Student Information

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Answer 1

a)

$a_n^g = a_n^p + a_n^h$

Since if we collect the terms with a_n an on one side, the equations becomes homogeneous, the general solution of this equation is equal to homogeneous solution.
Let's do the homogeneous solution

$a_n - 3a_{n-1} - 4a_{n-2} = 0$ for $n \geq 2$
$r^2 - 3r - 4 = 0$ (characteristic equation)
If we factorize the characteristic equation:
$(r-4) \cdot (r+1) = 0$

The roots of the equation are $r_1 = 4$ and $r_2 = -1$
The characteristic equation has 2 distinct reel roots. So:
$a_n^g = a_n^h = K(4)^n + L(-1)^n$ (eq1)

Let's substitute $a_0 = 2$ and $a_1 = 5$ to find coefficients K and L
$a_0 = K + L = 2$
$a_1 = 4K - L = 5$
$K = \frac{7}{5}$ and $L = \frac{3}{5}$
Substitute them into the eq1 to find the final solution.

$a_n^g = \frac{7}{5}(4)^n + \frac{3}{5}(-1)^n$

b)

Let $A(x)=\sum_{n=0}^{\infty} a_n x^n$

Multiply both sides with x^n and sum for $n \geq 2$

$\sum_{n=2}^{\infty} a_n x^n = 3 \cdot \sum_{n=2}^{\infty} a_{n-1} x^n + 4 \cdot \sum_{n=2}^{\infty} a_{n-2} x^n$

Let's adjust the lower limits

$\sum_{n=2}^{\infty} a_n x^n = 3x \cdot \sum_{n=1}^{\infty} a_n x^n + 4x^2 \cdot \sum_{n=0}^{\infty} a_n x^n$

$A(x) - a_0 - a_1 x = 3x(A(x) - a_0) + 4x^2 A(x)$

Since $a_0 = 2$ and $a_1 = 5$

$A(x) - 2 - 5x = 3xA(x) - 6x + 4x^2 A(x)$

$A(x)(1 - 3x - 4x^2) = 2 - x$

$A(x) = \frac{2-x}{1-3x-4x^2} = \frac{B}{1-4x} + \frac{C}{1+x}$

$\frac{B+Bx+C-4Cx}{(1-4x)(1+x)} = \frac{2-x}{1-3x-4x^2}$

$B = 7/5$
$C = 3/5$

$A(x) = \frac{7}{5}(\frac{1}{1-4x}) + \frac{3}{5}(\frac{1}{1+x})$

Since

$< 1, 1, 1, 1, 1, \cdots > = \frac{1}{1-x}$

$< r^0, r^1, r^2, r^3, \cdots > = \frac{1}{1-rx}$

$< 4^0, 4^1, 4^2, 4^3, \cdots > = \frac{1}{1-4x} = \sum_{n=0}^{\infty} 4^n x^n$

Again since

$< 1, 1, 1, 1, 1, \cdots > = \frac{1}{1-x}$

$< 1, -1, 1, -1, 1, \cdots > = \frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

$A(x)=\frac{7}{5} \sum_{n=0}^{\infty} 4^n x^n + \frac{3}{5} \sum_{n=0}^{\infty}(-1)^n x^n$

$A(x)=\sum_{n=0}^{\infty}(\frac{7}{5}4^n + \frac{3}{5}(-1)^n)x^n$

Since $A(x)=\sum_{n=0}^{\infty} a_n x^n$

$a_n=\frac{7}{5}4^n + \frac{3}{5}(-1)^n$

Answer 2

For $n = 2^m$, Let's define a new relation b_m by using the relation between n and m :

If $b_m = a_n$
$b_{m-1} = a_{\frac{n}{2}}$
$b_{m-1} = a_{\frac{n}{4}}$

If we substitute b_m's our new relation would become:

$b_m = b_{m-1} + 6b_{m-1}$

Since it is already homogeneous, the general solution is equal to homogeneous solution

$b_m - b_{m-1} - 6b_{m-1} = 0$
$r^2 - r - 6 = 0$ (characteristic equation)

The roots of the characteristic equation are $r_1 = -2$ and $r_2 = 3$. Since the equation has 2 distinct reel roots, the template of the solution is:

$b_m = K(-2)^m + L(3)^m$

Let's substitute a_1 and a_2
For $n = 1$ from the equation $n = 2^m$, $m = 0$.
Hence, if $a_1 = 3$, then $b_0 = 3$

For $n = 2$ from the equation $n = 2^m$, $m = 1$.
Hence, if $a_2 = 4$, then $b_1 = 4$

$b_0 = K + L = 3$
$b_1 = -2K + 3L = 4$
$K = 1$ and $L = 2$
$b_m = (-2)^m + 2(3)^m$

Again by using the equation $n = 2^m$, we can conclude that $log_2 n = m$. So we can write $log_2 n$ instead of m.

