Assignment1

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[2]:
```
import numpy as np
from matplotlib import pyplot
import MNISTtools
```

$ **Q1** $

[61]:
```
#1 Loading dataset and checking shape of the training dataset

#help(MNISTtools.load)
#help(MNISTtools.show)
xtrain, ltrain = MNISTtools.load(dataset="training", path=None)
print("Answer 1. Shape of xtrain is ", np.shape(xtrain))
print("          Shape of ltrain is ", np.shape(ltrain))
print("          Size of training dataset is ",xtrain.shape[1])
print("          Feature dimension is ",xtrain.shape[0])
```

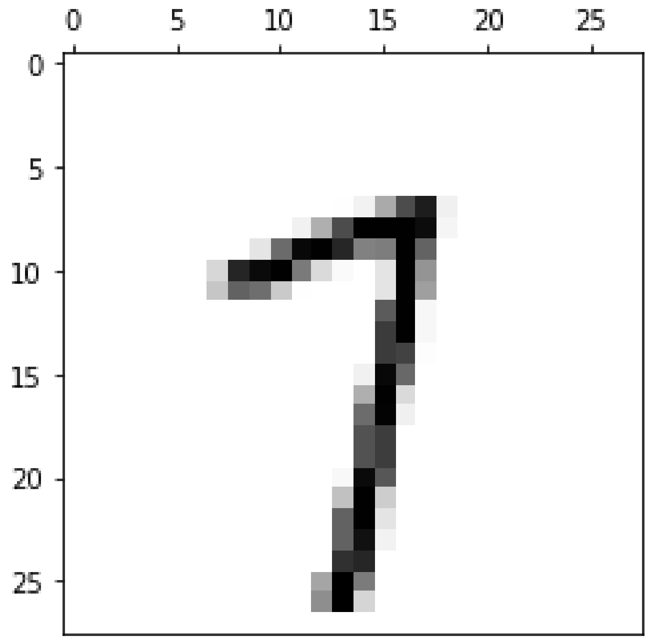
```
Answer 1. Shape of xtrain is  (784, 60000)
          Shape of ltrain is  (60000,)
          Size of training dataset is  60000
          Feature dimension is  784
```

$ **Q2** $

[4]:
```
#2 Displaying Image and corresponding label

print("Answer 2. Displaying image at index 42:")
MNISTtools.show(xtrain[:, 42])
print("          Label of image is ", ltrain[42])
```

```
Answer 2. Displaying image at index 42:
```




```
Label of image is  7
```

$ **Q3** $

[5]:
```python
#3 Finding xtrain range and type

#xtrain_max = np.max(xtrain)
#xtrain_min = np.min(xtrain)
print("Answer 3. xtrain range is [", np.min(xtrain), ", ", np.max(xtrain), "]")
print("          xtrain type is ",type(xtrain))
```

```
Answer 3. xtrain range is [ 0 ,  255 ]
          xtrain type is  <class 'numpy.ndarray'>
```

$ **Q4** $

[6]:
```python
#4 Normalizing and Updating xtrain

xtrain = xtrain.astype(np.float32)
def normalize_MNIST_images(x):
    x = -1 + (2*x/255)
    return x
```

[7]:
```python
#xtrain = (np.interp(xtrain, (np.min(xtrain), np.max(xtrain)), (-1, +1)))
xtrain = normalize_MNIST_images(xtrain)
print("Answer 4. Min of normalized xtrain", np.min(xtrain))
print("          Max of normalized xtrain", np.max(xtrain))
```

```
print("          Range of normalized xtrain is [", np.min(xtrain), ", ", np.
 ↪max(xtrain), "]")
```

```
Answer 4. Min of normalized xtrain -1.0
          Max of normalized xtrain 1.0
          Range of normalized xtrain is [ -1.0 ,  1.0 ]
```

$ **Q5** $

[8]:
```
#5 Converting label to one hot code.

def label2onehot(lbl):
    d = np.zeros((lbl.max() + 1, lbl.size))
    for i in range(lbl.max()):
        d[lbl, np.arange(lbl.size)] = 1
    return d
dtrain = label2onehot(ltrain)
print("Checking Shape of dtrain as ", np.shape(dtrain))
print("Answer 5. One hot code for index 42 is ", dtrain[:,42])
print("          Label for index 42 is ", ltrain[42])
```

```
Checking Shape of dtrain as  (10, 60000)
Answer 5. One hot code for index 42 is  [0. 0. 0. 0. 0. 0. 0. 1. 0. 0.]
          Label for index 42 is  7
```

