Convex and Nonsmooth Optimization
HW3: Mostly about Duality

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Spring 2020

1. Consider the primal problem of minimizing $f_0(x) = \frac{1}{2}x^T Qx$ subject to $h(x) = Ax - b$, where Q is a symmetric positive definite $n \times n$ matrix and A is $m \times n$, with $m \leq n$, with full rank m (in other words, A has m linearly independent rows. There are no inequality constraints.
   (a) Write down the Lagrangian $L(x, \nu)$.
   (b) Since $L(x, \nu)$ is convex, differentiable and bounded below in x, set its gradient to zero to find its minimizer and write down a formula for the Lagrange dual function $g(\nu) = \inf_x L(x, \nu)$ (as inf can be replaced by min, in this case).
   (c) Find the maximizer ν^* of the Lagrange dual function $g(\nu)$ (which is concave) by setting its gradient to zero. What is the dual optimal value $d^* = g(\nu^*)$?
   (d) Find the associated $\hat{x}$ attaining the minimizer of the Lagrangian $L(x, \nu^*)$.
   (e) Check whether $\hat{x}$ is feasible for the primal problem (whether it satisfies $Ax = b$).
   (f) Find the primal value $f_0(\hat{x})$. If $\hat{x}$ is primal feasible, then the optimal primal value $p^* \leq f_0(\hat{x})$.
   (g) Do you conclude that there is no duality gap, i.e., that $d^* = p^*$?

   Note: we did exactly this computation in class in the special case that Q is the identity matrix.
2. BV Ex 5.1
3. BV Ex 5.21 (hint: to verify that $f_1(x, y) = x^2/y$ is convex, derive its Hessian matrix and verify that it is positive definite or semidefinite)
4. Show that for convex problems of the form (5.25), that is (5.1) with $f_0, f_1, ..., f_m$ convex and $h_1, ..., h_p$ affine, the set $\mathcal{A}$ defined in BV (5.37) is convex. This is crucial to the proof of strong duality, because the separating hyperplane theorem is applied to separate $\mathcal{A}$ and $\mathcal{B}$.
5. BV Ex 4.15 (remember that when BV write $Ax \preceq b$, they mean the ordinary componentwise inequality $Ax \leq b$).