Homework set 1 **MAE281A**

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Linearization & phase portrait

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1. For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium. Use Matlab to compute the eigenvalues.

(1)
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \tfrac{1}{6}x_1^3 - x_2\end{aligned}$$

(2)
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3\end{aligned}$$

(3)
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2(1 + x_1)\\ \dot{x}_2 &= -x_1(1 + x_1)\end{aligned}$$

(4)
$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2\\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

Solution 1

(1)

$$\begin{aligned}0 &= x_2\\ 0 &= -x_1 + \frac{1}{6}x_1^3 - x_2\end{aligned}$$

There are three equilibrium points at $a = (0,0), b = (\sqrt{6}, 0)$, *and* $c = (-\sqrt{6}, 0)$.

$$\begin{aligned}
f(x) &= \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{6}x_1^3 - x_2 \end{bmatrix}\\
\frac{\partial f}{\partial x} &= \begin{bmatrix} 0 & 1 \\ -1 + \frac{1}{2}x_1^2 & -1 \end{bmatrix}\\
\left.\frac{\partial f}{\partial x}\right|_a &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},\ \lambda = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} \Rightarrow (0,0)\ \textit{is a stable focus.}\\
\left.\frac{\partial f}{\partial x}\right|_b &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix},\ \lambda = 1, -2 \Rightarrow (\sqrt{6}, 0)\ \textit{is a saddle point.}
\end{aligned}$$

Similarly, $(-\sqrt{6}, 0)$ *is a saddle point.*

(2)

$$\begin{aligned}0 &= -x_1 + x_2\\ 0 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3\end{aligned}$$

Hence, $x_2 = x_1$ *and*

$$0 = x_1(1.9 + x_1 + 0.1x_1^2) \Rightarrow x_1 = 0, -2.55, or - 7.45.$$

There are three equilibrium points at $a = (0,0), b = (-2.55, -2.55)$, *and* $c = (-7.45, -7.45)$.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \begin{bmatrix} -1 & 1 \\ 0.1 - 2x_1 - 0.3x_1^2 & -2 \end{bmatrix}\\
\left.\frac{\partial f}{\partial x}\right|_a &= \begin{bmatrix} -1 & 1 \\ 0.1 & -2 \end{bmatrix},\ \lambda = -.91, -2.1 \Rightarrow (0,0)\ \textit{is a stable node.}\\
\left.\frac{\partial f}{\partial x}\right|_b &= \begin{bmatrix} -1 & 1 \\ 3.25 & -2 \end{bmatrix},\ \lambda = .37, -3.37 \Rightarrow (-2.55, -2.55)\ \textit{is a saddle point.}\\
\left.\frac{\partial f}{\partial x}\right|_c &= \begin{bmatrix} -1 & 1 \\ -1.87 & -2 \end{bmatrix},\ \lambda = -1.5 \pm j1.18 \Rightarrow (-7.45, -7.45)\ \textit{is a stable focus.}
\end{aligned}$$

(3)

$$
\begin{aligned}
0 &= -x_1 + x_2(1+x_1)\\
0 &= -x_1(1+x_1)
\end{aligned}
$$

At $x_1 = 0, x_2 = 0$. At $x_1 = -1, \dot{x}_1 = 1 \neq 0$. Hence, there is a unique equilibrium point at $a = (0,0)$.

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \begin{bmatrix} -1+x_2 & 1+x_1 \\ -1-2x_1 & 0 \end{bmatrix}\\
\left.\frac{\partial f}{\partial x}\right|_a &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},\ \lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \Rightarrow (0,0)\ \textit{is a stable focus.}
\end{aligned}
$$

(4)

$$
\begin{aligned}
0 &= -x_1^3 + x_2\\
0 &= x_1 - x_2^3\\
x_2 = x_1^3 \Rightarrow x_1(1-x-1^8) = 0 \quad &\Rightarrow \quad x_1 = 0,\ \textit{or}\ x_1^8 = 1
\end{aligned}
$$

The equation $x_1^8 = 1$ has two real roots at $x_1 = 1$, and $x_1 = -1$. Thus, there are three equilibrium points at $a = (0,0), b = (1,1)$, and $c = (-1,-1)$.

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}\\
\left.\frac{\partial f}{\partial x}\right|_a &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \lambda = \pm 1 \Rightarrow (0,0)\ \textit{is a saddle point.}\\
\left.\frac{\partial f}{\partial x}\right|_b &= \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix},\ \lambda = -2, -4 \Rightarrow (1,1)\ \textit{is a stable node.}
\end{aligned}
$$

Similarly $(-1,-1)$ is a stable node.

2. The phase portrait of the following four systems are shown in Figure 1: parts (a), (b), (c), and (d), respectively. Mark the arrowheads and discuss the qualitative behavior of each system.

(1)
$$
\begin{aligned}
\dot{x}_1 &= -x_2\\
\dot{x}_2 &= x_1 - x_2(1 - x_1^2 + 0.1x_1^4)
\end{aligned}
$$

(2)
$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= x_1 + x_2 - 3\tan^{-1}(x_1 + x_2)
\end{aligned}
$$

(3)
$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -(0.5x_1 + x_1^3)
\end{aligned}
$$

(4)
$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -x_2 - \psi(x_1 - x_2)
\end{aligned}
$$

Where $\psi(y) = y^3 + 0.5y$ if $|y| \leq 1$ and $\psi(y) = 2y - 0.5$ if $|y| > 1$.

