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Assignment #3

ECE 269
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Problem 1: Orthogonal Complement of a Subspace

Solution

(a) Suppose $x_1, x_2 \in \mathcal{V}^\perp$, for any $y \in \mathcal{V}$

$$(x_1 + x_2)^T y = (x_1^T + x_2^T)y = x_1^T y + x_2^T y = 0$$

For any $\alpha \in \mathbb{R}$,

$$(\alpha x_1)^T y = \alpha x_1^T y = 0$$

Hence $\mathcal{V}^\perp$ is a subspace of $\mathbb{R}^n$.

(b) Because $\mathcal{V} = span(v_1, v_2, \dots, v_k)$, for any $y \in \mathcal{V}$,

$$y = \sum_{i=1}^{k} \alpha_i v_i = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \ where\ \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

Hence $\mathcal{V} = R(A)$.
For any $y \in \mathcal{V}$,

$$x^T y = x^T A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = 0, \quad where\ \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

$$\Rightarrow \left(A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \right)^T x = 0$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \end{bmatrix} A^T x = 0$$

$$\Rightarrow A^T x = 0$$

Hence $\mathcal{V}^\perp = N(A^T)$.

(c) $(\mathcal{V}^\perp)^\perp$ can be represented as the following:

$$(\mathcal{V}^\perp)^\perp = \{y \in \mathbb{R}^n : y^T x = 0, \forall x \in \mathcal{V}^\perp\}$$

Meanwhile, for any $x \in \mathcal{V}^\perp$,

$$x^T y = 0,\ \forall y \in \mathcal{V} \Rightarrow y^T x = 0,\ \forall y \in \mathcal{V}$$

Hence $(\mathcal{V}^\perp)^\perp = \mathcal{V}$

(d)

$$dim(\mathcal{V}) = dim[R(A)] = rank(A) = rank(A^T) = dim[R(A^T)]$$

Meanwhile

$$dim(\mathcal{V}^\perp) = dim[N(A^T)]$$

Hence

$$dim(\mathcal{V}) + dim(\mathcal{V}^\perp) = dim[R(A^T)] + dim[N(A^T)] = n$$

(e) Accoding to the definition,

$$\mathcal{W}^\perp = \{x_2 \in \mathbb{R}^n : \; x_2^T y_2 = 0, \; \forall y_2 \in \mathcal{W}\}$$

For any $y_1 \in \mathcal{V}$. Because $\mathcal{V} \subseteq \mathcal{W}$, $y_1 \in \mathcal{W}$, hence

$$x_2^T y_1 = 0 \;\Rightarrow\; x_2 \in \mathcal{V}^\perp$$

Hence $\mathcal{V} \subseteq \mathcal{W}$ for another subspace $\mathcal{W}$ implies $\mathcal{W}^\perp \subseteq \mathcal{V}^\perp$

(f) Suppose v is the projection of x onto subspace $\mathcal{V}$. Then

$$< x - v, y >= 0, \; \forall y \in \mathcal{V}$$

$$\Rightarrow y^T(x - v) = 0, \; \forall y \in \mathcal{V}$$
$$\Rightarrow (x - v)^T y = 0, \; \forall y \in \mathcal{V}$$

So $(x - v) \in \mathcal{V}^\perp$. Suppose there is a vector $v^\perp$, S.T.

$$x - v = v^\perp$$

$$\Rightarrow x = v + v^\perp$$

Meanwhile, because v is the projection of x onto subspace $\mathcal{V}$, v must be unique and thus $v^\perp$ is also unique. Hence any $x \in \mathbb{R}^n$ could be expressed uniquely as $x = v + v^\perp$.

Problem 2: Rank of a Product

Solution Suppose matrix A is

$$\begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \\ a_4^T \end{bmatrix}$$
$$\Rightarrow AB = \begin{bmatrix} B^T a_1 & B^T a_2 & B^T a_3 & B^T a_4 \end{bmatrix}$$

Because $Rank(A) = 2$, there should be $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $a_3 = \alpha_1 a_1 + \alpha_2 a_2$. Hence $B^T a_3 = \alpha_1 B^T a_1 + \alpha_2 B^T a_2$. The same could be show that $B^T a_4 = \beta_1 B^T a_1 + \beta_2 B^T a_2$. Hence there is two linear independent vectors at most in a_1, a_2, a_3, a_4. Hence

$$rank(AB) \leq 2$$

The folloing will show that $B^T a_1$ and $B^T a_2$ is linear independent. Suppose $\alpha_1, \alpha_2 \in \mathbb{R}$. a_1 and a_2 are linear independent.

