International Journal of Computer Vision, 13, 3, 331–356 (1994)

Systems and Replication

Review and Analysis of Solutions of the Three Point Perspective Pose Estimation Problem

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Received October 30, 1990; revised March 27, 1992 and October 29, 1993

Abstract. In this paper, the major direct solutions to the three point perspective pose estimation problems are reviewed from a unified perspective beginning with the first solution which was published in 1841 by a German mathematician, continuing through the solutions published in the German and then American photogrammetry literature, and most recently in the current computer vision literature. The numerical stability of these three point perspective solutions are also discussed. We show that even in case where the solution is not near the geometric unstable region, considerable care must be exercised in the calculation. Depending on the order of the substitutions utilized, the relative error can change over a thousand to one. This difference is due entirely to the way the calculations are performed and not due to any geometric structural instability of any problem instance. We present an analysis method which produces a numerically stable calculation.

1 Introduction

Given the perspective projection of three points constituting the vertices of a known triangle in 3D space, it is possible to determine the position of each of the vertices. There may be as many as four possible solutions for point positions in front of the center of perspectivity and four corresponding solutions whose point positions are behind the center of perspectivity. In photogrammetry, this problem is called the three point space resection problem.

This problem is important in photogrammetry as well as in computer vision, because it has a variety of applications, such as camera calibration (Tsai 1987), object recognition, robot picking, and robot navigation (Linnainmaa et al. 1988; Horaud 1987; Lowe 1987; Dhome 1988) in computer vision and the determination of the location in space from a set of landmarks appearing in the image in photogrammetry (Fischler and Bolles 1981). Three points is the minimal information to solve such a problem. It was solved by a direct solution first by a German mathematician in 1841 (Grunert 1841) and then refined by German photogrammatrists in 1904 and 1925 (Müller 1925). Then it was independently solved by an American photogrammatrist in 1949 (Merritt 1949).

The importance of the direct solution became less important to the photogrammetry community with the advent of iterative solutions which could be done by computer. The iterative solution technique which was first published by Church (1945, 1948), needs a good starting value which constitutes an approximate solution. In most photogrammetry situations scale and distances are known to within 10% and angle is known to within 15°. This is good enough for the iterative technique which is just a repeated adjustment to the linearized equations. The technique can be found in many photogrammetry books such as Wolf (1974) or the Manual of Photogrammetry (Slama 1980).

The exact opposite is true for computer vision problems. Most of the time approximate solutions are not known so that the iterative method cannot be used. This makes the direct solution method more important in computer vision. Indeed, in 1981 the computer vision community independently derived its first direct solution (Fischler and Bolles 1981). And the community has produced a few more direct solutions since then.

In this paper, first, we give a consistent treatment of all the major direct solutions to the three point pose estimation problem. There is a bit of mathematical tedium in describing the various solutions, and perhaps it is worthwhile to put them all in one place so that another vision researcher can be saved from having to redo the tedium himself or herself. Then, we compare the differences of the algebraic derivations and discuss the singularity of all the solutions. In addition to determining the positions of the three vertices in the 3D camera coordinate system, it is desirable to determine the transformation function from a 3D world coordinate system to the 3D camera coordinate system, which is called the absolute orientation in photogrammetry. Though many solutions to the absolute orientation can be found in photogrammetry literature (Schut 1960; Schonemann 1966; 1970; Wolf 1974; Slama 1980; Horn 1988; and Haralick et al. 1989) we present a simple linear solution in Appendix I to make the three point perspective pose estimation solution complete.

Second, we run experiments to study the numerical stability of each of these solutions and to evaluate some analysis methods described in Appendix II to improve the numerical stability of the calculation. It is well-known that rounding errors accumulate with increasing amounts of calculation and significantly magnify in some kinds of operations. Furthermore, we find that the order of using these equations to derive the final solution affects the accuracy of numerical results. The results show that the accuracy can be improved by a factor of about 10^3. Since the accumulation of rounding errors will be propagated into the calculation of the absolute orientation problem. As a result the error would be very serious at the final stage. In the advent of better sensors and higher image resolution, the numerical stability will play a more dominant role in the errors of computer vision problem.

Finally, we summarize the result of hundreds of thousands experiments which study the numerical behaviors of the six different solution techniques, the effect of the order of equation manipulation, and the effectiveness of analysis methods. These results show that the analysis techniques in Appendix II are effective in determining equation order manipulation. The source codes and documentation used for the experiments in the paper is available on a CDROM. The interested readers can send a request to the Intelligent Systems Laboratory at the University of Washington.

2 The Problem Definition

Grunert (1841) appears to have been the first one to solve the problem. The solution he gives is outlined by Müller (1925). The problem can be set up in the following way which is illustrated in Figure 1.

Let the unknown positions of the three points of the known triangle be

$$p_1, p_2, \text{ and } p_3;\ p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \quad i = 1,2,3.$$

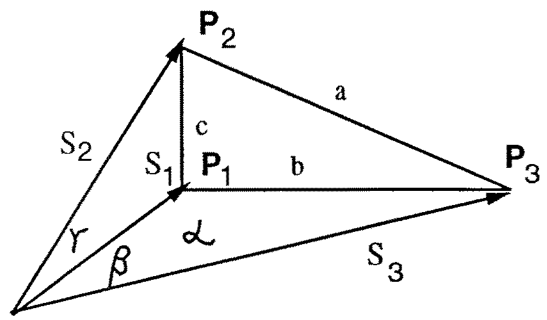

Fig. 1. Illustrates the geometry of the three point space resection problem. The triangle side lengths a, b and c are known and the unit vectors j_1, j_2, and j_3 are known. The problem is to determine the lengths s_1, s_2, and s_3 from which the 3D vertex point positions p_1, p_2, and p_3 can be immediately determined.

Let the known side lengths of the triangle be

$$\begin{aligned} a &= \|p_2 - p_3\| \\ b &= \|p_1 - p_3\| \\ c &= \|p_1 - p_2\|. \end{aligned}$$

We take the origin of the camera coordinate frame to be the center of perspectivity and the image projection plane to be a distance f in front of the center of perspectivity. Let the observed perspective projection of p_1, p_2, p_3 be q_1, q_2, q_3, respectively;

$$q_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad i = 1, 2, 3.$$

By the perspective equations,

$$\begin{aligned} u_i &= f\frac{x_i}{z_i} \\ v_i &= f\frac{y_i}{z_i}. \end{aligned}$$

The unit vectors pointing from the center of perspectivity to the observed points p_1, p_2, p_3 are given by

$$\frac{1}{\sqrt{u_i^2 + v_i^2 + f^2}} \begin{pmatrix} u_i \\ v_i \\ f \end{pmatrix}, \quad i = 1, 2, 3$$

respectively. The center of perspectivity together with the three points of the 3D triangle form a tetrahedron. Let the angles at the center of perspectivity opposite sides a, b, c be α, β, and γ. These angles are given by

$$\begin{aligned} \cos\alpha &= j_2 \cdot j_3 \\ \cos\beta &= j_1 \cdot j_3 \\ \cos\gamma &= j_1 \cdot j_2 \end{aligned}$$

where j_1, j_2, and j_3 are unit vectors given by

$$j_1 = \frac{1}{\sqrt{u_1^2 + v_1^2 + f^2}} \begin{pmatrix} u_1 \\ v_1 \\ f \end{pmatrix}$$

$$j_2 = \frac{1}{\sqrt{u_2^2 + v_2^2 + f^2}} \begin{pmatrix} u_2 \\ v_2 \\ f \end{pmatrix}$$

$$j_3 = \frac{1}{\sqrt{u_3^2 + v_3^2 + f^2}} \begin{pmatrix} u_3 \\ v_3 \\ f \end{pmatrix}$$

Let the unknown distances of the points p_1, p_2, p_3 from the center of perspectivity be s_1, s_2, and s_3, respectively. Then $s_i = \|p_i\|, i = 1, 2, 3$. To determine the position of the points p_1, p_2, p_3 with respect to the camera reference frame, it is sufficient to determine s_1, s_2, and s_3 since

$$p_i = s_i j_i, \quad i = 1, 2, 3.$$

3 The Solutions

There are six solutions presented by Grunert (1841), Finsterwalder (1937), Merritt (1949), Fischler and Bolles (1981), Linnainmaa et al. (1988), and Grafarend et al. (1989), respectively. In this section we first give the derivation of the Grunert solution to show how the problem can be solved. Then, we analyze what are the major differences among the algebraic derivation of these solutions before we give the detailed derivations for the rest of solutions. Finally, we will give comparisons of algebraic derivation among six solutions.

Grunert's Solution

Grunert proceeded in the following way. By the law of cosines,

$$s_2^2 + s_3^2 - 2s_2s_3\cos\alpha = a^2 \qquad (1)$$

$$s_1^2 + s_3^2 - 2s_1 s_3 \cos\beta = b^2 \tag{2}$$
$$s_1^2 + s_2^2 - 2s_1 s_2 \cos\gamma = c^2 \tag{3}$$

Let

$$s_2 = us_1 \text{ and } s_3 = vs_1. \tag{4}$$

Then

$$s_1^2(u^2 + v^2 - 2uv\cos\alpha) = a^2$$
$$s_1^2(1 + v^2 - 2v\cos\beta) = b^2$$
$$s_1^2(1 + u^2 - 2u\cos\gamma) = c^2.$$

Hence,

$$s_1^2 = \frac{a^2}{u^2 + v^2 - 2uv\cos\alpha} = \frac{b^2}{1 + v^2 - 2v\cos\beta} = \frac{c^2}{1 + u^2 - 2u\cos\gamma} \tag{5}$$

from which

$$u^2 + \frac{b^2 - a^2}{b^2}v^2 - 2uv\cos\alpha + \frac{2a^2}{b^2}v\cos\beta - \frac{a^2}{b^2} = 0 \tag{6}$$

$$u^2 - \frac{c^2}{b^2}v^2 + 2v\frac{c^2}{b^2}\cos\beta - 2u\cos\gamma + \frac{b^2 - c^2}{b^2} = 0. \tag{7}$$

From equation (6)

$$u^2 = -\frac{b^2 - a^2}{b^2}v^2 + 2uv\cos\alpha - \frac{2a^2}{b^2}v\cos\beta + \frac{a^2}{b^2}.$$

This expression for u^2 can be substituted into equation (7). This permits a solution for u to be obtained in terms of v.

$$u = \frac{\left(-1 + \frac{a^2 - c^2}{b^2}\right)v^2 - 2\left(\frac{a^2 - c^2}{b^2}\right)\cos\beta v + 1 + \frac{a^2 - c^2}{b^2}}{2(\cos\gamma - v\cos\alpha)} \tag{8}$$

This expression for u can then be substituted back into the equation (6) to obtain a fourth order polynomial in v.

$$A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0 \tag{9}$$

where

$$A_4 = \left(\frac{a^2 - c^2}{b^2} - 1\right)^2 - \frac{4c^2}{b^2}\cos^2\alpha$$

$$A_3 = 4\left[\frac{a^2 - c^2}{b^2}\left(1 - \frac{a^2 - c^2}{b^2}\right)\cos\beta - \left(1 - \frac{a^2 + c^2}{b^2}\right)\cos\alpha\cos\gamma + 2\frac{c^2}{b^2}\cos^2\alpha\cos\beta\right]$$

$$A_2 = 2\left[\left(\frac{a^2 - c^2}{b^2}\right)^2 - 1 + 2\left(\frac{a^2 - c^2}{b^2}\right)^2\cos^2\beta + 2\left(\frac{b^2 - c^2}{b^2}\right)\cos^2\alpha - 4\left(\frac{a^2 + c^2}{b^2}\right)\cos\alpha\cos\beta\cos\gamma + 2\left(\frac{b^2 - a^2}{b^2}\right)\cos^2\gamma\right]$$

$$A_1 = 4\left[-\left(\frac{a^2 - c^2}{b^2}\right)\left(1 + \frac{a^2 - c^2}{b^2}\right)\cos\beta + \frac{2a^2}{b^2}\cos^2\gamma\cos\beta - \left(1 - \left(\frac{a^2 + c^2}{b^2}\right)\right)\cos\alpha\cos\gamma\right]$$

$$A_0 = \left(1 + \frac{a^2 - c^2}{b^2}\right)^2 - \frac{4a^2}{b^2}\cos^2\gamma.$$

This fourth order polynomial equation can have as many as four real roots. By equation (8), to every solution for v there will be a corresponding solution for u. Then having values for u and v it is an easy matter to determine a value for s_1 from equation (5). The values for s_2 and s_3 are immediately determined from equation (4). Most of the time it gives two solutions (Wolfe et al. 1991).

