Section 5.2 Summations and Closed Forms

A *closed form* is an expression that can be computed by applying a fixed number of familiar operations to the arguments. For example, the expression $2 + 4 + \ldots + 2n$ is not a closed form, but the expression $n(n+1)$ is a closed form.

Summation Notation: $\sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$.

Summation Facts

(1) $\sum ca_k = c \sum a_k$. (2) $\sum (a_k + b_k) = \sum a_k + \sum b_k$.

(3) $\sum a_k x^{i+k} = x^i \sum a_k x^k$. (4) $\sum_{k=m}^{n} a_{k+i} = \sum_{k=m+i}^{n+i} a_k$.

(5) Collapsing Sums) $\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$ and $\sum_{k=1}^{n} (a_{k-1} - a_k) = a_0 - a_n$.

Some Useful Closed Forms

(1) $\sum_{k=m}^{n} c = (n - m + 1)c$. (2) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(3) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. (4) $\sum_{k=0}^{n} a^k = \frac{a^{n+1} - 1}{a - 1}$ (where $a \neq 1$).

(5) $\sum_{k=1}^{n} ka^k = \frac{a - (n+1)a^{n+1} + na^{n+2}}{(a-1)^2}$ (where $a \neq 1$).

Example. Find a closed form for the expression $\sum_{k=2}^{n}(k-1)2^{k+1}$.

Solution:

$$\sum_{k=2}^{n}(k-1)2^{k+1} = \sum_{k=1}^{n-1}k2^{k+2} \qquad \text{(Fact 4)}$$

$$= 2^2\sum_{k=1}^{n-1}k2^{k} \qquad \text{(Fact 3)}$$

$$= 2^2(2 - n2^n + (n-1)2^{n+1}) \qquad \text{(Form 5)}$$

$$= 2^3 - (2-n)2^{n+2}.$$

Example. Find a closed form for $2 + 2^2 \cdot 7 + 2^3 \cdot 14 + \cdots + 2^n(n-1)\cdot 7$.

Solution: The sum has the form $2 + \sum_{k=2}^{n}2^k(k-1)\cdot 7$

$$= 2 + 7\sum_{k=2}^{n}(k-1)2^k \qquad \text{(Fact 1)}$$

$$= 2 + 7\sum_{k=1}^{n-1}k2^{k+1} \qquad \text{(Fact 4)}$$

$$= 2 + 14\sum_{k=1}^{n-1}k2^{k} \qquad \text{(Fact 3)}$$

$$= 2 + 14(2 - n2^n + (n-1)2^{n+1}). \qquad \text{(Form 5)}$$