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Assignment #3

Problem 1: Orthogonal Complement of a Subspace

Solution

(a) Suppose $x_1, x_2 \in \mathcal{V}^{\perp}$, for any $y \in \mathcal{V}$

$$(x_1 + x_2)^T y = (x_1^T + x_2^T) y = x_1^T y + x_2^T y = 0$$

For any $\alpha \in \mathbb{R}$,

$$(\alpha x_1)^T y = \alpha x_1^T y = 0$$

Hence $\mathcal{V}^{\perp}$ is a subspace of $\mathbb{R}^n$.

(b) Because $\mathcal{V} = span(v_1, v_2, \ldots, v_k)$, for any $y \in \mathcal{V}$,

$$y = \sum_{i=1}^{k} \alpha_i v_i = \begin{bmatrix} v_1 & v_2 & \ldots & v_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \ where\ \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$

Hence $\mathcal{V} = R(A)$.
For any $y \in \mathcal{V}$,

$$x^T y = x^T A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = 0, \quad where\ \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$

$$\Rightarrow \left(A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \right)^T x = 0$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_k \end{bmatrix} A^T x = 0$$

$$\Rightarrow A^T x = 0$$

Hence $\mathcal{V}^{\perp} = N(A^T)$.

(c) $(\mathcal{V}^{\perp})^{\perp}$ can be represented as the following:

$$(\mathcal{V}^{\perp})^{\perp} = \{y \in \mathbb{R}^n : y^T x = 0, \forall x \in \mathcal{V}^{\perp}\}$$

Meanwhile, for any $x \in \mathcal{V}^{\perp}$,

$$x^T y = 0,\ \forall y \in \mathcal{V} \Rightarrow y^T x = 0,\ \forall y \in \mathcal{V}$$

Hence $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}^{\perp}$

(d)
$$dim(\mathcal{V}) = dim[R(A)] = rank(A) = rank(A^T) = dim[R(A^T)]$$

Meanwhile
$$dim(\mathcal{V}^{\perp}) = dim[N(A^T)]$$

Hence
$$dim(\mathcal{V}) + dim(\mathcal{V}^{\perp}) = dim[R(A^T)] + dim[N(A^T)] = n$$

(e) Accoding to the definition,
$$\mathcal{W}^{\perp} = \left\{x_2 \in \mathbb{R}^n :\ x_2^T y_2 = 0,\ \forall y_2 \in \mathcal{W}\right\}$$

For any $y_1 \in \mathcal{V}$. Because $\mathcal{V} \subseteq \mathcal{W}$, $y_1 \in \mathcal{W}$, hence
$$x_2^T y_1 = 0 \ \Rightarrow\ x_2 \in \mathcal{V}^{\perp}$$

Hence $\mathcal{V} \subseteq \mathcal{W}$ for another subspace $\mathcal{W}$ implies $\mathcal{W}^{\perp} \subseteq \mathcal{V}^{\perp}$

(f) Suppose v is the projection of x onto subspace $\mathcal{V}$. Then
$$< x - v, y >= 0,\ \forall y \in \mathcal{V}$$
$$\Rightarrow y^T(x - v) = 0,\ \forall y \in \mathcal{V}$$
$$\Rightarrow (x - v)^T y = 0,\ \forall y \in \mathcal{V}$$

So $(x - v) \in \mathcal{V}^{\perp}$. Suppose there is a vector $v^{\perp}$, S.T.
$$x - v = v^{\perp}$$
$$\Rightarrow x = v + v^{\perp}$$

Meanwhile, because v is the projection of x onto subspace $\mathcal{V}$, v must be unique and thus $v^{\perp}$ is also unique. Hence any $x \in \mathbb{R}^n$ could be expressed uniquely as $x = v + v^{\perp}$.

Problem 2: Rank of a Product

Solution

(a) Suppose

(b) Suppose

Problem 3: An Inequality for Orthonormal Matrices

Solution

Problem 4: Householder Reflections

Solution

(a)

$$\begin{aligned} QQ^T &= (I - 2uu^T)(I - 2uu^T)^T \\ &= (I - 2uu^T)^T - 2uu^T(I - 2uu^T)^T \\ &= I - 2uu^T - 2uu^T + 4uu^Tuu^T \\ &= I - 4uu^T + 4u(u^Tu)u^T \end{aligned}$$

Because u is unit vector, $u^Tu = ||u||^2 = 1$. Hence $u(u^Tu)u^T = uu^T$, so $QQ^T = I$. Therefore, Q is orthogonal.

