lxiii)] See the proof of Theorem 4.1.2.

$$\leq N\mathbb{E}\int_0^{t-s}\frac{d\gamma}{\gamma^{2+(\alpha-\beta)p}}\int_0^{\gamma}\int_0^t \|g(r)\|_{n-1,p}^p dr d\rho$$
$$= N(t-s)^{(\beta-\alpha)p}\mathbb{E}\int_0^t \|g(r)\|_{n-1,p}^p dr.$$

Finally, again by Lemma 7.0.3 and Theorem 4.1.2, we conclude

$$\begin{aligned} I_2(t,s) &\leq N\mathbb{E}\int_0^{t-s}\frac{d\gamma}{\gamma^{1+(\alpha-\beta)p}}\int_s^{t-\gamma}\|u_2(r)\|_{n,p}^p dr \\ &\leq N(t-s)^{(\beta-\alpha)p}\mathbb{E}\int_0^t \|u_2(r)\|_{n,p}^p dr \\ &\leq N(t-s)^{(\beta-\alpha)p}\mathbb{E}\int_0^t \|g(r)\|_{n-1,p}^p dr. \end{aligned}$$

Collecting all these estimates, we get eq. (72) at least for $\eta = T(=\tau)$. In the general case, observe that if $u(t\wedge\eta) - u(s\wedge\eta)$ is not zero, then $s\wedge\eta = s$ and $s \leq s\wedge\eta \leq t\wedge\eta \leq t$. After this it suffices to notice that, instead of points t and s on the left in eq. (77), we can obviously take any two points between them.

The proof of eq. (73) goes exactly the same way, the only difference being that this time we use eq. (78) and the formula

$$\begin{aligned} \mathbb{E}\sup_{t\leq T}\|h(t)\|_p^p &\leq N\mathbb{E}\|h(0)\|_p^p + N\mathbb{E}\sup_{t\leq T}\|h(t)-h(0)\|_p^p \\ &\leq N\mathbb{E}\|h(0)\|_p^p + NT^{\alpha p-1}\int_0^T\int_0^{T-\gamma}\frac{\mathbb{E}\|h(r+\gamma)-h(r)\|_p^p}{\gamma^{1+\alpha p}}drd\gamma \end{aligned}$$

because of eq. (77) by taking $s = 0$, supremum over t, and then take an expectation.

To prove assertion (ii) notice that an interpolation theorem (Theorem 2.4.2 in [23]) we have

$$\|u\|_{m(\theta)-d/p+d/q,q} \leq N(d,p,q,m(0),m(1),\theta)\|u\|_{m(0),p}^{1-\theta}\|u\|_{m(1),p}^{\theta}$$

whenever $2 \leq p \leq q < \infty$, $\theta \in (0,1)$, $m(\theta) := (1-\theta)m(0) + \theta m(1) \neq m(0)$, $m(i) \leq n$, and $u \in H_p^n$. Indeed, Theorem 2.4.2 in [23] gives that $(H_p^{m(0)}, H_p^{m(1)})_{\theta,2} = B_{p,2}^{m(\theta)} \subset H_p^{m(\theta)}$.(lxiv) Now if $q > p$, by the Sobolev embedding theorem gives $H_p^{m(\theta)} \subset H_q^{m(\theta)-d/p+d/q}$.(lxv) Now use the basic property of real interpolation spaces (see theorem 9.8.2).

Note also that, under the conditions in (ii), there is a β such that $1/2 > \beta > 1/p$ and $m \leq n - 2\beta(1-\theta) - d/p + d/q = m(\theta) - d/p + d/q$, where $m(\theta) := (1-\theta)(n-2\beta) + \theta n$. Therefore,

$$\begin{aligned} \mathbb{E}\left(\int_0^T \|u(t,\cdot)\|_{m,q}^{p/\theta}dt\right)^{\theta} &\leq \mathbb{E}\left(\int_0^T \|u(t,\cdot)\|_{m(\theta)-d/p+d/q,q}^{p/\theta}dt\right)^{\theta} \\ &\leq N\mathbb{E}\left(\int_0^T \|u(t,\cdot)\|_{n-2\beta,p}^{(1-\theta)p/\theta}\|u(t,\cdot)\|_{n,p}^p dt\right)^{\theta} \\ &\leq N\mathbb{E}\sup_{t\leq T}\|u(t,\cdot)\|_{n-2\beta,p}^{(1-\theta)p}\left(\int_0^T \|u(t,\cdot)\|_{n,p}^p dt\right)^{\theta}. \end{aligned}$$

To prove (ii), it remains only to apply Hölder's inequality, eq. (14), and eq. (73) with an α such that $\beta > \alpha > 1/p$. The theorem is proved.

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(lxiv) Recall that $F_{p,2}^s = H_p^s$, and $B_{p,p\wedge q}^s \subset F_{p,q}^s$. Here, remember also that $p \geq 2$.

(lxv) Recall that if $1 < p \leq q < \infty$ and $s - d/p = t - d/q$, we have $H_p^s \subset H_q^t$.

8 Applications

8.1 Filtering Equation

8.2 On the Notion of Stochastic Integral

To consider applications to equations with infinitely dimensional Wiener processes, we want to discuss the notion of stochastic integral and show that ‘‘basically’’ there is nothing more general than series of usual one-dimensional stochasic integrals. This will show that equations like eq. (1), containing series of integrals with respect to Wiener processes, are of a quite general nature.

The first notion was introduced by Paley, Wiener, and Zygmund in [16], where the stochastic integral of a *nonrandom smooth* function $f(t)$ against a one dimensional Wiener process w_t is defined as

$$\int_0^1 f(s)dw_s = f(1)w_1 - \int_0^1 w_s f'(s)ds. \tag{80}$$

Then it is verified that the $L_2(\Omega)$-norm of the stochastic integral is equal to the $L_2(0,1)$-norm of f, which allows one to extend the stochastic integral from smooth functions to all $f \in L_2(0,1)$. One obtains the same integral if, instead of eq. (80), one starts with

$$\int_0^1 f(s)dw_s = \sum_{i=1}^n a_i(w_{s_i} - w_{s_{i-1}}) \tag{81}$$

for functions f such that $f(t) = a_i$ on $(s_{i-1}, s_i]$ and $0 = s_0 \le s_1 \le \cdots \le s_n = 1$.

The definition based on eq. (81) has an advantage that it can be easily generalized to define an integral of a *nonrandom* function against a random orthogonal measure on a σ-finite measure space $(X, \mathcal{X}, m)$. More precisely, assume that we are given a π-system Π of subsets of X such that $\sigma(\Pi) = \mathcal{X}$, and a random (complex-valued) variable $\mu(\gamma)$ defined for each $\gamma \in \Pi$ (and perhaps not for all $\gamma \in \mathcal{X}$). Assume that $\mu(\gamma) \in L_2(\Omega)$ and $\mathbb{E}\mu(\gamma_1)\bar{\mu}(\gamma_2) = m(\gamma_1 \cap \gamma_2)$ for each $\gamma, \gamma_1, \gamma_2 \in \Pi$. Then for functions

$$f(x) = \sum_{j=1}^n a_j \mathbb{1}_{\gamma_j}(x),$$

where a_j's are constant and $\gamma_j \in \Pi$, one defines the stochastic integral of f against μ as

$$\int_X f(x)\mu(dx) = \sum_{j=1}^n a_j \mu(\gamma_j), \tag{82}$$

and, again by isometry, one extends the stochastic integral to all $f \in L_2(X, m)$. Such integrals are used in the theory of stationary processes. Suprisingly enough, as we will see, one can also say that this is the most general stochastic integral in Itô's stochastic calculus.

Another advantage of eq. (81) is that one can allow f to depend on ω, and if a_j are independent of the process $w_{t+s_{j-1}} - ws_{j-1}$, $t \ge 0$, then eq. (81) is again an isometry between a part of $L_2(\Omega \times (0,1])$ and a part of $L_2(\Omega)$. Closing this isometry, K. Itô defines his famous integral.

It turns out that Itô's integral is a particular case of the integral based on eq. (82). To be more precise, let w_t be (as usual) a Wiener process with respect to a filtration $\mathcal{F}_t$, $\mathcal{P}$ be the predictable σ-field on $\Omega \times (0,1]$, and Π be the set of all stochastic intervals $(\!(0,\tau]\!]$, where τ are stopping times ≤ 1. Then one gets Itô's integral by taking $X = \Omega \times (0,1]$, $\mathcal{X} = \mathcal{P}$, $\mu(\!(0,\tau]\!] = w_\tau$. In the same way one defines stochastic integrals with respect to any locally square integrable martingales.

K. Itô [4] was also the first to consider integration against measure-valued processes, which is a particular case of integration against martingale measures. Let $p(t,\Gamma)$, $t \ge 0$, be a square integrable process as a function of t with independent increments in time and a random orthogonal measures as a function of Γ for any t. Define $p((s,t],\Gamma) = p(t,\Gamma) - p(s,\Gamma)$. Itô's way of introducing the integral with respect to p is to replace the expression $a_j(w_{s_j}, w_{s_{j-1}})$ in eq. (81) with

$$\int_X f_j(x)p((s_{j-1},s_j],dx) = \int_X f_j(x)p(s_j,dx) - \int_X f_j(x)p(s_{j-1},dx), \tag{83}$$

where f_j are assumed to be independent of the processes $p((s_{j-1},t],\Gamma)$, $t \geq s_{j-1}$.

More generally, for any $\gamma \in \Pi$ let a process $p(t,\gamma)$ be given, which is a square integrable martingale with respect to a given filtration $\{\mathcal{F}_t\}_t$. Let $\langle p(\cdot,\gamma_1), p(\cdot,\gamma_2)\rangle_t = q(t,\gamma_1 \cap \gamma_2)$, where $q(t,\cdot)$ is a σ-finite measure on $(X,\mathcal{X})$ for any ω, t and $q(t,\Gamma)$ increases in t for any $\Gamma \in \mathcal{X}$ and ω. Then there is a measure $q(dt,dx)$ such that

$$q(t,\gamma) = \int_0^t \int_\gamma q(ds,dx).$$

By following Itô's method based on eq. (83), for any $\mathcal{P}\otimes\mathcal{X}$-measurable $f = f(\omega,t,x)$ such that

$$\int_0^1 \int_X f^2(s,x)q(ds,dx) < \infty,$$

one defines the stochastic integral

$$\int_0^1 \int_X f(s,x)p(ds,dx). \tag{84}$$

This integral is also a particular case of the integral of a *nonrandom* function against a random orthogonal measure. Indeed, define $\bar{X} = \Omega \times (0,1] \times X$, $\bar{\mathcal{X}} = \mathcal{P}\otimes\mathcal{X}$, and let $m(d\omega dt dx) = \mathbb{P}(d\omega)q(dt,dx)$. Also let

$$\bar{\Pi} = \{(\!(0,\tau]\!] \times \gamma : \mathbb{E}q((\!(0,\tau]\!] \times \gamma) < \infty\}, \quad \mu((\!(0,\tau]\!] \times \gamma) = p(\tau,\gamma).$$

Then on functions

$$f = \sum_{j\leq n} a_j \mathbb{1}_{t\leq\tau_j}\mathbb{1}_{\gamma_j}(x), \tag{85}$$

where $(\!(0,\tau]\!] \times \gamma_j \in \bar{\Pi}$ and a_j are some constants, integral eq. (84) equals

$$\sum_{j\leq n} a_j p(\tau_j,\gamma_j) = \sum_{j\leq n} a_j \mu((\!(0,\tau_j]\!] \times \gamma_j),$$

which agrees with eq. (82). Finally, by functions of type eq. (85) one can approximate any $\mathcal{P}\otimes\mathcal{X}$-measurable function for which eq. (84) can be defined.

It is worth mentioning that there are also other notions of martingale measures with respect to which one can define stochastic integration (see [25], where the martingale measures discussed abvove are called orthogonal martingale measures).

Even though the notion of integral of nonrandom functions with respect to random orthogonal measures is very convenient for the purpose of introducing Itô's stochastic integrals (cf. [**?**]), one works almost always with stochastic integrals with variable limits, and a different notation is more appropriate.

In connection with this notice that it is shown in [3] how to reduce the stochastic integral with respect to a martingale measure to a series of usual stochastic integrals. This was further used in [3] to treat *stochastic equations* containing integrals against martingale measures using *the same notation* as in the case of equations containing just usual stochastic integrals.

To be more precise, it is assumed in [3] that $\mathcal{X}$ is countably generated and

$$q(t,\Gamma) = \int_0^t q_s(\Gamma)dV_s,$$

where V_s is a predictable increasing process and $q_s(\Gamma)$ is a measure in Γ for any s and predictable in s for any $\Gamma \in \mathcal{X}$. Then it is shown that

$$\int_0^t \int_X f(s,x)p(ds,dx) = \sum_{k=1}^{\infty} \int_0^t f_k(s)dp_s^k, \tag{86}$$

where

$$p_t^k = \int_0^t \int_X \eta_k(s,x)p(ds,dx), \quad f_k(s) = \int_X \eta_k(s,x)f(s,x)q_s(dx),$$

and for any ω, s the system of functions $\{\eta_k(s,\cdot)\}_k$ forms an orthonormal basis in $L_2(X,q_s)$.

A particular case of the stochastic integral with respect to a martingale measure is the stochastic integral with respect to the two-dimensional Brownian sheet $W(t,x)$ defined for $t \geq 0$, $x \in \mathbb{R}$. In this case, one takes $\mathcal{F}_t$ so that the random variables $W(t,x)$ are $\mathcal{F}_t$-measurable, and defines

$$p((\!(0,\tau]\!] \times (a,b]) = W(\tau,b) - W(\tau,a).$$

This integral got very popular thanks to the article [25]. One can construct $W(s,x)$ by taking independent one-dimensional Wiener processes w_t^k, $k \geq 1$, and an orthonormal basis $\{\eta_k(x)\}_{k\geq 1}$ in $L_2(\mathbb{R})$, and letting

$$W(t,x) = \sum_{k=1}^{\infty} w_t^k \int_0^x \eta_k(y)dt \quad t \geq 0,\ x \in \mathbb{R}.$$

Incidentally, observe that thus defined W is a Gaussian field and

$$\mathbb{E}W(s,y)W(t,x) = (s \wedge t)\sum_{k=1}^{\infty} \int_0^x \eta_k(z)dz \int_0^y \eta_k(z)dz,$$

where[(lxvi)]

$$\sum_{k=1}^{\infty} \int_0^x \eta_k(z)dz \int_0^y \eta_k(z)dz = \begin{cases} \int_{\mathbb{R}} \mathbb{1}_{(0,x)}(z)\mathbb{1}_{(0,y)}(z)dz = x \wedge y & \text{if } x,y \geq 0, \\ \int_{\mathbb{R}} \mathbb{1}_{(x,0)}(z)\mathbb{1}_{(y,0)}(z)dz = |x| \wedge |y| & \text{if } x,y \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular case eq. (86) becomes

$$\int_0^t \int_{\mathbb{R}} f(s,x)W(ds,dx) = \sum_{k=1}^{\infty} \int_0^t \left\{ \int_{\mathbb{R}} \eta_k(x)f(s,x)dx \right\} dw_s^k. \tag{87}$$

By the way, general one-dimensional equations driven by the cylindrical space-time white noise $\dot{B}_t$ were considered in [2], where the right hand side of eq. (87) is taken by definition as $\int_0^t \langle f(s,\cdot), dB_t\rangle$. There the series was introduced from the very begining.

The last integral we want to discuss is the integral against a Hilbert-space valued Wiener process (see, for instance, [19]). Let H be a Hilbert space and w_t be a H-valued Wiener proces with covariance operator Q. This operator is known to be nuclear (see [26] for its definition). If h_k are its unit eigenvectors with nonzero eigenvalues, then $w_t^k := (h^k, w_t)(Qh^k, h^k)^{-1/2}$ are independent standard Wiener processes and, for any H-valued process f_t for which the integral $\int_0^t f_t \cdot dw_t$ is defined, the integral can be written as

$$\sum_k \int_0^t f_s^k dw_s^k,$$

where $f_s^k = (f_s, h^k)(Qh^k, h^k)^{1/2}$ (see, for instance, [19]).

(lxvi) Let H be a Hilbert space and $\mathfrak{B}$ an orthonormal basis for H. Then we have

$$\langle g,h\rangle = \sum_{f\in\mathfrak{B}} \langle g,f\rangle\langle f,h\rangle,$$

where the sum converges in the net sense.

8.3 Equations Driven by Space-Time White Noise

In this subsection, we consider one-dimensional equations with space-time white noise. Thus $d = 1$.(lxvii) Very often (see, for instance [2]) one writes these equations in the form

$$du(t,x) = [a(t,x)u''(t,x) + b(t,x)u'(t,x) + f(t,x,u(t,x))]dt + h(t,x,u(t,x))dB_t, \tag{88}$$

where B_t is a cylindrical Wiener process on L_2. There are also very many articles where instead of dB_t one writes $(\partial^2 W/\partial t \partial x)dt$. As we explained in Section 8.2 we may as well take $\sum_k \eta_k(x)dw_t^k$, where $\{\eta_k(x)\}_{k\geq 1}$ is an orthonormal basis in L_2 and w_t^k are independent $\mathcal{F}_t$-adapted one-dimensional Wiener process. Thus, instead of eq. (88), we will be considering the equation

$$du(t,x) = [a(t,x)u''(t,x) + b(t,x)u'(t,x) + f(t,x,u(t,x))]dt + g^k(t,x,u(t,x))dw_t^k \tag{89}$$

on a time interval $[0,\tau]$, where $g^k := h\eta_k$ and τ is a bounded stopping time.

