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Assignment #1

Problem 1: Vector Spaces other than $\mathbb{R}^N$

a) Suppose A is a set of rational numbers defined over $\mathbb{R}$. $A_1 \in A$. According to the definition of rational numbers, $A_1 = \frac{b}{a}$, where a and b are integers and $a \neq 0$. Suppose $B_1 \in \mathbb{R}$, and

$$B_1 = \pi \cdot A_1 = \frac{b\pi}{a}$$

Obviously, B_1 is not a rational number, that is $B_1 \notin A$. So the set of rational numbers defined over $\mathbb{R}$ doesn't satisfy closure of scalar multification. So the set is not valid vector field.

b) Suppose set $A = \{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{R}^+\}$. $A_1 \in A$ and $A_1 = a_0 + a_1x + a_2x^2$, $a_0, a_1, a_2 \in \mathbb{R}^+$. Let

$$B_1 = (-1) \cdot A_1 = (-a_0) + (-a_1)x + (-a_2)x^2$$

Obviously, $(-a_0), (-a_1), (-a_2) \notin \mathbb{R}^+$, so $B_1 \notin A$. Hence set A doesn't satisfy closure of scalar multification. So set A is not valid vector field.

c) i. Let

$$A = (\alpha_1 + \alpha_2) \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (\alpha_1 + \alpha_2)a \\ b \end{bmatrix}, \alpha_1, \alpha_2 \in \mathbb{R}$$

$$B = \alpha_1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (\alpha_1 + \alpha_2)a \\ 2b \end{bmatrix}, \alpha_1, \alpha_2 \in \mathbb{R}$$

Because $B \neq A$, so this set doesn't satisfy the following vector space property:

$$(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v, \alpha, \beta \in \mathcal{F} \; and \; v \in \mathcal{V}$$

So this set is not a valid vector field.

ii. Let $r = 1 \in \mathbb{R}$

$$r \cdot A = r \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \neq A$$

So this set doesn't satisfy the following vector space property:

$$1 \cdot v = v, where \; 1 \in \mathcal{F} \; and \; v \in \mathcal{V}$$

So this set is not a valid vector field.

iii. Suppose $B = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Let

$$A_1 = (\alpha_1 + \alpha_2) \cdot B = (\alpha_1 + \alpha_2) \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (\alpha_1 + \alpha_2)a \\ (\alpha_1 + \alpha_2)b \end{bmatrix}$$

$$A_2 = \alpha_1 \cdot B + \alpha_2 \cdot B = 0$$

Because $A_1 \neq A_2$, so this set doesn't satisfy the following vector space property:

$$(\alpha_1 + \alpha_2) \cdot v = \alpha_1 \cdot v + \alpha_2 \cdot v, where\ \alpha_1, \alpha_2 \in \mathcal{F}\ and\ v \in \mathcal{V}$$

So this set is not a valid vector field.

iv. Suppose

$$A = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}, B = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}, C = \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix}$$

Let:

$$A + (B + C) = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + (\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix}) = \begin{bmatrix} \alpha_1 - \alpha_2 + \alpha_3 \\ \beta_1 - \beta_2 + \beta_3 \end{bmatrix})$$

$$(A + B) + C = (\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}) + \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 - \alpha_2 - \alpha_3 \\ \beta_1 - \beta_2 - \beta_3 \end{bmatrix})$$

Because $A + (B + C) \neq (A + B) + C$, so this set doesn't satisfy the following vector space property:

$$A + (B + C) = (A + B) + C, where\ A, B, C \in \mathcal{V}$$

So this set is not a valid vector field.

Problem 2: Adjacency graph

Problem 3: Vector Spaces of Polynomials

a) A vector space should satisfy properties (A1)-(A5) and (M1)-(M5)

(A1).

Problem 4: Symmetric and Hermitian matrices

a) "There is a student in Gryffindor who has taken all elective classes."

Solution:

$$\exists x \forall y \forall z (H(x, \text{Gryffindor}) \wedge P(x, y)) \tag{1}$$

where

$H(x, z)$ is "x is of z house"

$P(x, y)$ is "x has taken y,"

the domain for x consists of all students in Hogwarts
the domain for y consists of all elective classes,
and the domain for z consists of all Hogwarts houses.

b) Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."
Solution:

1.
$$\forall n(P(n) \rightarrow Q(n)), \tag{2}$$
where
$P(n)$ is "n is an odd integer" and
$Q(n)$ is "n^2 is odd."
2. Assume $P(n)$ is true.
3. By definition, an odd integer is $n = 2k + 1$, where k is some integer.
4.
$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$
5. $\therefore n^2$ is an odd integer. $\square$

c) Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, \{1, 2, 3\}\}$:
Then, $A \in B$ and $A \subseteq B$.

d) Let $A = \{1, 3, 5\}$, $B = \{1, 2, 3, \}$, and universe $U = \{1, 2, 3, 4, 5\}$:
$$\begin{aligned} A \cup B &= \{1, 2, 3, 5\}, \\ A \cap B &= \{1, 3\}, \\ A - B &= \{5\}, \\ \bar{A} &= \{2, 4\}, \\ A - A &= \emptyset. \end{aligned}$$

Problem 5: Properties of Vector Spaces

Problem 6: Linear Independence

Problem 7: Finding Basis