The Left Null Space of a Vector

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Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ be any vector, let $\mathbf{e}_1 = (1, 0, \ldots, 0)^\top$, and let $\mathbf{v} = \mathbf{x} \pm \|x\|\mathbf{e}_1$. Calculate the Householder matrix $\mathtt{H}_\mathbf{v}$ such that $\mathtt{H}_\mathbf{v}\mathbf{x} = \mp\|x\|\mathbf{e}_1 = (\mp\|x\|, 0, \ldots, 0)^\top$.

$$\mathtt{H}_\mathbf{v} = \mathtt{I} - 2\frac{\mathbf{v}\mathbf{v}^\top}{\mathbf{v}^\top\mathbf{v}}$$

Note that $\mathtt{H}_\mathbf{v}$ is symmetric and orthogonal. $\mathbf{v}$ is called the Householder vector for $\mathbf{x}$. For stability reasons, the sign ambiguity in defining $\mathbf{v}$ should be resolved by setting

$$\mathbf{v} = \mathbf{x} + \operatorname{sign}(x_1)\|\mathbf{x}\|\mathbf{e}_1$$

In $[\mathbf{x}]^\perp\mathbf{x} = \mathbf{0}$, $[\mathbf{x}]^\perp$ is the left null space of $\mathbf{x}$ and may be calculated from $\mathtt{H}_\mathbf{v}$ as follows.

$$\mathtt{H}_\mathbf{v} = \begin{bmatrix} \mathbf{h}_\mathbf{v}^{1\top} \\ [\mathbf{x}]^\perp \end{bmatrix}$$

(i.e., $[\mathbf{x}]^\perp$ is $\mathtt{H}_\mathbf{v}$ with the first row omitted). Further, note that $([\mathbf{x}]^\perp)^\top$ is the (right) null space of $\mathbf{x}^\top$ (i.e., $\mathbf{x}^\top([\mathbf{x}]^\perp)^\top = \mathbf{0}^\top$.