Outline of the Differences of Algebraic Derivation

As we can find in the Grunert solution, the procedure to solve the problem is first to reduce three unknown variables s_1, s_2, and s_3 of three quadratic equations (1), (2) and (3) into two variables u and v, and then further reduce two variables u and v into one variable v from which we find the solutions of v and substitute them back into equation (5) and equation (4) to obtain s_1, s_2, and s_3. Though all six solutions mainly follow the outline mentioned above, there are several differences from the algebraic derivation point of view. We classify the differences from the following aspects.

Change of variables
: Linnainmaa et al. use $s_2 = u + \cos\gamma s_1$ and $s_3 = v + \cos\beta s_1$ instead of $s_2 = us_1$ and $s_3 = vs_1$ which are used by others.

Different pairs of equations
: There are three unknowns in the three equations (1), (2), and (3). After the change of variables is used, any two pairs of equations can be used to eliminate the third variable. For example, Grunert uses the pair of equations (1) and (2) and the pair of equations (2) and (3) and Merritt uses the pair of equations (1) and (2) and the pair of equations (1) and (3).

Approaches of further variables reduction
: When reducing two variables into one variable, Grunert and Merritt use substitution. Fischler and Bolles and Linnainmaa et al. use directly elimination to reduce the variables. Finsterwalder and Grafarend et al. introduce a new variable λ before reducing the variables.

The flow chart shown in Figure 2 gives a summary of the differences of algebraic derivation of six solutions in a unified frame. In the flow chart we start from the three equations (1), (2), and (3), make different change of variables, use different pairs of equations, do further variable reduction by different approaches, if necessary, solve the new variable, then we have six different solution techniques.

Finsterwalder's Solution

Finsterwalder (1903) as summarized by Finsterwalder and Scheufele (1937) proceeded in a manner which required only finding a root of a cubic polynomial and the roots of two quadratic polynomials rather than finding all the roots of a fourth order polynomial. Finsterwalder multiplies equation (7) by λ and adds the result to Equation (6) to produce

$$Au^2 + 2Buv + Cv^2 + 2Du + 2Ev + F = 0 \tag{10}$$

where the coefficients depend on λ:

$$\begin{aligned}
A &= 1 + \lambda \\
B &= -\cos\alpha \\
C &= \frac{b^2 - a^2}{b^2} - \lambda\frac{c^2}{b^2} \\
D &= -\lambda\cos\gamma \\
E &= \left(\frac{a^2}{b^2} + \lambda\frac{c^2}{b^2}\right)\cos\beta \\
F &= \frac{-a^2}{b^2} + \lambda\left(\frac{b^2 - c^2}{b^2}\right).
\end{aligned}$$

Finsterwalder considers this as a quadratic equation in v. Solving for v,

$$\begin{aligned}
v &= \frac{-2(Bu+E)\pm\sqrt{4(Bu+E)^2-4C(Au^2+2Du+F)}}{2C} \\
&= \frac{-(Bu+E)\pm\sqrt{(B^2-AC)u^2+2(BE-CD)u+E^2-CF}}{C}.
\end{aligned} \tag{11}$$

The numerically stable way of doing this computation is to determine the small root in terms of the larger root.

$$\begin{aligned}
v_{large} &= \frac{-\operatorname{sgn}(Bu+E)}{C}\Big[|Bu+E| \\
&\quad + \sqrt{(B^2-AC)u^2+2(BE-CD)u+E^2-CF}\Big] \\
v_{small} &= \frac{C}{Av_{large}}
\end{aligned}$$

Now Finsterwalder asks, can a value for λ be found which makes $(B^2 - AC)u^2 + 2(BE - CD)u + E^2 - CF$ be a perfect square. For in this case v can be expressed as a first order polynomial in terms of u. The geometric meaning of this case is that the solution to (10) corresponds to two intersecting lines. This first order polynomial can then be substituted back into equation (6) or (7) either one of which yields a quadratic equation which can be solved for u, and then using the just determined value for u in the first order expression for v, a value for v can be determined. Four solutions are produced since there are two first order expressions for v and when each of them is substituted back into equation (6) or (7) the resulting quadratic in u has two solutions.

The value of λ which produces a perfect square satisfies

$$\begin{aligned}
(B^2 - AC)u^2 &+ 2(BE - CD)u + E^2 - CF \\
&= (up + q)^2.
\end{aligned} \tag{12}$$

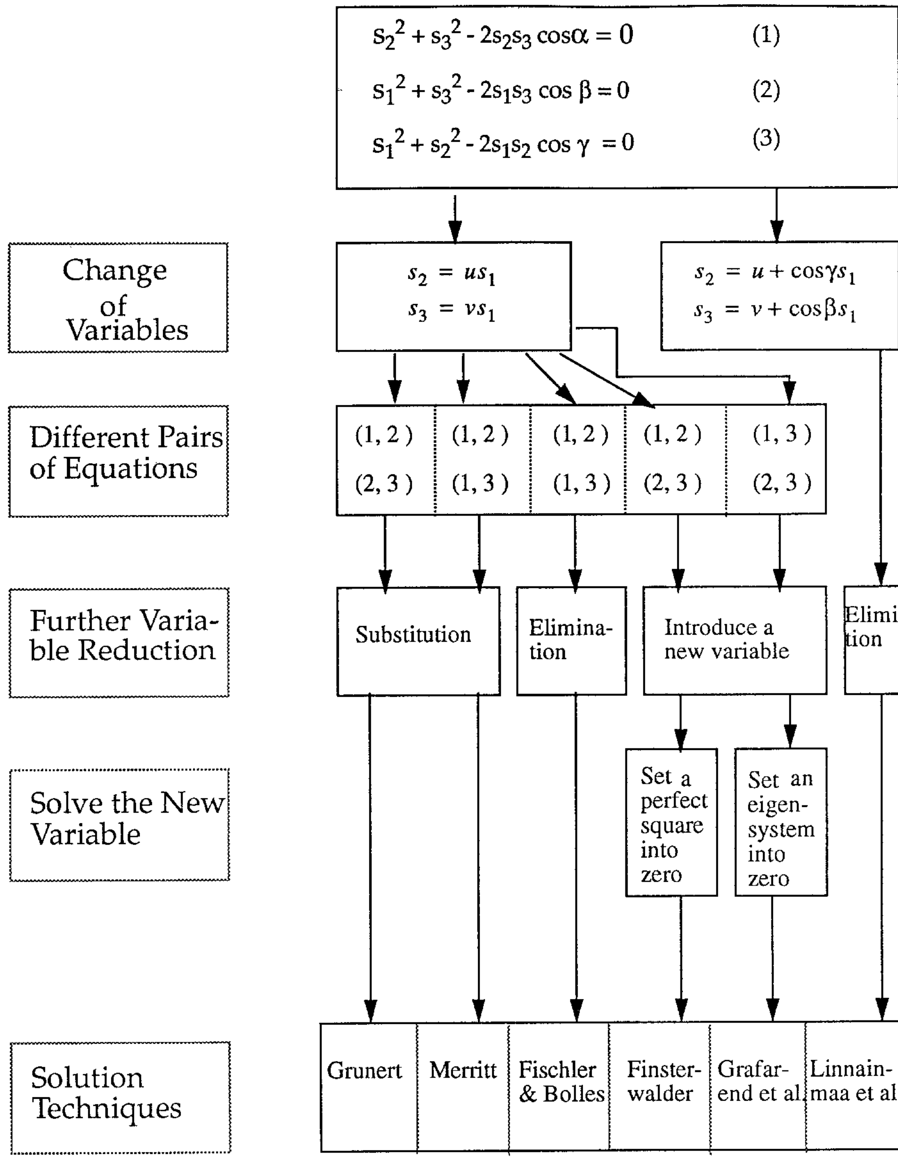

Fig. 2. Shows the differences of a algebraic derivations among six solution techniques.

Hence,

$$B^2 - AC = p^2$$
$$BE - CD = pq$$
$$E^2 - CF = q^2.$$

Since $p^2q^2 = (pq)^2$,

$$(B^2 - AC)(E^2 - CF) = (BE - CD)^2$$

After expanding this out, canceling a B^2E^2 on each side and dividing all terms by a common C there results

$$C(AF - D^2) + B(2DE - BF) - AE^2 = 0, \tag{13}$$

or expressed as a determinant

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0.$$

This is a cubic equation for λ:

$$G\lambda^3 + H\lambda^2 + I\lambda + J = 0 \tag{14}$$

where

$$\begin{aligned}
G &= c^2(c^2 \sin^2\beta - b^2 \sin^2\gamma) \\
H &= b^2(b^2 - a^2)\sin^2\gamma + c^2(c^2 + 2a^2)\sin^2\beta \\
&\quad + 2b^2c^2(-1 + \cos\alpha\cos\beta\cos\gamma) \\
I &= b^2(b^2 - c^2)\sin^2\alpha + a^2(a^2 + 2c^2)\sin^2\beta \\
&\quad + 2a^2b^2(-1 + \cos\alpha\cos\beta\cos\gamma) \\
J &= a^2(a^2\sin^2\beta - b^2\sin^2\alpha).
\end{aligned}$$

Solve this equation for any root λ_0. This determines p and q:

$$\begin{aligned}
p &= \sqrt{B^2 - AC} \\
&= \sqrt{\cos^2\alpha - (1 + \lambda_0)\left(\frac{b^2 - a^2}{b^2} - \lambda_0 \frac{c^2}{b^2}\right)} \\
q &= \operatorname{sgn}(BE - CD)\sqrt{E^2 - CF} \\
&= \operatorname{sgn}\left(-\cos\alpha\left(\frac{a^2}{b^2} + \lambda_0\frac{c^2}{b^2}\right)\cos\beta\right. \\
&\quad \left. - \left(\frac{b^2 - a^2}{b^2} - \lambda_0\frac{c^2}{b^2}\right)(-\lambda_0\cos\gamma)\right) \\
&\quad \bullet \sqrt{\left(\tfrac{a^2}{b^2} + \lambda_0\tfrac{c^2}{b^2}\right)^2\cos^2\beta - \left(\tfrac{b^2-a^2}{b^2} - \lambda_0\tfrac{c^2}{b^2}\right)\left(\tfrac{-a^2}{b^2} + \lambda_0\left(\tfrac{b^2-c^2}{b^2}\right)\right)}.
\end{aligned} \tag{15}$$

Then from equation (11)

$$\begin{aligned}
v &= [-(Bu + E) \pm (pu + q)]/C \\
&= [-(B \mp p)u - (E \mp q)]/C \\
&= um + n,
\end{aligned}$$

where

$$m = [-B \pm p]/C$$

and

$$n = [-(E \mp q)]/C.$$

Substituting this back into equation (7) and simplifying there results

$$\begin{aligned}
&(b^2 - mc^2)u^2 + 2(c^2(\cos\beta - n)m - b^2\cos\gamma)u \\
&\quad -c^2n^2 + 2c^2n\cos\beta + b^2 - c^2 = 0.
\end{aligned} \tag{16}$$

