Solution to Homework10

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1. (a) In this case, we know when $b \geq 0$, $t = \infty$. Otherwise by definition we know $\frac{1}{t}x + b \geq 0 \Rightarrow t \leq 1/\max\{-\frac{b_i}{x_i} : 1 \leq i \leq n\}$. So the largest t is $1/\max\{-\frac{b_i}{x_i} : 1 \leq i \leq n\}$

vecSolver.m

```
function [t] = vecSolver(x,b)
[n,~]=size(x);
t=inf;
if sum(b>=0)~=n
    t=max(-b./x);
    t=1/t;
end
end
```

(b) In this case, we know when $B \succeq 0$, $t = \infty$. Otherwise, Let $X = L^T L$, we will show the largest t is $1/\lambda_{\max}(-(L^{-1})^T BL^{-1})$. Firstly, we will show $\lambda_{\max}(-(L^{-1})^T BL^{-1}) > 0$, we know B is not positive semidefinite, which means $\exists x$ satisfying $-x^T Bx > 0$. Now consider Lx, we have $(Lx)^T(-(L^{-1})^T BL^{-1})(Lx) = -x^T Bx > 0$, which means $\lambda_{\max}(-(L^{-1})^T BL^{-1}) > 0$. Denote $1/\lambda_{\max}(-(L^{-1})^T BL^{-1})$ as t^*. Then we will show t^* satisfying $X + t^* B \succeq 0$:

$$
\begin{aligned}
& X + t^* B \succeq 0 \\
\Leftrightarrow & \frac{1}{t^*} X + B \succeq 0 \\
\Leftrightarrow & \frac{1}{t^*} x^T X x + x^T B x \geq 0 \quad \forall x \neq 0 \\
\Leftrightarrow & \frac{1}{t^*} \geq -\frac{x^T B x}{x^T X x} \geq 0 \quad \forall x \neq 0 \\
\Leftrightarrow & \frac{1}{t^*} \geq -\frac{(Lx)^T((L^{-1})^T B L^{-1})(Lx)}{(Lx)^T(Lx)} \quad \forall x \neq 0 \\
\Leftrightarrow & \frac{1}{t^*} \geq \lambda_{\max}(-(L^{-1})^T B L^{-1}) \\
\Leftrightarrow & \frac{1}{\lambda_{\max}(-(L^{-1})^T B L^{-1})} \geq t^*
\end{aligned}
$$

So we know t^* satisfying $X + t^* B \succeq 0$. By the proof, we can know if $t > 0$ and $X + tB \succeq 0$, then $t \leq 1/\lambda_{\max}(-(L^{-1})^T BL^{-1})$. So t^* is the largest.

matSolver.m

```
function [t] = matSolver(X,B)
[n,~]=size(X);
b=eig(B);
t=inf;
if sum(b>=0)~=n
    L=chol(X);
    I=eye(n);
    L=I\L;
    B=-B;
    t=max(eig(L'*B*L));
    t=1/t;
end
end
```

2. Suppose when $\|z\|_2 \le \epsilon$, $f(x+z) \ge f(x)$. By the definition of regular subdifferential, we know for any sequence $\{z_n\}$ satisfying $\lim_{n\to\infty} z_n = 0$, $z_n \ne 0$. $\exists N$, when $n \ge N$, $\|z_n\|_2 < \epsilon$. So we know:

$$\liminf_{n\to\infty} \frac{f(x+z_n) - f(x) - 0\cdot z_n}{\|z_n\|_2} \ge \inf_{n\ge N} \left\{ \frac{f(x+z_n)-f(x)}{\|z_n\|_2} \right\} \ge \inf_{n\ge N} \{0\} = 0$$

This means $0 \in \hat{\partial} f(x)$.

3. Suppose sequence $\{z_n\}$ satisfying $\lim_{n\to\infty} z_n = 0$, $z_n \ne 0$.

(a) $f(x) = |x^3|$.
In the class, we know if f is convex and $f \in \mathcal{C}^1$, then we have $\hat{\partial} f(x) = \partial f(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$, $\partial^\infty f(x) = \{0\}$. We know $f(x) = |x^3|$ is convex, and $f'(x) = 3x|x|$ is continuous. So $f(x) \in \mathcal{C}^1$.