Assumption 8.3.1 The functions $a(t,x) = a(\omega,t,x)$ and $b(t,x) = b(\omega,t,x)$ are real-valued functions defined on $(\!(0,\tau]\!] \times \mathbb{R}$.

(i) For any ω and $t \leq \tau(\omega)$, $a(\omega,t,\cdot) \in C^{1,1}(= B^2)$ and $b(\omega,t,\cdot) \in C^{0,1}(= B^1)$ and $\|a\|_{C^{1,1}} + \|b\|_{C^{0,1}} \leq K$. Also $K \geq a \geq \delta$.

(ii) For any $x \in \mathbb{R}$, the processes a and b are predictable.

To state the next assumption, take a fix $s \leq \infty$ and finite $\varkappa, p, r$ such that

$$\varkappa \in (0,1/2], \quad p \geq 2r \geq 2, \quad s \leq \infty, \quad \frac{1}{r} + \frac{1}{s} = 1, \quad r < \frac{1}{1-2\varkappa}. \tag{90}$$

Assumption 8.3.2 The functions $f(t,x,u)$ and $h(t,x,u)$ are real-valued functions on $(\!(0,\tau]\!] \times \mathbb{R}^2$ such that

(i) for any x and u, the processes $f(t,x,u)$ and $h(t,x,u)$ are predictable;

(ii) for any ω, t, x, u, and v,

$$|f(t,x,u) - f(t,x,v)| \leq K|u-v|, \quad |h(t,x,u) - h(t,x,v)| \leq \xi(t,x)|u-v|, \tag{91}$$

where ξ is certain function of ω, t, x satisfying $\|\xi(t,\cdot)\|_{2s} \leq K$.

Observe that one of possibilities is $r = 1$, and then $s = \infty$ and eq. (91) just means that both f and h satisfy the Lipschitz condition in u with the constant K.

To fit eq. (89) in our general scheme, we need to find an appropriate n such that the assumptions of Theorem 5.0.7 are satisfied. Define

$$n = -\varkappa - 3/2.$$

In the following lemma, we also set

$$R(x) = \chi|x|^{-(1-2\varkappa)/2} \int_0^\infty t^{-(5-2\varkappa)/4} e^{tx^2 - 1/(4t)} dt.$$

In Section 9.5, it is proved that R is the kernel of the operator $(1-\Delta)^{(n+1)/2}$, infinitely differentiable everywhere except the origin, decreases exponentially fast as $|x| \to \infty$, and behaves near the origin like $|x|^{-(1-2\varkappa)/2}$ if $\varkappa < 1/2$ and like $-\log|x|$ if $\varkappa = 1/2$. Finally, notice that generally speaking, the notation h, ξ, g is used in the lemma for functions different from the ones introduced in above.

(lxvii) We will discuss why such assumption is in here. We want u be a function. See Remark 8.3.9.

Lemma 8.3.3 Take some functions $h \in L_p$, $\xi \in L_{2s}$, and set $g^k = \xi h \eta_k$. Then $g = (g^k)_{k\geq 1} \in H_p^{n+1}(l_2)$ and

$$\|g\|_{n+1,p} = \|\bar{h}\|_p \leq (N\|\xi\|_{2s}\|h\|_p) \wedge (\|\xi h\|_2\|R\|_p), \tag{92}$$

where $N = \|R\|_{2r} < \infty$ and

$$\bar{h}(x) := \left(\int_{\mathbb{R}} R^2(x-y)\xi^2(y)h^2(y)dy\right)^{1/2}. \tag{93}$$

In addition, if $p(1-2\kappa) > 2$ and $\xi = 1$, then

$$\|g\|_{n+1,p} \leq N(\kappa)\|h\|_2^{2\kappa p/(p-2)}\|h\|_p^{1-2\kappa p/(p-2)}. \tag{94}$$

- -

proof. We know that

$$(1-\Delta)^{(n+1)/2}(\xi h\eta_k)(x) = \int_{\mathbb{R}} R(x-y)\xi(y)h(y)\eta_k(y)dy.$$

It follows that by Parseval's theorem (remember that $\{\eta_k\}_k$ forms an orthonormal basis in L_2)

$$\begin{aligned}
|(1-\Delta)^{(n+1)/2}g(x)|_{l_2}^2 &= \sum_{k=1}^{\infty}|(1-\Delta)^{(n+1)/2}g^k(x)|^2 \\
&= \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}} R(x-y)\xi(y)h(y)\eta_k(y)dy\right)^2 \\
&= \int_{\mathbb{R}} R^2(x-y)\xi^2(y)h^2(y)dy \\
&= \bar{h}^2(x).
\end{aligned}$$

We thus get the equality in eq. (92). To prove the inequality, notice that $\bar{h}^2$ is a convolution and by Minkowski's inequality, This immediately gives

$$\|\bar{h}\|_p = \|(\bar{h})^2\|_{p/2}^{1/2} \leq (\|\xi^2h^2\|_1\|R^2\|_{p/2})^{1/2} = \|\xi h\|_2\|R\|_p.$$

Also, by Hölder's inequality,

$$\bar{h}^2(x) \leq \|\xi\|_{2s}^2\left(\int_{\mathbb{R}} R^{2r}(x-y)h^{2r}(y)dy\right)^{1/r},$$

which leads to the second inequality in eq. (92): $\|\bar{h}\|_p \leq N\|\xi\|_{2s}\|h\|_p$, again by Minkowski's inequality and by the assumption $p \geq 2r$. The finiteness of N follows from the assumption that $r < (1-2\kappa)^{-1}$.[lxviii]

To prove eq. (94), we use that $R^2(y) \leq N|y|^{2\kappa-1}$[lxix] and we minimize with respect to $\epsilon > 0$ after the following computations:

$$\begin{aligned}
\left(\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} R^2(y)h^2(x-y)dy\right\}^{p/2}dx\right)^{2/p} &\leq \left(\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \mathbb{1}_{|y|\leq\epsilon}R^2(y)h^2(x-y)dy\right\}^{p/2}dx\right)^{2/p} \\
&\quad + \left(\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \mathbb{1}_{|y|\geq\epsilon}R^2(y)h^2(x-y)dy\right\}^{p/2}dx\right)^{2/p}
\end{aligned}$$

(lxviii) Recall the behavior of R at the origin because R decreases exponentially fast as $|x| \to \infty$.

(lxix) Notice that R behaves like $|x|^{(1-2\kappa)/2}$ near the origin, and R is exponentially decreases as $|x| \to \infty$.

$$\leq \|\mathbb{1}_{|y|\leq\epsilon}R^2\|_1\|h\|_p^2 + \|\mathbb{1}_{|y|\geq\epsilon}R^p\|_1^{2/p}\|h\|_2^2$$
$$\leq N\epsilon^{2\varkappa}\|h\|_p^2 + N\epsilon^{-(1-2\varkappa-2/p)}\|h\|_2^2.$$

This proves the lemma.

Theorem 8.3.4 Let $\varkappa \in (0, 1/2)$ and $u_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{(1/2)-\varkappa-2/p})$. Assume that

$$I^p(\tau) := \mathbb{E}\int_0^\tau [\|f(t,\cdot,0)\|_{-3/2-\varkappa,p}^p + \|\bar{h}(t,\cdot,0)\|_p^p]dt < \infty, \tag{95}$$

where

$$\bar{h}(t,x,0) := \left(\int_{\mathbb{R}} R^2(x-y)h^2(t,y,0)dy\right)^{1/2}.$$

Then in the space $\mathcal{H}_p^{1/2-\varkappa}(\tau)$, eq. (89) with the initial condition u_0 has a unique solution u. Moreover,

$$\|u\|_{\mathcal{H}_p^{1/2-\varkappa}(\tau)} \leq N\{I(\tau) + (\mathbb{E}\|u_0\|_{1/2-\varkappa-2/p,p}^p)^{1/p}\},$$

where the constant N depends only on $\varkappa, p, \delta, K$, and τ.

- -

proof. We will apply Theorem 5.0.7. Its assumptions concering a and $\sigma(\equiv 0)$ are obviously satisfied. Next, if $u \in H_p^{n+2}$, then $bu' \in H_p^{n+1} \subset H_p^n$, $u \in L_p$[lxx], $f(u) - f(0) \in L_p \subset H_p^n$ (because $n < 0$). Also. by eq. (95) we have $f(t,\cdot,0) \in H_p^n$ (a.e. on $(\!(0,\tau]\!]$). Furthermore, by Lemma 5.0.8,

$$\|bu'\|_{n,p} \leq \|bu'\|_{-1,p} \leq N\|u'\|_{-1,p} \leq N\|u\|_p = N\|u\|_{n+2-((1/2)-\varkappa),p},$$
$$\|f(u) - f(v)\|_{n,p} \leq \|f(u) - f(v)\|_p \leq K\|u - v\|_p.$$

We emphasize that $\|\cdot\|_p = \|\cdot\|_{n+2-((1/2)-\varkappa),p}$ and $n + 2 - ((1/2) - \varkappa) < n + 2$ for $\varkappa < 1/2$.[lxxi] Consequently (see Remark 5.0.11), the assumptions of Theorem 5.0.7 concerning $bu' + f(u)$ are satisfied. To check the remaining assumptions about $g(u)$, it suffices to notice that, by Lemma 8.3.3 we have

$$\|g(0)(t,\cdot)\|_{n+1,p} = \|\bar{h}(t,\cdot,0)\|_p,$$
$$\|g(u)(t,\cdot) - g(v)(t,\cdot)\|_{n+1,p} = \|\bar{h}(t,\cdot,u(t,\cdot),v(t,\cdot))\|_p \leq \|\bar{\bar{h}}(t,\cdot,u(t,\cdot),v(t,\cdot))\|_p \leq N\|u(t,\cdot) - v(t,\cdot)\|_p,$$

where

$$\bar{h}(t,x,u,v) := \left(\int_{\mathbb{R}} R^2(x-y)(h(t,y,u) - h(t,y,v))^2 dy\right)^{1/2},$$
$$\bar{\bar{h}}(t,x,u,v) := \left(\int_{\mathbb{R}} R^2(x-y)\xi^2(t,y)(u(t,y) - v(t,y))^2 dy\right)^{1/2}.$$

The theorem is proved.

Remark 8.3.5 We have obtained Theorem 8.3.4 by checking that all assumptions of Theorem 5.0.7 for $n = -\varkappa - 3/2$. After this, of course, all other results from Section 5 are available. For example, by the approximation theorem (Theorem 5.0.13) and Remark 5.0.15, we have $\|u - u_m\|_{\mathcal{H}_p^{n+2}(\tau)} \to 0$, where u_m is a unique solution to the following version of eq. (60)

$$du(t,x) = [a(t,x)u''(t,x) + b(t,x)u'(t,x) + f(t,x,u(t,x))]dt + \sum_{k=1}^m h(t,x,u(t,x))\eta_k dw_t^k$$

with initial condition u_0.

[lxx] Remember that $n = -\varkappa - 3/2$, so that $n + 2 = -\varkappa + 1/2 > 0$.

[lxxi] See Remark 5.0.11

Remark 8.3.6 Additional information about Hölder continuity properties of the soluion is readily obtained from the properties of elements of $\mathcal{H}_p^n(\tau)$ listed in Theorem 7.0.2. For example, in the case of Theorem 8.3.4 (the space $\mathcal{H}_p^{n+2}(\tau)$),

$$u \in C^{\alpha-1/p}([0,\tau], H_p^{n+2-2\beta}) \quad (a.s.) \tag{96}$$

provided $p > 2$ and $1/2 > \beta > \alpha > 1/p$. Here $H_p^{n+2-2\beta} \subset C^\gamma$ if

$$\gamma := n + 2 - 2\beta - 1/p = 1/2 - \kappa - 2\beta - 1/p > 0.$$

Hence, if the inequality $\mathbb{E}\|u_0\|^p_{1/2-\kappa-2/p,p} < \infty$ and eq. (95) are satisfied for any $\kappa \in (0,1/2)$ and $p \geq 2$ (say $f(t,x,0) = h(t,x,0) = 0$, $\xi = K$, and u_0 is a deterministic smooth function with compact support), then, after taking p large enough and κ, α, and β small, we see that u satisfies the Hölder condition in x of order $1/2 - \epsilon$ uniformly with respect to $t \in [0,\tau]$ (a.s.) for any $\epsilon > 0$.

On the other hand, by taking p large enough, both α and β close to $1/4$, and κ small, we get that u satisfies the Hölder condition in t of order $1/4 - \epsilon$ uniformly with respect to $x \in \mathbb{R}$ (a.s.) for any $\epsilon > 0$. In terms of parabolic Hölder spaces, this means that

$$u \in C^{1/4-\epsilon,1/2-\epsilon}_{t,x}([0,\tau] \times \mathbb{R}) \quad (a.s.).$$

Notice that using standard PDE methods, one can prove interior (with respect to t) estimates which would give similar continuity of u away from $t = 0$ under weaker assumptions on u_0. One of results which can be obtained is discussed in the following remark.

Remark 8.3.7 There are smoothing properties of equations in the sense that solutions may be much smoother than the initial data. For example, assume that $f(t,x,0) = h(t,x,0) = 0$, $\tau = T$, where T is a constant, $r = 1$, and $s = \infty$. Let $u_0 \in L_2(\Omega, \mathcal{F}_0, H_2^{-1+\epsilon})$ with $\epsilon \in (0,1/2)$, so that u_0 may be a finite measure or just a delta-function. We claim then that, for any $\epsilon \in (0,1/4)$,

$$u \in C^{1/4-\epsilon,1/2-\epsilon}_{t,x}([\epsilon,T] \times \mathbb{R}) \quad \text{(a.s.)}.$$

Indeed, by Theorem 8.3.4 with $p = 2$ and $\kappa = 1/2 - \epsilon$, there is a unique solution $u \in \mathcal{H}_2^\epsilon(T)$ of eq. (89) and

$$\|u\|^2_{\mathcal{H}_2^\epsilon(T)} \leq N\mathbb{E}\|u_0\|^2_{-1/2-\kappa,2}.$$

By eq. (14) we have

$$\int_0^T \mathbb{E}\|u(t,\cdot)\|^2_{\epsilon,2}dt = \|u\|^2_{\mathbb{H}_2^\epsilon(T)} \leq N\mathbb{E}\|u_0\|^2_{-1/2-\kappa,2} =: I.$$

By Chebyshev's inequality, for any $\gamma \in (0, T/2)$, Lebesgue measure of the set on $(\gamma, 2\gamma)$, where $\mathbb{E}\|u(t,\cdot)\|^2_{\epsilon,2} > 2I/\gamma$, is less than $\gamma/2$. This implies that for any $\gamma(0,T/2)$ there is a point $t_\gamma \in (\gamma, 2\gamma)$ such that

$$u(t_\gamma, \cdot) \in L_2(\Omega, \mathcal{F}_{t_\gamma}, H_2^\epsilon), \quad \mathbb{E}\|u(t_\gamma,\cdot)\|^2_{\epsilon,2} \leq 2I/\gamma.$$

Indeed, first of all, as u is predictable, it is $\mathcal{F}_t$-adapted. Also, as the Lebesgue measure of $(\gamma, 2\gamma)$ is γ, obviously such t_γ exists.

If we now consider eq. (89) after time t_γ instead of 0 and if we define the spaces $\mathcal{H}_2^{1/2-\kappa}(t_\gamma, T)$ in an obvious way, we get by Theorem 8.3.4 that $u \in \mathcal{H}_2^{1/2-\kappa}(t_\gamma, T)$ for *any* $\kappa \in (0,1/2)$.[lxxii] In addition,

$$\int_{t_\gamma}^T \mathbb{E}\|u(t,\cdot)\|^2_{1/2-\kappa,2}dt \leq N\mathbb{E}\|u(t_\gamma,\cdot)\|^2_{\epsilon,2} \leq NI/\gamma, \quad \mathbb{E}\|u(s_\gamma,\cdot)\|^2_{1/2-\kappa,2} \leq NI/\gamma^2, \tag{97}$$

where s_γ is a certain point in $(t_\gamma + \gamma, t_\gamma + 2\gamma)$ and $\gamma \leq T/4$.

[lxxii] Since $1/2 - \kappa - 1 = -1/2 - \kappa \in (-1/2, 0)$ for every $\kappa \in (0,1/2)$, and $u(t_\gamma,\cdot) \in L_2(\Omega, \mathcal{F}_{t_\gamma}, H_2^\epsilon)$ where $\epsilon \in (0,1/2)$, we can apply Theorem 8.3.4 for every such κ.

