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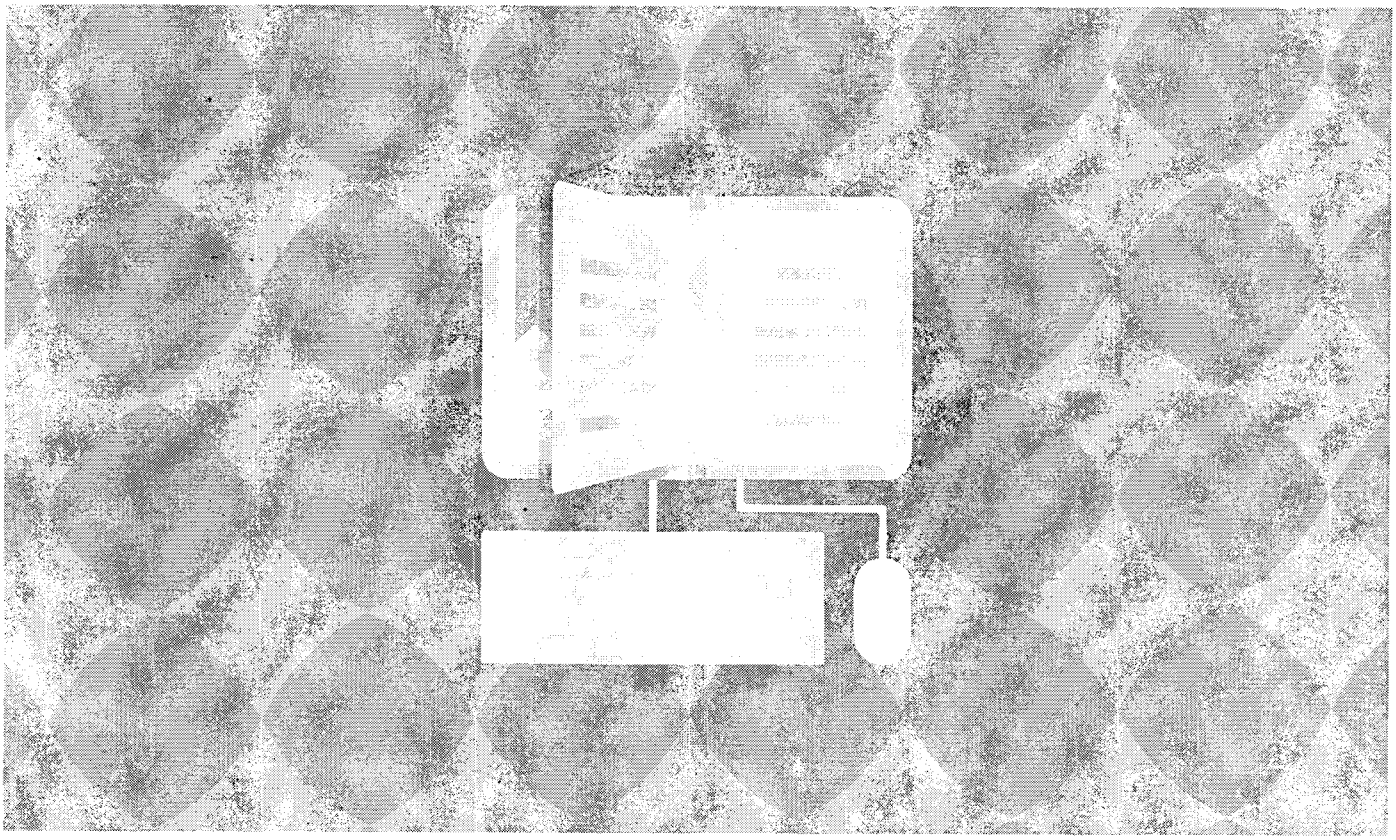

Study Manual for

SOA Exam FM
Financial Mathematics

12th Edition

Harold Cherry, FSA, MAAA

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Introduction

To the student: Please read this Introduction. It contains important information.

The 12th edition of this manual has been revised and updated for the new syllabus in effect for the June 2017 and later exams. The major changes in the syllabus were:

- The topic of derivatives (except for interest rate swaps) was removed.
- A new study note on interest rate swaps was added.
- A new study note on duration and convexity was added.[1]
- A new study note on determinants of interest rates was added.

The 12th edition consists of 11 sections covering all of the material on the syllabus (Part I of the manual), followed by six original practice exams (Part II of the manual). Following each section or subsection in Part I, there are illustrative examples called "Stepping Stones" (see below). These examples are followed by a summary of the key concepts and formulas in the section, then by problems and solutions from actual past SOA/CAS examinations.

Goals of this Manual

- To explain the concepts of financial mathematics in a way that appeals to your intuition and common sense.
- To point out shortcuts and tricks that can get you to the answer more quickly.
- To warn you about common traps that students fall into and help you to avoid them.
- To provide you with hundreds of problems from past exams, with solutions.
- To provide you with original practice exams that will help prepare you for the real thing.

To highlight the concepts, tricks, shortcuts, and traps, you will see special symbols such as the following throughout the manual:

Concept Alert! **Shortcut Alert!**

Trick Alert! **Trap Alert!**

Problems

There is an old cliché in the real estate industry: What are the three most important factors in evaluating property? Answer: Location! Location! Location! If we had to say what the three most important factors are in studying for (and passing) this exam, they would be Problems! Problems! Problems! You must do a great variety of problems, preferably under time pressure, and especially as you get closer to the date of the examination.

[1] The syllabus says that only Sections 1–4 of this study note are required reading. However, the author of this manual recommends that you read Section 5 as well.

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Many of the problems in this manual are taken from past SOA and CAS examinations. We want to thank the two actuarial societies for their kind permission to publish these questions.

There are a number of points about past exam questions that you should be aware of:

- These questions, which date from the early 1980's, were created by different exam committees, under different syllabi, for exams of different length, etc. Thus, they can vary greatly in style, difficulty and emphasis of topics. In spite of this, you should find that solving prior exam problems, even very old ones, is helpful in preparing for the exam. It will expose you to a great variety of types of questions and approaches to solutions.
- This manual does not contain the questions and solutions available on www.BeAnActuary.org. Currently, this website includes the actual May 2005 and November 2005 exams (but note that these two exams were very easy) and well over 100 sample questions and solutions, many of which are from past exams. ***You should definitely get these additional practice problems.*** You can download them for free from the BeAnActuary website. We have not included them in this manual because we didn't want to waste space and expense by duplicating material that is freely available from another source.
- The actual exam consists of 35 questions in CBT (computer based testing) format, with a 3-hour time limit.
- The order of the past exam questions that follow each topic in this manual is from the most recent to the oldest.
- There is a code following each question in this manual from a past exam that tells you which exam the question came from.
- Some of the questions have answer choices in ranges, such as "(A) Less than 11.3% (B) At least 11.3% but less than 11.5%," etc. These questions almost always come from CAS exams, which used this style for many years. Virtually all questions on the SOA/CAS Exam FM/2 since 2000 have had specific answer choices, rather than ranges.

To make it easier to locate them, the sections of the manual with past exam questions and solutions have running heads that are shaded like this:

Past Exam Questions on Sections 1b to 1f

and

Solutions to Past Exam Questions on Section 1b to 1f

"Stepping Stones"

Following each section of the manual, you will find illustrative examples, with solutions, that are called "Stepping Stones." They are indicated by this heading:

Stepping Stones

These examples are original to the manual and have a level of difficulty that runs from easy to moderate. They have two purposes:

- To make sure you understand the material that you have just read.
- To serve as "stepping stones" to more difficult exam-level problems.

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Original Practice Exams

In addition to the questions from past exams, the manual contains six original full-length practice exams, with solutions. These exams consist of 35 questions each, with a 3-hour time limit, just like the actual exam. The distribution of questions by topic follows the SOA syllabus.

Take these exams in the order given. We believe that Practice Exam 6 is the most difficult. As we receive feedback from students about the difficulty of the six exams, we will post the information on the ASM website at www.studymanuals.com.

Calculators

Begin using your calculator immediately. Become thoroughly familiar with its operation. It should become like a trusted friend to you after awhile.

Our advice is to get a financial calculator for the exam, such as the BA-35©, BA II Plus©, or BA II Plus Professional©. These calculators have special keys that are very useful for solving problem in financial mathematics. Which one should you use? Any of them will serve you well for the exam. However, Texas Instruments discontinued manufacturing the BA-35 in 2005. You really don't need the Professional model, unless you are willing to spend extra money for some additional features that aren't essential for the exam. For these reasons, we have based the Calculator Notes in this manual on the BA II Plus. You will find these notes at appropriate points in the manual, identified by this icon:

Please note that many students prefer to use one of the TI-30 MultiView series as their primary calculator because of the two-line display and superior algebraic capabilities. These students feel that they can even calculate certain interest functions, such as $a_{\overline{n}|}$, more quickly on the TI-30 series, even though the BA series has special keys for this function. They use the BA series as a backup calculator, mainly to calculate an unknown interest rate when the amounts of payments and their present value are given.