- $\hat{\partial} f(0) = \{0\}$
- $\partial f(0) = \{0\}$
- $\partial^\infty f(0) = \{0\}$
- By the definition of regular, we know $|x^3|$ is regular at 0 because $\partial f(0) = \hat{\partial} f(0) = \{0\}$, $\partial^\infty f(0) = (\hat{\partial} f(0))^\infty = \{0\}$

(b) $f(x) = |x|^{\frac{1}{3}}$

- $\hat{\partial} f(0) = \mathbb{R}$.
Proof: Firstly, we show $0 \in \hat{\partial} f(0)$. By the definition, we have:

$$\liminf_{n\to\infty} \frac{f(0+z_n) - f(0)}{\|z_n\|_2} = \liminf_{n\to\infty} |z_n|^{-\frac{2}{3}} \ge 0$$

So we have $0 \in \hat{\partial} f(0)$. Then consider $\forall \gamma \ne 0$. Because $\lim_{n\to\infty} z_n = 0$, we know $\exists N$, when $n \ge N$, $|z_n| < |\gamma|^{-\frac{3}{2}}$. By the definition we have:

$$\begin{aligned}\liminf_{n\to\infty} \frac{f(0+z_n) - f(0) - \gamma z_n}{\|z_n\|_2} &= \liminf_{n\to\infty} |z_n|^{-\frac{2}{3}} - \operatorname{sign}(z_n)\gamma \\ &\ge \inf_{n\ge N} |z_n|^{-\frac{2}{3}} - \operatorname{sign}(z_n)\gamma \\ &> \inf_{n\ge N} |\gamma|(1 - \operatorname{sign}(\gamma z_n)) \ge 0\end{aligned}$$

So we know $\hat{\partial} f(0) = \mathbb{R}$
- $\partial f(0) = \mathbb{R}$.
Proof: We know $\mathbb{R} = \hat{\partial} f(0) \subseteq \partial f(0)$, and $\partial f(0) \subseteq \mathbb{R}$. So $\partial f(0) = \mathbb{R}$
- $\partial^\infty f(0) = \mathbb{R}$
Proof: $\forall \gamma \in \mathbb{R}$, consider $\{x_n = 0\}$, $\{y_n = \frac{\gamma}{t_n}\}$, $\{t_n\} \in \mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$, we have:

$$\lim_{n\to\infty} t_n y_n = \gamma$$
$$y_n \in \hat{\partial} f(0) = \mathbb{R}$$

We have $\forall \gamma \in \mathbb{R}$, $\gamma \in \partial^\infty f(0)$. So $\partial^\infty f(0) = \mathbb{R}$.

- By the definition of regular, we know $|x|^{\frac{1}{3}}$ is regular at 0 because $\partial f(0) = \hat{\partial} f(0) = \mathbb{R}$, $\partial^\infty f(0) = (\hat{\partial} f(0))^\infty = \mathbb{R}$

(c) $f(x) = a|x|$ where $a \geq 0$

- $\hat{\partial} f(0) = [-a, a]$
  Proof: By the definition, $\gamma \in \hat{\partial} f(0)$ is equivalent to:

$$\liminf_{n\to\infty} \frac{f(0+z_n) - f(0) - \gamma z_n}{\|z_n\|_2} \geq 0$$
$$\Leftrightarrow \liminf_{n\to\infty} a - \text{sign}(z_n)\gamma \geq 0$$

  If $\gamma \in [-a, a]$, we know the inequality is always true for any sequence $\{z_n\}$ satisfying $\lim_{n\to\infty} z_n = 0$, $z_n \neq 0$.
  If $|\gamma| > a$, we can find a sequence which has $\text{sign}(z_n) = \text{sign}(\gamma)$ satisfying $\lim_{n\to\infty} z_n = 0$, $z_n \neq 0$. In this time, the inequality is wrong.
  So we know $\hat{\partial} f(0) = [-a, a]$.
- $\partial f(0) = [-a, a]$
  Proof: we know $[-a, a] = \hat{\partial} f(0) \subseteq \partial f(0)$. Then we will show $\partial f(0) \subseteq [-a, a]$. Suppose $\gamma \in \partial f(0)$. By the definition, we know there exist $\{x_n\}$, $\{y_n\}$ satisfying:

$$\lim_{n\to\infty} x_n = 0$$
$$\lim_{n\to\infty} y_n = \gamma$$
$$y_n \in \hat{\partial} f(x_n)$$

