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Assignment #1

ECE 269
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Problem 1: Convolution as Linea Map

Solution

(a) Obviously the size of matrice T must be $(N+1) \times (N+1)$. Let $T_{i1}, T_{i2}, \ldots, T_{i(N+1)}$ be the i-th row of matrice T. Since $y = Tx$, we can get

$$v(i-1) = \sum_{j=1}^{N+1} T_{ij} u(j-1) = T_{i1}u(0) + T_{i2}u(1) + \ldots + T_{i(N+1)}u(N) \tag{1}$$

Accoring to the convolution function,

$$v(i-1) = \sum_{k=-\infty}^{+\infty} h(k)u(i-1-k) = \sum_{k=(i-1-N)}^{i-1} h(k)u(i-1-k) \tag{2}$$

Comparing function (1) and function (2), it is easy to conclude that

$$T_{ij} = h(i-j),\ where\ 1 \leq i \leq (N+1),\ 1 \leq j \leq (N+1)$$

So the matrice is a matrice where each element could be represented as $T_{ij} = h(i - j),\ where\ 1 \leq i \leq (N+1),\ 1 \leq j \leq (N+1)$.

(b) The structure of matrice T could be described as follow:

$$T_{i,j} = T_{i+1,j+1}$$

Problem 2: Affine Funciton

Solution

(a) *Proof.* For any $\alpha, \beta \in \mathbb{R}$ and any $x, y \in \mathbb{R}^n$

$$\alpha f(x) + \beta f(y) = \alpha(Ax+b) + \beta(Ay+b) = \alpha Ax + \beta Ay + (\alpha+\beta)b = \alpha Ax + \beta Ay + b$$

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) + b = \alpha Ax + \beta Ay + (\alpha+\beta)b = \alpha Ax + \beta Ay + b$$

Hence, $\alpha f(x) + \beta f(y) = f(\alpha x + \beta y)$. So function $f(x) = Ax + b$ is affine. □

(b) *Proof.* First we can show that b is unique. Because $f(0) = b$, so b must be unique, otherwise $f(0)$ will be mapped as different values in $\mathbb{R}^m$, which conflicts with the function definition. Then we can show that A is unique. Let function

$$g(x) = f(x) - f(0) = Ax + b - b = Ax$$

Suppose $b_1, b_2, \ldots, b_n$ are the basis of $\mathbb{R}^n$ and $B = \sum_{i=1}^{n} \alpha_i b_i,\ \alpha_i \in \mathbb{R}$.

$$g(B) = g(\sum_{i=1}^{n} \alpha_i b_i) = A(\sum_{i=1}^{n} \alpha_i b_i) = \sum_{i=1}^{n} \alpha_i A b_i = \sum_{i=1}^{n} \alpha_i g(b_i)$$

Suppose $b_i = e_i,\ i = 1, 2, \ldots, n$, then $g(B)$ above could be represented as

$$\begin{aligned} g(B) &= \begin{bmatrix} g(b_1) & g(b_2) & \ldots & g(b_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} \\ &= \begin{bmatrix} g(e_1) & g(e_2) & \ldots & g(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \\ &= Ax \end{aligned}$$

where

$x \in \mathbb{R}^n$.

Hence A is unique. So any affine funciton f could be represented uniquely as $f(x) = Ax + b$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times n}$ $\square$

Problem 3: Matrix Multification

Solution

(a) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq 0$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0$$

Then

$$AB = 0$$

Hence the statement is incorrect.

(b) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$$

Then

$$A^2 = 0$$

Hence the statement is incorrect.

(c) Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then

$$A^T A = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 \\ \vdots & \vdots & & \vdots \\ a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \end{bmatrix}$$

So if $A^T A = 0$, then $a_{ij} = 0,\ 1 \leq i \leq n,\ 1 \leq j \leq n.\ A = 0$. So the statement is correct.

Problem 4: Linear Maps and Differentiation of polynomials

Solution

(a) For any $p_1(x),\ p_2(x) \in \mathcal{P}_n$,

$$T(p_1(x) + p_2(x)) = \frac{d(p_1(x) + p_2(x))}{dx} = \frac{dp_1(x)}{dx} + \frac{dp_2(x)}{dx} = T(p_1(x)) + T(p_2(x)) \quad (3)$$

$$T(\alpha p_1(x)) = \frac{d(\alpha p_1(x))}{dx} = \alpha \frac{dp_1(x)}{dx} = \alpha T(p_1(x)) \quad (4)$$

Since equations (3) and (4) are valid, T is linear.

(b) For