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Assignment #1

Problem 1: Vector Spaces other than $\mathbb{R}^N$

Solution

a) Suppose A is a set of rational numbers defined over $\mathbb{R}$. $A_1 \in A$. According to the definition of rational numbers, $A_1 = \frac{b}{a}$, where a and b are integers and $a \neq 0$. Suppose $B_1 \in \mathbb{R}$, and

$$B_1 = \pi \cdot A_1 = \frac{b\pi}{a}$$

Obviously, B_1 is not a rational number, that is $B_1 \notin A$. So the set of rational numbers defined over $\mathbb{R}$ doesn't satisfy closure of scalar multification. So the set is not valid vector field.

b) Suppose set $A = \{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{R}^+\}$. $A_1 \in A$ and $A_1 = a_0 + a_1x + a_2x^2$, $a_0, a_1, a_2 \in \mathbb{R}^+$. Let

$$B_1 = (-1) \cdot A_1 = (-a_0) + (-a_1)x + (-a_2)x^2$$

Obviously, $(-a_0), (-a_1), (-a_2) \notin \mathbb{R}^+$, so $B_1 \notin A$. Hence set A doesn't satisfy closure of scalar multification. So set A is not valid vector field.

c) i. Let

$$A = (\alpha_1 + \alpha_2) \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (\alpha_1 + \alpha_2)a \\ b \end{bmatrix}, \alpha_1, \alpha_2 \in \mathbb{R}$$

$$B = \alpha_1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (\alpha_1 + \alpha_2)a \\ 2b \end{bmatrix}, \alpha_1, \alpha_2 \in \mathbb{R}$$

Because $B \neq A$, so this set doesn't satisfy the following vector space property:

$$(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v, \alpha, \beta \in \mathcal{F} \ and \ v \in \mathcal{V}$$

So this set is not a valid vector field.

ii. Let $r = 1 \in \mathbb{R}$

$$r \cdot A = r \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \neq A$$

So this set doesn't satisfy the following vector space property:

$$1 \cdot v = v, where \ 1 \in \mathcal{F} \ and \ v \in \mathcal{V}$$

So this set is not a valid vector field.

iii. Suppose $B = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Let

$$A_1 = (\alpha_1 + \alpha_2) \cdot B = (\alpha_1 + \alpha_2) \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (\alpha_1 + \alpha_2)a \\ (\alpha_1 + \alpha_2)b \end{bmatrix}$$

$$A_2 = \alpha_1 \cdot B + \alpha_2 \cdot B = 0$$

Because $A_1 \neq A_2$, so this set doesn't satisfy the following vector space property:

$$(\alpha_1 + \alpha_2) \cdot v = \alpha_1 \cdot v + \alpha_2 \cdot v, where\ \alpha_1, \alpha_2 \in \mathcal{F}\ and\ v \in \mathcal{V}$$

So this set is not a valid vector field.

iv. Suppose

$$A = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}, B = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}, C = \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix}$$

Let:

$$A + (B + C) = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + (\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix}) = \begin{bmatrix} \alpha_1 - \alpha_2 + \alpha_3 \\ \beta_1 - \beta_2 + \beta_3 \end{bmatrix})$$

$$(A + B) + C = (\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}) + \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 - \alpha_2 - \alpha_3 \\ \beta_1 - \beta_2 - \beta_3 \end{bmatrix})$$

Because $A + (B + C) \neq (A + B) + C$, so this set doesn't satisfy the following vector space property:

$$A + (B + C) = (A + B) + C, where\ A, B, C \in \mathcal{V}$$

So this set is not a valid vector field.

Problem 2: Adjacency graph

Solution

A path in a graph is any sequence of vertices such that every consecutive pair of vertices in the sequence is connected by an edge in the graph*(P129, Networks:An Introduction, Mark Newman)*. The length of a path is the number of edges traversed along the path (not the number of vertices). B_{ij} is the number of paths of length k from node i to node j in the original graph.

Proof. For $k = 1$, the result is true since $A_{ij} = 1$ when there is an edge from node i to node j and $A_{ij} = 0$ when there is no an edge.

