

## Problem 1: Orthogonal Complement of a Subspace

### Solution

(a) Suppose  $x_1, x_2 \in \mathcal{V}^\perp$ , for any  $y \in \mathcal{V}$

$$(x_1 + x_2)^T y = (x_1^T + x_2^T)y = x_1^T y + x_2^T y = 0$$

For any  $\alpha \in \mathbb{R}$ ,

$$(\alpha x_1)^T y = \alpha x_1^T y = 0$$

Hence  $\mathcal{V}^\perp$  is a subspace of  $\mathbb{R}^n$ .

(b) Because  $\mathcal{V} = \text{span}(v_1, v_2, \dots, v_k)$ , for any  $y \in \mathcal{V}$ ,

$$y = \sum_{i=1}^k \alpha_i v_i = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \text{ where } \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

Hence  $\mathcal{V} = R(A)$ .

For any  $y \in \mathcal{V}$ ,

$$\begin{aligned} x^T y &= x^T A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = 0, \text{ where } \alpha_1, \dots, \alpha_k \in \mathbb{R} \\ &\Rightarrow \left( A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \right)^T x = 0 \\ &\Rightarrow [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_k] A^T x = 0 \\ &\Rightarrow A^T x = 0 \end{aligned}$$

Hence  $\mathcal{V}^\perp = N(A^T)$ .

(c) For any  $x \in \mathcal{V}$  and any  $y \in \mathcal{V}^\perp$ , because  $\mathcal{V} = R(A)$ , then

$$x^T y = (Az)^T y = z^T A^T y, \text{ where } z \in \mathbb{R}^k$$

Because  $y \in \mathcal{V}^\perp = N(A^T)$ , hence

$$x^T y = z^T A^T y = z^T (A^T y) = 0$$

So  $x \in (\mathcal{V}^\perp)^\perp$ . So for any  $x \in \mathcal{V}$ ,  $x \in (\mathcal{V}^\perp)^\perp$ . Conversely, for any  $x \in (\mathcal{V}^\perp)^\perp$

## Problem 2: Affine Function

### Solution

(a) *Proof.* For any  $\alpha, \beta \in \mathbb{R}$  and any  $x, y \in \mathbb{R}^n$

$$\alpha f(x) + \beta f(y) = \alpha(Ax + b) + \beta(Ay + b) = \alpha Ax + \beta Ay + (\alpha + \beta)b = \alpha Ax + \beta Ay + b$$

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) + b = \alpha Ax + \beta Ay + (\alpha + \beta)b = \alpha Ax + \beta Ay + b$$

Hence,  $\alpha f(x) + \beta f(y) = f(\alpha x + \beta y)$ . So function  $f(x) = Ax + b$  is affine.  $\square$

(b) *Proof.* Suppose

$$g(x) = f(x) - f(0)$$

The following will show that  $g(x)$  is linear.

$$\begin{aligned} g(x) + g(y) &= f(x) - f(0) + (f(y) - f(0)) \\ &= 2[0.5f(x) + 0.5f(y)] - 2f(0) \\ &= 2[f(0.5x + 0.5y)] - 2f(0) \\ &= 2[f(0.5x + 0.5y) + 0.5 * 0] - 2f(0) \\ &= 2[0.5f(x + y) + 0.5f(0)] - 2f(0) \\ &= f(x + y) + f(0) - 2f(0) \\ &= f(x + y) - f(0) \\ &= g(x + y) \end{aligned}$$

$$\begin{aligned} g(\alpha x) &= f(\alpha x) - f(0) \\ &= f(\alpha x + (1 - \alpha) * 0) - f(0) \\ &= \alpha f(x) + (1 - \alpha)f(0) - f(0) \\ &= \alpha f(x) - \alpha f(0) \\ &= \alpha g(x) \end{aligned}$$

So  $g(x)$  is linear. Hence  $g(x)$  could be represented as  $g(x) = Ax$ ,  $A \in \mathbb{R}^{m \times n}$ . The following will show that  $A$  is unique.

If there exists another  $A_1$  such that  $g(x) = A_1x$ ,  $A \in \mathbb{R}^{m \times n}$ , for any  $x \in \mathbb{R}^n$ ,

$$A_1x = Ax$$

Then

$$(A_1 - A)x = 0 \Rightarrow A_1 - A = 0 \Rightarrow A_1 = A$$

Hence  $A$  is unique. Meanwhile  $b$  is unique. Because  $f(0) = b$ , so  $b$  must be unique, otherwise  $f(0)$  will be mapped as different values in  $\mathbb{R}^m$ , which conflicts with the function definition. In conclusion, any affine function  $f$  could be represented uniquely as  $f(x) = Ax + b$  for some  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ .  $\square$

## Problem 3: Matrix Multiflication

### Solution

(a) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq 0$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0$$

Then

$$AB = 0$$

Hence the statement is incorrect.

