

1. (a) $L(x, \nu) = \frac{1}{2}x^T Qx + \nu^T (Ax - b)$
 (b) $\nabla_x L(x, \nu) = Qx + A^T \nu$, and $\nabla_x L(x, \nu) = 0 \Rightarrow x = -Q^{-1}A^T \nu$. For this minimizer x , the Lagrange dual function is: $g(\nu) = \min_x L(x, \nu) = \frac{1}{2}(-Q^{-1}A^T \nu)^T Q(-Q^{-1}A^T \nu) + \nu^T A(-Q^{-1}A^T \nu) - \nu^T b$

$$g(\nu) = -\frac{1}{2}\nu^T A Q^{-1} A^T \nu - \nu^T b$$

(c)

$$\begin{aligned}\nabla_\nu g(\nu) &= -A Q^{-1} A^T \nu - b \\ \nabla_\nu g(\nu) = 0 &\Rightarrow \nu^* = -(A Q^{-1} A^T)^{-1} b\end{aligned}$$

The dual optimal value d^* is:

$$d^* = g(\nu^*) = -\frac{1}{2}b^T [-(A Q^{-1} A^T)^{-1}]^T A Q^{-1} A^T [-(A Q^{-1} A^T)^{-1}] b - b^T [-(A Q^{-1} A^T)^{-1}]^T b$$

$$d^* = \frac{1}{2}b^T (A Q^{-1} A^T)^{-1} b$$

(d) The associated \hat{x} is:

$$\hat{x} = -Q^{-1}A^T \nu^* = -Q^{-1}A^T [-(A Q^{-1} A^T)^{-1} b] = Q^{-1}A^T (A Q^{-1} A^T)^{-1} b$$

(e) \hat{x} is feasible for the primal problem since it verifies the equality constraint $h(x) = Ax - b = 0$ since $h(\hat{x}) = A Q^{-1} A^T (A Q^{-1} A^T)^{-1} b - b = 0$.

(f)

$$f_0(\hat{x}) = \frac{1}{2}\hat{x}^T \hat{x} = \frac{1}{2}b^T (A Q^{-1} A^T)^{-1} A Q^{-1} Q Q^{-1} A^T (A Q^{-1} A^T)^{-1} b$$

$$f_0(\hat{x}) = \frac{1}{2}b^T (A Q^{-1} A^T)^{-1} b = p^* = d^*$$

(g) $d^* = p^*$ therefore there is no duality gap.

2. BV Ex 5.1

- (a) Given $f_0(x) = x^2 + 1$, the feasible set is the interval $[2, 4]$. $f'_0(x) = 2x$ so f_0 is increasing on $[2, 4]$, and the optimal value is $f_0(2) = 5$ and the optimal solution is 2.

(b)

$$L(x, \lambda) = x^2 + 1 + \lambda(x - 1)(x - 4) = (\lambda + 1)x^2 - 6\lambda x + 1 + 8\lambda$$

The feasible set is indicated by the grey area between the two vertical lines at $x = 2$ and $x = 4$, $L(x, \lambda)$ is plotted for $\lambda = 1, 2, 3$ (ref ??). We can verify on the graph that $\inf_x L(x, \lambda) \leq p^*, p^* = 5$. $L(x, \lambda)$ is convex (quadratic) and differentiable, we can set its gradient to zero to find its minimizer and write the Lagrange dual function $g(\nu)$. $\nabla_x L(x, \lambda) = 2(\lambda + 1)x - 6\lambda$, $\nabla_x L(x, \lambda) = 0 \Rightarrow x = \frac{3\lambda}{\lambda + 1}$ for $\lambda \neq -1$. $g(\nu) = \inf_x L(x, \lambda)$ is not bounded when $\lambda \leq -1$ thus

$$g(\lambda) = \begin{cases} \frac{-9\lambda^2}{\lambda + 1} + 1 + 8\lambda & \text{when } \lambda > -1 \\ -\infty & \text{otherwise} \end{cases}$$

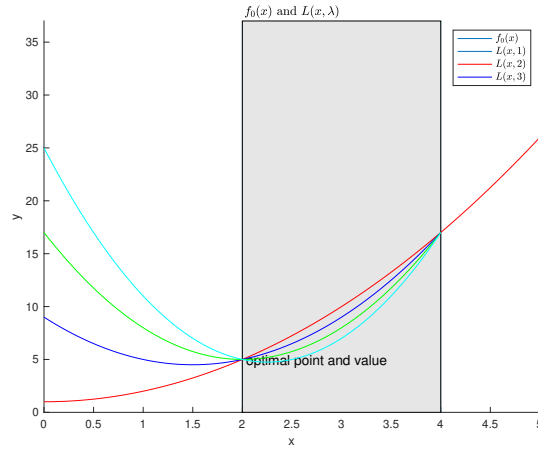


Figure 1: Plot of f_0 and the Lagrangian $L(x, \lambda)$ for $\lambda = 1, 2, 3$.

