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# Assignment #2

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## Problem 1: Convolution as Linea Map

**Solution**

(a) Obviously the size of matrice $T$ must be $(N+1) \times (N+1)$. Let $T_{i1}, T_{i2}, \ldots, T_{i(N+1)}$ be the i-th row of matrice $T$. Since $y = Tx$, we can get

$$v(i-1) = \sum_{j=1}^{N+1} T_{ij} u(j-1) = T_{i1}u(0) + T_{i2}u(1) + \ldots + T_{i(N+1)}u(N) \tag{1}$$

Accoring to the convolution function,

$$v(i-1) = \sum_{k=-\infty}^{+\infty} h(k)u(i-1-k) = \sum_{k=(i-1-N)}^{i-1} h(k)u(i-1-k) \tag{2}$$

Comparing function (1) and function (2), it is easy to conclude that

$$T_{ij} = h(i-j),\ where\ 1 \le i \le (N+1),\ 1 \le j \le (N+1)$$

So the matrice is a matrice where each element could be represented as $T_{ij} = h(i-j),\ where\ 1 \le i \le (N+1),\ 1 \le j \le (N+1)$.

(b) The structure of matrice $T$ could be described as follow:

$$T_{i,j} = T_{i+1,j+1}$$

## Problem 2: Affine Funciton

**Solution**

(a) *Proof.* For any $\alpha, \beta \in \mathbb{R}$ and any $x, y \in \mathbb{R}^n$

$$\alpha f(x) + \beta f(y) = \alpha(Ax+b) + \beta(Ay+b) = \alpha Ax + \beta Ay + (\alpha+\beta)b = \alpha Ax + \beta Ay + b$$

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) + b = \alpha Ax + \beta Ay + (\alpha+\beta)b = \alpha Ax + \beta Ay + b$$

Hence, $\alpha f(x) + \beta f(y) = f(\alpha x + \beta y)$. So function $f(x) = Ax + b$ is affine. □

(b) *Proof.* First we can show that $b$ is unique. Because $f(0) = b$, so $b$ must be unique, otherwise $f(0)$ will be mapped as different values in $\mathbb{R}^m$, which conflicts with the function definition. Then we can show that $A$ is unique. Let function

$$g(x) = f(x) - f(0) = Ax + b - b = Ax$$

Suppose $b_1, b_2, \dots, b_n$ are the basis of $\mathbb{R}^n$ and $B = \sum_{i=1}^{n} \alpha_i b_i,\ \alpha_i \in \mathbb{R}$.

$$g(B) = g(\sum_{i=1}^{n} \alpha_i b_i) = A(\sum_{i=1}^{n} \alpha_i b_i) = \sum_{i=1}^{n} \alpha_i A b_i = \sum_{i=1}^{n} \alpha_i g(b_i)$$

Suppose $b_i = e_i,\ i = 1, 2, \dots, n$, then $g(B)$ above could be represented as

$$\begin{aligned} g(B) &= \begin{bmatrix} g(b_1) & g(b_2) & \dots & g(b_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix} \\ &= \begin{bmatrix} g(e_1) & g(e_2) & \dots & g(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \\ &= Ax \end{aligned}$$

where

$x \in \mathbb{R}^n$.

Hence $A$ is unique. So any affine funciton $f$ could be represented uniquely as $f(x) = Ax + b$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times n}$ $\square$

# Problem 3: Matrix Multification

### Solution

(a) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq 0$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0$$

Then

$$AB = 0$$

Hence the statement is incorrect.

(b) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$$

Then

$$A^2 = 0$$

Hence the statement is incorrect.

(c) Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then

$$A^T A = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 \\ \vdots & \vdots & & \vdots \\ a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \end{bmatrix}$$

So if $A^T A = 0$, then $a_{ij} = 0,\ 1 \leq i \leq n,\ 1 \leq j \leq n.\ A = 0$. So the statement is correct.

# Problem 4: Linear Maps and Differentiation of polynomials

### Solution

(a) For any $p_1(x),\ p_2(x) \in \mathcal{P}_n$,

$$T(p_1(x) + p_2(x)) = \frac{d(p_1(x) + p_2(x))}{dx} = \frac{dp_1(x)}{dx} + \frac{dp_2(x)}{dx} = T(p_1(x)) + T(p_2(x)) \tag{3}$$

$$T(\alpha p_1(x)) = \frac{d(\alpha p_1(x))}{dx} = \alpha \frac{dp_1(x)}{dx} = \alpha T(p_1(x)) \tag{4}$$

Since equations (3) and (4) are valid, $T$ is linear.

(b) For any $p_(x) \in \mathcal{P}_n$, by using $\{1\ x\ x^2\ \dots\ x^n\}$ as basis, $p_(x)$ could be represented as

$$p_(x) = \sum_{i=0}^{n} \alpha_i x^i = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Similarily

$$T(p_(x)) = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix}$$

So we can find a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ that transforms the $p_(x)$ coefficient matrix to $T(p_(x))$ coefficient matrix. It means that

$$\begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix} = A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Obviously $rank(A) = n$.

