point of view. We classify the differences from the following aspects.

Change of variables

Linnainmaa et al. use $s_2 = u + \cos\gamma s_1$ and $s_3 = v + \cos\beta s_1$ instead of $s_2 = us_1$ and $s_3 = vs_1$ which are used by others.

Different pairs of equations

There are three unknowns in the three equations (1), (2), and (3). After the change of variables is used, any two pairs of equations can be used to eliminate the third variable. For example, Grunert uses the pair of equations (1) and (2) and the pair of equations (2) and (3) and Merritt uses the pair of equations (1) and (2) and the pair of equations (1) and (3).

Approaches of further variables reduction

When reducing two variables into one variable, Grunert and Merritt use substitution. Fischler and Bolles and Linnainmaa et al. use directly elimination to reduce the variables. Finsterwalder and Grafarend et al. introduce a new variable λ before reducing the variables.

The flow chart shown in Figure 2 gives a summary of the differences of algebraic derivation of six solutions in a unified frame. In the flow chart we start from the three equations (1), (2), and (3), make different change of variables, use different pairs of equations, do further variable reduction by different approaches, if necessary, solve the new variable, then we have six different solution techniques.

Finsterwalder's Solution

Finsterwalder (1903) as summarized by Finsterwalder and Scheufele (1937) proceeded in a manner which required only finding a root of a cubic polynomial and the roots of two quadratic polynomials rather than finding all the roots of a fourth order polynomial. Finsterwalder multiplies equation (7) by λ and adds the result to Equation (6) to produce

$$Au^2 + 2Buv + Cv^2 + 2Du + 2Ev + F = 0 \tag{10}$$

where the coefficients depend on λ:

$$\begin{aligned}
A &= 1 + \lambda \\
B &= -\cos\alpha \\
C &= \frac{b^2 - a^2}{b^2} - \lambda\frac{c^2}{b^2} \\
D &= -\lambda\cos\gamma \\
E &= \left(\frac{a^2}{b^2} + \lambda\frac{c^2}{b^2}\right)\cos\beta \\
F &= \frac{-a^2}{b^2} + \lambda\left(\frac{b^2 - c^2}{b^2}\right).
\end{aligned}$$

Finsterwalder considers this as a quadratic equation in v. Solving for v,

$$\begin{aligned}
v &= \frac{-2(Bu+E)\pm\sqrt{4(Bu+E)^2-4C(Au^2+2Du+F)}}{2C} \\
&= \frac{-(Bu+E)\pm\sqrt{(B^2-AC)u^2+2(BE-CD)u+E^2-CF}}{C}.
\end{aligned} \tag{11}$$

The numerically stable way of doing this computation is to determine the small root in terms of the larger root.

$$\begin{aligned}
v_{large} &= \frac{-\mathrm{sgn}(Bu+E)}{C}\left[|Bu+E|\right. \\
&\quad \left. + \sqrt{(B^2-AC)u^2+2(BE-CD)u+E^2-CF}\right] \\
v_{small} &= \frac{C}{Av_{large}}
\end{aligned}$$

Now Finsterwalder asks, can a value for λ be found which makes $(B^2 - AC)u^2 + 2(BE - CD)u + E^2 - CF$ be a perfect square. For in this case v can be expressed as a first order polynomial in terms of u. The geometric meaning of this case is that the solution to (10) corresponds to two intersecting lines. This first order polynomial can then be substituted back into equation (6) or (7) either one of which yields a quadratic equation which can be solved for u, and then using the just determined value for u in the first order expression for v, a value for v can be determined. Four solutions are produced since there are two first order expressions for v and when each of them is substituted back into equation (6) or (7) the resulting quadratic in u has two solutions.

The value of λ which produces a perfect square satisfies

$$\begin{aligned}
&(B^2 - AC)u^2 + 2(BE - CD)u + E^2 - CF \\
&\quad = (up + q)^2.
\end{aligned} \tag{12}$$

