Kindergarten

Volume II: Real Analysis

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Preface

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Chapter 0

Preliminaries

0.1 Basic notations

Let X be a set and $\mathbf{P}_j(x)$, $j = 1, 2, \cdots, n$ propositions constraint to $x \in X$, then we sometimes simply write $\{\mathbf{P}_1(x), \cdots, \mathbf{P}_n(x)\}$ instead of $\{x : \mathbf{P}_1(x), \cdots, \mathbf{P}_n(x)\}$. If a proposition $\mathbf{P}(x)$ is $x \in A$ for some subset $A \in X$, then we write $\{A, \mathbf{P}_1(x), \cdots, \mathbf{P}_n(x)\}$ simply instead of $\{x : x \in A, \mathbf{P}_1(x), \cdots, \mathbf{P}_n(x)\}$, and so on. For example, if $f : X \to \mathbb{R}$ is a function then $\{f > a\}$ means $\{x : f(x) > a\}$, and $\{A, B, f \leq b\}$ means $\{x : x \in A, x \in B, f(x) \leq b\}$.

However, if we write $\{A, B\}$ then it can be interpret either a doubleton containing sets A and B, or the set $\{x : x \in A, x \in B\}$. If there is no additional proposition that makes the context clear, then we always consider such case by the former one.

0.2 Orderings

Definition 0.2.1
Let X be a nonempty set. An *(partial) ordering* is a relation $\leq$ satisfies the following properties.

(i) $a \leq a$ for every $a \in X$.

(ii) For $a, b \in X$, if $a \leq b$ and $b \leq a$, then $a = b$.

(iii) For $a, b, c \in X$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Then we say $(X, \leq)$ an *ordered set*. We write $b \geq a$ if $a \leq b$, $a < b$ if $a \leq b$ and $a \neq b$, and $b > a$ if $b \geq a$ and $a \neq b$.
If an ordering is called by *totally ordered* if

(iv) For $a, b \in X$, either $a < b$, $a = b$, or $a > b$ is satisfied.

In this case, we say that an ordered set as a *totally ordered set*.

Definition 0.2.2
Let $(X, \leq)$ be an ordered set. Then we define

$$\begin{aligned} (a, b) &:= \{x : a < x < b\}, & [a, b] &:= \{x : a \leq x \leq b\}, \\ (a, b] &:= \{x : a < x \leq b\}, & [a, b) &:= \{x : a \leq x < b\}. \end{aligned}$$

All of four sets are called an *interval*. Especially, (a, b) is called an *open inverval* and $[a, b]$ is called a *closed interval*.

0.3 Product of sets

Definition 0.3.1 (Product set)
Let A be a nonempty set and for each $\alpha \in A$ consider some *nonempty* set X_α. Then define a *product* $\prod_{\alpha \in A} X_\alpha$ of sets $\{X_\alpha\}_{\alpha \in A}$ by a collection of all functions $f : A \to \bigcup_{\alpha \in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for each $\alpha \in A$.
If at least one set X_α is empty or $A = \varnothing$, define $\prod_{\alpha \in A} X_\alpha := \varnothing$.

In mathematics, if every sets are nonempty, their product is also not empty and such fact is an axiom. Formally,

Axiom 0.3.2 (Axiom of choice)
Let A be a nonempty set and for each $\alpha \in A$ consider some *nonempty* set X_α. Then

$$\prod_{\alpha \in A} X_\alpha \neq \varnothing.$$

Chapter 1

User Guides for General Topology

1.1 Basic terminologies related to the topology

In Volume 1, we explored basic terminologies and their properties of metric spaces. In this chapter, we will extend those by introducing the concept called "topology".

Definition 1.1.1
Let X be a set. A *topology* is a collection τ of some subsets of X satisfies the following conditions.

(i) $\varnothing, X \in \tau$;

(ii) If $\mathscr{U} \subset \tau$, then $\bigcup \mathscr{U} \in \tau$;

(iii) If $\mathscr{F} \subset \tau$ be a finite subcollection, then $\bigcap \mathscr{F} \in \tau$.

Every element of a topology is called an *open set*, and complement of open sets are called *closed sets*.

Everywhere in this book, **topological spaces are not empty**. By the definition of closed sets, we can clearly obtain the following.

Proposition 1.1.2

(i) If $\mathscr{U}$ is a collection of some closed sets, then $\bigcap \mathscr{U}$ is also closed.

(ii) If $\mathscr{F}$ is a finite collection of some closed sets, then $\bigcup \mathscr{F}$ is closed.

If X is a topological space, we should define closures and interiors differently. However, if one studied well about closures and interiors in metric spaces, below definition makes sense.

Definition 1.1.3 (Closures and interiors)
Let X be a topological space and A a subset of X. Then a *closure* of A is an intersection of all closed sets each of which contains A, and an *interior* of A is a union of all open sets each of which is contained in A.
We write a closure of A by $\mathrm{cl}A$, and an interior of A by $\mathrm{int}A$.

Remark 1.1.4
Notation $\overline{A}$ is also used for denoting the closure of A, and A° for writting the interior of A.

Now we define neighborhoods of a point in topological spaces. Although we defined this to be open in metric spaces, for some purpose, we define neighborhoods *need not to be open*.

Definition 1.1.5 (Neighborhoods)
Let X be a topological space and $x \in X$. Then a subset $N \subset X$ is called a *neighborhood of* x if its interior contains x. That is, $x \in \text{int}N$. For each $x \in X$, define a collection of all *open* neighborhoods of x.

1.2 Separation Axioms

Definition 1.2.1 (Separation Axioms)
Let X be a topological space. We say that X is

(i) T_0 space if $x, y \in X$ there exists an open set U such that either $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.

(ii) T_1 space if $x, y \in X$ there exist open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

(iii) T_2 space if $x, y \in X$ there exist *disjoint* open sets U and V such that $x \in U$ and $y \in V$.

(iv) T_3 space if $x \in X$ and a closed set C in X there exist *disjoint* open sets U and V such that $x \in U$ and $C \subset V$.

(v) T_4 space if C and D are closed sets in X there exist *disjoint* open sets U and V such that $C \subset U$ and $D \subset V$.

(vi) T_5 space if C and D are *separated* sets in X there exist *disjoint* open sets U and V such that $C \subset U$ and $D \subset V$. Here, we say that two sets C and D are separated if $\text{cl}C \cap D = C \cap \text{cl}D = \varnothing$.

T_2 space is also called by *Hausdorff space*. A *regular space* is both T_1 space and T_3 space, a *normal space* is both T_1 and T_4 space, and a *completely normal* is both T_1 and T_5 space.

1.3 Filters and ultrafilters

In the basis analysis, one is taught that if X is a metric space and A is a subset of X then $x \in \overline{A}$ if and only if there is a sequence $(x_n)_n$ in A that converges to x. However, such fact does not hold for general topological spaces. Actually, several properties of sequences in metric spaces break when we generalize those in the arbitrary topological spaces. This makes a reason introducing an object which acts like a sequence more general sense. There are two things that generalize sequences: filters, and nets. In here, we are going to study about filters. Net theory is introduced in the appendix. Also, one can see the filter theory at [1]

First, we need to see what are filters. Note that filters are defined even if X has no topology.

Definition 1.3.1 (Definition of filters)
Let X be a nonempty set. A *filter* $\mathcal{F}$ is a nonempty collection of some subsets of X satisfies the following conditions.

(i) $\varnothing \notin \mathcal{F}$;

(ii) If $A \in \mathcal{F}$, then $B \in \mathcal{F}$ for every $B \supset A$;

(iii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Clearly $\{X\}$ is a filter in X, so there exists a filter for any X. In addition, $\{X\}$ is the smallest filter we can ever make. Then, is there exists a maximal filter in X with respect to the set theoretical inclusion? Below proposition shows that indeed such filter exists, so this proposition justifies that Definition 1.3.3 is well-defined.

Proposition 1.3.2
Let X be a nonempty set and $\mathcal{F}$ be a filter in X. Then there exists a maximal filter $\mathcal{U} \supset \mathcal{F}$ in X meaning that there is no filter $\mathcal{G}$ in X which contains $\mathcal{U}$.

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proof. Fix a filter $\mathcal{F}$ in X and define an ordered set

$$\mathscr{C} := \{\mathcal{B} \supset \mathcal{F} : \mathcal{B} \text{ is a filter in } X\}$$

with an inclusion order. Then by the Hausdorff maximal theorem, there exists a maximal total ordered subset $\mathscr{D}$ of $\mathscr{C}$. Claim that $\bigcup \mathscr{D}$ is a filter and $\bigcup \mathscr{D} \supset \mathcal{F}$. Notice that the second one is trivial, so we are going to focus the first assertion.
Clearly $\varnothing \notin \bigcup \mathscr{D}$ and if $E \in \bigcup \mathscr{D}$ and $F \supset E$, then $F \in \bigcup \mathscr{D}$. If $E_1, E_2 \in \bigcup \mathscr{D}$, we can two filters $\mathcal{F}_1, \mathcal{F}_2 \in \mathscr{D}$ such that $E_1 \in \mathcal{F}_1$ and $E_2 \in \mathcal{F}_2$, respectively. Since $\mathscr{D}$ is a chain, it suffices to assume that $\mathcal{F}_1 \subset \mathcal{F}_2$. In this case, we have $E_1 \cap E_2 \in \mathcal{F}_2 \subset \bigcup \mathscr{D}$, proving that $\bigcup \mathscr{D}$ is a filter.
Notice that $\bigcup \mathscr{D} \in \mathscr{C}$ so $\bigcup \mathscr{D}$ is the desired one, otherwise we can take a filter $\mathcal{G} \supset \bigcup \mathscr{D}$ but this makes a contradiction of the maximality of $\mathscr{D}$. $\square$

Definition 1.3.3 (Definition of ultrafilters)
Let X be a set. An *ultrafilter* $\mathcal{U}$ in X is a maximal filter.