Solution 2

(1) This system has a unique equilibrium at the origin. The Jacobian at the origin is given by

$$
\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \lambda = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}.
$$

Hence, the origin is a stable focus. The phase portrait is shown in Figure 1(a) with the arrow heads. The direction of the arrow heads can be determined by inspection of the vector field. In particular, since $f_1(x) = -x_2$, we see that f_1 is negative in the upper half of the plane, and positive in the lower half. The system has two limit cycles. The inner limit cycle is unstable, while the outer limit cycle is stable. All trajectories starting inside the inner limit cycle spiral toward the origin. All trajectories starting outside the inner limit cycle approach the stable (outer) limit cycle.

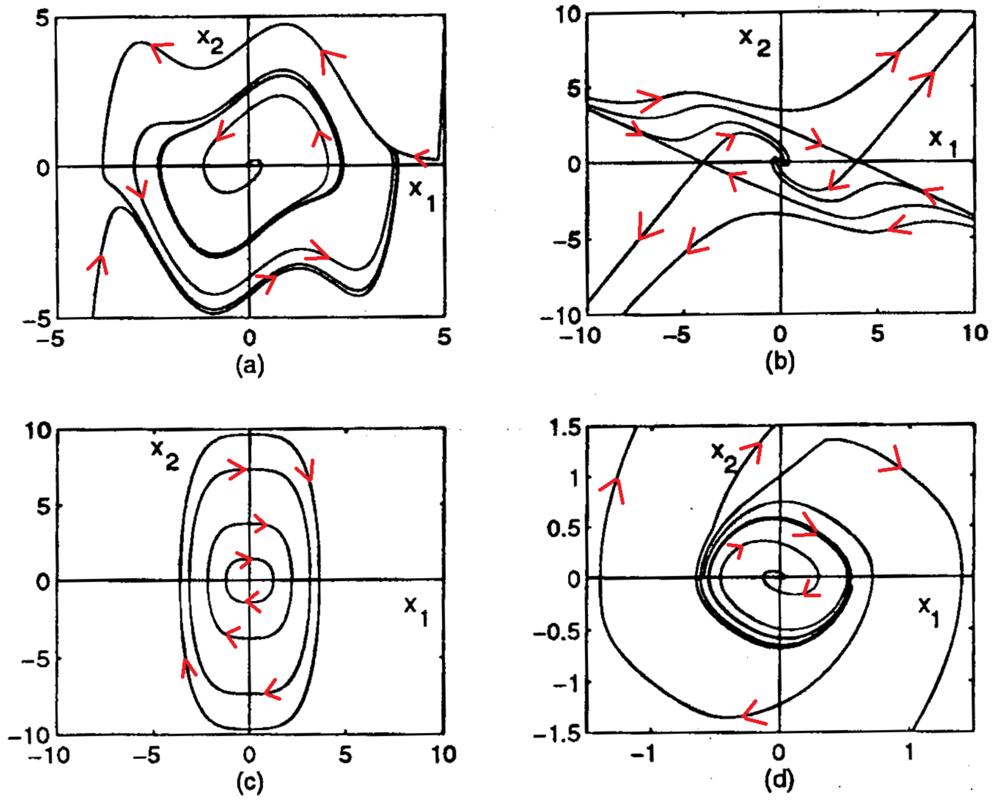

Figure 1: Exercise 2

(2) *This system has three equilibrium points at* $a = (0,0), b = (p,0)$, *and* $c = (-p,0)$, *where* p *is the root of*

$$p - \tan(\frac{p}{3}) = 0$$

which is approximated by $p = 3.9726$. *The Jacobian matrix is*

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \begin{bmatrix} 0 & 1 \\ 1 - \frac{3}{1+(x_1+x_2)^2} & 1 - \frac{3}{1+(x_1+x_2)^2} \end{bmatrix} \\
\left.\frac{\partial f}{\partial x}\right|_a &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \Rightarrow \lambda = -1 \pm j \Rightarrow (0,0) \textit{ is a stable focus.} \\
\left.\frac{\partial f}{\partial x}\right|_{b,c} &= \begin{bmatrix} 0 & 1 \\ .82 & .82 \end{bmatrix} \Rightarrow \lambda = 1.4 \ \& -.58 \Rightarrow b \ \& \ c \textit{ are saddle points.}
\end{aligned}$$

The phase portrait is shown in Figure 1(b) with the arrow heads. The direction of the arrow heads can be determined by inspection of the vector field. In particular, since $f_1(x) = x_2$, *we see that* f_1 *is positive in the upper half of the plane, and negative in the lower half. The stable trajectories of the saddle points form two separatrices which divide the plane into three regions. Trajectories starting in the middle region spiral toward the origin, which those starting in the outer regions approach infinity.*

(3) *This system has one equilibrium point at the origin.*

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix}$$

Linearization fails to determine the type of the equilibrium point. The phase portrait is shown in Figure 1(c) with the arrow heads. The direction of the arrow heads can be determined by inspection *of the vector field. In particular, since $f_1(x) = x_2$, we see that f_1 is positive in the upper half of the plane, and negative in the lower half. All trajectories are closed orbits centered at the origin.From this we see that the origin is a center.*

(4) This system has one equilibrium point at the origin. Since around the origin $|x_1 - x_2| < 1$ then $\psi(y) = y^3 + 0.5y$ is applicable.

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix} \Rightarrow \lambda = -0.25 \pm j0.66 \Rightarrow (0,0) \textit{ is a stable focus.}$$

The phase portrait is shown in Figure 1(d) with the arrow heads. The direction of the arrow heads can be determined by inspection of the vector field. In particular, since $f_1(x) = x_2$, we see that f_1 is positive in the upper half of the plane, and negative in the lower half. The system has an unstable limit cycle. All trajectories inside the limit cycle spiral toward the origin, while all trajectories outside the limit cycle diverge to infinity.