solutions prepared by Azad Ghaffari
Averaging theory

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1. Using averaging theory, analyze the following system:

$$\dot{x} = \epsilon\big[-x+1-2(y+\sin(t))^2\big] \tag{1}$$
$$\dot{y} = \epsilon z \tag{2}$$
$$\dot{z} = \epsilon\left[-z-\sin(t)\Big(\frac{1}{2}x+(y+\sin(t))^2\Big)\right]. \tag{3}$$

Solution 1 *First we calculate the average system*

$$\begin{aligned} f_{av} &= \frac{1}{2\pi}\int_0^{2\pi} f(t,x,0)\mathrm{d}t \\ &= \begin{bmatrix} -x+1-2y^2-\frac{1}{\pi}\int_0^{2\pi}\sin^2(t)\mathrm{d}t \\ z \\ -z-y\frac{1}{\pi}\int_0^{2\pi}\sin^2(t)\mathrm{d}t \end{bmatrix} \\ &= \begin{bmatrix} -x-2y^2 \\ z \\ -z-y \end{bmatrix}. \end{aligned}$$

The Jacobian

$$J_{av} = \frac{\partial f_{av}}{\partial x}\Big|_{(0,0,0)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

is Hurwitz. Note that $(0,0,0)$ *is not an equilibrium of (1)-(3). Hence, there exists* $\epsilon^* > 0$ *such that for all* $\epsilon \in (0,\epsilon^*)$ *there exists a locally exponentially stable solution* $(x^{2\pi}(t), y^{2\pi}(t), z^{2\pi}(t))$ *of period* 2π *and such that* $|(x^{2\pi}(t), y^{2\pi}(t), z^{2\pi}(t))| < O(\epsilon), \forall t \geq 0$.

2. Analyze the following system using the method of averaging for large ω:

$$\dot{x}_1 = \big(x_2\sin(\omega t)-2\big)x_1-x_3 \tag{4}$$
$$\dot{x}_2 = -x_2+\big(x_2^2\sin(\omega t)-2x_3\cos(\omega t)\big)\cos(\omega t) \tag{5}$$
$$\dot{x}_3 = 2x_2-\sin(x_3)+\big(4x_2\sin(\omega t)+x_3\big)\sin(\omega t) \tag{6}$$

Solution 2 *Let* $\tau = \omega t$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \frac{1}{\omega}\begin{bmatrix} \Big(x_2\sin(\tau)-2\Big)x_1-x_3 \\ -x_2+\Big(x_2^2\sin(\tau)-2x_3\cos(\tau)\Big)\cos(\tau) \\ 2x_2-\sin(x_3)+\Big(4x_2\sin(\tau)+x_3\Big)\sin(\tau) \end{bmatrix} \\ f_{av} &= \frac{1}{2\pi}\int_0^{2\pi} f(\tau,x,0)\mathrm{d}\tau \\ &= \begin{bmatrix} -2x_1-x_3 \\ -x_2-x_3 \\ 2x_2-\sin(x_3)+2x_2 \end{bmatrix}. \end{aligned}$$

The Jacobian

$$J_{av} = \left.\frac{\partial f_{av}}{\partial x}\right|_{(0,0,0)} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & -1 \end{bmatrix}$$

is Hurwitz and $f(\tau, 0, \epsilon) = 0$. *Hence, there exists* $\omega^* > 0$ *such that for all* $\omega > \omega^*$ *the origin is exponentially stable.*

3. Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= \sin(\omega t) y_1 && (7) \\ \dot{x}_2 &= \cos(\omega t) y_2 && (8) \\ y_1 &= \big[x_1 + \sin(\omega t)\big]\big[x_2 + \cos(\omega t) - x_1 - \sin(\omega t)\big] && (9) \\ y_2 &= \big[x_2 + \cos(\omega t)\big]\big[x_1 + \sin(\omega t) - x_2 - \cos(\omega t)\big]. && (10) \end{aligned}$$

Show that for sufficiently large ω there exists an exponentially stable periodic orbit in an $O(1/\omega)$ neighborhood of the origin $x_1 = x_2 = 0$.