As we know that such definition is quite difficult to draw some image what ultrafilters look like. There is a equivalent statement that describes ultrafilters.

Proposition 1.3.4
Let X be a nonempty set and $\mathcal{U}$ be a filter in X. Then $\mathcal{U}$ is an ultrafilter if and only if for any $A \subset X$, either A or $X \setminus A$ belongs to $\mathcal{U}$.

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proof. To prove the necessity, assume that there exists a subset $A \subset X$ such that both A and $X \setminus A$ are not in $\mathcal{U}$. Then the collection

$$\mathcal{U}' := \mathcal{U} \cup \{A \cap E : E \in \mathcal{U}\}$$

is a filter which contains $\mathcal{U}$ by Exercise 1.3.5. This makes a contradiction because $\mathcal{U}$ is an ultrafilter.
Conversely, suppose that $\mathcal{U}$ is a filter which satisfies the property in above. If $\mathcal{U}$ is not maximal, then there exists an ultrafilter $\mathcal{U}' \supset \mathcal{U}$ by Proposition 1.3.2. We are going to finish the proof by showing that $\mathcal{U} = \mathcal{U}'$.
Fix $A \in \mathcal{U}'$. By the property of $\mathcal{U}$, either A or $X \setminus A$ is contained in $\mathcal{U}$. In here, if

$X \setminus A \in \mathcal{U}$ then since $\mathcal{U} \subset \mathcal{U}'$, we have $X \setminus A \in \mathcal{U}'$ hence

$$\varnothing = A \cap (X \setminus A) \in \mathcal{U}'$$

because $\mathcal{U}'$ is a filter, but this is nonsense. This derives to have $A \in \mathcal{U}$, therefore $\mathcal{U}' \subset \mathcal{U}$. This finishes the proof. □

Exercise 1.3.5*
Prove that the collection $\mathcal{U}'$ defined in Proposition 1.3.4 is a filter.

Remark 1.3.6
If $\mathcal{U}$ is an ultrafilter in X for some nonempty set X, clearly both A and $X \setminus A$ do not lie in $\mathcal{U}$ for any $A \subset X$, otherwise by the definition of the filter,

$$\varnothing = A \cap (X \setminus A) \in \mathcal{U},$$

which is a contradiction.

We have learned what is a filter and what is an ultrafilter. To deal with filters, we need to know how we can make some appropriate filter. If X be a nonempty set and $\mathcal{C}$ a some collection of subsets of X, then we can generate a filter $\mathcal{F}$ in X which contains $\mathcal{C}$.

Definition 1.3.7 (Finite intersection property)
Let X be a set. A collection $\mathcal{C}$ of subsets of X has a *finite intersection property* if for any finite set $\mathcal{F} \subset \mathcal{C}$, the intersection $\bigcap \mathcal{F}$ is not empty. We also use the abbriviation FIP for the term "finite intersection property".

Proposition 1.3.8
Let X be a nonempty set and $\mathcal{C}$ a collection of subsets of X that has FIP. Then there exists a filter $\mathcal{F} \supset \mathcal{C}$ in X.

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proof. We can find one of such filter explicitly:

$$\mathcal{F} = \left\{ E \subset X : E \supset \bigcap \mathcal{D},\ \mathcal{D} \text{ a finite subset of } \mathcal{C} \right\}. \tag{1.1}$$

Since $\mathcal{C}$ has FIP, the emptyset does not contained in $\mathcal{F}$. It is clear that both $E \subset \mathcal{F}$ and $F \supset E$ imply $F \in \mathcal{F}$ by the definition of $\mathcal{F}$.
Fix $E_1, E_2 \in \mathcal{F}$ and take finite subsets $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{C}$ such that $\bigcap \mathcal{D}_1 \subset E_1$ and $\bigcap \mathcal{D}_2 \subset E_2$, respectively. By the basic set theory knowledge, we obtain $E_1 \cap E_2 \supset \bigcap(\mathcal{D}_1 \cup \mathcal{D}_2)$ and notice that $\mathcal{D}_1 \cup \mathcal{D}_2$ is a finite subset of $\mathcal{C}$, implying that $E_1 \cap E_2 \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a filter. It is clear by definition that $\mathcal{F} \supset \mathcal{C}$. □

Sometimes, we will deal with not only filters but their subsets which acts like a filter. Thanks to Proposition 1.3.8 we can consider the following objects.

Definition 1.3.9 (Filter base)
Let X be a set. A *filter base* $\mathcal{B}$ is a nonempty collection of subsets of X such that

(a) If $A, B \in \mathcal{B}$, then there exists $N \in \mathcal{B}$ such that $N \subset A \cap B$.

(b) $\varnothing \notin \mathcal{B}$.

If $\mathcal{B}$ is a filter base, then the filter $\mathcal{F}$ generated by eq. (1.1) substituted of $\mathcal{B}$ for $\mathcal{C}$ is called *generated filter* from $\mathcal{B}$.

Remark 1.3.10
Clearly every filter base satisfies FIP, so the generated filter is well-defined by Proposition 1.3.8. Every filter is clearly a filter base and the generated filter is itself. By the property (a) in Definition 1.3.9, eq. (1.1) and the below filter are identical.

$$\mathcal{F} = \{E \subset X : E \supset F,\ F \in \mathcal{B}\}.$$

After this definition, we will see why sequences can replaced into filters in the general topology theories. This definition explains the way how filters are transformed by functions.

Definition 1.3.11 (Pushforward and pullback of filters)
Let X and Y be nonempty sets and $f : X \to Y$ a function. If $\mathcal{F}$ is a filter in X, the filter

$$f_*\mathcal{F} := \{E \supset f[F] : F \in \mathcal{F}\}$$

is called a *pushforward of a filter* $\mathcal{F}$. If $\mathcal{B}$ is a filter base in X, then the filter base

$$f_*\mathcal{B} := \{f[F] : F \in \mathcal{B}\}$$

is called a *pushforward of a filter base* $\mathcal{B}$.
If f is surjective and $\mathcal{F}$ is a filter in Y, the filter

$$f^*\mathcal{F} := \{E \supset f^{-1}[F] : F \in \mathcal{F}\}$$

is called a *pullback of a filter* $\mathcal{F}$. If $\mathcal{B}$ is a filter base and the surjectivity of f may drop, then the filter base

$$f^*\mathcal{B} := \{f^{-1}[F] : F \in \mathcal{B}\}$$

is called a *pullback of a filter base* $\mathcal{B}$.

The reader should check that Definition 1.3.11 makes sense.

Exercise 1.3.12
Prove that $f_*\mathcal{F}$ and $f^*\mathcal{F}$ defined in Definition 1.3.11 are really a filter, respectively. Furthermore, show that $f_*\mathcal{B}$ and $f^*\mathcal{B}$ are really a filter base, respectively.

Remark 1.3.13
If the surjectivity is dropped, then $f^*\mathcal{F}$ might not be a filter if $\mathcal{F}$ is a filter in Y.

The pushforward of a filter has a remarkable property: every pushforward of ultrafilters is again an ultrafilter. Notice that such property does not hold for pullbacks in general.

Proposition 1.3.14
Let X, Y be nonempty sets, $f : X \to Y$ a function, and $\mathcal{U}$ an ultrafilter in X. Then $f_*\mathcal{U}$ is an ultrafilter in Y.

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proof. Since $f_*\mathcal{U}$ is a filter, it suffices that it satisfies the sufficiency condition of Proposition 1.3.4. Let $A \subset Y$ be fixed. Since $\mathcal{U}$ is an ultrafilter, by Proposition 1.3.4, either

$f^{-1}[A] \in \mathcal{U}$ or $X \setminus f^{-1}[A] \in \mathcal{U}$ holds. As $X \setminus f^{-1}[A] = f^{-1}[Y \setminus A]$, we can assume that $f^{-1}[A] \in \mathcal{U}$. In this case, since $A \supset f[f^{-1}[A]]$ and $f[f^{-1}[A]] \in f_*\mathcal{U}$, we therefore have $A \in f_*\mathcal{U}$. The proof of this theorem is now completed by applying Proposition 1.3.4. □

Now let X be a topological space. Then we can define a convergence of filters. Here is the definition.

Definition 1.3.15 (Convergence of filters)
Let X be a topological space. A filter $\mathcal{F}$ in X *converges* to $x \in X$ if every open neighborhood of x belongs to $\mathcal{F}$. In this case, we write $\mathcal{F} \to x$. A filter base $\mathcal{B}$ in X *converges* to $x \in X$ if the generated filter converges to x. In this case, we write $\mathcal{B} \to x$.

Below lemma tells us the equivalent statement of the convergence of a filter base. It will be useful to test the convergence.