You can do a search on the Discussion Forum of the Actuarial Outpost (www.actuarialoutpost.com) to get other students' opinions on which calculators to use, and then make your own decision.

Studying for the Exam

Everyone is different. Some people like to study in the morning, some late at night. Some people can only study for a couple of hours before their brain begins to fry; others can study for long stretches. Some people can cram very effectively, but others need to cover the material over a period of several months. Almost everyone finds ways to procrastinate from time-to-time. You know your own study habits best, so build on your strengths and be aware of your weaknesses.

We recommend that you set up a study schedule right now. Determine how many hours you will realistically be able to put in between now and the exam. Many students use this rule-of-thumb: devote 100 hours of study time for each hour of exam time. For a 3-hour exam like FM/2, this would mean 300 hours. Of course, people vary considerably in how quickly they learn new material, so use this rule-of-thumb only as a rough guide.

When you set up your study schedule, be very aware of the SOA's allocation of questions by topic. The following table shows how many questions, on the average, you should expect for each topic, assuming that the SOA sticks to its own allocation when it constructs your particular computer-based exam. The table also shows which sections of the manual cover each topic.

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Topic	Allocation by SOA	Midpoint of Allocation	Number of Questions (Based on a 35-Question Exam)	Sections of the Manual that Cover the Topic
A. Time Value of Money (Interest & discount rates, force of interest, equations of value, etc.)	10–15%	12.5%	4.375	Sections 1 & 2
B. Annuities (including arithmetic and geometric)	15–20%	17.5%	6.125	Sections 3 & 4
C. Loans (including interest and principal portion of payments, final payment questions, amortization, sinking funds, etc.)	15–20%	17.5%	6.125	Section 6
D. Bonds	15–20%	17.5%	6.125	Section 7
E. General Cash Flows and Portfolios (yield rates, dollar-weighted and time-weighted rates of returns, duration, convexity, spot rates, forward rates, yield curve, dividend discount model, etc.)	10–15%	12.5%	4.375	Sections 5, 8, 10a to 10g, 11a
F. Immunization	10–15%	12.5%	4.375	Sections 10h to 10k
G. Interest Rate Swaps	0–10%	5%	1.75	Section 11b
H. Determinants of Interest Rates	0–10%	5%	1.75	Section 9
Grand Total		100.0%	35	

Note especially that topics B, C, and D comprise 52.5% of the exam and that topics A, E, and F comprise 37.5% of the exam, for a total of 90%. This tells you where you should spend most of your time studying.

The above table is based on the syllabus for the June 2017 exam as posted on the BeAnActuary website in mid-January 2017, when this manual was published. Please note that the final syllabus will be posted on that website at the end of January 2017 or in early February. You should check the final syllabus to see whether there have been any changes. Also, if you are taking the exam in August 2017 or later, check the syllabus for the particular month and year of your exam, since the exam committee does make changes from time to time.

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As you follow your schedule, will you fall behind from time-to-time? Of course. But if you have a schedule, at least you will know how far behind you are. This should spur you on to catch up.

Should you do all of the problems in a sub-section before you move on to the next sub-section? Our advice is to do as many problems as you can, but to move on if you are falling behind schedule; hopefully, you will catch up later.

You should leave room at the end of your schedule for at least three or four weeks of doing nothing but solving problems (and maybe a little bit of review of the topics you are having difficulty with). Be strict with yourself and work "by the clock." It's not important that you get the correct answer to a problem the first time that you do it. What *is* important is that you try the problem again a couple of days later and get it right the second (or even the third) time. (Mark off the problems that you don't get right and do batches of these problems a few days later.) This shows that you have really mastered the points being tested.

"Hitting a Brick Wall": Sections 3 and 4

Some students find that when they attempt to cover the topics in Section 3 and 4, it's like "hitting a brick wall." There is a lot of material in these two sections and it can be daunting if you are encountering it for the first time. If this happens to you, we have a couple of suggestions:

- Don't knock yourself out if you don't understand something. Move on to the next section and come back later on. You will find that as you spend more time on the material, you will mature in your understanding. Concepts that seemed difficult at first will seem easier when you revisit them.
- Don't try to do all of the problems at the end of each section the first time around. Do the first few, since they are the most recent, or do only the odd or even problems. Come back later and do the rest of the problems (or most of them). They will seem easier the second time around.

It's true that there are a lot of problems in these two sections, about 200 in all, but this shows how important these topics have been on past exams.

Taking the Exam

The single most important rule about taking the exam is to *keep moving*. Don't get bogged down on any one question. Try to look at every question at least once. Be aware of your progress throughout the exam. Try to spend your time on the problems that are easier for you and drop the ones that are giving you too much trouble.

Since this exam has 35 questions and a 3-hour time limit, you have an average of about 5 minutes to spend on each question. You may be tempted to continue spending time on a problem because you have already invested, say, 5 or 6 minutes in it. We strongly suggest that you develop the discipline to drop the question at this point and move on. There may be an easier question waiting for you later on—and getting it right counts just as much as getting a hard question right. And you may have time to go back to the question later on and knock it off.

Your Comments

We welcome your comments, criticism, suggestions, and reports of any errata that you may find. Please e-mail us at *mail@studymanuals.com*

Acknowledgments

My thanks to Wafaa Shaban, Lecturer and Director of the Actuarial Program at the University of North Carolina at Charlotte, who meticulously reviewed all of the new material in the 12th edition, noted many corrections and made valuable suggestions for improvements. Of course, any remaining errors and defects are entirely my own responsibility. Thank you, Wafaa.

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My thanks to Rick Gorvett, currently Staff Actuary at the Casualty Actuarial Society, who was my co-author for previous editions of this manual. In addition to making significant contributions to these editions, Rick was the point man for fielding questions from students about the manual, a task he handled with agility, clarity, and good humor. Thank you, Rick.

My thanks to Paul Anagnostoupoulos of Windfall Software, who, as usual, rapidly keyboarded my messy scribblings into intelligible English and precise mathematical notation. Thank you, Paul.

Finally, my thanks go to my long-suffering wife, Maida, who had to compete with this manual for my time, usually unsuccessfully, over a period of several months. I'm sorry, Maida, that I had to take time out from my supposed "retirement" to once again work on the manual. I will convey your exhortations to the SOA that they not change the syllabus again for at least 25 years, but based on their track record, I think it's a vain hope. Thank you so much for your patience, Maida.

I wish you good studying and good luck on the exam. May the force of interest be with you.

Harold Cherry

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PART I

Financial Mathematics

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1

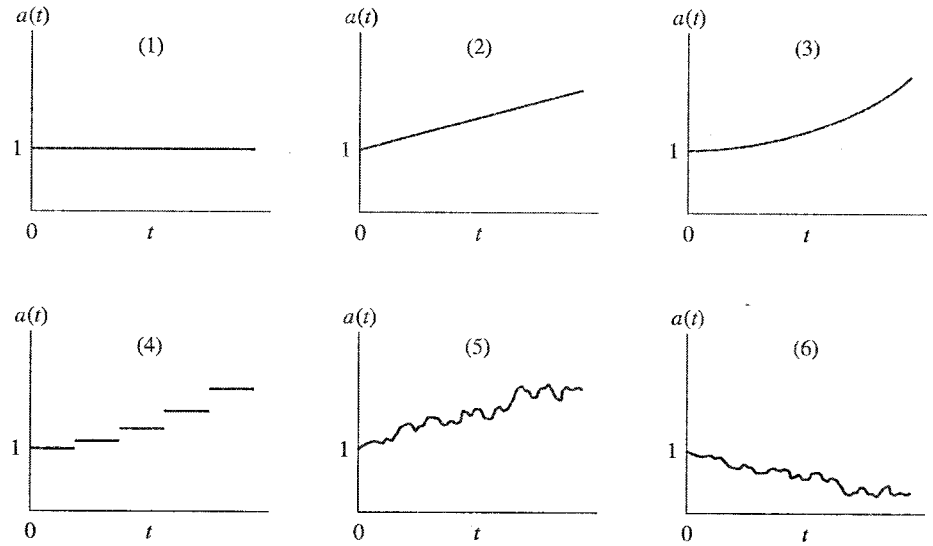
Interest Rates and Discount Rates

§ 1a. Basic Concepts

§ 1a(i) Accumulation Function and Effective Rate of Interest

Imagine a fund growing at interest. It would be very convenient to have a function representing the amount in the fund at any time t. The function $a(t)$ is defined as the accumulated value (AV) of the fund at time t of an initial investment of \$1.00 at time 0. $a(t)$ is called the "accumulation function."

Consider the following accumulation functions. Can you think of any real-life situations where you might encounter them?



(1) This is the accumulation function for money put in a piggy bank or under the mattress. It might also represent "a friendly loan from my father-in-law" or a checking account where you get no interest.

(2) This is so-called "simple interest," where the accumulation function is linear. But we will see that "simple interest" isn't so simple at all.

(3) This accumulation function is an exponential. As we shall see, this is referred to as "compound interest," where the fund earns interest on the interest at a constant rate.

(4) This is the accumulation function for an account where you are credited with interest only at the end of each interest period.

