Homework 2 Solutions, MAE281A 2016

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1

Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3] d\tau \tag{1}$$

where $k_1 \geq 0$, $k_2 \geq 0$, $k_3 \geq 0$, and $\alpha > k_2$. Using the Gronwall-Bellman inequality, show that

$$y(t) \leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\left[1 - e^{-(\alpha-k_2)(t-t_0)}\right] \tag{2}$$

Solution :

Introduce $z(t) = y(t)e^{\alpha(t-t_0)}$. Then by (1), we have

$$z(t) \leq k_1 + \frac{k_3}{\alpha}\left(e^{\alpha(t-t_0)} - 1\right) + \int_{t_0}^{t} k_2 z(\tau) d\tau. \tag{3}$$

Define $\lambda(t) = k_1 + \frac{k_3}{\alpha}\left(e^{\alpha(t-t_0)} - 1\right)$ and $\mu = k_2$. Applying Gronwall-Bellman Inequality to $z(t)$ ($\because z(t) \geq 0$), we have

$$\begin{aligned} z(t) \leq& \lambda(t) + \int_{t_0}^{t} \mu\lambda(\tau) e^{\int_\tau^t \mu ds} d\tau \\ =& k_1 + \frac{k_3}{\alpha}\left(e^{\alpha(t-t_0)} - 1\right) \\ &+ k_2\left\{\left(k_1 - \frac{k_3}{\alpha}\right)\int_{t_0}^{t} e^{k_2(t-\tau)} d\tau + \frac{k_3}{\alpha}\int_{t_0}^{t} e^{(\alpha-k_2)\tau + k_2 t - \alpha t_0} d\tau\right\} \end{aligned} \tag{4}$$

Calculating the integration,

$$\int_{t_0}^{t} e^{k_2(t-\tau)} d\tau = \frac{1}{k_2}\left(e^{k_2(t-t_0)} - 1\right), \tag{5}$$

$$\int_{t_0}^{t} e^{(\alpha-k_2)\tau + k_2 t - \alpha t_0} d\tau = \frac{1}{\alpha - k_2}\left(e^{\alpha(t-t_0)} - e^{k_2(t-t_0)}\right) \tag{6}$$

Substituting (5) and (6) into (4), we have

$$z(t) \leq \frac{k_3}{\alpha - k_2} e^{\alpha(t-t_0)} + \left(k_1 - \frac{k_3}{\alpha - k_2}\right) e^{k_2(t-t_0)} \tag{7}$$

Recalling $y(t) = z(t)e^{-\alpha(t-t_0)}$, we obtain (2) which completes the proof.

2

Using the comparison principle, show that if v, l_1, and l_2 are functions that satisfy

$$\dot{v} \le -cv + l_1(t)v + l_2(t), \quad v(0) \ge 0 \tag{8}$$

and if $c > 0$, then

$$v(t) \le (v(0)e^{-ct} + ||l_2||_1)e^{||l_1||_1}. \tag{9}$$

Suppose $l_1 > 0$. Using Gronwall's lemma, show that

$$v(t) \le (v(0)e^{-ct} + ||l_2||_1)(1 + ||l_1||_1 e^{||l_1||_1}) \tag{10}$$

Which of the two bound is less conservative?

Notes for inequalities

- In general, $|a+b| \le |a| + |b|$. (Triangle inequality).
- If $f_1(t) \le f_2(t)$ for $\forall t \ge 0$, then $\int_0^t f_1(s)ds \le \int_0^t f_2(s)ds$.
- In general, $\int_0^t f(s)g(s)ds \le \int_0^t |f(s)||g(s)|ds$ is true, but $\int_0^t f(s)g(s)ds \le \int_0^t f(s)|g(s)|ds$ is not true when the sign of $f(t)$ is not sure.
- In general, $\int_0^t f(s)ds \le \int_0^t |f(s)|ds \le \int_0^\infty |f(s)|ds$.