Now we can go to large powers. Take any $p > 4$[(lxxiii)] and define

$$A(p,\gamma,R) := \{\omega : \|u(s_\gamma,\cdot)\|_p \le R\}.$$

By Sobolev's embedding theorem, $H_2^{1/2-\kappa} \subset H_p^m$ if $-\kappa = m - 1/p$. For $m = 0$ we have $\kappa = 1/p$. Since in eq. (97) we can take $\kappa = 1/p$, we get by using Chebyshev's inequality that

$$\mathbb{P}(A(p,\gamma,R)) = \mathbb{P}\{\|u(s_\gamma,\cdot)\|_p \le R\} \ge \mathbb{P}\{\|u(s_\gamma,\cdot)\|_{1/2-\kappa,2} \le NR\} \ge 1 - NI\gamma^{-2}R^{-2}.$$

Next, obviously, on the set $A(p,\gamma,R)$ and the time interval (s_γ, T) the assumptions of Theorem 8.3.4 are satisfied with $\kappa = 1/2 - 2/p \in (0, 1/2)$ (so that $p > 4$).[(lxxiv)] Therefore $u\mathbb{1}_{A(p,\gamma,R)} \in \mathcal{H}_p^{2/p}(s_\gamma, T)$ and

$$\int_{s_\gamma}^T \mathbb{E}\mathbb{1}_{A(p,\gamma,R)}\|u(t,\cdot)\|_{2/p,p}^p dt \le N\mathbb{E}\mathbb{1}_{A(p,\gamma,R)}\|u(s_\gamma,\cdot)\|_p^p \le NR^p,$$

$$\mathbb{E}\mathbb{1}_{A(p,\gamma,R)}\|u(r_\gamma^1,\cdot)\|_{2/p,p}^p \le NR^p/\gamma,$$

for an $r_\gamma^1 \in (s_\gamma + \gamma, s_\gamma + 2\gamma)$ and $\gamma \le T/6$.

For $p > 8$, we can represent $2/p$ as $1/2 - \kappa - 2/p$ with $\kappa = 1/2 - 4/p \in (0, 1/2)$. For those p, by Theorem 8.3.4,

$$\int_{r_\gamma^1}^T \mathbb{E}\mathbb{1}_{A(p,\gamma,R)}\|u(t,\cdot)\|_{4/p,p}^p dt \le N\mathbb{E}\mathbb{1}_{A(p,\gamma,R)}\|u(r_\gamma^1,\cdot)\|_{2/p}^p \le NR^p/\gamma,$$

$$\mathbb{E}\mathbb{1}_{A(p,\gamma,R)}\|u(r_\gamma^2,\cdot)\|_{4/p,p}^p \le NR^p/\gamma^2,$$

for an $r_\gamma^2 \in (r_\gamma^1 + \gamma, r_\gamma^1 + 2\gamma)$ and $\gamma \le T/8$. One can keep going this way and, for $p > 4n$, where $n = 1, 2, \cdots$, find r_γ^n that are close to zero if γ is small, such that $\mathbb{1}_{A(p,\gamma,R)}u \in \mathcal{H}_p^{2n/p}(r_\gamma^n, T)$. Here p can be taken arbitrary large, r_γ^n small, and $2n/p$ can be made as close to $1/2$ as we wish by taking appropriate n. Also, the probability of the set $A(p,\gamma,R)$ can be chosen close to 1. Therefore, we obtain our claim as in Remark 8.3.6.

Remark 8.3.8 If $\epsilon \ge 0$ and $\kappa + 2\epsilon < 1/2$, then in eq. (88) we could take a noise like $(1-\Delta)^\epsilon B_t$, which is even ‘‘whiter’’ than B_t.[(lxxv)] This would only lead to replacing η_k in eq. (89) with $(1-\Delta)^\epsilon \eta_k$, and, under natural additional smoothness assumptions on h, would give the assertions of Theorem 8.3.4 with $\kappa + 2\epsilon$ in place of κ.

Remark 8.3.9 We prove the existence and the uniqueness of the one-dimensional equation eq. (88) with space-time white noise. Now we are going to observe the case when $d \ge 2$. In Section 9.5, we have calculated the kernel of the operator $(1-\Delta)^{(n+1)/2}$, which is

$$R(x) = \chi |x|^{-n-d-1} \int_0^\infty t^{-(n+d+3)/2} e^{-|x|^2 t - 1/(4t)} dt \quad x \in \mathbb{R}^d.$$

It is proved that R is infinitely differentiable except the origin, $|R(x)| = o(e^{-|x|/2})$ as $|x| \to \infty$, and

$$|R(x)| = \begin{cases} O(|x|^{-n-d-1}) & \text{if } n > -d-1 \\ O(-\log|x|) & \text{if } n = -d-1 \end{cases} \quad \text{as} \quad |x| \to 0.$$

Now we are going to track out the required assumption which n should have. In order that

(lxxiii) Later we will see why such assumption appears.

(lxxiv) For that κ, we have $(1/2) - \kappa - 2/p = 0$. Since $u_{s_\gamma} \in L_p(\Omega, \mathcal{F}_{s_\gamma}, L_p)$ on $A(p,\gamma,R)$, the assumption of Theorem 8.3.4 is satisfied.

(lxxv) Formally written, by taking the Fourier transform, one have $\mathcal{F}((1-\Delta)^\epsilon B_t) = (1+|\xi|^2)^\epsilon \widehat{B_t}$. Because $(1+|\xi|^2)^\epsilon$ amplifies the frequencies, the resulting field has more energy at large $|\xi|$. Thus the term ‘‘whiter’’ comes into the place.

$\|R\|_{2r} < \infty$, with the assumption $n > -d-1$,(lxxvi) we have

$$r < \frac{d}{2(n+d+1)}.$$

By the way, since $r \geq 1$, we have $n < -d/2 - 1$.(lxxvii) From our assumption $d \geq 2$, we find $n+2 < 0$. Hence, the solution to eq. (89) can be a generalized function since the solution space $\mathcal{H}_p^{n+2}(\tau) \supset \mathcal{H}_p^{-1/2}(\tau)$ contains non-functional distributions. See Section 9.12 about solvability of eq. (88) when $d \geq 2$.

8.4 Non–Explosion for a Nonlinear Equation

Take a, b, h satisfying Assumption 8.3.1 and 8.3.2 from Section 8.3 for $\tau = \infty$, $r = 1$, $s = \infty$, and $\xi \equiv 1$ and take a bounded real-valued $\mathcal{P} \otimes \mathfrak{B}(\mathbb{R})$-measurable function $c(t,x) = c(\omega, t, x)$. Assume that $h(t,x,u) = 0$ for $u \leq 0$. Fix a number $\lambda \in [0, 1/2)$ and let

$$g^k(t,x,u) = h(t,x,u) u_+^\lambda \eta_k(x),$$

where, as usual, $\{\eta_k\}_k$ form an orthonormal basis in L_2. Here it is convenient to assume additionally that each η_k is bounded.

The results from Section 8.3 can be easily applied to prove the following theorem.

Theorem 8.4.1 Consider the following equation

$$du(t,x) = [a(t,x)u''(t,x) + b(t,x)u'(t,x) + c(t,x)u(t,x)]dt + g^k(t,x,u(t,x))dw_t^k. \tag{98}$$

If the initial condition u_0 is nonnegative and, say, is nonrandom and belongs to C_0^∞, then the equation eq. (98) has a solution defined for all t in the class of functions such that $\sup_{t \leq T, x} |u(t,x)| < \infty$ (a.s.) for any $T < \infty$.

These facts for eq. (98) considered on a finite space interval with $a \equiv 1$, $b \equiv c \equiv 0$, and $h(u) = u_+$ and with zero boundary data were discovered in [**?**] with the help of a quite different approach. By using the maximum principle, one can show that our assertion implies the result of [**?**].

First, let us explain why eq. (98) is solvable despite the high growth of g in u. It turns out that eq. (98) possesses a kind of integral or conservation law. Observe that $u \geq 0$. Indeed, one notices that the solution u of eq. (98) also satisfies the equation with $g^k(t,x,u(t,x))$ replaced by $\nu^k(t,x)u(t,x)$, where $\nu^k(t,x) = g^k(t,x,u(t,x))u^{-1}(t,x)$ and $|\nu^k| \leq N|u|^\lambda$. (lxxviii) Then apply Theorem 8.4.1 and the maximum principle (see Theorem 5.0.18) on eq. (99). Moreover, if, for instance, $a = 1$, $b = c = 0$, then, by integrating eq. (98) formally with respect to x, one obtains that $\|u(t,\cdot)\|_1$ is a local martingale. It is nonnegative, therefore its trajectories are bounded (a.s.).

[Prove that $\|u(t,\cdot)\|_1$ is really a local martingale]

This takes care of "almost $u^{1/2}$" in the diffusion term. Indeed, one can rewrite eq. (98) with $\xi(t,x)u\eta_k(x)$ in place of $g^k(t,x,u)$, where

$$\xi(t,x) = h(t,x,u(t,x))u^{\lambda-1}(t,x)$$

(lxxvi) Notice that $-d-1 \leq -3$ because we assume $d \geq 2$. Thus if $n \leq -d-1$, then $n+2 \leq -1$. In this case, we can conclude that there is no "functional" solution of eq. (89) because the solution space is $\mathcal{H}_p^{n+2}(\tau)$.

(lxxvii) One should satisfy $1/(2(n+d+1)) > 1$.

(lxxviii) Actually, one consider

$$dv = [av'' + bv' + cv]dt + \nu^k v dw_t^k \tag{99}$$

where $nu^k(t,x) = g^k(t,x,u(t,x))u^{-1}(t,x)$ where u solves eq. (98). Then clearly u also solves eq. (99).

and $|\xi| \leq |u|^\lambda$. By the above, the latter is summable to power $2s = 1/\lambda$, and $s > 1$ (which is required in Theorem 8.3.4) for $\lambda < 1/2$. Therefore, u satisfies a linear equation with coefficients under control, and we get that $u(t,x)$ is a bounded continuous function on $[0,T]\times\mathbb{R}$ for any $T < \infty$ as in Remark 8.3.6 (by letting $p \to \infty$). The rigorous treatment below follows this idea.

By Theorem 8.3.4, for any $m = 1,2,3,\cdots$, the equation

$$du_m = (au_m'' + bu_m' + cu_m)dt + g^k(u_m \wedge m)dw_t^k \tag{100}$$

with the initial condition $u_m(0,\cdot) = u_0$ has a unique solution $u_m \in \mathcal{H}_p^{1/2-\kappa}(T)$ for any $\kappa \in (0,1/2)$, $p \geq 2$, and $T < \infty$. By Remark 8.3.6, the function $u_m(t,x)$ is continuous in (t,x) (a.s.). To proceed with the argument, we need the following lemma to be proved later.

Lemma 8.4.2 For any $T < \infty$,

$$\lim_{R\to\infty} \sup_m \mathbb{P}\{\sup_{t\leq T,x} |u_m(t,x)| \geq R = 0\}. \tag{101}$$

Define

$$\tau_m^R = \inf\{t \geq 0 : \sup_x |u_m(t,x)| \geq R\}.$$

Observe that if $m \geq R$ (and R is an integer), then the functions u_m and u_R satisfy the same equation on $[0,\tau_m^R)$ and therefore coincide by uniqueness. In particular, $\tau_m^m \geq \tau_m^R = \tau_R^R$ (a.s.). Therefore, there is no ambiguity in the definition

$$u(t,x) = u_m(t,x) \quad \text{on} \quad [0,\tau_m^m].$$

Of course, u satisfies eq. (98) on $(0,\lim\tau_m^m)$ and $|u(t,x)| \leq m$ for $t \leq \tau_m^m$. To finish the proof of Theorem 8.4.1, it only remains to notice that $\lim\tau_m^m = \infty$ (a.s.), since, by eq. (101),

$$\mathbb{P}\{\tau_m^m \leq T\} = \mathbb{P}\{\sup_{t\leq T,x} |u_m(t,x)| \geq m\} \leq \sup_n \mathbb{P}\{\sup_{t\leq T,x} |u_n(t,x)| \geq m\} \to 0$$

as $m \to \infty$.

- -

proof (Lemma 8.4.2). Define

$$\xi_m(t,x) = h(t,x,u_m(t,x)\wedge m)(u_m(t,x)\wedge m)_+^\lambda u_m^{-1}(t,x) \quad (0\cdot 0^{-1} := 0)$$

and notice that ξ_m is a bounded function. It follows from eq. (100) that u_m is a solution of the equation

$$dv = (av'' + bv' + cv)dt + v\xi_m\eta_k dw_t^k. \tag{102}$$

Obviouisly, Assumption 8.3.1 and 8.3.2 from Section 8.3 are satisfied for eq. (102). By Remark 8.3.5 and by virtue of our assumptionm about boundedness of η_k, Theorem 5.0.18 is valid for eq. (102). Thus $u_m \geq 0$ (a.s.).

Next, take $\zeta_k(x)$ from Theorem 5.0.13,[lxxix] multiply eq. (100) by $\zeta_k e^{-Kt}$, where $K = \sup(|a''| + |b'| + |c|)$, integrate by parts (that is, use the definition of solutions and Itô's formula[lxxx]), and take expectations. Then, for any constant T and stopping time $\tau \leq T$, we obtain[lxxxi]

[lxxix] Assuming further that ζ_k is nonnegative does not harm Theorem 5.0.13.

[lxxx] Notice that

$$d(u_m e^{-Kt}) = (au_m'' + bu_m' + (c-K)u_m)e^{-Kt}dt + g^k(u_m \wedge m)dw_t^k.$$

[lxxxi] In the equation, we used the fact that $((a'' - b' + c - K)\zeta_k, u_m) \leq 0$ (a.s.) (here, $\zeta_k \geq 0$ is assumed) because notice that $u_m \geq 0$ (a.s.).

$$
\begin{aligned}
e^{-KT}\mathbb{E}(\zeta_k, u_m(\tau,\cdot)) &\le \mathbb{E}(\zeta_k, u_m(\tau,\cdot))e^{-K\tau} \\
&= (\zeta_k, u_0) + \mathbb{E}\int_0^\tau (a\zeta_k'' + (2a'-b)\zeta_k' + (a''-b'+c-K)\zeta_k, u_m)e^{-Kt}dt \\
&\le N + N\mathbb{E}\int_0^\tau (|\zeta_k''| + |\zeta_k'|, u_m)dt \\
&\le N + \frac{N}{k}k^{1-1/p}\mathbb{E}\int_0^T \|u_m(t,\cdot)\|_p dt,
\end{aligned}
$$

$$
\mathbb{E}(\zeta_k, u_m(\tau,\cdot)) \le N + Nk^{-1/p}\left(\mathbb{E}\int_0^T \|u_m(t,\cdot)\|_p^p dt\right)^{1/p} \le N + Mk^{-1/p},
$$

where the last constant N is independent of m, k, and τ, and M is independent of k. Since this inequality is true for any stopping time $\tau \le T$, with the same N and M, for any $\gamma \in (0,1)$ (see, for instance, Theorem III.6.8 of [8]),

$$
\mathbb{E}\sup_{t\le T}\left(\int_{\mathbb{R}} \zeta_k(x)u_m(t,x)dx\right)^\gamma \le 1 + \gamma\frac{N+Mk^{-1/p}}{1-\gamma}, \quad \mathbb{E}\sup_{t\le T}\|u_m(t,\cdot)\|_1^\gamma \le \gamma\frac{N+1}{1-\gamma},
$$

where the latter relation is obtained from the former one by the monotone convergence theorem. It follows that

$$
\mathbb{P}\{\sup_{t\le T}\|u_m(t,\cdot)\|_1 > S\} \le \frac{N}{\sqrt{S}}, \tag{103}
$$

where N is independent of m and $S > 0$.

Now fix $m, S > 0$ and define

$$
\tau_m(S) = \inf\{t \ge 0 : \|u_m(t,\cdot)\|_1 \ge S\}, \quad h_m(t,x,u) = \xi_m(t,x)u.
$$

Observe that $\xi_m \le u_m^\lambda$, which yields that for $t \le \tau_m(S)$ we have $\|\xi_m(t,\cdot)\|_{2s} \le S^{1/(2s)}$ with $s = 1/(2\lambda)$. Also u_m satisfies eq. (102). By Theorem 8.3.4, we obtain that, for any $T < \infty$, $\|u_m\|_{\mathcal{H}_p^{1/2-\kappa}(T\wedge\tau_m(S))}$ is bounded by a constant independent of m, whenever $p \ge 2r = 2/(1-2\lambda)$ and $1/2 > \kappa > \lambda$.(lxxxii) The embedding theorems (cf. Remark 8.3.6) imply that $\mathbb{E}\sup_{t\le T\wedge\tau_m(S),x}|u_m(t,x)|^p$ is bounded independently of m for all large p (and hence for small p as well).(lxxxiii)

Thus

$$
\begin{aligned}
\mathbb{P}\{\sup_{t\le T,x}|u_m(t,x)| \ge R\} &\le \mathbb{P}\{\sup_{t\le T\wedge\tau_m(S),x}|u_m(t,x)| \ge R\} + \mathbb{P}\{\tau_m(S) < T\} \\
&\le \frac{N}{R} + \mathbb{P}\{\sup_{t\le T}\|u_m(t,\cdot)\|_1 > S\} \\
&\le \frac{N}{R} + \frac{N}{\sqrt{S}}
\end{aligned}
$$

with the constants N independent of m, S, R. This leads to eq. (101), and the lemma is proved.

