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Assignment #4

Problem 1: Moore–Penrose Pseudoinverse

Solution

(a) According to the difinition,

$$AA^+A = A \;\Rightarrow$$

(b) Denote $(A^TA)^{-1}A^T$ as A^+. Because matrix A is tall, hence $n \leq m$.

$$A^+A = (A^TA)^{-1}A^TA = (A^TA)^{-1}(A^TA) = I$$

So $(A^TA)^{-1}A^T$ a left inverse of matrix A.

$$AA^+A = A(A^+A) = AI = A$$

$$A^+AA^+ = (A^+A)A^+ = IA^+ = A^+$$

$A^+A = I$, so A^+A is symmetric. Meanwhile,

$$(AA^+)^T = (A^+)^TA^T = [(A^TA)^{-1}A^T]^TA^T = A(A^TA)^{-1}A^T = AA^+$$

So AA^+ is symmetric. In conclusion, $(A^TA)^{-1}A^T$ is the pseudoinverse and a left inverse of matrix A.

(c) Denote $A^T(AA^T)^{-1}$ as A^+.

$$AA^+ = AA^T(AA^T)^{-1} = (AA^T)(AA^T)^{-1} = I$$

So $A^T(AA^T)^{-1}$ is a right inverse of matrix A.

$$AA^+A = (AA^+)A = IA = A$$

$$A^+AA^+ = A^+(AA^+) = A^+I = A^+$$

$AA^+ = I$, so AA^+ is symmetric. Meanwhile,

$$(A^+A)^T = A^T(A^+)^T = A^T[A^T(AA^T)^{-1}]^T = A^T(AA^T)^{-1}A = A^+A$$

so A^+A is symmetric. In conclusion, $A^T(AA^T)^{-1}$ is the pseudoinverse and a right inverse of matrix A.

(d)

$$AA^{-1}A = IA = A \; and \; A^{-1}AA^{-1} = IA^{-1} = A^{-1}$$

Also $AA^{-1} = A^{-1}A = I$ is symmetric. So in conclusion, A^{-1} is the pseudoinverse of a full-rank square matrix A.

(e) For a projection matrix A, $A^2 = A$ and $A^T = A$. Hence

$$AAA = AA = A$$

Because $A^T = A$, so AA is symmetric. In conclusion, A is the pseudoinverse of itself for a projection matrix A.

(f)

$$A^T(A^+)^T A^T = [AA^+A]^T = A^T$$
$$(A^+)^T A^T (A^+)^T = [A^+AA^+]^T = (A^+)^T$$

Meanwhile,

$$[A^T(A^+)^T]^T = A^+A \;\Rightarrow\; \textit{symmetric}$$
$$[(A^+)^T A^T]^T = AA^+ \;\Rightarrow\; \textit{symmetric}$$

So in conclusion, $(A^T)^+ = (A^+)^T$.

(g) i.

(h)

(i) First, both AA^+ and A^+A are symmetric. Second,

$$P^2 = AA^+AA^+ = AA^+$$

$$Q^2 = A^+AA^+A = A^+A$$

Hence P and Q are projection matrix.

(j) Recall the result of problem 5 in Homework 3, $y = Px$ is the projection of x onto $R(P)$. For $\forall y \in R(P)$, there must exists $x \in \mathbb{R}^m$ S.T.

$$y = AA^+x \;\Rightarrow\; y = A(A^+x)$$

Hence for $\forall y \in R(P)$, there must exists $z = A^+x \in \mathbb{R}^n$, S.T. $y = Az$. Hence $R(P) = R(A)$. So $y = Px$ is the projection of x onto $R(A)$.

Similarly, $y = Qx$ is the projection of x onto $R(Q)$. For $\forall y \in R(Q)$, there must exists $x \in \mathbb{R}^n$ S.T.

$$y = A^+Ax \;\Rightarrow\; y = A^+(Ax)$$

Hence for $\forall y \in R(Q)$, there must exists $z = Ax \in \mathbb{R}^m$, S.T. $y = A^+z$. Hence $R(Q) = R(A^+) = R(A^T)$. So $y = Qx$ is the projection of x onto $R(A^T)$.

(k) The solution x^* must satiffies that Ax^* is the orthogonal projection of b onto $R(A)$. Recall the result in problem (j), the projection matrix onto $R(A)$ is $P = AA^+$, hence

$$Ax^* = AA^+b \;\Rightarrow\; x^* = A^+b$$

(l)

Problem 2: Eigenvalues

Solution

(a) The characteristic polynomial of A is

$$p(\lambda) = det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$

Hence

$$p(\lambda = 0) = (-1)^n det(A) = (-1)^n \lambda_1 \lambda_2 \dots \lambda_n$$

So $det(A) = \lambda_1 \lambda_2 \dots \lambda_n$.

(b) Suppose matrix A is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{bmatrix}$$

Meanwhile

$$\lambda I - A^T = \begin{bmatrix} \lambda - a_{11} & -a_{21} & \dots & -a_{n1} \\ -a_{12} & \lambda - a_{22} & \dots & -a_{n2} \\ \vdots & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & \dots & \lambda - a_{nn} \end{bmatrix}$$

According to the definition of determinant, we can easily see that $det(\lambda I - A) = det(\lambda I - A^T)$. A^T and A have the same characteristic polynomial. Hence the eigenvalues of A^T and A are the same.

(c)

Problem 3: Trace

Solution

(a) The

Problem 4: More on Eigenvalues

Solution

(a) The

Problem 5: Limit

Solution

(a) The