  It is easy to know if $x_n \neq 0$, $y_n = \text{sign}(x_n)a$. If $x_n = 0$, $y_n \in [-a, a]$. So we have $\forall n \in \mathbb{N}^+$, $y_n \in [-a, a]$. We know $[-a, a]$ is closed, so any limit of a sequence in a closed set will be in that closed set, which means $\lim_{n\to\infty} y_n = \gamma \in [-a, a]$. We know $\partial f(0) \subseteq [-a, a]$. So we have $\partial f(0) = [-a, a]$
- $\partial^\infty f(0) = \{0\}$
  Proof: Because $\hat{\partial} f(0) \neq \emptyset$, so it is easy to know $0 \in \partial^\infty f(0)$. We will show $\forall \gamma \neq 0$, $\gamma \notin \partial^\infty f(0)$. By the definition, we know if $\gamma \in \partial^\infty f(0)$. There exist $\{x_n\}$, $\{y_n\}$, $\{t_n > 0\}$ satisfying:

$$\lim_{n\to\infty} x_n = 0$$
$$\lim_{n\to\infty} t_n y_n = \gamma$$
$$\lim_{n\to\infty} t_n = 0$$
$$y_n \in \hat{\partial} f(x_n)$$

  By the general subdifferential part, we know $\forall n$, $y_n \in [-a, a]$. So we have $-at_n \leq t_n y_n \leq at_n$. By the limitation, we know:

$$\lim_{n\to\infty} -at_n = 0 \leq \lim_{n\to\infty} t_n y_n = \gamma \leq \lim_{n\to\infty} at_n = 0$$

  So we know $\gamma = 0$, which means $\partial^\infty f(0) = \{0\}$.

- By the definition of regular, we know $a|x|$ where $a \geq 0$ is regular at 0 because $\partial f(0) = \hat{\partial} f(0) = [-a, a]$, $\partial^\infty f(0) = (\hat{\partial} f(0))^\infty = \{0\}$

(d) $f(x) = a|x|$ where $a < 0$

- $\hat{\partial} f(0) = \emptyset$
  Proof: Suppose $\exists \gamma \in \mathbb{R}$, $\gamma \in \hat{\partial} f(0)$. By the defition, we know the following inequality is right for any sequence $\{z_n\}$ satisfying $\lim_{n\to\infty} z_n = 0$, $z_n \neq 0$.

$$\liminf_{n\to\infty} \frac{f(0+z_n) - f(0) - \gamma z_n}{\|z_n\|_2} \geq 0$$
$$\Leftrightarrow \liminf_{n\to\infty} a - \text{sign}(z_n)\gamma \geq 0$$

  If $\gamma \neq 0$, we can choose $\{z_n\}$ satisfying $\text{sign}(z_n) = \text{sign}(\gamma)$. Then we know $a - \text{sign}(z_n)\gamma = a - |\gamma| < 0$. If $\gamma = 0$, we know $\liminf_{n\to\infty} a < 0$. So there doesn't exist $\gamma \in \mathbb{R}$ and $\gamma \in \hat{\partial} f(0)$. So we know $\hat{\partial} f(0) = \emptyset$.
- $\partial f(0) = \{\pm a\}$
  Proof: consider $\{x_n = -\frac{1}{n}\}$, $\{y_n = -a\}$, we know this sequence satisfying the condition of general subdifferential. So $-a \in \partial f(0)$. Similarly, consider $\{x_n = \frac{1}{n}\}$, $\{y_n = a\}$, we know $a \in \partial f(0)$. Then we will show any other values are not in $\partial f(0)$. Consider $\gamma \neq \pm a$. If $\gamma \in \partial f(0)$, then we know $\exists \{x_n\}$, $\{y_n\}$ satisfying:

$$\lim_{n\to\infty} x_n = 0$$
$$\lim_{n\to\infty} y_n = \gamma$$
$$y_n \in \hat{\partial} f(x_n)$$

