Solutions for HW1

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1 Problem 1

For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium. Use Matlab to compute the eigenvalues.

In order to find the equilibrium points, we need to solve the equation $\dot{x} = f(x) = 0$. In most cases, x represents a vector, which means there are more than one variable in the state space. Then we will linearize the system near each equilibrium points and we only need to reserve the first-order term and ignore the higher orders. We could use another equation to approximate the system near the equilibrium like this:$\dot{x} = Ax$. We could measure the performance of each system near equilibrium points by analyzing matrix A.

1.1 1

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1 + \tfrac{1}{6}x_1^3 - x_2 \end{aligned}$$

For the above system, there are three different equilibrium points:

$$(x_1, x_2) = (0, 0) \quad (x_1, x_2) = (\sqrt{6}, 0) \quad (x_1, x_2) = (-\sqrt{6}, 0)$$

The Taylor expansion for this system is:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 + \frac{1}{2}x_1^2 & -1 \end{pmatrix} x$$

I take different equilibrium points into the above equation and obtain three matrix A:

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of each matrix A_1, A_2, A_3.

1. $\lambda_1 = -\frac{\sqrt{3}}{2}i - 0.5, \lambda_2 = \frac{\sqrt{3}}{2}i - 0.5$
2. $\lambda_1 = -2, \lambda_2 = 1$:
3. $\lambda_1 = -2, \lambda_2 = 1$:

1. The eigenvalues of matrix A_1 are complex values and the real component is smaller than 0, so the type of equilibrium point is stable focus.
2. The eigenvalues of matrix A_2 are real values with one of them larger than 0, so the type of equilibrium point is unstable node.
3. The eigenvalues of matrix A_3 are real values with one of them larger than 0, so the type of equilibrium point is saddle point.

1.2 2

$$\dot{x_1} = -x_1 + x_2$$
$$\dot{x_2} = 0.1x_1 - 2x_2 - x1^2 - 0.1x1^3$$

For the above system, there are three different equilibrium points:

$$(x_1, x_2) = (0,0) \quad (x_1, x_2) = (-7.45, -7.45) \quad (x_1, x_2) = (-2.55, -2.55)$$

The Taylor expansion for this system is:

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ -0.3x_1^2 - 2x_1 + 0.1 & -2 \end{pmatrix} x$$

I take different equilibrium points into the above equation and obtain three matrix A:

$$A_1 = \begin{pmatrix} -1 & 1 \\ 0.1 & -2 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1 & 1 \\ -1.65 & -2 \end{pmatrix} \quad A_3 = \begin{pmatrix} -1 & 1 \\ 3.25 & -2 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of each matrix A_1, A_2, A_3.

1. $\lambda_1 = -0.91, \lambda_2 = -2.09$
2. $\lambda_1 = -1.5 + 1.18i, \lambda_2 = -1.5 - 1.18i$:
3. $\lambda_1 = 0.37, \lambda_2 = -3.37$:

1. The eigenvalues of matrix A_1 are real values with both of them smaller than 0, so the type of equilibrium point is stable node.
2. The eigenvalues of matrix A_2 are complex values and the real component is smaller than 0, so the type of equilibrium point is stable focus.
3. The eigenvalues of matrix A_3 are real values with one of them larger than 0, so the type of equilibrium point is saddle point.

1.3 3

$$\dot{x_1} = -x_1 + x_2(1 + x_1)$$
$$\dot{x_2} = -x_1(1 + x_1)$$

For the above system, there are only one equilibrium points:

$$(x_1, x_2) = (0,0)$$

The Taylor expansion for this system is:

$$\dot{x} = \begin{pmatrix} x_2 - 1 & x_1 + 1 \\ -2x_1 - 1 & 0 \end{pmatrix} x$$

I take the equilibrium point into the above equation and obtain the matrix A:

$$A_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of the matrix A_1.

1. $\lambda_1 = -0.5 + \frac{\sqrt{3}}{2}i, \lambda_2 = -0.5 - \frac{\sqrt{3}}{2}i$:

1. The eigenvalues of matrix A_1 are complex values and the real component is smaller than 0, so the type of equilibrium point is stable focus.

