

Problem 1: Orthogonal Complement of a Subspace

Solution

(a) Suppose $x_1, x_2 \in \mathcal{V}^\perp$, for any $y \in \mathcal{V}$

$$(x_1 + x_2)^T y = (x_1^T + x_2^T)y = x_1^T y + x_2^T y = 0$$

For any $\alpha \in \mathbb{R}$,

$$(\alpha x_1)^T y = \alpha x_1^T y = 0$$

Hence \mathcal{V}^\perp is a subspace of \mathbb{R}^n .

(b) Because $\mathcal{V} = \text{span}(v_1, v_2, \dots, v_k)$, for any $y \in \mathcal{V}$,

$$y = \sum_{i=1}^k \alpha_i v_i = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \text{ where } \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

Hence $\mathcal{V} = R(A)$.

For any $y \in \mathcal{V}$,

$$\begin{aligned} x^T y &= x^T A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = 0, \text{ where } \alpha_1, \dots, \alpha_k \in \mathbb{R} \\ &\Rightarrow \left(A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \right)^T x = 0 \\ &\Rightarrow [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_k] A^T x = 0 \\ &\Rightarrow A^T x = 0 \end{aligned}$$

Hence $\mathcal{V}^\perp = N(A^T)$.

(c) $(\mathcal{V}^\perp)^\perp$ can be represented as the following:

$$(\mathcal{V}^\perp)^\perp = \{y \in \mathbb{R}^n : y^T x = 0, \forall x \in \mathcal{V}^\perp\}$$

Meanwhile, for any $x \in \mathcal{V}^\perp$,

$$x^T y = 0, \forall y \in \mathcal{V} \Rightarrow y^T x = 0, \forall y \in \mathcal{V}$$

Hence $(\mathcal{V}^\perp)^\perp = \mathcal{V}$

(d)

$$\dim(\mathcal{V}) = \dim[R(A)] = \text{rank}(A) = \text{rank}(A^T) = \dim[R(A^T)]$$

Meanwhile

$$\dim(\mathcal{V}^\perp) = \dim[N(A^T)]$$

Hence

$$\dim(\mathcal{V}) + \dim(\mathcal{V}^\perp) = \dim[R(A^T)] + \dim[N(A^T)] = n$$

(e) According to the definition,

$$\mathcal{W}^\perp = \{x_2 \in \mathbb{R}^n : x_2^T y_2 = 0, \forall y_2 \in \mathcal{W}\}$$

For any $y_1 \in \mathcal{V}$. Because $\mathcal{V} \subseteq \mathcal{W}$, $y_1 \in \mathcal{W}$, hence

$$x_2^T y_1 = 0 \Rightarrow x_2 \in \mathcal{V}^\perp$$

Hence $\mathcal{V} \subseteq \mathcal{W}$ for another subspace \mathcal{W} implies $\mathcal{W}^\perp \subseteq \mathcal{V}^\perp$

(f) Suppose v is the projection of x onto subspace \mathcal{V} . Then

$$\langle x - v, y \rangle = 0, \forall y \in \mathcal{V}$$

$$\Rightarrow y^T(x - v) = 0, \forall y \in \mathcal{V}$$

$$\Rightarrow (x - v)^T y = 0, \forall y \in \mathcal{V}$$

So $(x - v) \in \mathcal{V}^\perp$. Suppose there is a vector v^\perp , S.T.

$$x - v = v^\perp$$

$$\Rightarrow x = v + v^\perp$$

Meanwhile, because v is the projection of x onto subspace \mathcal{V} , v must be unique and thus v^\perp is also unique. Hence any $x \in \mathbb{R}^n$ could be expressed uniquely as $x = v + v^\perp$.

Problem 2: Rank of a Product

Solution

(a) Suppose

(b) Suppose

Problem 3: An Inequality for Orthonormal Matrices

Solution

Problem 4: Householder Reflections

Solution

(a)

$$\begin{aligned} QQ^T &= (I - 2uu^T)(I - 2uu^T)^T \\ &= (I - 2uu^T)^T - 2uu^T(I - 2uu^T)^T \\ &= I - 2uu^T - 2uu^T + 4uu^Tuu^T \\ &= I - 4uu^T + 4u(u^Tu)u^T \end{aligned}$$

Because u is unit vector, $u^Tu = \|u\|^2 = 1$. Hence $u(u^Tu)u^T = uu^T$, so $QQ^T = I$. Therefore, Q is orthogonal.