Note : We can see that One hot code of label at index 42 corresponds to value of label at index 42

$ **Q6** $

[9]:
```
#6 Converting one hot code to label

def onehot2label(d):
    lbl = d.argmax(axis=0)
    return lbl

print("Comparing ltrain to onehot2label(dtrain)")
print("Answer 6.", set(onehot2label(dtrain))==set(ltrain))
```

```
Comparing ltrain to onehot2label(dtrain)
Answer 6. True
```

Note : In Set comparison, True means arrays are same

$ **Q7** $

[10]:
```
#7
#Defining softmax function - An activation function
def softmax(a):
    M = a.max(axis=0)
    y = np.exp(a-M)/(np.exp(a-M).sum(axis=0))
```

```
    return y
```

```
[11]: y = softmax(xtrain)
print("Answer 7. Verifying Softmax fn.")
#print(np.shape(y[59]))
#print(np.shape(y))
print("          Sum of probabilities at some location:",np.sum(y[:,59]), "\n
 ↪       1.0 implies no numerical loss")
```

```
Answer 7. Verifying Softmax fn.
          Sum of probabilities at some location: 1.0
          1.0 implies no numerical loss
```

$ **Q8, Q9, Q10 : PROOFS solved by hand** $

$ **Q10** $

```
[13]: #10 Defining softmaxp fn. - Also an activation fn. that is the derivative of
 ↪softmax fn.

def softmaxp(a, e):
    y = softmax(a)
    d = np.multiply(y, e) - ((np.multiply(y, e)).sum(axis=0))*(y) #Derivative
 ↪of softmax()
    return d
```

$ **Q11** $

```
[37]: #11 Checking softmaxp fn. and its implementation by numerical approximations

eps = 1e-6  # finite difference step
a = np.random.randn(10, 200)# random inputs
e = np.random.randn(10, 200)# random directions

diff = softmaxp(a, e)
diff_approx = (softmax(a + eps*e) - softmax(a)) / eps
rel_error = np.abs(diff - diff_approx).mean() / np.abs(diff_approx).mean()
print("Answer 11. Checking softmaxp () and its implementation by numerical
 ↪approximations")
print("\t  ",rel_error,'should be smaller than 1e-6')
```

```
Answer 11. Checking softmaxp () and its implementation by numerical
approximations
           5.180789640314606e-07 should be smaller than 1e-6
```

As error found is smaller than 1e-6, the softmaxp() works as intended

$ **Q12** $

[15]:
```python
#12 Defining relu() and its directional derivative - called as relup()
#   Both are activation functions - REctified Linear Units(RELU)

def relu(a):
    return np.maximum(a, 0)       # RELU fn. gives a for a>0 and 0 for a<=0

def relup(a,e):
    c = np.maximum(a, 0)
    c[c>0] = 1
    c = np.multiply(c,e)          # RELUp fn. gives e for a>0 and 0 for a<=0
    return c
```

[38]:
```python
eps = 1e-6  # finite difference step
a = np.random.randn(10, 200)# random inputs
e = np.random.randn(10, 200)# random directions

diff = relup(a, e)
diff_approx = (relu(a + eps*e) - relu(a)) / eps
rel_error = np.abs(diff - diff_approx).mean() / np.abs(diff_approx).mean()
print("Answer 12. Checking relup () and its implementation by numerical␣
 ↪approximations")
print("\t  ",rel_error,'should be smaller than 1e-6')
```

```
Answer 12. Checking relup () and its implementation by numerical approximations
           3.6920519469948386e-11 should be smaller than 1e-6
```