Solution :

(Step 1) To apply comparison principle, let $u(t)$ be a function satisfying the following

$$\dot{u}(t) = -cu(t) + l_1(t)u(t) + l_2(t), \quad u(0) = v(0) \tag{11}$$

Then, because this is the form of LTV system, we can write the solution of $u(t)$ as

$$u(t) = e^{-ct + \int_0^t l_1(s)ds} u(0) + \int_0^t e^{-c(t-\tau) + \int_0^{t-\tau} l_1(s)ds} l_2(s)ds \tag{12}$$

The second term satisfies (see "Notes for inequalities" as a reference),

$$
\begin{aligned}
\int_0^t e^{-c(t-\tau)+\int_0^{t-\tau} l_1(s)ds} l_2(s)ds &\le \int_0^t \left| e^{-c(t-\tau)+\int_0^{t-\tau} l_1(s)ds} \right| |l_2(s)|ds \\
&= \int_0^t e^{-c(t-\tau)+\int_0^{t-\tau} l_1(s)ds} |l_2(s)|ds \\
&\le \int_0^t e^{\int_0^{t-\tau} l_1(s)ds} |l_2(s)|ds \quad (\because 0 \le e^{-c(t-\tau)} \le 1) \\
&\le \int_0^t e^{||l_1||_1} |l_2(s)|ds \quad \left(\because \int_0^{t-\tau} l_1(s)ds \le \int_0^{\infty} |l_1(s)|ds\right) \\
&\le e^{||l_1||_1} ||l_2||_1 \qquad (13)
\end{aligned}
$$

In addition, $\int_0^t l_1(s)ds \le \int_0^\infty |l_1(s)|ds$, so (12) is bounded by

$$u(t) \le (e^{-ct}u(0) + ||l_2||_1)e^{||l_1||_1} \qquad (14)$$

By comparison principle, we have $v(t) \le u(t)$ with $u(0) = v(0)$, which leads to (9).

(Step 2) Introduce $z(t) = v(t)e^{ct}$. Taking time derivative and using (8), we have

$$\dot{z}(t) \le l_1(t)z(t) + l_2(t)e^{ct} \qquad (15)$$

Integrating from 0 to t and taking some bounds,

$$
\begin{aligned}
z(t) \le& z(0) + \int_0^t l_2(\tau)e^{c\tau}d\tau + \int_0^t l_1(\tau)z(\tau)d\tau \\
\le& z(0) + e^{ct}\int_0^t |l_2(\tau)|d\tau + \int_0^t l_1(\tau)z(\tau)d\tau \quad (\because e^{c\tau} \le e^{ct}, \quad \forall \tau \le t) \\
\le& z(0) + e^{ct}||l_2||_1 + \int_0^t l_1(\tau)z(\tau)d\tau \qquad (16)
\end{aligned}
$$

Define $\lambda(t) = z(0) + e^{ct}||l_2||_1$. Applying Gronwall-Bellman Inequality to $z(t)$, we have

$$
\begin{aligned}
z(t) \le& \lambda(t) + \int_0^t \lambda(\tau)l_1(\tau)e^{\int_\tau^t l_1(s)ds}d\tau \\
\le& \lambda(t)\left(1 + \int_0^t |l_1(\tau)|e^{\int_\tau^t l_1(s)ds}d\tau\right) \quad (\because \lambda(\tau) \le \lambda(t), \quad \forall \tau \le t) \\
\le& \lambda(t)\left(1 + ||l_1||_1 e^{||l_1||_1}\right) \quad \left(\because \int_\tau^t l_1(s)ds \le ||l_1||_1 \quad 0 \le \forall \tau \le t\right) \quad (17)
\end{aligned}
$$

Recalling $v(t) = z(t)e^{-ct}$, we obtain (10).
Set $x = ||l_1||_1 \geq 0$ and $f(x)$ be a subtraction of R.H.S of (9) from R.H.S. of (10) (R.H.S. stands for "right hand side"), i.e.

$$f(x) = (v(0)e^{-ct} + ||l_2||_1)(1 + xe^x - e^x). \tag{18}$$

Then, $f(0) = 0$. Taking derivative, we have

$$f'(x) = (v(0)e^{-ct} + ||l_2||_1)xe^x \geq 0, \quad \forall x \geq 0. \tag{19}$$

Hence $f(x) \geq 0$ for $\forall x \geq 0$, which concludes that (9) is less conservative than (10).

