MATH-GA.2012.001 Selected Topics in Numerical Analysis :
Convex and Nonsmooth Optimization, Spring 2020
Homework Assignment 2
Yves Greatti - yg390

1. Prove that a function is convex if and only if its epigraph is a convex set. Suppose f is a convex function, $f : \mathbf{R}^n \to \mathbf{R}$ then $\forall (x, t_1), (y, t_2) \in \mathbf{epi}f$, and $\forall \theta \in [0, 1]$, we want to show that $\theta(x, t_1) + (1 - \theta)(y, t_2)$ is in $\mathbf{epi}f$. we have:

$$\begin{aligned} f(\theta x + (1-\theta)y) &\leq \theta f(x) + (1-\theta) f(y) \\ &\leq \theta t_1 + (1-\theta) t_2 \end{aligned}$$

thus $\mathbf{epi}f$ is convex. The other direction is similar $\forall (x, t_1), (y, t_2) \in \mathbf{epi}f$, $\mathbf{epi}f$ is a convex set, and $\forall \theta \in [0, 1]$: Let $t_1 = f(x)$, $t_2 = f(y)$ thus $\theta(x, t_1) + (1-\theta)(y, t_2) = (\theta x + (1-\theta)y, \theta t_1 + (1-\theta)t_2)$ is in $\mathbf{epi}f$ which implies: $f(\theta x + (1-\theta)y) \leq \theta t_1 + (1-\theta)t_2 \Rightarrow f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y) \Rightarrow$ f is convex.

2. BV Ex. 2.31 Properties of dual cones. Let K^* be the dual cone of a convex cone K. Prove the following.

   (a) K^* is indeed a convex cone. $\forall y_1, y_2 \in K^*, \theta_1, \theta_2 \geq 0$, and $\forall x \in K$, $x^T(\theta_1 y_1 + \theta_2 y_2) = \theta_1 x^T y_1 + \theta_2 x^T y_2 \geq 0$ thus K^* is a convex cone.

   (b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$. Suppose $y \in K_2^*$, $\forall x \in K_1$, $x^T y \geq 0$, and since $x \in K_2$ also, then $y \in K_1^*$ and $K_2^* \subseteq K_1^*$.

3. Show that if a convex cone K is closed, then $(K^*)^*$, the dual cone of the dual cone of K, is equal to K.

4. BV Ex. 233 Find the dual cone of $\{A\, x | x \geq 0\}$, where $A \in \mathbf{R}^{m \times n}$. The dual of $K = \{A\, x | x \geq 0\}$ is $K^* = \{y | (Ax)^T y \geq 0, \forall x \geq 0\}$ or $K^* = \{y | x^T(A^T y) \geq 0, x \geq 0\} = \{y | (A^T y)^T x \geq 0, x \geq 0\}$. Given $u = A^T y$, we are looking for vectors u such that the inner product is non-negative for any $x \geq 0$. Let $\{e_1, \cdots, e_n\}$ the canonical basis for $\mathbf{R}^n$, for any vector $u = A^T y, y \in K^*$, we have $u^T e_i \geq 0 \Rightarrow u_i \geq 0, i \in [1, n]$. Thus $K^* = \{y | A^T y \geq 0, x \geq 0\}$, this is sufficient as if $x \geq 0$ then $x^T A^T y \geq 0$.

5. Show that the second-order cone defined on p.31 of BV is self-dual, that is, it satisfies $K^* = K$. Let C the second-order cone, $C = \{(x, t) \in \mathbf{R}^n | \|x\|_2 \leq t\}$. $C^* = \{(y, s) | \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} y \\ s \end{bmatrix} \geq 0, \forall (x, t) \in C\}$. if $(y, s) \in C$ then $x^T y \leq \|x\|_2 \|y\|_2$ using Cauchy-Schwarz or $x^T y \leq t\ s$. $\begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} y \\ s \end{bmatrix} = x^T y + ts$, and by the triangle inequality, $\|x^T y + ts\| \geq t\ s - |x^T y| \geq 0 \Rightarrow y \in C^*$. Suppose $(y, s) \notin C$, then $\|y\|_2 > s$ and let m the index of the largest component of y, thus $\|y\|_2 = (\sum_{i=1,n} y_i^2)^{\frac{1}{2}} \leq (n^2 |y_m|^2)^{\frac{1}{2}} = n|y_m| \Rightarrow$. WLOG $|y_m| = y_m$,