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(lxxxii) Observe that $\|u_m\|_{\mathcal{H}_p^{1/2-\kappa}(T\wedge\tau_m(S))} \le N\|u_0\|_{1/2-\kappa-2/p,p}$ because u_0 is nonrandom.

(lxxxiii) Recall that Hölder norm contains a sup norm. Thus $\mathbb{E}\sup_t\|u(t,\cdot)\|_\infty^p \le \mathbb{E}\sup_t\|u(t,\cdot)\|_{C^\gamma}^p \le N\|u\|_{\mathcal{H}_p^n}$ (here, detailed condition in t is omited).

9 Appendix

9.1 Sobolev-Slobodeckij space

In this subsection we fix $p \in (1, \infty)$, m a nonnegative integer, and r a nonnegative real number, and put $r = r_{\mathbb{Z}} + r_{\mathbb{R}}$ where $r_{\mathbb{Z}}$ an integer, and $r_{\mathbb{R}} \in [0, 1)$.

Recall that the space $W_p^m(\Omega)$ consists of the elements of $u \in L_p(\Omega)$ such that $D^\alpha u \in L_p(\Omega)$, and norm on $W_p^m(\Omega)$ is

$$\|u\|_{W_p^m(\Omega)} := \sum_{j=1}^{m} \langle\langle u \rangle\rangle^{(j)}_{W_p^m(\Omega)}, \quad \langle\langle u \rangle\rangle^{(j)}_{W_p^m(\Omega)} := \sum_{|\alpha|=j} \|D^\alpha u\|_{L_p(\Omega)}.$$

The Sobolev-Slobodeckij space $W_p^r(\Omega)$ where $r_{\mathbb{R}} \neq 0$ consists of all $u \in W_p^{r_{\mathbb{Z}}}$ such that $\|u\|_{W_p^r(\Omega)}$ is finite, where

$$\|u\|_{W_p^r(\Omega)} := \|u\|_{W_p^{r_{\mathbb{Z}}}(\Omega)} + \langle\langle u \rangle\rangle_{W_p^r(\Omega)}, \quad \langle\langle u \rangle\rangle^p_{W_p^r(\Omega)} := \sum_{|\alpha|=r_{\mathbb{Z}}} \int_\Omega \int_\Omega \frac{|D_x^\alpha u(x) - D_y^\alpha u(y)|^p}{|x-y|^{d+pr_{\mathbb{R}}}} dxdy.$$

Finally, $W_p^{1,2}((0,T)\times\Omega)$ consists of the elements of $u \in L_p((0,T)\times\Omega)$ such that $D_x^\alpha D_t^k u \in L_p((0,T)\times\Omega)$ where $2|\alpha| + k \leq 2$.

Denote $W_p^r(T) := W_p^r((0,T) \times \mathbb{R}^d)$ and $W_p^{1,2}(T) := W_p^{1,2}((0,T) \times \mathbb{R}^d)$. Below theorem is from Theorem IV.9.2 in [15].

Theorem 9.1.1 For any $f \in L_p(T)$ and $u_0 \in W_p^{2-2/p}$ there exists a unique solution $u \in W_p^{1,2}(T)$ of the heat equation eq. (6) with initial data $u(0) = u_0$. In addition,

$$\|u\|_{W_p^{1,2}(T)} \leq N(d, p, T)(\|f\|_{L_p(T)} + \|u_0\|_{W_p^{2-2/p}}).$$

It is clear that $W_p^{1,2}(T) = H_p^{1,2}(T)$. However the space of initial value is different. Using the notation in [23], we have $W_p^r = \Lambda^r_{p,p} = B^r_{p,p} = F^r_{p,p}$. On the other hand, $H_p^r = F^r_{p,2}$ is true. Therefore, $W_p^r \supset H_p^r$ holds if $p \geq 2$. However, $W_p^r \subset H_p^r$ if $p \in [1, 2]$. For this sense, theorem 2.2.1 makes sense only when $p \geq 2$.

9.2 Checking whether two spaces are same

Define two spaces

$$\bar{H}_p^{1,2}(T) := \{u = u(t,x) : \|u\|^p_{1,2,p} := \int_0^T \left\|\frac{\partial u}{\partial t}(t,\cdot)\right\|_p^p dt + \int_0^T \|u(t,\cdot)\|^p_{2,p} dt < \infty\},$$

$$\tilde{H}_p^{1,2}(T) := \{u : u(t,x) = u(0,x) + \int_0^t f(s,x)ds,\ u, u_x, u_{xx}, f \in L_p((0,T) \times \mathbb{R}^d)\}.$$

In this section, we shall show that two definitions are same. Fix $u \in \tilde{H}_p^{1,2}(T)$ first and take f in the definition of $\tilde{H}_p^{1,2}(T)$. Since $f \in L_p((0,T) \times \mathbb{R}^d)$, there exists a Lebesgue measurable set $E \subset \mathbb{R}^d$ such that $|\mathbb{R}^d \setminus E| = 0$ and for each $x \in E$,

$$\int_0^T f(s,x)ds \leq T^{1-1/p} \|f(\cdot, x)\|_{L_p((0,T))} < \infty.$$

Hence by the fundamental theorem of calculus, for every $\phi \in C_0^\infty((0,T))$ and $x \in E$,

$$\int_0^T \phi'(t)u(t,x)dt = u(0,x)\int_0^T \phi'(t)dt + \int_0^T \int_0^t f(s,x)\phi'(t)dsdt = -\int_0^T \phi(t)f(t,x)dt.$$

This yields that $(\partial/\partial t)u(\cdot, x) = f(\cdot, x)$ for each $x \in E$. As $f \in L_p((0,T)\times\mathbb{R}^d)$, this therefore implies $u \in \bar{H}_p^{1,2}(T)$.

Conversely, if $u \in \bar{H}_p^{1,2}(T)$, fix $t \in (0,T)$ and apply Corollary 11.7.5 in [12] to obtain that for any $\phi \in C_0^\infty$,

$$\int_{\mathbb{R}^d} \phi(x)[u(t,x) - u(0,x)]dx = \int_{\mathbb{R}^d} \phi(x) \int_0^t \frac{\partial}{\partial t} u(s,x) ds dx,$$

which implies $u(t,x) - u(0,x) = \int_0^t u_t(s,x)ds$ (a.e.) x, and this proves $u \in \tilde{H}_p^{1,2}(T)$.

9.3 Rigorous proof of uniqueness in the page 44

We already proved the uniqueness of eq. (50) when τ is nonrandom, and proved the existence of the one for every bounded stopping time τ. To prove the uniqueness for eq. (50) for bounded stopping time τ, first take a nonrandom constant $T \in (0,\infty)$ such that $\tau \le T$, and consider $u \in \mathcal{H}_{p,0}^{n+2}(\tau)$ satisfies eq. (50). Now define $\bar{f}(t,x) = (\mathbb{D}u - \Delta u)(t,x)\mathbb{1}_{t\le\tau}$, $\bar{g}(t,x) = (\mathbb{S}u)(t,x)\mathbb{1}_{t\le\tau}$, and consider the following SPDE with zero initial condition:

$$dv = [\Delta v + \bar{f}]dt + \bar{g}^k dw_t^k. \tag{104}$$

Clearly u solves eq. (104). We know that eq. (104) has a solution $v \in \mathcal{H}_{p,0}^{n+2}(T)$. In addition, one can find that $u - v$ solves the heat equation on $[0,\tau]$ with zero initial condition. Hence by the uniqueness theorem (see Theorem 2.2.1), $u(t,\cdot) = v(t,\cdot)$ for every $t \in [0,\tau]$ (and for (a.a.) $\omega \in \Omega$). This fact yields that v is also a solution to

$$dv = [\Delta v + \tilde{f}(v,t,x)]dt + \tilde{g}^k(v,t,x)dw_t, \tag{105}$$

where

$$\tilde{f}(v,t,x) := [a^{ij}(t,x)v_{x^ix^j}(t,x) + f(v,t,x) - \Delta v(t,x)]\mathbb{1}_{t\le\tau}, \quad \tilde{g}^k(v,t,x) := [\sigma^{ik}(t,x)v_{x^i}(t,x) + g(v,t,x)]\mathbb{1}_{t\le\tau}.$$

Since we know that eq. (105) has the unique solution on $[0,T]$. We can conclude that every solution to eq. (50) coinsides with the one of the solution to eq. (105) at $[0,\tau]$. This proves the uniqueness for the general case.

9.4 The Heat Kernel

Let $f \in C_0^\infty(\mathbb{R}^d)$, $0 \le a < b < \infty$, and $h(t,x) = \mathbb{1}_{(a,b]}(t)f(x)$.

Consider [(lxxxiv)]

$$v(t,x) = \int_0^t T_{t-s}h(s,\cdot)(x)ds = \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y)h(s,y)dyds,$$

where p is the heat kernel, defined by

$$p(t,x) := \mathbb{1}_{t>0}\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}}.$$

One can easily check that $p_t = \Delta p$.

Below figure shows the singularity of the heat kernel. From this, if $f : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is integrable and there exists a bounded open set $V \in \mathbb{R}^d$ such that $f(t,x) = 0$ for every $t \in [0,\infty)$ and $x \notin V$, then one can easily checked that for fixed $T \in (0,\infty)$,

$$\lim_{|x|\to\infty} \sup_{t\in[0,T]} \int_0^t \int_{\mathbb{R}^d} p(t-s,x-y)f(s,y)dyds = 0.$$

[(lxxxiv)] The form of which comes from the Duhamel's principle (see [15]) with considering $\mathbb{1}_{t>0}h$.

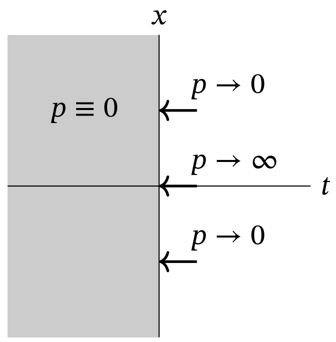

Figure 1: Singularity of the heat kernel p

The function v is differentiable once in t, infinitely differentiable in x, and satisfies the equation

$$\frac{\partial z}{\partial t} = \Delta z + f.$$

Since f is in C_0^∞, clearly v is infinitely differentiable in x and we have

$$D_x^\alpha y(t,x) = \int_0^t \int_{\mathbb{R}^d} T_{t-s}(D_x^\alpha h(s,\cdot))(x)dyds.$$

9.5 The Kernel of $(1-\Delta)^{-\alpha}$

Let $\alpha > 0$ be fixed and $\phi \in C_0^\infty$. By eq. (3),

$$\begin{aligned}(1-\Delta)^{-\alpha}\phi(x) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^\alpha e^{-t}T_t\phi(x)\frac{dt}{t} \\ &= \frac{1}{\Gamma(\alpha)}\int_{\mathbb{R}^d}\int_0^\infty t^\alpha e^{-t}\frac{1}{(4\pi t)^{d/2}}e^{-|x-y|^2/(4t)}\phi(y)\frac{dt}{t}dy \\ &= \chi\int_{\mathbb{R}^d}\int_0^\infty t^{\alpha-d/2-1}e^{-t-|x-y|^2/(4t)}\phi(y)dtdy \quad (\chi^{-1} = \Gamma(\alpha)(4\pi)^{d/2}).\end{aligned}$$

By changing the variable we therefore obtain the kernel R of $(1-\Delta)^{-\alpha}$ by

$$R(x) = \chi\int_0^\infty t^{\alpha-d/2-1}e^{-t-|x|^2/(4t)}dt = \chi|x|^{2\alpha-d}\int_0^\infty t^{\alpha-d/2-1}e^{-|x|^2t-1/(4t)}dt.$$

Fix two numbers $C > c > 0$. On $c < |x| < C$, for any multiindex β,

$$|t^{\alpha-d/2-1}(D_x^\beta e^{-|x|^2t})e^{-1/(4t)}| \le p_\beta(t,C)t^{\alpha-d/2-1}e^{-tc^2-1/(4t)} \in L_1(0,\infty)$$

where $p_\beta(t,x)$ is some polynomial since $D_x^\beta e^{-|x|^2t}$ is the product of $e^{-|x|^2t}$ and some polynomial with variables in t and x. Thus by the dominated convergence theorem, $|x|^{d-2\alpha}R(x)$ is infinitely differentiable for x on which $c < |x| < C$, and thus R is infinitely differentiable on there. Since c and C are arbitrary, we find that R is infinitely differentiable on $|x| > 0$. In addition, if $|x| > c > 1$, we have

$$\begin{aligned}e^{|x|/2}t^{\alpha-d/2-1}e^{-t-|x|^2/(4t)} &\le t^{\alpha-d/2-1}e^{-t-c^2/(8t)}\left(\sup_{|x|>c} e^{|x|/2}e^{-|x|^2/(8t)}\right) \\ &\le t^{\alpha-d/2-1}e^{-t-c^2/(4t)}e^{c^2/(8t)} \\ &= t^{\alpha-d/2-1}e^{-t-c^2/(8t)} \in L_1(0,\infty).\end{aligned}$$

By applying the dominated convergence theorem, we have $|R(x)| = o(e^{-|x|/2})$ as $|x| \to \infty$.

Now we are going to observe the behavior of R near zero. If $\alpha < d/2$, we have

$$\chi^{-1}|x|^{d-2\alpha}|R(x)| \le \int_0^\infty t^{\alpha-d/2-1}e^{-|x|^2t-1/(4t)}dt \le \int_0^1 t^{\alpha-d/2-1}e^{-1/(4t)}dt + \int_1^\infty t^{\alpha-d/2-1}dt < \infty,$$

and thus $|R(x)| = O(|x|^{2\alpha-d})$ as $|x| \to 0$. If $\alpha = d/2$, by the L'hospital's rule,

$$\lim_{|x|\to 0}\frac{|R(x)|}{-\log|x|} = \lim_{r\to 0} 2r^2\chi\int_0^\infty e^{-r^2t-1/(4t)}dt = \lim_{r\to 0} 2\chi\int_0^\infty e^{-t-r^2/(4t)}dt = 2\chi,$$

and we have $|R(x)| = O(-\log|x|)$ as $|x| \to 0$.

9.6 Auxiliary Lemmas

Lemma 9.6.1 Let $\mathcal{F}_t = \mathcal{W}_t \vee \mathcal{B}_t$ for $t \ge 0$, assume that σ-fields $\mathcal{W}_t$ and $\mathcal{B}_t$ form independent increasing filtrations, and $\mathcal{P}_\mathcal{F}, \mathcal{P}_\mathcal{W}$ are $\mathcal{F}_t$-predictable and $\mathcal{W}_t$-predictable σ-fields, respectively. Furthermore, $(w_t, \mathcal{W}_t)$ is an one-dimensional Wiener process. Then for any $u \in \bigcap_{T>0} L_2((0,T]], \mathcal{P}_\mathcal{F}, \mathbb{R})$, there exists a $\mathcal{P}_\mathcal{W}$-measurable process $\bar{u}$ such that

$$\bar{u}(t) = \mathbb{E}[u(t)|\mathcal{W}_t] \quad (\mathbb{P}\times\ell\text{-a.e.}),$$

and for almost all t,

$$\int_0^t \bar{u}(s)ds = \mathbb{E}\left[\int_0^t u(s)ds\middle|\mathcal{W}_t\right] \quad \text{(a.s.)}, \qquad \int_0^t \bar{u}(s)dw_s = \mathbb{E}\left[\int_0^t u(s)dw_s\middle|\mathcal{W}_t\right] \quad \text{(a.s.)}.$$

proof. Following proof is inspired from [19].

[I don't know that this theorem is right because this one is made by myself]

9.7 Separable Measure Space

Let $(X, \mathfrak{M}, \mu)$ be a measure space. We say that this space is *separable* if the subspace {

9.8 Interpolation Theory

Let A_0 and A_1 be two complex Banach spaces, both linearly and continuously embedded in a linear complex Hausdorff space $\mathcal{A}$,

$$A_0 \subset \mathcal{A}, \quad A_1 \subset \mathcal{A}.$$

"$\subset$" must be understood in the set-theoretical and in the topological sense. Two such Banach spaces are said to be an interpolation couple $\{A_0, A_1\}$.

Let $\{A_0, A_1\}$ be an interpolation couple. If $0 < t < \infty$ then

$$K(t,a) = K(t,a;A_0,A_1) := \inf_{\substack{a=a_0+a_1\\ a_0\in A_0, a_1\in A_1}} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1, \tag{106}$$

is an equivalent norm in the space $A_0 + A_1$. Now we define a real interpolation space.

Definition 9.8.1 Let $\{A_0, A_1\}$ be an interpolation couple. Let $0 < \theta < 1$. If $1 \leq q < \infty$, then

$$(A_0, A_1)_{\theta,q} := \left\{ a \in A_0 + A_1 \,:\, \|a\|_{(A_0,A_1)_{\theta,q}} := \left(\int_0^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

and if $q = \infty$, then

$$(A_0, A_1)_{\theta,\infty} := \left\{ a \in A_0 + A_1 \,:\, \|a\|_{(A_0,A_1)_{\theta,\infty}} := \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty \right\}.$$

Now we are going to study the basic property of the spaces $(A_0, A_1)_{\theta,q}$. Below theorem is in [22, 20] but it is written also in here for ease reading purpose.

Theorem 9.8.2

(a) $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$ is a Banach space.