As error found is smaller than 1e-6, the relup() works as intended

$ **Q13** $

[39]:
```python
# Backpropagation
#13 Initializing Shallow Networks

def init_shallow(Ni, Nh, No):
    b1 = np.random.randn(Nh, 1) / np.sqrt((Ni+1.)/2.)
    W1 = np.random.randn(Nh, Ni) / np.sqrt((Ni+1.)/2.)
    b2 = np.random.randn(No, 1) / np.sqrt((Nh+1.))
    W2 = np.random.randn(No, Nh) / np.sqrt((Nh+1.))
    return W1, b1, W2, b2
Ni = xtrain.shape[0]
Nh = 64
No = dtrain.shape[0]
netinit = init_shallow(Ni, Nh, No)
```

The shallow network is initialized

$ **Q14** $

[40]:
```
#14 Defining Forward propagation for the shallow network

def forwardprop_shallow(x, net):
    W1 = net[0]
    b1 = net[1]
    W2 = net[2]
    b2 = net[3]

    a1 = W1.dot(x) + b1
    h1 = relu(a1)
    a2 = W2.dot(h1) + b2
    y = softmax(a2)

    return y

yinit = forwardprop_shallow(xtrain, netinit)
```

Forward propagation is defined and yinit is obtained.

$ **Q15** $

[41]:
```
#15 Defining the evaluation loss using average of cross entropy using␣
↪predictions(y) and desired one-hot code(d)

def eval_loss(y, d):

    e = -((np.multiply(d,np.log(y))))
    e = np.mean(e)
    return e

print("Answer 15. ",eval_loss(yinit, dtrain), 'should be around .26')
```

```
Answer 15.  0.24909539618697962 should be around .26
```

eval_loss has been defined and the loss is close to the expected value.

$ **Q16** $

[51]:
```
#16 Evaluating the performance of the network

def eval_perfs(y, lbl):
    y = onehot2label(y)
    n = 0
    n = np.equal(y,lbl)
    p = sum(n)                              #Number of correctly labelled␣
↪sample is the number of true values in n

    performance = ((y.shape[0]-p)/y.shape[0]) #Percentage of misclassified␣
↪samples is
```

```
    performance = 100*performance             #Number of wrongly labelled␣
↪samples / total no of samples
    return performance

print("Answer 16. Percentage of initial misclassified samples" ,␣
↪eval_perfs(yinit, ltrain), "%")
```

```
Answer 16. Percentage of initial misclassified samples 89.28833333333334 %
```

Interpretation: The initial misclassified samples are about 89% which means that about 53573 samples are wrongly classified initially. Only about 6427 samples are classified correcltly. Hence, without any training, this network can classify correctly less than 11% of the time. The network is not suitable for classification without training.

$ **Q17** $

[52]:
```
#17
#Defining update_shallow to complete one backpropagation.

def update_shallow(x, d, net, gamma=.05):

    W1 = net[0]
    b1 = net[1]
    W2 = net[2]
    b2 = net[3]
    Ni = W1.shape[1]
    Nh = W1.shape[0]
    No = W2.shape[0]

    gamma = gamma / x.shape[1] # normalized by the training dataset size

    #Forward propagation
    a1 = W1.dot(x) + b1
    h1 = relu(a1)
    a2 = W2.dot(h1) + b2
    y = softmax(a2)

    delta2 = softmaxp(a2, -d/y) # -d/y is the error to be backpropagated.
    delta1 = relup(a1, W2.T.dot(delta2))

    # Weights and biases are updated

    W2 = W2 - gamma * delta2.dot(h1.T)
    W1 = W1 - gamma * delta1.dot(x.T)
    b2 = b2 - gamma * delta2.sum(axis = 1, keepdims = True)
    b1 = b1 - gamma * delta1.sum(axis = 1, keepdims = True)

    return W1, b1, W2, b2
```

Single backpropagation update for shallow network is obtained using the above function

See at end for error derivation proof.