then $y_m > \frac{n}{s^2}$ and let x the vector with the only component non-zero $x_m = -\frac{n}{s^2}$ then $x^T y = -\frac{n}{s^2}\, y_m \leq -1$ so $y \notin C^*$. In conclusion, $C = C^*$, C is self-dual.

6. "Chebyshev center" problem

   (a) function chebyshev_center(A, b) takes a matrix A of dimension $(2, n)$ and a vector $b(n)$ to find the largest Euclidean ball that lies in a polyhedron described by n linear inequalities.

```
% Compute the Chebyshev center of a polyhedron
% Boyd & Vandenberghe "Convex Optimization"
function [x_sol, r_sol] = chebyshev_center(A, b)
% The goal is to find the largest Euclidean ball (i.e. its center and
% radius) that lies in a polyhedron described by linear inequalites in t
% fashion: P = {x : a_i'*x <= b_i, i=1,...,m}

% Generate the data
[~,n]=size(A);

% Build and execute model
fprintf(1,'Computing Chebyshev center...');
cvx_begin
    variable r(1)
    variable x_c(2)
    maximize ( r )
    for k=1:n
        A(:, k)'*x_c + r*norm(A(:, k),2) <= b(k);
    end
cvx_end
fprintf(1,'Done! \n');
x_sol = x_c;
r_sol = r;

% Display results
fprintf(1,'The Chebyshev center coordinates are: \n');
disp(x_c);
fprintf(1,'The radius of the largest Euclidean ball is: \n');
disp(r);

% Generate the figure
x = linspace(-2,2);
for k=1:n
    plot(x, -x * A(1,k)./A(2,k) + b(k)./A(2,k),"b-");
    hold on
end
theta = 0:pi/100:2*pi;
plot( x_c(1) + r*cos(theta), x_c(2) + r*sin(theta), "r" );
```

```
plot(x_c(1),x_c(2),'b*');
xlabel("x_1")
ylabel("x_2")

txt = "# inequalities:" + num2str(n);
title({"Largest Euclidean ball lying in a 2D polyhedron", txt});
text(x_c(1), x_c(2), "\leftarrow  center")
axis([-1 1 -1 1])
axis equal
hold off
txt = "chebyshev_center_" + num2str(n);
saveas(gcf,txt,'epsc')
```

For the same example on the web page where matrix $A = \begin{bmatrix} 2 & 2 & -1 & -1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$ and vector $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we obtain the same circle which is tangent to the four hyperplanes $a_i^T x = b_i$ (see figure 1):

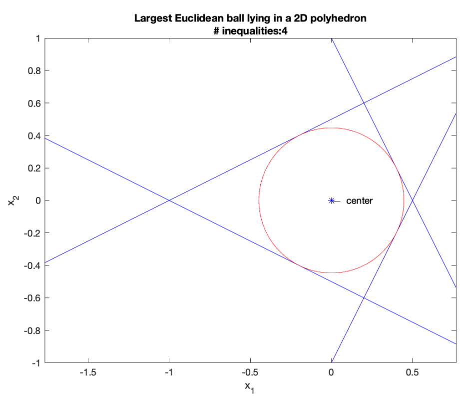

Figure 1: Sample example

We solve the same optimization problem with more inequalities and an interior center inside the polyhedron (see figure 2):

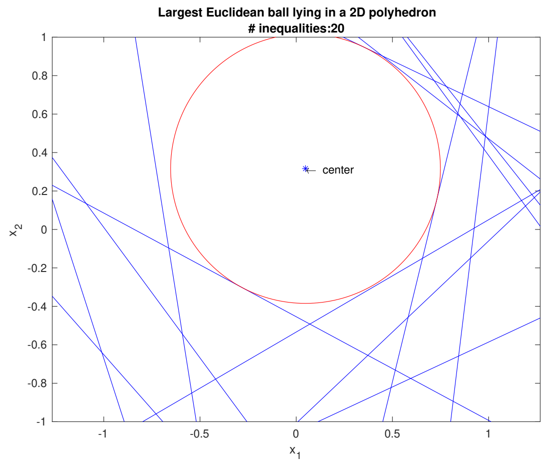

Figure 2: Example with 20 inequalities

(b) If we choose A and b such that the hyperplanes intersect at the same point then we CVX finds a center with radius zero.

