HW2 Theory Problems Solution

November 2018

Problem 1: Steradians

Let $(\theta_{\mathbf{v}}, \phi_{\mathbf{v}})$ be the spherical coordinate of $\mathbf{v}$. We have

$$\theta_{\mathbf{v}} = \arccos \frac{z_{\mathbf{v}}}{1} = \frac{\pi}{6}, \text{ and}$$

$$\phi_{\mathbf{v}} = \arctan \frac{y_{\mathbf{v}}}{x_{\mathbf{v}}} = \frac{\pi}{4}.$$

Therefore, the spherical wedge's integration range is

$$\theta' \in \left[\theta_{\mathbf{v}} - \frac{\theta}{2}, \theta_{\mathbf{v}} + \frac{\theta}{2}\right] = \left[\frac{\pi}{12}, \frac{\pi}{4}\right], \text{ and}$$

$$\phi' \in \left[\phi_{\mathbf{v}} - \frac{\phi}{2}, \phi_{\mathbf{v}} + \frac{\phi}{2}\right] = \left[\frac{\pi}{6}, \frac{\pi}{3}\right].$$

So we can calculate the solid angle as

$$\begin{aligned}
\Omega &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin\theta' d\theta' d\phi' \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(-\cos\theta'\right)\Big|_{\frac{\pi}{12}}^{\frac{\pi}{4}} d\phi' \\
&= \left(-\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\right) \cdot \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\
&= \frac{\sqrt{6} - \sqrt{2}}{4} \cdot \frac{\pi}{6} \\
&\approx 0.136\,\mathrm{sr}.
\end{aligned}$$

Problem 2: Irradiance

Let

- $\theta_1 = 60°$ be the angle between the optical axis and the wall's surface normal,
- $\theta_2 = 0$ be the angle between the optical axis and the image plane's surface normal, and
- $p = 1$ mm be the length of the pixel.

1.A.

Denote the area by dA_0. We have $\frac{dA_0}{p^2} = \frac{d^2}{f^2}$. So, $dA_0 = \frac{d^2}{f^2} \cdot p^2 = 400\,\text{mm}^2$.

1.B.

As stated in the question, we neglect perspective distortion, so dA_1 is a rectangle. After dA_1 is foreshortened by $60°$, it will look like parallel to the pixel. The parallel area is dA_0, so we have $dA_1 \cos\theta_1 = dA_0$.

Therefore, $dA_1 = \frac{dA_0}{\cos\theta_1} = \frac{2d^2}{f^2} \cdot p^2 = 800\,\text{mm}^2$.

1.C.

Radiance transfer from dA_1 to dA_2 is

$$L = \frac{P}{dA_1 \cos\theta_1 d\omega_{1\to 2}},$$

where the solid angle $d\omega_{1\to 2}$ is

$$d\omega_{1\to 2} = \frac{dA_2 \cos\theta_2}{d^2}.$$

Therefore,

$$\begin{aligned} P &= L dA_1 \cos\theta_1 \frac{dA_2 \cos\theta_2}{d^2} \\ &= L \frac{2\cancel{d^2} p^2}{f^2} \cdot \frac{1}{2} \cdot \frac{dA_2}{\cancel{d^2}} \\ &= L \frac{p^2}{f^2} dA_2 \\ &= \frac{L dA_2}{2500} \text{ Watts.} \end{aligned} \tag{1}$$

1.D.

The power P received at dA_2 passes through the pinhole and is received by the central pixel, whose area is p^2. So the irradiance is

$$E = \frac{P}{p^2} = \frac{P}{(0.001\,\mathrm{m})^2} = 400 L dA_2 \text{ Watts/m}^2.$$

2. & 3.

Note that in **1.C.** step (1), the distance d is cancelled out. So the power P remains the same when d changes to 2000 mm and 4000 mm. Also, the area of pixel is unchanged. So we have

$$E = \frac{P}{p^2} = \frac{L\frac{p^2}{f^2}dA_2}{p^2} = \frac{LdA_2}{f^2} = \frac{LdA_2}{(0.05\,\mathrm{m})^2} = 400 L dA_2 \text{ Watts/m}^2$$

for both **2.** and **3.**.

4.

The area on the wall dA_1 is proportional to d^2, while the solid angle $d\omega_{1\to 2}$ is proportional to $\frac{1}{d^2}$. Together they cancel out in image irradiance, so the irradiance is independent of d. This explains why the wall won't get brighter or darker when the camera backs away from it.