  We know if $x_n = 0$, there isn't a $y_n \in \hat{\partial} f(x_n)$ due to $\hat{\partial} f(x_n) = \emptyset$. However, if $x_n \neq 0$, we know $\gamma = \pm a$. But $\gamma \neq \pm a$. This is a contradiction. So we know if $\gamma \in \partial f(0)$, $\gamma = \pm a$, which means $\partial f(0) = \{\pm a\}$
- $\partial^\infty f(0) = \{0\}$.
  Proof: consider the similar proof when $f(x) = a|x|$ where $a \geq 0$. We know in this time, if $\gamma \in \partial^\infty f(0)$, γ must be 0. Consider $\{x_n = -\frac{1}{n}\}$, $\{y_n = -a\}$, $\{t_n = \frac{1}{n}\}$. We know in this time $\lim_{n\to\infty} t_n y_n = 0$. So $\partial^\infty f(0) = \{0\}$
- By the definition of regular, we know $a|x|$ where $a < 0$ is not regular at 0 because $\hat{\partial} f(0) = \emptyset$

(e) $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

In this case, we know $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$ when $x \neq 0$. When $x = 0$, $f'(0) = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x - 0} = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$. However, we know $\lim_{x\to 0} f'(x)$ doesn't exist. So $f(x) \notin \mathcal{C}^1$.

- $\hat{\partial} f(0) = \{0\}$

Proof: If there exists $\gamma \in \mathbb{R}$, $\gamma \in \hat{\partial} f(0)$. By the defition, we know the following inequality is right for any sequence $\{z_n\}$ satisfying $\lim_{n\to\infty} z_n = 0$, $z_n \neq 0$.

$$\liminf_{n\to\infty} \frac{f(0+z_n) - f(0) - \gamma z_n}{\|z_n\|_2} \geq 0$$
$$\Leftrightarrow \liminf_{n\to\infty} |z_n| \sin(\frac{1}{z_n}) - \operatorname{sign}(z_n)\gamma \geq 0$$

If $\gamma \neq 0$, let $z_n = \{\frac{\operatorname{sign}(\gamma)}{2\pi n}\}$, we know:

$$\liminf_{n\to\infty} |z_n| \sin(\frac{1}{z_n}) - \operatorname{sign}(z_n)\gamma = \liminf_{n\to\infty} -|\gamma| = -|\gamma| < 0$$

Then we will show $\gamma = 0$ is right. We know $\lim_{n\to\infty} z_n = 0$, so $\forall \epsilon > 0$, $\exists N$ when $n \geq N$, $|z_n| < \epsilon$. For such ϵ, we know:

$$\inf_{n\geq N} |z_n| \sin(\frac{1}{z_n}) \geq \inf_{n\geq N} -\epsilon = -\epsilon.$$

which means we can know:

$$\liminf_{n\to\infty} |z_n| \sin(\frac{1}{z_n}) \geq 0$$

So $\hat{\partial} f(0) = \{0\}$.

- $\partial f(0) = [-1, 1]$
  Proof: We know $f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$ when $x \neq 0$. So $f''(x) = 2\sin(\frac{1}{x}) - 2\cos(\frac{1}{x})/x - \sin(\frac{1}{x})/x^2$ when $x \neq 0$. For $\gamma \in [-1, 1]$, we consider a sequence $\{\gamma_n \in (-1, 1)\}$, and $\lim_{n\to\infty} \gamma_n = \gamma$. For $\gamma_i \in (-1, 1)$, consider $\theta(i) \in (-\pi, 0)$ and $\cos(\theta(i)) = \gamma_i$. Let $\{p_n^i = \frac{1}{2\pi n + \theta(i)}\}$, we know $f''(p_n^i) = \sqrt{1-\gamma_i^2}[(2\pi n + \theta(i))^2 - 2] - 2\gamma_i(2\pi n + \theta(i)) > 0$ when n is greater than some N denoted such N as $N(i)$ and we can always make $N(i+1) > N(i)$. So we know $\hat{\partial} f(p_{N(i)}^i) = f'(p_{N(i)}^i) = -\gamma_i - \frac{2\sqrt{1-\gamma_n^2}}{2\pi N(i) + \theta(i)}$. So we can take $\{x_n = p_{N(n)}^n\}$, $\{y_n = -\gamma_n - \frac{2\sqrt{1-\gamma_n^2}}{2\pi N(n) + \theta(n)}\}$. We know in this case:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2\pi N(n) + \theta(n)} = 0$$
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} -\gamma_n - \frac{2\sqrt{1-\gamma_n^2}}{2\pi N(n) + \theta(n)} = -\gamma$$
$$y_n \in \hat{\partial} f(x_n)$$

