ECE 269: Linear Algebra and Applications
Fall 2019

Homework # 2
Due: Tuesday, October 22 11:59pm,
via Gradescope

Collaboration Policy: This homework set allows limited collaboration. You are expected to try to solve the problems on your own. You may discuss a problem with other students to clarify any doubts, but you must fully understand the solution that you turn in and write it up entirely on your own. Blindly copying another student's result will be considered a violation of academic integrity. *

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1. **Suggested Readings**. Following sections of Carl D. Meyer's book "Matrix Analysis and Applied Linear Algebra":
   - Sections 4.5, 4.7: Properties of Rank, Linear Maps.
   - Sections 3.5, 3.6, 3.7: Matrix multiplication, matrix inverse and properties.

   Also review Summary Slides 1 and video lectures.

2. **Problem 1: Convolution as Linear Map.** Suppose that real-valued sequences $\{u(n)\}_{n=-\infty}^{\infty}$ and $\{v(n)\}_{n=-\infty}^{\infty}$ represent the input and output signals of a discrete-time linear time-invariant system with impulse response $h(n) \in \mathbb{R}$, $n \in \mathbb{Z}$. Then, $\{u(n)\}$ and $\{v(n)\}$ are related via convolution as

$$v(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k), \quad n \in \mathbb{Z}.$$

   Suppose that $u(n) = 0$ for $n < 0$ or $n > N$, and define

$$\mathbf{x} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N) \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(N) \end{bmatrix}.$$

   Thus $\mathbf{x}$ and $\mathbf{y}$ are vectors that capture $N+1$ values of the input and output signals, respectively.

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(a) Find the matrix $\mathbf{T}$ such that

$$\mathbf{y} = \mathbf{T}\mathbf{x}$$

in terms of $h(n)$.

(b) Describe the structure of $\mathbf{T}$. Matrices of this structure are said to be *Toeplitz*.

3. **Problem 2: Affine functions.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *affine* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, we have

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}).$$

Note that without the restriction $\alpha + \beta = 1$, this would be the definition of linearity.

(a) Suppose that $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that the function $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is affine.

(b) Prove the converse, namely, show that any affine function f can be represented uniquely as $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

(Hint: Consider the linearity of the function $g(\mathbf{x}) = f(\mathbf{x}) - f(0)$.)

4. **Problem 3: Matrix multiplication.** Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Prove or provide a counterexample to each of the following statements.

(a) If $\mathbf{A}\mathbf{B} = 0$, then $\mathbf{A} = 0$ or $\mathbf{B} = 0$.

(b) If $\mathbf{A}^2 = 0$, then $\mathbf{A} = 0$.

(c) If $\mathbf{A}^T\mathbf{A} = 0$, then $\mathbf{A} = 0$.

5. **Problem 4: Linear Maps and Differentiation of polynomials.** Let $\mathcal{P}_n$ be the vector space consisting of all polynomials of degree $\leq n$ with real coefficient.

(a) Consider the transformation $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T(p(x)) = \frac{dp(x)}{dx}.$$

For example, $T(1 + 3x + x^2) = 3 + 2x$. Show that T is linear.

(b) Using $\{1, x, \ldots, x^n\}$ as a basis, represent the transformation in part (b) by a matrix $\mathbf{A} \in \mathbb{R}^{(n+1)\times(n+1)}$. Find the rank of $\mathbf{A}$.

6. **Problem 5: Rank of $\mathbf{A}\mathbf{A}^T$.** Let $\mathbf{A} \in \mathbb{F}^{m\times n}$.

(a) Suppose that $\mathbb{F} = \mathbb{R}$. Prove that $\operatorname{rank}(\mathbf{A}\mathbf{A}^T) = \operatorname{rank}(\mathbf{A})$ or provide a counterexample.

(b) Suppose that $\mathbb{F} = \mathbb{C}$. Repeat part (a).

(c) Suppose that $\mathbb{F} = \mathbb{C}$. Prove that $\text{rank}(\mathbf{A}\mathbf{A}^H) = \text{rank}(\mathbf{A})$ or provide a counterexample.

7. **Problem 6: Left and Right Inverses.**

   (a) Show that if $\mathbf{A} \in \mathbb{R}^{m \times n}$ is full-rank and tall, then $\mathbf{A}^T\mathbf{A}$ is nonsingular.

   (b) Show that $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is a left inverse of a full-rank tall matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$.

   (c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be full-rank and strictly tall ($m > n$). Does it have a unique left inverse? Prove or provide a counterexample.

   (d) Show that if $\mathbf{A} \in \mathbb{R}^{m \times n}$ is full-rank and fat, then $\mathbf{A}\mathbf{A}^T$ is nonsingular.

   (e) Show that $\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$ is a right inverse of a full-rank fat matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$.

   (f) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be full rank and strictly fat ($m < n$). Does it have a unique right inverse? Prove or provide a counterexample.