$ **Q18** $

[53]:
```
#18 Updating Shallow network using Backpropagation.

def backprop_shallow(x, d, net, T, gamma=.05):
    y = forwardprop_shallow(xtrain, netinit)
    #print(eval_loss(y,d))
    lbl = onehot2label(d)
    for t in range(T):
        net=update_shallow(x,d,net)     # Updating the net for T iterations by␣
 ↪changing Weights and Biases using BackProp
        y = forwardprop_shallow(x,net)  # Computing the Forward propagation␣
 ↪after updation
        print("            Loss after T =",t," is ", eval_loss(y,d))    # Loss␣
 ↪after each iteration
        print("            Perf after T =",t," is ", eval_perfs(y,lbl))#␣
 ↪Performance after each iteratioon
    return net

print(" Answer 18. Loss and Percentage Training errors after backpropagation")
nettrain = backprop_shallow(xtrain, dtrain, netinit, 20)
```

```
Answer 18. Loss and Percentage Training errors after backpropogation
           Loss after T = 0  is  0.22504495967666144
           Perf after T = 0  is  81.28333333333333
           Loss after T = 1  is  0.2131964059396235
           Perf after T = 1  is  75.565
           Loss after T = 2  is  0.20365422114956025
           Perf after T = 2  is  68.88833333333334
           Loss after T = 3  is  0.19510487776796626
           Perf after T = 3  is  62.84166666666666
           Loss after T = 4  is  0.18707135626126045
           Perf after T = 4  is  57.86666666666667
           Loss after T = 5  is  0.1796354028747958
           Perf after T = 5  is  52.70333333333333
           Loss after T = 6  is  0.17307341940382925
           Perf after T = 6  is  52.61333333333334
           Loss after T = 7  is  0.16816917525073555
           Perf after T = 7  is  48.555
           Loss after T = 8  is  0.16513336725392155
           Perf after T = 8  is  53.64666666666666
           Loss after T = 9  is  0.16552390523302832
           Perf after T = 9  is  50.068333333333335
           Loss after T = 10  is  0.161377493269914
           Perf after T = 10  is  54.653333333333333
```

```
Loss after T = 11  is  0.15723808247017806
Perf after T = 11  is  46.62833333333333
Loss after T = 12  is  0.14658085851945635
Perf after T = 12  is  46.89833333333333
Loss after T = 13  is  0.1412203014376136
Perf after T = 13  is  40.471666666666664
Loss after T = 14  is  0.13554678047170413
Perf after T = 14  is  41.75833333333333
Loss after T = 15  is  0.13280282143145117
Perf after T = 15  is  38.32833333333333
Loss after T = 16  is  0.1279188942713922
Perf after T = 16  is  38.89
Loss after T = 17  is  0.1255576685031616
Perf after T = 17  is  36.25
Loss after T = 18  is  0.120926847853332
Perf after T = 18  is  36.29833333333333
Loss after T = 19  is  0.11874797784535145
Perf after T = 19  is  34.35333333333334
```

We can observe that the error reduces steadily more or less from 81% at T=0 to 34% error after T=19, there are some exceptions in the variation, but generally as the iteration increases, we can see that error decreases along with the loss which reduces from 0.22 to 0.11. Hence, we are getting a better trained network as T increases.

$ **Q19** $

[54]:
```
#19 LOADING TESTING SETS
xtest, ltest = MNISTtools.load(dataset="testing", path=None) #Loading testing␣
↪sets into xtest and ltest
xtest = normalize_MNIST_images(xtest)                        #Normalizing xtest␣
↪values from [0, 255] to [-1, 1] values
dtest = label2onehot(ltest)
print("Answer 19. Size of Testing set is","\n           xtest shape is", xtest.
↪shape)
print("           ltest shape is", ltest.shape)
#Testing Performance of network on testing dataset
y = forwardprop_shallow(xtest, nettrain)
print("           Testing Performance of our network on testing dataset")
print("           Testing Loss", eval_loss(y,dtest))
print("           Testing Perf", eval_perfs(y,ltest))
```

```
Answer 19. Size of Testing set is
           xtest shape is (784, 10000)
           ltest shape is (10000,)
           Testing Performance of our network on testing dataset
           Testing Loss 0.15961520709861643
           Testing Perf 46.07
```

The trained network which reaches an error of 34.35% in our training set produces an error of

46.07% in testing set. Also the training loss is 0.11 and the testing loss is 0.159.

