MATH-GA.2012.001 Selected Topics in Numerical Analysis :
Convex and Nonsmooth Optimization, Spring 2020
Homework Assignment 2
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1. Prove that a function is convex if and only if its epigraph is a convex set. Suppose f is a convex function, $f : \mathbf{R}^n \to \mathbf{R}$ then $\forall (x, t_1), (y, t_2) \in \mathbf{epi} f$, and $\forall \theta \in [0, 1]$, we want to show that $\theta(x, t_1) + (1 - \theta)(y, t_2)$ is in $\mathbf{epi} f$. we have:

$$\begin{aligned} f(\theta x + (1-\theta)y) &\leq \theta f(x) + (1-\theta) f(y) \\ &\leq \theta t_1 + (1-\theta) t_2 \end{aligned}$$

thus $\mathbf{epi} f$ is convex. The other direction is similar $\forall (x, t_1), (y, t_2) \in \mathbf{epi} f$, $\mathbf{epi} f$ is a convex set, and $\forall \theta \in [0, 1]$: Let $t_1 = f(x)$, $t_2 = f(y)$ thus $\theta(x, t_1) + (1 - \theta)(y, t_2) = (\theta x + (1 - \theta)y, \theta t_1 + (1 - \theta)t_2)$ is in $\mathbf{epi} f$ which implies: $f(\theta x+(1-\theta)y) \leq \theta t_1+(1-\theta)t_2 \Rightarrow f(\theta x+(1-\theta)y) \leq \theta f(x)+(1-\theta)f(y) \Rightarrow$ f is convex.

2. BV Ex. 2.31 Properties of dual cones. Let K^* be the dual cone of a convex cone K. Prove the following.

   (a) K^* is indeed a convex cone. $\forall y_1, y_2 \in K^*, \theta_1, \theta_2 \geq 0$, and $\forall x \in K$, $x^T(\theta_1 y_1 + \theta_2 y_2) = \theta_1 x^T y_1 + \theta_2 x^T y_2 \geq 0$ thus K^* is a convex cone.

   (b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$. Suppose $y \in K_2^*$, $\forall x \in K_1$, $x^T y \geq 0$, and since $x \in K_2$ also, then $y \in K_1^*$ and $K_2^* \subseteq K_1^*$.

3. Show that if a convex cone K is closed, then $(K^*)^*$, the dual cone of the dual cone of K, is equal to K. source: class notes on the web pointing that $(K^*)^*$ can be seen as the intersection of halfspaces.

   Let K a convex closed cone, then $(K^*)^* = \{x^* | y^T x^* \geq 0, \forall y \in K^*\}$. We can consider $(K^*)^*$ as the intersection of halfspaces $H_{x^* \in K} = \{y^T x^* \geq 0, \forall y \in K^*\}$. If $x \in K$ then $\forall y \in K^*, x^T y = y^T x \geq 0 \Rightarrow x \in (K^*)^*$. K being convex and closed by the corollary of the separating hyperplanes, $K = (K^*)^*$.

4. BV Ex. 233 Find the dual cone of $\{A\,x | x \geq 0\}$, where $A \in \mathbf{R}^{m \times n}$. The dual of $K = \{A\,x | x \geq 0\}$ is $K^* = \{y | (Ax)^T y \geq 0, \forall x \geq 0\}$ or $K^* = \{y | x^T(A^T y) \geq 0, x \geq 0\} = \{y | (A^T y)^T x \geq 0, x \geq 0\}$. Given $u = A^T y$, we are looking for vectors u such that the inner product is non-negative for any $x \geq 0$. Let $\{e_1, \cdots, e_n\}$ the canonical basis for $\mathbf{R}^n$, for any vector $u = A^T y, y \in K^*$, we have $u^T e_i \geq 0 \Rightarrow u_i \geq 0, i \in [1, n]$. Thus $K^* = \{y | A^T y \geq 0, x \geq 0\}$, this is sufficient as if $x \geq 0$ then $x^T A^T y \geq 0$.