Lemma 1.3.16
Let X be a topological space and $\mathcal{B}$ is a filter base in X. Then $\mathcal{B} \to x \in X$ if and only if for every open neighborhood U of x, there exists $V \subset \mathcal{B}$ such that $V \subset U$.

- -

proof. If $\mathcal{B} \to x$, then by the definition the generated filter $\mathcal{F}$ converges to x, and this means $\mathcal{N}_x$ is contained in $\mathcal{F}$. Then each $U \in \mathcal{N}_x$ associates an element $V \subset \mathcal{B}$ such that $V \subset U$ by the definition of the generated filter.
Conversely, if $\mathcal{B} \not\to x$ and $\mathcal{F}$ a generated filter of $\mathcal{B}$, there is an open neighborhood U of x that does not belongs to $\mathcal{F}$. Again, by the definition of the generated filter, this implies that $V \not\subset U$ for every $V \in \mathcal{B}$. □

Remark 1.3.17
One can ask about that if $x \in X$ is fixed then does there exist a filter converges to x. Let $\mathcal{N}$ be a collection of all open neighborhoods of x. Then one can easily checked that $\mathcal{N}$ is a filter base and $\mathcal{N} \to x$.

We are ready to see that why filters can be used instead of sequences in arbitrary topological spaces. The theorem uses a filter bases instead of filters. But recall that every filter is a filter base.

Theorem 1.3.18
Let X, Y be topological spaces. Then

(a) For any nonempty set $A \subset X$, $x \in \mathrm{cl}A$ if and only if there is a filter base $\mathcal{B}$ such that $A \in \mathcal{B}$ and $\mathcal{B} \to x$.

(b) Let $f : X \to Y$ be a function. Then f is continuous at $x \in X$ if and only if for every filter base $\mathcal{B}$ in X with $\mathcal{B} \to x$, $f_*\mathcal{B} \to f(x)$ holds.

- -

proof. (a) If $x \in \mathrm{cl}A$, then $N \cap A \neq \varnothing$ for every open neighborhood N of x. Now consider

$$\mathcal{C} := \{A\} \cup \mathcal{N}_x.$$

Then notice that $\mathcal{C}$ satisfies FIP because every finite intersection of open sets is also open. Thus Proposition 1.3.8 gives a filter $\mathcal{F}$ that contains $\mathcal{C}$. Observe that by the definition of $\mathcal{C}$, $A \in \mathcal{F}$ and $\mathcal{F} \to x$. Finally, recall that $\mathcal{F}$ is clearly a filter base.

Conversely, suppose that there is a point $x \in X$ and a filter base $\mathcal{B}$ in X such that $A \in \mathcal{B}$ and $\mathcal{B} \to x$. In order to show $x \in \mathrm{cl}A$, it suffices to show that $N \cap A$ is not empty for every open neighborhood N of x. For fixed $N \in \mathcal{N}_x$, Lemma 1.3.16 gives a set $V \in \mathcal{B}$ such that $V \subset N$. Since $\mathcal{B}$ is a filter base and $A \in \mathcal{B}$, we have

$$\varnothing \neq A \cap V \subset A \cap N.$$

This proves $x \in \mathrm{cl}A$ as N can be any open neighborhood of x.

(b) Let f be continuous at $x \in X$ and $\mathcal{B}$ a filter base in X that converges to x. Fix any open neighborhood N of $f(x)$. Being continuous, there exists an open set U containing x such that $f[U] \subset N$. Notice that there exists $V \in \mathcal{B}$ such that $V \subset U$ by Lemma 1.3.16. Then we have

$$N \supset f[U] \supset f[V].$$

By the definition of the pushforward of a filter base, $f[V] \in f_*\mathcal{B}$. Hence Lemma 1.3.16 shows that $f_*\mathcal{B} \to f(x)$.

Conversely, suppose that f is not continuous at x. Then there exists an open set $W \ni f(x)$ such that no open neighborhood of U of x satisfies $f[U] \subset W$. Claim that $f_*\mathcal{N}_x \not\to f(x)$. Since clearly $\mathcal{N}_x \to x$, it suffices to prove the claim to finish the proof.

By Lemma 1.3.16, in order that $f_*\mathcal{N}_x \to f(x)$, there should exist an element $E \in f_*\mathcal{N}_x$ such that $E \subset W$. Notice that every element of $f_*\mathcal{N}_x$ is the type

$$f[U], \quad \forall U \in \mathcal{N}_x.$$

However we just take an open set W such that no open neighborhood U of x satisfies $f[U] \subset W$. This proves the claim and therefore the theorem. □

If X is a metric space and $A \subset X$, then we know that $x \in \mathrm{cl}A$ if and only if there exists a sequence $(x_n)_n$ in A that converges to x as discussed in the introduction of this section. In Theorem 1.3.18, $A \in \mathcal{F}$ corresponds to the sentence $(x_n)_n \subset A$, and clearly $\mathcal{F} \to x$ means $x_n \to x$ as $n \to \infty$. In addition, statement Theorem 1.3.18 (ii) is quite clear to explain whether a function is continuous at x by considering the assertion related to a sequence.

Furthermore, note that on a Hausdorff space, every sequence converges at most one point. This is also true for filters, and we can say more.

Theorem 1.3.19
Let X be a topological space. Then X is Hausdorff if and only if every filter base converges at most one point. That is, if a filter base $\mathcal{B}$ in X converges in both $x \in X$ and $y \in X$, then $x = y$.

- -

proof. Let X be Hausdorff and assume that there exists a filter base $\mathcal{B}$ converges to different points x and y. Take two disjoint open sets U and V such that $x \in U$ and $y \in V$. Since $\mathcal{B} \to x$, by Lemma 1.3.16, there exists $W_1 \in \mathcal{B}$ such that $W_1 \subset U$. Similarly, as $\mathcal{B} \to y$, we can take $W_2 \in \mathcal{B}$ satisfies $W_2 \subset V$. Since $\mathcal{B}$ is a filter base,

$W_1 \cap W_2$ cannot be empty. However, we find

$$W_1 \cap W_2 \subset U \cap V = \varnothing,$$

which is a contradiction. Hence, there is no such filter base if X is Hausdorff.
Conversely, if X is not Hausdorff, then we can take two distict points x and y such that there is no pair (U, V) of open sets such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. Then define

$$\mathcal{B} := \{F \cap G : F \in \mathcal{N}_x,\ G \in \mathcal{N}_y\}$$

By the choice of x and y, $\mathcal{B}$ does not have an emptyset as its element, so it is easy to show that $\mathcal{B}$ is a filter base. Notice by the definition of $\mathcal{B}$ that $\mathcal{B} \to x$ and $\mathcal{B} \to y$. This proves the theorem. □

1.4 Compact sets

In this section, we will learn about compact sets in arbitrary topological spaces.

Definition 1.4.1 (Compact sets)
Let X be a topological space. Then a set X is said to be *compact* if for every collection $\mathscr{U}$ of open sets which cover X, we can find a finite subcollection $\mathscr{F} \subset \mathscr{U}$ such that $\bigcup \mathscr{F} \supset X$. The collection of type $\mathscr{U}$ described in above is called an *open cover of* X. A subset $K \subset X$ is said to be compact if K is compact within the subspace topology. A set is said to be *precompact* if the closure of which is compact.

We can show that the space is compact by using filters. Here is the theorem about.

Theorem 1.4.2
Let X be a topological space. Then X is compact if and only if every ultrafilter in X converges.

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proof. Suppose that there is an ultrafilter $\mathcal{U}$ that converges nowhere in X. For each $x \in X$, take an open set U_x which is not contained in $\mathcal{U}$. Note that clearly $\{U_x\}_{x \in X}$ is an open cover of X. If X were compact, there are finitely many points $x_1, \cdots, x_n$ such that

$$U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n} \supset X. \tag{1.2}$$

However as $\mathcal{U}$ is an ultrafilter, $X \setminus U_x \in \mathcal{U}$ holds for each $x \in X$ by Proposition 1.3.4. This and eq. (1.2) gives

$$\varnothing = (X \setminus U_{x_1}) \cap (X \setminus U_{x_2}) \cap \cdots \cap (X \setminus U_{x_n}) \in \mathcal{U},$$

which is a contradiction. Such contradiction occurs as we assumed that X is compact. Hence X cannot be compact.
Conversely, suppose that X is not compact. Then there is an open cover $\mathscr{U}$ which does not have a finite subset of $\mathscr{U}$ that covers X. Put

$$\mathscr{V} := \{X \setminus E : E \in \mathscr{U}\}.$$

Observe that $\mathscr{V}$ satisfies FIP. Take an ultrafilter $\mathcal{U}$ which is a superset of $\mathscr{V}$ by Proposition 1.3.2 and Proposition 1.3.8. Since $\mathscr{V} \subset \mathcal{U}$ and $\mathscr{U}$ is an open cover of X, there is no

$x \in X$ such that $\mathcal{U} \to x$. Indeed, if $x \in X$ then there is $U \in \mathscr{U}$ that contains x. Since $X \setminus U \in \mathcal{U}$, U does not belong to $\mathcal{U}$ and this implies that $\mathcal{U}$ does not converges to x. $\square$

Corollary 1.4.3
Let X be a topological space. Then $\varnothing \neq K \subset X$ is compact if and only if every ultrafilter $\mathcal{U}$ in X containing K converges to some point in K. Notice that the convergent point can differ.