Hence,

$$u = \frac{-(c^2(\cos\beta - n)m - b^2\cos\gamma)}{(b^2 - m^2c^2)} \frac{\pm\sqrt{(c^2(\cos\beta-n)m-b^2\cos\gamma)^2-(b^2-m^2c^2)(-c^2n^2+2c^2n\cos\beta+b^2-c^2)}}{(b^2-m^2c^2)}. \tag{17}$$

The numerically stable way to calculate u is to compute the smaller root in terms of the larger root. Let

$$\begin{aligned}
A &= b^2 - mc^2 \\
B &= c^2(\cos\beta - n)m - b^2\cos\gamma \\
C &= -cn^2 + 2c^2n\cos\beta + b^2 - c^2
\end{aligned}$$

then

$$\begin{aligned}
u_{large} &= \frac{-\operatorname{sgn}(B)}{A}\left[|B| + \sqrt{B^2 - AC}\right] \\
u_{small} &= \frac{C}{Au_{large}}.
\end{aligned}$$

Merritt's Solution

Merritt (1949) unaware of the German solutions also obtained a fourth order polynomial. Smith (1965) gives the following derivation for Merritt's polynomial. He multiplies equation (1) by b^2, multiplies equation (2) by a^2 and subtracts to obtain

$$\begin{aligned}
&a^2s_1^2 - b^2s_2^2 + (a^2 - b^2)s_3^2 - 2a^2s_1s_3\cos\beta \\
&\quad + 2b^2s_2s_3\cos\alpha = 0.
\end{aligned}$$

Similarly, after multiplying equation (1) by c^2, and equation (3) by a^2 and subtracting there results

$$\begin{aligned}
&a^2s_1^2 + (a^2 - c^2)s_2^2 - c^2s_3^2 - 2a^2s_1s_2\cos\gamma \\
&\quad + 2c^2s_2s_3\cos\alpha = 0.
\end{aligned}$$

Then using the substitution of equation (4) we obtain the following two equations.

$$\begin{aligned}
&-b^2u^2 + (a^2 - b^2)v^2 - 2a^2\cos\beta v \\
&\quad + 2b^2\cos\alpha uv + a^2 = 0
\end{aligned} \tag{18}$$

$$\begin{aligned}
&(a^2 - c^2)u^2 - c^2v^2 - 2a^2\cos\gamma u \\
&\quad + 2c^2\cos\alpha uv + a^2 = 0.
\end{aligned} \tag{19}$$

From equation (18),

$$v^2 = \frac{2a^2\cos\beta v - 2b^2\cos\alpha uv + b^2u^2 - a^2}{a^2 - b^2}. \tag{20}$$

Substituting this expression for v^2 into equation (19) and simplifying to obtain

$$\begin{aligned}(a^2 - b^2 - c^2)u^2 &+ 2c^2\cos\alpha uv \\ &+ 2(b^2 - a^2)\cos\gamma u - 2c^2\cos\beta v \\ &+ a^2 - b^2 + c^2 = 0. \qquad (21)\end{aligned}$$

From (21),

$$v = \frac{b^2-a^2-c^2+(b^2+c^2-a^2)u^2+2(a^2-b^2)\cos\gamma u}{2c^2(u\cos\alpha-\cos\beta)}.$$

Substituting this expression for v into equation (19) produces the fourth order polynomial equation

$$B_4u^4 + B_3u^3 + B_2u^2 + B_1u + B_0 = 0 \qquad (22)$$

where

$$\begin{aligned}B_4 &= (b^2 + c^2 - a^2)^2 - 4b^2c^2\cos^2\alpha \\ B_3 &= K - 2B_4\cos\gamma \\ B_2 &= B_4 + B_0 - 2K\cos\gamma \\ &\quad + 4c^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma \\ &\quad - 2\cos\alpha\cos\beta\cos\gamma - 1) \\ B_1 &= K - 2B_0\cos\gamma \\ B_0 &= (a^2 + c^2 - b^2)^2 - 4a^2c^2\cos^2\beta\end{aligned}$$

and

$$\begin{aligned}K &= 2(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)\cos\gamma \\ &\quad + 4c^2(a^2 + b^2 - c^2)\cos\alpha\cos\beta.\end{aligned}$$

Merritt solves for the roots of a fourth order polynomial

$$x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0$$

in the following way. Add

$$\left(\frac{c_3}{2}\right)^2 x^2$$

to each side

$$\begin{aligned}x^4 + c_3x^3 &+ \left(\frac{c_3}{2}\right)^2 x^2 \\ &= -c_2x^2 - c_1x - c_0 + \left(\frac{c_3}{2}\right)^2 x^2\end{aligned}$$

or

$$\begin{aligned}x^2&\left(x^2 + \frac{c_3}{2}\right)^2 \\ &= \left(-c_2 + \left(\frac{c_3}{2}\right)^2\right)x^2 - c_1x - c_0.\end{aligned}$$

Now add

$$\lambda x\left(x + \frac{c_3}{2}\right) + \frac{\lambda^2}{4}$$

to each side. There results

$$\begin{aligned}&\left[x\left(x + \frac{c_3}{2}\right) + \frac{\lambda}{2}\right]^2 \\ &\quad = \left[-c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda\right]x^2 + \left[-c_1 + \lambda\frac{c_3}{2}\right]x \\ &\qquad - c_0 + \frac{\lambda^2}{4}.\end{aligned}$$

Choose λ so that the right hand side is a perfect square.

$$\begin{aligned}&\left[-c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda\right]x^2 + \left[-c_1 + \lambda\frac{c_3}{2}\right]x \\ &\qquad - c_0 + \frac{\lambda^2}{4} = (mx + n)^2.\end{aligned}$$

This means that

$$\begin{aligned}-c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda &= m^2 \\ -c_1 + \lambda\frac{c_3}{2} &= 2mn \\ -c_0 + \frac{\lambda^2}{4} &= n^2\end{aligned}$$

or

$$\begin{aligned}&\left[-c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda\right]\left[-c_0 + \frac{\lambda^2}{4}\right] \\ &\quad = \left(\frac{-c_1 + \lambda c_3/2}{2}\right)^2.\end{aligned}$$

This is a cubic which can be solved for any real root λ_0.

Substituting the root λ_0 into the equation produces

$$\left[x\left(x + \frac{c_3}{2}\right) + \frac{\lambda_0}{2}\right]^2 = (mx + n)^2$$

from which there arises the two quadratics

$$x\left(x + \frac{c_3}{2}\right) + \frac{\lambda_0}{2} = \pm(mx + n)$$

which each can be solved for the two roots.

Fischler and Bolles' Solution

Fischler and Bolles (1981) were apparently not aware of the earlier American or earlier German solutions to the problem. From Equation (5), they obtain

$$\left(1-\frac{a^2}{b^2}\right)v^2+2\left(\frac{a^2}{b^2}\cos\beta-\cos\alpha u\right)v + u^2-\frac{a^2}{b^2}=0 \tag{23}$$

$$v^2+2(-\cos\alpha u)v+\left(1-\frac{a^2}{c^2}\right)u^2 + \frac{2a^2}{c^2}\cos\gamma u-\frac{a^2}{c^2}=0. \tag{24}$$

Equation (23) is identical to equation (6) but equation (24) is different from equation (7) since it arises by manipulating a different pair of equations than was used to obtain equation (6).

Multiply (23) by

$$\left(1-\frac{a^2}{c^2}\right)u^2+\frac{2a^2}{c^2}\cos\gamma u-\frac{a^2}{c^2},$$

multiply (24) by

$$u^2-\frac{a^2}{b^2}$$

and subtract to produce

$$\begin{aligned}&[(a^2-b^2-c^2)u^2+2(b^2-a^2)\cos\gamma u\\ &\quad+(a^2-b^2+c^2)]v+2b^2\cos\alpha u^3\\ &\quad+\left(2(c^2-a^2)\cos\beta-4b^2\cos\alpha\cos\gamma\right)u^2\\ &\quad+[4a^2\cos\beta\cos\gamma+2(b^2-c^2)\cos\alpha]u\\ &\quad-2a^2\cos\beta=0.\end{aligned} \tag{25}$$

Multiply (24) by

$$\left(1-\frac{a^2}{b^2}\right)$$

and subtract from (23).

$$\begin{aligned}&2c^2(\cos\alpha u-\cos\beta)v+(a^2-b^2-c^2)u^2\\ &\quad+2(b^2-a^2)\cos\gamma u+a^2-b^2+c^2=0\end{aligned} \tag{26}$$

Finally, multiply (25) by

$$2c^2(\cos\alpha u-\cos\beta),$$

multiply (26) by

$$[(a^2-b^2-c^2)u^2+2(b^2-a^2)\cos\gamma u+(a^2-b^2+c^2)]$$

and subtract to eliminate v. This produces the fourth order polynomial equation

$$D_4u^4+D_3u^3+D_2u^2+D_1u+D_0=0 \tag{27}$$

where

$$\begin{aligned}D_4&=4b^2c^2\cos^2\alpha-(a^2-b^2-c^2)^2\\ D_3&=-4c^2(a^2+b^2-c^2)\cos\alpha\cos\beta\\ &\quad-8b^2c^2\cos^2\alpha\cos\gamma\\ &\quad+4(a^2-b^2-c^2)(a^2-b^2)\cos\gamma\\ D_2&=4c^2(a^2-c^2)\cos^2\beta\\ &\quad+8c^2(a^2+b^2)\cos\alpha\cos\beta\cos\gamma\\ &\quad+4c^2(b^2-c^2)\cos^2\alpha\\ &\quad-2(a^2-b^2-c^2)(a^2-b^2+c^2)\\ &\quad-4(a^2-b^2)^2\cos^2\gamma\\ D_1&=-8a^2c^2\cos^2\beta\cos\gamma\\ &\quad-4c^2(b^2-c^2)\cos\alpha\cos\beta\\ &\quad-4a^2c^2\cos\alpha\cos\beta\\ &\quad+4(a^2-b^2)(a^2-b^2+c^2)\cos\gamma\\ D_0&=4a^2c^2\cos^2\beta-(a^2-b^2+c^2)^2\end{aligned}$$

Corresponding to each of the four roots of equation (27) for u there is an associated value for v through equation (26) or equation (25).

Grafarend, Lohse, and Schaffrim's Solution

Grafarend, Lohse, and Schaffrim (1989) aware of all the previous work, except for the Fischler-Bolles solution, proceed in the following way. They begin with equations (1), (2), and (3) and seek to reduce them to a homogeneous form. After multiplying equation (3) by

$$\frac{-a^2}{c^2}$$

and adding the result to equation (1) there results

$$-\frac{a^2}{c^2}s_1^2+\left(1-\frac{a^2}{c^2}\right)s_2^2+s_3^2 + 2\frac{a^2}{c^2}s_1s_2\cos\gamma-2\cos\alpha s_2s_3=0. \tag{28}$$

After multiplying equation (3) by

$$-\frac{b^2}{c^2}$$

and adding the result to equation (2), there results

$$\left(1-\frac{b^2}{c^2}\right)s_1^2-\frac{b^2}{c^2}s_2^2+s_3^2+2\frac{b^2}{c^2}\cos\gamma s_1s_2 -2\cos\beta s_1s_3=0. \tag{29}$$

Next they use the same idea as Finsterwalder. They multiply equation (29) by λ and from it subtract equation (28) to produce

$$(s_1 \quad s_2 \quad s_3)A\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix}=0 \tag{30}$$

where

$$A=\begin{pmatrix}\frac{a^2-\lambda(b^2-c^2)}{c^2} & \frac{(\lambda b^2-a^2)\cos\gamma}{c^2} & -\lambda\cos\beta\\ \frac{(\lambda b^2-a^2)\cos\gamma}{c^2} & \frac{a^2-c^2-\lambda b^2}{c^2} & \cos\alpha\\ -\lambda\cos\beta & \cos\alpha & -1+\lambda\end{pmatrix}.$$

Now, as Finsterwalder did, they seek a value of λ which makes the determinant of A zero. Setting the determinant of A to zero produces a cubic for λ. For this value of λ the solution to equation (30) becomes a pair of planes intersecting at the origin.