Because $-\gamma \in [-1, 1]$, we know $[-1, 1] \subseteq \partial f(0)$. Consider $\gamma \notin [-1, 1]$, by the definition of general subdifferential. We know when $x_n = 0$, $y_n = 0 \in [-1, 1]$. When $x_n \neq 0$, $\hat{\partial} f(x_n) = \{f'(x_n)\}$ or $\emptyset$ due to $f'(x)$ is continuous when $x \neq 0$, which means if y_n exists, $y_n = f'(x_n)$. So once y_n converges, we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty} x_n \sin(\frac{1}{x_n}) - \cos(\frac{1}{x_n}) = \lim_{n\to\infty} -\cos(\frac{1}{x_n}) \in [-1, 1]$ or $\lim_{n\to\infty} y_n = 0$. So we know $\partial f(0) \subseteq [-1, 1]$. Finally we have $\partial f(0) = [-1, 1]$

- $\partial^\infty f(0) = \{0\}$
  Proof: By the similar reason in general subdifferential, we know $y_n = f'(x_n) = 2x_n \sin(\frac{1}{x_n}) - \cos(\frac{1}{x_n})$ or 0, which means when $|x_n| < \epsilon$, y_n is bounded. So we know for any $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} t_n y_n = 0$. So $\partial^\infty f(0) = \{0\}$.
- By the definition of regular, we know $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not regular at 0 because $\partial f(0) \neq \hat{\partial} f(0)$

We can see the first three functions are regular at 0.

4. $x_{[k]}$ means the k-th largest entry of x. e^i $(1 \le i \le n)$ means the i-th standard unit vector in $\mathbb{R}^n$.

   (a) $f(x) = x_{[3]}$

   - $\hat{\partial} f(0) = \emptyset$
     Proof: We know $f(0) = 0_{[3]} = 0 = 0_{[2]}$, in class we know if $x_{[2]} = x_{[3]}$, then $\hat{\partial} x_{[3]} = \emptyset$. So $\hat{\partial} f(0) = \emptyset$
   - $\partial f(0) = \{y : y \in \text{conv}\{e^i : 1 \le i \le n\} \text{ and } \#\{y_i > 0\} \le n-2\}$
     Proof: By the theorem from class, we know $\partial f(x) = \{y : y \in \text{conv}\{e^i : x_i = f(x)\} \text{ and } \#\{y_i > 0\} \le \#\{x_i \ge f(x)\} - 3 + 1\}$. So $\partial f(0) = \{y : y \in \text{conv}\{e^i : 1 \le i \le n\} \text{ and } \#\{y_i > 0\} \le n-2\}$.
   - $\partial^\infty f(0) = \{0\}$
     Proof: we know whether $\{x_n\}$ takes what values, $\hat{\partial} f(x_n)$ is bounded. So if exists horizon subdifferential at 0, it must be 0. Consider $\{x_n = (\frac{1}{n}, \frac{1}{n}, 0, \cdots)\}$, in this time $\hat{\partial} f(x_n) \ne \emptyset$, so we can choose $\{y_n \in \hat{\partial} f(x_n)\}$. In this case, we know $\lim_{n\to\infty} t_n y_n = 0$. So we have $\partial^\infty f(0) = \{0\}$.
   - By the definition of regular, we know $f(x)$ is not regular at 0 because $\hat{\partial} f(0) = \emptyset$.