# Problem 5: Rank of $AA^T$

### Solution

(a) For any $A \in \mathbb{R}^{m\times n}$,

$$dim[R(AA^T)] + dim[N(AA^T)] = m$$

$$dim[R(A^T)] + dim[N(A^T)] = m$$

Since $rank(A) = rank(A^T) = dim[R(A^T)]$, so if we can show that $dim[N(AA^T)] = dim[N(A^T)]$, then $rank(A) = rank(AA^T)$ is valid. The following will proof $dim[N(AA^T)] = dim[N(A^T)]$.
For any $x \in N(A^T)$,

$$A^Tx = 0 \Rightarrow A(A^Tx) = 0 \Rightarrow AA^Tx = 0$$

So for any $x \in N(A^T)$, $x$ also satisfies $x \in N(AA^T)$. Conversely, for any $x \in N(AA^T)$,

$$AA^Tx = 0 \Rightarrow x(AA^Tx) = 0 \Rightarrow xAA^Tx = 0 \Rightarrow (A^Tx)^TA^Tx = 0$$

According to the conclusion of problem3 (c), $A^T x = 0$. It means that for for any $x \in N(AA^T)$, $x \in N(A^T)$. Hence $N(A^T) = N(AA^T)$ and $dim[N(AA^T)] = dim[N(A^T)]$. Further we can get

$$dim[R(AA^T)] = m - dim[N(AA^T)] = m - dim[N(A^T)] = dim[R(A^T)]$$

Since $rank(A) = rank(A^T) = dim[R(A^T)]$ and $rank(AA^T) = dim[R(AA^T)]$, So

$$rank(A)rank(AA^T)$$

(b) The statement is invalid. Suppose

$$A = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix},\ A^T = \begin{bmatrix} 1 & 1 \\ -i & -i \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Obviously $rank(AA^T) \neq rank(A)$.

(c) Similarily to problem(a), we can show that $N(A^H) = N(AA^H)$. For any $A \in \mathbb{C}^{m\times n}$, if $x \in N(A^H)$, then

$$A^H x = 0 \Rightarrow AA^H x = 0 \Rightarrow x \in N(AA^H)$$

Conversely, if $x \in N(AA^H)$, then

$$AA^H x = 0 \Rightarrow xAA^H x = 0 \Rightarrow (A^H x)^H A^H x = 0$$

Suppose $[A^H x]_{jk} = a_{jk} + b_{jk} i$, it is easy to get that $[(A^H x)^H A^H x]_{jk} = a_{jk}^2 + b_{jk}^2$. Hence if $(A^H x)^H A^H x = 0$, then $A^H x = 0$. So if $x \in N(AA^H)$, then $x \in N(A^H)$. In conclusion, $N(A^H) = N(AA^H)$ and $dim[N(A^H)] = dim[N(AA^H)]$. Since

$$dim[R(AA^H)] + dim[N(AA^H)] = m$$

$$dim[R(A^H)] + dim[N(A^H)] = m$$

So

$$dim[R(A^H)] = dim[R(AA^H)] \Rightarrow rank[A] = rank[A^H] = rank[AA^H]$$

## Problem 6: Left and Right Inverses

### Solution

(a) For any $x \in N(A^T A)$,

$$A^T Ax = 0 \Rightarrow x^T A^T Ax = 0 \Rightarrow (Ax)^T Ax = 0 \Rightarrow Ax = 0$$

Since $A$ is full-rank and tall, $rank(A) = n$. Then

$$dim[N(A)] = n - dim[R(A)] = n - rank(A) = 0$$

So for $Ax = 0$, there must be an unique solution that is $x = 0$, since $dim[N(A)] = 0$. Hence

$$dim[N(A^T A)] = 0 \Rightarrow dim[R(A^T A)] = n$$

So $A^T A$ is nonsingular.

(b)

$$\begin{aligned}(A^T A)^{-1} A^T A &= A^{-1}(A^T)^{-1} A^T A \\ &= A^{-1} I A \\ &= I\end{aligned}$$

So $(A^T A)^{-1} A^T$ is a left inverse of a full-rank tall matrix $A$.

(c) Suppose

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then

$$A_1 A = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I$$

$$A_2 A = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I$$

So $A$ doesn't have unique left inverse.

(d) Similar to problem (a), for any $x \in N(AA^T)$,

$$AA^T x = 0 \Rightarrow x^T AA^T x = 0 \Rightarrow (A^T x)^T A^T x = 0 \Rightarrow A^T x = 0$$

Since $A$ is full-rank and fat, $rank(A) = m$. Then

$$dim[N(A^T)] = m - dim[R(A^T)] = m - rank(A^T) = 0$$

So for $A^T x = 0$, there must be an unique solution that is $x = 0$, since $dim[N(A^T)] = 0$. Hence

$$dim[N(AA^T)] = 0 \Rightarrow dim[R(AA^T)] = m$$

So $A^T A$ is nonsingular.

(e)

$$
\begin{aligned}
AA^T(AA^T)^{-1} &= AA^T(A^T)^{-1}A^{-1} \\
&= AIA^{-1} \\
&= I
\end{aligned}
$$

So $A^T(AA^T)^{-1}$ is a right inverse of a full-rank tall matrix $A$.

(f) Suppose

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then

$$AA_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I$$

$$AA_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = I$$

So $A$ doesn't have unique right inverse.