$a_n = (-2)^{log_2 n} + 2(3)^{log_2 n}$

Answer 3

First of all, let's break $< 3, 9, 18, 39, 96, 261 \cdots >$ into more comprehensible sub-series.

By making some predictions and tries, it can be observed that $< 3, 9, 18, 39, 96, 261 \cdots >$ is the sum of the series $< 0, 3, 9, 27, 81, 243 \cdots >$ and $< 3, 6, 9, 12, 15, 18 \cdots >$.
Let's start with $< 0, 3, 9, 27, 81, 243 \cdots >$

$< 1, 1, 1, 1, 1, 1, 1 \cdots > \longleftrightarrow \frac{1}{1-x}$

$< r^0, r^1, r^2, r^3, \cdots > \longleftrightarrow \frac{1}{1-rx}$

$< 1, 3, 9, 27, 81, 243, \cdots > \longleftrightarrow < 3^0, 3^1, 3^2, 3^3, 3^4, \cdots > \longleftrightarrow \frac{1}{1-3x}$

Multiply with 3:

$< 3, 9, 27, 81, 243, \cdots > \longleftrightarrow \frac{3}{1-3x}$

Shift Right (Multiply with x):

$< 0, 3, 9, 27, 81, 243 \cdots > \longleftrightarrow \frac{3x}{1-3x}$

As we can obtain $< 0, 3, 9, 27, 81, 243 \cdots >$ we can continue with obtaining $< 3, 6, 9, 12, 15, 18 \cdots >$

$< 1, 1, 1, 1, 1, 1, \cdots > \longleftrightarrow \frac{1}{1-x}$

Take the derivative of it:

$< 1, 2, 3, 4, 5, 6, \cdots > \longleftrightarrow \frac{1}{(1-x)^2}$

Multiply with 3:

$< 3, 6, 9, 12, 15, 18 \cdots > \longleftrightarrow \frac{3}{(1-x)^2}$

Hence we can derive both part of the summation:

$< 3, 9, 18, 39, 96, 261 \cdots > \longleftrightarrow \frac{3x}{1-3x} + \frac{3}{(1-x)^2}$

Answer 4

a)

TRUE

Assume given a set A.
$P(A)$ is all subsets of A.
Let $S \subseteq P(A)$
S is a partition of A.
We will use the representation of S_i for the elements of S.
By the definition of partitioning:
$\bigcup_{i=1}^{n} S_i = A$
If $S_i \neq S_j$, then $S_i \cap S_j = \emptyset$
$S_i \neq \emptyset$
Let's use the partition S for proof of the theorem.
Let's define a relation R on A.
$R \subseteq A \times A$
If $(a, b) \in R$, then $a, b \in S_i \in S$
Now check whether R is an equivalence relation or not?
By the definition of equivalence relations R should satisfy the conditions:
Reflexivity
Symmetry
Transitivity

Reflexivity
To be reflexive it should satisfy $(a, a) \in R$.
$(a, a) \in R$ because $a \in S_i$ for some $S_i \in S$. This is because $\bigcup_{i=1}^{n} S_i = A$.

Symmetry
Assume $(a, b) \in R$, $a, b \in S_i \in S$ This implies that $b, a \in S_i \in S$.
Hence, if $b, a \in S_i$, then $(b, a) \in R$.

Transitivity
Assume:
$(a, b) \in R$ and $(b, c) \in R$
If $(a, b) \in R$, then $a, b \in S_i \in S$
If $(b, c) \in R$, then $b, c \in S_j \in S$
From the definition of partitioning we already knew:
If $S_i \neq S_j$, then $S_i \cap S_j = \emptyset$
However, in our case above $S_i \cap S_j = b$. So we can conclude that $i = j$.
$a, b, c \in S_i$
Since $a, b, c \in S_i$, $(a, c) \in R$.
Hence R is an equivalence relation

b)