1.4 4

$$\begin{aligned}\dot{x_1} &= -x_1^3 + x_2 \\ \dot{x_2} &= x_1 - x_2^3\end{aligned}$$

For the above system, there are three different equilibrium points in real number(there are some other solutions with complex components, but I think it is not necessary for us to analyze the behaviour near these points because they are not reachable):

$$(x_1, x_2) = (0, 0) \quad (x_1, x_2) = (-1, -1) \quad (x_1, x_2) = (1, 1)$$

The Taylor expansion for this system is:

$$\dot{x} = \begin{pmatrix} -3 * x_1^2 & 1 \\ 1 & -3 * x_2^2 \end{pmatrix} x$$

I take different equilibrium points into the above equation and obtain three matrix A:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \quad A_3 = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of each matrix A_1, A_2, A_3.

1. $\lambda_1 = -1, \lambda_2 = 1$
2. $\lambda_1 = -4, \lambda_2 = -2$:
3. $\lambda_1 = -4, \lambda_2 = -2$:

1. The eigenvalues of matrix A_1 are real values with one of them larger than 0, so the type of equilibrium point is saddle point.
2. The eigenvalues of matrix A_1 are real values with both of them smaller than 0, so the type of equilibrium point is stable node.
3. The eigenvalues of matrix A_1 are real values with both of them smaller than 0, so the type of equilibrium point is stable node.

2 Problem 2

The phase portrait of the following systems. Mark the arrowheads and discuss the qualitative behaviour of each system.

2.1 1

$$\dot{x_1} = -x_2$$
$$\dot{x_2} = x_1 - x_2(1 - x_1^2 + 0.1x1^4)$$

For the above system, there are only one equilibrium points:

$$(x_1, x_2) = (0, 0)$$

The Taylor expansion for this system is:

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ x_2(-0.4x_1^3 + 2x1) + 1 & -0.1x_1^4 + x_1^2 - 1 \end{pmatrix} x$$

I take the equilibrium point into the above equation and obtain the matrix A:

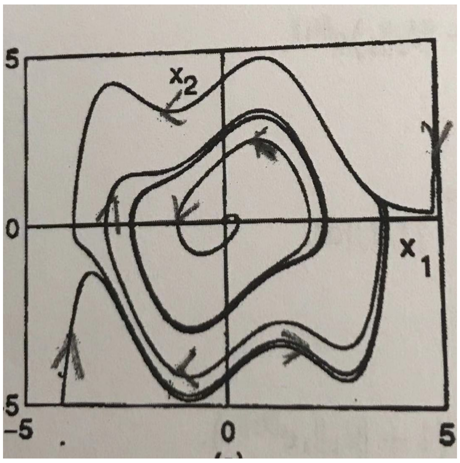
$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of the matrix A_1.

1. $\lambda_1 = -0.5 + 0.86i, \lambda_2 = -0.5 - 0.86i$

1. The eigenvalues of matrix A_1 are complex values and the real component is smaller than 0, so the type of equilibrium point is stable focus.

This could be proved by the phase portrait in the figure. The complex component could bring the sin and cos into the system output, and the negative real component could reduce the energy of these parts to zero gradually.



I mark the arrowheads shown in the above figure.

2.2 2

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_1 + x_2 - 3tan^{-1}(x_1 + x_2)$$

For the above system, there are only one equilibrium points:

$$(x_1, x_2) = (0, 0)$$

The Taylor expansion for this system is:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 - \frac{3}{(x_1+x_2)^2+1} & 1 - \frac{3}{(x_1+x_2)^2+1} \end{pmatrix} x$$

I take the equilibrium point into the above equation and obtain the matrix A:

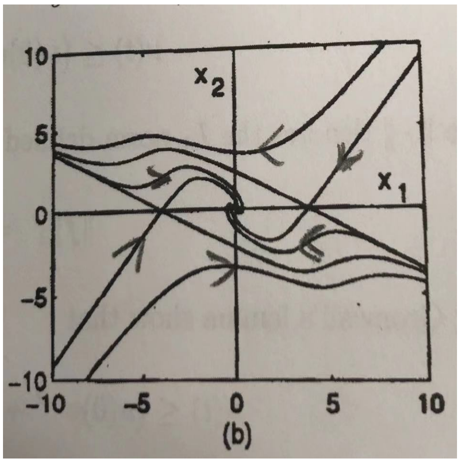
$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of the matrix A_1.

1. $\lambda_1 = -1 + 1i, \lambda_2 = -1 - 1i$

1. The eigenvalues of matrix A_1 are complex values and the real component is smaller than 0, so the type of equilibrium point is stable focus.

And this could be proved from the figure.



I mark the arrowheads shown in the above figure.

