Homework 4 Solutions, MAE281A 2016

Prepared by Shumon Koga, 2/12/2016

Lyapunov Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let V be a functional of x. Then, $x = 0$ is stable if

- V is positive definite("pdf"), i.e. $V(0) = 0$, and $V(x) > 0$ for $\forall x \neq 0$.
- $\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$ is negative semidefinite("nsdf"), i.e. $\dot{V}(x) \leq 0$ for $\forall x$.

In addition, $x = 0$ is asymptotically stable(a.s.) if

- $\dot{V}$ is negative definite, i.e. $\dot{V}(0) = 0$ and $\dot{V}(x) < 0$ for $\forall x \neq 0$.

These stability conditions hold globally if

- V is radially unbounded, i.e. $V(x) \to \infty$ as $|x| \to \infty$.

1

Prove global stability of the origin of the system

$$\dot{x}_1 = x_2, \tag{1}$$
$$\dot{x}_2 = -\frac{x_1}{1 + x_2^2} \tag{2}$$

Solution

Let $x = (x_1, x_2)^T$, and $V(x)$ be a Lyapunov candidate s.t.

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_2^4. \tag{3}$$

Then, $V(0) = 0$ and $V(x) > 0$ for $\forall x \neq 0$, and thus V is "pdf". Taking time derivative and from (1) (2), we obtain

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 + x_2^3\dot{x}_2 = x_1x_2 + x_2(1 + x_2^2) \times \left(-\frac{x_1}{1 + x_2^2}\right) \\ &= x_1x_2 - x_1x_2 = 0,\end{aligned} \tag{4}$$

Thus $\dot{V}$ is "nsdf".
In addition, by (3) $V(x) \to \infty$ as $|x| \to \infty$, and thus V is radially unbounded.
Therefore, this system is globally stable at the origin $x = 0$.

2

Prove global asymptotic stability of the origin of the system

$$\dot{x}_1 = -x_2^3, \tag{5}$$
$$\dot{x}_2 = x_1 - x_2. \tag{6}$$

How to find a good Lyapunov function

To show a.s, we need $\dot{V}$ "ndf".

- (1st Step)
  Looking at (5) and (6), $\dot{x}_1$ has 3rd power term, while $\dot{x}_2$ has 1st power term. Thus we guess $V = x_1^2/2 + x_2^4/4$ would work at first.

- (2nd Step)
  However, $\dot{V} = -x_2^4$, which is not "ndf" but "nsdf" ($\dot{V} = 0$ for $(x_1, 0)$). Thus, we need negative term of x_1, such as $-x_1^2$ in $\dot{V}$.

- (3rd Step)
  Looking at (5) and (6) again, x_1 shows up only in $\dot{x}_2$, thus we want $-x_1\dot{x}_2$ in $\dot{V}$ as additional term, which means $-x_1x_2$ is added to V.

- (4th Step)
  In addition, to keep V "pdf", x_2^2 should be added too, thus we guess $V = x_1^2/2 - x_1x_2 + x_2^2/2 + x_2^4/4 = (x_1 - x_2)^2/2 + x_2^4/4 > 0$ would work. (completion of square)

- (5th Step)
  Then, we have $\dot{V} = -(x_1 - x_2)^2$, and unfortunately this is "nsdf" too. However, if we add the first trial of Lyapunov candidate (i.e. define $V = [(x_1 - x_2)^2/2 + x_2^4/4] + [x_1^2/2 + x_2^4/4]$), it becomes $\dot{V} = -(x_1 - x_2)^2 - x_2^4$, and thus finally this is "ndf"!

Solution 1

Let $x = (x_1, x_2)^T$, and $V(x)$ be a Lyapunov candidate s.t.

$$V(x) = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4. \tag{7}$$

Then, $V(0) = 0$ and $V(x) > 0$ for $\forall x \neq 0$, and thus V is "pdf". Taking time derivative and from (5) (6), we obtain

$$\begin{aligned}
\dot{V} &= (x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + x_1\dot{x}_1 + 2x_2^3\dot{x}_2 \\
&= (x_1 - x_2)(-x_2^3 - (x_1 - x_2)) - x_1x_2^3 + 2x_2^3(x_1 - x_2) \\
&= -x_2^3(x_1 - x_2) - (x_1 - x_2)^2 - x_1x_2^3 + 2x_2^3(x_1 - x_2) \\
&= -(x_1 - x_2)^2 - x_2^4 < 0, \quad \forall x \neq 0,
\end{aligned} \tag{8}$$

Thus $\dot{V}$ is "ndf".
In addition, by (7) $V(x) \to \infty$ as $|x| \to \infty$, and thus V is radially unbounded.
Therefore, this system is g.a.s. at the origin $x = 0$.

Barbashin-Krasovskii's Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let V be a functional of x. Then, $x = 0$ is a.s. if

- V is "pdf" and $\dot{V} \leq 0$ for $\forall x \in D$.
- no solution can stay forever in $S := \left\{ x \in D | \dot{V} = 0 \right\}$ other than $x(t) = 0$.

