MATH-GA.2012.001 Selected Topics in Numerical Analysis :
Convex and Nonsmooth Optimization, Spring 2020
Homework Assignment 2
Yves Greatti - yg390

1. Prove that a function is convex if and only if its epigraph is a convex set. Suppose f is a convex function, $f : \mathbf{R}^n \to \mathbf{R}$ then $\forall (x, t_1), (y, t_2) \in \mathbf{epi} f$, and $\forall \theta \in [0, 1]$, we want to show that $\theta(x, t_1) + (1 - \theta)(y, t_2)$ is in $\mathbf{epi} f$. we have:

$$\begin{aligned} f(\theta x + (1-\theta)y) &\leq \theta f(x) + (1-\theta) f(y) \\ &\leq \theta t_1 + (1-\theta) t_2 \end{aligned}$$

thus $\mathbf{epi} f$ is convex. The other direction is similar $\forall (x, t_1), (y, t_2) \in \mathbf{epi} f$, $\mathbf{epi} f$ is a convex set, and $\forall \theta \in [0, 1]$: Let $t_1 = f(x)$, $t_2 = f(y)$ thus $\theta(x, t_1) + (1 - \theta)(y, t_2) = (\theta x + (1 - \theta)y, \theta t_1 + (1 - \theta)t_2)$ is in $\mathbf{epi} f$ which implies: $f(\theta x + (1-\theta)y) \leq \theta t_1 + (1-\theta)t_2 \Rightarrow f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y) \Rightarrow$ f is convex.

2. BV Ex. 2.31 Properties of dual cones. Let K^* be the dual cone of a convex cone K. Prove the following.

   (a) K^* is indeed a convex cone. $\forall y_1, y_2 \in K^*, \theta_1, \theta_2 \geq 0$, and $\forall x \in K$, $x^T(\theta_1 y_1 + \theta_2 y_2) = \theta_1 x^T y_1 + \theta_2 x^T y_2 \geq 0$ thus K^* is a convex cone.

   (b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$. Suppose $y \in K_2^*$, $\forall x \in K_1$, $x^T y \geq 0$, and since $x \in K_2$ also, then $y \in K_1^*$ and $K_2^* \subseteq K_1^*$.

3. Show that if a convex cone K is closed, then $(K^*)^*$, the dual cone of the dual cone of K, is equal to K.

4. BV Ex. 233 Find the dual cone of $\{A\,x | x \geq 0\}$, where $A \in \mathbf{R}^{m \times n}$. The dual of $K = \{A\,x | x \geq 0\}$ is $K^* = \{y | (Ax)^T y \geq 0, \forall x \geq 0\}$ or $K^* = \{y | x^T (A^T y) \geq 0, x \geq 0\} = \{y | (A^T y)^T x \geq 0, x \geq 0\}$. Given $u = A^T y$, we are looking for vectors u such that the inner product is non-negative for any $x \geq 0$. Let $\{e_1, \cdots, e_n\}$ the canonical basis for $\mathbf{R}^n$, for any vector $u = A^T y, y \in K^*$, we have $u^T e_i \geq 0 \Rightarrow u_i \geq 0, i \in [1, n]$. Thus $K^* = \{y | A^T y \geq 0, x \geq 0\}$, this is sufficient as if $x \geq 0$ then $x^T A^T y \geq 0$.

5. Show that the second-order cone defined on p.31 of BV is self-dual, that is, it satisfies $K^* = K$. Let C the second-order cone, $C = \{(x, t) \in \mathbf{R}^n | \|x\|_2 \leq t\}$. $C^* = \{(y, s) | \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} y \\ s \end{bmatrix} \geq 0, \forall (x, t) \in C\}$. if $(y, s) \in C$ then $x^T y \leq \|x\|_2 \|y\|_2$ using Cauchy-Schwarz or $x^T y \leq t\ s$. $\begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} y \\ s \end{bmatrix} = x^T y + ts$, and by the triangle inequality, $\|x^T y + ts\| \geq t\ s - |x^T y| \geq 0 \Rightarrow y \in C^*$. Suppose $(y, s) \notin C$, then $\|y\|_2 > s$ and let m the index of the largest component of y, thus $\|y\|_2 = (\sum_{i=1,n} y_i^2)^{\frac{1}{2}} \leq (n^2 |y_m|^2)^{\frac{1}{2}} = n|y_m| \Rightarrow$. WLOG $|y_m| = y_m$,

then $y_m > \frac{n}{s^2}$ and let x the vector with the only component non-zero $x_m = -\frac{n}{s^2}$ then $x^T y = -\frac{n}{s^2}\ y_m \leq -1$ so $y \notin C^*$. In conclusion, $C = C^*$, C is self-dual.