- -

proof. Suppose that $K \subset X$ is compact, and fix an ultrafilter $\mathcal{U}$ that contains K. Then the set

$$\mathcal{U}' := \{F \cap K : F \in \mathcal{U}\} \tag{1.3}$$

is an ultrafilter in K. Being compact, $\mathcal{U}' \to x \in K$ by Theorem 1.4.2. Now fix an open set N that contains x. Since $N \cap K$ is open in K and $x \in N \cap K$, we have $N \cap K \in \mathcal{U}'$. This implies $N \in \mathcal{U}$ by the definition of $\mathcal{U}'$, proving that $\mathcal{U} \to x$ since N is arbitrary.
To prove the sufficency, claim that if $\mathcal{U}'$ is any ultrafilter in K, then there exists an ultrafilter $\mathcal{U}$ in X such that both $\mathcal{U}$ and $\mathcal{U}'$ satisfies eq. (1.3). But indeed $\mathcal{U}'$ is a filter in X. By Proposition 1.3.2, there exists an ultrafilter $\mathcal{U}$ which contains $\mathcal{U}'$. The one side of eq. (1.3) is clear. Take any $F \in \mathcal{U}$. Since $\mathcal{U}' \subset \mathcal{U}$, F and K are intersect. In addition, as $\mathcal{U}'$ is an ultrafilter in K, either $F \cap K$ or $K \setminus F$ belongs to $\mathcal{U}'$ by Proposition 1.3.4. However if $K \setminus F \in \mathcal{U}'$, we have

$$\varnothing = (K \setminus F) \cap F \in \mathcal{U},$$

which is a contradiction. This shows that $F \cap K \in \mathcal{U}'$, so both $\mathcal{U}$ and $\mathcal{U}'$ satisfies eq. (1.3). From this fact, we are ready to prove that K is compact provided that every ultrafilter in X containing K converges to some point in K. Fix any ultrafilter $\mathcal{U}'$ in K and take an ultrafilter $\mathcal{U}$ in X satisfies eq. (1.3). By our assumption, $\mathcal{U}$ converges to some $x \in K$. By eq. (1.3) and the definition of the subspace topology, it is easy to verify that $\mathcal{U}' \to x$. Therefore, K is compact because of Theorem 1.4.2. $\square$

Below propositions can be proved without using any filter theory. In here, however, we will see how Theorem 1.4.2 (and Corollary 1.4.3) can be used to prove several compactness properties. One may solve Exercise 1.4.5.

Proposition 1.4.4
Let X be a topological space and $K \subset X$ be a nonempty set.

(a) If X is Hausdorff and K is compact, then K is closed.

(b) If X is compact and K is closed, then K is compact.

(c) Let Y be another topological space and $f : X \to Y$ a continuous map, then $f[K]$ is compact in Y whenever K is compact in X.

- -

proof. (a) Fix $x \in \operatorname{cl}K$ and take a filter $\mathcal{F}$ in X such that $K \in \mathcal{F}$ and $\mathcal{F} \to x$ by Theorem 1.3.18. Then take an ultrafilter $\mathcal{U} \supset \mathcal{F}$ by Proposition 1.3.2, and finally, consider

$$\mathcal{U}' := \{F \cap K : F \in \mathcal{U}\}.$$

Since $\mathcal{U}'$ is an ultrafilter in K, there is a point $y \in K$ such that $\mathcal{U}' \to y$ by Corollary 1.4.3. Fix any open set N in X that contains y. Then $N \cap K$ is open in K, we have $N \cap K \in \mathcal{U}'$ as $\mathcal{U}' \to y$. Then the definition of $\mathcal{U}'$ and Proposition 1.3.4 imply $N \in \mathcal{U}$ (otherwise, $X \setminus N$ must be in $\mathcal{U}$. This makes a contradiction), hence $\mathcal{U} \to y$.

What we have is that the ultrafilter $\mathcal{U}$ converges in both x and y where $x \in \mathrm{cl}K$ and $y \in K$. Since X is Hausdorff, $x = y$ by Theorem 1.3.19, proving that $x = y \in K$. Hence, K is closed because $K = \mathrm{cl}K$ which is what we proved in here.

(b) Fix an ultrafilter $\mathcal{U}$ in X which contains K. Being compact, $\mathcal{U}$ converges to some point $x \in X$. By Theorem 1.3.18, $x \in \mathrm{cl}K = K$. Hence, K is compact because of Corollary 1.4.3 as $\mathcal{U}$ is arbitrary.

(c) Fix an ultrafilter $\mathcal{U}$ in $f[K]$. Then take an ultrafilter $\mathcal{U}'$ in K such that $f^*\mathcal{U} \subset \mathcal{U}'$. Being compact, we can take a point $x \in K$ such that $\mathcal{U}' \to x$ by Theorem 1.4.2. Since f is continuous, $f_*\mathcal{U}' \to f(x)$ because of Theorem 1.3.18. Observe that $\mathcal{U} \subset f_*\mathcal{U}'$, hence both should be same since $\mathcal{U}$ is an ultrafilter. Therefore, $\mathcal{U}$ converges to $f(x)$, and we proved that $f[K]$ is compact by Theorem 1.4.2 as $\mathcal{U}$ is arbitrary. $\square$

Exercise 1.4.5
Prove Proposition 1.4.4 *without using any filter theories*.

1.5 Product spaces

If $\{X_\alpha\}_{\alpha\in A}$ is a collection of topological spaces, we have defined their product $X = \prod_{\alpha\in A} X_\alpha$ but yet we do not know what kind of topology it has naturally. For each $\alpha \in A$ consider the function $\pi_\alpha : X \to X_\alpha$ defined by $\pi_\alpha(x) = x(\alpha)$.[1] Since each X_α equipped a topology, we want to give a topology on X such that every map π_α is continuous. Thus the definition of the product topology is following.

Definition 1.5.1 (Product topology)
Let A be a nonempty set and

Theorem 1.5.2 (Tychonoff's Theorem)
Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of compact topological spaces where A is an index set. Then the space $X = \prod_{\alpha\in A} X_\alpha$ equipped with the product topology is compact.

- -

proof. Let us define a projection $\pi_\alpha : X \to X_\alpha$ for each $\alpha \in A$, and fix an ultrafilter $\mathcal{U}$ in X. By Proposition 1.3.14, $(\pi_\alpha)_*\mathcal{U}$ is an ultrafilter in X_α, so we can take a point $x_\alpha \in X_\alpha$ such that $(\pi_\alpha)_*\mathcal{U} \to x_\alpha$. Such points exists because X_α is compact and recall Theorem 1.4.2. Finally, take the point $x \in X$ such that $\pi_\alpha(x) = x_\alpha$ for every $\alpha \in A$. Claim that $\mathcal{U} \to x$.
To prove this, fix an open neighborhood N of x in X. Since

$$\{\pi_\alpha^{-1}[U] : \alpha \in A,\ U \in X_\alpha\}$$

forms a subbasis of X, there exists $\alpha_1, \cdots, \alpha_n \in A$, and $U_{\alpha_1}, \cdots, U_{\alpha_n}$ are open sets in

[1] Recall the definition of the product of sets.

X_{α_j}, $j = 1, 2, \cdots, n$, respectively, such that

$$x \in \pi_{\alpha_1}^{-1}[U_{\alpha_1}] \cap \pi_{\alpha_2}^{-1}[U_{\alpha_2}] \cap \cdots \cap \pi_{\alpha_n}^{-1}[U_{\alpha_n}] \subset N.$$

Notice that $x_{\alpha_j} \in U_{\alpha_j}$ for $j = 1, 2, \cdots, n$. This gives that $U_{\alpha_j} \in (\pi_\alpha)_* \mathcal{U}$, hence

$$\pi_{\alpha_j}^{-1}[U_{\alpha_j}] \in \pi_\alpha^* (\pi_\alpha)_* \mathcal{U} \subset \mathcal{U}.$$

Thus,

$$\pi_{\alpha_1}^{-1}[U_{\alpha_1}] \cap \pi_{\alpha_2}^{-1}[U_{\alpha_2}] \cap \cdots \cap \pi_{\alpha_n}^{-1}[U_{\alpha_n}] \in \mathcal{U} \quad \Longrightarrow \quad N \in \mathcal{U}.$$

Since N is arbitrary open neighborhood of x, we have proved that $\mathcal{U} \to x$. Therefore, X is compact because of Theorem 1.4.2. $\square$

Chapter 2

Measures and Integrations

2.1 Introduction

What is the definition of the "area" of some figures? In the *real* kindergarten, the reader is taught about the area of a rectangle is calculated by a multiple of the width and the height of the rectangle. In the high school, the reader learn how to calculate the circle by approximating areas of rectangles which fill the whole circle from inside. Generally speaking, one can infer that every figure has an area and it can be calculated with similar mannar.

In our intuition, a length —an area in $\mathbb{R}$— is translation invariant, and it should be in our real life. Let us denote ℓ a length of sets in $\mathbb{R}$ and assume that we can take a length from arbitrary sets in $\mathbb{R}$. For instance, $\ell([a, b]) = b - a$ and such definition comes from our intuition.

The Entire settings of this chapter is heavly inspired from [3].