2.3 3

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -(0.5x_1 + x_1^3)$$

For the above system, there are only one equilibrium points:

$$(x_1, x_2) = (0, 0)$$

The Taylor expansion for this system is:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3x_1^2 - 0.5 & 0 \end{pmatrix} x$$

I take the equilibrium point into the above equation and obtain the matrix A:

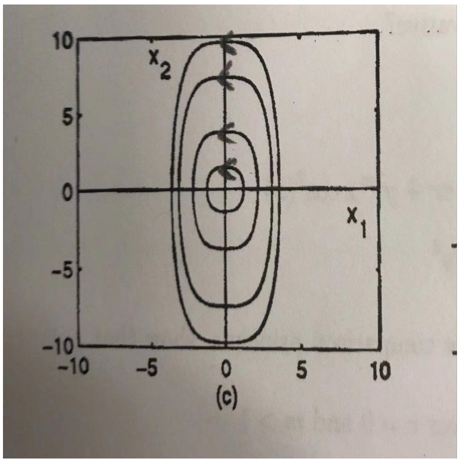
$$A_1 = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of the matrix A_1.

1. $\lambda_1 = -0.71i, \lambda_2 = 0.71i$

1. The eigenvalues of matrix A_1 are pure complex values, so the type of equilibrium point is center.

And this could be proved from the figure. It would never converge to the equilibrium point.



I mark the arrowheads shown in the above figure.

2.4 4

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_2 - \psi(x_1 - x_2)$$

In the above equations, where $\psi(y) = y^3 + 0.5y$ if $|y| \leq 1$ and $\psi(y) = 2y - 0.5$ if $|y| \geq 1$. For the above system, there is only one equilibrium points:

$$(x_1, x_2) = (0, 0)$$

The Taylor expansion for this system is(for this case, $\psi(y) = y^3 + 0.5y$):

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3(x_1 - x_2)^2 - 0.5 & 3(x_1 - x_2)^2 - 0.5 \end{pmatrix} x$$

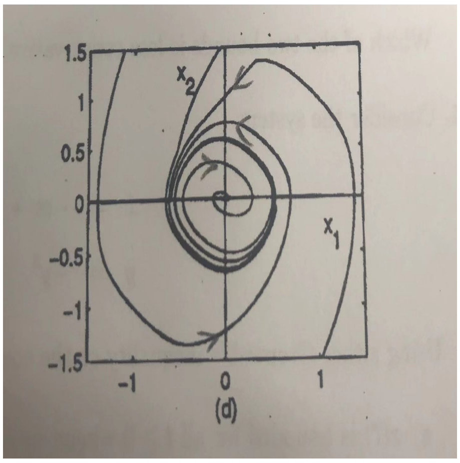
I take the equilibrium point into the above equation and obtain the matrix A:

$$A_1 = \begin{pmatrix} 0 & 1 \\ -0.5 & -0.5 \end{pmatrix}$$

I use Matlab to compute the eigenvalues of the matrix A_1.

1. $\lambda_1 = -0.25 + 0.66i, \lambda_2 = -0.25 - 0.66i$

1. The eigenvalues of matrix A_1 are complex values and the real component is smaller than 0, so the type of equilibrium point is stable focus.



I mark the arrowheads shown in the above figure.

Matlab Code

```
%%automatically compute the eigenvalues of different systems.
clear all;
clc;
syms x1 x2
A='x2=0';
B='-x2-2*(x1-x2)+0.5=0';
x1_dot='x2';
x2_dot='-x2-2*(x1-x2)+0.5';
[solution_x1,solution_x2]=solve(A,B);
disp('All the equilibrium points are:');
disp(solution_x1');
disp(solution_x2');
number_equ=size(solution_x1);
number_equ=number_equ(1,1);
A_11=diff(x1_dot,x1);
A_12=diff(x1_dot,x2);
A_21=diff(x2_dot,x1);
A_22=diff(x2_dot,x2);
A_symbol=[A_11,A_12;A_21,A_22];
disp('The matrix A is:')
disp(A_symbol)
for i=1:number_equ
    A_11_exp=subs(A_11,[x1,x2],[solution_x1(i,1),solution_x2(i,1)]);
    A_12_exp=subs(A_12,[x1,x2],[solution_x1(i,1),solution_x2(i,1)]);
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    A_21_exp=subs(A_21,[x1,x2],[solution_x1(i,1),solution_x2(i,1)]);
    A_22_exp=subs(A_22,[x1,x2],[solution_x1(i,1),solution_x2(i,1)]);
    disp('The matrix A is:')
    A_exp=[A_11_exp,A_12_exp;A_21_exp,A_22_exp];
    disp(A_exp);
    disp('The eigenvalues are:')
    disp(eig(A_exp));
end
```