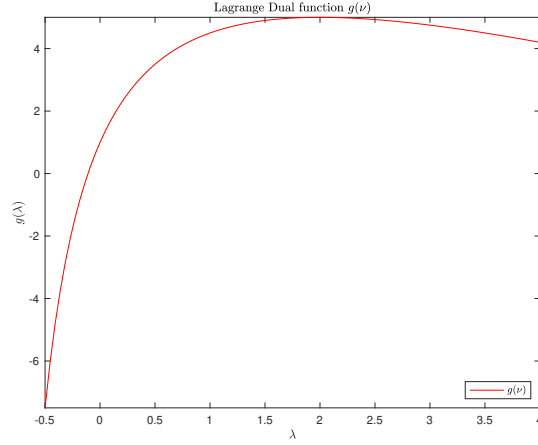


Figure 2: Plot of $g(\lambda)$ for $\lambda \in [-0.5, 5]$.

(c) The dual problem is:

$$\begin{aligned} & \text{maximize } g(\lambda) = \frac{-9\lambda^2}{\lambda + 1} + 1 + 8\lambda \\ & \text{subject to } \lambda > -1 \end{aligned}$$

Based on the graph of $g(\lambda)$, it is concave. Setting the derivative of $g(\lambda)$ to zero, leads to $g'(\lambda) = 8 - 9\frac{\lambda(\lambda+2)}{(\lambda+1)^2} = 0$ which gives the dual optimal solution $\lambda^* = 2$ and the dual optimal value $d^* = 5 = p^*$ thus strong duality holds.

- (d) Expanding $(x - 2)(x - 4) - u \leq 0$, we have $x^2 - 6x + 8 - u \leq 0$. The determinant $\Delta = 4(1 + u)$ is valid only when $u \geq -1$ and the roots of the quadratic equation are $3 - \sqrt{1 + u}$ and $3 + \sqrt{1 + u}$, thus the feasible set is $[3 - \sqrt{1 + u}, 3 + \sqrt{1 + u}]$, $u \geq -1$. We notice that $\min_x x^2 - 6x + 8 = -1$ and $p^*(u)$ is not defined when $u < -1$. And for $p(u^*) = f(3 - \sqrt{1 + u}) = 11 + u - 6\sqrt{1 + u}$ reaches its global minimizer point u_0^* when $p'(u_0^*) = 1 - \frac{3}{\sqrt{1+u}} = 0 \Rightarrow u_0^* = 8, p(u_0^*) = 1$. For $u \in [-1, 8]$, the solution to the problem in x is decreasing to the minimum value 1 to increase again when $u^* > 8$.

$$p(u^*) = \begin{cases} \infty & \text{when } u < -1 \\ 11 + u - 6\sqrt{1 + u} & \text{when } u \in [-1, 8] \\ 1 & \text{when } u > 8 \end{cases}$$

We verify that $\frac{dp(u^*)}{du} = 1 - \frac{3}{\sqrt{1+u}}$ and $\frac{dp(0)}{du} = -2 = -\lambda^*$.

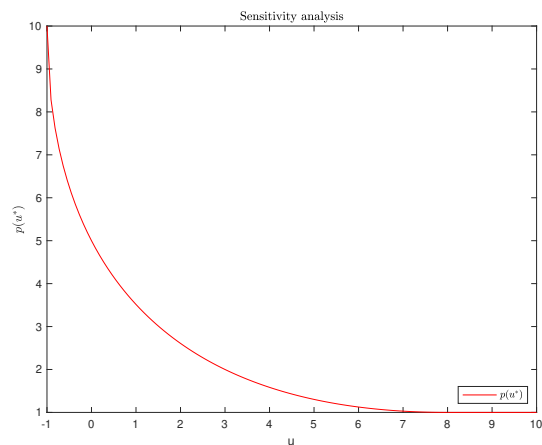


Figure 3: $p(u^*)$.

3. BV Ex 5.21 The exponential function e^{-x} is convex on \mathbf{R} , it then remains to determine if the Hessian of $f_1(x, y) = \frac{x^2}{y}$ is positive definite or semi definite.

$$\nabla^2 f_1(x, y) = \begin{bmatrix} \frac{2}{y} & \frac{-2x}{y^2} \\ \frac{-2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}$$

The polynomial characteristic is $p(\lambda) = \frac{\lambda}{y^3}(y^3\lambda - 2(x^2 + y^2))$ so the eigenvalues are 0 and $2\frac{x^2 + y^2}{y^3} > 0, y > 0$. Hence the Hessian is positive semidefinite and $f_1(x, y)$ is convex. The problem is a convex optimization problem. The minimizer of e^{-x} satisfying the inequality constraint is $x = 0$ and the optimal value is 1.