They let $p=s_2/s_1$ and $q=s_3/s_1$ and rewrite the homogeneous equation (30) in s_1, s_2, and s_3 as a non-homogeneous equation in p and q.

$$(a^2-c^2-\lambda b^2)p^2+2c^2\cos\alpha pq+c^2(-1+\lambda)q^2 +2(-a^2+\lambda b^2)\cos\gamma p-2\lambda c^2\cos\beta q +a^2-\lambda(b^2-c^2)=0 \tag{31}$$

Now since $|A|=0$, and assuming

$$\begin{vmatrix}c^2\cos\alpha & c^2(-1+\lambda)\\ a^2-c^2-\lambda b^2 & c^2\cos\alpha\end{vmatrix}\neq 0$$

a value for (p_0, q_0) exists such that (31) can be written in the homogeneous form

$$(a^2-c^2-\lambda b^2)(p-p_0)^2 +2c^2\cos\alpha(p-p_0)(q-q_0) +c^2(-1+\lambda)(q-q_0)^2=0 \tag{32}$$

where

$$p_0=\frac{\begin{vmatrix}\lambda c^2\cos\beta & c^2(-1+\lambda)\\ -\cos\gamma(-a^2+\lambda b^2) & c^2\cos\alpha\end{vmatrix}}{\begin{vmatrix}c^2\cos\alpha & c^2(-1+\lambda)\\ a^2-c^2-\lambda b^2 & c^2\cos\alpha\end{vmatrix}}$$

$$q_0=\frac{\begin{vmatrix}c^2\cos\alpha & \lambda c^2\cos\beta\\ a^2-c^2-\lambda b^2 & -\cos\gamma(-a^2+\lambda b^2)\end{vmatrix}}{\begin{vmatrix}c^2\cos\alpha & c^2(-1+\lambda)\\ a^2-c^2-\lambda b^2 & c^2\cos\alpha\end{vmatrix}}.$$

Rotating the coordinate system by angle θ so that the cross term can be eliminated, θ must satisfy

$$\tan 2\theta=\frac{2c^2\cos\alpha}{a^2-\lambda(b^2+c^2)}. \tag{33}$$

Define the new coordinate (p', q') by

$$\begin{pmatrix}p'\\ q'\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}p-p_0\\ q-q_0\end{pmatrix}. \tag{34}$$

Then in terms of the new coordinate system (p', q'), equation (32) becomes

$$Ap'^2+Bq'^2=0 \tag{35}$$

where

$$A=\frac{a^2-2c^2+\lambda(c^2-b^2)\pm\sqrt{(a^2-\lambda(b^2+c^2))^2+(2c^2\cos\alpha)^2}}{2}$$

$$B=\frac{a^2-2c^2+\lambda(c^2-b^2)\mp\sqrt{(a^2-\lambda(b^2+c^2))^2+(2c^2\cos\alpha)^2}}{2}.$$

We choose the negative root square term for A and the positive root square term for B when the value of 2θ falls in the first and third quadrant and choose the positive root square term for A and the negative root square term for B when the value of 2θ falls in the second and fourth quadrant.

Assuming $B/A<0$, (35) results in

$$p'=\pm Kq' \tag{36}$$

where

$$K=\sqrt{\frac{-B}{A}}.$$

Using (34) there results

$$p[\cos\theta\pm K\sin\theta]+q[\sin\theta\pm K(-\cos\theta)] +[-p_0(\cos\theta\pm K\sin\theta) +q_0(-\sin\theta\pm K\cos\theta)]=0. \tag{37}$$

Equation (36) is a function of λ. For any λ equation (36) degenerates into a pair of straight lines intercepting in p, q plane. All possible combinations of any two λs out of λ_1, λ_2, and λ_3 will give a real solution for p' and q'. Then we solve p and q. Finally, from equation (1), (2), and (3)

$$s_1 = \sqrt{\frac{c^2}{1 + p^2 - 2p\cos\gamma}} \tag{39}$$

$$s_2 = \sqrt{\frac{a^2}{1 + \left(\frac{q}{p}\right)^2 - 2\left(\frac{q}{p}\right)\cos\alpha}} \tag{40}$$

$$s_3 = \sqrt{\frac{b^2}{1 + \left(\frac{1}{q}\right)^2 - 2\left(\frac{1}{q}\right)\cos\beta}}. \tag{41}$$

However, there is a simple method proposed by Lohse (1989). Instead of translating and rotating equation (31), one can solve the quadratic equation in (31) to get a p and q relation by using different λ. Once the relation of p and q is obtained it can be substituted into (28) to solve for s_1. There are 15 possible solutions. Since we are only interested in real solutions, we only use real λ to solve (31).

Linnainmaa, Harwood, and Davis' Solution

Linnainmaa, Harwood, and Davis (1988) give another direct solution. They begin with equations (1), (2), and (3) and make a change of variables.

$$s_2 = u + \cos\gamma s_1 \tag{42}$$

$$s_3 = v + \cos\beta s_1 \tag{43}$$

equations (2) and (3) become

$$(1 - \cos^2\beta)s_1^2 + v^2 = b^2 \tag{44}$$

$$(1 - \cos^2\gamma)s_1^2 + u^2 = c^2. \tag{45}$$

Substituting (42), (43), (44), and (45) into (1) there results

$$\begin{aligned} s_1^2(2\cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma + 2\cos^2\beta - 2) \\ - 2\cos\alpha uv + c^2 + b^2 - a^2 \\ + 2us_1(\cos\gamma - \cos\alpha\cos\beta) \\ + 2vs_1(\cos\beta - \cos\alpha\cos\gamma) = 0. \end{aligned} \tag{46}$$

Letting

$$\begin{aligned} q_1 &= 1 - \cos^2\gamma \\ q_2 &= 1 - \cos^2\beta \\ q_3 &= 2(\cos^2\gamma - \cos\alpha\cos\beta\cos\gamma \\ &\quad + \cos^2\beta - 1) \\ q_4 &= c^2 + b^2 - a^2 \\ q_5 &= 2(\cos\alpha\cos\beta - \cos\gamma) \\ q_6 &= 2(\cos\alpha\cos\gamma - \cos\beta) \end{aligned} \tag{47}$$

there results

$$q_1 s_1^2 + u^2 = c^2 \tag{48}$$

$$q_2 s_1^2 + v^2 = b^2 \tag{49}$$

$$q_3 s_1^2 - 2\cos\alpha uv + q_4 = q_5 u s_1 + q_6 v s_1. \tag{50}$$

Then they square equation (50) and simplify, obtaining

$$r_1 s_1^4 + r_2 s_1^2 + r_3 = \left(r_4 s_1^2 + r_5\right)uv \tag{51}$$

where

$$\begin{aligned} r_1 &= q_3^2 + 4q_1q_2\cos^2\alpha + q_1q_5^2 + q_2q_6^2 \\ r_2 &= 2q_3q_4 - 4\left(c^2q_2 + b^2q_1\right)\cos^2\alpha \\ &\quad - c^2q_5 - b^2q_6 \\ r_3 &= q_4^2 + 4\cos^2\alpha b^2c^2 \\ r_4 &= 4\cos\alpha q_3 + 2q_5q_6 \\ r_5 &= 4\cos\alpha q_4. \end{aligned}$$

Then to eliminate the uv term, they square equation (51) and simplify to obtain

$$t_8 s_1^8 + t_6 s_1^6 + t_4 s_1^4 + t_2 s_1^2 + t_0 = 0 \tag{52}$$

where

$$\begin{aligned} t_8 &= r_1^2 - r_4^2 q_1 q_2 \\ t_6 &= (b^2q_1 + c^2q_2)r_4^2 - 2r_4r_5q_1q_2 + 2r_1r_2 \\ t_4 &= r_2^2 - b^2c^2r_4^2 - r_5^2q_1q_2 + 2r_4r_5b^2q_1 \\ &\quad + 2r_4r_5c^2q_2 + 2r_1r_3 \\ t_2 &= \left(b^2q_1 + c^2q_2\right)r_5^2 + 2r_2r_3 - 2b^2c^2r_4r_5 \\ t_0 &= r_3^2 - r_5^2b^2c^2. \end{aligned}$$

Equation (52) is considered as a 4th degree equation in s_1^2. Since s_1 must be positive, there are at most 4 solutions to equation (52). Once a value for s_1 has been determined, equations (48) and (49) can be solved for two values of u and v. Each of these can then be substituted into equation (42) and (43) to obtain the positive solutions for s_2 and s_3.

Comparisons of the Algebraic Derivations

The main difference between the Grunert solution and the Merritt solution is that they use different pairs of equations. As a result, the coefficients in their fourth order polynomials are different. However, if we replace b with c, c with b, β with γ, and γ with β in equation (9), then we can obtain equation (22). Therefore, from the algebraic point of view, their solutions are identical. But Merritt converts the fourth order polynomial into two quadratics instead of solving it directly. The difference between Fischler and Bolles' and Grunert's solution is that the former just multiplies some terms to two pairs of equations and then subtracts each other without expressing one variable in terms of the other.

Grunert and Merritt use the substitution to reduce the two variables into one variable. The advantage of the substitution approach is that it is pretty trivial. But there exists a singular region when the denominator is zero in equations (8) and (21). This is discussed more fully in the next subsection. Fischler and Bolles and Lin nainmaa et al. use direct elimination to reduce the variables. Though the approaches are not trivial, it does not generate any singular point during the derivation.

Linnainmaa et al. use $s_2 = u + \cos\gamma s_1$ and $s_3 = v + \cos\beta s_1$ as the change of variables. Naturally, this leads to another different derivation to the problem. Although we consider equation (52) as a fourth order equation in s_1^2, the complexity of the coefficients is much higher than that of Grunert's fourth order equation.

Finsterwalder and Grafarend et al. introduce the same variable, but they use different approaches to solve λ. Finsterwalder solves equation (10) for v and seeks a λ to make the term inside the square root be a perfect square. Grafarend et al. actually rewrite the quadratic equations into matrix form $(s_1\ s_2\ s_3)P(s_1\ s_2\ s_3)^t = 0$ and $(s_1\ s_2\ s_3)Q(s_1\ s_2\ s_3)^t = 0$, then try to solve the eigensystem $(s_1\ s_2\ s_3)(P-\lambda Q)(s_1\ s_2\ s_3) = 0$, which is another form of equation (30). At this point these two approaches are algebraically equivalent.

Singularity of Solutions

It is well-known that there exist some geometric structures for the three point space resection, on which the resection is unstable (Thompson 1966) or indeterminate (Smith 1965). For the unstable geometric structure, a small change in the position of the center of perspectivity will result in a large change in the position of three vertices. For the indeterminate geometric structure, the position of three vertices cannot be solved. Besides the singularity caused by geometric structures, there also exist some singularities caused by the algebraic derivation of solutions. In the following paragraphs we will give detailed explanations and examples.

The danger cylinder is a typical case for the unstable geometric structure and refers to the geometric structure where the center of perspectivity is located on a circular cylinder passing through the three vertices of a triangle and having its axis normal to the plane of the triangle. An illustration of the danger cylinder is shown in Figure 3.a. The reason for the instability can be explained as follows. Instead of determining the position of three vertices, we fix them and let the coordinates of the center of perspectivity be unknown, (x, y, z), as in the resection problem. Since the problem is mainly to solve the three unknown variables s_1, s_2, and s_3, there is actually no difference between fixing the vertices or the center of perspectivity. Now the value of s_1, s_2, and s_3 are functions of x, y, z. Rewrite equations (1), (2), and (3) into $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$, and $f_3(x, y, z) = 0$, and take total derivatives. We then have

$$\frac{1}{s_1 s_2 s_3} AB \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} df_1 \\ df_2 \\ df_3 \end{pmatrix} \tag{53}$$

where

$$A = \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{pmatrix},$$

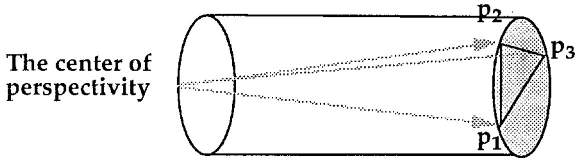

Fig. 3a. An illustration of the danger cylinder.