(b)

$$\begin{aligned} Qu &= (I - 2uu^T)u \\ &= Iu - 2uu^Tu \\ &= u - 2u(u^Tu) \\ &= u - 2u \\ &= -u \end{aligned}$$

$$\begin{aligned} Qv &= (I - 2uu^T)v \\ &= v - 2uu^Tv \\ &= v - 2u < v, u > \\ &= v \end{aligned}$$

(c) Known $Q \in \mathbb{R}^{n \times n}$, suppose $u_1, u_2, \ldots, u_n$ is the column of Q. Because Q is orthogonal, then $u_1, u_2, \ldots, u_n$ are orthogonal, hence $u_1, u_2, \ldots, u_n$ are linearly independent. Thus matrix Q is full-rank and inversible. Thus given y,

$$x = Q^{-1}y$$

(d) Obviously matirx uu^T is square, hence

$$det(Q) = det(I - 2uu^T) = det(I - 2u^TuI) = det(I - 2I) = det(-I) = -1$$

(e)

Problem 5: Projection Matrices

Solution

(a) Obviously $I - P$ is also a symmetric matrix.

$$\begin{aligned}(I-P)(I-P) &= (I-P) - P(I-P) \\ &= I - P - P + P^2 \\ &= I - 2P + P \\ &= I - P\end{aligned}$$

So $I - P$ is also a projection matrix.

(b) Obviously UU^T is symmetric matirx.

$$\begin{aligned}(UU^T)^2 &= UU^TUU^T \\ &= U(U^TU)U^T\end{aligned}$$

Because the columns of U is orthonormal, so $U^TU = I_{k\times k}$. Hence

$$(UU^T)^2 = U(U^TU)U^T = UIU^T = UU^T$$

So UU^T is a projection matrix.

(c) First we should show $P = A(A^TA)^{-1}A^T$ is a symmetric matrix.

$$\begin{aligned}[A(A^TA)^{-1}A^T]^T &= A[(A^TA)^{-1}]^TA^T \\ &= A[(A^TA)^T]^{-1}A^T \\ &= A(A^TA)^{-1}A^T\end{aligned}$$

So $A(A^TA)^{-1}A^T$ is a symmetric matrix. Second we should show that $P = P^2$.

$$\begin{aligned}P^2 &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}IA^T \\ &= A(A^TA)^{-1}A^T\end{aligned}$$

Hence $A(A^TA)^{-1}A^T$ is a projection matrix.

(d) Suppose $P = [p_1, p_2, \ldots, p_n]$, where p_i is the column of matrix P. Hence $R(P)$ could be represented as $span\{p_1, p_2, \ldots, p_n\}$. For any $p_i \in \{p_1, p_2, \ldots, p_n\}$,

$$< x - y, p_i >= p_i^T(x - Px) = (p_i^T - p_i^TP)x \tag{1}$$

If P is a projection matrix, then $P = P^2$ and $P = P^T$must be valid. Therefore,

$$P^2 = \begin{bmatrix} p_1^T p_1 & p_1^T p_2 & \dots & p_1^T p_n \\ p_2^T p_1 & p_2^T p_2 & \dots & p_2^T p_n \\ \vdots & \vdots & & \vdots \\ p_n^T p_1 & p_n^T p_2 & \dots & p_n^T p_n \end{bmatrix} = \begin{bmatrix} < p_1, p_1 > & < p_2, p_1 > & \dots & < p_n, p_1 > \\ < p_1, p_2 > & < p_2, p_2 > & \dots & < p_n, p_2 > \\ \vdots & \vdots & & \vdots \\ < p_1, p_n > & < p_2, p_n > & \dots & < p_n, p_n > \end{bmatrix}$$

Hence

$$p_i = [< p_1, p_1 >, < p_1, p_2 >, \dots, < p_1, p_n >]^T$$

Because $p_i \in \mathbb{R}^n$,

$$p_i = [< p_1, p_1 >, < p_2, p_1 >, \dots, < p_n, p_1 >]^T$$

Using the result above into equation (1), then we get

$$p_i^T - p_i^T P = p_i^T - [< p_1, p_1 >, < p_2, p_1 >, \dots, < p_n, p_1 >] = 0 \Rightarrow < x - y, p_i >= 0$$

Hence y is the point in $R(P)$ closest to x. y is the projection of x. In conclusion, if P is a projection matrix, then $y = Px$ is the projection of x onto $R(P)$.

(e) Obviously, the basis of *spanu* is $\{u\}$. Hence there is only one solution of the Normal Equation, that is:

$$\alpha = (u^T u)^{-1} u^T x$$

Therefore

$$y = u(u^T u)^{-1} u^T x \ \Rightarrow \ P = u(u^T u)^{-1} u^T = uu^T$$