   (b) $f(x) = (Ax)_{[1]}$

   Suppose $g(x) = x_{[1]}$, we know $f(x) = g(Ax)$. By the theorem from class, we know $\hat{\partial} g(0) = \partial g(0) = \text{conv}\{e^i : 1 \le i \le n\}$, $\partial^\infty g(0) = \{0\}$. So $g(x)$ is regular at 0. Moreover, we have $\ker(A^T) \cap \partial^\infty g(0) = \{0\}$. So we can use the chain rule to get the following results.

   - $\hat{\partial} f(0) = A^T \partial g(0) = \text{conv}\{A_i^T : 1 \le i \le n, \text{where } A_i^T \text{ is the i-th column vector of } A^T\}$
   - $\partial f(0) = \hat{\partial} f(0) = \text{conv}\{A_i^T : 1 \le i \le n, \text{where } A_i^T \text{ is the i-th column vector of } A^T\}$
   - $\partial^\infty f(0) = A^T \partial^\infty g(0) = \{0\}$
   - By the definition of regular, we know $f(x)$ is regular at 0 because $\partial f(0) = \hat{\partial} f(0)$, $\partial^\infty f(0) = (\hat{\partial} f(0))^\infty$

5. We know:

$$F'(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(a) In this time we know $F(\bar{x}) = (0,0)^T$. It is easy to know $\hat{\partial} g(0) = \partial g(0) = \{(a,b)^T : a, b \in [-1,1]\}$, $\partial^\infty g(0) = \{0\}$. So the assumption I is right. Whether what the values of the null space of $F'(0)^T$ are, we always have $\ker(F'(0)^T) \cap \partial^\infty g(0) = \{0\}$. So the assumption II is right.
The RHS $F'(0)^T \partial g(0) = \{(a,0)^T : a \in [-1,1]\}$.
The LHS $\partial f(0) = \{(a,0)^T : a \in [-1,1]\}$. Use the definition to calculate it. Firstly, it is easy to know:

$$\hat{\partial} f(x) = \begin{cases} \{(\text{sign}(x_1), 2x_2)^T\} & x_1 \neq 0 \\ \{(\alpha, 2x_2)^T : \alpha \in [-1,1]\} & x_1 = 0 \end{cases}$$

Suppose $\{x_n = (x_{n1}, x_{n2})^T\}$, $\{y_n = (y_{n1}, y_{n2})^T\}$ satisfying:

$$\begin{aligned} &\lim_{n \to \infty} x_n = 0 \\ &\lim_{n \to \infty} y_n = \gamma = (\gamma_1, \gamma_2)^T \\ &y_n \in \hat{\partial} f(x_n) \end{aligned}$$

We know $\gamma_2 = \lim_{n \to \infty} y_{n2} = \lim_{n \to \infty} 2x_{n2} = 0$, $\gamma_1 = \lim_{n \to \infty} y_{n1} \in [-1,1]$, which means $\partial f(0) \subseteq \{(a,0)^T : a \in [-1,1]\}$. If we choose $\{x_n = 0\}$, $\{y_n = (a,0)^T\}$ where $a \in [-1,1]$, we know $\gamma \in \{(a,0)^T : a \in [-1,1]\}$, which means $\{(a,0)^T : a \in [-1,1]\} \subseteq \partial f(0)$. So we know the LHS $\partial f(0) = \{(a,0)^T : a \in [-1,1]\}$

(b) In this time we know $F(\bar{x}) = (0,0)^T$. It is easy to know $\hat{\partial} g(0) = \partial g(0) = \mathbb{R}^2$, $\partial^\infty g(0) = \mathbb{R}^2$. So the assumption I is right. However, we know $\ker(F'(0)^T) = \{(0,b)^T : b \in \mathbb{R}\}$, this makes $\ker(F'(0)^T) \cap \partial^\infty g(0) \neq \{0\}$. So the assumption II is wrong.
The RHS $F'(0)^T \partial g(0) = \{(a,0)^T : a \in \mathbb{R}\}$.
The LHS $\partial f(0) = \mathbb{R}^2$. Use the definition to calculate it. We know $f(x) = |x_1|^{\frac{1}{3}} + |x_2|^{\frac{2}{3}}$. We firstly show $\hat{\partial} f(0) = \mathbb{R}^2$. By the definition, $\forall \gamma \in \mathbb{R}^2$, let $\{z_n = (a_n, b_n)^T\}$ we just need to show:

$$\liminf_{n \to \infty} \frac{|a_n|^{\frac{1}{3}} + |b_n|^{\frac{2}{3}} - \gamma^T z_n}{\|z_n\|_2} \geq 0$$

