《有限元》读书报告

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2023 年 5 月 26 日

考虑如下方程

$$-u''(x) + \pi^2 \cos^2(\pi x) u(x) = f(x). \quad x \in [0, 1] \tag{1}$$

with boundary conditions:

$$\begin{aligned} u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{2}$$

i) Consider $f(x) = \pi^2 \sin(\pi x) \cosh(\sin(\pi x))$, and check that the function $u(x) = \sinh(\sin(\pi x))$ is the solution to the boundary value problem (BVP) (1)+(2).

ii) We want to solve this BVP numerically. We begin by discretizing the interval $[0, 1]$. For this, consider the gridpoints:

$$x_i = ih, \quad i = 0, 1, \ldots, n+1, \; h = \frac{1}{n+1} \tag{3}$$

Note that $h_i = x_{i+1} - x_i = h$ for all i. Now we approximate the second derivative. Show that if g has four continuous derivatives, then

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + O(h^2), \tag{4}$$

where $g_i = g(x_i)$.

iii) Consider now the linear system of equations

$$-\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + \pi^2 \cos^2(\pi x_i) g_i = f(x_i), \quad i = 1, 2, \ldots, n. \tag{5}$$

Show that this can be rewritten in matrix form as

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f},$$

where $\boldsymbol{g} = (g_1, \ldots, g_n)^T$, $\boldsymbol{f} = (f_1, \ldots, f_n)^T$, and the matrix $\boldsymbol{A}$ is tridiagonal, with entries:

$$a_{ij} = \begin{cases} -\frac{1}{h^2} & , \quad |i-j| = 1 \\ \frac{2}{h^2} + \pi^2 \cos^2(\pi x_i) & , \quad i = j \\ 0 & , \quad \textit{otherwise.} \end{cases} \tag{6}$$

iv) Show that Scheme (5) is second-order accurate.

v) Solve the system of equation (5). Use following values: $n = 10, 20, 40, 80, 160, 320$. For each $h = 1/(n+1)$, compute the error

$$e(h) = \sup_{1 \leqslant i \leqslant n} |g_i - u(x_i)|. \tag{7}$$

and do a log-log plot of $e(h)$, that is, plot $\log(e(h))$ as a function of $\log(h)$. Show, using this plot, that $e(h) = O(h^2)$, consistent with iv.

Solution:

i) Since

$$\sinh' x = \cosh x,$$
$$\cosh' x = \sinh x$$

we get

$$\begin{aligned} & u'(x) = \pi \cosh(\sin \pi x), \\ \Rightarrow \quad & u''(x) = \pi^2 \left[\sinh(\sin \pi x) \cos^2 \pi x - \cosh(\sin \pi x) \sin \pi x\right]. \end{aligned}$$

Thus

$$-u''(x) + \pi^2 \cos^2(\pi x) u(x) = \pi^2 \cosh(\sin \pi x) \sin \pi x = f(x).$$

Besides, the boundary value conditions can be easily checked. So the function $u(x) = \sinh(\sin \pi x)$ is the solution of BVP (1) + (2).

ii) Expanding the function $g(x)$ using Taylor series at points x_i and x_{i+1},

$$
\begin{aligned}
g_{i+1} - g_i &= g_i' h + \frac{g_i''}{2} h^2 + \frac{g_i'''}{3!} h^3 + O(h^4), \\
g_{i-1} - g_i &= -g_i' h + \frac{g_i''}{2} h^2 - \frac{g_i'''}{3!} h^3 + O(h^4),
\end{aligned}
$$

Subtracing the above two equations, then divideing both sides by h^2, we get

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + O(h^2).$$

where $g_i = g(x_i)$.

iii) Writing formula (5) as following form

$$-\frac{1}{h^2} g_{i-1} + \left[\frac{2}{h^2} + \pi^2 \cos^2(\pi x_i)\right] g_i - \frac{1}{h^2} g_{i+1} = f(x_i), i = 2, \ldots, n-1. \tag{8}$$

We have known $g_0 = g_{n+1} = 0$ because of the boundary value conditions. So (5) has two special form when $i = 1, n$.