2.2 Measures and measurable sets

In this section, we will learn about measures and their properties. First two definitions introduce what is a measure and what is a measurable set.

Definition 2.2.1 (Measures and measure spaces)
Let X be a nonempty set. A *measure* μ on X is a function $\mu : \mathcal{P}(X) \to [0, \infty]$ such that

(i) $\mu(\varnothing) = 0$;

(ii) $\mu(A) \leq \mu(B)$ if $A \subset B$;

(iii) $\mu(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu(A_j)$.

We say the pair (X, μ) a *measure space*.

Below exercise is almost trivial, so we use this fact without mention it. However, we should prove the following property because it is not implied from the definition directly.

Exercise 2.2.2*
Let X be a nonempty set and μ a measure on X. Then for every positive integer n,

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) \leq \sum_{j=1}^{n} \mu(A_j).$$

Now we are going to define a term "measurable" of sets.

Definition 2.2.3 (Measurable sets)
Let (X, μ) be a measure space. A set $A \subset X$ is called *μ-measurable* if for every set $E \subset X$, the following relation holds.

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A).$$

Define $\mathfrak{M}_\mu$ the collection of all μ-measurable sets.

Definition 2.2.4 (Null set)
Let (X, μ) be a measure space. A subset $E \subset X$ is called a μ-null set or just null set in short if $\mu(E) = 0$.

Exercise 2.2.5*
On a measure space (X, μ), $A \in \mathfrak{M}_\mu$ if and only if for every $E \subset X$ with $\mu(E) < \infty$,

$$\mu(E) \geq \mu(E \cap A) + \mu(E \setminus A).$$

Remark 2.2.6
Actually, the object μ defined in Definition 2.2.1 is so called an *outer measure* in several references, and reserve the term *measure* for different manner. Furthermore, the term *measure space* stands for a triple $(X, \mathfrak{M}, \mu)$ where $\mathfrak{M}$ is so-called σ-field (definition of which is introduced in Definition 2.2.7) and μ a measure (not an outer measure). Although there is some confusion while reading this textbook, it has some benefits when we define terms *measure* and *measure space* by Definition 2.2.1. So reader sould beware from misunderstanding.

So far we have defined a measure μ and the collection of μ-measurable sets $\mathfrak{M}_\mu$ on a set X. Below definition gives a term which $\mathfrak{M}_\mu$ will satisfy, and the theorem proves it.

Definition 2.2.7
Let X be a nonempty set. A subcollection $\mathfrak{M} \subset \mathcal{P}(X)$ is called a *σ-field* if

(i) $\varnothing, X \in \mathfrak{M}$.

(ii) If $A \in \mathfrak{M}$, then $X \setminus A \in \mathfrak{M}$.

(iii) If $\{A_n\}_{n=1}^\infty \subset \mathfrak{M}$ is a countable subcollection, then $\bigcup_{n=1}^\infty A_n \in \mathfrak{M}$. Notice that $\{A_n\}_n$ need not to be mutually disjoint.

If the condition *countable* is replaced by *finite* in (iii), then the collection $\mathfrak{M}$ is called by a *field*.

Remark 2.2.8
Terms *field* and *σ-field* are usually called by probabilists. In other people who study analysis, *algebra* and *σ-algebra* is used, repectively.
However, personally, the former terms is the *right* term in the sense that *fields* (resp. *σ-fields*) are closed under all set theoretical operations $\cup$, $\cap$, $\setminus$, and the complement $\cdot^c$.

Exercise 2.2.9*
Let X be a nonempty set. A subcollection $\mathfrak{M} \subset \mathcal{P}(X)$ satisfies

(i) $\varnothing, X \in \mathfrak{M}$.

(ii) If $A \in \mathfrak{M}$, then $X \setminus A \in \mathfrak{M}$.

(iii) If $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{M}$ is a *mutually disjoint* countable subcollection, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{M}$.

Then prove that $\mathfrak{M}$ is a σ-field.

Theorem 2.2.10 (Carathéodory's Theorem)
Let (X, μ) be a measure space. Then $\mathfrak{M}_\mu$ is a σ-field. Furthermore, if $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{M}_\mu$ is a *mutually disjoint* countable collection, then we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

- -

proof. Obviously, $\varnothing$ and X belongs to $\mathfrak{M}_\mu$. If $A \in \mathfrak{M}_\mu$, then for every $E \subset X$, we have

$$\mu(E \cap (X \setminus A)) + \mu(E \setminus (X \setminus A)) = \mu(E \setminus A) + \mu(E \cap A) = \mu(E),$$

so $X \setminus A \in \mathfrak{M}_\mu$.
Now, we are going to prove the following: For $A, B \in \mathfrak{M}_\mu$ with $A \cap B = \varnothing$, then $A \cup B = \mathfrak{M}_\mu$ and for any $E \subset X$,

$$\mu(E \cap (A \cup B)) = \mu(E \cap A) + \mu(E \cap B). \tag{2.1}$$

Since $A \in \mathfrak{M}_\mu$ and both A, B are disjoint, by the definition of measurable sets, we have

$$\begin{aligned}\mu(E \cap (A \cup B)) &= \mu((E \cap A) \cup (E \cap B)) \\ &= \mu([(E \cap A) \cup (E \cap B)] \cap A) + \mu([(E \cap A) \cup (E \cap B)] \setminus A) \\ &= \mu(E \cap A) + \mu(E \cap B).\end{aligned}$$

We will use Exercise 2.2.5 to show $A \cup B \in \mathfrak{M}_\mu$. Fix any $E \subset X$ with $\mu(E) < \infty$. Then since A, B are μ-measurable, and A and B are disjoint,

$$\begin{aligned}\mu(E) &\geq \mu(E \cap A) + \mu(E \setminus A) \\ &\geq \mu(E \cap A) + \mu((E \setminus A) \cap B) + \mu((E \setminus A) \setminus B) \\ &= \mu(E \cap A) + \mu(E \cap (B \setminus A)) + \mu(E \setminus (A \cup B)) \\ &= \mu(E \cap A) + \mu(E \cap B) + \mu(E \setminus (A \cup B)) \\ &\geq \mu(E \cap (A \cup B)) + \mu(E \setminus (A \cup B)).\end{aligned}$$

This proves $A \cup B \in \mathfrak{M}_\mu$.
To show the last condition of the σ-field, we will use Exercise 2.2.9 and prove the last statement at once. Fix $\{A_n\}_{n=1}^{\infty}$ be a mutually disjoint countable subset of $\mathfrak{M}_\mu$, and $E \subset X$ with $\mu(E) < \infty$. We already proved this case when the collection has only two sets. By the induction over the number of the sets, we find for every $n \geq 2$ that

$$\bigcup_{j=1}^{n} A_j \in \mathfrak{M}_\mu, \quad \text{and} \quad \sum_{j=1}^{n} \mu(E \cap A_j) = \mu\left(E \cap \bigcup_{j=1}^{n} A_j\right) \leq \mu\left(E \cap \bigcup_{j=1}^{\infty} A_j\right).$$

Therefore,

$$\sum_{j=1}^{\infty} \mu(E \cap A_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E \cap A_j) \le \mu\left(E \cap \bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu(E \cap A_j), \tag{2.2}$$

and we find

$$\begin{aligned}
\mu(E) &\ge \overline{\lim_{n\to\infty}} \left[\mu\left(E \cap \bigcup_{j=1}^{n} A_j\right) + \mu\left(E \setminus \bigcup_{j=1}^{n} A_j\right)\right] \\
&\ge \overline{\lim_{n\to\infty}} \left[\mu\left(E \cap \bigcup_{j=1}^{n} A_j\right) + \mu\left(E \setminus \bigcup_{j=1}^{\infty} A_j\right)\right] \\
&= \mu\left(E \setminus \bigcup_{j=1}^{\infty} A_j\right) + \overline{\lim_{n\to\infty}} \mu\left(E \cap \bigcup_{j=1}^{n} A_j\right) \\
&= \mu\left(E \setminus \bigcup_{j=1}^{\infty} A_j\right) + \overline{\lim_{n\to\infty}} \sum_{j=1}^{n} \mu(E \cap A_j) \\
&= \mu\left(E \setminus \bigcup_{j=1}^{\infty} A_j\right) + \sum_{j=1}^{\infty} \mu(E \cap A_j) \\
&= \mu\left(E \setminus \bigcup_{j=1}^{\infty} A_j\right) + \mu\left(E \cap \bigcup_{j=1}^{\infty} A_j\right).
\end{aligned}$$

This shows that $\bigcup_1^\infty A_n \in \mathfrak{M}_\mu$, and therefore $\mathfrak{M}_\mu$ is a σ-field by Exercise 2.2.9. Observe that eq. (2.2) can be obtained without the constraint $\mu(E) < \infty$. This yields the last equality by replacing E into X. □

Corollary 2.2.11
Let (X, μ) be a measure space. Then

(i) If A, B are μ-measurable such that $A \subset B$ and $\mu(B) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(ii) If A_n are μ-measurable such that

$$A_1 \subset A_2 \subset A_3 \subset \cdots,$$

then $\mu(\bigcup_1^\infty A_n) = \lim_{n\to\infty} \mu(A_n)$.