(b) It holds that

$$(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}. \tag{107}$$

(c) There exists a positive number $N = N(\theta, q)$ such that for all $a \in (A_0, A_1)_{\theta,q}$ and for all t with $0 < t < \infty$

$$K(t,a) \leq N t^\theta \|a\|_{(A_0,A_1)_{\theta,q}}. \tag{108}$$

(d) If $0 < \theta < 1$ and $1 \leq q \leq q' \leq \infty$, then

$$(A_0, A_1)_{\theta,1} \subset (A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\theta,q'} \subset (A_0, A_1)_{\theta,\infty}. \tag{109}$$

(e) In additionaly, $A_0 \subset A_1$. Then for $0 < \theta < \theta' < 1$ and $1 \leq q, q' \leq \infty$

$$(A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\theta',q'}. \tag{110}$$

(f) If $A_0 = A_1$, then $(A_0, A_1)_{\theta,q} = A_0 = A_1$.

(g) There exists a positive number $N = N(\theta, q)$, $0 < \theta < 1$, $1 \leq q \leq \infty$, such that for all $a \in A_0 \cap A_1$,

$$\|a\|_{(A_0,A_1)_{\theta,q}} \leq N \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta}. \tag{111}$$

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proof.

9.9 Brief introduction to Bochner integral

Let $(X, \mathfrak{M}, \mu)$ be a measure space, $(Y, \|\cdot\|)$ be a Banach space.

$$s = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j},$$

where $n \in \mathbb{Z}_+$, $c_1, \cdots, c_n \in Y$ are distinct and the sets $E_j \subset X$ are $\mathfrak{M}$-measurable and mutually disjoint. Let $S(X, Y)$ denotes the family of all such simple functions.

A function $u : X \to Y$ is said to be *strongly measurable* if there exists a sequence $(s_n)_n$ in $S(X, Y)$ such that

$$\lim_{n\to\infty} \|u(x) - s_n(x)\| = 0, \quad (\mu - \text{a.e.})\ x \in X.$$

A simple function $s \in S(X, Y)$ is *Bochner integrable* if s attains nonzero values on each set of finite measure. For every $E \in \mathfrak{M}$, the *Bochner integral* of s over E is defined by

$$\int_E s d\mu := \sum_{j=1}^{n} c_j \mu(E_j \cap E) \quad (0 \cdot \infty := 0).$$

Now let u is strongly measurable. If there exists a sequence $(s_n)_n$ in $S(X, Y)$ such that

$$\lim_{n\to\infty} \|u(x) - s_n(x)\| = 0 \quad (\mu\text{-a.e. } x),$$

$$\lim_{n\to\infty} \int_X \|u - s_n\| d\mu = 0,$$

We define

$$\int_E u d\mu := \lim_{n\to\infty} \int_E s_n d\mu,$$

where the limit performs on the strong sense. We say that such sequence $(s_n)_n$ is called a *generating sequence* of u.

For $p \in [1, \infty)$ we define a space $L_p(X, Y)$ consists of all strongly measurable functions u such that

$$\|u\|^p_{L_p(X,Y)} := \int_X \|u\|^p d\mu < \infty.$$

Furthermore, define $L_\infty(X, Y)$ consists of all strongly measurable functions u such that

$$\operatorname*{esssup}_{x\in X} \|u(x)\| < \infty.$$

Theorem 9.9.1 For $1 \le p \le \infty$, $L_p(X, Y)$ is a Banach space and $S_p(X, Y) := S(X, Y) \cap L_p(X, Y)$ is dense in $L_p(X, Y)$.

If we further assume that μ is σ-finite, then

(a) if $1 \le p < \infty$, X is a separable metric space, and Y is separable, then $L_p(X, Y)$ is also separable.

(b) if $1 < p < \infty$ and Y is reflexive, then for $\Lambda \in (L_p(X, Y))^*$ there exists a unique $v \in L_q(X, Y^*)$ such that

$$\Lambda u = \int_X (v, u) d\mu$$

for every $u \in L_p(X, Y)$. In other words, $(L_p(X, Y))^* \simeq L_q(X, Y^*)$.

Theorem 9.9.2 Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite measure spaces. The for any $p \in [1, \infty)$, one can identify $L_p(X, \mathfrak{M}, L_p(\mathfrak{N}, \nu))$ and $L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$ by the map $T : L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu) \to L_p(X, \mathfrak{M}, L_p(\mathfrak{N}, \nu))$ by

$$(Tf)(x) := f(x, \cdot).$$

- -

proof. For any $f \in L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$, the Fubini's theorem gives that the function

$$x \in X \mapsto \|f(x, \cdot)\|^p_{L_p(\mathfrak{N},\nu)}$$

is $\mathfrak{M}$-measurable and $\int_X \|f(x, \cdot)\|^p_{L_p(\mathfrak{N},\nu)} dx = \|f\|_p < \infty$. This yields that $\|f(x, \cdot)\|_{L_p(\mathfrak{N},\nu)} < \infty$ for

(μ-a.e) x. For this sake, we can define $T : L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu) \to L_p(X, \mathfrak{M}, L_p(\mathfrak{N}, \nu))$ by

$$(Tf)(x) := f(x, \cdot).$$

Since both μ and ν are σ-finite, we have $L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu) = L_p(\mathfrak{M} \times \mathfrak{N}, \mu \times \nu)$ (see [8]), and this implies that Tf is strongly measurable. Furthermore, we have

$$\|Tf\|^p_{L_p(X,\mathfrak{M},L_p(\mathfrak{N},\nu))} = \int_X \|(Tf)(x)\|^p_{L_p(\mathfrak{N},\nu)}\mu(dx) = \int_X \|f(x,\cdot)\|^p_{L_p(\mathfrak{N},\nu)}\mu(dx) = \|f\|^p_{L_p} < \infty. \tag{112}$$

Hence the map T is well-defined. It is clearly linear, and by eq. (112), injectivity of T is obtained.

Now fix $g \in L_p(X, \mathfrak{M}, L_p(\mathfrak{N}, \nu))$. Then there exists a sequence $(s_n)_n$ of Bochner integrable simple functions $s_n : X \to L_p(\mathfrak{N}, \nu)$ such that

$$\lim_{n\to\infty} \|g(x) - s_n(x)\|_{L_p(\mathfrak{N},\nu)} = 0 \quad (\mu - \text{a.e.})\ x \in X,$$

$$\lim_{n\to\infty} \|g - s_n\|^p_{L_p(X,\mathfrak{M},L_p(\mathfrak{N},\nu))} = \lim_{n\to\infty} \int_X \|g(x) - s_n(x)\|^p_{L_p(\mathfrak{N},\nu)}\mu(dx) = 0.$$

Define

$$s_n = \sum_{j=1}^{m_n} c_{jn} \mathbb{1}_{E_{jn}},$$

where $c_{jn} \in L_p(\mathfrak{N}, \nu)$, $E_{jn} \in \mathfrak{M}$, and $\{E_{jn}\}_{j,n}$ is mutually disjoint. Now define

$$\tilde{s}_n(x, y) = \sum_{j=1}^{m_n} c_{jn}(y)\mathbb{1}_{E_{jn}}(x).$$

Then $\tilde{s}_n$ is $\mathfrak{M} \otimes \mathfrak{N}$-measurable function on $X \times Y$, and almost clearly, we obtain

$$\|s_n\|_{L_p(X,\mathfrak{M},L_p(\mathfrak{N},\nu))} = \|\tilde{s}_n\|_{L_p(\mathfrak{M}\otimes\mathfrak{N},\mu\times\nu)}, \quad \|s_n - s_m\|_{L_p(X,\mathfrak{M},L_p(\mathfrak{N},\nu))} = \|\tilde{s}_n - \tilde{s}_m\|_{L_p(\mathfrak{M}\otimes\mathfrak{N},\mu\times\nu)}.$$

This implies that $\tilde{s}_n \in L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$ and it is a Cauchy sequence on it. Then one can take $\tilde{g} \in L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$ such that $\tilde{s}_n \to \tilde{g}$ in $L_p(\mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$ as $n \to \infty$. On the other hand, by eq. (112) and observing that $T\tilde{s}_n = s_n$ for every n, one have

$$\|T\tilde{g} - s_n\|_{L_p(X,\mathfrak{M},L_p(\mathfrak{N},\nu))} = \|\tilde{g} - \tilde{s}_n\|_{L_p(\mathfrak{M}\otimes\mathfrak{N},\mu\times\nu)} \to 0.$$

Hence, $T\tilde{g} = g$ on $L_p(X, \mathfrak{M}, L_p(\mathfrak{N}, \nu))$, and thus T is surjective.

Therefore we proved that T is an isometry. Using this map, we can idenfity both spaces. $\square$

9.10 Stochastic Fubini's theorem

Following theorems are presented in [13]. The reader also refer section IV.6 of [17]. In this subsection, Γ is a Borel subset of $\mathbb{R}^d$ with nonzero finite Lebesgue measure.

Definition 9.10.1 Let $B_t(x)$ be a real-valued function on $\Omega \times [0, \infty) \times \Gamma$. We say that it is a regular field on Γ if

(a) It is measurable with respect to $\mathcal{F} \otimes \mathfrak{B}([0, \infty)) \otimes \mathfrak{B}(\Gamma)$;

(b) For each $x \in \Gamma$, there is an event Ω_x such that $\mathbb{P}(\Omega_x) = 1$ and for any $\omega \in \Omega_x$, the function $B_t(\omega, x)$ is a continuous function of t on $[0, \infty)$.

(c) It is $\mathcal{F}_t$-measurable for each $x \in \Gamma$ and $t \in [0, \infty)$.

We call it a regular martingale field on Γ if in addition

(d) For each $x \in \Gamma$ the process $B_t(x)$ is a local $\mathcal{F}_t$-martingale on $[0, \infty)$ starting at zero.

Lemma 9.10.2 Let $B_t^n(x)$, $n = 1, 2, \cdots$ be regular fields on Γ and let $B_t(x)$ be a real-valued function on $\Omega \times [0, \infty) \times \Gamma$. Assume that for each x we have $B_t^n(x) \to B_t(x)$ uniformly on finite intervals in probability as $n \to \infty$. Then there exists a regular field $A_t(x)$ on Γ such that, for each x, with probability one $A_t(x) = B_t(x)$ for all t and

(b′) For each $\omega \in \Omega$ and $x \in \Gamma$ the function $A_t(x)$ is continuous on $[0, \infty)$.

Definition 9.10.3 If a regular field on Γ possesses property (b') of lemma 9.10.2, then we call it strongly regular.

The argument in the last part of the proof of lemma 9.10.2 proves the following.

Lemma 9.10.4 If $B_t(x)$ is a regular field on Γ, then there exists a strongly regular field $A_t(x)$ on Γ such that, for each x, with probability one $A_t(x) = B_t(x)$ for all t.

Lemma 9.10.5 Let $p \in (0, \infty)$ and let $m_t(x)$ be a regular martingale field on Γ. Then there exists a nonnegative strongly regular field $A_t(x)$ on Γ such that, for each $x \in \Gamma$, with probability one $A_t(x) = \langle m(x) \rangle_t$ for all $t \in [0, \infty)$.

Moreover, if $A_t(x)$ is a function with the above described properties and such that

(i) It is $\mathcal{F}_t \otimes \mathfrak{B}(\Gamma)$-measurable for each $t \in \mathbb{R}_+$;

(ii) Almost surely

$$\int_\Gamma \sup_{t \in [0,\infty)} A_t^{p/2}(x) dx < \infty, \tag{113}$$

then for any countable set $\rho \subset [0, \infty)$ with probability one

$$\int_\Gamma \sup_{t \in \rho} |m_t(x)|^p dx < \infty \tag{114}$$

and for any $\epsilon, \delta > 0$ we have

$$\mathbb{P}\left(\int_\Gamma \sup_{t \in \rho} |m_t(x)|^p dx \geq \delta \right) \leq \mathbb{P}(C_\infty \geq \epsilon) + \frac{N}{\delta} \mathbb{E}(\epsilon \wedge C_\infty) \tag{115}$$

where the constant N depends only on p and

$$C_t := \int_\Gamma \sup_{s \leq t} A_s^{p/2}(x) dx.$$

Lemma 9.10.6 Let $f_t(x)$ be a real-valued function on $\Omega \times (0, \infty) \times \Gamma$ which is $\mathcal{P} \otimes \mathfrak{B}(\Gamma)$-measurable and such that

$$\int_0^\infty f_t^2(x) dt < \infty$$

for each $x \in \Gamma$ and ω. Then there exists a strongly regular martingale field $m_t(x)$ on Γ such that for each $x \in \Gamma$ with probability one

$$m_t(x) = \int_0^t f_s(x) dw_s \tag{116}$$

for all t. Furthermore, if

$$\int_\Gamma \left(\int_0^\infty f_t^2(x) dt \right)^{1/2} dx < \infty \quad \text{(a.s.)}, \tag{117}$$

then for any function $m_t(x)$ with the properties described above

$$\int_0^\infty \left(\int_\Gamma f_s(x) dx \right)^2 ds < \infty, \qquad \int_\Gamma \sup_t |m_t(x)| dx < \infty \quad \text{(a.s.)}, \tag{118}$$

the stochastic integral

$$\int_0^t \left(\int_\Gamma f_s(x) dx \right) dw_s \tag{119}$$

is well-defined, and with probability one

$$\int_\Gamma m_t(x) dx = \int_0^t \left(\int_\Gamma f_s(x) dx \right) dw_s \tag{120}$$

for all t.

Lemma 9.10.7 Let $T \in (0, \infty)$ and let $G_t(x)$ be real-valued and $H_t(x) = (H_t^k(x))_{k=1}^\infty$ be l_2-valued functions defined on $\Omega \times (0, T] \times \Gamma$ and possessing the following properties:

(i) The functions $G_t(x)$ and $H_t(x)$ are $\mathcal{P}_T \otimes \mathfrak{B}(\Gamma)$-measurable, where $\mathcal{P}_T$ is the restriction of $\mathcal{P}$ to $\Omega \times (0, T]$;

(ii) There is an event Ω' of full probability such that for each $\omega \in \Omega'$ and $x \in \Gamma$ we have

$$\int_0^T \left(|G_t(x)| + |H_t(x)|_{l_2}^2 \right) dt < \infty;$$

(iii) We have (a.s.)

$$\int_0^T \int_\Gamma |G_t(x)| dx dt + \int_\Gamma \left(\int_0^T |H_t(x)|_{l_2}^2 dx \right)^{1/2} dx < \infty.$$

Under these assumptions we claim that

(a) There is a function $F_t(x)$ on $\Omega \times [0, T] \times \Gamma$, which is $\mathcal{F} \otimes \mathfrak{B}([0, T]) \otimes \mathfrak{B}(\Gamma)$ -measurable, continuous in t, and such that for any $x \in \Gamma$ with probability one we have

$$F_t(x) = \int_0^t G_t(x) ds + \sum_{k=1}^\infty \int_0^t H_s^k(x) dw_s^k \tag{121}$$

for all $t \in [0, T]$, where the series converges uniformly on $[0, T]$ in probability;

(b) For any $k = 1, 2, \cdots$, the stochastic integrals (no summation in k)

$$\int_0^t \int_\Gamma H_t^k(x) dx dw_s^k$$

are well-defined for $t \in [0, T]$;

(c) If we are given a function $F_t(x)$ on $\Omega \times [0,T] \times \Gamma$ with somewhat weaker properties, namely, such that

(iv) For each $t \in [0,T]$ the function $F_t(x)$ is measurable in (ω, x) with respect to the completion $\overline{\mathcal{F} \otimes \mathfrak{B}(\Gamma)}$ of $\mathcal{F} \otimes \mathfrak{B}(\Gamma)$ with respect to the product measure;

(v) For each $t \in [0,T]$ and $x \in \Gamma$ eq. (121) holds almost surely,

then for any countable subset ρ of $[0,T]$

$$\int_\Gamma \sup_{t\in\rho} |F_t(x)| dx < \infty \quad \text{(a.s.)}, \tag{122}$$

and for each $t \in [0,T]$ almost surely

$$\int_\Gamma F_t(x) dx = \int_0^t \int_\Gamma G_s(x) dx ds + \sum_{k=1}^\infty \int_0^t \int_\Gamma H_s^k(x) ds dw_s^k, \tag{123}$$

where the series converges uniformly on $[0,T]$ in probability.

(d) If for a function $F_t(x)$ as in (c), for almost all (ω, x), $F_t(x)$ is continuous in t on $[0,T]$ (like the one from assertion (a)), then with probability one eq. (123) holds for all $t \in [0,T]$.

9.11 Recipies to prove lemma 4.1.1

Following settings, theorems are presented in [11].