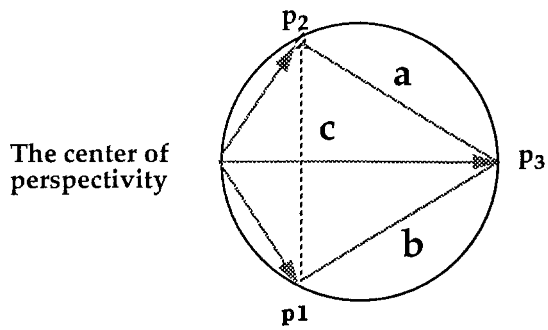

Fig. 3b. An illustration of the concylic case.

$$B = \begin{pmatrix} 0 & s_2 - s_3\cos\alpha & s_3 - s_2\cos\alpha \\ s_1 - s_3\cos\beta & 0 & s_3 - s_1\cos\beta \\ s_1 - s_2\cos\gamma & s_2 - s_1\cos\gamma & 0 \end{pmatrix},$$

and $p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$, $i = 1,2,3$ as defined before are the position of three vertices of the triangle in the camera coordinate frame.

To make the system stable the dx, dy, dz must have no solutions other than zeros; that is, matrices A and B must be non-singular. The determinant of matrix A is proportional to the volume of a tetrahedron formed by three vertices and the center of perspectivity. As long as these four points are not coplanar the matrix A is nonsingular. When the matrix B is singular; that is, where the determinant of B is zero, we can expand the determinant of B and express s_1, s_2, s_3, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ in terms of x, y, z. Then we can obtain an equation that represents the equation of a circular cylinder circumscribing three vertices of the triangle with its axis normal to the plane of the triangle. For example, let the center of perspectivity be located at the origin, $p_1 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$, $p_2 = \begin{pmatrix} 2 \\ 2 \\ 10 \end{pmatrix}$, and $p_3 = \begin{pmatrix} -2 \\ 2 \\ 10 \end{pmatrix}$. Then the second row of the matrix B will be a zero vector; that is, the matrix B is singular.

When the center of perspectivity and the vertices of a triangle are concylic as shown in Figure 3.b, the resection problem is indeterminate. Note that the problem cannot be solved when the five coefficients of equation (9) are all equal to zeros; that is, the four point are concylic. For example, let three side lengths $a = b = c$ and three angles $\alpha = \gamma = 60°$ and $\beta = 120°$, then all coefficients of polynomials of six solutions will be equal to zeros.

The singularity in the algebraic derivation can occur in the Grunert, Finsterwalder, Merritt, Grafarend et al. solutions when the denominator term in the formula equals zero. For example, let three side lengths $a = b = c$ and three angles $\alpha = \gamma = \beta = 60°$, i.e., an equilateral triangle parallel to the image plane with the triangle center at z axis, then s_1, s_2, and s_3 must equal one and thus v or u equals to one. As a result the denominator $(\cos\gamma - v \cdot \cos\alpha)$ in the Grunert solution and $(u\cos\alpha - \cos\beta)$ in the Merritt solution equal zero. Hence, both solutions have a singularity.

Determination of the Absolute Orientation

Once the position of three vertices of the triangle is determined, the transformation function which governs where the 3D camera coordinate system is with respect to the 3D world coordinate system can be calculated.

The problem can be stated as follows. Given three points in the 3D camera coordinate system and their corresponding three points in the 3D world coordinate system, we want to determine a rotation matrix R and translation vector T which satisfies

$$p_i = Rp'_i + T \qquad i = 1,2,3 \tag{54}$$

where $p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ $i = 1,2,3$ are the points in the 3D camera coordinate system, $p'_i = \begin{pmatrix} x'_i \\ y'_i \\ z'_i \end{pmatrix}$ $i = 1,2,3$ are the points in the 3D world coordinate

Table I. The summary of characteristic of six solutions.

Authors	Features	Algebraic singularity
Grunert 1841	Direct solution, solve a fourth order polynomial	Yes
Finsterwalder 1903	Form a cubic polynomial and find the roots of two quadratics	Yes
Merritt 1949	Direct solution, solve a fourth order polynomial	Yes
Fischler and Bolles 1981	Another approach to form a fourth order polynomial	No
Linnainmaa et al. 1988	Generate an eighth order polynomial	No
Grafarend et al. 1989	Form a cubic polynomial and find intersection of two quadratics	Yes

system, R is a 3 by 3 orthonormal matrix, i.e., $RR^t = I$, and $T = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$. The problem can be solved by a linear (Schut 1960), an iterative (Wolf 1974; Slama 1980), or noniterative closed-form solution (Horn 1988). We give a simple linear solution in Appendix I.

4 The Experiments

To characterize the numerical sensitivity of the six different 3 point resection solutions we perform experiments. The experiments study the effects of rounding errors and numerical instability of each of these six different solutions in both single and double precision mode. In addition, we examine the relation of the equation manipulation order. This is accomplished by changing the order in which the three corresponding point pairs are given to the resection procedure.

Since singularities and unstable structures exist in the three point perspective pose estimation problem, we wanted to know how often it can happen in the testing data. To isolate these singularities and unstable structures, we ran 100000 experiments on the Grunert solution, because it has both algebraic and geometric singularities. Then we screened out the singular cases by picking those trials whose error is larger than a certain value.

4.1 Test Data Generation

The coordinates of the vertices of the 3D triangle are randomly generated by a uniform random number generator. The range of the x, y, and z coordinates are within [−25, 25], [−25, 25], and $[f + a, b]$ respectively. Since the image plane is located in front of camera at the distance of focal length, f, the z coordinate must be larger than the focal length. So $a \geq 0$ and $b \geq f + a$. The a and b are used as parameters to test the solution under different sets of depth. Projecting the 3D spatial coordinates into the image frame we obtain the perspective image coordinates u and v.

4.1.1 Permutation of Test Data. To test the numerical stability of each resection technique we permute the order of the three vertices of a triangle and the order of the perspective projection of the 3D triangle vertices. Assume the original order of vertices is 123 for vertex one, vertex two and vertex three, respectively, then the other five permutations are 312, 231, 132, 321, and 213. The permutation of triangle vertices means permuting in a consistent way the 3D triangle side lengths, the 3D vertices and the corresponding 2D perspective projection vertices.

4.2 The Design of Experiments

In this section we will summarize the parameters in the experiments discussed in Appendix II. The experimental procedure of experiments will be presented too. The parameters and methods involved in accuracy and picking the best permutation are denoted by

N_1 – the number of trials = 10000
N_2 – the number of trials = 100000
P – different number of precisions = 2
d_1 – the first set of depths along z axis
d_2 – the second set of depths along z axis
$S_w = \sum_{i=0}^{4} |S_i|$ – the worst sensitivity value for all coefficients.
$S_{wn} = \sum_{i=0}^{4} |S_{a_i}^x|$ – the worst normalized sensitivity value for all coefficients.
$\epsilon_{ware} = \sum_{i=0}^{4} \epsilon_{ware_i}$ – the worst absolute error for all coefficients
$\epsilon_{wrre} = \sum_{i=0}^{4} \epsilon_{wrre_i}$ – the worst relative error for all coefficients
$\epsilon_{sware} = \sum_{i=0}^{4} |S_i \times \epsilon_{ware_i}|$ – the worst polynomial zero drift due to the absolute error
$\epsilon_{swrre} = \sum_{i=0}^{4} |S_{a_i}^x \times \epsilon_{wrre_i}|$ – the worst polynomial zero drift due to the relative error

where $S_i = \frac{dx}{da_i}\Big|_{x=z_j} = \frac{-\frac{\partial P}{\partial a_i}}{\frac{\partial P}{\partial x}}\Big|_{x=z_j}$, $P = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, and $S_{a_i}^x = \frac{a}{x}\frac{\partial x}{\partial a_i}$. The ϵ_{wrre_i} and ϵ_{ware_i} are the total relative and absolute rounding errors propagated from the first to the last mathmetical operations of each coefficient of the polynomial P.

4.2.1 The Design Procedures. The experimental procedures and the characteristics to be studied are itemized as follows:

Step 0. Do the following steps N times.
Step 1. Generate the coordinates of vertices of the 3D triangle.
$-25 < x_i < 25$ where $i = 1, 2, 3$
$-25 < y_i < 25$
For z_i coordinate there are several sets to be tested.
1. $d_1 = \{(a, b) \mid (a, b) \in \{(0, 5), (4, 20)\}, f = 1\}$
2. $d_2 = \{(a, b) \mid (a, b) \in \{(0, 5), (4, 20), (24, 75)\}, f = 1\}$
Step 2. For single and double precision do the resection calculation.
Step 3. Permutation of the vertices. Let the original vertex order be 123 (vertex one, vertex two and vertex three, respectively), then we permute the order as 312, 231, 132, 321, and 213.
Step 4. For each of the resection techniques, determine the location of the 3D vertices if the calculation can succeed.

Step 4.1. For any calculation which has succeeded record the absolute distance error (ADE) associated with each permutation. The mean absolute distance error (MADE) is defined as follows:

$$\epsilon_\mu = \sum_{i=1}^{n} \frac{\epsilon_i}{n}$$

where n is the number of experiments and

$$\epsilon_i = \sum_{i=1}^{n} (\epsilon_{i1} + \epsilon_{i2} + \epsilon_{i3})$$
$$\epsilon_{i1} = \sqrt{((x_{ci1} - x_{i1})^2 + (y_{ci1} - y_{i1})^2 + (z_{ci1} - z_{i1})^2)}$$
$$\epsilon_{i2} = \sqrt{((x_{ci2} - x_{i2})^2 + (y_{ci2} - y_{i2})^2 + (z_{ci2} - z_{i2})^2)}$$
$$\epsilon_{i3} = \sqrt{((x_{ci3} - x_{i3})^2 + (y_{ci3} - y_{i3})^2 + (z_{ci3} - z_{i3})^2)}$$

and $(x_{ci}, y_{ci}, z_{ci})^t$ is the calculated point coordinates and the $(x_i, y_i, z_i)^t$ is the correct generated point coordinates. The error standard deviation is expressed as follows:

$$sd = \sqrt{\frac{\sum_{i=1}^{n} (\epsilon_i - \epsilon_\mu)^2}{(n-1)}}$$

Step 5. This procedure is only applied to Grunert's solution

Step 5.1. Calculate the sensitivity of zero w.r.t. each coefficient and total sensitivity for all coefficients based on the discussion in A.2.3.
Step 5.2. Calculate the worst absolute and relative rounding errors for each coefficient based on the discussion in A.2.4. The number of significant digits is the same as the mantissa representation of machine for multiplication and division. For addition and subtraction the possibly lost significant digits in each operation must be checked.

Step 5.3. Calculate the polynomial zero drift.

Step 5.4. Record the values of the sensitivity S_w, the normal sensitivity S_{wn}, the worst absolute rounding error ϵ_{ware}, the worst relative rounding error ϵ_{wrre}, the worst polynomial zero drift due to absolute rounding error, and the worst polynomial zero drift due to relative rounding error ϵ_{swrre} for each permutation.

Step 5.5. Based on the smallest value of ϵ_{ware}, ϵ_{wrre}, S_w, S_{wn}, ϵ_{sware}, or ϵ_{swrre} picks the corresponding error generated by the corresponding permutation and accumulate the number of its rank in the six permutation. Rank each permutation in terms of the error associated with the permutation. The rank one is associated with the smallest error and the rank six is associated with the largest error.

