Middle East Technical University

Department: Computer Science and Engineering

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Course: Discrete Computational Structures

Student's Solution

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1 Question 1

1.

$$\sum_{j=1}^{n} j \cdot (j+1) \cdot \dots \cdot (j+k-1) = \frac{n(n+1) \cdot \dots \cdot (n+k)}{k+1}$$

We can observe that the expression inside the summation operator can be expressed in another way, because for any j:

$$j(j+1)(j+2)\dots(j+k-1) = \frac{(j+k-1)!}{(j-1)!}$$

However, we will multiply both the numerator and denominator by $k!$ to obtain a simpler equation:

$$\sum_{j=1}^{n} j(j+1)(j+2)\dots(j+k-1) = \sum_{j=1}^{n} \left(\frac{(j+k-1)!}{(j-1)!} \cdot \frac{k!}{k!} \right)$$

Now we have a new representation of this equation:

$$\sum_{j=1}^{n} \left(\frac{(j+k-1)!}{(j-1)!} \cdot \frac{k!}{k!} \right) = \sum_{j=1}^{n} \binom{j+k-1}{k} k!$$

Thus, we can factor out $k!$:

$$k! \sum_{j=1}^{n} \binom{j+k-1}{k} = k! \left(\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{k+n-2}{k} + \binom{k+n-1}{k} \right)$$

Now, applying Pascal's Identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ repeatedly, for example:

$$k!\left(\binom{k+2}{k+1} + \binom{k+2}{k} + \cdots + \binom{k+n-2}{k} + \binom{k+n-1}{k}\right)$$

Since $\binom{k}{k} = \binom{k+1}{k+1} = 1$, and if we substitute $\binom{k+1}{k+1}$ instead of $\binom{k}{k}$, we can obtain $\binom{k+2}{k+1}$ from the summation of $\binom{k+1}{k+1} + \binom{k+1}{k}$.

Thus, we get:

$$k!\left(\binom{k+3}{k+1} + \binom{k+3}{k} + \cdots + \binom{k+n-2}{k} + \binom{k+n-1}{k}\right)$$

Using the identity $\binom{k+2}{k+1} + \binom{k+2}{k} = \binom{k+3}{k+1}$, and substituting it in place of $\binom{k+2}{k+1} + \binom{k+2}{k}$, and continuing this process, we reach:

$$k!\binom{k+n}{k+1}$$

For the right-hand side of the equation, we have:

$$\frac{(k+n)!}{(n-1)!(k+1)} = \frac{(k+n)!}{(n-1)!(k+1)!}k!$$

Hence, we obtain the equation:

$$k!\binom{k+n}{k+1} = \frac{(k+n)!}{(n-1)!(k+1)!}k!$$

Finally, we can cancel the $k!$ from both sides:

$$\binom{k+n}{k+1} = \frac{(k+n)!}{(n-1)!(k+1)!}$$

Thus, the equality is true for any n, as proven by using direct proof.

2. Since $p-1$ is divisible by y, let's assume that

$$p - 1 = yk + r \quad \text{where} \quad 0 \leq r < y, \quad r \in \mathbb{Z}.$$

We already know that $x^{p-1} \equiv 1 \pmod{p}$ by Fermat's Theorem. Then, if we substitute our expression for $p-1$, we get:

$$x^{p-1} \equiv x^{yk+r} = x^{yk} \cdot x^r \equiv 1 \pmod{p}.$$

We are given that $x^y \equiv 1 \pmod{p}$. So, substituting 1 for x^y, we obtain:

$$x^{yk} \cdot x^r \equiv 1 \pmod{p}.$$

We are also given that y is the smallest positive integer that satisfies $x^y \equiv 1 \pmod{p}$. Therefore, using $0 \leq r < y$ and $x^r \equiv 1 \pmod{p}$, we can conclude that if $r = 0$, then:

$$p - 1 = yk.$$

Therefore, it follows that y divides $p - 1$, or equivalently,

$$y \mid (p - 1).$$

3. Base Case:
For $n = 0$, we have

$$\binom{0}{0} = 2^0 = 1,$$

which is true.