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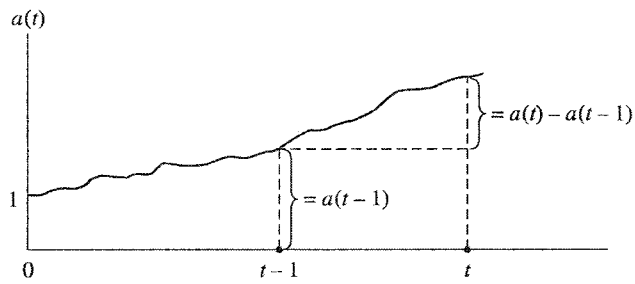
(5) This is an accumulation function for a fund that grows at a varying rate. An investment in stocks is often given as an example.

(6) This is the stock you bought last month.

What are the properties of an accumulation function? By definition, $a(0) \equiv 1$. Other than that, anything goes. But the accumulation functions that we will generally deal with in this course will also have the properties of being (1) continuous and (2) increasing.

Suppose we want to measure the rate of growth of a fund in, say, the tth year. Let's say that the accumulation function looks like the following graph in the tth year. (Remember that the tth year runs from time $(t-1)$ to time t, just as the first year runs from time 0 to time 1.)



The *amount* of growth in the tth year (i.e., the interest earned) is $a(t) - a(t-1)$. The *rate* of growth (based on the amount in the fund at the beginning of the year) is:

$$\frac{a(t) - a(t-1)}{a(t-1)}$$

(Note that the amount in the fund at the beginning of the tth year is the same as the amount in the fund at the end of the $(t-1)^{st}$ year, namely, $a(t-1)$, as long as no new investments are made.) This rate of growth is called the "effective rate of interest" and has the symbol i_t. So we have:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

We will also define an "amount function," $A(t)$, as the AV at time t of k invested at time 0 (rather than 1 invested at time 0). Obviously, $A(t) = ka(t)$. Mathematically, we need only one of these two functions, but having both can be handy. Given the definition of $A(t)$, the effective rate of interest can also be defined as:

$$i_t = \frac{A(t) - A(t-1)}{A(t-1)}$$

because substituting $A(t) = ka(t)$ and $A(t-1) = ka(t-1)$, the k's cancel and we're right back to the original definition.

Using this version of the definition of i_t, go ahead and solve for $A(t)$. You should get the following:

$$A(t) = A(t-1) + i_t A(t-1)$$

or $A(t) = (1 + i_t)A(t-1)$

How would you explain this?

The answer is as follows: The fund at the end of the tth year is equal to the fund at the beginning of the year plus the interest earned during the year.

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Note that the interest earned during the year is the effective rate of interest multiplied by the fund at the beginning of the year. This is consistent with the definition of the effective rate of interest.

Stepping Stones

EXAMPLE 1

You are given that the amount function is $A(t) = 4t^2 + 8t + 4$. What is the accumulation function?

SOLUTION

First, determine k: $A(0) = k = 4$. Then $A(t) = ka(t) = 4a(t)$ and $a(t) = (1/4)A(t) = (1/4)(4t^2 + 8t + 4) = t^2 + 2t + 1$.

EXAMPLE 2

For the amount function in Example 1, what is the total interest earned in the 2nd, 3rd, and 4th years?

SOLUTION

Total interest $= A(4) - A(1) = 100 - 16 = 84$.

EXAMPLE 3

For the amount function in Example 1, what is the effective rate of interest in the 5th year?

SOLUTION

$$i_5 = [A(5) - A(4)]/A(4) = (144 - 100)/100 = 44\%$$

EXAMPLE 4

John invests X in a fund growing in accordance with the accumulation function implied by the amount function in Example 1. Edna invests X in another fund growing in accordance with the accumulation function implied by the amount function $A(t) = 4t^2 + 2$. When does Edna's investment exceed John's?

SOLUTION

We want to compare the two funds assuming that equal investments of X are made at time 0. From Example 1, John's accumulation function is $t^2 + 2t + 1$. For Edna, $k = 2$ and the accumulation function is $2t^2 + 1$. To determine when Edna's investment exceeds John's, we set:

$$X(2t^2 + 1) > X(t^2 + 2t + 1)$$

which reduces to:

$$t^2 - 2t > 0 \quad \text{or} \quad t(t - 2) > 0$$

Thus, Edna's fund exceeds John's after 2 years.

§ 1a(ii) Simple Interest

Let's consider so-called "simple interest." What are the effective rates of interest for years $1, 2, 3, \ldots, t$? Under simple interest, the accumulation function is linear. We will use i as the linear constant:

$$a(t) = 1 + it$$

We have the following by definition of i_t:

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$$i_1 = \frac{a(1) - a(0)}{a(0)} = \frac{(1+i) - 1}{1} = i$$

$$i_2 = \frac{a(2) - a(1)}{a(1)} = \frac{(1+2i) - (1+i)}{1+i} = \frac{i}{1+i}$$

$$i_3 = \frac{a(3) - a(2)}{a(2)} = \frac{(1+3i) - (1+2i)}{1+2i} = \frac{i}{1+2i}$$

$$\vdots$$

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} = \frac{(1+it) - [1+i(t-1)]}{1+i(t-1)} = \frac{i}{1+i(t-1)}$$

For each year, the numerator (the interest earned) is a constant i, since $a(t)$ is linear. The denominator (the amount in the fund at the beginning of the year) is increasing, which means that the effective rate of interest decreases with time.

So we see that "simple interest" isn't simple at all. A linear accumulation function implies a decreasing effective rate of interest. (In fact, it decreases hyperbolically!)

In financial transactions, simple interest is often used for fractions of an interest period for convenience. (It is a bit confusing to use i as the linear constant, since it is the effective rate of interest for the first year only. It would be better to use another constant, as in $a(t) = 1 + kt$, but the general practice is to use i.)

Stepping Stones

EXAMPLE 1

Money doubles in 10 years at a simple interest rate of i per annum. Determine i.

SOLUTION

We are given that 1 accumulates to 2 in 10 years at a simple interest rate i:

$$1 + 10i = 2$$
$$i = .10 = 10\%$$

EXAMPLE 2

John deposits \$100 at a simple interest rate of i per annum. At the end of 10 years, he has \$150 in his account. Lolita deposits \$200 at a simple interest rate of $2i$ per annum. How much is in her account at the end of 20 years?

SOLUTION

First, determine i:

$$100(1 + 10i) = 150$$
$$i = .05 \quad \text{and} \quad 2i = .10$$

At the end of 20 years, Lolita has $200[1 + (20)(.10)] = \$600$ in her account.

§ 1a(iii) The Accumulation Function in Terms of i_t

Suppose we are given that the effective rate of interest is 5% in the first year and 6% in the second year. We invest \$1.00 at time 0. How much is in the fund at the end of one year? Obviously, \$1.05. How much is in the fund at the end of two years? We start with \$1.05 at the beginning of the second year and we earn interest on it at 6%, so:

$$a(2) = 1.05 + .06(1.05)$$

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or $a(2) = (1.05)(1.06)$

In symbols:

$$a(2) = (1 + i_1)(1 + i_2)$$

If we continue this process for t years, we have:

$$a(t) = (1 + i_1)(1 + i_2) \ldots (1 + i_t)$$

or using the symbol $\prod$ for product:

$$a(t) = \prod_{j=1}^{t} (1 + i_j)$$

Note that each factor in this chain multiplication has the effect of (a) bringing the fund at the beginning of the year to the end of the year without interest (this is the "1" in $(1 + i_j)$) and (b) adding interest on this amount at the effective rate (this is the "i_j" in $(1 + i_j)$).

Stepping Stones

EXAMPLE 1

Elsie deposits \$500 in a bank account that earns interest at a 5% effective rate in year 1. At the end of year 2, Elsie has \$552.30 in her account. What effective rate of interest did she earn in year 2?

SOLUTION

$$(500)(1.05)(1 + i_2) = 552.30$$
$$i_2 = 5.20\%$$

EXAMPLE 2

The effective rate of interest in year t is given as $i_t = k + .01(t - 1)$, where k is a constant. You are given that the accumulated value of \$100 at the end of 2 years is \$112.35. Determine the effective rate of interest for year 2.

SOLUTION

$$100(1 + i_1)(1 + i_2) = 112.35$$
$$(1 + k)(1 + k + .01) - 1.1235 = 0$$
$$k^2 + 2.01k - .1135 = 0$$

Using the quadratic formula, you should get the positive root of k as $k = .0550$ (to 4 decimals). Then $i_2 = k + .01 = .0650 = 6.50\%$.

§ 1a(iv) Compound Interest

Let's take the case where i_j is a constant i. The chain product collapses into:

$$a(t) = (1 + i)^t$$

This special case is called "compound interest." (Any chain multiplication, even where the i_j's are not constant, involves earning interest on the previous interest, or "compounding." But you will often see the term "compound interest" used for the case of a constant effective rate.) Most of the problems for this exam are based on the assumption of compound interest.

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Stepping Stones

EXAMPLE 1

Mary deposits money in a bank that compounds interest at i effective per annum. At the same time, Sam deposits the same amount as Mary in another bank that compounds interest at $(i + .01)$ effective per annum. At the end of 5 years, Sam has as much in his account as Mary will have at the end of 10 years. Determine i.