Step 6. Check for singular cases.
Redo the whole procedure again by changing N_1 to N_2 and d_1 to d_2 and use Grunert's solution only. If the largest absolute distance error is greater than 10^{-7} redo steps 5 and record the corresponding values for the large error cases.

5 Results and Discussion

In this section we discuss the results of the experiments. The software is coded in the C language and the experiments are run on both a Sun 3/280 workstation and a Vax 8500 computer. Unless stated otherwise, the results in the following paragraphs are obtained from the Sun 3/280. Table II shows the results of random permutation of six different solutions. From Table II we find that Finsterwalder's solution (solution two) gives the best accuracy and Merritt's solution gives the worst result. Grunert's solution (solution one), Fischler's solution (solution four) and Grafarend's (solution six) are about the same order and give the second best accuracy. The reasons for the better results can be explained in terms of the order of polynomial and the complexity of computation. Linnainmaa's solution (solution five) generates an eighth order polynomial. Though it doesn't have to solve the eighth order polynomial, the complexity of the coefficients of Linnainmaa's solution is still higher than that of others. Finsterwalder's solution only needs to solve a third order polynomial. The higher order polynomial and higher complexity calculations tend to be less numerically stable. However, Merritt's solution also converts the fourth order polynomial problem into a third order polynomial problem, but it gives a worse result. This is because the conversion process itself is not the most numerically stable. An experiment which directly solves Merritt's fourth order polynomial was conducted. A Laguerre's method was used to find the zeros of a polynomial. The results are similar to that of Grunert's solution.

The histogram of the absolute distance errors

Table II. Results of random permutation of six solutions in double precision and single precision.

Algorithms	Precision	Mean absolute distance error	Standard deviation
Sol. 1 (Grunert)	D.P.	0.19e−08	0.16e−06
	S.P.	0.31e−01	0.88e−00
Sol. 2 (Finsterwalder)	D.P.	0.22e−10	0.90e−09
	S.P.	0.89e−02	0.51e−01
Sol. 3 (Merritt)	D.P.	0.11e−05	0.64e−04
	S.P.	0.28e−01	4.15e−00
Sol. 4 (Fischler)	D.P.	0.62e−08	0.59e−06
	S.P.	0.14e−01	0.34e−00
Sol. 5 (Linnainmaa)	D.P.	0.74e−07	0.61e−05
	S.P.	0.32e−01	0.82e−00
Sol. 6 (Grafarend)	D.P.	0.46e−08	0.43e−06
	S.P.	0.20e−01	0.75e−01

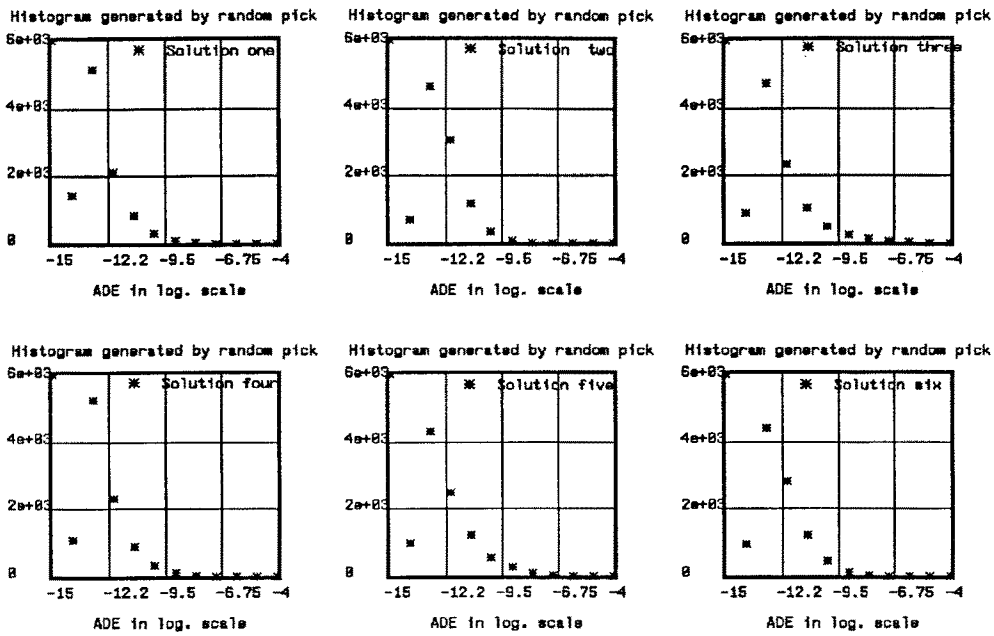

Fig. 4. Shows histograms of the absolute distance error of random permutations in log. scale for six solution techniques.

Table III. The best and the worst mean absolute distance error in single precision.

Algorithms	The best MADE	Standard deviation	The worst MADE	Standard deviation
Sol. 1 (Grunert)	0.10e−03	0.25e−02	0.81e−01	1.45e−00
Sol. 2 (Finsterwalder)	0.74e−04	0.12e−02	0.59e−01	1.71e−00
Sol. 3 (Merritt)	0.17e−02	0.54e−01	1.29e−00	8.53e−00
Sol. 4 (Fischler)	0.87e−04	0.47e−03	0.40e−01	0.47e−00
Sol. 5 (Linnainmaa)	0.16e−02	0.14e−00	0.11e−00	2.16e−00
Sol. 6 (Grafarend)	0.77e−04	0.14e−02	0.94e−01	2.75e−00

(ADE) of 10000 trials are shown in Figure 4. From the histogram of the ADE we can see all the solutions can give an accuracy to the order of 10^{-13} in double precision. The populations of high accuracy results of solution one and solution four are larger than that of solution two. But the population of less accuracy for solution one and solution four also is a little bit more than that of solution two.

As we can expect, the double precision calculation gives a much better results than the single precision calculation. For single precision most of the solutions give the accuracy of the ADE to the order of 10^{-5}. Generally speaking, the results of double precision are about 10^7 times better than the results of single precision. In the single precision mode the root finder subroutine fails in several cases and thus brings up the MADE. Therefore, if possible, double precision calculation is recommended for the 3-points perspective projection calculation.

The best MADE and the worst MADE of six permutations for the double precision and the single precision are shown in Table III and Table IV. The best results are about 10^4 times better than the worst results. Finsterwalder's solution, Grunert's solution and Fischler's solution give the same best accuracy.

Because Grunert's solution has the second best accuracy and is easier to analyze, we use it

Table IV. The best and the worst mean absolute distance error in double precision.

Algorithms	The best MADE	Standard deviation	The worst MADE	Standard deviation
Sol. 1 (Grunert)	0.41e−12	0.90e−11	0.60e−08	0.26e−06
Sol. 2 (Finsterwalder)	0.34e−12	0.73e−11	0.20e−09	0.51e−08
Sol. 3 (Merritt)	0.26e−10	0.15e−08	0.18e−04	0.13e−02
Sol. 4 (Fischler)	0.69e−12	0.19e−10	0.13e−07	0.69e−06
Sol. 5 (Linnainmaa)	0.35e−11	0.24e−09	0.36e−06	0.23e−04
Sol. 6 (Grafarend)	0.44e−12	0.16e−10	0.88e−08	0.48e−06

to demonstrate how analysis methods can discriminate the worst and the best from the six permutations. The analysis methods can be applied to the other solution techniques as well. In the following paragraphs we discuss the results of analysis.

For each trial there are six permutation by which the data can be presented to the resection technique. In the controlled experiments where the correct answer are known, the six resection results can be ordered from least error (best pick) to the highest error (worst pick) using the square error distance between the correct 3D position of the triangle vertices and the calculated 3D position of the triangle vertices. The fraction of times each selection technique selects the data permutation giving the best (least) error to the worst (most) error for two different depths are plotted in Figures 5 and 6. The histogram of the absolute distance error of the six selection methods is shown in Figure 7. Figures 5 and 6 show that the drift of zeros is not affected by the absolute error (i.e. WS) or the relative error (i.e. WRRE). The worst sensitivity (i.e. WS) and the worst relative error (i.e. WRRE) do not permit an accurate choice to be made for the picking order. The worst normalized sensitivity produces the best results and can effectively stabilize the calculation of the coefficients of the polynomial.

The absolute drift of polynomial zeros is changed by both the absolute error of coefficients and the sensitivity of the polynomial zero with respect to the coefficients. Thus, the ϵ_{sware} methods can suppress the probability of picking the worst result from the six permutations. Both the relative error of coefficients and the worst normalized sensitivity of the polynomial zero with respect to the coefficients give the relative drift of the zeros. Hence, the ϵ_{swrre} method also gives a pretty good accuracy. The comparisons of the MADE of randomly order, the best and the worst and the MADE picked by the ϵ_{ware}, ϵ_{wrre}, S_w, S_{wn}, ϵ_{swrre} and ϵ_{sware} for two different depths are shown in Table V.

Table V. The comparison of the mean absolute distance error of randomly order, the best and the worst and the mean absolute distance error picked by the ϵ_{ware}, ϵ_{wrre}, S_w, S_{wn}, ϵ_{swrre} and ϵ_{sware} for two different depths.

Picking methods	Mean absolute	Distance error
Depth	$1 < z < 5$	$5 < z < 20$
Random order	0.19e−08	0.16e−06
The best	0.41e−12	0.19e−11
The worst	0.60e−08	0.87e−08
ϵ_{ware}	0.99e−11	0.34e−09
ϵ_{wrre}	0.40e−08	0.31e−08
Sw	0.15e−08	0.75e−09
Swn	0.89e−12	0.58e−11
ϵ_{swrre}	0.90e−12	0.11e−10
ϵ_{sware}	0.93e−12	0.11e−10

The goal is to achieve the best accuracy. The accuracy of the best permutation is about a ten thousand times better than the accuracy obtained by the worst case and the accuracy obtained by choosing a random permutation. The S_{wn}, ϵ_{sware}, and ϵ_{swrre} methods have approximately a half of the accuracy obtained by the best permutation. Any of these three methods can be used to choose a permutation order which gives reasonably good accuracy. However, the worst normalized sensitivity only involves the sensitivity calculation. So it is a good method to quickly pick the right permutation. Although the histograms of probability of S_{wn}, ϵ_{sware}, and ϵ_{swrre} do not have very high value around the

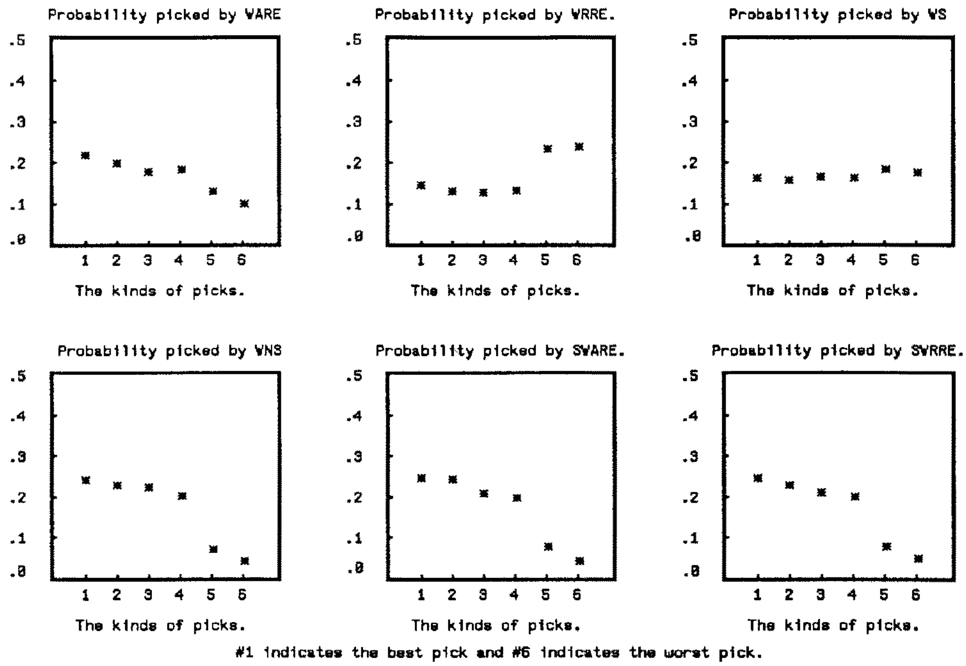

Fig. 5. Shows, for each of the six selection techniques, the fraction of times the technique selected the data permutation giving the best (least) error to the worst (most) error for all 10000 experimental cases for which the depth z is in the range $1 < z < 5$.

