Jiaming Lai
A53314574

Assignment #4

Problem 1: Moore–Penrose Pseudoinverse

Solution

(a) Suppose A^+ is not unique and there is A_1^+ and A_2^+. According to the difinition, for any matrix $A \in \mathbb{R}^{m\times n}$,

$$AA_1^+A = A$$

$$AA_2^+A = A$$

$$\Rightarrow (AA_1^+ - AA_2^+)A = A(A_1^+A - A_2^+A) = 0$$

Because the above equation is valid for any matrix $A \in \mathbb{R}^{m\times n}$, hence there must be $AA_1^+ = AA_2^+$ and $A_1^+A = A_2^+A$. So

$$A_1^+ = A_1^+AA_1^+ = A_1^+AA_2^+ = A_2^+AA_2^+ = A_2^+$$

Hence A^+ must be unique.

(b) Denote $(A^TA)^{-1}A^T$ as A^+. Because matrix A is tall, hence $n \le m$.

$$A^+A = (A^TA)^{-1}A^TA = (A^TA)^{-1}(A^TA) = I$$

So $(A^TA)^{-1}A^T$ a left inverse of matrix A.

$$AA^+A = A(A^+A) = AI = A$$

$$A^+AA^+ = (A^+A)A^+ = IA^+ = A^+$$

$A^+A = I$, so A^+A is symmetric. Meanwhile,

$$(AA^+)^T = (A^+)^TA^T = [(A^TA)^{-1}A^T]^TA^T = A(A^TA)^{-1}A^T = AA^+$$

So AA^+ is symmetric. In conclusion, $(A^TA)^{-1}A^T$ is the pseudoinverse and a left inverse of matrix A.

(c) Denote $A^T(AA^T)^{-1}$ as A^+.

$$AA^+ = AA^T(AA^T)^{-1} = (AA^T)(AA^T)^{-1} = I$$

So $A^T(AA^T)^{-1}$ is a right inverse of matrix A.

$$AA^+A = (AA^+)A = IA = A$$

$$A^+AA^+ = A^+(AA^+) = A^+I = A^+$$

$AA^+ = I$, so AA^+ is symmetric. Meanwhile,

$$(A^+A)^T = A^T(A^+)^T = A^T[A^T(AA^T)^{-1}]^T = A^T(AA^T)^{-1}A = A^+A$$

so A^+A is symmetric. In conclusion, $A^T(AA^T)^{-1}$ is the pseudoinverse and a right inverse of matrix A.

(d)
$$AA^{-1}A = IA = A \ and \ A^{-1}AA^{-1} = IA^{-1} = A^{-1}$$

Also $AA^{-1} = A^{-1}A = I$ is symmetric. So in conclusion, A^{-1} is the pseudoinverse of a full-rank square matrix A.

(e) For a projection matrix A, $A^2 = A$ and $A^T = A$. Hence

$$AAA = AA = A$$

Because $A^T = A$, so AA is symmetric. In conclusion, A is the pseudoinverse of itself for a projection matrix A.

(f)
$$A^T(A^+)^TA^T = [AA^+A]^T = A^T$$
$$(A^+)^TA^T(A^+)^T = [A^+AA^+]^T = (A^+)^T$$

Meanwhile,
$$[A^T(A^+)^T]^T = A^+A \ \Rightarrow \ symmetric$$
$$[(A^+)^TA^T]^T = AA^+ \ \Rightarrow \ symmetric$$

So in conclusion, $(A^T)^+ = (A^+)^T$.

(g) i.

(h)

(i) First, both AA^+ and A^+A are symmetric. Second,

$$P^2 = AA^+AA^+ = AA^+$$

$$Q^2 = A^+AA^+A = A^+A$$

Hence P and Q are projection matrix.

(j) Recall the result of problem 5 in Homework 3, $y = Px$ is the projection of x onto $R(P)$. For $\forall y \in R(P)$, there must exists $x \in \mathbb{R}^m$ S.T.

$$y = AA^+x \ \Rightarrow \ y = A(A^+x)$$

Hence for $\forall y \in R(P)$, there must exists $z = A^+x \in \mathbb{R}^n$, S.T. $y = Az$. Hence $R(P) = R(A)$. So $y = Px$ is the projection of x onto $R(A)$.

