Jacobian Matrices for Imaging Geometry

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February 21, 2018

This paper lists analytical expressions of Jacobian matrices for several operations and mappings used in imaging geometry.

Notation

The following notation is used throughout this paper.

- Homogeneous coordinates in 2D are represented as 3-vectors and shown as lowercase letters, e.g., $\mathbf{x} = (x, y, w)^\top$. Additionally, normalized coordinates include a hat, e.g., $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{w})^\top$.
- Inhomogeneous coordinates in 2D are represented as 2-vectors and shown as lowercase letters with a tilde, e.g., $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})^\top = (x/w, y/w)^\top$, and may also include normalized coordinates, e.g., $\hat{\tilde{\mathbf{x}}} = (\hat{\tilde{x}}, \hat{\tilde{y}})^\top = (\hat{x}/\hat{w}, \hat{y}/\hat{w})^\top$.
- Homogeneous coordinates in 3D are represented as 4-vectors and shown as uppercase letters, e.g., $\mathbf{X} = (X, Y, Z, T)^\top$.
- Inhomogeneous coordinates in 3D are represented as 3-vectors and shown as uppercase letters with a tilde, e.g., $\widetilde{\mathbf{X}} = (\widetilde{X}, \widetilde{Y}, \widetilde{Z})^\top = (X/T, Y/T, Z/T)^\top$.
- If an upper case letter is used to denote a matrix, then the vector denoted by the corresponding lower case letter is composed of the entries of the matrix by

$$\mathtt{A} \in \mathbb{R}^{m\times n} \Leftrightarrow \mathtt{A} = \begin{bmatrix} \mathbf{a}^{1\top} \\ \mathbf{a}^{2\top} \\ \vdots \\ \mathbf{a}^{m\top} \end{bmatrix}, \mathbf{a} = \begin{pmatrix} \mathbf{a}^{1} \\ \mathbf{a}^{2} \\ \vdots \\ \mathbf{a}^{m} \end{pmatrix} \in \mathbb{R}^{mn}$$

  where $\mathbf{a}^{i\top} \in \mathbb{R}^n$ is the ith row of $\mathtt{A}$ (i.e., $\mathbf{a} = \mathrm{vec}(\mathtt{A}^\top)$).

1 Sinc function

The sinc function

$$\mathrm{sinc}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin(x)}{x} & \text{otherwise} \end{cases} \tag{1}$$

The derivative is given by

$$\frac{\mathrm{d}\,\mathrm{sinc}(x)}{\mathrm{d}x} = \begin{cases} 0 & \text{if } x = 0 \\ \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} & \text{otherwise} \end{cases} \tag{2}$$

2 Parameterization of a homogeneous vector

Let the homogeneous vector $\bar{\mathbf{v}} = (a, \mathbf{b}^\top)^\top \in \mathbb{R}^n$, where $\|\bar{\mathbf{v}}\| = 1$ (i.e., $\bar{\mathbf{v}}$ is a unit vector), be parameterized as

$$\mathbf{v} = \frac{2}{\mathrm{sinc}(\cos^{-1}(a))}\mathbf{b} \in \mathbb{R}^{n-1} \tag{3}$$

then, if $\|\mathbf{v}\| > \pi$, normalized by

$$\mathbf{v} = \left(1 - \frac{2\pi}{\|\mathbf{v}\|}\left\lceil \frac{\|\mathbf{v}\| - \pi}{2\pi} \right\rceil\right)\mathbf{v} \tag{4}$$

The parameterized homogeneous vector $\mathbf{v}$ is deparameterized as the homogeneous vector

$$\bar{\mathbf{v}} = \left(\cos\left(\frac{\|\mathbf{v}\|}{2}\right), \frac{\mathrm{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{2}\mathbf{v}^\top\right)^\top \in \mathbb{R}^n \tag{5}$$

$$\bar{\mathbf{v}} = \left(a, \mathbf{b}^\top\right)^\top, \text{ where } a = \cos\left(\frac{\|\mathbf{v}\|}{2}\right) \text{ and } \mathbf{b} = \frac{\mathrm{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{2}\mathbf{v}$$

where $\|\bar{\mathbf{v}}\| = 1$ and a is nonnegative. For the deparameterization,

$$\frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{v}} = \frac{\partial(a, \mathbf{b}^\top)}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\mathrm{d}a}{\partial \mathbf{v}} \\ \frac{\partial \mathbf{b}}{\partial \mathbf{v}} \end{bmatrix} \in \mathbb{R}^{n \times (n-1)} \tag{6}$$

where

$$\frac{\mathrm{d}a}{\partial \mathbf{v}} = \begin{cases} \mathbf{0}^\top & \text{if } \|\mathbf{v}\| = 0 \\ -\frac{1}{2}\mathbf{b}^\top & \text{otherwise} \end{cases}$$

and

$$\frac{\partial \mathbf{b}}{\partial \mathbf{v}} = \begin{cases} \frac{1}{2}\mathtt{I} & \text{if } \|\mathbf{v}\| = 0 \\ \frac{\mathrm{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{2}\mathtt{I} + \frac{1}{4\|\mathbf{v}\|}\frac{\mathrm{d}\,\mathrm{sinc}\left(\frac{\|\mathbf{v}\|}{2}\right)}{\mathrm{d}\frac{\|\mathbf{v}\|}{2}}\mathbf{v}\mathbf{v}^\top & \text{otherwise} \end{cases}$$