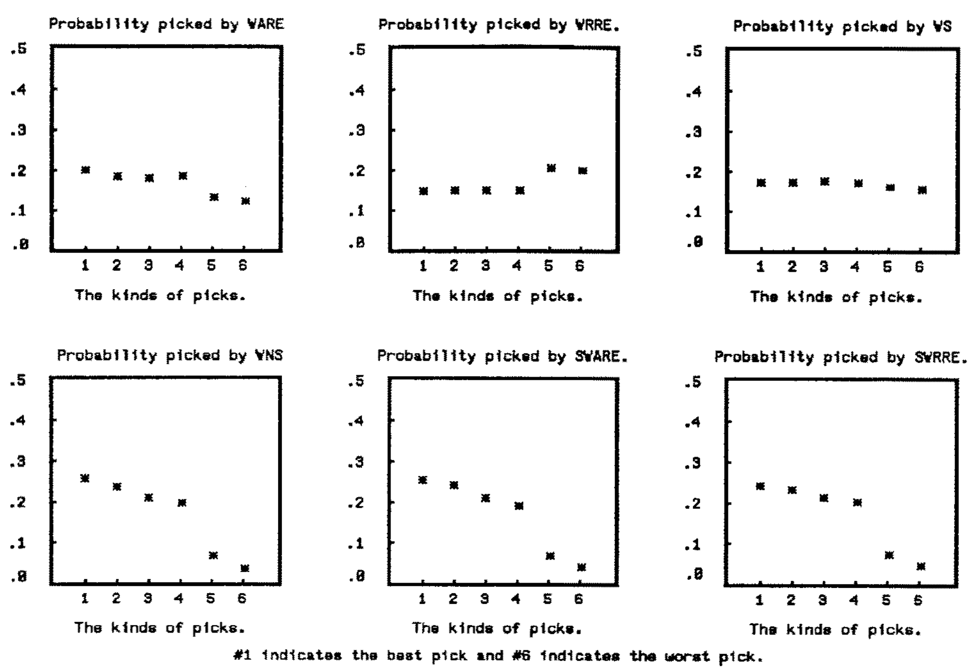

Fig. 6. Shows, for each of the six selection techniques, the fraction of times the technique selected the data permutation giving the best (least) error to the worst (most) error for all 10000 experimental cases for which the depth z is in the range $5 < z < 20$.

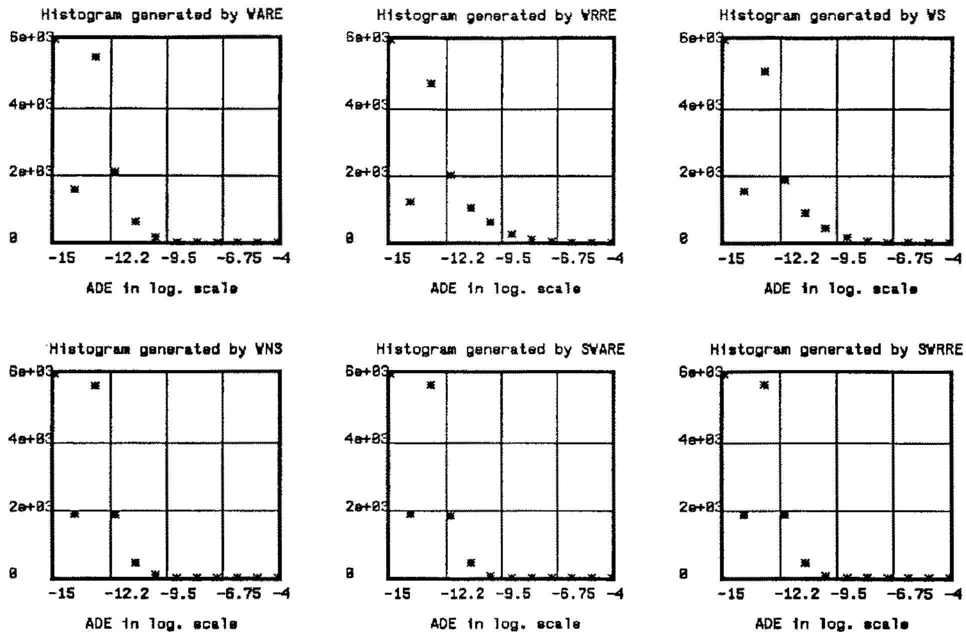

Fig. 7. Shows histograms of the absolute distance error of six selection methods in log. scale.

best pick, they still have a very accurate absolute distance error compared to the best absolute distance error. This reveals that in many cases the accuracy of six permutations are too close to be discriminated.

In order to study the frequency with which singularities and instabilities may happen we pick the large error cases whose absolute distance error is greater than 10^{-7}, run more trials and add different depths for Grunert's technique. Around each singularity we find a region within the parameter space leading to large absolute distance errors in the Grunert solution, diverging with decreasing distance to the point of singularity. Because the real singularities may seldomly happen in the numerical calculation, most cases we only have to deal with very large errors in the vicinity of singular points in the parameter space. Because the set of the vicinities of all singularities in the parameter space does not have the full symmetry of permutation group, we always can find a better parametrization of our experiment. Our task is to define an objective function on the parameter space, which allows us to select a parametrization from the six possible parametrizations, which has the smallest absolute distance error to the exact solution. The results are shown in Table VII.

Table VI and Table VII whose results are obtained from the Vax 8500 running VMS operating system contain the statistics of the absolute distance error of the different selection methods for the three different depth cases, based on the sample of all 100000 experiments in Table VI and based on the subsample of large error cases in Table VII. The sample size for this cases is about 69 for the first depths, about 96 for the second depth and about 495 for the large depth. Table VII shows that the singular cases do not really happen in these experiments because the mean ADE is about 10^{-2}. However, in the vicinity of singular points the error is much larger compared to that of Table VI. The results in Table VII also show that the selection methods work fine in these cases.

When the experiments of Table VI and Table VII are run in the Sun 3/280, results are similar to these obtained from the VAX8500 and the magnitude differences in numerical accuracy of results between two systems are within

Table VI. The same as Table V. But it runs 100000 trials and with three different depths.

Picking	Depth [1...5] MADE	Depth [1...5] Std. dev.	Depth [5...20] MADE	Depth [5...20] Std. dev.	Depth [25...75] MADE	Depth [25...75] Std dev.
Random	2.22e−07	6.58e−05	4.49e−09	6.11e−07	1.44e−07	1.72e−05
Best	6.69e−12	1.79e−09	2.01e−12	2.41e−10	4.18e−11	3.88e−09
Worst	1.69e−06	4.39e−04	7.14e−07	1.90e−04	1.61e−04	5.50e−02
ϵ_{ware}	1.06e−08	3.31e−06	4.49e−10	6.24e−08	3.70e−07	1.14e−04
ϵ_{wrre}	1.68e−06	4.39e−04	6.31e−07	1.88e−04	1.83e−06	3.81e−04
Sw	5.99e−09	1.33e−06	5.98e−07	1.88e−04	1.34e−08	2.13e−06
Swn	9.18e−12	1.88e−09	3.76e−12	3.55e−10	2.43e−10	3.89e−08
ϵ_{sware}	7.64e−12	1.80e−09	3.66e−12	4.17e−10	1.21e−10	1.18e−08
ϵ_{swrre}	7.57e−12	1.80e−09	4.16e−12	4.53e−10	1.21e−10	1.17e−08

Table VII. The same as Table VI. But it only considers large error cases.

Picking	Depth [1...5] MADE	Depth [1...5] Std. dev.	Depth [5...20] MADE	Depth [5...20] Std. dev.	Depth [25...75] MADE	Depth [25...75] Std dev.
Random	1.35e−05	6.35e−05	4.03e−06	2.57e−05	1.22e−04	2.43e−03
Best	7.23e−08	5.16e−07	2.43e−09	1.64e−08	1.37e−08	2.38e−07
Worst	1.18e−04	5.78e−04	2.59e−03	2.52e−02	5.43e−02	1.20e−00
ϵ_{ware}	6.76e−07	4.30e−06	2.94e−07	1.21e−06	9.56e−06	2.08e−04
ϵ_{wrre}	6.02e−05	3.18e−04	2.58e−03	2.52e−02	1.29e−04	2.43e−03
Sw	6.21e−06	4.70e−05	1.14e−07	5.25e−07	1.09e−04	2.42e−03
Swn	5.42e−07	4.23e−06	8.02e−09	4.49e−08	2.18e−08	2.97e−07
ϵ_{sware}	5.20e−07	4.23e−06	7.93e−09	4.43e−08	1.78e−08	2.87e−07
ϵ_{swrre}	5.20e−07	4.23e−06	8.01e−09	4.49e−09	1.73e−08	2.87e−07

an order of one except in worst cases with depth[5...20] and depth[25...75] and in *Sw* case with depth[5...20] whose magnitude differences are an order of two and three, respectively.

6 Conclusions

We have reviewed the six solutions of the three point perspective pose estimation problem from a unified perspective. We gave the comparisons of the algebraic derivations among the six solutions and observed the situations in which there may be numerical instability and indeterminate solutions. We ran hundreds of thousands of experiments to analyze the numerical stability of the solutions. The results show that the Finsterwalder solution gives the best accuracy, about 10^{-10} in double precision and about 10^{-2} in single precision. We have shown that the use of different pairs of equations and change of variables can produce different numerical behaviors. We have described an analysis method to almost always produce a numerically stable calculation for the Grunert solution. The analysis method described here can pick the solution's accuracy about 0.9×10^{-12}, which is very close to the best accuracy 0.41×10^{-12} that can be achieved by picking the best permutation each trial and about thousand times better than 0.19×10^{-8} which is achieved by picking the random permutation.

Acknowledgment

The authors wish to thank the reviewers for their helpful suggestions.

Appendix I: A Simple Linear Solution for the Absolute Orientation

Let us restate the problem. Given three points in the 3D camera coordinate system and their corresponding three points in the 3D world coordinate system, we want to determine a rotation matrix R and translation vector T which satisfies

$$p_i = Rp'_i + T \qquad i = 1,2,3 \qquad \text{(a.1)}$$

where $p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ $i = 1,2,3$ are the points in the 3D camera coordinate system, $p'_i = \begin{pmatrix} x'_i \\ y'_i \\ z'_i \end{pmatrix}$ $i = 1,2,3$ are the points in the 3D world coordinate system, R is a 3 by 3 orthonormal matrix, i.e., $RR^t = I$, and $T = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$.