5. Show that the second-order cone defined on p.31 of BV is self-dual, that is, it satisfies $K^* = K$. Let C the second-order cone, $C = \{(x, t) \in \mathbf{R}^n | \|x\|_2 \leq t\}$. $C^* = \{(y, s) | \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} y \\ s \end{bmatrix} \geq 0, \forall (x, t) \in C\}$. if $(y, s) \in C$ then $x^T y \leq \|x\|_2 \|y\|_2$

using Cauchy-Schwarz or $x^T y \leq t\ s$. $\begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} y \\ s \end{bmatrix} = x^T y + ts$, and by the triangle inequality, $\|x^T y + ts\| \geq t\ s - |x^T y| \geq 0 \Rightarrow y \in C^*$. Suppose $(y, s) \notin C$, then $\|y\|_2 > s$ and let m the index of the largest component of y, thus $\|y\|_2 = (\sum_{i=1,n} y_i^2)^{\frac{1}{2}} \leq (n^2 |y_m|^2)^{\frac{1}{2}} = n|y_m| \Rightarrow$. WLOG $|y_m| = y_m$, then $y_m > \frac{n}{s^2}$ and let x the vector with the only component non-zero $x_m = -\frac{n}{s^2}$ then $x^T y = -\frac{n}{s^2}\ y_m \leq -1$ so $y \notin C^*$. In conclusion, $C = C^*$, C is self-dual.

6. "Chebyshev center" problem

   (a) function chebyshev_center(A, b) takes a matrix A of dimension $(2, n)$ and a vector $b(n)$ to find the largest Euclidean ball that lies in a polyhedron described by n linear inequalities.

```
% Compute the Chebyshev center of a polyhedron
% Boyd & Vandenberghe "Convex Optimization"
function [x_sol, r_sol] = chebyshev_center(A, b)
% The goal is to find the largest Euclidean ball (i.e. its center and
% radius) that lies in a polyhedron described by linear inequalites in t
% fashion: P = {x : a_i'*x <= b_i, i=1,...,m}

% Generate the data
[~,n]=size(A);

% Build and execute model
fprintf(1,'Computing Chebyshev center...');
cvx_begin
    variable r(1)
    variable x_c(2)
    maximize ( r )
    for k=1:n
        A(:, k)'*x_c + r*norm(A(:, k),2) <= b(k);
    end
cvx_end
fprintf(1,'Done! \n');
x_sol = x_c;
r_sol = r;

% Display results
fprintf(1,'The Chebyshev center coordinates are: \n');
disp(x_c);
fprintf(1,'The radius of the largest Euclidean ball is: \n');
disp(r);

% Generate the figure
x = linspace(-2,2);
for k=1:n
```

```
    plot(x, -x * A(1,k)./A(2,k) + b(k)./A(2,k),"b-");
    hold on
end
theta = 0:pi/100:2*pi;
plot( x_c(1) + r*cos(theta), x_c(2) + r*sin(theta), "r" );
plot(x_c(1),x_c(2),'b*');
xlabel("x_1")
ylabel("x_2")

txt = "# inequalities:" + num2str(n);
title({"Largest Euclidean ball laying in a 2D polyhedron", txt});
text(x_c(1), x_c(2), "\leftarrow  center")
axis([-1 1 -1 1])
axis equal
hold off
txt = "chebyshev_center_" + num2str(n);
saveas(gcf,txt,'epsc')
```

For the same example on the web page where matrix $A = \begin{bmatrix} 2 & 2 & -1 & -1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$ and vector $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we obtain the same circle which is tangent to the four hyperplanes $a_i^T x = b_i$ (see figure 1):

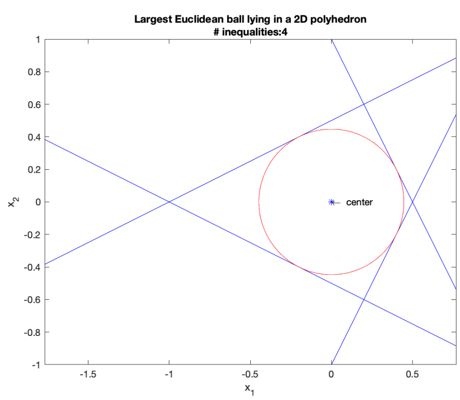

Figure 1: Sample example

We solve the same optimization problem with more inequalities and an interior center inside the polyhedron (see figure 2):