$$\left[\frac{2}{h^2} + \pi^2 \cos^2(\pi x_1)\right] g_1 - \frac{1}{h^2} g_2 = f(x_1). \tag{9}$$

$$-\frac{1}{h^2} g_{n-1} + \left[\frac{2}{h^2} + \pi^2 \cos^2(\pi x_n)\right] g_n = f(x_n). \tag{10}$$

Write the system of equations (8) (9) (10) in matrix form $\boldsymbol{A}\boldsymbol{g} = \boldsymbol{f}$. Where matrix $\boldsymbol{A}$ is as following

$$
\begin{bmatrix}
\frac{2}{h^2} + \pi^2 \cos^2(\pi x_1) & -\frac{1}{h^2} & & & \\
-\frac{1}{h^2} & \frac{2}{h^2} + \pi^2 \cos^2(\pi x_2) & -\frac{1}{h^2} & & \\
& \ddots & \ddots & \ddots & \\
& & & & -\frac{1}{h^2} \\
& & & -\frac{1}{h^2} & \frac{2}{h^2} + \pi^2 \cos^2(\pi x_n)
\end{bmatrix}
$$

and $\boldsymbol{g} = (g_1, \ldots, g_n)^T, \boldsymbol{f} = (f_1, \ldots, f_n)^T$.

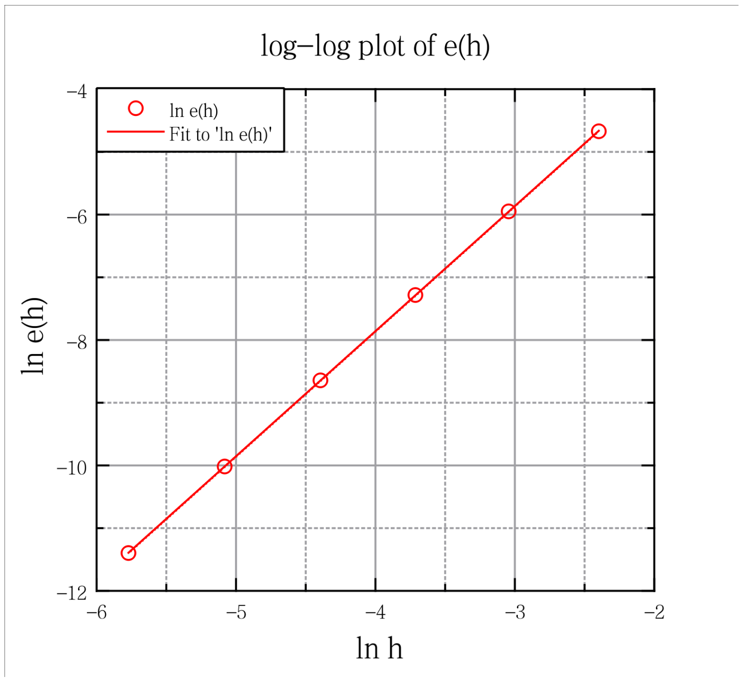
iv)

$$
\begin{aligned}
\boldsymbol{\tau} &= \boldsymbol{A}\boldsymbol{u} - \boldsymbol{f} \\
&= \boldsymbol{A}\boldsymbol{u} - \boldsymbol{A}\boldsymbol{g} \\
&= \boldsymbol{A}\boldsymbol{e}
\end{aligned}
$$

where $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{g},\ \boldsymbol{u} = (u(x_1), \ldots, u(x_n))^T$. Thus

$$\|\boldsymbol{e}\| \leqslant \|\boldsymbol{A}^{-1}\|\|\boldsymbol{\tau}\| \leqslant c\|\boldsymbol{A}^{-1}\|h^2$$

v) Solving this linear systems using chase method, then computing $e(h)$ when $n = 10, 20, 40, 80, 160, 320$. A log-log plot of $e(h)$ is presented as follows.



Fitting the data poings using a linear function, the result is

$$\ln e(h) = 1.99474 \ln h + 0.119827.$$

Based on the above results, it can be determined that scheme (5) has second-order accuracy.