# CAGD - Abgabe 1: Results

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## 1 Aufgabe 1

The algorithm of de Casteljau is illustrated in Figure 1. Figure 2 shows that a Bezier curve is well approximated by the control polygon created by consecutively applying the algorithm of the de Casteljau. Note that only three iterations are sufficient to compute a good approximation.

## 2 Aufgabe 2

**Proposition.** A Bezier curve $f(t)$ of degree $n$ with control points $P_0, P_1, ...P_n$ can be written as a Bezier curve $g(t)$ of degree $n+1$ with control points $Q_0, Q_1, ...Q_{n+1}$, where

$$Q_0 = P_0 \tag{1}$$

$$Q_j = \frac{j}{n+1} P_{j-1} + \left(1 - \frac{j}{n+1}\right) P_j \tag{2}$$

**Proof.** We want to show that $g(t) = f(t)$ for all $t \in R$ if we define the control points $Q_i, i = 0...n+1$ as in Equations 1 and 2 (where Equation 1 follows from Equation 2). Using the parametric representation of the Bezier

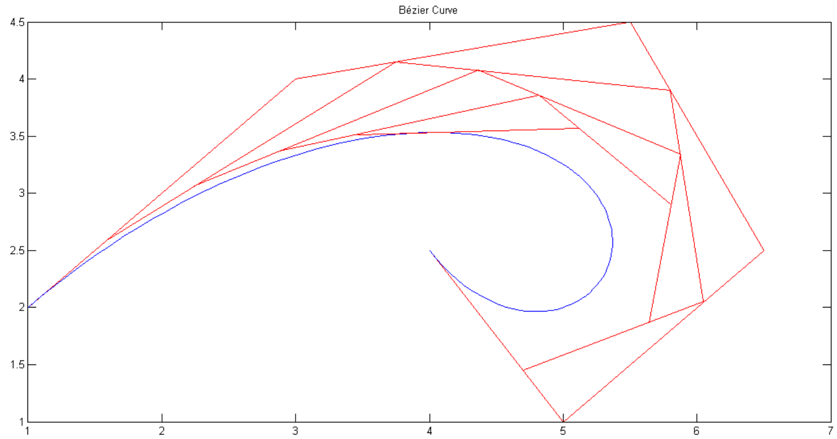


Figure 1: The algorithm of de Casteljau for t=0.3

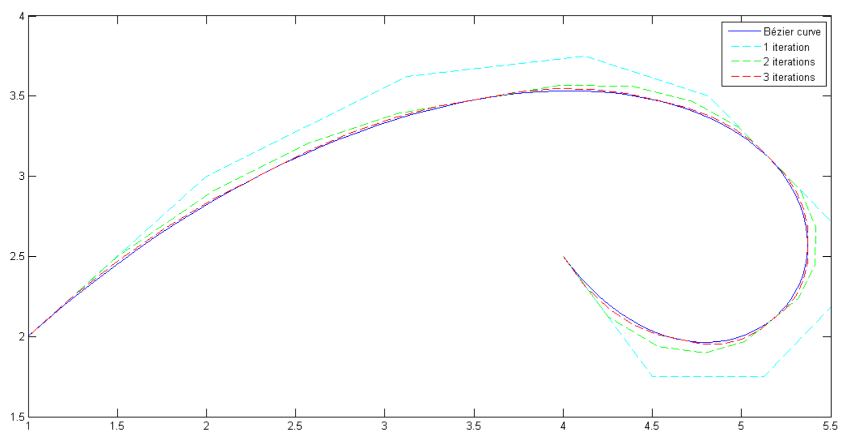


Figure 2: Comparison of the Bezier curve and its approximations applying the algorithm of de Casteljau one, two and three times, respectively.

curve - and writing $B_j^n := B_j^n(t)$ for easier notation - we write

$$
\begin{aligned}
g(t) &= \sum_{j=0}^{n+1} B_j^{n+1} \left( \frac{j}{n+1} P_{j-1} + \left(1 - \frac{j}{n+1}\right) P_j \right) \\
&= \sum_{j=0}^{n+1} B_j^{n+1} \frac{j}{n+1} P_{j-1} + \sum_{j=0}^{n+1} B_j^{n+1} \left(1 - \frac{j}{n+1}\right) P_j
\end{aligned}
$$

Note that the first and last term of the first and second sum, respectively, equals 0. Therefore we can write

$$
\begin{aligned}
& \sum_{j=0}^{n+1} B_j^{n+1} \frac{j}{n+1} P_{j-1} + \sum_{j=0}^{n+1} B_j^{n+1} \left(1 - \frac{j}{n+1}\right) P_j \\
=& \sum_{j=1}^{n+1} B_j^{n+1} \frac{j}{n+1} P_{j-1} + \sum_{j=0}^{n} B_j^{n+1} \left(1 - \frac{j}{n+1}\right) P_j \\
=& \sum_{j=0}^{n} B_{j+1}^{n+1} \frac{j+1}{n+1} P_j + \sum_{j=0}^{n} B_j^{n+1} \left(1 - \frac{j}{n+1}\right) P_j
\end{aligned}
$$

Recalling that the Bernstein polynomials are given by

$$
B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j}
$$

we can deduce that

$$
\begin{aligned}
B_{j+1}^{n+1} \frac{j+1}{n+1} &= \frac{(n+1)!}{(j+1)!(n+1) - (j+1))!} t^{j+1} (1-t)^{n+1-(j+1)} \frac{j+1}{n+1} \\
&= \binom{n}{j} t^{j+1} (1-t)^{n-j} \\
&= t B_j^n \\
B_j^{n+1} \left(1 - \frac{j}{n+1}\right) &= \frac{(n+1)!}{j!(n+1-j)!} t^j (1-t)^{n+1-j} \frac{n+1-j}{n+1} \\
&= \binom{n}{j} t^j (1-t)^{n+1-j} \\
&= (1-t) B_j^n
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\sum_{j=0}^{n} B_{j+1}^{n+1}\frac{j+1}{n+1}P_j + \sum_{j=0}^{n} B_j^{n+1}\left(1-\frac{j}{n+1}\right)P_j &= \\
&= \sum_{j=0}^{n} tB_j^n P_j + \sum_{j=0}^{n}(1-t)B_j^n P_j \\
&= t\sum_{j=0}^{n} B_j^n P_j + (1-t)\sum_{j=0}^{n} B_j^n P_j \\
&= \sum_{j=0}^{n} B_j^n P_j \\
&= f(t)
\end{aligned}
$$

$\square$.