Induction Hypothesis:
Let's assume that the statement is true for some arbitrary n. That is,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Inductive Step:
We need to show that if the statement holds for n, it also holds for $n+1$. We want to prove that

$$\sum_{k=0}^{n+1} \binom{n+1}{k}) = 2^{n+1}$$

By applying **Pascal's Identity**,

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r},$$

we can break down the sum:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1} + .$$

We will split the sum into two parts: one part involving n and another part involving $n+1$. Using the induction hypothesis, we get:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Now, we can apply Pascal's Identity to each of the middle terms. For example:
$\binom{n+1}{1} = \binom{n}{0} + \binom{n}{1}$

$\binom{n+1}{2} = \binom{n}{1} + \binom{n}{2}$
and so on.

Thus, the sum becomes:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) + \binom{n+1}{n+1}.$$

Now, since $C(n,0) = 1$ and $C(n+1,0) = 1$, we can substitute $C(n+1,0)$ for $C(n,0)$, and similarly for $C(n,n)$ and $C(n+1,n+1)$:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1}$$

By summing the two separate sums, we can write:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k}.$$

From the induction hypothesis, we know that:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Therefore, the sum becomes:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = 2^n + 2^n = 2^{n+1}.$$

Thus, by the principle of mathematical induction, the result is proven.

2 Question 2

1.

2. Fundamental theorem of arithmetic is defined as every integer which is greater than 1 can be written uniquely as a product of prime numbers.
   Let $P(n)$ be a function that writes n as a product of primes.

   Base Case:
   For $n = 2$, $P(2) = 2$. Since 2 is already a prime number, its factorization is already unique. Because 2 cannot be represented with smaller primes, the base case holds true for $n = 2$.

   Induction Hypothesis:
   Assume that it is true for every integer $2 \leq n \leq k$. This means that for every k such that $2 \leq n \leq k$, k has a unique factorization. This is true for $P(1), P(2), P(3), \ldots, P(k)$.

   Induction Step:
   We need to show that if $P(1), P(2), P(3), \ldots, P(k)$ are true, then it is also true for $P(k+1)$.

   Since the prime factorization depends on whether $k + 1$ is prime or not, we divide the proof into two cases.

   Case 1: $k + 1$ is prime.
   In this case, there is no need to prove further, as $k + 1$ already has a unique prime factorization. There is no other way to represent it.

   Case 2: $k + 1$ is a composite number.
   If $k + 1$ is a composite number, it can be written as a product of two positive integers. Let us assume r, m satisfy the equation $k + 1 = r \cdot m$. They also must satisfy the condition $2 \leq r \leq m \leq k + 1$.
   By the inductive hypothesis, we already know that both r and m have their own unique factorizations:

   $$r = s_1 \ldots s_l, \quad m = t_1 \ldots t_b$$

   where s_j and t_j are primes. Hence, by multiplying the two unique factorizations, we can obtain a new unique factorization for k+1. Thus, by strong induction, k+1 can be represented by a new unique factorization.

The Fundamental Theorem of Arithmetic is proven by strong induction.

3. In this proof, we use 2-D induction because we have two variables, n and k, and we need to prove the statement for both variables.

   To better understand the base case and the inductive hypothesis, we represent the cases in a matrix-like format: To simplify the representation Let P(n,k) be a funtion which takes n and k as parameters and map them

$$\begin{array}{ccccc} (1,1) & (1,2) & \dots & (1,k) & \\ \vdots & \vdots & \ddots & \vdots & \\ \vdots & \vdots & \ddots & \vdots & \\ (n,1) & (n,2) & \dots & (n,k) & (n,k+1) \\ \vdots & \vdots & \dots & (n+1,k) & (n+1,m+n) \end{array}$$

   Base Case:

   The base case is $P(1,1)$, and we define it as:

$$P(1,1) = 1$$

   Inductive Hypothesis:

   Assume that the statment holds for $P(n,k)$, $P(n,k+1)$, and $P(n+1,k)$.

   Inductive Step:

   We should show that if the statement is true for $P(n,k)$, $P(n,k+1)$, and $P(n+1,k)$, it follows that it must also hold for $P(n+1,k+1)$.