SOLUTION

$$(1 + i + .01)^5 = (1 + i)^{10}$$

Taking the 5th root of each side, we get:

$$(1.01 + i) = (1 + i)^2$$

$$i^2 + i - .01 = 0$$

Solving the quadratic for the positive root of i, you should get $i = .0099$ to 4 decimals, or $i = .99\%$.

EXAMPLE 2

Chauncey makes an investment of \$1,000 that will earn interest at an effective annual rate of 5% for 5 years. Abigail makes an investment of \$1,000 that will earn effective annual rates of 3%, 4%, 5%, 6%, and 7% in years 1, 2, 3, 4, and 5, respectively. Abigail boasts to Chauncey, "My investment is better than yours. I will have more than you 5 years from now." Is Abigail right?

SOLUTION

At the end of 5 years, Chauncey will have $(1{,}000)(1.05)^5 = \$1{,}276.28$ and Abigail will have $(1{,}000)(1.03)(1.04)(1.05)(1.06)(1.07) = \$1{,}275.70$. Abigail is wrong: She will have slightly less (\$.58) than Chauncey.

§ 1a(v) Present Value

So far, we've been talking about the AV of a fund, i.e., how much is in the fund after t years, if we invest a given amount today. Consider the "opposite" question: How much should we invest today in order to have a given amount, say \$1.00, at the end of t years? The amount that we should invest is called the ***present value*** (PV) of \$1.00 due in t years.

Why would we be interested in present values? Isn't the real-life question, "How much is in a fund at a time t if we invest a given amount today?"

Suppose that in your personal financial life, you need to save or invest money that will grow to a specified amount in a specified number of years. For example, you may want to start saving now to buy a car for \$20,000 in two years, or to make a down payment of \$25,000 on a home in 3 years, or to pay a year's college expenses of \$30,000 in 8 years. All of these questions involve finding the PV, i.e., the amount invested today that will grow to the desired amount in the desired time.

Also, in your work as an actuary, you will often have to determine how much someone should pay now in order to receive future benefits, such as death benefits under a life insurance policy, or lifetime income under an annuity contract or pension plan. (The payments, or "investments," and the benefits, or "accumulated values," may be paid in installments in many cases. Also, these examples involve contingencies, i.e., probabilities of future events. For this exam, we are only concerned with payments that are certain to be made.) So you can see that the concept of present value is *very important* in actuarial work.

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Consider the general accumulation function $a(t)$. We want to determine how much to invest today in order to have \$1.00 in t years. We will designate this amount as (PV). Since we require that (PV) grows to \$1.00 in t years, we have:

$$(PV)a(t) = 1$$

$$(PV) = \frac{1}{a(t)}$$

The function $\frac{1}{a(t)}$ is called the "discount function." As we will see in the next section, "discounting" a future amount means the same thing as finding its present value. In particular, if $a(t) = (1+i)^t$ (the compound interest case), we have:

$$(PV) = \frac{1}{a(t)} = \frac{1}{(1+i)^t} = (1+i)^{-t}$$

Because the term $(1+i)^{-1}$ comes up so often (i.e., the PV of \$1.00 due in a year), a special symbol has been invented for it, namely, v. So we have:

$$v = \frac{1}{1+i} = (1+i)^{-1}$$

(PV) of \$1.00 due in t years$= v^t = (1+i)^{-t}$.

Stepping Stones

EXAMPLE 1

What deposit made today will provide for a payment of \$1,000 in 1 year and \$2,000 in 3 years, if the effective rate of interest is 7.5%?

SOLUTION

$$PV = 1{,}000v + 2{,}000v^3 \text{ at } 7.5\% = \$2{,}540.15$$

$$(v = 1.075^{-1})$$

EXAMPLE 2

A fund credits interest at a constant effective rate. The present value of \$1.00 due in n years is \$.50. Three deposits of \$1.00 each are made at times 0, n, and 2n. What is the accumulated value of these deposits at time $3n$?

SOLUTION

Since $v^n = 0.5$, $(1+i)^n = 2$. The AV of the 3 deposits is $(1+i)^{3n} + (1+i)^{2n} + (1+i)^n = 2^3 + 2^2 + 2 = 14$.

EXAMPLE 3

The accumulation function is given as $a(t) = .1t^2 + .2t + 1$. Determine the present value of \$5 paid at time 5 and \$10 paid at time 7.

SOLUTION

$$PV = 5/a(5) + 10/a(7) = 5/4.5 + 10/7.3 = \$2.48$$

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Calculator Notes

It is essential that you use a financial calculator to get numerical results. As we noted in the Introduction, Calculator Notes for the BA II Plus will appear at appropriate points in this manual. If you are using another calculator, do the problems in these notes anyway, and check that you got the same answers.

Calculator Notes #1 begins on the following page.

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Calculator Notes #1: Formatting; Present Values and Future Values

"Calculator Notes" will appear at appropriate points in this manual. These notes will cover the functions of the Texas Instruments BA II Plus calculator that are essential for Exam FM/2. (As noted in the Introduction, the BA-35 was discontinued by TI in 2005, although it is still an authorized calculator for the actuarial exams.)

These notes will not cover functions that you don't need for the exam (for example, the trig, statistical or depreciation functions).

In addition to the Calculator Notes in this manual, you should print out the review of calculator functions for the BA II Plus written by Sam Broverman and available on the BeAnActuary.org website. Also, you should have the TI guidebook as a general reference. If you don't have it, you can download it from the TI website.

Primary and Secondary Functions

The calculator has primary and secondary functions. The primary functions are printed on the keys themselves, for example, [×] for "multiply." The secondary functions are shown in yellow *above* the keys. For example, the secondary function of the [×] key is x!.

The secondary functions are accessed by pressing the [2nd] key (which is the only yellow key) and then the desired key. For example, to compute 5!, use the following sequence of keystrokes: 5 [2nd] [x!]. You should get 120.

Note that we have indicated the primary function [×] by using a box and the secondary function [x!] by using brackets.

Formatting the Calculator

The calculator was pre-set to certain defaults when it left the factory, but these settings may have been changed since then. Let's format the calculator so that we can all start from the same place.

To access the Format mode, press [2nd] [FORMAT]. (The [FORMAT] key is the middle key in the bottom row.)

You can scroll the various formats by pressing the [↓] or [↑] keys in the top row. Scrolling down, note that the formats are "DEC," "DEG," "US" (for dates), "US" (for number separation), "Chn," and back to "DEC." We are only interested in a couple of these formats for Exam FM/2.

Scroll to "DEC." This controls the number of decimal places shown in the display. The factory default is two decimal places but you can set the calculator to show from 0 to 9 places. (The calculator uses 13 places internally, regardless of what is shown in the display.)

Let's say we want the display to show 4 decimal places. While "DEC" is in the display, press 4 [ENTER]. Let's check that this has worked. Leave the [FORMAT] mode by pressing [2nd] [QUIT]. (The [QUIT] key is in the upper left-hand corner.) This puts us in the standard calculator mode. Now multiply two numbers, such as 2.6835 [×] 5.625 [=]. The answer shows to 4 decimals (rounded) as 15.0947.

Leave your calculator set to display 4 decimals. This is enough accuracy for most questions on the exam. (Remember that internally, 13 places are carried.) But you can set it for fewer or more places if you wish. It's a matter of personal preference: After you've used the calculator for awhile, you can decide for yourself.

Let's return to the Format mode by pressing [2nd] [FORMAT] again. Now scroll down to "DEG." This controls whether angles are expressed in degrees or radians, which doesn't concern us for this

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exam. So we scroll down to "US 12-31-1990." This controls how dates are shown (i.e., whether month or day comes first). We're not going to be using dates very much, so scroll down to "US 1,000.0000." This controls how numbers are separated. For example, in US format, "one thousand" with 4 decimals is shown as 1,000.0000. In European format, commas become decimal points and decimal points become commas, so "one thousand" is shown as 1.000,0000. You should set the calculator in US format. In case it's in European format (shown as "EUR"), you can change it as follows: [2nd] [SET]. (The [SET] key is secondary to the [ENTER] key.) If you keep pressing [2nd] [SET], the display will alternate between "US" and "EUR." Leave it at "US."

Finally, scroll down to "Chn." This controls the calculation method, i.e., either "Chain" or "AOS" (Algebraic Operating System). To alternate between "Chn" and "AOS," press [2nd] [SET]. Leave the setting at "Chn."

What's the difference between the Chain and AOS calculation methods? The Chain method is probably what you are most familiar with in a calculator. For example, if you press 2 [+] 5 [×] 3 [=], you expect the answer to be 21, i.e., the calculator first adds 5 to 2 and then multiples the result by 3. The AOS method follows the hierarchy of mathematical operations: multiplication and division are completed before addition and subtraction. Thus, for the same keystrokes in the AOS mode, the answer will be 17: The calculator multiples 5 by 3 first and then adds 2. You should leave the calculator in "Chn" mode.

You now have completed formatting the calculator. Press [2nd] [QUIT] to return to standard calculator mode.