Similarly, $y = Qx$ is the projection of x onto $R(Q)$. For $\forall y \in R(Q)$, there must exists $x \in \mathbb{R}^n$ S.T.
$$y = A^+Ax \ \Rightarrow \ y = A^+(Ax)$$

Hence for $\forall y \in R(Q)$, there must exists $z = Ax \in \mathbb{R}^m$, S.T. $y = A^+z$. Hence $R(Q) = R(A^+) = R(A^T)$. So $y = Qx$ is the projection of x onto $R(A^T)$.

(k) The solution x^* must satiffies that Ax^* is the orthogonal projection of b onto $R(A)$. Recall the result in problem (j), the projection matrix onto $R(A)$ is $P = AA^+$, hence

$$Ax^* = AA^+b \;\Rightarrow\; x^* = A^+b$$

(l)

Problem 2: Eigenvalues

Solution

(a) The characteristic polynomial of A is

$$p(\lambda) = det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$

Hence

$$p(\lambda = 0) = det(-A) = (-1)^n det(A) = (-1)^n \lambda_1\lambda_2\dots\lambda_n$$

So $det(A) = \lambda_1\lambda_2\dots\lambda_n$.

(b) Because

$$\lambda I - A^T = (\lambda I - A)^T$$

and

$$det((\lambda I - A)^T) = det(\lambda I - A)$$

So $det(\lambda I - A) = det(\lambda I - A^T)$. A^T and A have the same characteristic polynomial. Hence the eigenvalues of A^T and A are the same.

(c) Give the fact that $Av = \lambda_i v$, where $i = 1, 2, \dots, n$,

$$A^k v = A^{k-1}\lambda_i v = A^{k-2}\lambda_i^2 v = \dots = \lambda_i^k v$$

Hence $\lambda_i^k,\ i = 1, 2, \dots, n$ are eigenvalues of matrix A^k.

(d) If matrix A is invertible, then suppose matrix A has a zero eigenvalue λ. There must be a vector $v \neq 0$, S.T.

$$Av = \lambda v = 0$$

Obviously $v \in N(A)$. Because A is invertible, so $dim(N(A) = 0) \Rightarrow N(A) = 0$. Hence $v = 0$. But this contradicts the fact that $v \neq 0$. So if A is invertible, it does not have a zero eigenvalue.

If matrix A does not have a zero eigenvalue, then there is no a vector $v \neq 0$ S.T.

$$Av = \lambda v,\ \lambda = 0 \;\Rightarrow\; Av = 0$$

The only solution to $Av = 0$ is $v = 0$, which means $dim[N(A)] = 0$. Hence matrix A is full-rank and invertible.

In conclusion, A is invertible if and only if it does not have a zero eigenvalue.

(e) Accoding to the definition,

$$Av = \lambda_i v \ \Rightarrow \ A^{-1}Av = \lambda_i A^{-1}v \ \Rightarrow \ v = \lambda_i A^{-1}v \ \Rightarrow \ \lambda_i^{-1}v = A^{-1}v$$

Hence $\lambda_i^{-1}, \ i = 1, 2, \ldots, n$ are eigenvalues of A^{-1}.

(f) The characteristic polynomial of $T^{-1}AT$ is

$$\begin{aligned} det(T^{-1}AT - \lambda I) &= det(T^{-1}AT - \lambda T^{-1}IT) \\ &= det[T^{-1}(A - \lambda I)T] \\ &= det(T^{-1})det(A - \lambda I)det(T) \end{aligned}$$

Because $det(T^{-1})det(T) = det(T^{-1}T) = 1$. Hence

$$det(T^{-1}AT - \lambda I) = det(A - \lambda I)$$

So A and $T^{-1}AT$ have the same eigenvalues.

Problem 3: Trace

Solution

(a) The

(b) Using the result from problem 2(c), $\lambda_i^k, \ i = 1, 2, \ldots, n$ are eigenvalues of matrix A^k. Hence

$$tr(A^k) = \sum_{i=1}^{n} \lambda_i^k$$

Problem 4: More on Eigenvalues

Solution

(a)

Problem 5: Limit

Solution

(a) The