(iii) If A_n are μ-measurable such that

$$A_1 \supset A_2 \supset A_3 \supset \cdots,$$

and $\mu(A_1) < \infty$, then $\mu(\bigcap_1^\infty A_n) = \lim_{n\to\infty} \mu(A_n)$.

- -

proof. Define $A_0 = \varnothing$ and $B_n = A_n \setminus A_{n-1}$ for $n = 1, 2, 3, \cdots$. Then $\bigcup_1^\infty B_n = \bigcup_1^\infty A_n$ and the collection $\{B_n\}_1^\infty$ is mutually disjoint. Thus by applying Theorem 2.2.10,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

$$= \lim_{n\to\infty}\sum_{j=1}^{n}\mu(B_j) = \lim_{n\to\infty}\mu\left(\bigcup_{j=1}^{n}B_j\right) = \lim_{n\to\infty}\mu(A_n).$$

This proves (ii). To prove (iii), define $B_n = A_1 \setminus A_n$. Then $\bigcup_1^\infty B_n = A_1 \setminus \bigcap_1^\infty A_n$, and

$$B_1 \subset B_2 \subset \cdots.$$

Hence, by (ii), we obtain

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty}A_n\right) = \mu\left(\bigcup_{n=1}^{\infty}B_n\right) = \lim_{n\to\infty}\mu(B_n) = \lim_{n\to\infty}\mu(A_1\setminus A_n),$$

and since $\mu(A_1) < \infty$, we can apply (i) and this yields (iii).
(i) is trivial from Theorem 2.2.10 with the simple identity

$$B = A \cup (B \setminus A).$$

This proves the corollary. □

Corollary 2.2.12 (Borel-Cantelli's Lemma)
Let (X, μ) be a measure space and $A_1, A_2, \cdots$ be a sequence of μ-measurable subsets of X such that

$$\sum_{n=1}^{\infty}\mu(A_n) < \infty.$$

Then the set

$$A = \overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}A_m$$

is μ-null.

- -

proof. Since

$$\mu\left(\bigcup_{n=1}^{\infty}A_n\right) \le \sum_{n=1}^{\infty}\mu(A_n) < \infty,$$

we can apply Corollary 2.2.11 (iii) to obtain

$$\mu(A) = \lim_{n\to\infty}\mu\left(\bigcup_{m=n}^{\infty}A_m\right) = 0.$$

This proves the corollary. □

We will end this section for more informations about σ-fields. First of all, if $\mathcal{C}$ is a collection of some subsets of X we can generate a σ-field from it.

Definition 2.2.13
Let X be a set and $\mathcal{C}$ be a subset of $\mathcal{P}(X)$. Then the σ-field $\sigma(\mathcal{C})$ generated from $\mathcal{C}$ is defined by

$$\sigma(\mathcal{C}) := \bigcap\{\mathfrak{M} \subset \mathcal{P}(X) : \mathfrak{M} \supset \mathcal{C},\ \mathfrak{M} \text{ is a } \sigma\text{-field}\}.$$

That is, $\sigma(\mathcal{C})$ is the smallest σ-field which contains $\mathcal{C}$.

Exercise 2.2.14*
Let X be a nonempty set and $\mathcal{M}$ be a collection of some σ-fields on X. Then the intersection $\bigcap \mathcal{M}$ is also a σ-field. From this fact, $\sigma(\mathcal{C})$ defined in Definition 2.2.13 is really a σ-field.

Exercise 2.2.15*
Let X be a set, $\mathcal{C}$ be some collection of subsets of X and $\mathfrak{M}$ a σ-field such that $\mathcal{C} \subset \mathfrak{M}$. Prove that $\sigma(\mathcal{C}) \subset \mathfrak{M}$.

Exercise 2.2.16
Let X and Y be nonempty sets and $f : X \to Y$ a function. If $\mathfrak{M}$ is a σ-field on Y then the collection

$$f^{-1}[\mathfrak{M}] := \{f^{-1}[E] : E \in \mathfrak{M}\}$$

is a σ-field on X.

Exercise 2.2.17
Prove that every σ-field is either finite or uncountable.

2.3 Measurable functions

Before defining an integration, we need to say that what kind of functions are "measurable to integrate". There is a general way to define it, but we first focus for real-valued functions. Everywhere in this section, (X, μ) denotes a measure space. Furthermore, after this moment, we use the fact that $\mathfrak{M}_\mu$ is σ-field (because of Theorem 2.2.10) without any mention.

Definition 2.3.1 (Measurable functions)
Let Y be a topological space. We say that a function $f : X \to Y$ is *μ-measurable* if $\{f \in V\}$ is μ-measurable for every open set V in Y. Recall that $\{f \in V\}$ is an abbriviation of $\{x : f(x) \in V\}$.

Below proposition gives an equivalent statement of μ-measurable functions.

Proposition 2.3.2
Let $f : X \to \mathbb{R}$ be a function, Then the followings are equivalent.

(i) f is μ-measurable.

(ii) $\{f \in F\}$ is μ-measurable for every closed subset F of $\mathbb{R}$.

(iii) $\{f > a\}$ is μ-measurable for every $a \in \mathbb{R}$.

(iv) $\{f \leq a\}$ is μ-measurable for every $a \in \mathbb{R}$.

(v) $\{f < a\}$ is μ-measurable for every $a \in \mathbb{R}$.

(vi) $\{f \geq a\}$ is μ-measurable for every $a \in \mathbb{R}$.

(vii) $\{f \in I\}$ is μ-measurable for every interval $I \subset \mathbb{R}$.

(viii) $\{f \in I\}$ is μ-measurable for every open interval $I \subset \mathbb{R}$.

(ix) $\{f \in I\}$ is μ-measurable for every closed interval $I \subset \mathbb{R}$.

- -

proof. Since $\mathfrak{M}_\mu$ is a σ-field, (iii) and (iv), (v) and (vi) are automatically equivalent, respectively. Now observe that

$$\{f \geq a\} = \bigcap_{n=1}^{\infty} \{f > a - n^{-1}\}, \quad \{f > a\} = \bigcup_{n=1}^{\infty} \{f \geq a + n^{-1}\}.$$

As $\mathfrak{M}_\mu$ is a σ-field, upper relation yields that (iii) and (vi) are equivalent. Hence, (iii)-(vi) are all equivalent.
Notice that clearly all (iii)-(vi) imply (vii), and (vii) implies (viii) and (ix). In addition, since

$$\{f > a\} = \bigcup_{n=1}^{\infty} \{a < f < n\}, \quad \{f \geq a\} = \bigcup_{n=1}^{\infty} \{a \leq f \leq n\},$$

(viii) implies (iii), and (ix) implies (vi).
Every open set in $\mathbb{R}$ can be expressed by a countable union of open intervals in $\mathbb{R}$. Thus (viii) implies (i). Also, clearly (i) implies (viii). Finally, observe that clearly (i) and (ii) are equivalent. This finishes the proof. □

Exercise 2.3.4 is clear since $\mathfrak{M}_\mu$ is a σ-field. From Proposition 2.3.2 and Exercise 2.3.4, we obtain the analogous statement when the codomain is $[-\infty, \infty]$.

Proposition 2.3.3
Let $f : X \to [-\infty, \infty]$ be a function, Then the followings are equivalent.

(i) f is μ-measurable.

(ii) $\{f \leq a\}$ is μ-measurable for every $a \in [-\infty, \infty]$.

(iii) $\{f < a\}$ is μ-measurable for every $a \in (-\infty, \infty]$.

(iv) $\{f \geq a\}$ is μ-measurable for every $a \in [-\infty, \infty]$.

(v) $\{f \in I\}$ is μ-measurable for every interval $I \subset [-\infty, \infty]$.

(vi) $\{f \in I\}$ is μ-measurable for every open interval $I \subset [-\infty, \infty]$, and both $\{f = -\infty\}$ and $\{f = \infty\}$ are μ-measurable.

(vii) $\{f \in I\}$ is μ-measurable for every closed interval $I \subset [-\infty, \infty]$.

- -

proof. See Exercise 2.3.5. □

Exercise 2.3.4*
A function $f : X \to [-\infty, \infty]$ is μ-measurable if and only if $\{f > a\}$ is μ-measurable for every $a \in \mathbb{R}$ and $\{f = -\infty\}$ is μ-measurable.

Exercise 2.3.5*
Prove Proposition 2.3.3.

We already know that if f and g are real-valued continuous function on the real line, then both $f+g$ and fg are also continuous. Then one can question about for measurability. Similar for continuity, if f and g are real-valued μ-measurable functions, then both $f+g$ and fg are μ-measurable.

Proposition 2.3.6
Let $f, g : X \to \mathbb{R}$ are μ-measurable functions. Then $-f$, $f+g$ and fg are μ-measurable.

