Middle East Technical University   Department of Computer Engineering

CENG 223

Discrete Computational Structures

Fall 2024-2025

Take Home Exam 2

Due date: November 13, 2024, Wednesday, 23:59

Question 1 - Sets (5+5+15+15+10= 50 pts)

1. Show that the symmetric difference operation is associative or not, ie. if A, B, C are sets, is it true that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$?

2. Let $f : B \to C, g : A \to B$. Prove whether g is 1-to-1 when $f, f \circ g$ are 1-to-1.

3. Show that if S is a non-empty set, there does not exist an onto map from S to $P(S)$, power set of S, by using proof by contradiction as described below.

   Assume there exists a function $f : S \to P(S)$ that is onto, meaning every subset of S appears as $f(s)$ for some $s \in S$. Define the subset $T = \{s \in S \mid s \notin f(s)\}$.

   1. Using the definition of T, explain why there can be no element $s_T \in S$ for which $f(s_T) = T$.
   2. How does this lead to a contradiction, showing that no such onto function f can exist?

   Hint: Think about whether $s_T \in T$ or $s_T \notin T$ could hold if $f(s_T) = T$, and relate this to the elements included in T by definition.

4. Determine whether the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto. If the function is onto, provide values for m, n to generate all $\mathbb{Z}$. If not, provide a counter example and show that a certain number or a set of numbers cannot be expressed.

   (a) $f(m, n) = 2m + n$
   (b) $f(m, n) = m^2 - n^2$
   (c) $f(m, n) = m + n + 1$
   (d) $f(m, n) = |m| - |n|$
   (e) $f(m, n) = m^2 - 4$

5. (a) Find the result of $\bigcap_{i=2}^{\infty} (0 - \frac{1}{i}, 5 + \frac{1}{i})$. Are 0 and 5 in the resulting set? If not, at what index they are eliminated?

   (b) Find the result of $\bigcup_{i=2}^{\infty} [0 + \frac{1}{i}, 5 - \frac{1}{i}]$. Are 0 and 5 in the resulting set? If yes, at what index they are added?

Question 2 - Algorithms (10+5+10= 25 pts)

1. Is $\sin x = O(\cos x)$? Prove your answer.
2. Show that if $f(x)$ is $O(x)$ then $f(x)$ is $O(x^2(2 + \cos x))$ by providing witnesses (do not make simplifications without showing a witness).
3. Show that $x \ \log x$ is $O(x^2)$ but x^2 is not $O(x \ \log \ x)$.

Question 3 - Divisibility (10+10+5=25 pts)

1. Prove that $\sqrt{7}$ is not a rational number.
   Hint: Use fundamental theorem of arithmetic.
2. Prove that the set of primes of the form $3k + 2$ are not finite.
   Hint: This is similar to the proof of prime numbers are not finite. Take $3q_1q_2..q_n - 1$ and show that this has a prime factor of the form $3k + 2$, where $\{q_i\}$ is the finite sequence of all prime numbers of the form $3k + 2$.
3. Show that if a, b, and m are integers such that $m \geq 2$ and $a \equiv b (\text{mod } m)$, then $\gcd(a, m) = \gcd(b, m)$.

Regulations

1. You have to write your answers to the provided sections of the template answer file given in LaTeX. **Handwritten solutions will not be accepted.**
2. Do not write any extra stuff like question definitions to the answer file. Just give your solution to the question. Otherwise you will get 0 from that question.
3. **Late Submission: Not allowed!**
4. **Cheating: We have zero tolerance policy for cheating.** People involved in cheating will be punished according to the university regulations. .tex file will be checked for plagiarism.
5. Submit a single PDF file named eXXXXXXX.pdf (7-digit student number). Submission that are not in the specified format will recieve a penalty of 10 points.
6. You may ask your questions in the course forum or by sending a mail to oguzhan@ceng.metu.edu.tr