We know $-\gamma^T z_n \geq -\|z_n\|_2 \|\gamma\|_2$. So we just need to show:

$$\liminf_{n \to \infty} \frac{|a_n|^{\frac{1}{3}} + |b_n|^{\frac{2}{3}}}{\|z_n\|_2} - \|\gamma\|_2 \geq 0$$

When $\|\gamma\|_2 = 0$, this inequality is true obviously. When $\|\gamma\|_2 \neq 0$ We know $\exists N$ when $n \geq N$, we have $|a_n| < \min(1, \|\gamma\|_2^{-3})$, $|b_n| < \min(1, \|\gamma\|_2^{-3})$ so:

$$
\begin{aligned}
\liminf_{n\to\infty} \frac{|a_n|^{\frac{1}{3}} + |b_n|^{\frac{2}{3}}}{\|z_n\|_2} - \|\gamma\|_2 &\geq \inf_{n\geq N} \frac{|a_n|^{\frac{1}{3}} + |b_n|^{\frac{2}{3}}}{\|z_n\|_2} - \|\gamma\|_2 \\
&\geq \inf_{n\geq N} \frac{|a_n|^{\frac{2}{3}} + |b_n|^{\frac{2}{3}}}{|a_n| + |b_n|} - \|\gamma\|_2 \\
&= \inf_{n\geq N} \frac{|a_n|}{|a_n| + |b_n|}|a_n|^{-\frac{1}{3}} + \frac{|b_n|}{|a_n| + |b_n|}|b_n|^{-\frac{1}{3}} - \|\gamma\|_2 \\
&\geq \inf_{n\geq N} \min(|a_n|^{-\frac{1}{3}}, |b_n|^{-\frac{1}{3}}) - \|\gamma\|_2 \\
&\geq 0
\end{aligned}
$$

So we know $\mathbb{R}^2 \subseteq \hat{\partial} f(0)$, which means $\hat{\partial} f(0) = \mathbb{R}^2$. We also have $\mathbb{R}^2 = \hat{\partial} f(0) \subseteq \partial f(0)$. So we have $\partial f(0) = \mathbb{R}^2$

6. The following functions are locally Lipschitz at 0:

- $f(x) = |x|^3$, $\partial^C f(0) = \{0\}$
  Proof: when $x, y \in [-\epsilon, \epsilon]$
  $$\frac{|f(x) - f(y)|}{|x - y|} = \frac{||x| - |y||}{|x - y|}(|x|^2 + |x||y| + |y|^2) \le 3\epsilon^2$$
  $$\begin{aligned}\partial^C f(0) &= \text{conv}(\partial f(0)) \\ &= \text{conv}(\{0\}) \\ &= \{0\}\end{aligned}$$

- $f(x) = a|x|$ where $a \ge 0$, $\partial^C f(0) = [-a, a]$
  Proof: when $x, y \in [-\epsilon, \epsilon]$
  $$\frac{|f(x) - f(y)|}{|x - y|} = a\frac{||x| - |y||}{|x - y|} \le a$$
  $$\begin{aligned}\partial^C f(0) &= \text{conv}(\partial f(0)) \\ &= \text{conv}([-a, a]) \\ &= [-a, a]\end{aligned}$$

- $f(x) = a|x|$ where $a < 0$, $\partial^C f(0) = [a, -a]$
  Proof: when $x, y \in [-\epsilon, \epsilon]$
  $$\frac{|f(x) - f(y)|}{|x - y|} = |a|\frac{||x| - |y||}{|x - y|} \le |a|$$
  $$\begin{aligned}\partial^C f(0) &= \text{conv}(\partial f(0)) \\ &= \text{conv}(\{\pm a\}) \\ &= [a, -a]\end{aligned}$$