In addition, $x = 0$ is g.a.s. if V is radially unbounded.

Solution 2

Let $V(x)$ be a Lyapunov candidate s.t.

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4. \tag{9}$$

Then, $V(0) = 0$ and $V(x) > 0$ for $\forall x \neq 0$, and thus V is "pdf". Taking time derivative and from (5) (6), we obtain

$$\begin{aligned} \dot{V} &= -x_1 x_2^3 + x_2^3(x_1 - x_2) \\ &= -x_2^4 \leq 0, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \tag{10}$$

Thus $\dot{V}$ is "nsdf".
In addition, by (9) $V(x) \to \infty$ as $|x| \to \infty$, and thus V is radially unbounded. Let $S := \left\{ (x_1, x_2) \in \mathbb{R}^2 | \dot{V} = 0 \right\} = \{x_2 = 0\}$. Then, substituting $x_2 = 0$ in (6), we obtain $x_1 = 0$. Therefore, no solution can stay forever in S other than the origin $(x_1, x_2) = (0, 0)$. By Barbashin-Krasovskii's Theorem, we conclude that the origin is g.a.s.

3

Prove global asymptotic stability of the origin of the system

$$\dot{x}_1 = x_2 - (2x_1^2 + x_2^2)x_1, \tag{11}$$
$$\dot{x}_2 = -x_1 - 2(2x_1^2 + x_2^2)x_2 \tag{12}$$

Is the origin locally exponentially stable and why or why not?

Solution

Let $x = (x_1, x_2)^T$, and $V(x)$ be a Lyapunov candidate s.t.

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{13}$$

Then, $V(0) = 0$ and $V(x) > 0$ for $\forall x \neq 0$, and thus V is "pdf". Taking time derivative and from (11) (12), we obtain

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(x_2 - (2x_1^2 + x_2^2)x_1) + x_2 \times (-x_1 - 2(2x_1^2 + x_2^2)x_2) \\ &= -x_1^2(2x_1^2 + x_2^2) - 2x_2^2(2x_1^2 + x_2^2) \\ &= -(2x_1^2 + x_2^2)(x_1^2 + 2x_2^2) < 0, \quad \forall x \neq 0 \end{aligned} \tag{14}$$

Thus $\dot{V}$ is "ndf".
In addition, by (13) $V(x) \to \infty$ as $|x| \to \infty$, and thus V is radially unbounded.
Therefore, this system is globally stable at the origin $x = 0$.

Computing the Jacobian matrix of (11) (12) and evaluating $x = 0$, we have

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{15}$$

($\because$ the system (11) (12) has only polynomial term, we can get (15) easily just by neglecting nonlinear term and remaining linear term of (11) (12)).
Because eigenvalues of (15) are $\pm j$, the origin is not locally exponentially stable.

4

Consider the system

$$\dot{x}_1 = -x_1 + x_1 x_2, \tag{16}$$

$$\dot{x}_2 = -\frac{x_1^2}{1 + x_1^2} \tag{17}$$

Show that the equilibrium $x_1 = x_2 = 0$ is globally stable and that

$$\lim_{t \to \infty} x_1(t) = 0 \tag{18}$$

Solution

Let $x = (x_1, x_2)^T$, and $V(x)$ be a Lyapunov candidate s.t. $V(x) = \phi(x_1) + x_2^2$. Then, to make V "pdf", we need

$$\phi(0) = 0, \quad \phi(x_1) > 0, \quad \forall x_1 \neq 0 \tag{19}$$

In addition, taking time derivative,

$$\begin{aligned}
\dot{V} &= \frac{\partial \phi(x_1)}{\partial x_1} \dot{x}_1 + 2 x_2 \dot{x}_2 \\
&= \frac{\partial \phi(x_1)}{\partial x_1} (-x_1 + x_1 x_2) - \frac{x_1^2}{1 + x_1^2} 2 x_2 \\
&= -x_1 \frac{\partial \phi(x_1)}{\partial x_1} + x_1 x_2 \left(\frac{\partial \phi(x_1)}{\partial x_1} - \frac{2 x_1}{1 + x_1^2} \right)
\end{aligned} \tag{20}$$

Thus, to make $\dot{V}$ "nsdf", we choose

$$\frac{\partial \phi(x_1)}{\partial x_1} = \frac{2 x_1}{1 + x_1^2} \tag{21}$$

By (19) and (21), we obtain $\phi(x_1)$ as

$$\phi(x_1) = \ln(1 + x_1^2) \tag{22}$$

which makes V "pdf" and $\dot{V}$ "nsdf". In addition, $V(x) \to \infty$ as $|x| \to \infty$, and thus V is radially unbounded.
Therefore, this system is globally stable at the origin $x = 0$.
In addition, by (20) and (21), we have $\dot{V} = -\frac{2x_1^2}{1+x_1^2}$, and thus $\dot{V} = 0$ for $x_1 = 0$ which competes the proof of (18).