Suppose for every i, j, $[A^{k-1}]_{ij}$ represents the number of paths of length $k - 1$ from node i to node j in the original graph. For each k length path from v_i to v_j, there exists a node h

such that the path could be thought of $k-1$ length path from v_i to v_h, combined with an edge from v_h to v_j. So $[A^k]_{ij}$ could be represented as:

$$[A^k]_{ij} = \sum_{h=1}^{n} [A^{k-1}]_{ih}[A]_{hj}$$

Hence, B_{ij} is the number of paths of length k from node i to node j in the original graph. $\square$

Problem 3: Vector Spaces of Polynomials

Solution

a) A vector space should satisfy properties (A1)-(A5) and (M1)-(M5). Suppose $A = \sum_{k=0}^{n} a_k x^k, a_k \in \mathbb{R}$, $B = \sum_{k=0}^{n} b_k x^k, b_k \in \mathbb{R}$ and $C = \sum_{k=0}^{n} c_k x^k, c_k \in \mathbb{R}$. $A, B, C \in \mathbb{P}_n(\mathbb{R})$.

(A1). Because

$$A + B = \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n} b_k x^k = \sum_{k=0}^{n} (a_k + b_k) x^k \in \mathbb{P}_n(\mathbb{R}), (a_k + b_k) \in \mathbb{R}$$

So $A + B \in \mathbb{P}_n(\mathbb{R})$. Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (A1).

(A2). Because

$$(A+B)+C = \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n} b_k x^k + \sum_{k=0}^{n} c_k x^k = \sum_{k=0}^{n} (a_k+b_k+c_k) x^k \in \mathbb{P}_n(\mathbb{R}), (a_k+b_k+c_k) \in \mathbb{R}$$

$$A+(B+C) = \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n} b_k x^k + \sum_{k=0}^{n} c_k x^k = \sum_{k=0}^{n} (a_k+b_k+c_k) x^k \in \mathbb{P}_n(\mathbb{R}), (a_k+b_k+c_k) \in \mathbb{R}$$

So $(A + B) + C = A + (B + C)$. Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (A2).

(A3). Because

$$A + B = \sum_{k=0}^{n} (a_k + b_k) x^k = B + A$$

So $A + B = B + A$. Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (A3).

(A4). Let $B = \sum_{k=0}^{n} b_k x^k, b_k = 0$. For any $A \in \mathbb{P}_n(\mathbb{R})$

$$A + B = \sum_{k=0}^{n} (a_k + 0) x^k = A$$

Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (A4).

(A5).

$$A + (-A) = \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n} (-a_k) x^k = \sum_{k=0}^{n} (a_k - a_k) x^k = \mathbf{0} \in \mathbb{P}_n(\mathbb{R})$$

Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (A5).

(M1). For $\alpha \in \mathbb{R}$

$$\alpha \cdot A = \sum_{k=0}^{n} (\alpha a_k) x^k \in \mathbb{P}_n(\mathbb{R})$$

Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (M1).

(M2). For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$

$$(\alpha\beta) \cdot A = \sum_{k=0}^{n} (\alpha\beta a_k) x^k$$

$$\alpha \cdot (\beta \cdot A) = \sum_{k=0}^{n} (\alpha\beta a_k) x^k$$

Because $(\alpha\beta) \cdot A = \alpha \cdot (\beta \cdot A)$. Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (M2).

(M3). For $\alpha \in \mathbb{R}$

$$\alpha \cdot (A + B) = \alpha \sum_{k=0}^{n} (a_k + b_k) x^k = \sum_{k=0}^{n} (\alpha a_k + \alpha b_k) x^k$$

$$\alpha \cdot A + \alpha \cdot B = \sum_{k=0}^{n} (\alpha a_k + \alpha b_k) x^k$$

Because $\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B$. Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (M3).

(M4). For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$

$$(\alpha + \beta) \cdot A = \sum_{k=0}^{n} [(\alpha + \beta) a_k] x^k$$

$$\alpha \cdot A + \beta \cdot A = \sum_{k=0}^{n} [(\alpha + \beta) a_k] x^k$$

Because $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A$. Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (M4).

(M5). For $1 \in \mathbb{R}$

$$1 \cdot A = \sum_{k=0}^{n} (a_k) x^k = A$$

Hence $\mathbb{P}_n(\mathbb{R})$ satisfy property (M5).

