Convex and Nonsmooth Optimization
HW9: Subgradients for Nonconvex Functions

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See my notes for the necessary background.

1. First, a question relevant to primal-dual interior point methods. Write MATLAB functions for the following:

   (a) Given a vector (an LP interior point) $x > 0$ and a vector (search direction) b, compute the largest scalar t such that $x + tb \geq 0$.

   (b) Given a positive definite symmetric matrix (an SDP interior point) X and a symmetric matrix (search direction) B, compute the largest scalar t such that $X + tB$ is positive semidefinite. Hint: using the Cholesky factorization of X, convert this into a question that is easy to solve using eigenvalues and implement this using `eig`.

   Note that the answer is `inf` in some cases — what exactly are these cases? It is easy to make a mistake in these codes, so check them carefully on some examples to see if they are correct.

2. We say that $x \in \mathbb{R}^n$ locally minimizes f if, for all sufficiently small $z \in \mathbb{R}^n$ (meaning, all z with sufficiently small $\|z\|$), $f(x + z) \geq f(x)$. Show that a necessary condition for x to locally minimize f is $0 \in \hat{\partial} f(x)$, i.e, zero is in the regular subdifferential of f at x, immediately from the definition on the first page of the notes.

3. Determine the regular subdifferential $\hat{\partial} f$, the (general) subdifferential ∂f, and the horizon subdifferential $\partial^\infty f$ of the following functions of one variable at $x = 0$:

   (a) $f(x) = |x|^3$

(b) $f(x) = |x|^{1/3}$

(c) $f(x) = a|x|$ where a is a nonnegative real number

(d) $f(x) = a|x|$ where a is a negative real number

(e) $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, with $f(0) = 0$. First show by using the definition of the derivative that f is differentiable at 0 with $f'(0) = 0$, and give a formula for $f'(x)$ at any other point by using the ordinary rules of calculus. Also show that f' is not continuous at 0, and hence that f is not C^1 at 0.

Using the definition on p. 6 of the notes, which of these five functions is regular at 0?

4. Same questions for the following two functions of n variables, at $x = 0$:

   (a) $f(x) =$ 3rd largest entry of x, assuming $n \geq 3$ (see p. 3–5 of the notes)

   (b) $f(x) =$ largest entry of Ax, where A is any $n \times n$ matrix (use the chain rule on p. 7 of the notes)

   Which of these two functions is regular at 0?

5. This question illustrates the chain rule given on p. 7 of the notes.[1] Let $n = m = 2$, so $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$. Let

$$F(x) = \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix} \quad \text{and} \quad \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   (a) Suppose $g(\xi) = |\xi_1| + |\xi_2|$ for $\xi \in \mathbb{R}^2$ and define $f = g \circ F$. Do assumptions (1) and (2) of the chain rule hold at $\bar{x}$? Why or why not? What are the left and right-hand sides of equation (†) on p. 7?

   (b) Suppose $g(\xi) = |\xi_1|^{1/3} + |\xi_2|^{1/3}$ for $\xi \in \mathbb{R}^2$ and define $f = g \circ F$. Do assumptions (1) and (2) of the chain rule hold at $\bar{x}$? Why or why not? What are the left and right-hand sides of equation (†) on p. 7?

6. Which of the seven functions given in questions 3 and 4 are locally Lipschitz at 0? In these cases, the horizon subdifferential $\partial^\infty f(0) = \{0\}$, and the Clarke subdifferential $\partial^C f(0)$ is the convex hull of the (general) subdifferential $\partial f(0)$: give explicit formulas for this in these cases.

[1] In assumption (2), $\mathcal{N}$ denotes null space.