(b) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$$

Then

$$A^2 = 0$$

Hence the statement is incorrect.

(c) Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then the diagonal elements of  $A^T A$  is

$$[A^T A]_{ii} = \sum_{k=1}^n a_{ki}^2, \text{ where } 1 \leq i \leq n$$

So if  $A^T A = 0$ , the diagonal elements should be zero. Then  $a_{ij} = 0$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n$ .  
 $A = 0$ . So the statement is correct.

## Problem 4: Linear Maps and Differentiation of polynomials

### Solution

(a) For any  $p_1(x), p_2(x) \in \mathcal{P}_n$ ,

$$T(p_1(x) + p_2(x)) = \frac{d(p_1(x) + p_2(x))}{dx} = \frac{dp_1(x)}{dx} + \frac{dp_2(x)}{dx} = T(p_1(x)) + T(p_2(x)) \quad (1)$$

$$T(\alpha p_1(x)) = \frac{d(\alpha p_1(x))}{dx} = \alpha \frac{dp_1(x)}{dx} = \alpha T(p_1(x)) \quad (2)$$

Since equations (3) and (4) are valid,  $T$  is linear.

(b) For any  $p(x) \in \mathcal{P}_n$ , by using  $\{1 \ x \ x^2 \ \dots \ x^n\}$  as basis,  $p(x)$  could be represented as

$$p(x) = \sum_{i=0}^n \alpha_i x^i = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Similarly

$$T(p(x)) = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix}$$

So we can find a matrix  $A \in \mathbb{R}^{(n+1) \times (n+1)}$  that transforms the  $p(x)$  coefficient matrix to  $T(p(x))$  coefficient matrix. It means that

$$\begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix} = A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Obviously  $\text{rank}(A) = n$ .

## Problem 5: Rank of $AA^T$

### Solution

(a) For any  $A \in \mathbb{R}^{m \times n}$ ,

$$\dim[R(AA^T)] + \dim[N(AA^T)] = m$$

$$\dim[R(A^T)] + \dim[N(A^T)] = m$$

Since  $\text{rank}(A) = \text{rank}(A^T) = \dim[R(A^T)]$ , so if we can show that  $\dim[N(AA^T)] = \dim[N(A^T)]$ , then  $\text{rank}(A) = \text{rank}(AA^T)$  is valid. The following will proof  $\dim[N(AA^T)] = \dim[N(A^T)]$ .

For any  $x \in N(A^T)$ ,

$$A^T x = 0 \Rightarrow A(A^T x) = 0 \Rightarrow AA^T x = 0$$

So for any  $x \in N(A^T)$ ,  $x$  also satisfies  $x \in N(AA^T)$ . Conversely, for any  $x \in N(AA^T)$ ,

$$AA^T x = 0 \Rightarrow x(AA^T x) = 0 \Rightarrow xAA^T x = 0 \Rightarrow (A^T x)^T A^T x = 0$$

According to the conclusion of problem3 (c),  $A^T x = 0$ . It means that for any  $x \in N(AA^T)$ ,  $x \in N(A^T)$ . Hence  $N(A^T) = N(AA^T)$  and  $\dim[N(AA^T)] = \dim[N(A^T)]$ . Further we can get

$$\dim[R(AA^T)] = m - \dim[N(AA^T)] = m - \dim[N(A^T)] = \dim[R(A^T)]$$

Since  $\text{rank}(A) = \text{rank}(A^T) = \dim[R(A^T)]$  and  $\text{rank}(AA^T) = \dim[R(AA^T)]$ , So

$$\text{rank}(A) = \text{rank}(AA^T)$$

(b) The statement is invalid. Suppose

$$A = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 1 \\ -i & -i \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Obviously  $\text{rank}(AA^T) \neq \text{rank}(A)$ .

