Convex Homework 1

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Throughout, assume θ is an arbitrary element of $[0, 1]$.

1. Suppose $(x_1, t_1), (x_2, t_2) \in C$ where C is the quadratic cone. Then $\|(1-\theta)x_1 + \theta x_2\|_2 \leq (1-\theta)\|x_1\|_2 + \theta\|x_2\| \leq (1-\theta)t_1 + \theta t_2$.

   This implies $(1-\theta)(x_1, t_1) + \theta(x_2, t_2) \in C$, so C is convex.

2. Suppose $x, y \in A \cap B$. By convexity of A and B, $(1-\theta)x + \theta y$ is an element of A and B, hence $A \cap B$. So $A \cap B$ is convex.

3. Start with $f(x) = Ax + b$ and C is convex.

   Say $\alpha, \beta \in f(C)$ such that $f(x) = \alpha$, $f(y) = \beta$ where $x, y \in C$. Then

   $$(1-\theta)\alpha + \theta\beta = A((1-\theta)x + \theta y) + b = f((1-\theta)x + \theta y) \in f(C)$$

   So $f(C)$ is convex.

   Now, say $\alpha, \beta \in f^{-1}(C)$ such that $f(\alpha) = x, f(\beta) = y$ for $x, y \in C$. Then

   $$f((1-\theta)\alpha + \theta\beta) = (1-\theta)(A\alpha + b) + \theta(A\beta + b) = (1-\theta)x + \theta y \in C$$

   So $(1-\theta)\alpha + \theta\beta \in f^{-1}(C)$, hence it's convex.

4. Consider $(\theta_1, \ldots, \theta_{k+1})$ such that $\theta_i \geq 0$ and $\sum_{i=1}^{k+1} \theta_i = 1$. Define $y := \sum_{i=1}^{k} \theta_i x_i$ and $z := \sum_{i=1}^{k+1} \theta_i x_i$.

   If $\theta_{k+1} = 1$, then $z = x_{k+1} \in C$. Otherwise, the inductive hypothesis gives $y/(1-\theta_{k+1}) \in C$ since $\sum_{i=1}^{k} \theta_i = 1 - \theta_{k+1}$. And by convexity $z = (1-\theta_{k+1}) * y/(1-\theta_{k+1}) + \theta_{k+1}x_{k+1} \in C$.

5. If C is convex, a line is convex so their intersection is also convex. Conversely, if $x, y \in C$, and the intersection of C with the line $L := \{(1-\alpha)x + \alpha y : \alpha \in \mathbb{R}\}$ is convex, then $(1-\theta)x + \theta y \in C \cap L \subseteq C$ and therefore C is convex.

   Similarly, if C is affine, a line is affine so their intersection is also affine. Conversely, if $x, y \in C$, and the intersection of C with the line $L := \{(1-\alpha)x + \alpha y : \alpha \in \mathbb{R}\}$ is affine, then for any $\beta \in \mathbb{R}$, $(1-\beta)x + \beta y \in C \cap L \subseteq C$ and therefore C is affine.

6. (a) Define $C_1 = \{(x,t) : (A^{1/2}x)^T(A^{1/2}x) \le t^2, t \ge 0\}$, $C_2 = \{(x,t) : t^2 \le -b^T x - c\}$, and $C_3 = \{(y,t) : t^2 \le y\}$. C_1 is the preimage of the second order cone under the affine map $(x,t) \to (A^{1/2}x, t)$ and therefore convex. If $(x_1,t_1),(x_2,t_2) \in C_3$, we observe

$$\begin{aligned}(\theta t_1 + (1-\theta)t_2)^2 &= \theta^2 t_1^2 + (1-\theta)^2 t_2^2 + 2\theta(1-\theta)t_1 t_2 \\ &\le \theta^2 t_1^2 + (1-\theta)^2 t_2^2 + \theta(1-\theta)(t_1^2 + t_2^2) \\ &\le \theta t_1^2 + (1-\theta)t_2^2 \\ &\le (1-\theta)y_1 + \theta y_2\end{aligned}$$

Therefore C_3 is convex. And C_2 is the preimage of C_3 under the map $(x,t) \to (-b^T x - c, t)$.

Finally, C is gotten by forgetting the last coordinate of $C_1 \cap C_2$, and is therefore convex.

To show the converse is not true, consider $A = -I$, $b = 0$, $c = 0$, then the constraint reduces to $-\|x\|_2^2 \le 0$, so $C = \mathbb{R}^n$ and is therefore convex.

