

Problem 1: Convolution as Linea Map

Solution

- (a) Obviously the size of matrix T must be $(N + 1) \times (N + 1)$. Let $T_{i1}, T_{i2}, \dots, T_{i(N+1)}$ be the i -th row of matrix T . Since $y = Tx$, we can get

$$v(i - 1) = \sum_{j=1}^{N+1} T_{ij}u(j - 1) = T_{i1}u(0) + T_{i2}u(1) + \dots + T_{i(N+1)}u(N) \quad (1)$$

Accoring to the convolution function,

$$v(i - 1) = \sum_{k=-\infty}^{+\infty} h(k)u(i - 1 - k) = \sum_{k=(i-1-N)}^{i-1} h(k)u(i - 1 - k) \quad (2)$$

Comparing function (1) and function (2), it is easy to conclude that

$$T_{ij} = h(i - j), \text{ where } 1 \leq i \leq (N + 1), 1 \leq j \leq (N + 1)$$

So the matrix is a matrix where each element could be represented as $T_{ij} = h(i - j)$, where $1 \leq i \leq (N + 1), 1 \leq j \leq (N + 1)$.

- (b) The structure of matrix T could be described as follow:

$$T_{i,j} = T_{i+1,j+1}$$

Problem 2: Affine Funtion

Solution

- (a) *Proof.* For any $\alpha, \beta \in \mathbb{R}$ and any $x, y \in \mathbb{R}^n$

$$\alpha f(x) + \beta f(y) = \alpha(Ax + b) + \beta(Ay + b) = \alpha Ax + \beta Ay + (\alpha + \beta)b = \alpha Ax + \beta Ay + b$$

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) + b = \alpha Ax + \beta Ay + (\alpha + \beta)b = \alpha Ax + \beta Ay + b$$

Hence, $\alpha f(x) + \beta f(y) = f(\alpha x + \beta y)$. So function $f(x) = Ax + b$ is affine. \square

- (b) *Proof.* First we can show that b is unique. Because $f(0) = b$, so b must be unique, otherwise $f(0)$ will be mapped as different values in \mathbb{R}^m , which conflicts with the function definition. Then we can show that A is unique. Let function

$$g(x) = f(x) - f(0) = Ax + b - b = Ax$$

Suppose b_1, b_2, \dots, b_n are the basis of \mathbb{R}^n and $B = \sum_{i=1}^n \alpha_i b_i$, $\alpha_i \in \mathbb{R}$.

$$g(B) = g\left(\sum_{i=1}^n \alpha_i b_i\right) = A\left(\sum_{i=1}^n \alpha_i b_i\right) = \sum_{i=1}^n \alpha_i A b_i = \sum_{i=1}^n \alpha_i g(b_i)$$

Suppose $b_i = e_i$, $i = 1, 2, \dots, n$, then $g(B)$ above could be represented as

$$\begin{aligned} g(B) &= \begin{bmatrix} g(b_1) & g(b_2) & \dots & g(b_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix} \\ &= \begin{bmatrix} g(e_1) & g(e_2) & \dots & g(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \\ &= Ax \end{aligned}$$

where

$$x \in \mathbb{R}^n.$$

Hence A is unique. So any affine function f could be represented uniquely as $f(x) = Ax + b$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. \square

Problem 3: Matrix Multification

Solution

(a) Suppose

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq 0 \\ B &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0 \end{aligned}$$

Then

$$AB = 0$$

Hence the statement is incorrect.

(b) Suppose

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$$

Then

$$A^2 = 0$$

Hence the statement is incorrect.

(c) Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then

$$A^T A = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 \\ \vdots & \vdots & & \vdots \\ a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \end{bmatrix}$$

So if $A^T A = 0$, then $a_{ij} = 0$, $1 \leq i \leq n$, $1 \leq j \leq n$. $A = 0$. So the statement is correct.

Problem 4: Linear Maps and Differentiation of polynomials

Solution

(a) For any $p_1(x)$, $p_2(x) \in \mathcal{P}_n$,

$$T(p_1(x) + p_2(x)) = \frac{d(p_1(x) + p_2(x))}{dx} = \frac{dp_1(x)}{dx} + \frac{dp_2(x)}{dx} = T(p_1(x)) + T(p_2(x)) \quad (3)$$

$$T(\alpha p_1(x)) = \frac{d(\alpha p_1(x))}{dx} = \alpha \frac{dp_1(x)}{dx} = \alpha T(p_1(x)) \quad (4)$$

Since equations (3) and (4) are valid, T is linear.