3

Consider the system

$$\dot{x} = -cx + y^{2m} x \cos^2(x), \tag{20}$$

$$\dot{y} = -y^3. \tag{21}$$

Using either Gronwall's inequality or the comparison principle, show that

a) $x(t)$ is bounded for all $t \geq 0$ whenever $c = 0$ and $m > 1$.
b) $x(t) \to 0$ as $t \to \infty$ whenever $c > 0$ and $m = 1$.

Solution : The solution of (21) is obtained by

$$y^2 = (2t + a)^{-1}, \quad a = y_0^{-2}, \tag{22}$$

Introduce $z = xe^{ct}$. Taking time derivative and using (20), we get

$$\begin{aligned} \dot{z} &= \dot{x}e^{ct} + cxe^{ct} = y^{2m} x \cos^2(x) e^{ct} \\ &= y^{2m} \cos^2(x) z \end{aligned} \tag{23}$$

Taking integration from 0 to t and substituting (22), we obtain

$$z(t) = z(0) + \int_0^t (2\tau + a)^{-m} \cos^2(x(\tau)) z(\tau) d\tau \tag{24}$$

Taking absolute value on (24) and considering the bound $\cos^2(x) \leq 1$ for $\forall x$, we have the following inequality

$$\begin{aligned} |z(t)| &\leq |z(0)| + \int_0^t (2\tau + a)^{-m} |z(\tau)| d\tau \\ &= \lambda + \int_0^t \mu(\tau) |z(\tau)| d\tau \end{aligned} \tag{25}$$

where we defined $\lambda = |z(0)|$ and $\mu(\tau) = (2\tau + a)^{-m}$. Applying Gronwall-Bellman Inequality to $|z(t)|$ (in the case λ is constant), we have

$$|z(t)| \leq \lambda e^{\int_0^t \mu(\tau) d\tau} = |z(0)| e^{\int_0^t (2\tau + a)^{-m} d\tau} \tag{26}$$

By the definition $z = xe^{ct}$, we have $|x| = |z|e^{-ct}$, and the following inequality of $|x|$ is obtained

$$|x(t)| \leq |x(0)| e^{\int_0^t (2\tau + a)^{-m} d\tau} e^{-ct} \tag{27}$$

a) Take $c = 0$ and $m > 1$. Then, the integration is solved by

$$\int_0^t (2\tau + a)^{-m} d\tau = \frac{1}{2(m-1)} \left\{ a^{-(m-1)} - (2t + a)^{-(m-1)} \right\} \tag{28}$$

Substituting this integration into (27) and $c = 0$, we have

$$|x(t)| \leq |x(0)| \exp \left(\frac{1}{2(m-1)} \left\{ a^{-(m-1)} - (2t + a)^{-(m-1)} \right\} \right) \tag{29}$$

Because the right hand side of the above is monotonically increasing in t, taking $t \to \infty$, we have

$$|x(t)| \leq |x(0)| \exp \left(\frac{a^{-2(m-1)}}{2(m-1)} \right) \tag{30}$$

Therefore, $|x(t)|$ is bounded.

b) Take $c > 0$ and $m = 1$. Then, the integration is solved by

$$\int_0^t (2\tau + a)^{-1} d\tau = \ln \left(1 + \frac{2t}{a} \right) \tag{31}$$

Substituting this integration into (27), we have

$$|x(t)| \leq |x(0)| \left(1 + \frac{2t}{a} \right) e^{-ct} \tag{32}$$

Therefore, $|x(t)| \to 0$ as $t \to \infty$, and thus $x(t) \to 0$.

