Convex and Nonsmooth Optimization
HW7: Subgradients

Michael Overton

Spring 2020

1. Let[1]
$$f(x) = \max_{1 \leq i \leq n} x_i.$$
Write down the general formula for $\partial f(x)$ at any $x \in \mathbb{R}^n$ and prove that your answer is correct. (See p. SG2 of the notes mentioned below.)

2. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex, proper and closed and let f^* be its Fenchel conjugate (see BV, p. 91). Prove the Fenchel-Young inequality
$$f(x) + f^*(y) \geq x^T y$$
and show that it holds with equality if and only if $y \in \partial f(x)$.

3. Under the same assumption on f, prove that $y \in \partial f(x)$ if and only if $y^T d \leq f'(x; d)$ for all $d \neq 0 \in \mathbb{R}^n$, where $f'(x; d)$ is the ordinary directional derivative of f at x in the direction d.

My notes on subgradients and the subdifferential are here.

[1] In BV notation, this is written $f(x) = x_{[1]}$.