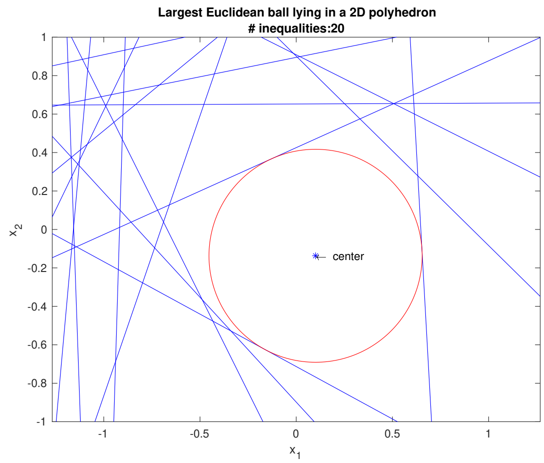

Figure 2: Example with 20 inequalities

(b) If we choose A and b such that there is no interior point : if the hyperplanes intersect at the same point then CVX finds a center with radius zero, if they do not intersect then CVX will not find a solution.

(c) The problem to solve now is to find the largest "scaled unit ball" $\mathcal{B} = \{x_c + u | \|u\|_p \leq r\}$ that lies in the polyhedron described by a set of linear inequalities: $\mathcal{P} = \{x \in \mathbf{R}^n | a_i^T x \leq b_i, i = 1, \cdots, m\}$. For any point of $\mathcal{B}$ laying in one halfspace $a_i^T x \leq b_i$, similarly to the euclidean space, we have $\|u\|_p \leq r \Rightarrow a_i^T x_c + r\|a_i\|_p \leq b_i$ since $g_i = \sup \{a_i^T u | \|u\|_p \leq r\} = r\|a_i\|_p$. And the Chebyshev center can be determined by solving the problem:

$$\begin{aligned} &\text{maximize } r \\ &\text{subject to } a_i^T x_c + r\|a_i\|_p \leq b_i, i = 1, \cdots, m \end{aligned}$$

This is still an LP problem since the inequalities are linear. We obtain different solutions for the same matrix A and vector b, corresponding to different p-norm, $p = 1$ is the diamond shape ball (fig. 3), for $p = 1.5$ the ball has a shape between the diamond and the circle $p = 2$ (fig. 4), and as we increase p, for $p = \infty$ the ball becomes a square (where either $\|x\| = 1$ and $\|y\| \leq 1$ or $\|x\| \leq 1$ and $\|y\| = 1$, (fig. 5)).