In conclusion, $\mathbb{P}_n(\mathbb{R})$ satisfy properties (A1)-(A5) and (M1)-(M5). The dimension of $\mathbb{P}_n(\mathbb{R})$ is $n+1$.

b) $\cup_{n=1}^{m}\mathbb{P}_n$ is also a vector space. Actually, for all $\mathbb{P}_k, k = 1\dots m$, $\mathbb{P}_k$ is a subspace of $\mathbb{P}_m$. So the union $\cup_{n=1}^{m}\mathbb{P}_n$ again is a subspace of $\mathbb{P}_m$. Hence $\cup_{n=1}^{m}\mathbb{P}_n$ is also a vector space.

c) $\{1, x, x^2, x^3, x^4\}$ is a basis of $\mathbb{P}_4$. So the union set of $\{1, x, x^2, x^3, x^4\}$ and $\{1+x^2, 1-x^2\}$ spans $\mathbb{P}_4$. Since $dim(\mathbb{P}_4) = 5$, so we can reduce the number elements in union set of $\{1, x, x^2, x^3, x^4\}$ and $\{1+x^2, 1-x^2\}$ to 5 linearly independent vecotrs. Obviously, both $\{1, 1+x^2, 1-x^2\}$ and $\{x^2, 1+x^2, 1-x^2\}$ are linearly dependent. For set A of$\{x, x^2+1, x^2-1, x^3, x^4\}$, the only solution of $k_1x + k_2(x^2+1) + k_3(x^2-1) + k_4x^3 + k_5x^4 = 0$ is $k_1 = k_2 = k_3 = k_4 = k_5 = 0$, so the set A is linearly independent set.
So the basis is $\{x, x^2+1, x^2-1, x^3, x^4\}$.

d) Since $dim(\mathbb{P}_2) = 3$, so the basis contains three vecotrs. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow E_A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the basis is $\{1+x, x+x^2, x+2x^2\}$.

Problem 4: Symmetric and Hermitian matrices

Solution

a) The set of all $n \times n$ real-valued symmetric matrices over $\mathbb{R}$ is subset of $\mathbb{R}^{n\times n}$.

$$(A+B)^T = A^T + B^T = A + B$$

$$(\alpha \cdot A)^T = \alpha \cdot A^T = \alpha \cdot A, \alpha \in \mathbb{R}$$

Obviously the set of all $n \times n$ real-valued symmetric matrices over $\mathbb{R}$ satisfies closure of addition and scalar multification. Hence set of all $n \times n$ real-valued symmetric matrices over $\mathbb{R}$ is a vecotr space.

b) The set of all $n \times n$ complex-valued symmetric matrices over $\mathbb{C}$ is a vecotr space. For any A, B in this set

$$(A+B)^T = A^T + B^T = A + B$$

$$(\alpha \cdot A)^T = \alpha \cdot A^T = \alpha \cdot A, \alpha \in \mathbb{C}$$

Hence set of all $n \times n$ complex-valued symmetric matrices over $\mathbb{C}$ satisfies closure of addition and scalar multification.So it is a vecotr space.

c) The set of all $n \times n$ complex-valued hermitian matrices over $\mathbb{R}$ is a vecotr space. For any A, B in this set

$$(A+B)^H = (\overline{A+B})^T = (\overline{A}+\overline{B})^T = \overline{A}^T + \overline{B}^T = A+B$$

$$(\alpha \cdot A)^H = \alpha \cdot A^H = \alpha \cdot A, \alpha \in \mathbb{R}$$

Hence set of all $n \times n$ complex-valued hermitian matrices over $\mathbb{R}$ satisfies closure of addition and scalar multification.So it is a vecotr space.

d) The set $\mathcal{V}$ of all $n \times n$ complex-valued hermitian matrices over $\mathbb{C}$ is not a vecotr space.

$$v = (1+i) \cdot \begin{bmatrix} 1 & 1-i \\ 1-i & 1 \end{bmatrix} = \begin{bmatrix} 1+i & 2 \\ 2i & 1+i \end{bmatrix} \notin \mathcal{V}$$

So set $\mathcal{V}$ doesn't satisfies closure of scalar multification. So set of all $n \times n$ complex-valued hermitian matrices over $\mathbb{C}$ is not a vecotr space.

Problem 5: Properties of Vector Spaces

Solution

a) *Proof.* Suppose v_{I1} and v_{I2} are two different additve inverse of an element v.

$$v + v_{I1} = 0, v + v_{I2} = 0$$

hence

$$v + v_{I1} = v + v_{I2}$$

Add v_{I1} and v_{I2} to the both side, so we get

$$(v_{I1} + v) + v_{I1} = (v_{I1} + v) + v_{I2}$$

hence

$$\mathbf{0} + v_{I1} = \mathbf{0} + v_{I2}$$

$$v_{I1} = v_{I2}$$

Hence addtive inverse of an element in unique. □

b) *Proof.* If $S_{ext} = \{w_1, w_2, \ldots, w_N, v\}$ will not add new vectors to $span(S)$, then $v \in span(S)$, for otherwise $S_{ext} = \{w_1, w_2, \ldots, w_N, v\}$ will add at least new vector v to $span(S)$. Conversely, suppose $v \in span(S)$. Then

$$span(S_{ext}) = \{\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_N w_N + \alpha_{N+1} v | \alpha_1, \ldots, \alpha_{N+1} \in \mathcal{F}\}$$