$$\alpha_1 B^T a_1 + \alpha_2 B^T a_2 = 0 \Rightarrow B^T(\alpha_1 a_1 + \alpha_2 a_2) = 0$$

Because B is full-rank, so $dim(N(B^T)) = 3 - 3 = 0$. Hence $N(B^T) = 0$ and $\alpha_1 a_1 + \alpha_2 a_2 = 0$. Because a_1 and a_2 are linear independent, $\alpha_1 = \alpha_2 = 0$. So $B^T a_1$ and $B^T a_2$ are linear independent. Hence

$$rank(AB) = rank(B^T A^T) \geq 2$$

Hence $r_{min} = r_{max} = 2$. The following is the special case:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3: An Inequality for Orthonormal Matrices

Solution

$$||U^T x||_2 \leq ||x||_2 \Leftrightarrow ||U^T x||_2^2 \leq ||x||_2^2 \Leftrightarrow (U^T x)^T U^T x \leq x^T x$$

Hence we only have to show that

$$x^T x - x^T u u^T x \geq 0$$

$$\Rightarrow x^T (I - uu^T) x \geq 0$$

Suppose $v = (I - uu^T)$,

$$\begin{aligned} v^T v &= (I - uu^T)^T (I - uu^T) \\ &= (I - uu^T)(I - uu^T) \\ &= I - uu^T - uu^T + uu^T uu^T \\ &= I - uu^T - uu^T + uIu^T \\ &= I - uu^T \\ &= v \end{aligned}$$

Hence

$$x^T(I - uu^T)x = x^T vx = x^T v^T vx =< vx, vx >$$

Because $< vx, vx > \geq 0$, hence $x^T(I - uu^T)x \geq 0$. So $||U^T x||_2 \leq ||x||_2$ is valid.

Problem 4: Householder Reflections

Solution

(a)

$$\begin{aligned} QQ^T &= (I - 2uu^T)(I - 2uu^T)^T \\ &= (I - 2uu^T)^T - 2uu^T(I - 2uu^T)^T \\ &= I - 2uu^T - 2uu^T + 4uu^T uu^T \\ &= I - 4uu^T + 4u(u^T u)u^T \end{aligned}$$

Because u is unit vector, $u^T u = ||u||^2 = 1$. Hence $u(u^T u)u^T = uu^T$, so $QQ^T = I$. Therefore, Q is orthogonal.

(b)

$$\begin{aligned} Qu &= (I - 2uu^T)u \\ &= Iu - 2uu^T u \\ &= u - 2u(u^T u) \\ &= u - 2u \\ &= -u \end{aligned}$$

$$\begin{aligned} Qv &= (I - 2uu^T)v \\ &= v - 2uu^T v \\ &= v - 2u < v, u > \\ &= v \end{aligned}$$

Suppose $\hat{x}$ is the orthogonal projection of x onto $span\{u\}$, and $x_1 = x - \hat{x}$ and $x_2 = \hat{x} = \alpha u$. In this case, $(x - \hat{x}) \perp \hat{x} \Rightarrow x_1 \perp x_2$ and $x = x_1 + x_2$. Hence

$$Qx_2 = Q\hat{x} = \alpha Qu = -\alpha u = -x_2$$

$$Qx_1 = x_1$$

Hence

$$y = Qx = Q(x_1 + x_2) = x_1 - x_2$$

$\hat{x}$ is the orthogonal projection of x onto $span\{u\}$. u is the normal vector of hyperplane $u^T x = 0$ and so x_1 should be the projection of x onto hyperplane $u^T x = 0$. Because $x = x_1 + x_2$ and $y = Qx = x_1 - x_2$, orthogonal projection of x onto $span\{u\}$ changes to negative but the projection of x onto hyperplane $u^T x = 0$ stay unchanged. So $y = Qx$ is actually reflect x through the hyperplane with normal vector u.

(c) Known $Q \in \mathbb{R}^{n\times n}$, suppose $u_1, u_2, \ldots, u_n$ is the column of Q. Because Q is orthogonal, then $u_1, u_2, \ldots, u_n$ are orthogonal, hence $u_1, u_2, \ldots, u_n$ are linearly independent. Thus matrix Q is full-rank and inversible. Hence there is only one solution. Meanwhile

$$QQ = (I - 2uu^T)(I - 2uu^T) = Q$$

So

$$Qy = QQx = x$$

So the unique solution is $x = Qy$

(d) Obviously matirx uu^T is square, hence

$$det(Q) = det(I - 2uu^T) = det(I - 2u^T uI) = det(I - 2I) = det(-I) = -1$$

(e) Following problem (b). Suppose $\hat{x}$ is the orthogonal projection of x onto $span\{u\}$, and $x_1 = x - \hat{x}$ and $x_2 = \hat{x} = \alpha u$. Because $Qx \in y$, suppose $Qx = \beta y$. Hence we get