FALSE

For $A = \{1, 2, 3, 4\}$ and the relation $R = \{(1, 4), (4, 3), (3, 2), (2, 1)\}$
R satisfies anti-symmetry conditions.
Transitive closure of a matrix $R' = R \cup R^2 \cup R^3 \dots$
Let's represent this with matrices:
R :

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

R^2 :

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

R^3 :

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

If we take $R' = R \cup R^2 \cup R^3$
R' :

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Hence, R' is symmetric, it is false.

c)

FALSE

For $A = \{1, 2, 3\}$ and the relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$
Conditions of equivalence relation:
Reflexivity
Symmetry
Transitivity

These conditions are satisfied by R
However, Conditions for partial ordering:

Reflexivity
Anti-symmetry
Transitivity
Since R is not anti-symmetric, ($(1,2) \in R$ and $(2,1) \in R$ but $1 \neq 2$) It is not a partial order.

d)

TRUE

Since A is an anti-symmetric relation $\forall (a,b) \in R$, if $(b,a) \in R$, then $a = b$.
$R^{-1} = \{(b,a) | (a,b) \in R\}$
If we take the intersection of R and R^{-1} $(R \cap R^{-1})$, it should contain the elements (a,a)
Hence, to be in this intersection elements should satisfy $a = b$
So, we can simply conclude that $R \cap R^{-1} \subseteq \Delta$=$\{(a,a) | a \in A\}$
This implies that for every element of $R \cap R^{-1}$ it should satisfy $a = b$.
Which is the definition of anti-symmetry.

e)

TRUE

Reflexive: if $x \in A$, then $(x,x) \in R$.
Transitive: for $x, y, z \in A$, if $(x,y) \in R$, $(y,z) \in R$, then $(x,z) \in R$
Base Case:
For $n = 1$, $R^1 = R$

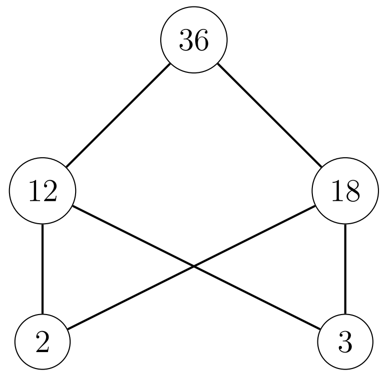
Inductive Hypothesis:
Assume it is true for an arbitrary k.
$R^k = R$

Induction Step:
Prove that it is true for R^{k+1}
$R^{k+1} = R^k \circ R$
From inductive hypothesis:
$R^{k+1} = R \circ R = R^2$
By the definition of transitivity $R^2 = R$
Since R^2 is defined as $R^2 = \{(x,z) | (x,y) \in R, (y,z) \in R\}$
Hence by mathematical induction $R^n = R$

Answer 5

a)

We can write
$R = \{(2,2),(2,12),(2,18),(2,36),(3,3),(3,12),(3,18),(3,36),(12,12),(12,36),(18,18),(18,36),(36,36)\}$
However we should ignore cycles, and transitions while drawing Hasse Diagram

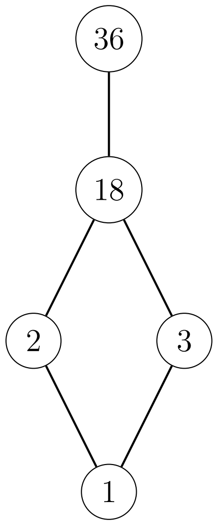


b)

$R_s = R \cup R^{-1}$ (from the definition of symmetric closure)
Since $R^{-1} = \{(b,a)|(a,b) \in R\}$
We can define R^{-1} like:
$R^{-1} = \{(2,2),(12,2),(18,2),(36,2),(3,3),(12,3),(18,3),(36,3),(12,12),(36,12),(18,18),(36,18),(36,36)\}$
We are trying to list the elements of $R_s - R$
$(R \cup R^{-1}) - R$ can be considered as $R^{-1} - R$
Hence the solution is $R_s - R$=$\{(12,2),(18,2),(36,2),(12,3),(18,3),(36,3),(36,12),(36,18)\}$

c)

From the definition of lattice, for being lattice:
$\forall x, y \in A$, $GLB(x,y) \neq \emptyset$
$\forall x, y \in A$, $LUB(x,y) \neq \emptyset$
However in this case $GLB(2,3) = \emptyset$
So we can add 1 to this Hasse diagram while removing 12.



Now for every pair, a greatest lower bound and a lowest upper bound exists.