   For simplicity, we will continue with the following notations:

   - $e_{\substack{(n)\\k}}$: represents e_k for the set $x_1, x_2, \dots, x_n$.
   - $e_{\substack{(n+1)\\k}}$: represents e_k for the set $x_1, x_2, \dots, x_n, x_{n+1}$.
   - $p_{\substack{(n)\\k}}$: represents p_k for the set $x_1, x_2, \dots, x_n$.
   - $p_{\substack{(n+1)\\k}}$: represents p_k for the set $x_1, x_2, \dots, x_n, x_{n+1}$.

   We also use the following equation (for simplicity it will be referred to as Theorem 1):

$$e_{\substack{(n+1)\\k}} = e_{\substack{(n)\\k}} + x_{n+1} \cdot e_{\substack{(n)\\k-1}}$$

   This equation expresses the relationship between $e_{\substack{(n+1)\\k}}$ and its previous form $e_{\substack{(n)\\k}}$, allowing us to incorporate the new term x_{n+1} into the product.

   Now, we will prove the statement for $P(n+1,k+1)$. To do this, we will use (Theorem 2), which states:

$$p_{\substack{(n+1)\\ i}} = p_{\substack{(n)\\ i}} + e_{\substack{(i)\\ n+1}}$$

Now, $p_{\substack{(n+1)\\ i}}$ is:

$$p_{\substack{(n+1)\\ i}} = x_1^i + x_2^i + \cdots + x_n^i + x_{n+1}^i = (x_1^i + x_2^i + \cdots + x_n^i) + x_{n+1}^i = p_i^n + x_{n+1}^i$$

There are two cases to consider:

1. $1 \leq k < n$ 2. $1 \leq n < k$

Case 1: $1 \leq k < n$

We assume the following equations for the various terms:

$$(n,k): \quad k \cdot e_{\substack{(n)\\ k}} = \sum_{i=1}^{k} (-1)^{i-1} \cdot e_{\substack{(n)\\ k-i}} \cdot p_{\substack{(n)\\ i}} \tag{1}$$

$$(n+1,k): \quad k \cdot e_{\substack{(n+1)\\ k}} = \sum_{i=1}^{k} (-1)^{i-1} \cdot e_{\substack{(n+1)\\ k-i}} \cdot p_{\substack{(n+1)\\ i}} \tag{2}$$

$$(n,k+1): \quad (k+1) \cdot e_{\substack{(n)\\ k+1}} = \sum_{i=1}^{k+1} (-1)^{i-1} \cdot e_{\substack{(n)\\ k+1-i}} \cdot p_{\substack{(n)\\ i}} \tag{3}$$

We need to prove the equation for $P(n+1,k+1)$:

$$(n+1,k+1): \quad (k+1) \cdot e_{\substack{(n+1)\\ k+1}} = \sum_{i=1}^{k+1} (-1)^{i-1} \cdot e_{\substack{(n+1)\\ k+1-i}} \cdot p_{\substack{(n+1)\\ i}}$$

We use Theorem 1 for $e_{\substack{(n+1)\\ k+1-i}}$, but we must be cautious when the index i reaches $k+1$, as this would make $k - i = -1$, which is invalid. Therefore, the summation must go up to k.

$$\sum_{i=1}^{k+1} (-1)^{i-1} \cdot e_{\substack{(n+1)\\ k+1-i}} \cdot p_{\substack{(n+1)\\ i}} = \sum_{i=1}^{k} (-1)^{i-1} \cdot e_{\substack{(n+1)\\ k+1-i}} \cdot p_{\substack{(n+1)\\ i}} + (-1)^k \cdot e_{\substack{(n+1)\\ 0}} \cdot p_{\substack{(n+1)\\ k+1}}$$