Accumulated Values and Present Values

1. What is the accumulated value of $1,537 at the end of 8 years at an effective annual interest rate of 7%?

There are two ways of computing this:

a. Using the regular keys

The exponential key [y^x] is located in the 4th column, 5 keys down. You'll be using it a lot, so get to know where it is.

The answer to this question is $1{,}537(1.07)^8$. Use these keystrokes:

1.07 [y^x] 8 [×] 1,537 [=]

The display should show 2,640.8522.

Note that it is not necessary to press [=] after entering 1.07 [y^x] 8. When you press [×], the exponentiation is automatically done and the result is ready to be multiplied by the next value you enter. (Some guides include this equal sign in the keystrokes. This doesn't affect the final result, but it's not necessary.)

b. Using the TVM keys

"TVM" stands for "time value of money." This simply means that $1.00 in your greedy little hand today is worth more than $1.00 paid to you at a future time. (See the first page of Chapter 2 of this manual.)

The TVM keys are the keys in the 3rd row: [N], [I/Y], [PV], [PMT] and [FV]. For simple accumulation and present value problems, it really isn't necessary to use these keys: It's easy and direct to use the [y^x] key, as we did above. However, the TVM keys are very important when it comes to handling annuities, loans, mortgages, bonds, etc. (covered in Chapter 3 and later), so we may as well get familiar with them now.

[N] is the number of years (or periods).

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[I/Y] is the interest rate per year (or per period).
[PV] is the present value.
[PMT] is the amount of each payment of an annuity. (We won't use this key until Chapter 3.)
[FV] is the future value. (This is the same as what we have been calling the accumulated value, or AV.)

There are two very important "hidden" settings that affect the TVM calculations. Press [2nd] [P/Y] ([P/Y] is secondary to the [I/Y] key) and the display will show "P/Y = ." "P/Y" stands for the number of payments per year. Now press the arrow key [↓]. "C/Y = " will show in the display. "C/Y" stands for the number of times that interest is compounded per year.

We strongly recommend that you set P/Y equal to 1 and always keep it at 1.[1]

This is because it's very easy to forget what value is in the P/Y register. If you know that it's always set to 1, you are much less likely to make calculation errors. (This will become clearer when we get to Section 4.)

This is how to set P/Y equal to 1: Press [2nd] [P/Y], then press 1 [ENTER]. C/Y will also automatically be set to 1. (You can check this by pressing [↓].)

To get back to the standard calculation mode, press [2nd] [QUIT].

Trap Alert!

When you take the actual exam, the proctor will probably reset the calculator to its factory defaults by pressing [2nd] [RESET]. (The [RESET] key is secondary to the [+/−] key.) If your calculator is reset, make sure that [P/Y] = [C/Y] = 1, because the default for some older models of the BA-II Plus is 12. Also, resetting the calculator will set two decimal places to show in the display. If you want to display a different number of places, you will have to set that as well.

Now we're ready to do problem 1. We have to enter the values of N, I/Y and PV, and then compute FV. But first make sure that you clear any current values in the TVM registers by pressing [2nd] [CLR TVM]. ([CLR TVM] is secondary to the [FV] key. Pressing it will *not* change the settings of P/Y and C/Y.) Then use these keystrokes:

8 [N] 7 [I/Y] 1,537 [+/−] [PV] [CPT] [FV]

The answer should appear as 2,640.8522, which agrees with the previous answer. (Pressing the [+/−] key, which is in the bottom row, changes the sign of the value in the display. We will explain why we enter −1,537 as the PV in the comments below.)

Some important comments:

- The data can be entered in any order in the TVM keys. For example, we could have entered the interest rate first, then the number of years and then the PV. If we then press [CPT] [FV], we would get the same answer as above.
- The effective rate of interest is entered as a percent in [I/Y], so 7% is entered as 7, not .07. Similarly, a rate like 3.4562% would be entered as 3.4562.
- After we enter the data, we compute the unknown by pressing [CPT] and then the value that we want, in this case [FV].
- Here is an explanation of why we enter the PV as a negative amount: We will take the point of view of a depositor, a lender, etc., i.e., someone who has a cash flow *out* in the amount of 1,537 at time 0. At the end of 8 years, we expect to receive a cash flow *in*, as a withdrawal from the bank, the repayment of the loan by the borrower, etc. The calculator expects us to

[1] C/Y will automatically be set to whatever P/Y is set to. It can also be set independently of P/Y.

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enter a cash flow *out* as a negative value. When we press [CPT] [FV], the calculator computes the cash flow *in* that we receive at the end of 8 years as a positive value.

Of course, we could reverse roles. If we take the point of view of the bank, the borrower, etc., 1,537 is the amount we *receive* at time 0, i.e., it is a cash flow *in,* and we would enter it as a positive value. When we press [CPT] [FV], the calculator computes FV as a negative amount, since it is a cash flow *out*, i.e., the amount the bank, the borrower, etc., repays the investor.

It really doesn't matter which point of view we take, as long as we are consistent within the same problem. (In Calculator Notes, we will sometimes take one point of view and sometimes the other. If you happen to take the other point of view, our answers will have opposite signs.)

- Sometimes you may forget what values are in the TVM registers. You can check a register by pressing [RCL] and then the register you want. Try this now.
- If there is an incorrect value in a register, simply enter the correct value in the display and then press the register key.
- To check the value currently in the P/Y register, press [RCL] [2nd] [P/Y]. For the value of C/Y, press [↓].
- It's essential to clear the TVM registers before you start a new problem. You do this by pressing [2nd] [CLR TVM]. *Even if you turn the calculator off, the "garbage" that you left in TVM will still be there when you turn the calculator back on! Try it and see! So you must clear these registers for each new problem.*

Now that you have done this problem using the TVM keys, and have read all of the comments, we wouldn't blame you for asking, "Why go through all of this when we could simply use the [y^x] key?" The answer is that, as we noted before, you will be using TVM *big time* starting with Chapter 3, so you may as well get used to it now. With a little practice, using TVM will become second nature to you. Also, some problems, like finding an unknown interest rate or an unknown time are easier to do using the TVM keys.

Now let's continue using the calculator to solve problems.

2. What is the PV of $1,250 due in 8 years at an effective annual interest rate of 5.23%?

a. Using the regular keys

1.0523 [y^x] 8 [+/−] [×] 1,250 [=]

The answer is $831.37 to the nearest $.01.

b. Using the TVM keys

Make sure that you have kept P/Y and C/Y equal to 1, as we recommended.

[2nd] [CLR TVM] (to clear the TVM registers)

8 [N] 5.23 [I/Y] 1,250 [FV] [CPT] [PV]

The answer is −831.37. If you had entered 1,250 as a negative, the answer would show as a positive value.

Unknown interest rate

3. $10,250 accumulates to $23,237 in 13 years. What is the effective annual interest rate?

a. Using the regular keys

The equation of value for this problem is:

$10{,}250(1+i)^{13} = 23{,}237$

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Solving for i:

$$i = \left(\frac{23{,}237}{10{,}250}\right)^{1/13} - 1$$

Note: The calculator has 10 storage registers numbered 0 to 9. To store a value that appears in the display, let's say in register 0, simply press [STO] 0. The value will be stored in register 0 and will also continue to be shown in the display. If you want to recall this value later, press [RCL] 0 and it will be shown in the display. It will also remain in the register.

We can use the following keystrokes:

13 [1/x] [STO] 0 23,237 [÷] 10,250 [y^x] [RCL] 0 [−] 1 [=]

The answer is .0650 or 6.50%.

Note that we computed 1/13 as the reciprocal of 13, stored it, and later recalled it to use as the exponent in the [y^x] operation.

b. Using the TVM keys

[2nd] [CLR TVM] *(From now on, we will not show this step. We'll assume that you do it at the beginning of each new problem.)*

13 [N] 10,250 [+/−] [PV] 23,237 [FV] [CPT] [I/Y]

The answer is 6.50% to 2 decimals, as before.

Unknown time

4. At an effective rate of interest of 8% per annum, the present value of \$100,000 due in X years is \$65,322. Determine X.

a. Using the regular keys

The equation of value is $65{,}322(1.08)^X = 100{,}000$. Solving for X:

$$1.08^X = \frac{100{,}000}{65{,}322}$$

$$X = \frac{\ln\left(\frac{100{,}000}{65{,}322}\right)}{\ln 1.08}$$

The natural log key [LN] is located in the first column of keys. We can use the following keystrokes:

1.08 [LN] [STO] 0 100,000 [÷] 65,322 [=] [LN] [÷] [RCL] 0 [=]

The answer is 5.5332 years.

b. Using the TVM keys

8 [I/Y] 65,322 [+/−] [PV] 100,000 [FV] [CPT] [N]

The answer is $N = 5.5332$ years.

5. How long does it take money to double at 5% effective?

a. Using the regular keys

$1.05^x = 2$

$$x = \frac{\ln 2}{\ln 1.05}$$

1.05 [LN] [STO] 0 2 [LN] [÷] [RCL] 0 [=]

The answer is 14.2067 years.