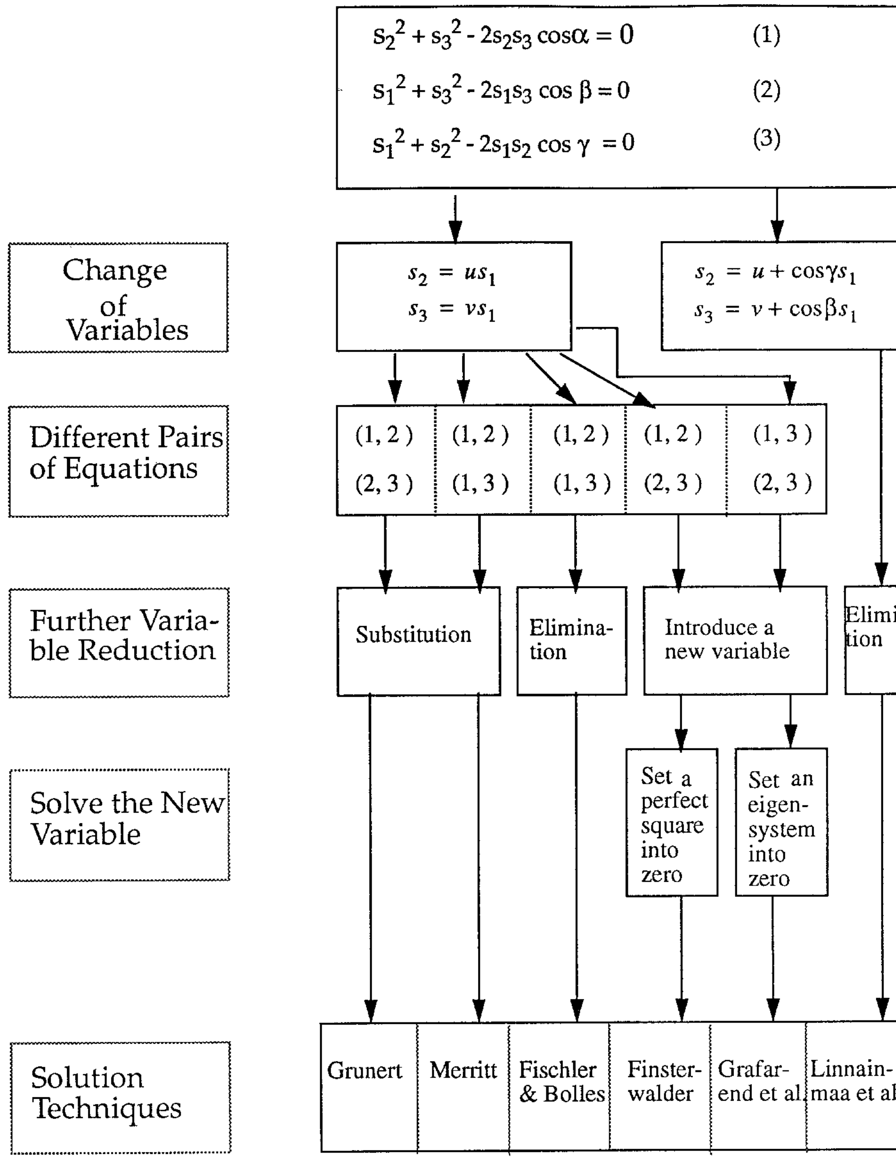

Fig. 2. Shows the differences of a algebraic derivations among six solution techniques.

Hence,

$$B^2 - AC = p^2$$
$$BE - CD = pq$$
$$E^2 - CF = q^2.$$

Since $p^2q^2 = (pq)^2$,

$$(B^2 - AC)(E^2 - CF) = (BE - CD)^2$$

After expanding this out, canceling a B^2E^2 on each side and dividing all terms by a common C there results

$$C(AF - D^2) + B(2DE - BF) - AE^2 = 0, \tag{13}$$

or expressed as a determinant

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0.$$

This is a cubic equation for λ:

$$G\lambda^3 + H\lambda^2 + I\lambda + J = 0 \tag{14}$$

where

$$\begin{aligned} G &= c^2(c^2 \sin^2\beta - b^2 \sin^2\gamma) \\ H &= b^2(b^2 - a^2)\sin^2\gamma + c^2(c^2 + 2a^2)\sin^2\beta \\ &\quad + 2b^2c^2(-1 + \cos\alpha\cos\beta\cos\gamma) \\ I &= b^2(b^2 - c^2)\sin^2\alpha + a^2(a^2 + 2c^2)\sin^2\beta \\ &\quad + 2a^2b^2(-1 + \cos\alpha\cos\beta\cos\gamma) \\ J &= a^2(a^2\sin^2\beta - b^2\sin^2\alpha). \end{aligned}$$