$ **Q20** $

[55]:
```
#20 Running backpropagation as minibatches for 5 epochs with 100 minibatches
def backprop_minibatch_shallow(x, d, net, T, B=100, gamma=.05):
    N = x.shape[1]
    NB = int((N+B-1)/B)
    lbl = onehot2label(d)
    for t in range(T):
        shuffled_indices = np.random.permutation(range(N))
        for l in range(NB):
            minibatch_indices = shuffled_indices[B*l:min(B*(l+1), N)]
            net = update_shallow(x[:, minibatch_indices], d[:,␣
 ↪minibatch_indices], net, gamma) #Using interger array indexing
        y = forwardprop_shallow(x, net)
        print("EPOCH -> ", t)
        print("Training Loss", eval_loss(y,d))        #Evaluating loss of␣
 ↪network after epoch
        print("Training Perf", eval_perfs(y,lbl))     #Evaluating Performance␣
 ↪of network after epoch
    return net

print("Answer 20. Testing performance on training dataset")
netminibatch = backprop_minibatch_shallow(xtrain, dtrain, netinit, 5, B=100)
```

```
Answer 20. Testing performance on training dataset
EPOCH ->  0
Training Loss 0.034129991870732214
Training Perf 10.135
EPOCH ->  1
Training Loss 0.023448948954816983
Training Perf 6.736666666666667
EPOCH ->  2
Training Loss 0.020612303260010786
Training Perf 5.946666666666667
EPOCH ->  3
Training Loss 0.016649122240158753
Training Perf 4.745
EPOCH ->  4
Training Loss 0.015190385082186416
Training Perf 4.34
```

$ **Q21** $

[60]:
```
#21
y = forwardprop_shallow(xtest, netminibatch)
print("Answer 21. Performance of the final network on testing dataset")
print("            Testing Loss", eval_loss(y,dtest))
```

```
print("            Testing Perf", eval_perfs(y,ltest))
```

```
Answer 21. Performance of the final network on testing dataset
           Testing Loss 0.038005452485664434
           Testing Perf 8.93
```

We can see that the training error is about 4.34% and the testing error is about 8.93%. Hence, our network can classify the numbers with an accuracy of 91.07% in the testing dataset.

Inference: We see that the testing error obtained by using minibatches is much lesser than the error obtained from backpropagation in lesser number of epochs and the computation is faster.

RESULT: Training accuracy = 95.66% ; Testing accuracy = 91.07%

Q.8 Show that $\dfrac{\partial g(a)_i}{\partial a_i} = g(a)_i(1 - g(a)_i)$

$$g(a)_i = \frac{e^{a_i}}{\sum_{j=1}^{10} e^{a_j}}$$

LHS =

$$\frac{\partial g(a_i)}{\partial a_i} = \frac{\left(\sum_{j=1}^{10} e^{a_j}\right) \cdot e^{a_i} - e^{a_i} \cdot e^{a_i}}{\left(\sum_{j=1}^{10} e^{a_j}\right)^2}$$

$$= \frac{e^{a_i}}{\left(\sum_{j=1}^{10} e^{a_j}\right)} - \frac{\left(e^{a_i}\right)^2}{\left(\sum_{j=1}^{10} e^{a_j}\right)^2}$$

$$= g(a)_i - \left(g(a)_i\right)^2$$

$$\boxed{\text{LHS} = \text{RHS} = g(a)_i\left(1 - g(a)_i\right)}$$

Hence Proved

Q.9. Show that $\frac{\partial g(a)_i}{\partial a_j} = -g(a)_i\, g(a)_j$ for $i \neq j$

$$LHS = \frac{\partial g(a_i)}{\partial a_j} = \frac{\partial}{\partial a_j}\left(\frac{e^{a_i}}{\sum_{j=1}^{10} e^{a_j}}\right)$$

$$= e^{a_i} \times \frac{\partial}{\partial a_j}\left(\frac{1}{\sum_{j=1}^{10} e^{a_j}}\right)$$

$$= e^{a_i} \times \frac{-1}{\left(\sum_{j=1}^{10} e^{a_j}\right)^2} \times e^{a_j}$$

$$= -\frac{e^{a_i}}{\sum_{j=1}^{10} e^{a_j}} \times \frac{e^{a_j}}{\sum_{j=1}^{10} e^{a_j}}$$

$$\boxed{LHS = RHS = -g(a)_i \cdot g(a_j)}$$

Hence Proved.