(c) First we will show that  $\text{rank}[A] = \text{rank}[A^H]$ . Suppose  $x_1, x_2, \dots, x_k$  is a basis of  $R(A^H)$ . Then  $Ax_1, Ax_2, \dots, Ax_k \in R(A)$ . Suppose  $\sum_{i=1}^k c_i(Ax_i) = A \sum_{i=1}^k c_i x_i = 0$ ,  $v = \sum_{i=1}^k c_i x_i$  then  $v \in N(A)$  and  $v \in R(A^H)$ . Meanwhile,

$$v^H v = v^H \left( \sum_{i=1}^k c_i x_i \right) = v^H A^H x = (Av)^H x$$

Because  $v \in N(A)$ , then  $v^H v = 0$ . So  $c_i x_i = 0$ , which means  $c_i = 0$  and thus  $Ax_1, Ax_2, \dots, Ax_k$  are linear independent. So  $\dim[R(A^H)] = k \leq \dim[R(A)] = \text{rank}(A)$ . Let  $B = A^H$ , then

$$\dim[R(B^H)] \leq \dim[R(B)] \Rightarrow \text{rank}(A) \leq k$$

So

$$\text{rank}(A) = k \Rightarrow \text{rank}(A) = \text{rank}(A^H)$$

The following is similar to problem(a), we can show that  $N(A^H) = N(AA^H)$ . For any  $A \in \mathbb{C}^{m \times n}$ , if  $x \in N(A^H)$ , then

$$A^H x = 0 \Rightarrow AA^H x = 0 \Rightarrow x \in N(AA^H)$$

Conversely, if  $x \in N(AA^H)$ , then

$$AA^H x = 0 \Rightarrow xAA^H x = 0 \Rightarrow (A^H x)^H A^H x = 0$$

Suppose  $[A^H x]_{jk} = a_{jk} + b_{jk}i$ , similar to problem3-(c), it is easy to get that if  $(A^H x)^H A^H x = 0$ , then  $A^H x = 0$ . So if  $x \in N(AA^H)$ , then  $x \in N(A^H)$ . In conclusion,  $N(A^H) = N(AA^H)$  and  $\dim[N(A^H)] = \dim[N(AA^H)]$ . Since

$$\dim[R(AA^H)] + \dim[N(AA^H)] = m$$

$$\dim[R(A^H)] + \dim[N(A^H)] = m$$

So

$$\dim[R(A^H)] = \dim[R(AA^H)] \Rightarrow \text{rank}[A] = \text{rank}[A^H] = \text{rank}[AA^H]$$

## Problem 6: Left and Right Inverses

### Solution

(a) For any  $x \in N(A^T A)$ ,

$$A^T A x = 0 \Rightarrow x^T A^T A x = 0 \Rightarrow (Ax)^T Ax = 0 \Rightarrow Ax = 0$$

Since  $A$  is full-rank and tall,  $\text{rank}(A) = n$ . Then

$$\dim[N(A)] = n - \dim[R(A)] = n - \text{rank}(A) = 0$$

So for  $Ax = 0$ , there must be a unique solution that is  $x = 0$ , since  $\dim[N(A)] = 0$ . Hence

$$\dim[N(A^T A)] = 0 \Rightarrow \dim[R(A^T A)] = n$$

So  $A^T A$  is nonsingular.

(b)

$$\begin{aligned}(A^T A)^{-1} A^T A &= A^{-1} (A^T)^{-1} A^T A \\ &= A^{-1} I A \\ &= I\end{aligned}$$

So  $(A^T A)^{-1} A^T$  is a left inverse of a full-rank tall matrix  $A$ .

(c) Suppose

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then

$$\begin{aligned}A_1 A &= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I \\ A_2 A &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I\end{aligned}$$

So  $A$  doesn't have unique left inverse.

(d) Similar to problem (a), for any  $x \in N(AA^T)$ ,

$$AA^T x = 0 \Rightarrow x^T AA^T x = 0 \Rightarrow (A^T x)^T A^T x = 0 \Rightarrow A^T x = 0$$

Since  $A$  is full-rank and fat,  $\text{rank}(A) = m$ . Then

$$\dim[N(A^T)] = m - \dim[R(A^T)] = m - \text{rank}(A^T) = 0$$

So for  $A^T x = 0$ , there must be a unique solution that is  $x = 0$ , since  $\dim[N(A^T)] = 0$ .  
Hence

$$\dim[N(AA^T)] = 0 \Rightarrow \dim[R(AA^T)] = m$$

So  $A^T A$  is nonsingular.

(e)

$$\begin{aligned}AA^T(AA^T)^{-1} &= AA^T(A^T)^{-1}A^{-1} \\ &= AIA^{-1} \\ &= I\end{aligned}$$

So  $A^T(AA^T)^{-1}$  is a right inverse of a full-rank tall matrix  $A$ .

(f) Suppose

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then

$$AA_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = I$$

$$AA_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = I$$

So  $A$  doesn't have unique right inverse.