(b) For any $p(x) \in \mathcal{P}_n$, by using $\{1 \ x \ x^2 \ \dots \ x^n\}$ as basis, $p(x)$ could be represented as

$$p(x) = \sum_{i=0}^n \alpha_i x^i = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Similarly

$$T(p(x)) = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix}$$

So we can find a matrix $A \in \mathbb{R}^{(n+1) \times (n+1)}$ that transforms the $p(x)$ coefficient matrix to $T(p(x))$ coefficient matrix. It means that

$$\begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ \vdots \\ n\alpha_n \\ 0 \end{bmatrix} = A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Obviously $\text{rank}(A) = n$.

Problem 5: Rank of AA^T

Solution

(a) For any $A \in \mathbb{R}^{m \times n}$,

$$\dim[R(AA^T)] + \dim[N(AA^T)] = m$$

$$\dim[R(A^T)] + \dim[N(A^T)] = m$$

Since $\text{rank}(A) = \text{rank}(A^T) = \dim[R(A^T)]$, so if we can show that $\dim[N(AA^T)] = \dim[N(A^T)]$, then $\text{rank}(A) = \text{rank}(AA^T)$ is valid. The following will proof $\dim[N(AA^T)] = \dim[N(A^T)]$.

For any $x \in N(A^T)$,

$$A^T x = 0 \Rightarrow A(A^T x) = 0 \Rightarrow AA^T x = 0$$

So for any $x \in N(A^T)$, x also satisfies $x \in N(AA^T)$. Conversely, for any $x \in N(AA^T)$,

$$AA^T x = 0 \Rightarrow x(AA^T x) = 0 \Rightarrow xAA^T x = 0 \Rightarrow (A^T x)^T A^T x = 0$$

According to the conclusion of problem3 (c), $A^T x = 0$. It means that for any $x \in N(AA^T)$, $x \in N(A^T)$. Hence $N(A^T) = N(AA^T)$ and $\dim[N(AA^T)] = \dim[N(A^T)]$. Further we can get

$$\dim[R(AA^T)] = m - \dim[N(AA^T)] = m - \dim[N(A^T)] = \dim[R(A^T)]$$

Since $\text{rank}(A) = \text{rank}(A^T) = \dim[R(A^T)]$ and $\text{rank}(AA^T) = \dim[R(AA^T)]$, So

$$\text{rank}(A)\text{rank}(AA^T)$$

(b) The statement is invalid. Suppose

$$A = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 1 \\ -i & -i \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Obviously $\text{rank}(AA^T) \neq \text{rank}(A)$.

(c) Similarly to problem(a), we can show that $N(A^H) = N(AA^H)$. For any $A \in \mathbb{C}^{m \times n}$, if $x \in N(A^H)$, then

$$A^H x = 0 \Rightarrow AA^H x = 0 \Rightarrow x \in N(AA^H)$$

Conversely, if $x \in N(AA^H)$, then

$$AA^H x = 0 \Rightarrow xAA^H x = 0 \Rightarrow (A^H x)^H A^H x = 0$$

Suppose $[A^H x]_{jk} = a_{jk} + b_{jk}i$, it is easy to get that $[(A^H x)^H A^H x]_{jk} = a_{jk}^2 + b_{jk}^2$. Hence if $(A^H x)^H A^H x = 0$, then $A^H x = 0$. So if $x \in N(AA^H)$, then $x \in N(A^H)$. In conclusion, $N(A^H) = N(AA^H)$ and $\dim[N(A^H)] = \dim[N(AA^H)]$. Since

$$\dim[R(AA^H)] + \dim[N(AA^H)] = m$$

$$\dim[R(A^H)] + \dim[N(A^H)] = m$$

So

$$\dim[R(A^H)] = \dim[R(AA^H)] \Rightarrow \text{rank}[A] = \text{rank}[A^H] = \text{rank}[AA^H]$$

Problem 6: Left and Right Inverses

Solution

(a)