3 Projection of a point under the camera projection matrix

The homogeneous 3D point $\mathbf{X}$ is projected to the homogeneous 2D point $\mathbf{x}$ under the (homogeneous) camera projection matrix $\mathtt{P}$ by

$$\mathbf{x} = \mathtt{P}\mathbf{X} \tag{7}$$

Dehomogenizing the 2D point results in the mapping $\mathbf{X} \mapsto \tilde{\mathbf{x}}$. For this mapping

$$\frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{p}} = \frac{1}{w} \begin{bmatrix} \mathbf{X}^\top & \mathbf{0}^\top & -\tilde{x}\mathbf{X}^\top \\ \mathbf{0}^\top & \mathbf{X}^\top & -\tilde{y}\mathbf{X}^\top \end{bmatrix} \tag{8}$$

and

$$\frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{X}} = \frac{1}{w} \begin{bmatrix} \mathbf{p}^{1\top} - \tilde{x}\mathbf{p}^{3\top} \\ \mathbf{p}^{2\top} - \tilde{y}\mathbf{p}^{3\top} \end{bmatrix} \tag{9}$$

where $w = \mathbf{p}^{3\top}\mathbf{X}$ and $\mathbf{p}^{i\top}$ is the ith row of $\mathtt{P}$.

4 Mapping of a vector under an affine transformation

An affine transformation consists of a linear transformation matrix $\mathtt{A}$ and a translation vector $\mathbf{t}$. The vector $\mathbf{v}$ is transformed to $\mathbf{v}'$ under an affine transformation by

$$\mathbf{v}' = \mathtt{A}\mathbf{v} + \mathbf{t} \tag{10}$$

For this transformation

$$\frac{\partial \mathbf{v}'}{\partial \mathbf{v}} = \mathtt{A} \tag{11}$$

$$\frac{\partial \mathbf{v}'}{\partial \mathbf{a}} = \mathtt{I} \otimes \mathbf{v}^\top \tag{12}$$

and

$$\frac{\partial \mathbf{v}'}{\partial \mathbf{t}} = \mathtt{I} \tag{13}$$

5 Vector norm

Given a vector $\mathbf{v}$, its norm is written as $\|\mathbf{v}\|$. The derivative is given by

$$\frac{\mathrm{d}\|\mathbf{v}\|}{\partial \mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}^\top \tag{14}$$

6 3D Rotation, angle-axis representation

The 3-vector $\mathbf{v}$ is rotated to $\mathbf{v}'$ under the angle-axis representation $\boldsymbol{\omega}$ by

$$\mathbf{v}' = \exp([\boldsymbol{\omega}]_\times)\mathbf{v} \tag{15}$$

$$\mathbf{v}' = \begin{cases} \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} & \text{if } \theta \text{ is 0 or nearly 0} \\ \mathbf{v} + \operatorname{sinc}(\theta)\boldsymbol{\omega} \times \mathbf{v} + \frac{1-\cos(\theta)}{\theta^2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{v}) & \text{otherwise} \end{cases} \tag{16}$$

where $\theta = \|\boldsymbol{\omega}\|$. For this rotation

$$\frac{\partial \mathbf{v}'}{\partial \boldsymbol{\omega}} = \begin{cases} [-\mathbf{v}]_\times & \text{if } \theta \text{ is 0 or nearly 0} \\ \operatorname{sinc}(\theta)[-\mathbf{v}]_\times + \boldsymbol{\omega} \times \mathbf{v}\dfrac{\mathrm{d}\operatorname{sinc}(\theta)}{\mathrm{d}\theta}\dfrac{\mathrm{d}\theta}{\partial \boldsymbol{\omega}} & \\ \quad + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{v})\dfrac{\mathrm{d}s}{\mathrm{d}\theta}\dfrac{\mathrm{d}\theta}{\partial \boldsymbol{\omega}} + s([\boldsymbol{\omega}]_\times[-\mathbf{v}]_\times + [-(\boldsymbol{\omega} \times \mathbf{v})]_\times) & \text{otherwise} \end{cases} \tag{17}$$

where

$$s = \frac{1-\cos(\theta)}{\theta^2} \text{ and } \frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{\theta\sin(\theta) - 2(1-\cos(\theta))}{\theta^3}$$

and

$$\frac{\partial \mathbf{v}'}{\partial \mathbf{v}} = \exp([\boldsymbol{\omega}]_\times) \tag{18}$$