Problem 3: Diffused Objects and Brightness

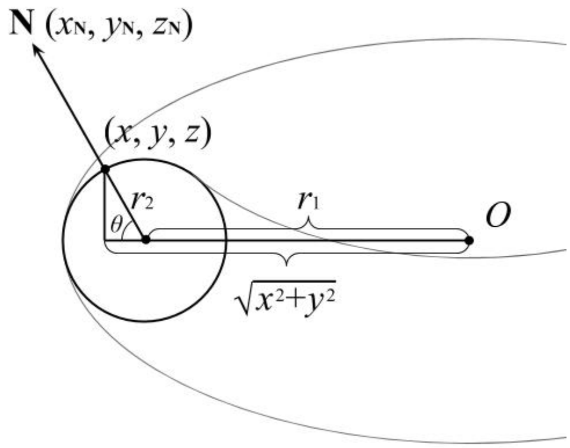

Figure 1: Problem 3 solution

Let $\mathbf{N} = (x_{\mathbf{N}}, y_{\mathbf{N}}, z_{\mathbf{N}})$ be the surface normal of the torus on point (x, y, z). The intensity on this point is

$$\begin{aligned} L &= \rho \mathbf{N} \cdot \mathbf{S} \\ &= \rho\,(x_{\mathbf{N}}, y_{\mathbf{N}}, z_{\mathbf{N}}) \cdot (0, 0, 1) \\ &= \rho z_{\mathbf{N}}. \end{aligned}$$

Figure 1 shows that (x, y, z) is on a cross section plane that rotates around z axis. Since $\mathbf{N}$ is a unit vector, we know that $z_{\mathbf{N}} = \sin\theta = \dfrac{z}{r_2}$.

To calculate z, we apply Pythagorean theorem to the triangle inside the cross section circle:

$$z^2 + \left(r_1 - \sqrt{x^2 + y^2}\right)^2 = r_2^2.$$

So $z = \sqrt{r_2^2 - \left(r_1 - \sqrt{x^2 + y^2}\right)^2}$. Therefore,

$$L = \rho z_{\mathbf{N}} = \rho \frac{z}{r_2} = \rho \frac{\sqrt{r_2^2 - \left(r_1 - \sqrt{x^2 + y^2}\right)^2}}{r_2}.$$

Since $r_1 = 4$ and $r_2 = 1$, finally $L = \rho \sqrt{1 - \left(4 - \sqrt{x^2 + y^2}\right)^2}$.

Problem 4: Occlusion, Umbra and Penumbra

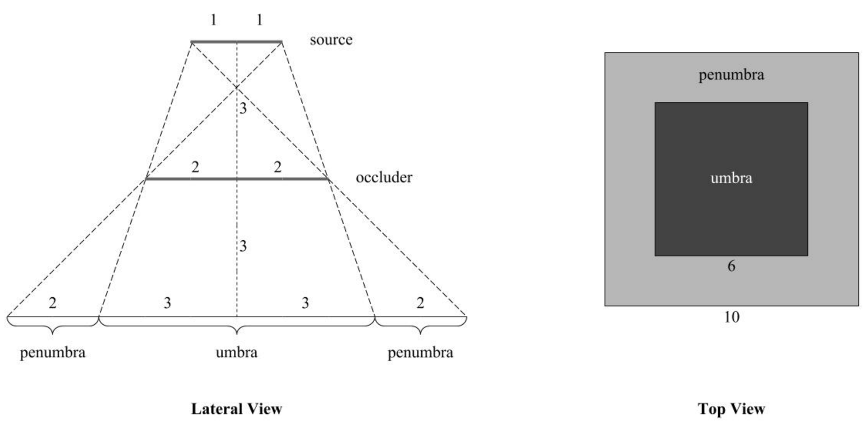

Figure 2: Problem 4 solution

As shown in Figure 2, the shape of umbra is a square whose side is 6 (by similar triangles). Therefore, $A_{\text{umbra}} = 6 \times 6 = 36$.

Similarly, the outer bound of penumbra is a square whose side is 10. The penumbra is the area around the umbra. Therefore, $A_{\text{penumbra}} = 10 \times 10 - A_{\text{umbra}} = 64$.