Homework 6 Solutions, MAE281A 2016

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Barbashin-Krasovskii's Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let V be a functional of x. Then, $x = 0$ is a.s. if

- V is "pdf" and $\dot{V} \leq 0$ for $\forall x \in D$.
- no solution can stay forever in $S := \left\{ x \in D | \dot{V} = 0 \right\}$ other than $x(t) \equiv 0$.

In addition, $x = 0$ is g.a.s. if V is radially unbounded.

La Salle's Invariance Principle

Let Ω be a compact positively invariant set of $\dot{x} = f(x)$, and let V be a functional of x. Suppose

- $\dot{V} \leq 0$ for $\forall x \in D$.
- $E := \left\{ x \in \Omega | \dot{V} = 0 \right\}$, and M be the largest invariant set contained in E.

Then, $x(t) \to M$ as $t \to \infty$ if $x(0) \in \Omega$.

1

Using the Lyapunov function candidate

$$V = \frac{1}{4}x^4 + \frac{1}{2}y^2 + \frac{1}{4}z^4 \tag{1}$$

study the stability of the origin of the system

$$\dot{x} = y, \tag{2}$$
$$\dot{y} = -x^3 - y^3 - z^3, \tag{3}$$
$$\dot{z} = -z + y. \tag{4}$$

Solution

By (1), V is "pdf" and radially unbounded. Taking time derivative and from (2) (4), we obtain

$$\begin{aligned}\dot{V} &= x^3\dot{x} + y\dot{y} + z^3\dot{z} = x^3y + y(-x^3 - y^3 - z^3) + z^3(-z + y) \\ &= -y^4 - z^4,\end{aligned} \tag{5}$$

thus $\dot{V} \leq 0$ for $\forall(x, y, z) \in \mathbb{R}^3$.
Let $S := \left\{(x, y, z) \in \mathbb{R}^3 | \dot{V} = 0\right\} = \{y = 0, z = 0\}$. Then, substituting $y = 0$ and $z = 0$ in (3), we obtain $x = 0$. Therefore, no solution can stay forever in S other than the origin $(x, y, z) \equiv 0$. By Barbashin-Krasovskii's Theorem, we conclude that $(x, y, z) = 0$ is g.a.s.

2

Consider the system

$$
\begin{aligned}
\dot{x} &= -x + yx + z\cos(x), &(6)\\
\dot{y} &= -x^2, &(7)\\
\dot{z} &= -x\cos(x). &(8)
\end{aligned}
$$

a) Determine all the equilibria of the system.
b) Show that the equilibrium $x = y = z = 0$ is globally stable.
c) Show that $x(t) \to 0$ as $t \to \infty$.
d) Show that $z(t) \to 0$ as $t \to \infty$.

Solution

a) Let (x^*, y^*, z^*) be an equilibrium of the system. Then, by (7), we have $x^* = 0$. Substituting $x^* = 0$ into (6), we have $z^* = 0$. y^* is arbitral in this system. Therefore, the equilibria of the system is written as

$$
(x^*, y^*, z^*) = (0, y^*, 0) \tag{9}
$$

with an arbitral point y^*.

b) Let $V(x, y, z)$ be a Lyapunov candidate s.t.

$$
V(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2. \tag{10}
$$

Then, $V(0) = 0$ and $V > 0$ for $\forall(x, y, z) \neq 0$, and thus V is "pdf". In addition, $V \to \infty$ as $|(x, y, z)| \to \infty$, thus V is radially unbounded.
Taking time derivative and from (6)– (8), we obtain

$$
\begin{aligned}
\dot{V} &= x\dot{x} + y\dot{y} + z\dot{z} = x(-x + yx + z\cos(x)) - x^2 y - xz\cos(x)\\
&= -x^2 \leq 0, \quad \forall(x, y, z) &(11)
\end{aligned}
$$

$\dot{V}$ is "nsdf". Therefore, the system is globally stable at the origin $x = y = z = 0$.

c) Let $E := \left\{(x, y, z) \in \mathbb{R}^3 | \dot{V} = 0\right\}$. Then, by (11), we have $E = \{x = 0\}$. By La Salle's theorem, the solution converges to the largest invariant set contained in E, which leads to at least $x(t) \to 0$ as $t \to \infty$.

d) To obtain the largest invariant set $M \in E = \{x = 0\}$, substituting $x = 0$ in (6), we have $z = 0$. Since $(0, y^*, 0)$ is the equillibrium set by problem a), it yields $M = \{x = 0, z = 0\}$. By La Salle's theorem, we can say $x(t) \to 0$ and $z(t) \to 0$ as $t \to \infty$.

3

Which of the state variables of the following system are guaranteed to converge to zero from any initial condition?

$$\dot{x}_1 = x_2 + x_1 x_3, \tag{12}$$
$$\dot{x}_2 = -x_1 - x_2 + x_2 x_3, \tag{13}$$
$$\dot{x}_3 = -x_1^2 - x_2^2. \tag{14}$$

Solution

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $x^* = (x_1^*, x_2^*, x_3^*)$ be an equilibrium of the system. Then, by (14) we have $x_1^* = 0$ and $x_2^* = 0$. This is an equilibrium of (12)– (14) for arbitral x_3^*. Let $V(x)$ be a Lyapunov candidate s.t.

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2. \tag{15}$$

Taking the time derivative and from (12)– (14), we obtain

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 = x_1(x_2 + x_1x_3) + x_2(-x_1 - x_2 + x_2x_3) - x_3(x_1^2 + x_2^2)\\ &= -x_2^2 \le 0, \quad \forall x \end{aligned} \tag{16}$$

Let $E := \left\{(x, y, z) \in \mathbb{R}^3 | \dot{V} = 0\right\}$. Then, by (16), we have $E = \{x_2 = 0\}$. To obtain the largest invariant set $M \in E$, substituting $x_2 = 0$ in (13), we have $x_1 = 0$. Since $(0, 0, x_3^*)$ is the equillibrium set, we have $M = \{x_1 = 0, x_2 = 0\}$. By La Salle's theorem, we can say $x_1(t) \to 0$ and $x_2(t) \to 0$ as $t \to \infty$.