Q.10. Given $\dfrac{\partial g(a)}{\partial(a)} = \begin{pmatrix} \dfrac{\partial g(a)_1}{\partial a_1} & \cdots & \dfrac{\partial g(a)_1}{\partial a_{10}} \\ \vdots & & \vdots \\ \dfrac{\partial g(a)_{10}}{\partial a_1} & \cdots & \dfrac{\partial g(a_{10})}{\partial a_{10}} \end{pmatrix}$

Prove that Jacobian of softmax is symmetric and

that $\delta = g(a) \otimes e - \langle g(a), e \rangle g(a)$

$\otimes \rightarrow$ Element wise product.

Jacobian of softmax.

$$\frac{\partial g(a)}{\partial(a)} = \begin{pmatrix} g(a)_1(1-g(a)_1) & -g(a)_1 g(a)_2 & \cdots & -g(a)_1 g(a)_{10} \\ -g(a)_1 g(a)_2 & g(a)_2(1-g(a)_2) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ -g(a)_1 g(a)_{10} & \cdots & \cdots & g(a)_{10}(1-g(a)_{10}) \end{pmatrix}$$

Using properties from Q.8 and Q.9. i.e.

$$\frac{\partial g(a)_i}{\partial a_i} = g(a)_i(1-g(a)_i)$$

and $\frac{\partial g(a)_i}{\partial a_j} = -g(a)_i \, g(a)_j$

We can see that $\frac{\partial g(a)_i}{\partial a_j} = \frac{\partial g(a)_j}{\partial a_i} = -g(a_i)\, g(a_j)$

Hence, the Jacobian matrix is symmetrical, i.e. $\frac{\partial g(a)}{\partial a} = \left(\frac{\partial g(a)}{\partial a}\right)^T$

$$\delta = \left(\frac{\partial g(a)}{\partial a}\right)^T \times e$$

$$= \frac{\partial g(a)}{\partial a} \times e \quad \text{as} \quad \frac{\partial g(a)}{\partial a} = \left(\frac{\partial g(a)}{\partial a}\right)^T$$

$$\frac{\partial g(a)}{\partial a} \times e = \begin{bmatrix} g(a)_1 \left(1 - g(a)_1\right) & -g(a)_1\, g(a_2) & \cdots & -g(a)_1\, g(a)_{10} \\ \vdots & g(a)_2 \left(1 - g(a)_2\right) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ -g(a)_1\, g(a_{10}) & \cdots & \cdots & \left(1 - g(a)_{10}\right) g(a)_{10} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{10} \end{bmatrix}$$

$$= \begin{bmatrix} g(a)_1\, e_1 \left(1 - g(a_1)\right) & -g(a)_1\, g(a_2)\, e_2 & \cdots & -g(a)_1 (a)_{10}\, e_{10} \\ \vdots & & & \vdots \\ -g(a_1)\, e_1 \left(1 - g(a_1)\right) & -g(a_2)\, g(a_{10})\, e_2 & \cdots & -g(a)_{10}\, g(a)_{10}\, e_{10} \end{bmatrix}$$

$$= \begin{bmatrix} g(a)_1 e_1 - \left(\sum_{j=1}^{10} g(a)_j e_j \right) g(a)_1 \\ g(a)_2 e_2 - \left(\sum_{j=1}^{10} g(a)_j e_j \right) g(a)_2 \\ \vdots \\ g(a)_{10} e_{10} - \left(\sum_{j=1}^{10} g(a)_j e_j \right) g(a)_{10} \end{bmatrix}$$

$$= g(a) \otimes e - \overbrace{\langle g(a), e \rangle}^{\text{Inner product}} g(a)$$

$$\boxed{\delta = g(a) \otimes e - \langle g(a), e \rangle g(a)}$$

Hence Proved

Q.17. Show that $(\nabla_y E)_i = \dfrac{-d_i}{y_i}$

$$E = -\sum_{j=1}^{N} \sum_{i=1}^{10} d_{ij} \log y_{ij}$$

$$(\nabla_y E) = \frac{\partial E}{\partial y} = -\sum_{j=1}^{N} \sum_{i=1}^{10} \left(\frac{d_{ij}}{y_{ij}} \right)$$

$$\boxed{(\nabla_y E)_i = -\frac{d_i}{y_i}}$$