MATH-GA.2012.001 Selected Topics in Numerical Analysis :
Convex and Nonsmooth Optimization, Spring 2020
Homework Assignment 1
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1. Prove that the quadratic cone is convex. Given the quadratic cone C = $\{$(x,t) $\in$ $\mathbf{R}^{n+1} | ||x||_2 \leq t$ $\}$. By triangle inequality, and homogeneity for any $x1, x2 \in C$ and $\theta \in [0, 1]$:

$$\begin{aligned} ||\theta \begin{bmatrix} x1 \\ t \end{bmatrix} + (1-\theta) \begin{bmatrix} x2 \\ t \end{bmatrix} ||_2 &\leq ||\theta \begin{bmatrix} x1 \\ t \end{bmatrix} ||_2 + ||(1-\theta) \begin{bmatrix} x2 \\ t \end{bmatrix} ||_2 \\ &= \theta|| \begin{bmatrix} x1 \\ t \end{bmatrix} ||_2 + (1-\theta)|| \begin{bmatrix} x2 \\ t \end{bmatrix} ||_2 \\ &\leq \theta t + (1-\theta)t \\ &= t \end{aligned}$$

2. Prove (using the definition of convexity) that the intersection of two convex sets is convex. (See BV p.36) Let C1, C2 two convex sets and $C3 = C1 \cap C2$. For any $x1, x2 \in C3$ and $\theta \in [0, 1]$:

$$\begin{aligned} \theta x1 + (1-\theta)x2 &\in C1 \text{ since } x1, x2 \in C1 \\ \theta x1 + (1-\theta)x2 &\in C2 \text{ since } x1, x2 \in C2 \\ \Rightarrow \theta x1 + (1-\theta)x2 &\in C1 \cap C2 = C3 \end{aligned}$$

   C3 is convex.

3. Prove that the image of a convex set under an affine function is convex, and that the inverse image is also convex. Given f an affine function, $f : \mathbf{R}^n \to \mathbf{R}^m$, with $f(x) = Ax + b$, where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$ and suppose that $C \subseteq \mathbf{R}^n$ is convex,

   $\forall y_1, y_2 \in f(C), \forall \theta \in [0, 1]$, and let $f(x_1) = y_1, f(x_2) = y_2$, we have:

$$\begin{aligned} \theta y_1 + (1-\theta)y_2 &= \theta f(x_1) + (1-\theta)f(x_2) \\ &= \theta(Ax_1 + b) + (1-\theta)(Ax_2 + b) \\ &= \theta A(x_1 + x_2) \end{aligned}$$

   which is a linear combination of x_1, x_2 with b =0, and since C is convex $x_1, x_2 \in C$, so $\theta y_1 + (1-\theta)y_2 \in f(C)$ and $f(C)$ is convex.

   Suppose now $\forall x_1, x_2 \in f^{-1}(C), \forall \theta \in [0, 1]$, and let $f(x_1) = y_1, f(x_2) = y_2$, with $y_1, y_2 \in C$, C is a convex set, we have:

$$\begin{aligned} f(\theta x_1 + (1-\theta)x_2) &= A[\theta x_1) + (1-\theta)x_2] \\ &= \theta(Ax_1 + b) + (1-\theta)(Ax_2 + b) \\ &= \theta y_1 + (1-\theta)y_2 \end{aligned}$$

Since C is convex then $y_3 = \theta y_1 + (1-\theta)y_2$ is also in C. Therefore we showed that there exist $y_3 \in C$ such that $f(\theta x_1 + (1-\theta)x_2) = y_3$ which proves that $f^{-1}(C)$ is convex.

4. BV Ex 2.1 Let $C \subseteq \mathbf{R}^n$ be a convex set, $x1_, \cdots, x_k \in C$ and $\theta 1, \cdots, \theta_k \in C$, with $\theta_i \geq 0$ and $\sum_i \theta_i = 1$. Then by definition of the convexity, for k=2, $\sum_{i=1}^{k} \theta_i x_i \in C$ holds. Assuming this is also true for k=n-1, then $\sum_{i=1}^{n} \theta_i x_i = (\sum_{i=1}^{n-1} \theta_i x_i) + \theta_n x_n$, which is the sum of two elements of C which is in C by induction.

5. BV Ex 2.2 Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

   If C is a convex set, a line being affine is also convex and the intersection will be convex. If the intersection of a set with a line is convex and non empty, any points of C will also be in the intersection therefore in C. The same applies to affine set since any affine set is convex.

6. BV Ex 2.10 Let $C \subseteq \mathbf{R}^n$, the solution set of a quadratic inequality,

$$C = \{x \in \mathbf{R}^n | x^T A x + b^T x + c \leq 0\}$$

   with $A \in \mathbf{S}^n$,$b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.

   (a) Show that C is convex if $A \succeq 0$

   (b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbf{R}$.

   Are the converses of these statements true?

   source: math.stackexchange.com

   (a) rewriting C as $C = \{x \in \mathbf{R}^n | (x^T A x) + (b^T x) \leq \alpha, \alpha \in \mathbf{R}\}$. Then the condition on x is the sum of two convex functions $x^T A x$ if $A \succeq 0$ and $b^T x$ and sublevels set of a function are convex (BV 3.1.6). If $A = -1, b = 0, c = -1$, $C = \{x \in \mathbf{R}^n | \|x\|_2^2 \geq 1\}$ is convex but A is not positive semi-definite so the converse is not true.