(b) Call the hyperplane H. Define $C' = \{x : x^T(A + \lambda gg^T)x - \lambda h^2 + b^T x + c \le 0\}$. By assumption $A + \lambda gg^T \succeq 0$, so by part (a), C' is convex. And for $x \in C \cap H$, $\lambda h^2 = \lambda x^T gg^T x$, so it follows $C \cap H = C' \cap H$, which is the intersection of convex sets and therefore convex.

For the converse, consider $A = -I$, $b = 0$, $c = 0$, $g = e_1$, $h = 0$, then $A + \lambda gg^T$ still has negative eigenvalues regardless of λ, but C is the hyperplane $\{x : g^T x = 0\}$ and therefore convex.

7. Say $(a,b),(c,d) \in S$, so there exists $(a,b'),(c,d') \in S_1$ and $(a, b-b'),(c, d-d') \in S_2$.

By convexity $((1-\theta)a + \theta c, ((1-\theta)b' + \theta d')) \in S_1$ and $((1-\theta)a + \theta c, (1-\theta)(b-b') + \theta(d-d')) \in S_2$

Then $(1-\theta)(a,b) + \theta(c,d) = ((1-\theta)a + \theta c, (1-\theta)b' + \theta d' + (1-\theta)(b-b') + \theta(d-d')) \in S$.

8. Choose $A = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ and $B = \{(x,y) \in \mathbb{R}^2 : y \ge 1/x, x \ge 1\}$. A is clearly affine and therefore convex and closed.

If $(x_1,y_1),(x_2,y_2) \in B$, $(1-\theta)x_1 + \theta x_2 \ge 1$. Note that $\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{x_1^2 + x_2^2}{x_1 x_2} \ge \frac{2x_1 x_2}{x_1 x_2} = 2$, and therefore

$$\begin{aligned}((1-\theta)y_1 + \theta y_2)((1-\theta)x_1 + \theta x_2) &\ge (1-\theta)^2 + \theta^2 + \theta(1-\theta)\left(\frac{x_1}{x_2} + \frac{x_2}{x_1}\right) \\ &\ge (1-\theta)^2 + \theta^2 + 2\theta(1-\theta) = 1\end{aligned}$$

So B is convex, and by continuity of $1/x$, B is also closed.

Note that any hyperplane not of the form $\{(x,y) \in \mathbb{R}^2 : y = c\}$ will intersect A. If $c < 0$ the hyperplane doesn't separate the sets, if $c = 0$ it intersects A and if $c > 0$ it intersects B, so there is no strictly seperating hyperplane.

9. Let $I = \{i \in [n] : |\hat{x}_i| = 1\}$. Since $\hat{x}$ is in the boundary of C, I is non-empty.

   Consider a potential supporting hyperplane $H(a) = \{x \in \mathbb{R}^n : a^T(x - \hat{x}) = 0\}$.

   Suppose $a \neq 0$ is such that $a_i = -1 * \delta_{i \in I} * \hat{x}_i * \alpha_i$ where $\alpha_i \geq 0$.

   If $x \in C$,

   $$a^T(x - \hat{x}) = \sum_i a_i(x_i - \hat{x}_i) = \sum_{i \in I} \alpha_i(1 - x_i\hat{x}_i) \geq 0$$

   and therefore those a define valid supporting hyperplanes.

   We show those are the only valid normal vectors. Call $\mathbf{e}_i$ the one-hot vector at coordinate i. Say there is $i \notin I$ such that, WLOG, $a_i > 0$. Since $i \notin I$, $\hat{x} - \epsilon\mathbf{e}_i \in C$ for some $\epsilon > 0$. Then $a^T(\hat{x} - \epsilon\mathbf{e}_i - \hat{x}) = -a_i\epsilon < 0$.

   Alternatively, say $i \in I$ and $sign(a_i) = sign(\hat{x}_i)$. Clearly $\hat{x} - \epsilon\hat{x}_i\mathbf{e}_i \in C$, but $a^T(\hat{x} - \epsilon\hat{x}_i\mathbf{e}_i - \hat{x}) = -\epsilon a_i\hat{x}_i < 0$.

10. Call H the hyperbolic cone and C the second-order cone.

    If $x \in H$, then $xP^Tx \leq (c^Tx)^2$ and $c^Tx \geq 0$. Call $z = P^{1/2}x$ and $t = c^Tx$, then $z^Tz \leq t^2$ and $t \geq 0$, hence $f(x) = (z, t) \in C$ and $x \in f^{-1}(C)$.

    Conversely, if $x \in f^{-1}(C)$, then $f(x) \in C$. So $xP^Tx = (P^{1/2}x)^TP^{1/2}x \leq (c^Tx)^2$ and $c^Tx \geq 0$, i.e. $x \in H$.