In order to solve the problem linearly we express the rotation matrix as follows

$$R = \begin{pmatrix} r_{11} r_{12} r_{13} \\ r_{21} r_{22} r_{23} \\ r_{31} r_{32} r_{33} \end{pmatrix}$$

Then, equation (a.1) is an underconstraint system of 9 equations in 12 unknowns. However, as stated in Ganapathy (1984) those unknowns in the rotation matrix are not independent. There exist some constraints as follows

$$r_{11}^2 + r_{12}^2 + r_{13}^2 = r_{21}^2 + r_{22}^2 + r_{23}^2 = r_{31}^2 + r_{32}^2 + r_{33}^2 = 1$$

$$\begin{aligned} r_{13} &= r_{21}r_{32} - r_{22}r_{33} \\ r_{23} &= r_{12}r_{31} - r_{11}r_{32} \\ r_{33} &= r_{11}r_{22} - r_{12}r_{21} \end{aligned} \qquad \text{(a.2)}$$

Since three vertices of the triangle are coplanar, with the constraints above we can assume $z_i = 0$, $i = 1,2,3$. Thus, equation (a.1) can be written as

$$\begin{aligned} x_i &= r_{11}x'_i + r_{12}y'_i + t_x \\ y_i &= r_{21}x'_i + r_{22}y'_i + t_y \qquad i = 1,2,3 \\ z_i &= r_{31}x'_i + r_{32}y'_i + t_z \end{aligned}$$

In terms of matrix form we have

$$AX = B$$

where

$$A = \begin{pmatrix} x'_1 & y'_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x'_1 & y'_1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x'_1 & y'_1 & 0 & 0 & 1 \\ x'_2 & y'_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x'_2 & y'_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x'_2 & y'_2 & 0 & 0 & 1 \\ x'_3 & y'_3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x'_3 & y'_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x'_3 & y'_3 & 0 & 0 & 1 \end{pmatrix}$$

$$X = [r_{11}\ r_{12}\ r_{21}\ r_{22}\ r_{31}\ r_{32}\ t_x\ t_x\ t_z]^t$$

$$B = [x_1\ y_1\ z_1\ x_2\ y_2\ z_2\ x_3\ y_3\ z_3]^t$$

The matrix A will not be singular as long as the three points are not collinear. Hence, it has a unique solution. After the vector X is solved, equation (a.2) can be used to solve r_{13}, r_{23} and r_{33}.

Appendix II: The Numerical Accuracy of the Solutions

A.1 The Problem Definition

In general, all the solutions given in Section 3 can be used to solve the three point perspective resection problem. However, the behavior of the numerical calculations are different for the different solution techniques. Furthermore, for each solution technique the numerical behavior will be different when the order of the equation manipulation or variables is different. For example, if we let $s_1 = us_2$ and $s_3 = vs_2$ instead of $s_2 = us_1$ and $s_3 = vs_1$, then the coefficients of equation (9) will be changed. These changes can be reflected by replacing a with b, b with a, α with β and β with α. As a result, it may affect the numerical accuracy of the final results.

The order of the equation manipulation combined with choosing different pairs of equations for substitution can produce six different numerical behaviors for each solution. To simulate these effects we preorder the 2D perspective projection and the corresponding 3D points in the six different possible permutations.

In this appendix we describe some analysis methods that can be used to determine the numerical stability of the solutions and truly aid in determining a good order in which to present the three corresponding point pairs to the re section procedure.

A.2 The Analysis—A Rounding Error Consideration

There are several sensitivity measures which can be used. They include the numerical relative and absolute errors, and the drift of polynomial zeros. We are mainly concerned about how the manipulation order affects the rounding error propagation and the computed roots of the polynomial. Since both the absolute rounding error and the relative rounding error may affect the final accuracy, we consider both factors. The sensitivity analysis focuses on the roots of the polynomial formed by the three-point perspective solutions. In contrast, the polynomial zero drift considers both the errors and the sensitivity of polynomial zero. However, all factors can affect the numerical results. Each of these measures will be used to predict sensitivity in terms of the mean absolute error.

A.2.1 The Effect of Significant Digits. In this analysis all computations are conducted in both single precision and double precision for the six techniques. The quantity measured is the mean absolute distance error for each precision.

A.2.2 The Histogram of the Mean Absolute Distance Error. The histogram analysis will give the distribution of the absolute distance error. A technique may give a large number of highly accurate results, but produce a few large errors due to degenerate cases; others may give accurate results to all trials without any degenerate cases. The analysis of the histogram of the errors will help us to discriminate between which techniques are uniformly good from those which are only good sometimes.

A.2.3 The Sensitivity Analysis of Polynomial Zeros. The global accuracy is affected by the side lengths, the angles at the center of perspectivity with respect to side lengths, and the permutation order in which the input data is given. These effects will appear in the coefficients of the computed polynomial and affect the stability of the zeros of the polynomial. For an ill-condition polynomial a small change in the value of a coefficient will dramatically change the location of one or more zeros. This change will then propagate to the solution produced by the 3 point perspective resection technique. The sensitivity of the zeros of a polynomial with respect to a change in the coefficients is best derived by assuming the zero location is a function of the coefficients (Vlach and Singhal 1983). Thus for j-th zero z_j of the polynomial $P(a_0, a_1, \ldots, a_n, x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ we represent

$$P(a_0, a_1, \ldots, a_n, x(a_0, a_1, \ldots, a_n))|_{x=z_j} = 0$$

Differentiating with respect to a_i gives

$$\frac{\partial P}{\partial a_i} + \frac{\partial P}{\partial x}\frac{dx}{da_i}\bigg|_{x=z_j} = 0$$

Rearranging the equation gives

$$S_i = \frac{dx}{da_i}\bigg|_{x=z_j} = \frac{-\frac{\partial P}{\partial a_i}}{\frac{\partial P}{\partial x}}\Bigg|_{x=z_j}$$

where $a_0, a_1, \ldots, a_n$ are the coefficients of the polynomial, z_j is the j-th zero of polynomial.

Consider the total sensitivity, S, of all the coefficients on a particular zero. We have

$$S = \sum_{i=0}^{n} S_i$$

To avoid the cancellation among positive and negative terms, we take the absolute value of each term and consider the worst case. We express the worst sensitivity S_w by

$$S_w = \sum_{i=0}^{n} |S_i|$$

A large sensitivity of the zero with respect to the coefficients may lead to a large error in the final result. Laguerre's method is used to find the zeros of polynomial. It has advantage of first extracting the zeros with small absolute values to better preserve accuracy in the deflation of the polynomial and can converge to a complex zero from a real initial estimate. The accuracy for the iterative stop criterion is the rounding error of the machine.

A.2.4 The Numerical Stability. Discussion in most numerical books show how calculations involving finite-digit arithmetic can lead to significant errors in some circumstances. For example, the division of a finite-digit result by a small number, i.e., multiplying by a relative large number, is numerically unstable. Another example is the subtraction of large and nearly equal numbers which can produce an unacceptable rounding error. In order to study how large a rounding absolute error can be produced by the mathematical operation, we will calculate the worst absolute and relative error for each kind of arithmetic operation. Let fl be the floating point mathematical operator. Hence, the rounding error produced by fl on two numbers which themselves have rounding error or truncation error (Wilkinson 1963) can be modeled as follows:

$$\begin{aligned}
fl(\hat{x}_1 + \hat{x}_2) &= (x_1(1+\epsilon_{x1}) + x_2(1+\epsilon_{x2}))(1+\epsilon_r) \\
&\cong (x_1+x_2)\left(1+\epsilon_r + \frac{x_1}{x_1+x_2}\epsilon_{x1} + \frac{x_2}{x_1+x_2}\epsilon_{x2}\right) \\
fl(\hat{x}_1 - \hat{x}_2) &= (x_1(1+\epsilon_{x1}) - x_2(1+\epsilon_{x2}))(1+\epsilon_r) \\
&\cong (x_1-x_2)\left(1+\epsilon_r + \frac{x_1}{x_1-x_2}\epsilon_{x1} - \frac{x_2}{x_1-x_2}\epsilon_{x2}\right) \\
fl(\hat{x}_1 \times \hat{x}_2) &= x_1x_2(1+\epsilon_{x1})(1+\epsilon_{x2})(1+\epsilon_r) \\
&\cong x_1x_2(1+\epsilon_r+\epsilon_{x1}+\epsilon_{x2}) \\
fl\left(\frac{\hat{x}_1}{\hat{x}_2}\right) &= \frac{x_1(1+\epsilon_{x1})}{x_2(1+\epsilon x2)}(1+\epsilon_r) \\
&\cong \left(\frac{x_1}{x_2}\right)(1+\epsilon_r+\epsilon_{x1}-\epsilon_{x2}) \\
&-0.5\times 10^{1-d} \le \epsilon_r \le 0.5\times 10^{1-d}
\end{aligned}$$

where d is the number of significant digits of $fl(\hat{x}_1 + \hat{x}_2)$; ϵ_r is relative error introduced by each operation; the relative errors of x_1 and x_2 are ϵ_{x1} and ϵ_{x2} respectively, and these are propagated from the previous operations. The higher order terms are very small, thus they are omitted.

DEFINITION. *A sequence $\langle OP_1, OP_2, \ldots, OP_{n-1}\rangle$ of binary mathematical operators from the class of addition, subtraction, multiplication and division applied to a series of numbers $(x_1, x_2, \ldots, x_n)$ two at a time is given as follows:*

$$\begin{aligned}
OP_{i=1}^{n-1}(x_i, x_{i+1})f(\epsilon_{x_i}, \epsilon_{x_{i+1}}, \epsilon_r) \\
= \hat{x}(1+\epsilon_{total})
\end{aligned}$$

where f is a function of $\epsilon_{x_i}, \epsilon_{x_{i+1}}$ and ϵ_r, $\hat{x}$ is the result of the operation assuming infinite precision computation and ϵ_{total} is the total relative error propagated from the first operation to the last operation. Hence, $\hat{x}(1+\epsilon_{total})$ is the result of the calculation using finite precision. Similarly, ϵ_{x_i} is the relative error of x_i; $\epsilon_{x_{i+1}}$ is the relative error of x_{i+1}.

We consider the worst case for each operation, i.e., $\epsilon_r = 0.5 \times 10^{1-d}$. Thus, the worst relative rounding error(ϵ_{wrre_i}) is expressed by

$$\epsilon_{wrre_i} = \epsilon_{total}$$

and the worst absolute rounding error(ϵ_{ware_i}) is given

$$\epsilon_{ware_i} = \hat{x} \times \epsilon_{total}$$

The ϵ_{ware_i} and ϵ_{wrre_i} will be accumulated for each of the coefficient. As in the sensitivity of zero section we expect a large relative or absolute error lead to a large final error.

A.2.5 Polynomial Zero Drift. The zero sensitivity helps us to understand how a permutation of the polynomial coefficients affects the zeros. The worst relative and absolute error provide a quantitative measurement of errors. The drift of a polynomial zero from its correct value depends on both sensitivity and error variation. In this paragraph we will give the definition of polynomial zero drift. Define the normalized sensitivity $S^x_{a_i}$ of zero with respect to a coefficient by

$$S^x_{a_i} = \frac{a}{x}\frac{\partial x}{\partial a_i}$$

and the function x

$$x|_{x=z_j} = x(a_0, a_1, \ldots, a_n)$$

Then, the worst normalized sensitivity (S_{wn}) is given by

$$S_{wn} = \sum_{i=0}^{n} |S_{a_i}^{x}|$$

The polynomial zero drift can be expressed as follows:

$$dx|_{x=z_j} = \sum_{i=0}^{n} \frac{\partial x}{\partial a_i} da_i$$

Divide both sides of the above equation by x and in terms of normalized sensitivity we obtain

$$\left.\frac{dx}{x}\right|_{x=z_j} = \sum_{i=0}^{n} S_{a_i}^{x} \frac{da_i}{a_i}$$

Consider the worst absolute drift case due to the absolute rounding error we have

$$\epsilon_{sware} = \sum_{i=0}^{n} |S_i \times \epsilon_{ware_i}|$$

and the worst relative drift case due to the relative rounding error we have

$$\epsilon_{swrre} = \sum_{i=0}^{n} |S_{a_i}^{x} \times \epsilon_{wrre_i}|$$

As discussed above the final error is expected in proportion to the value of the worst drift ϵ_{swrre} and ϵ_{sware}.

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