7 Projection of a point under the normalized camera projection matrix

The inhomogeneous 3D point $\tilde{\mathbf{X}}$ is projected to the homogeneous 2D point in normalized coordinates $\tilde{\mathbf{x}}$ under the normalized camera projection matrix $\hat{\mathtt{P}} = [\exp([\boldsymbol{\omega}]_\times) \,|\, \mathbf{t}] = [\mathtt{R} \,|\, \mathbf{t}]$, where $\mathtt{R} = \exp([\boldsymbol{\omega}]_\times)$, by

$$\hat{\mathbf{x}} = [\exp([\boldsymbol{\omega}]_\times) \,|\, \mathbf{t}] \begin{bmatrix} \tilde{\mathbf{X}} \\ 1 \end{bmatrix} \tag{19}$$

$$\hat{\mathbf{x}} = \exp([\boldsymbol{\omega}]_\times)\tilde{\mathbf{X}} + \mathbf{t} \tag{20}$$

$$\hat{\mathbf{x}} = \tilde{\mathbf{X}}_{\text{rotated}} + \mathbf{t} \tag{21}$$

where $\tilde{\mathbf{X}}_{\text{rotated}} = \exp([\boldsymbol{\omega}]_\times)\tilde{\mathbf{X}} = \mathtt{R}\tilde{\mathbf{X}}$. Dehomogenizing the 2D point results in the mapping $\tilde{\mathbf{X}} \mapsto \tilde{\hat{\mathbf{x}}}$. For this mapping

$$\frac{\partial \tilde{\hat{\mathbf{x}}}}{\partial \boldsymbol{\omega}} = \frac{\partial \tilde{\hat{\mathbf{x}}}}{\partial \tilde{\mathbf{X}}_{\text{rotated}}} \frac{\partial \tilde{\mathbf{X}}_{\text{rotated}}}{\partial \boldsymbol{\omega}} \tag{22}$$

$$\frac{\partial \tilde{\hat{\mathbf{x}}}}{\partial \tilde{\mathbf{X}}} = \frac{\partial \tilde{\hat{\mathbf{x}}}}{\partial \tilde{\mathbf{X}}_{\text{rotated}}} \frac{\partial \tilde{\mathbf{X}}_{\text{rotated}}}{\partial \tilde{\mathbf{X}}} \tag{23}$$

where

$$\frac{\partial \tilde{\hat{\mathbf{x}}}}{\partial \tilde{\mathbf{X}}_{\text{rotated}}} = \begin{bmatrix} 1/\hat{w} & 0 & -\tilde{\hat{x}}/\hat{w} \\ 0 & 1/\hat{w} & -\tilde{\hat{y}}/\hat{w} \end{bmatrix}$$

where $\hat{w} = \tilde{Z}_{\text{rotated}} + t_3$, $\frac{\partial \tilde{\mathbf{X}}_{\text{rotated}}}{\partial \boldsymbol{\omega}}$ is calculated using (17), and $\frac{\partial \tilde{\mathbf{X}}_{\text{rotated}}}{\partial \tilde{\mathbf{X}}} = \exp([\boldsymbol{\omega}]_\times) = \mathtt{R}$, and

$$\frac{\partial \tilde{\hat{\mathbf{x}}}}{\partial \mathbf{t}} = \begin{bmatrix} 1/\hat{w} & 0 & -\tilde{\hat{x}}/\hat{w} \\ 0 & 1/\hat{w} & -\tilde{\hat{y}}/\hat{w} \end{bmatrix} \tag{24}$$

where $\hat{w} = \mathbf{r}^{3\top}\tilde{\mathbf{X}} + t_3$.

8 Mapping of a point under a 2D projective transformation

The homogeneous 2D point $\mathbf{x}$ is mapped to the homogeneous 2D point $\mathbf{x}'$ under the (homogeneous) 2D projective transformation matrix $\mathtt{H}$ by

$$\mathbf{x}' = \mathtt{H}\mathbf{x} \tag{25}$$

Dehomogenizing $\mathbf{x}'$ results in the mapping $\mathbf{x} \mapsto \tilde{\mathbf{x}}'$. For this mapping

$$\frac{\partial \tilde{\mathbf{x}}'}{\partial \mathbf{h}} = \frac{1}{w'} \begin{bmatrix} \mathbf{x}^\top & \mathbf{0}^\top & -\tilde{x}'\mathbf{x}^\top \\ \mathbf{0}^\top & \mathbf{x}^\top & -\tilde{y}'\mathbf{x}^\top \end{bmatrix} \tag{26}$$

and

$$\frac{\partial \tilde{\mathbf{x}}'}{\partial \mathbf{x}} = \frac{1}{w'} \begin{bmatrix} \mathbf{h}^{1\top} - \tilde{x}'\mathbf{h}^{3\top} \\ \mathbf{h}^{2\top} - \tilde{y}'\mathbf{h}^{3\top} \end{bmatrix} \tag{27}$$

where $w' = \mathbf{h}^{3\top}\mathbf{x}$ and $\mathbf{h}^{i\top}$ is the ith row of $\mathtt{H}$.