9.11.1 Main results

Fix a constant $K \in (0,\infty)$ and let $\psi(x)$ be a $C^1(\mathbb{R}^d)$ integrable function such that

$$\int_{\mathbb{R}^d} \psi dx = 0, \quad \int_{\mathbb{R}^d} (|\psi(x)| + |\nabla \psi(x)| + |x||\psi(x)|) dx \le K.$$

Introduce

$$2\hat{\psi}(x) = \psi(x) d + (x, \nabla \psi(x))$$

and assume that there exists a continuously differentiable function $\bar{\psi}$ defined on $[0,\infty)$ such that

$$|\psi(x)| + |\nabla\psi(x)| + |\hat{\psi}(x)| \le \bar{\psi}(|x|), \quad \int_0^\infty |\bar{\psi}'(x)| dx \le K,$$

$$\bar{\psi}(\infty) = 0, \quad \int_r^\infty |\bar{\psi}'(x)| x^d dx \le K/r, \quad \forall r \ge 1.$$

Now define

$$\Psi_t h(x) := t^{-d/2} \psi(x/\sqrt{t}) * h(x).$$

The classical Littlewood-Paley inequality (see, for instance, Chapter 1 in [21]) says that for any $p \in (1,\infty)$ and $f \in L_p$ it holds that

$$\int_{\mathbb{R}^d} \left[\int_0^\infty |\Psi_t f(x)|^2 \frac{dt}{t} \right]^{p/2} dx \le N(d,p) \|f\|_p^p.$$

Here we want to generalize this fact by proving the following result in which H is a Hilbert space, $\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$. For $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$, $t > a \ge -\infty$, and $x \in \mathbb{R}^d$ we set

$$\mathcal{G}_a f(t,x) = \left[\int_a^t |\Psi_{t-s} f(s,\cdot)(x)|_H^2 \frac{ds}{t-s} \right]^{1/2}, \quad \mathcal{G} = \mathcal{G}_{-\infty}.$$

Theorem 9.11.1 Let $p \in [2,\infty)$, $-\infty \le a < b \le \infty$, $f \in C_0^\infty((a,b)\times\mathbb{R}^d, H)$. Then

$$\int_{\mathbb{R}^d}\int_a^b [\mathcal{G}_a f(t,x)]^p dtdx \le N \int_{\mathbb{R}^d}\int_a^b |f(t,x)|_H^p dtdx, \tag{124}$$

where the constant N depends only on d, p, and K.

The proof of this theorem is given in section 9.11.5 after we prove some elementary properties of partitions in section 9.11.2, prove deep albeit simple Fefferman-Stein theorem in section 9.11.3 and study few properties of the operator $\mathcal{G}$ in section 9.11.4.

9.11.2 Partitions

Let F be a Banach space. For a domain $\Omega \in \mathbb{R}^d$, by $L_p(\Omega, F)$ we denote the closure of the set of F-valued continuous functions compactly supported on Ω with respect to the norm $\|\cdot\|_{L_p(\Omega,F)}$ defined by

$$\|u\|^p_{L_p(\Omega,F)} := \int_\Omega |u(x)|_F^p dx.$$

We also stipulate that $L_p(\Omega) = L_p(\Omega, \mathbb{R})$, $L_p = L_p(\mathbb{R}^d)$. By $|\Omega|$ we denote the volume of Ω.

Definition 9.11.2 Let $(\mathbb{Q}_n)_{n\in\mathbb{Z}}$ be a sequence of partitions of $\mathbb{R}^d$ each consisting of disjoint bounded Borel subsets $Q \in \mathbb{Q}_n$. We call it *a filtration of partitions* if

(i) the partitions become finer as n increases:

$$\inf_{Q\in\mathbb{Q}_n} |Q| \to \infty \quad \text{as} \quad n \to -\infty, \qquad \sup_{Q\in\mathbb{Q}_n} \operatorname{diam} Q \to 0 \quad \text{as} \quad n \to \infty;$$

(ii) the partitions are nested: for each n and $Q \in \mathbb{Q}_n$ there is a (unique) $Q' \in \mathbb{Q}_{n-1}$ such that $Q \subset Q'$;

(iii) the following regularity property holds: for Q and Q' as in (ii) we have

$$|Q'| \le N_0|Q|,$$

where N_0 is independent of n, Q, Q'. Notice that because of (i), one must have $N_0 > 1$.

Example 9.11.3 In the application in this section we will be dealing with the filtration of parabolic dyadic cubes in

$$\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}^d\},$$

defined by

$$\mathbb{Q}_n = \{Q_n(i_0, i_1, \cdots, i_d) : i_0, \cdots, i_d \in \mathbb{Z}\},$$

$$Q_n(i_0, i_1, \cdots, i_d) = [i_0 4^{-n}, (i_0+1)4^{-n}) \times Q_n(i_1, \cdots, i_d), \tag{125}$$

$$Q_n(i_1, \cdots, i_d) = \prod_{j=1}^d [i_j 2^{-n}, (i_j+1)2^{-n}). \tag{126}$$

Definition 9.11.4 Let $\mathbb{Q}_n$, $n \in \mathbb{Z}$, be a filtration of partitions of $\mathbb{R}^d$.

(i) Let $\tau = \tau(x)$ be a function on $\mathbb{R}^d$ with values in $\mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$. We call τ *a stopping time* (relative to the filtration) if, for each $n \in \mathbb{Z}$, the set $\{\tau = n\}$ is the union of some elements of $\mathbb{Q}_n$.

(ii) For any $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}$, by $Q_n(x)$ we denote the (unique) $Q \in \mathbb{Q}_n$ containing x.

(iii) For a function $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d, F)$ and $n \in \mathbb{Z}$, we denote

$$f_{|n}(x) = \fint_{Q_n(x)} f(y)dy, \quad \left(\fint_\Gamma f dx := \frac{1}{|\Gamma|}\int_\Gamma f dx\right).$$

If we are also given a stopping time τ, we let $f_{|\tau}(x) = f_{|\tau(x)}(x)$ for those x for which $\tau(x) < \infty$ and $f_{|\tau}(x) = f(x)$ otherwise.

Remark 9.11.5 It is easy to see that in the case of real-valued functions $f \in L_2$, for each n, $f_{|n}$ provides the best approximation in L_2 of f by functions that are constant on each element of $\mathbb{Q}_n$.

- -

proof. Denote $\mathbb{M}$ a collection of all functions that are constant on each element of $\mathbb{Q}_n$. First to prove is that $\mathfrak{M}$ is closed linear subspace of L_2. It suffices to show that $\mathfrak{M}$ is closed, so fix $g \in \overline{\mathfrak{M}}$ and take a sequence $(g_n)_n$ in $\mathfrak{M}$ that converges to g in L_2. For fixed $Q \in \mathbb{Q}_n$ we have

$$|Q||g_n(x) - g_m(x)|^2 = \int_Q |g_n - g_m|^2 dy \to 0$$

as $m, n \to \infty$ for every $x \in Q$ by the definition of $\mathfrak{M}$. This implies there exists a number α such that $g_n(x) \to \alpha$ as $n \to \infty$ for all $x \in Q$ (Note that g_n's are constant on Q). We have

$$\frac{1}{2}\int_Q |\alpha - g(y)|^2 dy \le \int_Q |\alpha - g_n(y)|^2 dy + \int_{\mathbb{R}^d} |g_n(y) - g(y)|^2 dy \to 0.$$

Therefore we showed that g is constant on each $Q \in \mathbb{Q}_n$, which implies $g \in \mathfrak{M}$.

As L_2 is a Hilbert space, we can express $L_2 = \mathfrak{M} \oplus \mathfrak{M}^\perp$. Claim that $f_{|n} \in \mathfrak{M}$ and $f - f_{|n} \in \mathfrak{M}^\perp$. This immediately implies the main result of this remark.

Fix $Q \in \mathbb{Q}_n$. Since $\mathbb{Q}_n$ is a filtration of partition, $Q_n(x) = Q$ if and only if $x \in Q$. This implies

$$f_{|n}(x) = \fint_{Q_n(x)} f(y)dy = \fint_Q f(y)dy \tag{127}$$

for every $x \in Q$, which shows that $f_{|n}$ is constant on Q. As Q is arbitrary, this proves that $f_{|n} \in \mathfrak{M}$.

Now fix $g \in \mathfrak{M}$ and $Q \in \mathbb{Q}_n$. First of all, there is a constant c such that $g(x) = c$ for every $x \in Q$. By eq. (127), we have

$$\begin{aligned}
\int_Q (f(y) - f_{|n}(y))\bar{g}(y)dy &= \int_Q (f(y) - f_{|n}(y))\bar{g}(y)dy \\
&= \int_Q f(y)\bar{g}(y)dy - \int_Q f_{|n}(y)\bar{g}(y)dy \\
&= \int_Q f(y)\bar{c}dy - \int_Q f_{|n}(y)\bar{c}dy \\
&= \bar{c}\left(\int_Q f(y)dy - \int_Q f_{|n}(y)dy\right)
\end{aligned}$$

$$
\begin{aligned}
&= \bar{c}\left(\int_Q f(y)dy - \int_Q \fint_Q f(x)dxdy\right) \\
&= \bar{c}\left(\int_Q f(y)dy - \int_Q f(y)dy\right) \\
&= 0.
\end{aligned}
$$

This proves $f - f_{|n} \in \mathfrak{M}^{\perp}$.

Lemma 9.11.6 Let $\mathbb{Q}_n$, $n \in \mathbb{Z}$ be a filtration of partitions of $\mathbb{R}^d$.

(i) Let $p \in [1, \infty)$, $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d, F)$, and let τ be a stopping time. Then

$$
\int_{\mathbb{R}^d} |f_{|\tau}(x)|_F^p \mathbb{1}_{\tau<\infty} dx \le \int_{\mathbb{R}^d} |f(x)|_F^p \mathbb{1}_{\tau<\infty} dx. \tag{128}
$$

In addition, eq. (128) becomes an equality if $f \ge 0$ and $p = 1$.

(ii) Let $g \in L_1$, $g \ge 0$, and $\lambda > 0$. Then

$$
\tau(x) \coloneqq \inf\{n : g_{|n}(x) > \lambda\} \quad (\inf \varnothing \coloneqq \infty) \tag{129}
$$

is a stopping time. Furthermore, we have

$$
0 \le g_{|\tau}(x)\mathbb{1}_{\tau<\infty} \le N_0\lambda, \quad |\{x : \tau(x) < \infty\}| \le \lambda^{-1} \int_{\mathbb{R}^d} g(x)\mathbb{1}_{\tau<\infty} dx. \tag{130}
$$

- -

proof.

(i) By Hölder's inequality $|f_{|n}|_F^p \le (|f|_F^p)_{|n}$. Therefore we may concentrated on $p = 1$ and real-valued nonnegative f. In that case notice that, for any n and set Γ which is the union of some elements $Q_i \in \mathbb{Q}_n$, obviously

$$
\int_\Gamma f_{|n} dx = \sum_i \int_{Q_i} f_{|n} dx = \sum_i \int_{Q_i} f dx = \int_\Gamma f dx.
$$

Hence,

$$
\int_{\mathbb{R}^d} f_{|\tau}\mathbb{1}_{\tau<\infty} dx = \sum_{n\in\mathbb{Z}} \int_{\tau=n} f_{|n} dx = \sum_{n\in\mathbb{Z}} \int_{\tau=n} f dx = \int_{\mathbb{R}^d} f\mathbb{1}_{\tau<\infty} dx.
$$

(ii) First notice that $\tau > -\infty$ since $g_{|n} \to 0$ as $n \to -\infty$ due to $g \in L_1$. Next, observe that

$$
Q_n(x) \subset Q_m(x)
$$

for all $m \le n$ since the partitions are nested. It follows that, if $y \in Q_n(x)$, then

$$
Q_m(y) = Q_m(x), \quad g_{|m}(y) = g_{|m}(x), \quad \forall m \le n.
$$

9.11.3 Maximal and sharp functions

Having proved lemma 9.11.6 we derive the following.

Corollary 9.11.7 (maximal inequality) Let $f \in L_{1,\text{loc}}(\mathbb{R}^d, F)$. Define the (filtering) maximal function of f by

$$Mf(x) = \sup_{n<\infty} (|f|_F)_{|n}(x),$$

so that $Mf = M|f|_F$. Then, for nonnegative $g \in L_1$, the maximal inequality holds:

$$|\{Mg > \lambda\}| \le \lambda^{-1} \int_{\mathbb{R}^d} g(x) \mathbb{1}_{Mg>\lambda} dx, \quad \forall \lambda > 0. \tag{131}$$

- -

proof. Indeed, for τ as in eq. (129), we have $\{Mg > \lambda\} = \{\tau < \infty\}$.

Remark 9.11.8 Our interest in estimating $|\{Mg > \lambda\}|$ as in corollary 9.11.7 is based on the following formula valid for any $f \ge 0$

$$\int_{\mathbb{R}^d} f(x) dx = \int_0^\infty |\{x : f(x) > t\}| dt. \tag{132}$$

Corollary 9.11.9 Let $p \in (1, \infty)$, $g \in L_1$, $g \ge 0$. Then

$$\|Mg\|_{L_p} \le q \|g\|_{L_p},$$

where $q = p/(p-1)$.

- -

proof. Indeed, from eq. (131), eq. (132), and Fubini's theorem we conclude that, for any finite constant $\nu > 0$,

$$\begin{aligned}
\|\nu \wedge Mg\|_{L_p}^p &= \int_0^\infty |\{x : \nu \wedge Mg(x) > \lambda^{1/p}\}| d\lambda \\
&= \int_0^{\nu^p} |\{x : Mg(x) > \lambda^{1/p}\}| d\lambda \\
&\le \int_{\mathbb{R}^d} g(x) \int_0^{\nu^p} \lambda^{-1/p} \mathbb{1}_{Mg>\lambda^{1/p}} d\lambda dx \\
&= \int_{\mathbb{R}^d} g(x) \int_0^{(\nu \wedge Mg)^p} \lambda^{-1/p} d\lambda dx \\
&= q \int_{\mathbb{R}^d} g(x) (\nu \wedge Mg)(x)^{p-1} dx \\
&\le \nu^{p-1} q \int_{\mathbb{R}^d} g(x) dx.
\end{aligned}$$

This implies that $\|\nu \wedge Mg\|_{L_p} < \infty$. Then upon using Hölder's inequality we get

$$\|\nu \wedge Mg\|_{L_p}^p \le q \|g\|_{L_p} \|\nu \wedge Mg\|_{L_p}^{p-1}, \quad \|\nu \wedge Mg\|_{L_p} \le q \|g\|_{L_p}$$

and it only remains to let $\nu \to \infty$ and use Fatou's lemma.

Theorem 9.11.10 For any $p \in (1,\infty)$ and $g \in L_p(\mathbb{R}^d, F)$
$$\|Mg\|_{L_p} \le q\|g\|_{L_p(\mathbb{R}^d,F)}.$$

proof. Since
$$Mg = M|g|_F \quad \text{and} \quad \|g\|_{L_p(\mathbb{R}^d,F)} = \||g|_F\|_{L_p}$$
we may concentrated on real-valued $g \in L_p$, $g \ge 0$. For $r > 0$ define $g^r(x) = g(x)\mathbb{1}_{|x|\le r}$. Then $g^r \in L_1$ and
$$\|Mg^r\|_{L_p} \le q\|g^r\|_{L_p} \le q\|g\|_{L_p}$$
by corollary 9.11.9. It only remains to use Fatou's lemma along with the observation that for any x since $Q_n(x)$ is bounded, we have
$$(g^r)_{|n}(x) \to g_{|n}(x) \quad \text{as} \quad r \to \infty.$$
which implies
$$g_{|n}(x) \le \varliminf_{r\to\infty} \sup_m (g^r)_{|m}(x), \quad Mg \le \varliminf_{r\to\infty} Mg^r.$$
The theorem is proved.

Let $f \in L_{1,\text{loc}}(\mathbb{R}^d, F)$. Define the sharp function of f by
$$f^\#(x) := \sup_{n<\infty} \fint_{Q_n(x)} |f(y) - f_{|n}(y)|_F dy.$$
Obviously $f^\# \le 2Mf$. It turns out that f and hence Mf are also controlled by $f^\#$.

Before proving a lemma, we prove the following basic fact.

Lemma 9.11.11 Let $f \in L_{1,\text{loc}}(\mathbb{R}^d, F)$. Then $f_{|n}(x) \to f(x)$ (a.e.).

proof. It suffices to show that the lemma holds on each ball $|x| < N$. For this sake, we only prove the case when $f \in L_1(\mathbb{R}^d, F)$. Clearly the lemma holds for continuous functions (actually, it is true for every $x \in \mathbb{R}^d$). For fixed $\epsilon > 0$ take a continuous g such that $\|f - g\|_{L_1(\mathbb{R}^d,F)} < \epsilon$, and it is easy to show that such g really exists. Then
$$\varlimsup_{n\to\infty} |f_{|n}(x) - f(x)|_F \le |f(x) - g(x)|_F + \varlimsup_{n\to\infty} |f_{|n}(x) - g_{|n}(x)|_F \le |f(x) - g(x)|_F + M(f-g)(x).$$
Hence for every $c > 0$,
$$\{x : \varlimsup_{n\to\infty} |f_{|n}(x) - f(x)|_F > c\} \subset \{x : M(f-g)(x) > c/2\} \cup \{x : |f-g|_F(x) > c/2\}.$$

Since $Mf = M|f|_F$ and by corollary 9.11.7,
$$|\{x : M(f-g)(x) > c/2\}| = |\{x : M|f-g|_F(x) > c/2\}| \le \frac{2}{c}\|f-g\|_{L_1(\mathbb{R}^d,F)}.$$
This and the Chebyshev's inequality gives
$$|\{x : \varlimsup_{n\to\infty} |f_{|n}(x) - f(x)|_F > c\}| \le \frac{4\epsilon}{c}.$$
Taking $\epsilon \to 0$ implies that the left hand side of the set has zero measure for every $c > 0$, which finishes the proof of the lemma.

