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Input-to-state stability

1. Show that the following system is ISS and determine its gain function:

$$\dot{x} = -x^3 + xu. \tag{1}$$

Solution 1 *Let*

$$\begin{aligned} V &= \frac{1}{2}x^2, \qquad \alpha_1(|x|) = \alpha_2(|x|) \\ \dot{V} &= -x^2(x^2 - u) \leq -\frac{1}{2}x^4 - x^2\Big(\frac{1}{2}x^2 - |u|\Big). \end{aligned}$$

Since $\dot{V} < 0$ whenever $|x| > \sqrt{2|u|}$, this system is ISS with the gain function $\gamma(r) = \sqrt{2r}$.

2. Show that the following system is ISS and determine its gain function:

$$\dot{x} = -x + u^3. \tag{2}$$

Solution 2 *Consider*

$$\begin{aligned} V &= \frac{1}{2}x^2, \qquad \alpha_1(|x|) = \alpha_2(|x|) \\ \dot{V} &= -x^2 + xu^3. \end{aligned}$$

By using Young's inequality with $p = q = 2$, we have

$$\dot{V} \leq -\frac{1}{2}x^2 + \frac{1}{2}u^6,$$

so $\dot{V}$ is negative for $|x| \geq |u|^3$. This system is ISS with the gain function $\gamma(r) = r^3$.

3. Show that the following system is ISS and guess its gain function:

$$\dot{x} = -x^3 + xy \tag{3}$$

$$\dot{y} = -y + u^3. \tag{4}$$

Solution 3 *This system is a combination of system (1) and (2). From the first problem we conclude that subsystem (3) is ISS w.r.t. variable y with the gain function $\gamma_1(r) = \sqrt{2r}$. Also subsystem (4) is ISS w.r.t. input u with the gain function $\gamma_2(r) = r^3$. According to Lemma C4 from [KKK] the entire system is ISS from u to (x, y) with the gain function $\gamma(r) = \gamma_1\Big(\gamma_2(r)\Big) + \gamma_2(r) = \sqrt{2r^3} + r^3$.*

4. Consider the system

$$\begin{aligned}\dot{x} &= -x + y^3 &(5)\\ \dot{y} &= -y - \frac{x}{\sqrt{1+x^2}} + z^2 &(6)\\ \dot{z} &= -z + u. &(7)\end{aligned}$$

Show that this system is ISS using the Lyapunov function

$$V = \sqrt{1+x^2} - 1 + \frac{1}{4}y^4 + \frac{1}{2}z^8. \tag{8}$$

Solution 4 *We have*

$$\dot{V} = -\frac{x^2}{\sqrt{1+x^2}} - y^4 - 4z^8 + z^2y^3 + 4z^7u,$$

by applying Young's inequality twice, with $(p = 8/7, q = 8)$ and $(p = 4/3, q = 4)$ we get

$$\begin{aligned}\dot{V} &\leq -\frac{x^2}{\sqrt{1+x^2}} - \frac{1}{4}y^4 - \frac{1}{4}z^8 + \frac{1}{2}u^8\\ &\leq -\rho(|x|) - \rho(|y|) - \rho(|z|) + \frac{1}{2}u^8,\end{aligned}$$

where

$$\rho(r) = \min\left\{\frac{r^2}{\sqrt{1+r^2}}, \frac{r^4}{4}, \frac{r^8}{4}\right\}.$$

Note now that $\gamma(a+b+c) \leq \gamma(3a) + \gamma(3b) + \gamma(3c)$ for all $a, b, c > 0$ and any γ in class $\mathcal{K}$. Hence,

$$\dot{V} \leq -\rho\left(\frac{1}{3}(|x| + |y| + |z|)\right) + \frac{u^8}{2}.$$

According to Theorem C3 from [KKK] this system is ISS, with a gain function $\gamma_3(r) = 3\rho^{-1}\left(\frac{1}{2}r^8\right)$.

5. Show that the following system is ISS

$$\begin{aligned}\dot{x} &= -x + x^{1/3}y + p^2 &(9)\\ \dot{y} &= -y - x^{4/3} + p^3 &(10)\\ \dot{p} &= -p + u. &(11)\end{aligned}$$

Solution 5 *Consider*

$$\begin{aligned}V_1 &= \frac{1}{2}\left(x^2 + y^2\right)\\ \dot{V}_1 &= -x^2 - y^2 + xp^2 + yp^3\end{aligned}$$

for subsystem (x, y). Using Young's inequality we get

$$\dot{V}_1 = -\frac{1}{2}\left(x^2 + y^2\right) + \frac{p^4 + p^6}{2}$$

which shows that the (x, y) subsystem is ISS w.r.t. p with the gain function $\gamma_1(r) = r^2\sqrt{1+r^2}$. The subsystem p is ISS w.r.t. the input u with the gain function $\gamma_2(r) = r$. According to Lemma C4 from [KKK] this system is ISS. The overall gain from u to (x, y, p) is $\gamma(r) = \gamma_1(\gamma_2(r)) + \gamma_2(r) = r^2\sqrt{1+r^2} + r$.