- -

proof. The measurability of $-f$ is immediately follows from Proposition 2.3.2. Fix $a \in \mathbb{R}$. Then observe that

$$\{f+g>a\} = \{f > a-g\} = \bigcup_{q\in\mathbb{Q}} \{f>q\} \cap \{g > a-q\}.$$

Since f and g are μ-measurable, and $\mathfrak{M}_\mu$ is a σ-field, that identity implies that $\{f+g>a\}$ is μ measurable. As $a \in \mathbb{R}$ is arbitrary, we can conclude that $f+g$ is μ-measurable.
To prove that fg is μ-measurable, we use the following formula

$$fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2.$$

Hence, it suffices to show that f^2 is μ-measurable. Fix $a \in \mathbb{R}$. If $a<0$ then clearly $\{f^2 > a\} = X$, so we can further assume that $a \geq 0$. In this case,

$$\{f^2 > a\} = \{|f| > \sqrt{a}\} = \{f \geq 0, f > \sqrt{a}\} \cup \{f < 0, f < -\sqrt{a}\},$$

and the right hand side set is indeed μ-measurable by Proposition 2.3.2. □

In the proof of Proposition 2.3.6, we actually proved that f^2 and $|f|$ are μ-measurable provided that f is μ-measurable. One can notice that from Proposition 2.3.2, we can prove the following result.

Proposition 2.3.7
Let Y and Z are topological spaces, $f : X \to Y$ be μ-measurable function, and $\phi : Y \to Z$ be a continuous function. Then $\phi \circ f$ is μ-measurable.

- -

proof. Fix any open set U in Z. Being continuous, $\phi^{-1}[U]$ is also open in Y. Therefore, since

$$\{\phi \circ f \in U\} = \{f \in \phi^{-1}[U]\},$$

we can conclude that $\phi \circ f$ is μ-measurable. □

We have been studied the definition of μ-measurable functions. We end this section by showing one of the most important theorem about μ-measurable functions. Before that we need a definition.

Definition 2.3.8 (Simple functions)
We say that *simple functions* $s : X \to \mathbb{R}$ is the type

$$s = \sum_{j=1}^{n} c_j 1_{A_j},$$

where $c_j \in \mathbb{R}$ and A_j are μ-measurable sets for $j = 1, \cdots, n$.

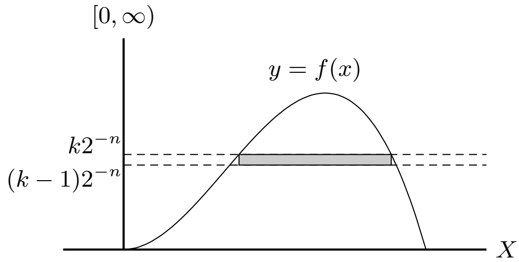

Figure 2.1: Definition of the Lebesgue integration

2.4 Integrations

Now we are ready to define a Lebesgue integration. We are going to define an integration using a measure. This section is inspired from [2]. Everywhere in this section, (X, μ) denotes a measure space.

First of all, we need to define the positive part and negative part of a function.

Definition 2.4.1
Let $f : X \to \mathbb{R}$ be a function (here, X need not to take a measure). We define the *positive part* f_+ and *negative part* f_- of a function f by

$$f_+ := \frac{|f| + f}{2}, \quad f_- := \frac{|f| - f}{2}.$$

Exercise 2.4.2
Let $f : X \to \mathbb{R}$ be a μ-measurable function. Then both f_+ and f_- are μ-measurable.

Now we are ready to define the Lebesgue integration. Let us fix a function $f : X \to [0, \infty)$. We want to define $\int_X f(x)\mu(dx)$ by the "area" of the set

$$\Gamma_f := \{(x, y) : y < f(x),\ x \in X\}.$$

Fix some positive integers n, k and consider a set $\{f > k2^{-n}\}$ where the set is nonempty. Observe that

$$\{f > k2^{-n}\} \times ((k-1)2^{-n}, k2^{-n}] \subset \Gamma_f.$$

Since X equipped a measure, we can "define" the area of the rectangle-like shape $\{f > k2^{-n}\} \times ((k-1)2^{-n}, k2^{-n}]$ by $2^{-n}\mu\{f > k2^{-n}\}$. Summing up all such rectangles over k and taking a limit as $n \to \infty$, one can define the value of the integral $\int_X f d\mu$ as $\int_0^\infty \mu\{f > t\}dt$. For this perspective, here is a definition of the Lebesgue integration.

Definition 2.4.3 (Lebesgue Integration)
Let $f : X \to \mathbb{R}$ be a function such that one of those Riemann integrations

$$\int_0^\infty \mu\{f_+ > t\}dt, \quad \int_0^\infty \mu\{f_- > t\}dt \tag{2.3}$$

exists and finite. Then we define a *Lebesgue integration* respect to μ by

$$\int_X f(x)\mu(dx) := \int_0^\infty \mu\{f_+ > t\}dt - \int_0^\infty \mu\{f_- > t\}dt.$$

If $E \subset X$, then we define

$$\int_E f(x)\mu(dx) := \int_X 1_E(x) f(x)\mu(dx).$$

Sometimes, we use the abbreviation of $\int_E f(x)\mu(dx)$ by $\int_E f d\mu$ if the variable of which is understood.

Remark 2.4.4

We need to justify Definition 2.4.3. For any $0 \le s \le t$,

$$\{f_+ > t\} \subset \{f_+ > s\}, \quad \{f_- > t\} \subset \{f_- > s\}.$$

Since μ is a measure, $t \mapsto \mu\{f_+ > t\}$ and $t \mapsto \mu\{f_- > t\}$ are decreasing functions, so they are Riemann integrable on $[\epsilon, N]$ for every $0 < \epsilon < N < \infty$. Now we are understood Riemann integrations described in Definition 2.4.3 by an improper integral, so

$$\int_0^\infty \mu\{f_+ > t\}dt = \lim_{\substack{N\uparrow\infty \\ \epsilon\downarrow 0}} \int_\epsilon^N \mu\{f_+ > t\}dt,$$

and similar for f_-.

Remark 2.4.5

To explain the intuition of the Lebesgue integral, we introduce the set Γ_f and "define" its area by $\int_0^\infty \mu\{f > t\}dt$. However, the "real" definition for the area of Γ_f is introduced in Chapter 3.

Below proposition gives some basic facts about integrations.

Proposition 2.4.6

(i) Let f and g are nonnegative functions such that $f \le g$. Then $\int_X f d\mu \le \int_X g d\mu$.

(ii) If $A \in \mathfrak{M}_\mu$, and $c \ge 0$,

$$\int_X c 1_A d\mu = c\mu(A).$$

- -

proof. Since $f \le g$, we have $\{f > a\} \subset \{g > a\}$ for every $a \ge 0$. Also as μ is a measure, $\mu\{f > a\} \le \mu\{g > a\}$ holds. Therefore by the property of the Riemann integral and the definition of the Lebesgue integral, we obtain (i).

For (ii), by observing that

$$\{c1_A > t\} = \begin{cases} \varnothing & \text{if } t \ge c, \\ A & \text{if } t < c; \end{cases}$$

we can obtain that

$$\int_X c1_A d\mu = \int_0^\infty \mu\{c1_A > t\}dt = \int_0^c \mu(A)dt = c\mu(A).$$

This proves the proposition. $\square$

Exercise 2.4.7
Let $A_1, A_2, \cdots, A_n$ be μ-measurable sets and $c_1, \cdots, c_n \geq 0$, then

Now we are going to study very important limit theorems.

Theorem 2.4.8 (Monotone Convergence Theorem)
Let f, f_n are μ-measurable functions such that $0 \leq f_1 \leq f_2 \leq \cdots \leq f$ and $f_n \uparrow f$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_X f_n(x)\mu(dx) = \int_X f(x)\mu(dx).$$

Theorem 2.4.9 (Fatou's Lemma)
Let f_n are nonnegative μ-measurable functions. Then

$$\int_X \varliminf_{n\to\infty} f_n(x)\mu(dx) \leq \varliminf_{n\to\infty} \int_X f_n(x)\mu(dx).$$

Theorem 2.4.10 (Dominated Convergence Theorem)
Let f, f_n are μ-measurable functions such that $f_n \to f$ as $n \to \infty$ and $|f_n| \leq g$ for some μ-measurable function g such that $\int_X g d\mu < \infty$. Then

$$\lim_{n\to\infty} \int_X f_n(x)\mu(dx) = \int_X f(x)\mu(dx).$$

There is the generalized version of Theorem 2.4.10.

Exercise 2.4.11*
Let f, f_n, g, g_n are μ-measurable functions such that $f_n \to f$, $g_n \to g$ as $n \to \infty$, $|f_n| \leq g_n$ for every n, and $\int_X g_n d\mu \to \int_X g d\mu < \infty$ as $n \to \infty$. Then the below relation holds.

$$\lim_{n\to\infty} \int_X f_n(x)\mu(dx) = \int_X f(x)\mu(dx).$$

2.5 Borel sets and Borel measures

If X is a topological space, the topology τ of X is one of subsets of $\mathcal{P}(X)$. This means we can consider the σ-field $\sigma(\tau)$. Such σ-field has a special name and used in serveral places in measure theory.

Definition 2.5.1 (Borel σ-fields, Borel measure)
Let X be a topological space and τ be its topology. Then the σ-field $\sigma(\tau)$ is called the *Borel σ-field* and denoted by $\mathfrak{B}(X)$.
Every set belongs to the Borel σ-field is called by a *Borel set*.
If X has a measure μ such that all Borel sets are μ-measurable, then we say that μ is a *Borel measure*, or μ is *Borel* in short.

Lemma 2.5.2
Let X be a topological space and μ is a measure on X. The μ is Borel if and only if every open set is μ-measurable if and only if every closed set is μ-measurable.