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b. Using the TVM keys

5 [I/Y] 1 [+/−] [PV] 2 [FV] [CPT] [N]

The answer is N = 14.2067 years.

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Summary of Concepts and Formulas in Sections 1a(i) to 1a(v)

(1) The accumulation function $a(t)$ is the AV at time t of \$1.00 invested at time 0.

(2) The effective rate of interest in the tth year is based on the amount in the fund at the beginning of the year:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

(3) Under "simple interest":

$$a(t) = 1 + it$$

$$i_t = \frac{i}{1 + i(t-1)}$$

(4) In terms of effective rates of interest:

$$a(t) = (1 + i_1)(1 + i_2) \cdots (1 + i_t)$$

(5) Under "compound interest," the effective rate of interest is a constant in all years:

$$i_t = i \quad \text{and} \quad a(t) = (1 + i)^t$$

(6) (a) For any accumulation function $a(t)$, the present value of \$1.00 due in t years is:

$$PV = \frac{1}{a(t)}$$

(b) Under compound interest:

$$PV = \frac{1}{(1+i)^t} = (1+i)^{-t} = v^t$$

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Past Exam Questions on Sections 1a(i) to 1a(v)

1. Sally has two IRA's. IRA #1 earns interest at 8% effective annually and IRA #2 earns interest at 10% effective annually. She has not made any contributions since January 1, 1985, when the amount in IRA #1 was twice the amount in IRA #2. The sum of the two accounts on January 1, 1993 was \$75,000. Determine how much was in IRA #2 on January 1, 1985. [CAS 5/93 #6]

   (A) <\$12,750 (B) \$12,750/\$13,000 (C) \$13,000/\$13,250 (D) \$13,250/\$13,500 (E) ≥ \$13,500

2. Money accumulates in a fund at an effective annual interest rate of i during the first 5 years, and at an effective annual interest rate of $2i$ thereafter.

   A deposit of 1 is made into the fund at time 0. It accumulates to 3.09 at the end of 10 years and to 13.62 at the end of 20 years.

   What is the value of the deposit at the end of 7 years? [SOA 5/93 #2]

   (A) 1.90 (B) 1.98 (C) 2.06 (D) 2.14 (E) 2.23

3. At an effective annual interest rate of i, $i > 0$, each of the following two sets of payments has present value K:

   (i) A payment of 121 immediately and another payment of 121 at the end of one year.
   (ii) A payment of 144 at the end of two years and another payment of 144 at the end of three years.

   Calculate K. [SOA 11/92 #2]

   (A) 237 (B) 232 (C) 227 (D) 222 (E) 217

4. A business permits its customers to pay with a credit card or to receive a percentage discount r for paying cash.

   For credit card purchases, the business receives 95% of the purchase price one-half month later.

   At an effective annual rate of 12%, the two payment methods are equivalent.

   Determine r. [SOA 11/92 #3]

   (A) 4.55 (B) 4.85 (C) 5.15 (D) 5.45 (E) 5.75

5. At an annual effective interest rate of i, $i > 0$, the following are all equal:

   (i) the present value of 10000 at the end of 6 years;
   (ii) the sum of the present values of 6000 at the end of year t and 56000 at the end of year $2t$; and
   (iii) 5000 immediately.

   Calculate the present value of a payment of 8000 at the end of year $t + 3$ using the same annual effective interest rate. [SOA 11/90 #4]

   (A) 1330 (B) 1414 (C) 1600 (D) 1775 (E) 2000

6. An investor puts 100 into Fund X and 100 into Fund Y. Fund Y earns compound interest at the annual rate of $j > 0$, and Fund X earns simple interest at the annual rate of $1.05j$.

   At the end of 2 years, the amount in Fund Y is equal to the amount in Fund X.

   Calculate the amount in Fund Y at the end of 5 years. [SOA 5/90 #1]

   (A) 150 (B) 153 (C) 157 (D) 161 (E) 165

7. (This question should actually appear after Section 1a(vi) to 1a(ix).) You are given:

   (i) Fund X accumulates at an interest rate of 8% compounded quarterly;
   (ii) Fund Y accumulates at an interest rate of 6% compounded semiannually;
   (iii) at the end of 10 years, the total amount in the two funds combined is 1000; and
   (iv) at the end of 5 years, the amount in Fund X is twice that in Fund Y.

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Calculate the total amount in the two funds at the end of 2 years. [SOA 5/90 #4]

(A) 560 (B) 570 (C) 580 (D) 590 (E) 600

8. Gertrude deposits 10,000 in a bank. During the first year, the bank credits an annual effective rate of interest i. During the second year, the bank credits an annual effective rate of interest $(i - 5\%)$. At the end of two years, she has 12,093.75 in the bank.

   What would Gertrude have in the bank at the end of three years, if the annual effective rate of interest were $(i + 9\%)$ for each of the three years? [SOA 11/83 #3]

   (A) 16,851 (B) 17,196 (C) 17,499 (D) 17,936 (E) 18,113

9. John borrows 1,000 from Jane at an annual effective rate of interest i. He agrees to pay back 1,000 after six years and 1,366.87 after another six years.

   Three years after his first payment, John repays the outstanding balance.

   What is the amount of John's second payment? [SOA 11/88 #4]

   (A) 1,020 (B) 1,027 (C) 1,048 (D) 1,073 (E) 1,094

10. Two funds, A and B, start with the same amount.

    Fund A grows at an annual interest rate of $i > 0$ for n years, and at an annual interest rate of $j > 0$ for the next n years.

    Fund B grows at an annual interest rate of $k > 0$ for $2n$ years.

    Fund A equals 1.5 times fund B after n years. The amount in the two funds are equal after $2n$ years.

    Which of the following are true?

    (i) $j < k < i$
    (ii) $k < \frac{i+j}{2}$
    (iii) $j = k\left(\frac{2}{3}\right)^{1/n}$

    [SOA 5/88 #3]

    (A) I and II only (B) I and III only (C) II and III only (D) I, II and III
    (E) The correct answer is not given by (A), (B), (C), or (D).

11. Fund A is invested at an effective annual interest rate of 3%.

    Fund B is invested at an effective annual interest rate of 2.5%.

    At the end of 20 years, the total in the two funds is 10,000. At the end of 31 years, the amount in Fund A is twice the amount in Fund B.

    Calculate the total in the two funds at the end of 10 years. [SOA 11/87 #18]

    (A) 5,732 (B) 6,602 (C) 7,472 (D) 7,569 (E) 8,123

12. You are given:

    (i) The sum of the present values of a payment of X at the end of 10 years and a payment of Y at the end of 20 years is equal to the present value of a payment of $X + Y$ at the end of 15 years.
    (ii) $X + Y = 100$
    (iii) $i = 5\%$

    Calculate X. [SOA 5/87 #8]

    (A) 44 (B) 48 (C) 50 (D) 52 (E) 57

13. The present value of 200 paid at the end of n years, plus the present value of 100 paid at the end of $2n$ years is 200.

    Determine the annual effective rate of interest. [SOA 11/85 #3]

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(A) $\left(\frac{\sqrt{3}+1}{2}\right)^{1/n} - 1$

(B) $1 - \left(\frac{\sqrt{3}-1}{2}\right)^{1/n}$

(C) $\left(\frac{\sqrt{3}-1}{2}\right)^{1/n} - 1$

(D) $\left(\frac{1+\sqrt{3}}{2}\right)^{1/2n} - 1$

(E) $1 - \left(\frac{\sqrt{3}-1}{2}\right)^{1/2n}$

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Solutions to Past Exam Questions on Sections 1a(i) to 1a(v)

1. $2X(1.08)^8 + X(1.1)^8 = 75{,}000$

$$X = \frac{75{,}000}{(2)(1.08)^8 + 1.1^8} = \$12{,}830 \qquad \text{ANS. (B)}$$

2. $(1+i)^5(1+2i)^5 = 3.09$

$(1+i)^5(1+2i)^{15} = 13.62$

Dividing the second equation by the first:

$$(1+2i)^{10} = \frac{13.62}{3.09} = 4.407767$$

$$i = 7.9952\%$$

$$X = (1.079952^5)(1.159904^2) = 1.98 \qquad \text{ANS. (B)}$$

3. $121(1+v) = 144(v^2 + v^3)$

$$= 144v^2(1+v)$$

$$\therefore\ 121 = 144v^2, \quad v^2 = \left(\frac{11}{12}\right)^2, \quad v = \frac{11}{12}$$

$$K = 121 + 121\left(\frac{11}{12}\right) = 231.9 \qquad \text{ANS. (B)}$$

4. Assume that the purchase price is \$1.00, since the price will cancel out anyway. If cash is paid, the business receives $(1-r)$ now. If payment is by credit card, the business receives .95 one-half a month later. The PV of this payment is $.95v^{1/24}$ at 12% effective. Setting these PV's equal:

$$1 - r = .95v^{\frac{1}{24}}$$

$$r = 1 - .95v^{\frac{1}{24}} = 5.45\% \qquad \text{ANS. (D)}$$

5. (i) $= 10{,}000v^6$

(ii) $= 6{,}000v^t + 56{,}000v^{2t}$

(iii) $= 5{,}000$

We are given that (i) = (ii) = (iii). Setting (ii) = (iii) and simplifying:

$$56v^{2t} + 6v^t - 5 = 0$$

This is a quadratic in v^t, which can be seen more clearly by setting $X = v^t$:

$$56X^2 + 6X - 5 = 0$$

This can be solved for X by using the quadratic formula or by factoring:

$$(14X + 5)(4X - 1) = 0$$

Taking the positive root:

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$$4X - 1 = 0$$
$$X = 0.25 = v^t$$

Setting (i) = (iii):

$$10{,}000v^6 = 5{,}000$$
$$v^6 = 0.5, \text{ so } v^3 = 0.5^{1/2}$$

We want $8{,}000v^{t+3} = 8{,}000v^t v^3 = (8{,}000)(0.25)(0.5^{1/2}) = 1{,}414$. ANS. **(B)**

6. $$(100)(1+j)^2 = 100[1 + (2)(1.05j)]$$
$$(1+j)^2 = 1 + 2.1j$$

Expanding the left side and solving for j, we get:

$$j = .1$$
$$AV = (100)(1.1)^5 = 161$$ ANS. **(D)**

7. (a) $1.02^{40}X + 1.03^{20}Y = 1{,}000$
(b) $1.02^{20}X = 1.03^{10}(2Y)$

From (b), $X = \frac{1.03^{10}}{1.02^{20}}(2Y)$

Substituting in (a):

$$\frac{1.03^{10}}{1.02^{20}} \cdot 1.02^{40}(2Y) + 1.03^{20}Y = 1{,}000$$
$$Y = 172.41$$
$$X = \frac{1.03^{10}}{1.02^{20}}(2Y) = 311.86$$
$$(AV) \text{ in 2 years } = 1.02^8(311.86) + 1.03^4(172.41)$$
$$= 559.44$$ ANS. **(A)**

8. $$10{,}000(1+i)(1+i-.05) = 12{,}093.75$$
$$(1+i)(.95+i) - 1.209375 = 0$$
$$i^2 + 1.95i - .259375 = 0$$

Solving the quadratic for the positive root:

$$i = .125$$
$$i + .09 = .125 + .09 = .215$$

AV in 3 years at a rate of .215

$$= 10{,}000(1.215)^3 = 17{,}936$$ ANS. **(D)**

9. Under the original terms of the loan:

$$1{,}000 = 1{,}000v^6 + 1{,}366.87v^{12}$$

This is a quadratic in v^6:

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$$1.36687\left(v^6\right)^2 + v^6 - 1 = 0$$

Using the quadratic formula, the positive root is:

$$v^6 = .5644736$$

Payment in 3 years in place of \$1,366.87 in 6 years

$$= 1{,}366.87v^3 = (1{,}366.87)(.5644736)^{.5}$$
$$= 1{,}027 \qquad \text{ANS. (B)}$$

10. We will use \$1.00 as the starting amount of Fund A and Fund B, since the starting amount will cancel out anyway.

We are given that:

$$(1+i)^n(1+j)^n = (1+k)^{2n} \quad \text{and}$$
$$(1+i)^n = 1.5(1+k)^n$$

From the first equation:

$$(1+j)^n = (1+k)^{2n}/(1+i)^n$$

Substituting for $(1+i)^n$ from the second equation:

$$(1+j)^n = (1+k)^{2n}/1.5(1+k)^n = (2/3)(1+k)^n$$

Taking the nth root of both sides:

$$(1+j) = (2/3)^{1/n}(1+k) \quad \text{and} \quad j = (2/3)^{1/n}(1+k) - 1$$

Therefore, (iii) is false.

The second equation is:

$$(1+i)^n = 1.5(1+k)^n$$

Taking the nth root of both sides:

$$(1+i) = (1.5)^{1/n}(1+k)$$

Since $(1.5)^{1/n}$ is greater than 1, $(1+i) > (1+k)$ and $i > k$.

From a previous result, we had $(1+j) = (2/3)^{1/n}(1+k)$. Since $(2/3)^{1/n} < 1$, $(1+j) < (1+k)$ and $j < k$.

Putting these results together, we have $j < k < i$. Therefore, (i) is true.

Taking the nth root of both sides of the first equation:

$$(1+i)(1+j) = (1+k)^2, \quad \text{or}$$
$$(1+k) = [(1+i)(1+j)]^{1/2}$$

The right hand side of this equation is the geometric mean of $(1+i)$ and $(1+j)$. The arithmetic mean of these values is $(1/2)[(1+i)+(1+j)] = 1 + (1/2)(i+j)$. The geometric mean is less than the arithmetic mean, so we have:

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$$1 + k < 1 + (1/2)(i + j) \quad \text{and}$$
$$k < (1/2)(i + j)$$

(ii) is true. ANS. **(A)**

11. Let A = starting amount in Fund A and B = starting amount in Fund B. We are given that:

$$1.03^{20}A + 1.025^{20}B = 10{,}000 \quad \text{and}$$
$$1.03^{31}A = (2)(1.025)^{31}B$$

Evaluating the exponentials in the first equation:

$$1.8061A + 1.6386B = 10{,}000$$

From the second equation:

$$A = (2)[(1.025/1.03)]^{31}B = 1.7200B$$

Substituting this value of A:

$$(1.8601)(1.7200)B + 1.6386B = 10{,}000$$
$$4.7451B = 10{,}000$$
$$B = 2{,}107.44$$
$$A = 1.7200B = 3{,}624.80$$

We want the total in the funds at the end of 10 years:

$$(3{,}624.80)(1.03)^{10} + (2{,}107.44)(1.025)^{10} = 7{,}569 \qquad \text{ANS. (D)}$$

Note: Your intermediate results may differ because of rounding.

12. We are given that:

$$Xv^{10} + Yv^{20} = (X + Y)v^{15} \quad \text{and} \quad X + Y = 100$$

The second equation gives us $Y = 100 - X$.

Substituting for Y and for $X + Y$ in the first equation:

$$Xv^{10} + (100 - X)v^{20} = 100v^{15}$$
$$X = 100\frac{v^{15} - v^{20}}{v^{10} - v^{20}} \text{ at } 5\%$$
$$= 43.93 \qquad \text{ANS. (A)}$$

13. $200v^n + 100v^{2n} = 200$

This is a quadratic in v^n. For clarity, let $X = v^n$:

$$X^2 + 2X - 2 = 0$$
$$X = \frac{-2 \pm \sqrt{12}}{2} = \sqrt{3} - 1 \quad \text{(taking the positive root)}$$

Since $X = v^n$, we have:

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$$v^n = \sqrt{3} - 1$$

$$(1+i)^n = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}$$

$$i = \left(\frac{\sqrt{3}+1}{2}\right)^{\frac{1}{n}} - 1 \qquad \text{ANS. (A)}$$

Note: If we had multiplied the original equation by $(1+i)^{2n}$ at the start, it would have simplified the solution a bit.

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§ 1a(vi) Effective Rate of Discount

The effective rate of interest tells us how fast a fund is growing based on the amount in the fund at the ***beginning*** of the year. But we could also define a rate of growth based on the amount in the fund at the ***end*** of the year. This rate has the symbol d_t and is called the "effective rate of discount":

$$d_t = \frac{a(t) - a(t-1)}{a(t)}$$

Mathematically, it is not necessary to define an effective rate of discount. It will be shown that d_t can be expressed in terms of i_t, and vice versa, so we only need one of these rates. But it's handy to have a discount rate in a situation where a specified amount will be payable on a specified date in the future, and we want to determine how much to pay for it (i.e., to determine its PV). This situation often arises in the financial world, including transactions between banks.

The process of finding the price that we would be willing to pay for a promise to receive a future amount is called "discounting." It's just another term for "finding the present value."

For example, suppose a note that promises to pay \$1,000 in one year is for sale. What would you be willing to pay for the note if you want to earn an effective rate of *discount* of 10%?

By definition of the effective rate of discount, the amount of interest to be earned in the year is 10% × \$1,000 = \$100. (Remember that the discount rate is based on the fund at the *end* of the year.) If \$100 is the interest earned, the PV (or the discounted amount) of the note must be \$1,000 − 100 = \$900. This is the price to earn $d = 10\%$.

In general, if we use the symbol d, what we have done is to compute the discounted value of \$1,000 (the price we would be willing to pay for the note), as \$1,000 − 1,000d, or \$1,000$(1 - d)$.

We could also have computed the discounted amount (or present value) of the note as $1{,}000v$ or $\frac{1{,}000}{1+i}$, where i is the effective rate of interest ***equivalent*** to d. In order for i and d to be equivalent, i.e., to produce the same present value, we must have:

$$1,000(1-d) = 1,000v$$

or $1 - d = v = \frac{1}{1+i}$.

Take a moment to solve for i in terms of d. You should get:

$$i = \frac{d}{1-d}$$

In the example, $d = 10\%$, so the equivalent i is $\frac{.10}{.90} = 11.1\dot{1}\%$.

(This agrees with the definition of i as the interest earned (\$100) divided by the fund at the beginning of the year (\$900): $\frac{100}{900} = 11.1\dot{1}\%$).