(c) The problem to solve now is to find the largest "scaled unit ball" $\mathcal{B} = \{x_c + u | \|u\|_p \leq r\}$ that lies in the polyhedron described by a set of linear inequalities: $\mathcal{P} = \{x \in \mathbf{R}^n | a_i^T x \leq b_i, i = 1, \cdots, m\}$. For any point of $\mathcal{B}$ lying in one halfspace $a_i^T x \leq b_i$, similarly to the euclidean space, we have $\|u\|_p \leq r \Rightarrow a_i^T x_c + r\|a_i\|_p \leq b_i$ since $g_i = \sup \{a_i^T u | \|u\|_p \leq r\} = r\|a_i\|_p$. And the Chebyshev center can be determined by solving the problem:

$$\begin{aligned} &\text{maximize } r \\ &\text{subject to } a_i^T x_c + r\|a_i\|_p \leq b_i, i = 1, \cdots, m \end{aligned}$$

This is still an LP problem since the inequalities are still linear. We obtain different solutions for the same matrix A and vector b, corresponding to different p-norm, $p = 1$ is the diamond shape ball (fig. 3), for $p = 1.5$ the ball has a shape between the diamond and the circle $p = 2$ (fig. 4), and as we increase p, for $p = \infty$ the ball becomes a square (where either $\|x_1\| = 1$ and $\|x_2\| \leq 1$ or $\|x_2\| \leq 1$ and $\|x_2\| = 1$, (fig. 5)).