- -

proof. By **??**, $\mathfrak{M}_\mu$ is a σ-field. Hence, every open set is μ-measurable means that every open set belongs to $\mathfrak{M}_\mu$, so $\mathfrak{B}(X) \subset \mathfrak{M}_\mu$ because of **??**. In addition, the fact that $\mathfrak{M}_\mu$ is a σ-field clearly implies that every open set is μ-measurable if and only if every closed set is μ-measurable. Finally, if μ is Borel measurable, then $\mathfrak{B}(X)$ is contained in $\mathfrak{M}_\mu$. By the definition of the Borel σ-field, clearly every open set is then μ-measurable. □

Checking whether a measure is Borel is difficult. However if the space has a metric, then there is an easy way to verify it. The statement of the theorem and its proof can be found also in [3].

Theorem 2.5.3
Let X be a metric space with a metric d, and μ be a measure on X. For any $x \in X$ and subsets A and B of X, define

$$d(x,B) := \inf\{d(x,y) : y \in B\}, \quad d(A,B) := \inf\{d(x,B) : y \in B\} \quad (\inf \varnothing := \infty).$$

If $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $d(A,B) > 0$, then μ is a Borel measure.

- -

proof. By Lemma 2.5.2 it suffices to show that every closed set is μ-measurable. Fix any closed set C in X and any subset $A \subset X$. By Exercise 2.2.5, we can further assume that $\mu(A) < \infty$. For each $n = 1, 2, 3, \cdots$, consider

$$C_n := \{x \in X : d(x,C) \leq 2^{-n}\}.$$

One can easily show that C_n is closed because the function $d(\cdot, C)$ is continuous, and $\bigcap_{n=1}^\infty C_n = C$ because C is closed. First of all, observe that $d(A \setminus C_n, A \cap C) > 0$ for all n. Indeed, for any $x \in A \setminus C_n$, we have

$$d(x, A \cap C) \geq d(x, C) > \frac{1}{2^n}.$$

Thus, taking an infimum over x, we obtain $d(A \setminus C_n, A \cap C) \geq 2^{-n} > 0$. Then our assumption about μ gives

$$\mu(A) \geq \mu((A \setminus C_n) \cup (A \cap C)) = \mu(A \setminus C_n) + \mu(A \cap C). \tag{2.4}$$

Claim that $\mu(A \setminus C_n) \to \mu(A \setminus C)$ as $n \to \infty$. Beware that we cannot use Corollary 2.2.11 because we do not know their measurability. Define E_n for $n = 1, 2, \cdots$ by

$$E_n := A \cap (C_n \setminus C_{n+1}) = \{x \in A : 2^{-n-1} < d(x,C) \leq 2^{-n}\}.$$

First of all, observe that $d(E_n, E_{n+2}) > 0$ for each $n = 1, 2, \cdots$. Indeed, for any $x \in E_n$, $y \in E_{n+2}$, and $c \in C$,

$$\frac{1}{2^{n+1}} < d(x,c) \leq d(x,y) + d(y,c).$$

Taking an infimum over extreme parts of the inequality, we obtain

$$\frac{1}{2^{n+1}} \leq d(x,y) + d(y,C) \leq d(x,y) + \frac{1}{2^{n+2}},$$

and thus $d(x,y) \geq 2^{-n-2}$. Taking infimums over x and y therefore gives $d(E_n, E_{n+2}) \geq 2^{-n-2} > 0$. Such fact and our assumption gives

$$\sum_{n=1}^N \mu(E_{2n}) + \mu(E_{2n-1}) = \mu\left(\bigcup_{n=1}^N E_{2n}\right) + \mu\left(\bigcup_{n=1}^N E_{2n-1}\right) \leq 2\mu(A).$$

Letting $N \to \infty$ gives $\sum_{n=1}^{\infty} \mu(E_n) \leq 2\mu(A)$. Recall that we assumed that $\mu(A)$ is finite. From this fact and since $C = \bigcap_{n=1}^{\infty} C_n$, we can now show our claim by

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \mu(A \setminus C_n) &\leq \mu(A \setminus C) \\
&\leq \underline{\lim_{n\to\infty}} \left[\mu(A \setminus C_n) + \sum_{m=n}^{\infty} \mu(E_m) \right] \\
&= \underline{\lim_{n\to\infty}} \mu(A \setminus C_n) + \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(E_m) \\
&= \underline{\lim_{n\to\infty}} \mu(A \setminus C_n).
\end{aligned}$$

Therefore, letting $n \to \infty$ on eq. (2.4) gives

$$\mu(A) \geq \mu(A \setminus C) + \mu(A \cup C),$$

hence C is μ-measurable because of Exercise 2.2.5. This finishes the proof. $\square$

From Theorem 2.5.3 we can construct a canonical Borel measure on any metric spaces.

Theorem 2.5.4
Let (X, d) be a metric space and $f : (0, \infty) \to [0, \infty)$ be a (monotonically) increasing function such that $f(0+) = 0$. Then the function $\ell_{X,f} : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\ell_{X,f}(E) := \inf \left\{ \sum_{n=1}^{\infty} f(r_n) : E \subset \bigcup_{n=1}^{\infty} B(a_n, r_n),\ a_n \in X,\ r_n > 0 \right\} \quad (\inf \varnothing := \infty)$$

is a Borel measure on X.

- -

proof. By Exercise 2.5.5, $\ell_{X,f}$ is a measure.
To show that $\ell_{X,f}$ is a Borel measure, we are going to use Theorem 2.5.3. Fix two subsets A and B such that $d(A, B) > 0$. If $\ell_{X,f}(A \cup B) = \infty$, there is nothing to prove, so assume that $\ell_{X,f}(A \cup B) < \infty$. Put $\delta = d(A, B)/4$ and take a sequence $(B(a_n, r_n))_1^{\infty}$ of open balls such that $A \cup B \subset \bigcup_{n=1}^{\infty} B(a_n, r_n)$. Define

$$\mathcal{A} := \{n : A \cap B(a_n, r_n) \neq \varnothing\}, \quad \mathcal{B} := \{n : B \cap B(a_n, r_n) \neq \varnothing\}.$$

Observe that $\mathcal{A}$ and $\mathcal{B}$ are disjoint. Indeed, if there exists $N \in \mathcal{A} \cap \mathcal{B}$, we can take $a \in A \cap B(a_N, r_N)$ and $b \in B \cap B(a_N, r_N)$. Then we have

$$4\delta = d(A, B) \leq d(a, b) \leq 2r_N \leq 2\delta,$$

which makes a contradiction.
Now we have

$$A \subset \bigcup_{n \in \mathcal{A}} B(a_n, r_n), \quad B \subset \bigcup_{n \in \mathcal{B}} B(a_n, r_n),$$

and

$$\sum_{n=1}^{\infty} f(r_n) = \sum_{n \in \mathcal{A}} f(r_n) + \sum_{n \in \mathcal{B}} f(r_n) \geq \ell_{X,f}(A) + \ell_{X,f}(B).$$

Since f is increasing, taking an infimum over sequence of such open balls, we obtain $\ell_{X,f}(A \cup B) \geq \ell_{X,f}(A) + \ell_{X,f}(B)$. Therefore $\ell_{X,f}$ is a Borel measure because of Theorem 2.5.3. $\square$

Exercise 2.5.5*
Prove that $\ell_{X,f}$ defined in Theorem 2.5.4 is a measure on X.

Definition 2.5.6
Let (X, d) be a metric space. We call $\ell_{X,f}$ defined in Theorem 2.5.4 by the *canonical Borel measure on X generated by f.*

Now we are going to look the specific measure on $\mathbb{R}$ in daily use, the Lebesgue measure. If ℓ denotes the Lebesgue measure on $\mathbb{R}$ whose definition is presented in later, then we obtain $\mathfrak{B}(\mathbb{R}) \subset \mathfrak{M}_\ell$ because of Theorem 2.5.3. However both σ-fields are not equal. Furthermore, we also going to prove that $\mathfrak{M}_\ell \neq \mathcal{P}(\mathbb{R})$ at the end of this section. First of all, we should learn what is the Lebesgue measure on $\mathbb{R}$.

Definition 2.5.7 (Lebesgue measure on $\mathbb{R}$)
The *Lebesgue measure* on $\mathbb{R}$ is denoted by ℓ_1 defined by $\ell_1 := \ell_{\mathbb{R},f}$ where $f(x) = x$.

Remark 2.5.8
Later, we will define the Lebesgue measure on $\mathbb{R}^d$ by ℓ_d. Like the Lebesgue measure on $\mathbb{R}$, we can define the Lebesgue measure on $\mathbb{R}^d$ by a canonical Borel measure generated by an appropriate function. But we postpone to introduce such function because at the first glance it is totally unnatural to think of.

For some simplicity, we denote ℓ for an one dimensional Lebesgue measure instead of ℓ_1 in this entire section. By Theorem 2.5.4 $\mathfrak{B}(\mathbb{R}) \subset \mathfrak{M}_\ell$ is automatically proved.

Chapter 3

The Fubini-Tonelli's theorem

Appendix A

Nets

A.1 Some properties about nets

A.2 Compact sets

A.3 Relation between nets and filters

We now learned both filters and nets. As it is mentioned in earlier, both concepts are made to generalize sequences. Then one might question about the correspondence between filters and nets.

Appendix B

Proofs of several difficult theorems

B.1 foo

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