Lemma 9.11.12 For $\alpha = (2N_0)^{-1}$ (so that $\alpha < 1/2$), any constant $c > 0$, and $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d, F)$, we have

$$|\{|f|_F > c\}| \leq \frac{2}{c} \int_{\mathbb{R}^d} \mathbb{1}_{Mf(x) > \alpha c} f^{\#}(x) dx.$$

proof. Define $g = |f|_F$ and

$$\tau(x) = \inf\{n : g_{|n}(x) > c\alpha\}.$$

Observe that

$$g_{|n}(x) \geq |f_{|n}(x)|_F, \quad |f(x)|_F - |f_{|n}(x)|_F \leq |f(x) - f_{|n}(x)|_F.$$

Also use lemma 9.11.6 (ii) and the fact that $f_{|n} \to f$ (a.e.), we find that (a.e.)

$$\begin{aligned}\{x : |f(x)|_F \geq c\} &= \{x : |f(x)|_F \geq c, \tau(x) < \infty\} \\ &= \{x : |f(x)|_F \geq c, \tau(x) < \infty, g_{|\tau}(x) \leq c/2\} \\ &\subset \{x : \tau(x) < \infty, |f(x) - f_{|\tau}(x)|_F \geq c/2\} =: A.\end{aligned}$$

Next, represent the set $\{\tau < \infty\}$ as the union $\bigcup_{n,k} Q_{nk}$ of disjoint Q_{nk}, satisfying $Q_{nk} \in \mathbb{Q}_n$ and $\tau = n$ on Q_{nk} for each n, k, and use lemma 9.11.6 (i) and Chebyshev's inequality to find

$$\begin{aligned}|A| &\leq \frac{2}{c} \int_{\mathbb{R}^d} \mathbb{1}_{\tau(x) < \infty} |f(x) - f_{|\tau}(x)|_F dx \\ &= \frac{2}{c} \sum_{n,k} \int_{Q_{nk}} |f(x) - f_{|n}(x)|_F dx \\ &= \frac{2}{c} \sum_{n,k} \int_{Q_{nk}} \left(\fint_{Q_n(z)} |f(x) - f_{|n}(x)|_F dx \right) dz \\ &\leq \frac{2}{c} \sum_{n,k} \int_{Q_{nk}} f^{\#}(z) dz \\ &= \frac{2}{c} \int_{\mathbb{R}^d} \mathbb{1}_{\tau(z) < \infty} f^{\#}(z) dz.\end{aligned}$$

Now it only remains to notice that $\{\tau(x) < \infty\} = \{Mf(x) > c\alpha\}$.

Theorem 9.11.13 Let $p \in (1, \infty)$. Then for any $f \in L_p(\mathbb{R}^d, F)$ we have

$$\|f\|_{L_p(\mathbb{R}^d, F)} \leq N \|f^{\#}\|_{L_p},$$

where $N = (2q)^p N_0^{p-1}$.

proof. As in the proof of corollary 9.11.9 we get from lemma 9.11.12 that if $f \in L_1(\mathbb{R}^d, F)$, then Hölder's inequality gives

$$\|f\|^p_{L_p(\mathbb{R}^d, F)} \leq N \int_{\mathbb{R}^d} f^{\#} (Mf)^{p-1} dx \leq N \|f^{\#}\|_{L_p} \|Mf\|^{p-1}_{L_p}.$$

If in addition $f \in L_p(\mathbb{R}^d, F)$, then it only remains to use theorem 9.11.10 and check that the resulting constant is right.

If we only have $f \in L_p(\mathbb{R}^d, F)$, then it suffices to take $f_n \in C_0(\mathbb{R}^d, F)$ converging to f in $L_p(\mathbb{R}^d, F)$ and observe that $f_n^{\#} \leq (f - f_n)^{\#} + f^{\#}$ and

$$\|(f - f_n)^{\#}\|_{L_p} \leq 2\|M(f - f_n)\|_{L_p} \leq 2q \|f - f_n\|_{L_p(\mathbb{R}^d, F)} \to 0.$$

This proves the theorem.

Remark 9.11.14 By Hölder's inequality, for any $p \in [1, \infty]$

$$f^{\#}(x) \le \sup_{n<\infty} \left(\fint_{Q_n(x)} |f(y) - f_{|n}(y)|_F^p dy \right)^{1/p}.$$

The maximal function introduced in corollary 9.11.7 is related to the underlying filtration of partitions. Below we are also using the following more traditional maximal function:

$$\mathbb{M}g(x) := \sup_{r>0} \fint_{B_r(x)} |g(y)| dy. \tag{133}$$

Let Mg be the maximal function associated with the filtration of dyadic cubes Q_n introduced in eq. (126). It turns out that, in a sense, Mg and $\mathbb{M}g$ are comparable.

First, since $Q_n(x) \subset B_{r_n}(x)$ with $r_n = 2^{-n}\sqrt{d}$, we have $|B_{r_n}(x)| = N(d)|Q_n(x)|$,

$$\fint_{Q_n(x)} |g| dy \le \frac{|B_{r_n}(x)|}{|Q_n(x)|} \fint_{B_{r_n}(x)} |g| dy \le N(d)\mathbb{M}g(x),$$

and $Mg \le N\mathbb{M}g$. On the other hand, we have the following.

Lemma 9.11.15 There is a constant $N = N(d)$ such that if $g \in L_1$, then for any $\lambda > 0$

$$|\{x : \mathbb{M}g(x) > N\lambda\}| \le N|\{x : Mg(x) > \lambda\}|. \tag{134}$$

Here is the classical maximal function estimate.(lxxxv)

Theorem 9.11.16 Let $p \in (1, \infty)$ and $g \in L_p$. Then $\mathbb{M}g \in L_p$ and

$$\|\mathbb{M}g\|_{L_p} \le N\|g\|_{L_p}, \tag{135}$$

where N is independent of g.

proof. Without loss of generality we assume that $g \ge 0$. If $g \in L_1$, then eq. (135) is obtained by replacing λ with $\lambda^{1/p}$ in eq. (134), integrating with respect to λ, remembering eq. (132), and using corollary 9.11.9.

If the additional assumption that $g \in L_1$ is not satisfied, it suffices to use the argument from the proof of theorem 9.11.10.

9.11.4 Preliminary estimates on $\mathcal{G}$

Throughout the subsection f is a fixed element of $C_0^\infty(\mathbb{R}^{d+1}, H)$ and $u = \mathcal{G}f$.

Lemma 9.11.17 For any $T \in (-\infty, \infty]$,

$$\|u\|_{L_2(\mathbb{R}^{d+1}\cap\{t\le T\})} \le N(d, K)\|f\|_{L_2(\mathbb{R}^{d+1}\cap\{t\le T\})}. \tag{136}$$

(lxxxv) The collection of open balls does not form a filtration of partitions. Indeed, there is no way to cover $\mathbb{R}^d$ with *disjoint* open balls.

proof. Since f is smooth, its values belong to a separable subspace of H. Then by using orthonomal basis, the Fubini's theorem, Fourier transformation, and Plancherel's theorem,(lxxxvi)

$$\begin{aligned}
\|u\|_{L_2(\mathbb{R}^{d+1}\cap\{t\le T\})} &= \int_{\mathbb{R}^d}\int_{-\infty}^{T}\int_{-\infty}^{t} |(t-s)^{-d/2}\psi(x/\sqrt{t-s}) * f(s,x)|_H^2 \frac{ds}{t-s}dtdx \\
&= \int_{-\infty}^{T}\int_{-\infty}^{t}\int_{\mathbb{R}^d} |(t-s)^{-d/2}\psi(x/\sqrt{t-s}) * f(s,x)|_H^2 dx\frac{ds}{t-s}dt \\
&= \int_{-\infty}^{T}\int_{-\infty}^{t}\int_{\mathbb{R}^d} |\tilde{\psi}(\xi\sqrt{t-s})\tilde{f}(s,\xi)|_H^2 d\xi\frac{ds}{t-s}dt \\
&= \int_{-\infty}^{T}\int_{\mathbb{R}^d}\left[\int_{-\infty}^{T-s}|\tilde{\psi}(\xi\sqrt{t})|^2\frac{dt}{t}\right]|\tilde{f}(s,\xi)|_H^2 d\xi ds \\
&=: I.
\end{aligned}$$

Here $\tilde{\psi}(0)=0$ ($\because \int\psi = 0$) and by the mean value theorem and basic facts about Fourier transformations,

$$|\tilde{\psi}(\xi)| \le |\xi|\sup|\nabla\tilde{\psi}| \le N(d)|\xi|\int_{\mathbb{R}^d}|x||\psi(x)|dx,$$

$$|\xi||\tilde{\psi}(\xi)| \le N(d)\int_{\mathbb{R}^d}|\nabla\psi(x)|dx,$$

so that with $\bar{\xi}=\xi/|\xi|$, by changing variables gives

$$\begin{aligned}
\int_0^\infty |\tilde{\psi}(\xi\sqrt{t})|^2\frac{dt}{t} &= \int_0^\infty |\tilde{\psi}(\bar{\xi}\sqrt{t})|^2\frac{dt}{t} \\
&= \int_0^1 |\tilde{\psi}(\bar{\xi}\sqrt{t})|^2\frac{dt}{t} + \int_1^\infty |\tilde{\psi}(\bar{\xi}\sqrt{t})|^2\frac{dt}{t} \\
&= N(d)\left(\int_{\mathbb{R}^d}|x||\psi(x)|dx\right)^2 + N(d)\int_1^\infty\frac{1}{t^2}\left(\int_{\mathbb{R}^d}|\nabla\psi(x)|dx\right)^2 dt \\
&\le N(d,K),
\end{aligned}$$

we have

$$I \le N\int_{-\infty}^{T}\int_{\mathbb{R}^d}|\tilde{f}(s,\xi)|_H^2 d\xi ds.$$

Since the last expression equals the right-hand side of eq. (136). The lemma is proved.

To process further we need some notation. According to eq. (133) introduce the maximal function of a real-valued function h given on $\mathbb{R}^d$ relative to balls. We denote this function $\mathbb{M}_x h$ to emphasize that this maximal function is taken with respect to x. Similarly, for functions h on $\mathbb{R}$ we introduce $\mathbb{M}_t h$ as the maximal function of h relative to symmetric intervals:

$$\mathbb{M}_t h(t) = \sup_{r>0}\frac{1}{2r}\int_{-r}^{r}|h(t+r)|dr.$$

For a function $h(t,x)$ set

$$\mathbb{M}_h h(t,x) = \mathbb{M}_x(h(t,\cdot))(x), \quad \mathbb{M}_t h(t,x) = \mathbb{M}_t(h(\cdot,x))(t).$$

Notice the following consequence of lemma 9.11.17, in which and below we denote by $B_r(x)$ the open ball of radius r centered at x and $B_r = B_r(0)$.

(lxxxvi) Separability is need to apply Plancherel's theorem.

Corollary 9.11.18 Set

$$Q_0 = [-4, 0] \times [-1, 1]^d \tag{137}$$

and assume that $f = 0$ outside of $[-12, 12] \times B_{3d}$.[lxxxvii] Then for any $(t, x) \in Q_0$

$$\int_{Q_0} |u(s, y)|^2 dsdy \le N\mathbb{M}_t\mathbb{M}_x|f|_H^2(t, x), \tag{138}$$

where N depends only on d and K.

- -

proof. By lemma 9.11.17 for $g := |f|_H^2$ the left-hand side is less than

$$\begin{aligned} N\int_{-\infty}^0 \int_{\mathbb{R}^d} gdyds &\le N\int_{-12}^0 \int_{|y|\le 3d} gdyds \\ &\le N\int_{-12}^0 \int_{|x-y|\le 4d} gdyds \\ &\le N\int_{-12}^0 \mathbb{M}_x g(s, x)ds \\ &\le N\mathbb{M}_t\mathbb{M}_x g(t, x). \end{aligned}$$

Here is a generalization of corollary 9.11.18.

Lemma 9.11.19 Assume that $f(t, x) = 0$ for $t \notin (-12, 12)$. Then eq. (138) holds again for any $(t, x) \in Q_0$.

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proof. We take a $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\zeta = 1$ in B_{2d} and $\zeta = 0$ outside of B_{3d}. Set $\alpha = \zeta f$ and $\beta = (1 - \zeta)f$. Since $\mathcal{G}f \le \mathcal{G}\alpha + \mathcal{G}\beta$ and $\mathcal{G}\alpha$ admits the stated estimate, it suffices to concentrate on $\mathcal{G}\beta$. In other words, in the rest of the proof we may assume that $f(t, x) = 0$ for $x \in B_{2d}$.

Introduce $\bar{f} = |f|_H$, tatke $0 > s > r > -12$, and recall that $\Psi_s f(x) = t^{-d/2}\phi(x/\sqrt{t}) * f(x)$. We write

$$\begin{aligned} |\Psi_{s-r}f(r, \cdot)(y)|_H &\le (s - r)^{-d/2}\int_{\mathbb{R}^d} |\psi(z/\sqrt{s - r})||f(r, y - z)|_H dz \\ &\le (s - r)^{-d/2}\int_{\mathbb{R}^d} \bar{\psi}(|z|/\sqrt{s - r})\bar{f}(r, y - z)dz. \end{aligned}$$

We transform the last integral by using the formula

$$\begin{aligned} \int_{R\ge|z|\ge\epsilon} F(z)G(|z|)dz &= \int_\epsilon^R \int_{|z|=\rho} F(z)G(|z|)\sigma(dz)d\rho \\ &= \int_\epsilon^R G(\rho)\frac{\partial}{\partial\rho}\left(\int_{|z|\le\rho} F(z)dz\right)d\rho \\ &= G(R)\int_{|z|\le R} f(z)dz - G(\epsilon)\int_{|z|\le\epsilon} f(z)dz - \int_\epsilon^R G'(\rho)\int_{|z|\le\rho} F(z)dzd\rho, \end{aligned} \tag{139}$$

where $0 \le \epsilon \le R \le \infty$ and F and G satisfy appropriate conditions. See [1] for polar coordinate

[lxxxvii] This $3d$ can be any positive number. In the proof, one need the fact that if $x \in [-1, 1]^d$ and $y \in B_a$, then one have $|x - y| \le a + \sqrt{d}$. If $a = 3d$, then we have $|x - y| \le 4d$, which makes calculations much easier.

integration properties. Also notice that if $(s,y) \in Q_0$ and $|z| \le \rho$ with a $\rho > 1$, then

$$|x-y| \le 2d =: \nu, \quad B_\rho(y) \subset B_{\nu+\rho}(x) \subset B_{\mu\rho}(x), \quad \mu = \nu + 1, \tag{140}$$

whereas if $|z| \le 1$, then $|y-z| \le 2d$ and $f(r, y-z) = 0$.(lxxxviii)

Then we see that for $0 > s > r > -12$ and $(s,y) \in Q_0$, by recalling properties of $\bar\psi$,eq. (139), and eq. (140),

$$\begin{aligned}
|\Psi_{s-r}f(r,\cdot)(y)|_H &\le (s-r)^{-d/2}\left(\int_{|z|<1} + \int_{|z|\ge 1}\right)\bar\psi(|z|/\sqrt{s-r})\bar f(r,y-z)dz \\
&= (s-r)^{-d/2}\int_{|z|\ge 1}\bar\psi(|z|/\sqrt{s-r})\bar f(r,y-z)dz \\
&= -(s-r)^{-(d+1)/2}\int_1^\infty \bar\psi'(\rho/\sqrt{s-r})\left(\int_{|z|\le\rho}\bar f(r,y-z)dz\right)d\rho \\
&\le (s-r)^{-(d+1)/2}\int_1^\infty |\bar\psi'(\rho/\sqrt{s-r})|\left(\int_{|z|\le\rho}\bar f(r,y-z)dz\right)d\rho \\
&= (s-r)^{-(d+1)/2}\int_1^\infty |\bar\psi'(\rho/\sqrt{s-r})|\left(\int_{B_\rho(y)}\bar f(r,z)dz\right)d\rho \\
&\le (s-r)^{-(d+1)/2}\int_1^\infty |\bar\psi'(\rho/\sqrt{s-r})|\left(\int_{B_{\mu\rho}(x)}\bar f(r,z)dz\right)d\rho \\
&\le N(d)\mathbb{M}_x\bar f(r,x)(s-r)^{-(d+1)/2}\int_1^\infty |\bar\psi'(\rho/\sqrt{s-r})|\rho^d d\rho \\
&= N(d)\mathbb{M}_x\bar f(r,x)\int_{(s-r)^{-1/2}}^\infty |\bar\psi'(\rho)|\rho^d d\rho \\
&\le N(d,K)(s-r)^{1/2}\mathbb{M}_x\bar f(r,x).
\end{aligned}$$

Also observe that by Hölder's inequality $(\mathbb{M}_x\bar f)^2 \le \mathbb{M}_x\bar f^2$. Then for $(s,y) \in Q_0$ we obtain(lxxxix)

$$|u(s,y)|^2 = \int_{-12}^s |\Psi_{s-r}f(r,\cdot)(y)|_H^2 \frac{dr}{s-r} \le N\int_{-12}^0 \mathbb{M}_x|f|_H^2(r,x)dr,$$

where the last expression is certainly less than the right-hand side of eq. (138). This finishes the proof.