We now substitute the terms and simplify:

$$S = \sum_{i=1}^{k} (-1)^{i-1} \cdot p_i^n \cdot e_{k+1-i}^n + x_{n+1} \sum_{i=1}^{k} (-1)^{i-1} \cdot p_i^n \cdot e_{k-i}^n + \sum_{i=1}^{k} (-1)^{i-1} \cdot x_{n+1}^i \cdot e_{k+1-i}^n + \sum_{i=1}^{k} (-1)^{i-1} \cdot x_{n+1}^{i+1} \cdot e_{k-i}^n$$

$$+ (-1)^k \cdot e_0^{n+1} \cdot p_{k+1}^n + (-1)^k \cdot e_0^{n+1} \cdot x_{n+1}^{k+1}$$

Let's simplify the expression:

$$S = (k+1) \cdot e_{k+1}^{n}$$

Conclusion:

Thus, we have proven that:

$$S = (k+1) \cdot e_{k+1}^{n+1}$$

Therefore, $P(n+1, k+1)$ is proven by the mathematical induction.

3 Question 3

(a) The solution of this problem can be obtained by using the concept of the distribution of indistinguishable objects to distinguishable boxes. To apply this concept size of the c-multiset is considered as indistinguishable objects let's say them balls and behave towards them like they are indistinguishable balls and we are separating them into c distinguishable bins. to achieve this separation c-1 divider is used. Then permutation with repetition is applied.

$$\frac{(n+c-1)!}{n!(c-1)!}$$

which equals to

$$\binom{n+c-1}{n}$$

by definition.

(b) the solution way of this problem has the same concept with part 1. 169 is chosen for indistinguishable objects and x's are behaven like distinguishable bins. To represent the solution process 169 is sembolyzed by 169 stars and 11 seperator are used as dividor. And permütation with repatition is applied

$$\frac{180!}{169!11!}$$

which equals to

$$1.18 \times 10^{17}$$

(c) Let's divide this problem into 2 cases. First one is the one which has the 9 as it's first digit, and second one is the one which has the different first digit which is not definitely 0 or 9. let's start with first case.

First Case: The number starts with 9

In this case, the number takes the form:

$$9x_1x_2x_3x_4x_5x_6$$

Since this 7-digit integer should be divisible by 3. By the definition that is given above we must have:

$$9 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3k$$

which simplifies to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3k$$

Now, we treat this problem as a stars and bars problem, where the sum $3k$ is represented by indistinguishable stars and the x_i's are distinguishable bins. Using stars and bars with $6 - 1 = 5$ dividers, to obtain the number of ways to distribute $3k$ among the $x_1, x_2, \ldots, x_6$, we should apply permutation with repetition which is given by:

$$\frac{(3k+5)!}{(3k)!5!} = \binom{3k+5}{3k}$$

This is the number of solutions for an arbitrary value of k. Since k can take values from 0 to 15 (because the sum of all the digits can be at most 45 when all $x_i = 9$), we sum over all possible values of k:

$$\sum_{k=0}^{15} \binom{3k+5}{3k}$$

Second Case: The number does not start with 9, but contains at least one 9

In this case, we must choose the position of the '9'. Since the first digit cannot be '9' or '0'(because if it would be zero, then this integer cannot be 7-digit), the '9' must appear in one of the last 6 positions. The number of ways to choose one of these positions is:

$$\binom{6}{1}$$

Next, we consider the sum of the remaining digits. Let the number be represented as $x_1x_2x_3x_4x_5x_6$, where the digit '9' has already been placed in one of the positions. However, the all the remaining solutions have 6 possibility. We then divide the problem into smaller cases based on the value of x_1.

Case a: $x_1 \equiv 0 \pmod{3}$

This means that, x_1 can be 3 or 6(because it cannot be 0 or 9 even if they satisfy the condition). To ensure the sum of the digits is divisible by 3, the sum of the remaining digits must satisfy:

$$x_2 + x_3 + x_4 + x_5 + x_6 = 3k$$

Again, using stars and bars, the number of ways to distribute $3k$ among x_2, x_3, x_4, x_5, x_6 is:

$$\binom{3k+4}{3k}$$

Since k ranges from 0 to 15, we sum over all values of k:

$$\sum_{k=0}^{15} \binom{3k+4}{3k}$$

Multiplying by 2 (since x_1 can be either 3 or 6), we get:

$$2\sum_{k=0}^{15}\binom{3k+4}{3k}$$

Case b: $x_1 \equiv 1 \pmod 3$

In this case, x_1 can be 1, 4, or 7. To be divisible by 3, the sum of the remaining digits must satisfy:

$$x_2 + x_3 + x_4 + x_5 + x_6 = 3k + 2$$

Again, using stars and bars, the number of ways to distribute $3k+2$ among x_2, x_3, x_4, x_5, x_6 is:

$$\binom{3k+6}{3k+2}$$

Summing over all values of k from 0 to 14 (since $3k+2$ must be at most 45), we get:

$$\sum_{k=0}^{14}\binom{3k+6}{3k+2}$$

Multiplying by 3 (since x_1 can be 1, 4, or 7), we get:

$$3\sum_{k=0}^{14}\binom{3k+6}{3k+2}$$

Case c: $x_1 \equiv 2 \pmod 3$

In this case, x_1 can be 2, 5, or 8. To be divisible by 3, the sum of the remaining digits must satisfy:

$$x_2 + x_3 + x_4 + x_5 + x_6 = 3k + 1$$

Using stars and bars, the number of ways to distribute $3k+1$ among x_2, x_3, x_4, x_5, x_6 is:

$$\binom{3k+5}{3k+1}$$

Summing over all values of k from 0 to 14, we get:

$$\sum_{k=0}^{14}\binom{3k+5}{3k+1}$$

Multiplying by 3 (since x_1 can be 2, 5, or 8), we get:

$$3\sum_{k=0}^{14}\binom{3k+5}{3k+1}$$

To obtain the total result of case 2, we should combine our case a,b,c results:

$$6\left(2\sum_{k=0}^{15}\binom{3k+4}{3k}+3\sum_{k=0}^{14}\binom{3k+6}{3k+2}+3\sum_{k=0}^{14}\binom{3k+5}{3k+1}\right)$$

To find the total number of valid seven-digit integers, we combine the results from both cases (case 1+case 2):

$$\sum_{k=0}^{15}\binom{3k+5}{3k}+6\left(2\sum_{k=0}^{15}\binom{3k+4}{3k}+3\sum_{k=0}^{14}\binom{3k+6}{3k+2}+3\sum_{k=0}^{14}\binom{3k+5}{3k+1}\right)$$

4 Question 4 or 5

(a) To define the set D_n, we should first look at how we can define a bijective function from V to V that maps the n-gon onto itself, with the condition of preserving its geometry because it is the set of these functions for n.
Let $F(n)$ be a function that maps an n-gon onto itself. To define $F(n)$, we should take into account different cases. We can derive this function by using 3 different ways. The first one is rotating, the second one is taking its symmetry according to the diagonals which pass through the origin, and the third one is taking its symmetry according to the lines which pass through the midpoint of an edge, the origin, and the opposite corner of that edge.

Let's define the first part of $F(n)$. (Rotating)
To visualize the mapping with rotation, let's take a 4-gon and a 5-gon.

The angle of rotation is defined by using the equation $2\pi/n$, where n is the number of edges of the n-gon. The reason behind why we use this equation for the angle of rotation is to preserve the original geometry, even if we rotate the object, it should look the same, so we place edges and vertices onto the same place. To achieve this aim we should rotate them with angle which is the angle between the lines which is obtained by combining the center and a vertex, and center and next vertex of the previous one.
For the 4-gon, the rotation angle is $\pi/2$.
For the 4-gon:
Rotation with the angle of 0 rad:
$F(n)$ maps A to A, B to B, C to C, D to D.
Rotation with the angle of $\pi/2$ rad:
$F(n)$ maps A to C, B to D, C to A, D to B.
Rotation with π rad:
$F(n)$ maps A to C, B to D, C to A, D to B.
Rotation with $3\pi/2$ rad:
$F(n)$ maps A to D, B to A, C to B, D to C.
Rotation with 2π rad:
Which is the same as 0 rad because the rotation is completed.

