Jiaming Lai
A53314574

Assignment #3

Problem 1: Orthogonal Complement of a Subspace

Solution

(a) Suppose $x_1, x_2 \in \mathcal{V}^\perp$, for any $y \in \mathcal{V}$

$$(x_1 + x_2)^T y = (x_1^T + x_2^T)y = x_1^T y + x_2^T y = 0$$

For any $\alpha \in \mathbb{R}$,

$$(\alpha x_1)^T y = \alpha x_1^T y = 0$$

Hence $\mathcal{V}^\perp$ is a subspace of $\mathbb{R}^n$.

(b) Because $\mathcal{V} = span(v_1, v_2, \ldots, v_k)$, for any $y \in \mathcal{V}$,

$$y = \sum_{i=1}^{k} \alpha_i v_i = \begin{bmatrix} v_1 & v_2 & \ldots & v_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \; where\ \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$

Hence $\mathcal{V} = R(A)$.
For any $y \in \mathcal{V}$,

$$x^T y = x^T A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = 0, \quad where\ \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$

$$\Rightarrow \left(A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \right)^T x = 0$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_k \end{bmatrix} A^T x = 0$$

$$\Rightarrow A^T x = 0$$

Hence $\mathcal{V}^\perp = N(A^T)$.

(c) For any $x \in \mathcal{V}$ and any $y \in \mathcal{V}^\perp$, because $\mathcal{V} = R(A)$, then

$$x^T y = (Az)^T y = z^T A^T y, \; where\ z \in \mathbb{R}^k$$

Because $y \in \mathcal{V}^\perp = N(A^T)$, hence

$$x^T y = z^T A^T y = z^T (A^T y) = 0$$

So $x \in (\mathcal{V}^\perp)^\perp$. So for any $x \in \mathcal{V}$, $x \in (\mathcal{V}^\perp)^\perp$. Conversely, for any $x \in (\mathcal{V}^\perp)^\perp$,

(d)
$$dim(\mathcal{V}) = dim[R(A)] = rank(A) = rank(A^T) = dim[R(A^T)]$$

Meanwhile
$$dim(\mathcal{V}^\perp) = dim[N(A^T)]$$

Hence
$$dim(\mathcal{V}) + dim(\mathcal{V}^\perp) = dim[R(A^T)] + dim[N(A^T)] = n$$

(e) Accoding to the definition,
$$\mathcal{W}^\perp = \left\{x_2 \in \mathbb{R}^n : \ x_2^T y_2 = 0, \ \forall y_2 \in \mathcal{W}\right\}$$

For any $y_1 \in \mathcal{V}$. Because $\mathcal{V} \subseteq \mathcal{W}$, $y_1 \in \mathcal{W}$, hence
$$x_2^T y_1 = 0 \ \Rightarrow \ x_2 \in \mathcal{V}^\perp$$

Hence $\mathcal{V} \subseteq \mathcal{W}$ for another subspace $\mathcal{W}$ implies $\mathcal{W}^\perp \subseteq \mathcal{V}^\perp$

(f) A

Problem 2: Affine Funciton

Solution

(a) *Proof.* For any $\alpha, \beta \in \mathbb{R}$ and any $x, y \in \mathbb{R}^n$
$$\alpha f(x) + \beta f(y) = \alpha(Ax+b) + \beta(Ay+b) = \alpha Ax + \beta Ay + (\alpha+\beta)b = \alpha Ax + \beta Ay + b$$
$$f(\alpha x + \beta y) = A(\alpha x + \beta y) + b = \alpha Ax + \beta Ay + (\alpha+\beta)b = \alpha Ax + \beta Ay + b$$
Hence, $\alpha f(x) + \beta f(y) = f(\alpha x + \beta y)$. So function $f(x) = Ax + b$ is affine. □

(b) *Proof.* Suppose
$$g(x) = f(x) - f(0)$$

The following will show that $g(x)$ is linear.
$$\begin{aligned} g(x) + g(y) &= f(x) - f(0) + (f(y) - f(0)) \\ &= 2[0.5f(x) + 0.5f(y)] - 2f(0) \\ &= 2[f(0.5x + 0.5y)] - 2f(0) \\ &= 2[f(0.5x + 0.5y) + 0.5 * 0] - 2f(0) \\ &= 2[0.5f(x+y) + 0.5f(0)] - 2f(0) \\ &= f(x+y) + f(0) - 2f(0) \\ &= f(x+y) - f(0) \\ &= g(x+y) \end{aligned}$$

$$
\begin{aligned}
g(\alpha x) &= f(\alpha x) - f(0) \\
&= f(\alpha x + (1-\alpha) * 0) - f(0) \\
&= \alpha f(x) + (1-\alpha) f(0) - f(0) \\
&= \alpha f(x) - \alpha f(0) \\
&= \alpha g(x)
\end{aligned}
$$