- $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \ne 0 \\ 0 & x = 0 \end{cases}$, $\partial^C f(0) = \text{conv}(\partial f(0)) = \text{conv}([-1, 1]) = [-1, 1]$
  Proof: when $x, y \in [-\epsilon, \epsilon]$, we know $f(x)$ is differentiable on this interval, so by mean value theorem
  $$\frac{|f(x) - f(y)|}{|x - y|} = |f'(z) = 2z\sin(\frac{1}{z}) - \cos(\frac{1}{z})| \le |2z\sin(\frac{1}{z})| + |\cos(\frac{1}{z})| \le 2\epsilon + 1$$
  $$\begin{aligned}\partial^C f(0) &= \text{conv}(\partial f(0)) \\ &= \text{conv}([-1, 1]) \\ &= [-1, 1]\end{aligned}$$

- $f(x) = x_{[3]}$, $\partial^C f(0) = \text{conv}\{e^i : 1 \le i \le n\}$ where e^i $(1 \le i \le n)$ means the i-th standard unit vector in $\mathbb{R}^n$.

  Proof: we will show $f(x)$ is Lipschitz everywhere with Lipschitz constant $L = 1$. For x and y, suppose $\{x_i : 1 \le i \le n\} = \{a_i : 1 \le i \le n, a_i \ge a_{i+1}\}$, $\{y_i : 1 \le i \le n\} = \{b_i : 1 \le i \le n, b_i \ge b_{i+1}\}$. We firstly show the following result:

$$\begin{aligned}
&\|x - y\| \ge \sqrt{\sum_{i=1}^{n}(a_i - b_i)^2} \\
\Leftrightarrow &\sum_{i=1}^{n}(x_i - y_i)^2 \ge \sum_{i=1}^{n}(a_i - b_i)^2 \\
\Leftrightarrow &\sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} x_i y_i
\end{aligned}$$

  The last inequality is true due to the rearrangement inequality. So we know:

$$\begin{aligned}
\frac{|f(x) - f(y)|}{\|x - y\|} &= \frac{|a_3 - b_3|}{\|x - y\|} \\
&\le \frac{|a_3 - b_3|}{\sqrt{\sum_{i=1}^{n}(a_i - b_i)^2}} \\
&\le 1
\end{aligned}$$

  So we know $x_{[3]}$ is 1-Lipschitz everywhere.

$$\begin{aligned}
\partial^C f(0) &= \text{conv}(\partial f(0)) \\
&= \text{conv}(\{y : y \in \text{conv}\{e^i : 1 \le i \le n\} \text{ and } \#\{y_i > 0\} \le n - 2\}) \\
&= \text{conv}\{e^i : 1 \le i \le n\}
\end{aligned}$$

  where e^i $(1 \le i \le n)$ means the i-th standard unit vector in $\mathbb{R}^n$.

- $f(x) = (Ax)_{[1]}$, $\partial^C f(0) = \text{conv}\{A_i^T : 1 \le i \le n, \text{where } A_i^T \text{ is the i-th column vector of } A^T\}$

  Proof: By the proof of $x_{[3]}$, we know $x_{[1]}$ is also 1-Lipschitz everywhere. So we have:

$$\begin{aligned}
\frac{|f(x) - f(y)|}{\|x - y\|} &= \frac{|(Ax)_{[1]} - (Ay)_{[1]}|}{\|x - y\|} \\
&= \frac{|(Ax)_{[1]} - (Ay)_{[1]}|}{\|Ax - Ay\|} \times \frac{\|Ax - Ay\|}{\|x - y\|} \\
&\le 1 \times \frac{\|A(x - y)\|}{\|x - y\|} \\
&\le \sigma_{\max}(A)
\end{aligned}$$

  where $\sigma_{\max}(A)$ is the largest singular value of A.

$$\begin{aligned}
\partial^C f(0) &= \text{conv}(\partial f(0)) \\
&= \text{conv}(\text{conv}\{A_i^T : 1 \le i \le n, \text{where } A_i^T \text{ is the i-th column vector of } A^T\}) \\
&= \text{conv}\{A_i^T : 1 \le i \le n, \text{where } A_i^T \text{ is the i-th column vector of } A^T\}
\end{aligned}$$