The Null Space of a Matrix

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Below is a summary of the (right) null space and left null space of a matrix, and how to compute them using singular value decomposition (SVD).

(Right) null space

The (right) null space of a matrix $\mathtt{A} \in \mathbb{R}^{m \times n}$ is the matrix $\mathtt{X} = \mathrm{null}(\mathtt{A})$ such that

$$\mathtt{AX} = \mathtt{0}$$

where $\mathtt{X} \in \mathbb{R}^{n \times (n-r)}$ and $r = \mathrm{rank}(\mathtt{A}) \leq \min(m, n)$.

Left null space

The left null space of a matrix $\mathtt{A} \in \mathbb{R}^{m \times n}$ is the matrix $\mathtt{Y}$ such that

$$\mathtt{YA} = \mathtt{0}$$

where $\mathtt{Y} \in \mathbb{R}^{(m-r) \times m}$ and $r = \mathrm{rank}(\mathtt{A}) \leq \min(m, n)$. The left null space may be calculated using the (right) null space as $\mathtt{Y} = (\mathrm{null}(\mathtt{A}^\top))^\top$.

Computation of the right and left null space using SVD

The singular value decomposition (SVD) of a matrix $\mathtt{A} \in \mathbb{R}^{m \times n}$ may be written as

$$\mathtt{A} = \mathtt{U\Sigma V}^\top$$

where the orthogonal matrix $\mathtt{U} \in \mathbb{R}^{m \times m}$, the diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\min(m,n)}) \in \mathbb{R}^{m \times n}$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$, and the orthogonal matrix $\mathtt{V} \in \mathbb{R}^{n \times n}$. σ_i is the i-th singular value of $\mathtt{A}$, and the i-th column of $\mathtt{U}$ and i-th column of $\mathtt{V}$ are the corresponding left singular vector and right singular vector, respectively, of $\mathtt{A}$. The rank r of $\mathtt{A}$ is the number of nonzero singular values. The (right) null space of $\mathtt{A}$ is the columns of $\mathtt{V}$ corresponding to singular values equal to zero. The left null space of $\mathtt{A}$ is the rows of $\mathtt{U}^\top$ corresponding to singular values equal to zero (or the columns of $\mathtt{U}$ corresponding to singular values equal to zero, transposed).