Lemma 9.11.20 Assume that $f(t,x) = 0$ for $t \ge -8$. Then for any $(t,x) \in Q_0$

$$\int_{Q_0} |u(s,y) - u(t,x)|^2 dsdy \le N\mathbb{M}_t\mathbb{M}_x|f|_H^2(t,x), \tag{141}$$

where the constant N depends only on K and d.

(lxxxviii) Recall that $(t,x) \in Q_0$ and $[-1,1]^d \subset B_{\sqrt{d}}$.

(lxxxix) Since $f(t,x) = 0$ for $t \notin (-12,12)$, $\Psi_{s-t}f(t,\cdot)(y) = (s-t)^{-d/2}\psi(y/\sqrt{s-t}) * f(t,y) = 0$ if $t \notin (-12,12)$.

proof. The left-hand side of eq. (141) is certainly less than a constant times

$$\sup_{Q_0}[|D_s u|^2 + |\nabla u|^2]. \tag{142}$$

Fix $(s, y) \in Q_0$. note that $s \geq -4$ and by Cauchy-Schwarz's inequality, observe that[xc]

$$\begin{aligned}
\nabla u^2(s,y) &= \nabla \int_{-\infty}^{s} |\Psi_{s-r} f(r,\cdot)(y)|_H^2 \frac{dr}{s-r} \\
&= \nabla \int_{-\infty}^{-8} |\Psi_{s-r} f(r,\cdot)(y)|_H^2 \frac{dr}{s-r} \\
&= \int_{-\infty}^{-8} \nabla |\Psi_{s-r} f(r,\cdot)(y)|_H^2 \frac{dr}{s-r} \\
&= 2\int_{-\infty}^{-8} \mathrm{Re}(\Psi_{s-r} f(r,\cdot)(y), \nabla[\Psi_{s-r} f(r,\cdot)(y)])_H \frac{dr}{s-r} \\
&\leq 2\int_{-\infty}^{-8} (|\Psi_{s-r} f(r,\cdot)(y)|_H)\,(|\nabla\Psi_{s-r} f(r,\cdot)(y)|_H)\,\frac{dr}{s-r} \\
&\leq 2\left(\int_{-\infty}^{-8} |\Psi_{s-r} f(r,\cdot)(y)|_H^2 \frac{dr}{s-r}\right)^{1/2} \left(\int_{-\infty}^{-8} |\nabla\Psi_{s-r} f(r,\cdot)(y)|_H^2 \frac{dr}{s-r}\right)^{1/2}.
\end{aligned}$$

Since $\nabla u^2 = 2u \cdot \nabla u$ we obtain

$$|\nabla u(s,y)|^2 \leq \int_{-\infty}^{-8} |\nabla \Psi_{s-r} f(r,\cdot)(y)|_H^2 \frac{dr}{s-r} =: \int_{-\infty}^{-8} I^2(r,s,y)\frac{dr}{s-r}.$$

It is easy to get

$$I(r,s,y) \leq (s-r)^{-(d+1)/2} \int_{\mathbb{R}^d} \bar{\psi}(|z|/\sqrt{s-r}) \bar{f}(r, y-s)dz,$$

where as before, $\bar{f} = |f|_H$.

Also use again eq. (139) and eq. (140). Then we see that for $s > r$

$$\begin{aligned}
I(r,s,y) &\leq (r-s)^{-(d+2)/2} \int_0^\infty |\bar{\psi}'(\rho/\sqrt{s-r})| \left(\int_{B_\rho(y)} \bar{f}(r,z)dz\right) d\rho \\
&\leq N(d)\mathbb{M}_x \bar{f}(r,x)(r-s)^{-(d+2)/2} \int_0^\infty |\bar{\psi}'(\rho/\sqrt{s-r})|(\nu+\rho)^d d\rho \\
&= N(d)\mathbb{M}_x \bar{f}(r,x)(r-s)^{-1/2} \int_0^\infty |\bar{\psi}'(\rho)|(\nu/\sqrt{s-r}+\rho)^d d\rho.
\end{aligned}$$

For $r \leq -8$ we have $s - r \geq 4$ and we conclude

$$\int_0^\infty |\bar{\psi}'(\rho)|(\nu/\sqrt{s-r}+\rho)^d d\rho \leq N, \quad I(r,s,y) \leq N(s-r)^{-1/2}\mathbb{M}_x \bar{f}(r,x),$$

$$|\nabla u(s,y)|^2 \leq N \int_{-\infty}^{-8} \mathbb{M}_x \bar{f}^2(r,x) \frac{dr}{(4+r)^2}.$$

We transform the last integral integrating by parts or using eq. (139) to find

$$|\nabla u(s,y)|^2 \leq N \int_{-\infty}^{-8} \frac{-1}{(4+r)^3} \left(\int_r^0 \mathbb{M}_x \bar{f}^2(p,x)dp\right) dr$$

[xc] If $f, g : \mathbb{R}^d \to H$ are differentiable, then by the chain rule (one can easily prove), $D_x(f(x), g(x))_H = (f'(x), g(x))_H + (f(x), g'(x))_H$. Since $|f(x)|_H^2 = (f(x), f(x))_H$, it is also differentiable.

$$\leq N\mathbb{M}_t\mathbb{M}_x\bar{f}^2(t,x)\int_{-\infty}^{-8}\frac{-1}{(4+r)^3}dr$$
$$= N\mathbb{M}_t\mathbb{M}_x\bar{f}^2(t,x).$$

We thus have estimated part of eq. (142).

To estimate $D_s u$, we process similarly:

$$|D_s u(s,y)|^2 \leq \int_{-\infty}^{-8}|D_s\Psi_{s-r}f(r,y)|_H^2\frac{dr}{s-r} = \int_{-\infty}^{-8}J^2(r,s,y)\frac{dr}{s-r},$$

where

$$J(r,s,y) := |D_s\Psi_{s-r}f(r,y)|_H \leq (s-r)^{-(d+2)/2}\int_{\mathbb{R}^d}\bar{\psi}(|z|/\sqrt{s-r})\bar{f}(r,y-z)dz.$$

For $r \leq -8$ we may further write (by recalling $s-r \geq 4$)

$$J(r,s,y) \leq N(s-r)^{-(d+1)/2}\int_{\mathbb{R}^d}\bar{\psi}(|z|/\sqrt{s-r})\bar{f}(r,y-z)dz$$

and then it only remains to refer to the above computations. This proves the lemma.

Remark 9.11.21 The behind technique in both Lemma 9.11.19 and 9.11.20 is by splitting regions of time space where the kernel behaves differently. For instance, we want to try proving the following proposition:

> Assume that $f(t,x) = 0$ for $t \leq -12$. Then for any $(t,x) \in Q_0$
>
> $$\int_{Q_0}|u(s,y)-u(t,x)|^2 dsdy \leq N(d,K)\mathbb{M}_t\mathbb{M}_x|f|_H^2(t,x).$$

If we use the same method in Lemma 9.11.20, then we eventually obtain the following inequality.

$$|\nabla u(s,y)|^2 \leq N\mathbb{M}_t\mathbb{M}_x\bar{f}^2(t,x)\int_{-12}^{s}\frac{-1}{(4+r)^3}dr.$$

Since $s \geq -4$, the integral of the right hand side diverges, so that we cannot obtain some appropriate estimate. Similarly, the opposite case also has some trouble.

By observing the behavior of the kernel and splitting each domains and applying different methods (L_2 estimate[xci], and Hölder estimate[xcii]) is well-used technique to obtain the estimate related to kernels.

9.11.5 Proof of Theorem 9.11.1

First note that for any $f \in C_0^\infty((a,b)\times\mathbb{R}^d, H)$ we have $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$ and equation eq. (124) with $-\infty$ and ∞ in place of a and b respectively is stronger than as is. Therefore, we may assume that $a = -\infty$ and $b = \infty$. Then our assertion is that for $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$ and $u \in \mathcal{G}f$ we have

$$\|u\|_{L_p(\mathbb{R}^{d+1})} \leq N(d,p,K)\|f\|_{L_p(\mathbb{R}^{d+1},H)}.$$

This estimate follows from Lemma 9.11.17 if $p = 2$. Hence we may concentrate on $p > 2$. We start considering this case by claiming that at each point in $\mathbb{R}^{d+1}$

$$(\mathcal{G}f)^\# \leq N(d,K)(\mathbb{M}_t\mathbb{M}_x|f|_H^2)^{1/2}, \tag{143}$$

[xci] Used in Lemma 9.11.19.
[xcii] Used in Lemma 9.11.20.

where the sharp function $(\mathcal{G}f)^{\#}$ is defined relative to the parabolic dyadic cubes of type eq. (125). By remark 9.11.14 shows that to prove eq. (143) it suffices to prove that for each $Q = Q_n(i_0, \cdots, i_d)$ (see eq. (125)) and $(t,x) \in Q$

$$\fint_Q |\mathcal{G}f - (\mathcal{G}f)_Q|^2 dyds \le N(d,K)\mathbb{M}_t\mathbb{M}_x|f|_H^2(t,x), \tag{144}$$

where

$$(\mathcal{G}f)_Q = \fint_Q \mathcal{G}f dyds.$$

To prove eq. (144), observe that if a constant $c \neq 0$, then $\Psi_t h(c\cdot)(x) = \Psi_{tc^2} h(cx)$, and

$$\begin{aligned}\mathcal{G}f(c^2\cdot, c\cdot)(t,x) &= \left[\int_{-\infty}^{t} |\Psi_{(t-s)c^2} f(c^2 s, \cdot)(cx)|_H^2 \frac{ds}{t-s}\right]^{1/2} \\ &= \left[\int_{-\infty}^{tc^2} |\Psi_{tc^2-s} f(s, \cdot)(cx)|_H^2 \frac{ds}{tc^2-s}\right]^{1/2} = \mathcal{G}f(c^2t, cx).\end{aligned}$$

This and the fact that dilations do not affect averages show that it suffices to prove eq. (144) for $Q = Q_{-1}(i_0, \cdots, i_d)$. In that case Q is just a shift of Q_0 from eq. (137). Furthermore, the shift is harmless since $\mathbb{M}_x$ and $\mathbb{M}_t$ are defined in terms of balls rather than dyadic cubes.(xciii)

Thus let $Q = Q_0$ and take a function $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta = 1$ on $[-8, 8]$, $\zeta = 0$ outside of $[-12, 12]$, and $0 \le \zeta \le 1$. Set

$$\alpha = f\zeta, \quad \beta = f - \alpha.$$

Observe that

$$\Psi_{t-s}\alpha(s,\cdot) = \zeta(s)\Phi_{t-s} f(s,\cdot), \quad \mathcal{G}f \le \mathcal{G}\alpha + \mathcal{G}\beta, \quad \mathcal{G}\beta \le \mathcal{G}f.$$

It follows that for any constant c

$$|\mathcal{G}f - c| \le |\mathcal{G}\alpha| + |\mathcal{G}\beta - c|$$

and it light of remark 9.11.5 the left hand side of eq. (144) is less than

$$\fint_Q |\mathcal{G}f - c|^2 dyds \le 2\fint_Q |\mathcal{G}\alpha|^2 dyds + 2\fint_Q |\mathcal{G}\beta - c|^2 dyds.$$

We finally take $c = \mathcal{G}\beta(t,x)$ and obtain eq. (144) from Lemma 9.11.19 and Lemma 9.11.20.

After having proved eq. (143), by considering the Fefferman-Stein theorem with the L_q, $q > 1$, boundedness of the maximal operators we conclude (recall that $p > 2$)

$$\begin{aligned}\|u\|_{L_p(\mathbb{R}^{d+1})}^p &\le N\|(\mathbb{M}_t\mathbb{M}_x|f|_H^2)^{1/2}\|_{L_p(\mathbb{R}^{d+1})}^p \\ &= N\int_{\mathbb{R}^d}\int_{\mathbb{R}} (\mathbb{M}_t\mathbb{M}_x|f|_H^2)^{p/2} dtdx \\ &\le N\int_{\mathbb{R}^d}\int_{\mathbb{R}} (\mathbb{M}_x|f|_H^2)^{p/2} dtdx \\ &= N\int_{\mathbb{R}}\int_{\mathbb{R}^d} (\mathbb{M}_x|f|_H^2)^{p/2} dxdt \\ &\le N\|f\|_{L_p(\mathbb{R}^{d+1}, H)}^p.\end{aligned}$$

This proves the theorem.

(xciii) Fix $x, y \in \mathbb{R}$. In case of balls, we have $B_r(x+y) = B_r(x) + y$. So it is easily obtained that

$$\mathbb{M}g(x+y) = \sup_{r<\infty}\fint_{B_r(x+y)} |g(z)|dz = \sup_{r<\infty}\fint_{y+B_r(x)} |g(z)|dz = \sup_{r<\infty}\fint_{B_r(x)} |g(z-y)|dz = \mathbb{M}g(\cdot - y)(x).$$

However, for the dyadic cubes, being disjoint, translation does not work well, that is, $Mg(x+y)$ and $Mg(\cdot - y)(x)$ may different.

9.12 More Possible Results for eq. (88)

We proved Theorem 8.3.4 when $d = 1$ and from Remark 8.3.9, we checked that we cannot use the kernel to prove Theorem 8.3.4 when $d \geq 2$.

Let us rewrite eq. (88):

$$du(t,x) = [a^{ij}(t,x)u_{x^ix^j}(t,x) + b^i(t,x)u_{x^i}(t,x) + f(t,x,u(t,x))]dt + h(t,x,u(t,x))dB_t, \tag{145}$$

where B_t is a cylindrical Wiener process on $L_2 = L_2(\mathbb{R}^d)$.

[Fill out the detail why it suffices to consider eq. (146)]

We may take $\sum_k \eta_k(x)dw_t^k$, where $\{\eta_k(x)\}$ is an orthonormal basis in L_2, and w_t^k are independent $\mathcal{F}_t$-adapted one-dimensional Wiener process. Then instead of eq. (145), we may consider the equation

$$du(t,x) = [a(t,x)u''(t,x) + b(t,x)u'(t,x) + f(t,x,u(t,x))]dt + g^k(t,x,u(t,x))dw_t^k, \tag{146}$$

on a time interval $[0,\tau]$ where $g^k = h\eta_k$ and τ is a bounded stopping time.

Assumptions for a, b, and f are similar with Assumption 8.3.1 and Assumption 8.3.2 but this time, we do not assume $d = 1$.

Assumption 9.12.1 The functions $a(t,x) = a(\omega,t,x)$ and $b(t,x) = b(\omega,t,x)$ are real-valued functions defined on $(\!(0,\tau]\!] \times \mathbb{R}^d$.

(i) For any ω and $t \leq \tau(\omega)$, $a(\omega,t,\cdot) \in C^{1,1}(= B^2)$ and $b(\omega,t,\cdot) \in C^{0,1}(= B^1)$ and $\|a\|_{C^{1,1}} + \|b\|_{C^{0,1}} \leq K$. Also $K(\delta^{ij}) \geq (a^{ij}) \geq \delta(\delta^{ij})$.

(ii) For any $x \in \mathbb{R}^d$, the processes a and b are predictable.

Assumption 9.12.2 The function $f(t,x,u)$ is a real-valued function on $(\!(0,\tau]\!] \times \mathbb{R}^d \times \mathbb{R}$ such that

(i) for any x and u, the process $f(t,x,u)$ is predictable;

(ii) for any ω, t, x, u, and v,

$$|f(t,x,u) - f(t,x,v)| \leq K|u - v|.$$

9.12.1 when h is independent of u

First of all, we are going to prove that eq. (146) has a unique solution if h is independent of u. More specifically, assume that h satisfies the below assumption.

Assumption 9.12.3 The $h(t,x)$ are real-valued functions on $(\!(0,\tau]\!] \times \mathbb{R}^d$ such that for any x the process $h(t,x)$ is predictable.

As h is independent of u, we also write $g^k(t,x)$ instead of $g^k(t,x,u)$.

Theorem 9.12.4 Let $\kappa \in (0,1/2)$ and $u_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{(1/2)-\kappa-2/p})$. Assume that

$$I^p(\tau) := \mathbb{E}\int_0^\tau [\|f(t,\cdot,0)\|_{-3/2-\kappa,p} + \|h(t,\cdot)\|_p^p]dt < \infty.$$

Then in the space $\mathcal{H}_p^{1/2-\kappa}(\tau)$, eq. (146) with the initial condition u_0 has a unique solution u. Moreover,

$$\|u\|_{\mathcal{H}_p^{1/2-\kappa}(\tau)} \leq N[I(\tau) + (\mathbb{E}\|u_0\|^p_{1/2-\kappa-2/p,p})^{1/p}],$$

where the constant N depends only on κ, p, δ, K, and τ.

- -

proof. As in the proof of Theorem 8.3.4, on the assumptions of assumption 9.12.1 and 9.12.2, $b \cdot Du + f$ satisfies assumptions of Theorem 5.0.7.

9.12.2 The case $h(u, t, x) = u(t, x)$

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