Hint: The following functions have a zero mean over the interval $[0, 2\pi]$: $\sin(\tau), \cos(\tau), \sin(\tau)\cos(\tau), \sin^3(\tau)$, $\cos^3(\tau), \sin^2(\tau)\cos(\tau)$, and $\sin(\tau)\cos^2(\tau)$.

Solution 3 *Let* $\tau = \omega t$,

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}\tau} &= \frac{1}{\omega} y_1 \sin(\tau) \\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau} &= \frac{1}{\omega} y_2 \cos(\tau) \\ f_{av} &= \frac{1}{2\pi}\begin{bmatrix} \int_0^{2\pi} y_1 \sin(\tau)\mathrm{d}\tau \\ \int_0^{2\pi} y_2 \cos(\tau)\mathrm{d}\tau \end{bmatrix} \\ &= \begin{bmatrix} -x_1 + \frac{1}{2}x_2 \\ \frac{1}{2}x_1 - x_2 \end{bmatrix}. \end{aligned}$$

Note that the origin is not an equilibrium of (7)-(10). Hence there exist $\omega^* > 0$ *such that for all* $\omega > \omega^*$ *there exist a locally exponentially stable solution* $(x^{2\pi/\omega}(t), y^{2\pi/\omega}(t))$ *of period* $2\pi/\omega$ *and such that* $|(x^{2\pi/\omega}(t), y^{2\pi/\omega}(t))| \le O(1/\omega), \forall t \ge 0$.

4. Consider Rayleigh's equation

$$m\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + ku = \lambda\Big[1 - \alpha\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2\Big]\frac{\mathrm{d}u}{\mathrm{d}t} \tag{11}$$

where m, k, λ, and α are positive constants.

a) Using the dimensionless variables $y = \frac{u}{u^*}, \tau = \frac{t}{t^*}$, and $\epsilon = \frac{\lambda}{\lambda^*}$, where $(u^*)^2\alpha k = \frac{m}{3}$, $t^* = \sqrt{\frac{m}{k}}$, and $\lambda^* = \sqrt{km}$, show that the equation can be normalized to

$$\ddot{y} + y = \epsilon\Big(\dot{y} - \frac{1}{3}\dot{y}^3\Big) \tag{12}$$

where $\dot{y}$ denotes the derivative of y with respect to τ.

b) Apply the averaging method to show that the normalized Rayleigh equation has a stable limit cycle. Estimate the location of the limit cycle in the plane $(y, \dot{y})$.

Solution 4 *By applying the chain rule*

$$m\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}\Big)\frac{\mathrm{d}\tau}{\mathrm{d}t}+ku \;=\; \lambda\Big[1-a\Big(\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}\Big)^2\Big]\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}$$

we get equation (12) from (11).

Assume $x_1 = y$ and $x_2 = \dot{y}$, and consider following transformation

$$\begin{aligned}\phi &= \tan^{-1}\Big(\frac{x_1}{x_2}\Big)\\ r &= \sqrt{x_1^2+x_2^2},\end{aligned}$$

then we have

$$\begin{aligned}\dot{\phi} &= 1-\epsilon\Big(\sin(\phi)\cos(\phi)-\frac{1}{3}r^2\sin(\phi)\cos^3(\phi)\Big)\\ \dot{r} &= \epsilon\Big(r\cos^2(\phi)-\frac{1}{3}r^3\cos^4(\phi)\Big),\\ \frac{\mathrm{d}r}{\mathrm{d}\phi} &= \epsilon\frac{r\cos^2(\phi)-\frac{1}{3}r^3\cos^4(\phi)}{1-\epsilon\Big(\sin(\phi)\cos(\phi)-\frac{1}{3}r^2\sin(\phi)\cos^3(\phi)\Big)}=\epsilon f(r,\phi,\epsilon)\\ f_{av} &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\phi,0)\mathrm{d}\phi\\ &= \frac{1}{2}r-\frac{1}{8}r^3=0\Rightarrow r=0 \quad or \quad r=2.\end{aligned}$$

Since $\dot{r}>0$ for $r<2$ and $\dot{r}<0$ for $r>0$, there is a stable limit cycle with $r=2$.