

## Problem 1: Orthogonal Complement of a Subspace

### Solution

- (a) Suppose  $x_1, x_2 \in \mathcal{V}^\perp$ , for any  $y \in \mathcal{V}$

$$(x_1 + x_2)^T y = (x_1^T + x_2^T)y = x_1^T y + x_2^T y = 0$$

For any  $\alpha \in \mathbb{R}$ ,

$$(\alpha x_1)^T y = \alpha x_1^T y = 0$$

Hence  $\mathcal{V}^\perp$  is a subspace of  $\mathbb{R}^n$ .

- (b) Because  $\mathcal{V} = \text{span}(v_1, v_2, \dots, v_k)$ , for any  $y \in \mathcal{V}$ ,

$$y = \sum_{i=1}^k \alpha_i v_i = [v_1 \ v_2 \ \dots \ v_k] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \text{ where } \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

Hence  $\mathcal{V} = R(A)$ .

For any  $y \in \mathcal{V}$ ,

$$\begin{aligned} x^T y &= x^T A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = 0, \text{ where } \alpha_1, \dots, \alpha_k \in \mathbb{R} \\ &\Rightarrow \left( A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \right)^T x = 0 \\ &\Rightarrow [\alpha_1 \ \alpha_2 \ \dots \ \alpha_k] A^T x = 0 \\ &\Rightarrow A^T x = 0 \end{aligned}$$

Hence  $\mathcal{V}^\perp = N(A^T)$ .

- (c)  $(\mathcal{V}^\perp)^\perp$  can be represented as the following:

$$(\mathcal{V}^\perp)^\perp = \{y \in \mathbb{R}^n : y^T x = 0, \forall x \in \mathcal{V}^\perp\}$$

Meanwhile, for any  $x \in \mathcal{V}^\perp$ ,

$$x^T y = 0, \forall y \in \mathcal{V} \Rightarrow y^T x = 0, \forall y \in \mathcal{V}$$

Hence  $(\mathcal{V}^\perp)^\perp = \mathcal{V}^\perp$

(d)

$$\dim(\mathcal{V}) = \dim[R(A)] = \text{rank}(A) = \text{rank}(A^T) = \dim[R(A^T)]$$

Meanwhile

$$\dim(\mathcal{V}^\perp) = \dim[N(A^T)]$$

Hence

$$\dim(\mathcal{V}) + \dim(\mathcal{V}^\perp) = \dim[R(A^T)] + \dim[N(A^T)] = n$$

(e) According to the definition,

$$\mathcal{W}^\perp = \{x_2 \in \mathbb{R}^n : x_2^T y_2 = 0, \forall y_2 \in \mathcal{W}\}$$

For any  $y_1 \in \mathcal{V}$ . Because  $\mathcal{V} \subseteq \mathcal{W}$ ,  $y_1 \in \mathcal{W}$ , hence

$$x_2^T y_1 = 0 \Rightarrow x_2 \in \mathcal{V}^\perp$$

Hence  $\mathcal{V} \subseteq \mathcal{W}$  for another subspace  $\mathcal{W}$  implies  $\mathcal{W}^\perp \subseteq \mathcal{V}^\perp$

(f) Suppose  $v$  is the projection of  $x$  onto subspace  $\mathcal{V}$ . Then

$$\begin{aligned} & \langle x - v, y \rangle = 0, \forall y \in \mathcal{V} \\ & \Rightarrow y^T(x - v) = 0, \forall y \in \mathcal{V} \\ & \Rightarrow (x - v)^T y = 0, \forall y \in \mathcal{V} \end{aligned}$$

So  $(x - v) \in \mathcal{V}^\perp$ . Suppose there is a vector  $v^\perp$ , S.T.

$$\begin{aligned} x - v &= v^\perp \\ \Rightarrow x &= v + v^\perp \end{aligned}$$

Meanwhile, because  $v$  is the projection of  $x$  onto subspace  $\mathcal{V}$ ,  $v$  must be unique and thus  $v^\perp$  is also unique. Hence any  $x \in \mathbb{R}^n$  could be expressed uniquely as  $x = v + v^\perp$ .

## Problem 2: Rank of a Product

### Solution

- (a) Suppose
- (b) Suppose

## Problem 3: An Inequality for Orthonormal Matrices

### Solution

## Problem 4: Householder Reflections

### Solution

(a)

$$\begin{aligned} QQ^T &= (I - 2uu^T)(I - 2uu^T)^T \\ &= (I - 2uu^T)^T - 2uu^T(I - 2uu^T)^T \\ &= I - 2uu^T - 2uu^T + 4uu^Tuu^T \\ &= I - 4uu^T + 4u(u^Tu)u^T \end{aligned}$$

Because  $u$  is unit vector,  $u^Tu = \|u\|^2 = 1$ . Hence  $u(u^Tu)u^T = uu^T$ , so  $QQ^T = I$ . Therefore,  $Q$  is orthogonal.