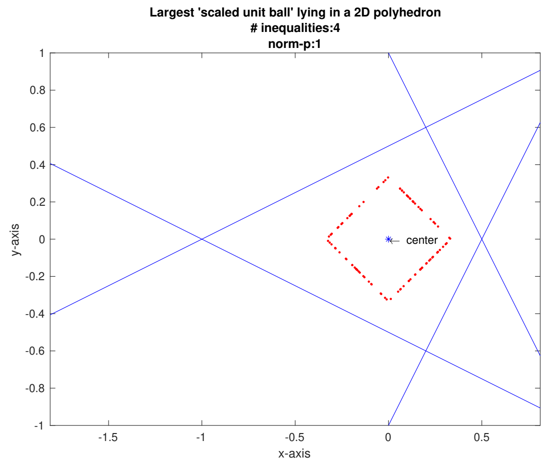

Figure 3: ”Scaled” unit ball for p=1

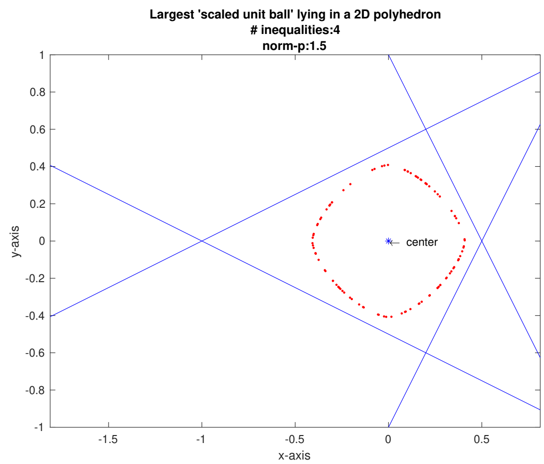

Figure 4: ”Scaled” unit ball for p=1.5

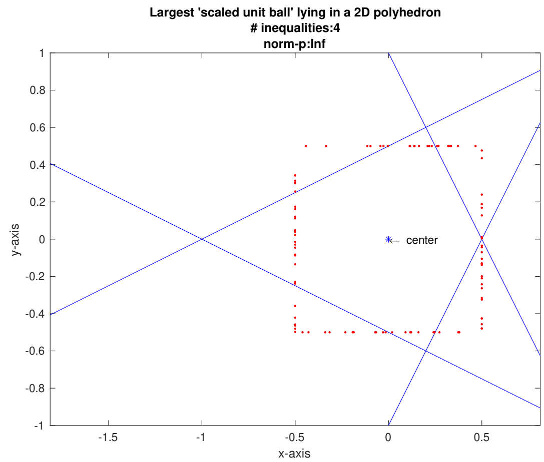

Figure 5: "Scaled" unit ball for p=∞

```
% Compute the Chebyshev center of a polyhedron
function [x_sol, r_sol] = chebyshev_center_with_norm(A, b, p)
% The goal is to find the largest Euclidean ball (i.e. its center and
% radius) that lies in a polyhedron described by linear inequalites in t
% fashion: P = {x : a_i'*x <= b_i, i=1,...,m}

rng('default')
format long g
[~,n]=size(A);

% Build and execute model
fprintf(1,'Computing Chebyshev center...');
cvx_begin
    variable r(1)
    variable x_c(2)
    maximize ( r )
    for k=1:n
        A(:, k)' * x_c + r * norm(A(:, k), p) <= b(k);
    end
cvx_end
fprintf(1,'Done! \n');
x_sol = x_c;
r_sol = r;


% Display results
fprintf(1,"The Chebyshev center coordinates are: \n");
disp(x_c);
```

```
txt = "Radius of largest 'scaled unit ball' using norm p:" + num2str(p)
fprintf(1,txt);
disp(r);

% Generate the figure
x = linspace(-2,2);
for k=1:n
    plot(x, -x * A(1,k)./A(2,k) + b(k)./A(2,k),"b-");
    hold on
end

n_vecs = 100;
[x, y] = gen_random_vectors(n_vecs, p);
x = x.* r + x_c(1);
y = y.* r + x_c(2);

for i=1:n_vecs
    plot(x(i), y(i), "r." );
    hold on
end
plot(x_c(1),x_c(2),'b*');
xlabel("x-axis")
ylabel("y-axis")

txt1 = "# inequalities:" + num2str(n);
tx2 = "norm-p:" + num2str(p);
title({"Largest 'scaled unit ball' laying in a 2D polyhedron", txt1, tx2
text(x_c(1), x_c(2), "\leftarrow  center")
axis([-1 1 -1 1])
axis equal
hold off
txt = "chebyshev_center_norm_" + num2str(p);
saveas(gcf,txt,'epsc')

function [x, y] = gen_random_vectors(n, p)
    r = randn(n, 2); % Use a large n
    for i=1:n
        norm_r = norm(r(i,:), p);
        r(i, :) = r(i, :) ./ norm_r;
    end
    x = r(:, 1);
    y = r(:, 2);
```

(d) As we decrease p, the p-norm of a_i grows exponentially (as shown in figure 6), thus CVX, to solve the problem, finds that the solution for the center is the intersection of the hyperplanes if it exists and r to be zero.

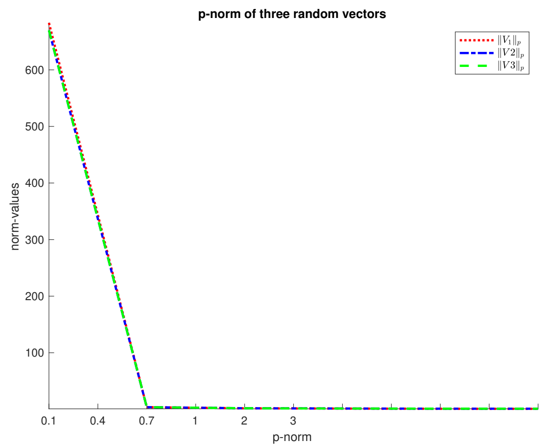

Figure 6: p-norms of three random vectors