   (b) First we show that the intersection of C with a line is convex when $A \succeq 0$. Let $l = \{x + tv | t \in \mathbf{R}\}$ an arbitrary line, replacing any point of this line in $C \cap l$, we have:

$$(x+tv)^T A(x+tv) + b^T(x+tv) + c = (v^T A v)t^2 + (2x^T A v + b^T v)t + x^T A x + b^T x + c$$

   $C \cap l = \{x | \alpha t^2 + \beta t + \gamma \leq 0$ where $\alpha = v^T A v, \beta = 2x^T A v + b^T v$ and $\gamma = x^T A x + b^T x + c\}$. It is the equation of a parabola, which opens upward towards $+\infty$ when $\alpha > 0$ and the points solution are all the points for which the quadratic equation is negative or zero; it is a bounded interval and convex. When $\alpha = 0$ the equation is $\beta t + \gamma$ which is affine and convex. And when $\alpha < 0$ the parabola is open downward towards $-\infty$ and the solutions are the union of two disjoint intervals and is not convex. Thus $C \cap l$ is convex when $\alpha = v^T Av \geq 0$ thus C is convex when $A \succeq 0$. WLOG we now consider $C \cap l \cap H$, and notice:

$$g^T \,.\, (x + t\ v) + h = 0$$
$$g^T\ v\ t = 0 \text{ since } g^T\ x + h = 0$$

So we are looking for points in $I = C \cap l \cap H = \{x | \alpha t^2 + \beta t + \gamma \leq 0, \epsilon t = 0,$ with the same as above $\alpha, \beta, \gamma, \epsilon = g^T\ \ v\}$. If $t = 0$ then the intersection reduces to the point $\{x\}$ assuming $\gamma \leq 0$ or the empty set, in both cases the intersection is convex. if $g^T v = 0$ then the intersection is now $I = \{x | \alpha t^2 + \beta t + \gamma \leq 0\}$ and this is verified when $A \succeq 0$. Since $gg^T \succeq 0$, we conclude that I is convex if $A \succeq 0 \Rightarrow (A + \lambda gg^T) \succeq 0, \lambda \geq 0$. The converse is not verified for the same counter-example as (a).

7. BV Ex 2.16

   Let S_1, S_2 two convex sets $\in \mathbf{R}^{m+n}$ and $S = \{(x, y_1 + y_2) | x \in \mathbf{R}^m, y_1, y_2 \in \mathbf{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}$. $\forall (x, y_1 + y_2) \in S, (x, z_1 + z_2) \in S, \forall \theta \in [0, 1]$, we have: $\theta(x, y_1 + y_2) + (1 - \theta)(x, z_1 + z_2) = (x, (\theta y_1 + (1 - \theta) y_2) + (\theta z_1 + (1 - \theta) z_2))$ which is in the form $(x, t + s)$ where $x \in \mathbf{R}^m, t = (\theta y_1 + (1 - \theta) y_2) \in S_1, s = (\theta z_1 + (1 - \theta) z_2) \in S_2$ since S_1, S_2 are convex. Thus S is convex.

8. BV Ex 2.23 Give an example of two closed convex sets that are disjoint but cannot be strictly separated. $S_1 = \{x \in \mathbf{R}^2 : x_1 > 0, x_2 \geq \frac{1}{x_1}\}$ and $S_2 = \{x \in \mathbf{R}^2 : x_2 = 0\}$. S_1 and S_2 are closed, convex, and disjoints. Any line of separating the two sets must be of the form $[01]^T x = \beta$ but $[01]^T b = 0$ for all $b \in S_2$, on the other hand $\inf_{a \in S_1} [01]^T a = 0$, this implies there cannot be strict separation.

9. BV Ex 2.24 (b) Supporting hyperplanes.
   Let $C = \{x \in \mathbf{R}^n | \|x\|_\infty \leq 1\}$ and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

   By definition if C is supported at $\hat{x}$ iff $\exists v \in \mathbf{R}^n, v \neq 0$ such that $v^T.a \geq v^T.\hat{x}$ for all $a \in C$. If $\|\hat{x}\| = 1$, and $\hat{x} = 1$ then we take v = -1, if $\|\hat{x}\| = 1$, and $\hat{x} = -1$ then we take v = 1, and $\|\hat{x}\| \leq 1$, with $-1 < \hat{x} < 1$ then we take v = 0.

   source: https://pages.wustl.edu/files/pages/imce/nachbar/convexityrn.pdf

10. Verify that as stated on BV p.39, the hyperbolic cone is the inverse image of the second order cone under the given affine transformation. Let C, the hyperbolic cone: $C = \{x | x^T Px \leq (c^T x)^2; c^t x \geq 0\}$ where $P \in \mathbf{S}^n_+$ and $c \in \mathbf{R}^n$, and S, the second-order cone: $S = \{(z, t) | z^T z \leq t^2; t \geq 0\}$. For any point x of C, we want to show that under affine function $f(x) = (P^{\frac{1}{2}} x, c^T x)$, $C = \{x | f(x) \in S\}$. $(P^{\frac{1}{2}} x)^T (P^{\frac{1}{2}} x) = x^T (P^{\frac{1}{2}})^T P^{\frac{1}{2}} x = x^T P^{\frac{1}{2}} P^{\frac{1}{2}} x$ since P is symmetric. So

$C = \{x | \|P^{\frac{1}{2}}x\|_2^2 \leq (c^T x)^2\}$ or $C = \{(x, ct) | \|P^{\frac{1}{2}}x\|_2^2 \leq (c^T x)^2, (c^T x) \geq 0\}$,
$C = \{(x, ct) | f(x) \in S\}$.