As it can be derived from above, if we rotate an n-gon the number of its edges times, we return to its original version. This can also be concluded from the angle of rotation equation: If we decide the angle of rotation by using $2\pi/n$ equation, to return back we should rotate n times, and then we can obtain 2π rad, which is the same as 0 rad. Hence, it can be concluded from the observations above that we can generate n different functions by using rotation for an n-gon.

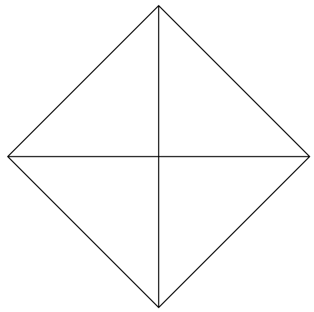

Figure 1

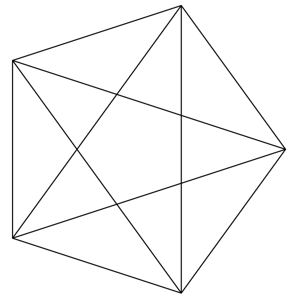

Figure 3

Part 2: (Taking symmetry according to diagonals which pass through the origin)
If we try to define a function which maps an n-gon onto itself according to its diagonals which pass through the origin, we can come up with some edge cases because if our n-gon has an odd number of edges (n is odd), there is no diagonal which passes through the origin. This is because if we take an arbitrary edge from an n-gon (n is odd), it has no opposite corner, so diagonals cannot pass through the origin.
If n is an even number, it has $n/2$ symmetries for this concept. We can think of this as combining opposite corners as symmetry lines.

As it can be analyzed from above, if n is an odd number, $F(n)$ cannot be generated. However, if n is even, $n/2$ different $F(n)$ functions can be generated.

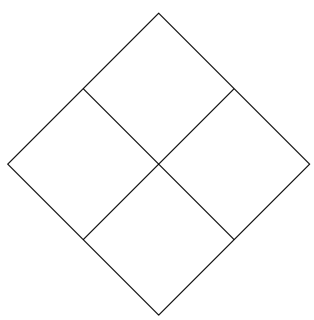

Figure 2

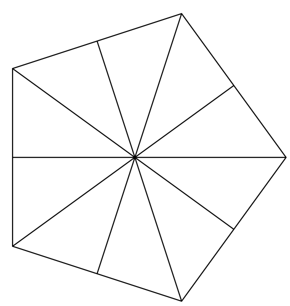

Figure 5

Part 3: (Taking symmetry according to lines which pass through the midpoint of an edge-origin-opposite corner of this edge)
If we try to define a function which maps an n-gon onto itself according to its symmetry lines, which are obtained by combining the midpoint of an edge, the origin, and the opposite corner of this edge, we can come up with some edge cases. If n is an even number, $n/2$ different $F(n)$ functions can be generated. Since for every edge, there is an opposite edge; for every corner, there is an opposite corner, $n/2$ of them satisfy a line which passes through a corner-origin and this corner's opposite edge. However, if n is an odd number, since there is an opposite edge for every corner, for every corner a line that is defined above can be drawn (n times).

As it can be analyzed from above, if n is an even number, $n/2$ different $F(n)$ functions can be generated.
However, if n is odd, n different $F(n)$ functions can be generated.

Hence, the number of $2n$ functions $F(n)$ can be defined for each scenario because the sum of all possibilities is equal to $2n$, whether n is odd or even.

If we can define the number of $2n$ functions for an n-gon, it can be concluded that the cardinality of D_n equals $2n$, which is finite.

(b) It easy to observe that we already define our F(n) function by using r and s functions. Because these are the ways that can be used for generating a bijective mapping function while avoiding deform the object.
If we come up with following these rules we can conclude that S is equivalent to our function. Hence we can use our already generated function above to proof.
To proof S is isomorphic to D$_n, weshouldproof$:
$Itpreservesgroupoperations$: ($S(a$·b)=S(a)·S(b))
This means that applying the function S two the pair of two operation such as, two rotation, one rotation and one symmetry gives the same result with applying S to them separately. Let's explain with two rotation:
Take r_k represents the 2k/n radians rotation. Then we have:
$S(r_1 \cdot\ r_2\ )=$