Since $v \in span(S)$, so $v = \beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_N w_N$, hence

$$span(S_{ext}) = \{(\alpha_1 + \beta_1) w_1 + \ldots + (\alpha_N + \beta_N) w_N | (\alpha_1 + \beta_1), \ldots, (\alpha_N + \beta_N) \in \mathcal{F}\}$$

$$span(S_{ext}) = span(S)$$

So, adding vector v to $span(S)$ will not add new vectors to $span(S)$. □

Problem 6: Linear Independence

Solution

a) i. Vectors $z_1, z_2, \ldots, z_n$ are linearly independent.

Proof. For equation:

$$k_1 z_1 + k_2 z_2 + \ldots + k_n z_n = 0$$
$$k_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + k_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \ldots + k_n \begin{bmatrix} x_n \\ y_n \end{bmatrix} = 0$$
$$\begin{bmatrix} k_1 x_1 \\ k_1 y_1 \end{bmatrix} + \begin{bmatrix} k_2 x_2 \\ k_2 y_2 \end{bmatrix} + \ldots + \begin{bmatrix} k_n x_n \\ k_n y_n \end{bmatrix} = 0$$
$$\begin{bmatrix} k_1 x_1 + k_2 x_2 + \ldots + k_n x_n \\ k_1 y_1 + k_2 y_2 + \ldots + k_n y_n \end{bmatrix} = 0 \tag{1}$$

Becauce $x_1, x_2, \ldots, x_n$ are linearly independent, the only solution for equation(1) is $k_1 + k_2 + \ldots + k_n = 0$. So Vectors $z_1, z_2, \ldots, z_n$ are linearly independent. □

ii. We can't not conclude that $z_1, z_2, \ldots, z_n$ are linearly dependent.

Proof. If $y_1, y_2, \ldots, y_n$ are linearly independent, then $z_1, z_2, \ldots, z_n$ are linearly independent. For example, suppose $x_1 = x_2 = 0$ and

$$z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; z_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Obviously, z_1, z_2 are linearly independent. So we can't not conclude that $z_1, z_2, \ldots, z_n$ are linearly dependent. □

Problem 7: Finding Basis

Solution

a) For every $v \in U \cap V$, there exist $a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$

$$v = a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$v = b_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So we get

$$\begin{cases} 2a_1 + a_2 = 0 \\ -a_1 = b_1 \\ 3a_1 - a_2 = b_2 \\ 0 = b_3 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = -2a_1 \\ b_1 = -a_1 \\ b_2 = 5a_1 \\ b_3 = 0 \end{cases}$$

So v could be represented as

$$v = \begin{bmatrix} 0 \\ -a_1 \\ 5a_1 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

So basis of subspace S is $\{[0, -1, 5, 0]^T\}$.

b) For a set of all vectors whose components are equal, the basis is $\{[1, 1, \dots, 1]^T\}$. Because for every vecotr v in set of all vectors whose components are equal,

$$v = [a, a, \dots, a]^T = a[1, 1, \dots, 1]^T$$

So the basis is $\{[1, 1, \dots, 1]^T\}$.

c) For every vecotr v in set of all vectors whose components sum to zero,

$$v = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ -\sum_{k=1}^{n-1} a_k \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ \dots \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ \dots \\ -1 \end{bmatrix} + \dots + a_{n-1} \begin{bmatrix} 0 \\ \dots \\ 1 \\ -1 \end{bmatrix}$$

The set of $\{[1, 0, \dots, 0, -1]^T, [0, 1, \dots, 0, -1]^T, \dots, [0, 0, \dots, 1, -1]^T\}$ is linearly independent. So the basis is $\{[1, 0, \dots, 0, -1]^T, [0, 1, \dots, 0, -1]^T, \dots, [0, 0, \dots, 1, -1]^T\}$.

d)

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow E_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So the basis is $\{[1, 1, 0, 0]^T, [0, 1, 1, 0]^T, [0, 0, 1, 1]^T\}$