So $g(x)$ is linear. Hence $g(x)$ could be represented as $g(x) = Ax,\ A \in \mathbb{R}^{m\times n}$. The following will show that A is unique.
If there exits another A_1 such that $g(x) = A_1 x,\ A \in \mathbb{R}^{m\times n}$, for any $x \in \mathbb{R}^n$,

$$A_1 x = Ax$$

Then

$$(A_1 - A)x = 0 \Rightarrow A_1 - A = 0 \Rightarrow A_1 = A$$

Hence A is unique. Meanwhile b is unique. Because $f(0) = b$, so b must be unique, otherwise $f(0)$ will be mapped as different values in $\mathbb{R}^m$, which conflicts with the function definition. In conclusion, any affine funciton f could be represented uniquely as $f(x) = Ax + b$ for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times n}$ $\square$

Problem 3: Matrix Multification

Solution

(a) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq 0$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0$$

Then

$$AB = 0$$

Hence the statement is incorrect.

(b) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$$

Then

$$A^2 = 0$$

Hence the statement is incorrect.

(c) Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then the diagonal elements of A^TA is

$$[A^TA]_{ii} = \sum_{k=1}^{n} a_{ki}^2, \ where \ 1 \leq i \leq n$$

So if $A^TA = 0$, the diagonal elements shoule be zero. Then $a_{ij} = 0, \ 1 \leq i \leq n, \ 1 \leq j \leq n$. $A = 0$. So the statement is correct.

Problem 4: Linear Maps and Differentiation of polynomials

Solution

(a) For any $p_1(x), \ p_2(x) \in \mathcal{P}_n$,

$$T(p_1(x) + p_2(x)) = \frac{d(p_1(x) + p_2(x))}{dx} = \frac{dp_1(x)}{dx} + \frac{dp_2(x)}{dx} = T(p_1(x)) + T(p_2(x)) \quad (1)$$

$$T(\alpha p_1(x)) = \frac{d(\alpha p_1(x))}{dx} = \alpha \frac{dp_1(x)}{dx} = \alpha T(p_1(x)) \quad (2)$$

Since equations (3) and (4) are valid, T is linear.

(b) For any $p_(x) \in \mathcal{P}_n$, by using $\{1 \ x \ x^2 \ \dots \ x^n\}$ as basis, $p_(x)$ could be represented as

$$p_(x) = \sum_{i=0}^{n} \alpha_i x^i = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Similarily

$$T(p_(x)) = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix}$$

So we can find a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ that transforms the $p_(x)$ coefficient matrix to $T(p_(x))$ coefficient matrix. It means that

$$\begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix} = A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Obviously $rank(A) = n$.

Problem 5: Rank of AA^T

Solution

(a) For any $A \in \mathbb{R}^{m\times n}$,

$$dim[R(AA^T)] + dim[N(AA^T)] = m$$
$$dim[R(A^T)] + dim[N(A^T)] = m$$

Since $rank(A) = rank(A^T) = dim[R(A^T)]$, so if we can show that $dim[N(AA^T)] = dim[N(A^T)]$, then $rank(A) = rank(AA^T)$ is valid. The following will proof $dim[N(AA^T)] = dim[N(A^T)]$.
For any $x \in N(A^T)$,

$$A^Tx = 0 \Rightarrow A(A^Tx) = 0 \Rightarrow AA^Tx = 0$$

So for any $x \in N(A^T)$, x also satisfies $x \in N(AA^T)$. Conversely, for any $x \in N(AA^T)$,

$$AA^Tx = 0 \Rightarrow x(AA^Tx) = 0 \Rightarrow xAA^Tx = 0 \Rightarrow (A^Tx)^TA^Tx = 0$$

According to the conclusion of problem3 (c), $A^Tx = 0$. It means that for for any $x \in N(AA^T)$, $x \in N(A^T)$. Hence $N(A^T) = N(AA^T)$ and $dim[N(AA^T)] = dim[N(A^T)]$. Further we can get

$$dim[R(AA^T)] = m - dim[N(AA^T)] = m - dim[N(A^T)] = dim[R(A^T)]$$

Since $rank(A) = rank(A^T) = dim[R(A^T)]$ and $rank(AA^T) = dim[R(AA^T)]$, So

$$rank(A) = rank(AA^T)$$

(b) The statement is invalid. Suppose

$$A = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix},\ A^T = \begin{bmatrix} 1 & 1 \\ -i & -i \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Obviously $rank(AA^T) \neq rank(A)$.