4

Consider the system

$$
\begin{aligned}
\dot{x} &= -x + yx\sin(x), && (33)\\
\dot{y} &= -y + zy\sin(y), && (34)\\
\dot{z} &= -z && (35)
\end{aligned}
$$

Using Gronwall's lemma (twice), show that

$$|x(t)| \le |x_0| e^{|y_0| e^{|z_0|}} e^{-t}, \quad \forall t \le 0 \tag{36}$$

Solution : The solution of (35) is obtained by

$$z(t) = z(0)e^{-t} \tag{37}$$

(Step 1) Introduce $v = ye^t$. Taking time derivative and substituting (34), we get

$$
\begin{aligned}
\dot{v} &= \dot{y}e^t + ye^t = zy\sin(y)e^t\\
&= \sin(y)zv && (38)
\end{aligned}
$$

Taking integration from 0 to t and substituting (37), we obtain

$$v(t) = v(0) + \int_0^t \sin(y(\tau))z(0)e^{-\tau}v(\tau)d\tau \tag{39}$$

Considering the bound $|\sin(y)| \le 1$ for $\forall y$, we have the following inequality

$$
\begin{aligned}
|v(t)| &\le |v(0)| + \int_0^t |z(0)|e^{-\tau}|v(\tau)|d\tau\\
&= \lambda_1 + \int_0^t \mu_1(\tau)|v(\tau)|d\tau && (40)
\end{aligned}
$$

where we defined $\lambda_1 = |v(0)|$ and $\mu_1(\tau) = |z(0)|e^{-\tau}$. Applying Gronwall-Bellman Inequality to $|v(t)|$ (in the case λ is constant), we have

$$
\begin{aligned}
|v(t)| \le \lambda_1 e^{\int_0^t \mu_1(\tau)d\tau} = |v(0)|e^{\int_0^t |z(0)|e^{-\tau}d\tau} &= |v(0)|e^{|z(0)|(1-e^{-t})}\\
&\le |v(0)|e^{|z(0)|} && (41)
\end{aligned}
$$

By the definition $v = ye^t$, we have $|y| = |v|e^{-t}$, and thus the following inequality of $|y|$ is obtained

$$|y(t)| \leq |y(0)|e^{|z(0)|}e^{-t} \tag{42}$$

(Step 2) Introduce $w = xe^t$. Taking time derivative and substituting (33), we get

$$\begin{aligned} \dot{w} &= \dot{x}e^t + xe^t = yx\sin(x)e^t \\ &= \sin(x)yw \end{aligned} \tag{43}$$

Taking integration from 0 to t, we obtain

$$w(t) = w(0) + \int_0^t \sin(x(\tau))y(\tau)w(\tau)d\tau \tag{44}$$

Considering the bound $|\sin(x)| \leq 1$ for $\forall x$ and (42), we have the following inequality

$$\begin{aligned} |w(t)| &\leq |w(0)| + \int_0^t |\sin(x(\tau))||y(\tau)||w(\tau)|d\tau \\ &\leq |w(0)| + \int_0^t |y(0)|e^{|z(0)|}e^{-\tau}|w(\tau)|d\tau \\ &= \lambda_2 + \int_0^t \mu_2(\tau)|w(\tau)|d\tau \end{aligned} \tag{45}$$

where we defined $\lambda_2 = |w(0)|$ and $\mu_2(\tau) = |y(0)|e^{|z(0)|}e^{-\tau}$. Applying Gronwall-Bellman Inequality to $|w(t)|$ (in the case λ is constant), we have

$$\begin{aligned} |w(t)| \leq \lambda_2 e^{\int_0^t \mu_2(\tau)d\tau} = |w(0)|e^{\int_0^t |y(0)|e^{|z(0)|}e^{-\tau}d\tau} &= |w(0)|e^{|y(0)|e^{|z(0)|}(1-e^{-t})} \\ &\leq |w(0)|e^{|y(0)|e^{|z(0)|}} \end{aligned} \tag{46}$$

By the definition $w = xe^t$, we have $|x| = |w|e^{-t}$, and thus the following inequality of $|x|$ is obtained

$$|x(t)| \leq |x(0)|e^{|y(0)|e^{|z(0)|}}e^{-t} \tag{47}$$

which completes the proof of (36).