Solve this equation for any root λ_0. This determines p and q:

$$\begin{aligned} p &= \sqrt{B^2 - AC} \\ &= \sqrt{\cos^2\alpha - (1 + \lambda_0)\left(\frac{b^2 - a^2}{b^2} - \lambda_0\frac{c^2}{b^2}\right)} \\ q &= \mathrm{sgn}(BE - CD)\sqrt{E^2 - CF} \\ &= \mathrm{sgn}\left(- \cos\alpha\left(\frac{a^2}{b^2} + \lambda_0\frac{c^2}{b^2}\right)\cos\beta \right. \\ &\quad \left. - \left(\frac{b^2 - a^2}{b^2} - \lambda_0\frac{c^2}{b^2}\right)(-\lambda_0\cos\gamma)\right) \\ &\quad \bullet \sqrt{\left(\frac{a^2}{b^2} + \lambda_0\frac{c^2}{b^2}\right)^2\cos^2\beta - \left(\frac{b^2-a^2}{b^2} - \lambda_0\frac{c^2}{b^2}\right)\left(\frac{-a^2}{b^2} + \lambda_0\left(\frac{b^2-c^2}{b^2}\right)\right)}. \end{aligned} \tag{15}$$

Then from equation (11)

$$\begin{aligned} v &= [-(Bu + E) \pm (pu + q)]/C \\ &= [-(B \mp p)u - (E \mp q)]/C \\ &= um + n, \end{aligned}$$

where

$$m = [-B \pm p]/C$$

and

$$n = [-(E \mp q)]/C.$$

Substituting this back into equation (7) and simplifying there results

$$\begin{aligned} &(b^2 - mc^2)u^2 + 2(c^2(\cos\beta - n)m - b^2\cos\gamma)u \\ &\quad -c^2n^2 + 2c^2n\cos\beta + b^2 - c^2 = 0. \end{aligned} \tag{16}$$

Hence,

$$u = \frac{-(c^2(\cos\beta - n)m - b^2\cos\gamma)}{(b^2 - m^2c^2)} \frac{\pm\sqrt{(c^2(\cos\beta-n)m-b^2\cos\gamma)^2-(b^2-m^2c^2)(-c^2n^2+2c^2n\cos\beta+b^2-c^2)}}{(b^2-m^2c^2)}. \tag{17}$$

The numerically stable way to calculate u is to compute the smaller root in terms of the larger root. Let

$$\begin{aligned} A &= b^2 - mc^2 \\ B &= c^2(\cos\beta - n)m - b^2\cos\gamma \\ C &= -cn^2 + 2c^2n\cos\beta + b^2 - c^2 \end{aligned}$$

then

$$\begin{aligned} u_{large} &= \frac{-\mathrm{sgn}(B)}{A}\left[|B| + \sqrt{B^2 - AC}\right] \\ u_{small} &= \frac{C}{Au_{large}}. \end{aligned}$$

Merritt's Solution

Merritt (1949) unaware of the German solutions also obtained a fourth order polynomial. Smith (1965) gives the following derivation for Merritt's polynomial. He multiplies equation (1) by b^2, multiplies equation (2) by a^2 and subtracts to obtain

$$\begin{aligned} &a^2s_1^2 - b^2s_2^2 + (a^2 - b^2)s_3^2 - 2a^2s_1s_3\cos\beta \\ &\quad + 2b^2s_2s_3\cos\alpha = 0. \end{aligned}$$

Similarly, after multiplying equation (1) by c^2, and equation (3) by a^2 and subtracting there results

$$\begin{aligned} &a^2s_1^2 + (a^2 - c^2)s_2^2 - c^2s_3^2 - 2a^2s_1s_2\cos\gamma \\ &\quad + 2c^2s_2s_3\cos\alpha = 0. \end{aligned}$$

Then using the substitution of equation (4) we obtain the following two equations.

$$\begin{aligned} &-b^2u^2 + (a^2 - b^2)v^2 - 2a^2\cos\beta v \\ &\quad + 2b^2\cos\alpha uv + a^2 = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} &(a^2 - c^2)u^2 - c^2v^2 - 2a^2\cos\gamma u \\ &\quad + 2c^2\cos\alpha uv + a^2 = 0. \end{aligned} \tag{19}$$

From equation (18),

$$v^2 = \frac{2a^2\cos\beta v - 2b^2\cos\alpha uv + b^2u^2 - a^2}{a^2 - b^2}. \tag{20}$$