$$\begin{cases} x = x_1 + x_2 \\ \beta y = x_1 - x_2 \end{cases}$$

so we get

$$\begin{cases} x_1 = 0.5x + 0.5\beta y \\ x_2 = 0.5x - 0.5\beta y \end{cases}$$

Since $x_1 \perp x_2$, so

$$< 2x_1, 2x_2 >=< x + \beta y, x - \beta y >= ||x||_2^2 - \beta^2 ||y||_2^2 = 0$$

$$\Rightarrow \beta^2 = \frac{||x||_2^2}{||y||_2^2}$$

$$\Rightarrow x_2 = 0.5x - 0.5\frac{||x||_2}{||y||_2} y$$

$$\Rightarrow u = \frac{x - \frac{||x||_2}{||y||_2} y}{||x - \frac{||x||_2}{||y||_2} y||_2}$$

If $0.5x - 0.5\frac{||x||_2}{||y||_2} y = 0$, then any unit vector $u \perp x$ will make sense.

Problem 5: Projection Matrices

Solution

(a) Obviously $I - P$ is also a symmetric matrix.

$$\begin{aligned}(I-P)(I-P) &= (I-P) - P(I-P)\\ &= I - P - P + P^2\\ &= I - 2P + P\\ &= I - P\end{aligned}$$

So $I - P$ is also a projection matrix.

(b) Obviously UU^T is symmetric matirx.

$$\begin{aligned}(UU^T)^2 &= UU^TUU^T\\ &= U(U^TU)U^T\end{aligned}$$

Because the columns of U is orthonormal, so $U^TU = I_{k\times k}$. Hence

$$(UU^T)^2 = U(U^TU)U^T = UIU^T = UU^T$$

So UU^T is a projection matrix.

(c) First we should show $P = A(A^TA)^{-1}A^T$ is a symmetric matrix.

$$\begin{aligned}[A(A^TA)^{-1}A^T]^T &= A[(A^TA)^{-1}]^TA^T\\ &= A[(A^TA)^T]^{-1}A^T\\ &= A(A^TA)^{-1}A^T\end{aligned}$$

So $A(A^TA)^{-1}A^T$ is a symmetric matrix. Second we should show that $P = P^2$.

$$\begin{aligned}P^2 &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T\\ &= A(A^TA)^{-1}IA^T\\ &= A(A^TA)^{-1}A^T\end{aligned}$$

Hence $A(A^TA)^{-1}A^T$ is a projection matrix.

(d) Suppose $P = [p_1, p_2, \ldots, p_n]$, where p_i is the column of matrix P. Hence $R(P)$ could be represented as $span\{p_1, p_2, \ldots, p_n\}$. For any $p_i \in \{p_1, p_2, \ldots, p_n\}$,

$$< x - y, p_i >= p_i^T(x - Px) = (p_i^T - p_i^TP)x \tag{-6}$$

If P is a projection matrix, then $P = P^2$ and $P = P^T$must be valid. Therefore,

$$P^2 = \begin{bmatrix} p_1^T p_1 & p_1^T p_2 & \dots & p_1^T p_n \\ p_2^T p_1 & p_2^T p_2 & \dots & p_2^T p_n \\ \vdots & \vdots & & \vdots \\ p_n^T p_1 & p_n^T p_2 & \dots & p_n^T p_n \end{bmatrix} = \begin{bmatrix} < p_1, p_1 > & < p_2, p_1 > & \dots & < p_n, p_1 > \\ < p_1, p_2 > & < p_2, p_2 > & \dots & < p_n, p_2 > \\ \vdots & \vdots & & \vdots \\ < p_1, p_n > & < p_2, p_n > & \dots & < p_n, p_n > \end{bmatrix}$$

Hence

$$p_i = [< p_1, p_1 >, < p_1, p_2 >, \dots, < p_1, p_n >]^T$$

Because $p_i \in \mathbb{R}^n$,

$$p_i = [< p_1, p_1 >, < p_2, p_1 >, \dots, < p_n, p_1 >]^T$$

Using the result above into equation (1), then we get

$$p_i^T - p_i^T P = p_i^T - [< p_1, p_1 >, < p_2, p_1 >, \dots, < p_n, p_1 >] = 0 \Rightarrow < x - y, p_i >= 0$$

Hence y is the point in $R(P)$ closest to x. y is the projection of x. In conclusion, if P is a projection matrix, then $y = Px$ is the projection of x onto $R(P)$.

(e) The thing is that we want to find a projection matrix P such that $R(P) = span\{u\}$