(c) First we will show that $rank[A] = rank[A^H]$. Suppose $x_1, x_2, \ldots, x_k$ is a basis of $R(A^H)$. Then $Ax_1, Ax_2, \ldots, Ax_k \in R(A)$. Suppose $\sum_{i=1}^k c_i(Ax_i) = A\sum_{i=1}^k c_i x_i = 0$, $v = \sum_{i=1}^k c_i x_i$ then $v \in N(A)$ and $v \in R(A^H)$. Meanwhile,

$$v^H v = v^H(\sum_{i=1}^k c_i x_i) = v^H A^H x = (Av)^H x$$

Because $v \in N(A)$, then $v^H v = 0$. So $c_i x_i = 0$, which means $c_i = 0$ and thus $Ax_1, Ax_2, \ldots, Ax_k$ are linear independent. So $dim[R(A^H)] = k \leq dim[R(A)] = rank(A)$. Let $B = A^H$, then

$$dim[R(B^H)] \leq dim[R(B)] \Rightarrow rank(A) \leq k$$

So

$$rank(A) = k \Rightarrow rank(A) = rank(A^H)$$

The following is similar to problem(a), we can show that $N(A^H) = N(AA^H)$. For any $A \in \mathbb{C}^{m\times n}$, if $x \in N(A^H)$, then

$$A^H x = 0 \Rightarrow AA^H x = 0 \Rightarrow x \in N(AA^H)$$

Conversely, if $x \in N(AA^H)$, then

$$AA^H x = 0 \Rightarrow xAA^H x = 0 \Rightarrow (A^H x)^H A^H x = 0$$

Suppose $[A^H x]_{jk} = a_{jk} + b_{jk} i$, similiar to problem3-(c), it is easy to get that if $(A^H x)^H A^H x = 0$, then $A^H x = 0$. So if $x \in N(AA^H)$, then $x \in N(A^H)$. In conclusion, $N(A^H) = N(AA^H)$ and $dim[N(A^H)] = dim[N(AA^H)]$. Since

$$dim[R(AA^H)] + dim[N(AA^H)] = m$$

$$dim[R(A^H)] + dim[N(A^H)] = m$$

So

$$dim[R(A^H)] = dim[R(AA^H)] \Rightarrow rank[A] = rank[A^H] = rank[AA^H]$$

Problem 6: Left and Right Inverses

Solution

(a) For any $x \in N(A^T A)$,

$$A^T Ax = 0 \Rightarrow x^T A^T Ax = 0 \Rightarrow (Ax)^T Ax = 0 \Rightarrow Ax = 0$$

Since A is full-rank and tall, $rank(A) = n$. Then

$$dim[N(A)] = n - dim[R(A)] = n - rank(A) = 0$$

So for $Ax = 0$, there must be an unique solution that is $x = 0$, since $dim[N(A)] = 0$. Hence

$$dim[N(A^T A)] = 0 \Rightarrow dim[R(A^T A)] = n$$

So $A^T A$ is nonsingular.

(b)

$$\begin{aligned}(A^T A)^{-1} A^T A &= A^{-1}(A^T)^{-1} A^T A \\ &= A^{-1} I A \\ &= I\end{aligned}$$

So $(A^T A)^{-1} A^T$ is a left inverse of a full-rank tall matrix A.

(c) Suppose

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then

$$A_1 A = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I$$

$$A_2 A = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I$$

So A doesn't have unique left inverse.

(d) Similar to problem (a), for any $x \in N(AA^T)$,

$$AA^T x = 0 \Rightarrow x^T AA^T x = 0 \Rightarrow (A^T x)^T A^T x = 0 \Rightarrow A^T x = 0$$

Since A is full-rank and fat, $rank(A) = m$. Then

$$dim[N(A^T)] = m - dim[R(A^T)] = m - rank(A^T) = 0$$

So for $A^Tx = 0$, there must be an unique solution that is $x = 0$, since $dim[N(A^T)] = 0$. Hence

$$dim[N(AA^T)] = 0 \Rightarrow dim[R(AA^T)] = m$$

So A^TA is nonsingular.

(e)

$$\begin{aligned} AA^T(AA^T)^{-1} &= AA^T(A^T)^{-1}A^{-1} \\ &= AIA^{-1} \\ &= I \end{aligned}$$

So $A^T(AA^T)^{-1}$ is a right inverse of a full-rank tall matrix A.

(f) Suppose

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then

$$AA_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I$$

$$AA_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = I$$

So A doesn't have unique right inverse.