(b)

$$\begin{aligned} Qu &= (I - 2uu^T)u \\ &= Iu - 2uu^Tu \\ &= u - 2u(u^Tu) \\ &= u - 2u \\ &= -u \end{aligned}$$

$$\begin{aligned} Qv &= (I - 2uu^T)v \\ &= v - 2uu^Tv \\ &= v - 2u \langle v, u \rangle \\ &= v \end{aligned}$$

(c) Known $Q \in \mathbb{R}^{n \times n}$, suppose u_1, u_2, \dots, u_n is the column of Q . Because Q is orthogonal, then u_1, u_2, \dots, u_n are orthogonal, hence u_1, u_2, \dots, u_n are linearly independent. Thus matrix Q is full-rank and invertible. Thus given y ,

$$x = Q^{-1}y$$

(d) Obviously matrix uu^T is square, hence

$$\det(Q) = \det(I - 2uu^T) = \det(I - 2u^TuI) = \det(I - 2I) = \det(-I) = -1$$

(e)

Problem 5: Projection Matrices

Solution

(a) Obviously $I - P$ is also a symmetric matrix.

$$\begin{aligned}(I - P)(I - P) &= (I - P) - P(I - P) \\ &= I - P - P + P^2 \\ &= I - 2P + P \\ &= I - P\end{aligned}$$

So $I - P$ is also a projection matrix.

(b) Obviously UU^T is symmetric matrix.

$$\begin{aligned}(UU^T)^2 &= UU^TUU^T \\ &= U(U^TU)U^T\end{aligned}$$

Because the columns of U is orthonormal, so $U^TU = I_{k \times k}$. Hence

$$(UU^T)^2 = U(U^TU)U^T = UIU^T = UU^T$$

So UU^T is a projection matrix.

(c) First we should show $P = A(A^TA)^{-1}A^T$ is a symmetric matrix.

$$\begin{aligned}[A(A^TA)^{-1}A^T]^T &= A[(A^TA)^{-1}]^TA^T \\ &= A[(A^TA)^T]^{-1}A^T \\ &= A(A^TA)^{-1}A^T\end{aligned}$$

So $A(A^TA)^{-1}A^T$ is a symmetric matrix. Second we should show that $P = P^2$.

$$\begin{aligned}P^2 &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}IA^T \\ &= A(A^TA)^{-1}A^T\end{aligned}$$

Hence $A(A^TA)^{-1}A^T$ is a projection matrix.

(d) Suppose $P = [p_1, p_2, \dots, p_n]$, where p_i is the column of matrix P . Hence $R(P)$ could be represented as $\text{span}\{p_1, p_2, \dots, p_n\}$. For any $p_i \in \{p_1, p_2, \dots, p_n\}$,

$$\langle x - y, p_i \rangle = p_i^T(x - Px) = (p_i^T - p_i^TP)x \quad (1)$$

If P is a projection matrix, then $P = P^2$ and $P = P^T$ must be valid. Therefore,

$$P^2 = \begin{bmatrix} p_1^T p_1 & p_1^T p_2 & \cdots & p_1^T p_n \\ p_2^T p_1 & p_2^T p_2 & \cdots & p_2^T p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_n^T p_1 & p_n^T p_2 & \cdots & p_n^T p_n \end{bmatrix} = \begin{bmatrix} \langle p_1, p_1 \rangle & \langle p_2, p_1 \rangle & \cdots & \langle p_n, p_1 \rangle \\ \langle p_1, p_2 \rangle & \langle p_2, p_2 \rangle & \cdots & \langle p_n, p_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle p_1, p_n \rangle & \langle p_2, p_n \rangle & \cdots & \langle p_n, p_n \rangle \end{bmatrix}$$

Hence

$$p_i = [\langle p_1, p_1 \rangle, \langle p_1, p_2 \rangle, \dots, \langle p_1, p_n \rangle]^T$$

Because $p_i \in \mathbb{R}^n$,

$$p_i = [\langle p_1, p_1 \rangle, \langle p_2, p_1 \rangle, \dots, \langle p_n, p_1 \rangle]^T$$

Using the result above into equation (1), then we get

$$p_i^T - p_i^T P = p_i^T - [\langle p_1, p_1 \rangle, \langle p_2, p_1 \rangle, \dots, \langle p_n, p_1 \rangle] = 0 \Rightarrow \langle x - y, p_i \rangle = 0$$

Hence y is the point in $R(P)$ closest to x . y is the projection of x . In conclusion, if P is a projection matrix, then $y = Px$ is the projection of x onto $R(P)$.

- (e) The thing is that we want to find a projection matrix P such that $R(P) = \text{span}\{u\}$