(b)

$$\begin{aligned} Qu &= (I - 2uu^T)u \\ &= Iu - 2uu^Tu \\ &= u - 2u(u^Tu) \\ &= u - 2u \\ &= -u \end{aligned}$$

$$\begin{aligned} Qv &= (I - 2uu^T)v \\ &= v - 2uu^Tv \\ &= v - 2u < v, u > \\ &= v \end{aligned}$$

- (c) Known  $Q \in \mathbb{R}^{n \times n}$ , suppose  $u_1, u_2, \dots, u_n$  is the column of  $Q$ . Because  $Q$  is orthogonal, then  $u_1, u_2, \dots, u_n$  are orthogonal, hence  $u_1, u_2, \dots, u_n$  are linearly independent. Thus matrix  $Q$  is full-rank and invertible. Thus given  $y$ ,

$$x = Q^{-1}y$$

- (d) Obviously matirx  $uu^T$  is square, hence

$$\det(Q) = \det(I - 2uu^T) = \det(I - 2u^TuI) = \det(I - 2I) = \det(-I) = -1$$

(e)

## Problem 5: Projection Matrices

### Solution

(a) Obviously  $I - P$  is also a symmetric matrix.

$$\begin{aligned}(I - P)(I - P) &= (I - P) - P(I - P) \\ &= I - P - P + P^2 \\ &= I - 2P + P \\ &= I - P\end{aligned}$$

So  $I - P$  is also a projection matrix.

(b) Obviously  $UU^T$  is symmetric matirx.

$$\begin{aligned}(UU^T)^2 &= UU^TUU^T \\ &= U(U^TU)U^T\end{aligned}$$

Because the columns of  $U$  is orthonormal, so  $U^TU = I_{k \times k}$ . Hence

$$(UU^T)^2 = U(U^TU)U^T = UIU^T = UU^T$$

So  $UU^T$  is a projection matrix.

(c) First we should show  $P = A(A^TA)^{-1}A^T$  is a symmetric matrix.

$$\begin{aligned}[A(A^TA)^{-1}A^T]^T &= A[(A^TA)^{-1}]^TA^T \\ &= A[(A^TA)^T]^{-1}A^T \\ &= A(A^TA)^{-1}A^T\end{aligned}$$

So  $A(A^TA)^{-1}A^T$  is a symmetric matrix. Second we should show that  $P = P^2$ .

$$\begin{aligned}P^2 &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}IA^T \\ &= A(A^TA)^{-1}A^T\end{aligned}$$

Hence  $A(A^TA)^{-1}A^T$  is a projection matrix.

(d) Suppose  $P = [p_1, p_2, \dots, p_n]$ , where  $p_i$  is the column of matrix  $P$ . Hence  $R(P)$  could be represented as  $\text{span}\{p_1, p_2, \dots, p_n\}$ . For any  $p_i \in \{p_1, p_2, \dots, p_n\}$ ,

$$\langle x - y, p_i \rangle = p_i^T(x - Px) = (p_i^T - p_i^T P)x \quad (1)$$

If  $P$  is a projection matrix, then  $P = P^2$  and  $P = P^T$  must be valid. Therefore,

$$P^2 = \begin{bmatrix} p_1^T p_1 & p_1^T p_2 & \dots & p_1^T p_n \\ p_2^T p_1 & p_2^T p_2 & \dots & p_2^T p_n \\ \vdots & \vdots & & \vdots \\ p_n^T p_1 & p_n^T p_2 & \dots & p_n^T p_n \end{bmatrix} = \begin{bmatrix} \langle p_1, p_1 \rangle & \langle p_2, p_1 \rangle & \dots & \langle p_n, p_1 \rangle \\ \langle p_1, p_2 \rangle & \langle p_2, p_2 \rangle & \dots & \langle p_n, p_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle p_1, p_n \rangle & \langle p_2, p_n \rangle & \dots & \langle p_n, p_n \rangle \end{bmatrix}$$

Hence

$$p_i = [\langle p_1, p_1 \rangle, \langle p_1, p_2 \rangle, \dots, \langle p_1, p_n \rangle]^T$$

Because  $p_i \in \mathbb{R}^n$ ,

$$p_i = [\langle p_1, p_1 \rangle, \langle p_2, p_1 \rangle, \dots, \langle p_n, p_1 \rangle]^T$$

Using the result above into equation (1), then we get

$$p_i^T - p_i^T P = p_i^T - [\langle p_1, p_1 \rangle, \langle p_2, p_1 \rangle, \dots, \langle p_n, p_1 \rangle] = 0 \Rightarrow \langle x - y, p_i \rangle = 0$$

Hence  $y$  is the point in  $R(P)$  closest to  $x$ .  $y$  is the projection of  $x$ . In conclusion, if  $P$  is a projection matrix, then  $y = Px$  is the projection of  $x$  onto  $R(P)$ .

- (e) The thing is that we want to find a projection matrix  $P$  such that  $R(P) = \text{span}\{u\}$