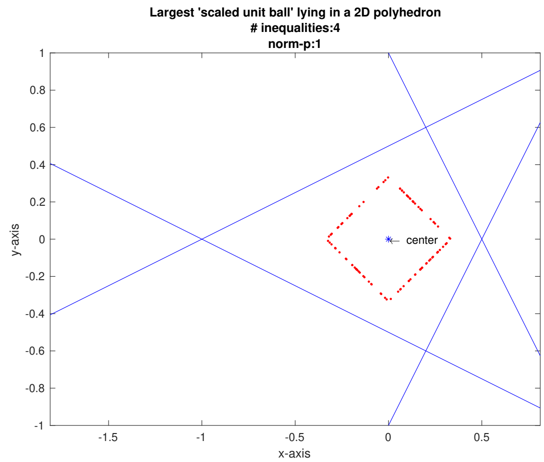

Figure 3: ”Scaled” unit ball for p=1

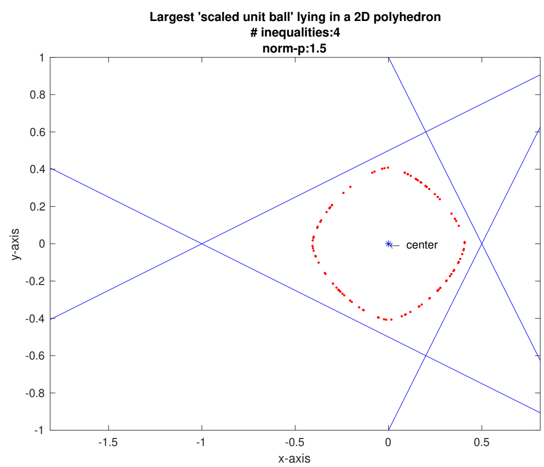

Figure 4: ”Scaled” unit ball for p=1.5

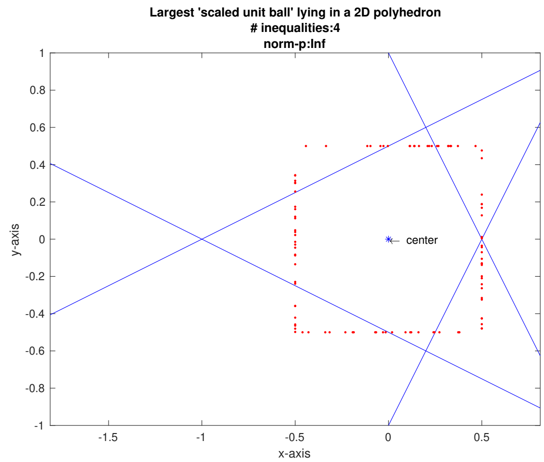

Figure 5: "Scaled" unit ball for p=∞

```
% Compute the Chebyshev center of a polyhedron
function [x_sol, r_sol, x, y] = chebyshev_center_with_norm(A, b, p)
% The goal is to find the largest Euclidean ball (i.e. its center and
% radius) that lies in a polyhedron described by linear inequalites in t
% fashion: P = {x : a_i'*x <= b_i, i=1,...,m}

rng('default')
format long g
[~,n]=size(A);

% Build and execute model
fprintf(1,'Computing Chebyshev center...');
cvx_begin
    variable r(1)
    variable x_c(2)
    maximize ( r )
    for k=1:n
        A(:, k)' * x_c + r * norm(A(:, k), p) <= b(k);
    end
cvx_end
fprintf(1,'Done! \n');
x_sol = x_c;
r_sol = r;


% Display results
fprintf(1,"The Chebyshev center coordinates are: \n");
disp(x_c);
```

```
txt = "Radius of largest 'scaled unit ball' using norm p:" + num2str(p)
fprintf(1,txt);
disp(r);

% Generate the figure
x = linspace(-2,2);
for k=1:n
    plot(x, -x * A(1,k)./A(2,k) + b(k)./A(2,k),"b-");
    hold on
end

n_vecs = 100;
[x, y] = gen_random_vectors(n_vecs, p);
x = x.* r + x_c(1);
y = y.* r + x_c(2);

for i=1:n_vecs
    plot(x(i), y(i), "r." );
    hold on
end
plot(x_c(1),x_c(2),'b*');
xlabel("x-axis")
ylabel("y-axis")

txt1 = "# inequalities:" + num2str(n);
tx2 = "norm-p:" + num2str(p);
title({"Largest 'scaled unit ball' lying in a 2D polyhedron", txt1, tx2}
text(x_c(1), x_c(2), "\leftarrow  center")
axis([-1 1 -1 1])
axis equal
hold off
txt = "chebyshev_center_norm_" + num2str(p);
saveas(gcf,txt,'epsc')

function [x, y] = gen_random_vectors(n, p)
    r = randn(n, 2); % Use a large n
    for i=1:n
        norm_r = norm(r(i,:), p);
        r(i, :) = r(i, :) ./ norm_r;
    end
    x = r(:, 1);
    y = r(:, 2);
```

(d) As we decrease p, the norm of a_i grows exponentially (as shown in figure 6), thus CVX, to solve the problem, quickly finds that the solution for the center is the intersection of the hyperplanes and for r to be zero.

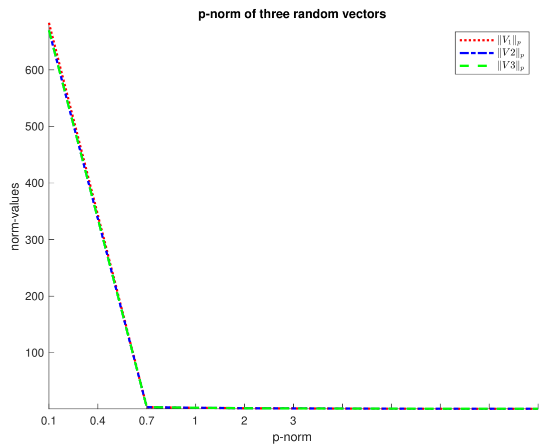

Figure 6: p-norms of three random vectors