Homework 5 Solutions, MAE281A 2016

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Chetaev's Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let V be a functional of x. Then, $x = 0$ is unstable if

- $V(0) = 0$, and $V(x_0) > 0$ for some arbitrary small $|x_0|$.
- $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) > 0$ for $\forall x \in U_0$, where U_0 was defined as $U_0 = \{x \in B_r | V(x) > 0\}$.

1

With Chetaev's theorem, show that the equilibrium at the origin of the following three systems is unstable :

a)

$$\dot{x} = x^3 + xy^3, \tag{1}$$
$$\dot{y} = -y + x^2 \tag{2}$$

Solution

The system (1) and (2) has an equilibrium at the origin $(x, y) = (0, 0)$. Let $V(x)$ be a functional s.t.

$$V(x, y) = \frac{1}{2}x^2 - \frac{1}{4}y^4. \tag{3}$$

Then, $V(0, 0) = 0$ and $V(x, y) > 0$ for $2x^2 > y^4$. Taking time derivative and from (1) (2), we obtain

$$\begin{aligned} \dot{V}(x, y) &= x\dot{x} - y^3\dot{y} = x(x^3 + xy^3) - y^3(-y + x^2) \\ &= x^4 + y^4. \end{aligned} \tag{4}$$

Defining $U_0 = \{(x, y)|V(x, y) > 0\} = \{2x^2 > y^4\}$, we can state that $\dot{V}(x, y) > 0$ for $\forall x, y \in U_0$.
Therefore, by Chetaev's theorem, the origin $(x, y) = (0, 0)$ is unstable.

b)

$$\dot{\xi} = \eta + \xi^3 + 3\xi\eta^2, \tag{5}$$
$$\dot{\eta} = -\xi + \eta^3 + 3\eta\xi^2 \tag{6}$$

Solution
The system (5) and (6) has an equilibrium at the origin $(\xi, \eta) = (0, 0)$. Let $V(\xi, \eta)$ be a functional s.t.

$$V(\xi, \eta) = \frac{1}{2}\xi^2 + \frac{1}{2}\eta^2. \tag{7}$$

Then, $V(0, 0) = 0$ and $V(\xi, \eta) > 0$ for $\forall(\xi, \eta) \in \mathbb{R}^2/\{(0, 0)\}$. Taking time derivative and from (5) (6), we obtain

$$\dot{V}(\xi, \eta) = \xi\dot{\xi} + \eta\dot{\eta} = \xi^4 + 6\xi^2\eta^2 + \eta^4. \tag{8}$$

Defining $U_0 = \{(\xi, \eta)|V(\xi, \eta) > 0\} = \mathbb{R}^2/\{(0, 0)\}$, we can state that $\dot{V}(\xi, \eta) > 0$ for $\forall(\xi, \eta) \in U_0 = \mathbb{R}^2/\{(0, 0)\}$.
Therefore, by Chetaev's theorem, the origin $(\xi, \eta) = (0, 0)$ is unstable.

c)

$$\dot{x} = |x|x + xy\sqrt{|y|}, \tag{9}$$
$$\dot{y} = -y + |x|\sqrt{|y|}. \tag{10}$$

Solution
The system (9) and (10) has an equilibrium at the origin $(x, y) = (0, 0)$. Let $V(x, y)$ be a functional s.t.

$$V(x, y) = x - \frac{y^2}{2}. \tag{11}$$

Then, $V(0, 0) = 0$ and $V(x, y) > 0$ for $x > y^2/2$. Taking time derivative and from (9) (10), we obtain

$$\begin{aligned}\dot{V}(x, y) = \dot{x} - y\dot{y} &= \left(|x|x + xy\sqrt{|y|}\right) - y\left(-y + |x|\sqrt{|y|}\right) \\ &= (|x|x + y^2) + (x - |x|)y\sqrt{|y|}. \end{aligned} \tag{12}$$

Define $U_0 = \{(x, y)|V(x, y) > 0\} = \{x > y^2/2\}$. Then, for $\forall(x, y) \in U_0$, we have $x > 0$, which leads to $|x| = x$ for $\forall(x, y) \in U_0$. Thus, we have

$$\dot{V}(x, y) = x^2 + y^2, \quad \forall(x, y) \in U_0 \tag{13}$$

Therefore, $\dot{V}(x, y) > 0$ for $\forall(x, y) \in U_0$. By Chetaev's theorem, the origin $(x, y) = (0, 0)$ is unstable.