When dealing with future amounts, like \$1,000 due in one year, it's natural to use discount rates, since they are defined in terms of future amounts.

Starting with the relationships:

$$1 - d = v$$

$$or\ \ i = \frac{d}{1-d}$$

you can derive a number of other relationships that you should eventually know by heart. The most important are:

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$$d = \frac{i}{1+i} = iv$$

$$i - d = id$$

$$\frac{1}{d} - \frac{1}{i} = 1$$

From the first of these three relationships, we see that $d < i$ for the "usual" range of i.

Trap Alert!

When using d to *discount*, the exponent of $(1-d)$ is ***positive***, e.g., the PV of \$1.00 due in t years is $(1-d)^t$. So to *accumulate* \$1.00 using d, the exponent is ***negative***, e.g., the AV is $(1-d)^{-t}$.

On the other hand, to discount \$1.00 using i, the exponent is negative: $(1+i)^{-t}$. To accumulate using i, the exponent is positive: $(1+i)^t$.

Be very careful when using d, since it's very easy to fall into the trap of using a positive exponent to accumulate investments.

An easy way to avoid the trap is to always remember that $1-d=v$. Since the present value is v^t, it is also $(1-d)^t$ (positive exponent). The accumulated value in terms of v is v^{-t} or $(1+i)^t$, so the AV in terms of d is $(1-d)^{-t}$ (negative exponent).

Stepping Stones

EXAMPLE 1

Given that $a(t) = .1t^2 + 1$, determine the effective rate of discount in the 5th year.

SOLUTION

$$d_5 = [a(5) - a(4)]/a(5) = (3.5 - 2.6)/3.5 = 25.71\% \quad \text{(2 decimals)}$$

EXAMPLE 2

Determine (a) the effective rate of discount equivalent to an effective rate of interest of 5% and (b) the effective rate of interest equivalent to an effective rate of discount of 5%.

SOLUTION

For (a), since $d = i/(1+i)$, $d = .05/1.05 = 4.76\%$ to 2 decimals.

For (b), since $i = d/(1-d)$, $i = .05/.95 = 5.26\%$.

Note that in both cases $i > d$.

EXAMPLE 3

You are given that the effective rate of discount is 10% per annum. What is the accumulated value of \$100 at the end of three years (to the nearest cent)?

SOLUTION

In general, using an effective rate of discount, the accumulated value of \$1.00 at the end of n years $= (1-d)^{-n}$. In this case, we have:

$$AV = 100(1-.10)^{-3} = 100/.9^3 = 100/.729 = \$137.17$$

§ 1a(vii) Nominal Rates of Interest

Two banks offer the following rates:

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Bank #1: An effective annual rate of interest of 6%

Bank #2: A nominal annual rate of interest of 6% compounded monthly

Which bank would you rather deposit your money in?

Even if you haven't had this course before, you probably know the answer from your general knowledge: Bank #2 offers the better deal.

If you deposit \$1.00 in Bank #1, it will grow to \$1.06 in one year. Bank #2 offers a "nominal" annual rate of 6% compounded monthly. What this means is that the effective rate for a ***one month period*** is $\frac{1}{12} \times 6\% = 0.5\%$. So \$1.00 deposited in Bank #2 will grow to $(1 + .005)^{12} = 1.0616778$, which is a little more than 1.06. Bank #2 indeed has the better rate.

The symbol for the nominal rate offered by Bank #2 is $i^{(12)}$. This is read "i upper 12." We are told that $i^{(12)} = 6\%$. Please note that (12) is ***not*** an exponent. It is a special symbol that tells you that 6% is a nominal rate and that the effective rate per *month* is $\frac{1}{12} \times 6\% = 0.5\%$. Thus, the AV of \$1 invested for one year is 1.005^{12}.

In general, if $i^{(m)}$ is the nominal annual rate compounded m times a year, $\frac{i^{(m)}}{m}$ is the effective rate for an mth of a year and the AV of \$1.00 in one year is $\left(1 + \frac{i^{(m)}}{m}\right)^m$. To find the equivalent effective annual rate i (the rate that will give the same result if compounded only once in the year), we set this equal to $(1 + i)$:

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i$$

You should also know this identity in the "other direction," i.e., solving for $i^{(m)}$, we have:

$$i^{(m)} = m[(1 + i)^{\frac{1}{m}} - 1].$$

In the example above, we found that the annual effective rate i equivalent to $i^{(12)} = 6\%$ is

$$1.0616778 - 1 = 6.16778\%$$

In general, $i^{(m)} < i$ for $m > 1$. It's no surprise that $\frac{1}{12} \times 6\%$ compounded 12 times a year results in an effective rate which is greater than 6%.

Stepping Stones

EXAMPLE 1

Investment A will earn a nominal rate of interest of 12% compounded monthly. Investment B will earn a nominal rate of interest of 12.2% compounded semiannually. Which is the better investment?

SOLUTION

We will determine which investment grows to a greater amount over the same time. It's simplest to use the growth of \$1.00 over a one-year period:

Investment A: $(1 + .12/12)^{12} = 1.01^{12} = 1.126825$

Investment B: $(1 + .122/2)^2 = 1.061^2 = 1.125721$

Investment A is the better one.

Note that although B's nominal rate of interest (12.2%) is greater than A's (12%), A is the better investment, due to the greater frequency of compounding.

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EXAMPLE 2

A bank offers a nominal rate of interest of 8% compounded quarterly. Determine the present value of \$100 due in 2 years.

SOLUTION

$$PV = 100(1 + .08/4)^{-8} = 100/1.02^8 = \$85.35$$

EXAMPLE 3

You are given that $i^{(m)}$ is the nominal annual rate of interest compounded m-thly. Determine $i^{(p)}$, the nominal annual rate of interest compounded p-thly that gives the same investment results, in terms of $i^{(m)}$.

SOLUTION

$$(1 + i^{(m)}/m)^m = (1 + i^{(p)}/p)^p$$

Solving for $i^{(p)}$, you should get:

$$i^{(p)} = p[(1 + i^{(m)}/m)^{m/p} - 1]$$

§ 1a(viii) Nominal Rates of Discount

Let $d^{(m)}$ represent the nominal annual rate of discount compounded m times a year. Thus, the PV of \$1.00 due in a year can be expressed as $\left(1 - \frac{d^{(m)}}{m}\right)^m$. If d is the equivalent annual effective rate of discount, we can set this expression equal to $(1 - d)$:

$$\left(1 - \frac{d^{(m)}}{m}\right)^m = 1 - d$$

Again, you should also know this identity in the "other direction." Solving for $d^{(m)}$, we get:

$$d^{(m)} = m[1 - (1 - d)^{\frac{1}{m}}]$$

Remember the trap we noted before. $\left(1 - \frac{d^{(m)}}{m}\right)^m$ is the PV of \$1.00 due in a year. If we want to determine the AV of \$1.00, we must use a negative exponent.

Stepping Stones

EXAMPLE 1

Express the following amounts in terms of the given interest and discount rates. (You don't have to get numerical values for this example.)

(1) AV of \$150 at the end of 9 years at a nominal annual rate of interest of 7% compounded semiannually for the first 4 years and 5% compounded quarterly for the next 5 years.

(2) PV of \$750 due 7 years from now, at a nominal annual rate of interest of 6% compounded monthly.

(3) AV of \$1,000 at the end of 6 years, at a nominal annual rate of discount of 5% compounded semiannually.

SOLUTION

(1) $AV = 150(1.035)^8(1.0125)^{20}$

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(2) $PV = 750(1.005)^{-84}$

(3) $AV = 1{,}000(.975)^{-12}$

(Note that the solution to (3) involves a negative exponent, since we are *accumulating* using a discount rate.)

EXAMPLE 2

The present value of \$1.00 due in six months is \$.97. Determine the nominal annual rate of discount compounded quarterly to 4 decimals.

SOLUTION

In terms of $d^{(4)}$, we have:

$$.97 = (1 - d^{(4)}/4)^2 \quad \text{(There are two quarterly interest period in six months.)}$$

$$d^{(4)} = 4(1 - .97^{1/2}) = 6.0457\%$$

§ 1a(ix) A Note About Nominal Rates at Unusual Frequencies

Occasionally, you will encounter nominal rates of interest (or discount) at unusual frequencies.

It's easy to understand that a 6% nominal rate of interest ***compounded quarterly*** means that the effective rate of interest for a quarter-year period is $1\frac{1}{2}\%$, and that 1 will grow to $\left(1 + \frac{.06}{4}\right)^4 = 1.015^4$ in one year. But what should we make of a 6% nominal rate of interest ***compounded once every two years?***

A nominal rate tells us what the equivalent effective rate is for a specified period of time. For example, as just noted, "6% nominal compounded quarterly" tells us that the equivalent effective rate is $1\frac{1}{2}\%$ for a $\frac{1}{4}$-year period. Let's see how we apply this to other frequencies.

The key to handling any nominal rate—at whatever frequency—is to determine the period for the equivalent effective rate. The period comes directly from the description of the nominal rate:

	6% nominal rate of interest compound quarterly	**6% nominal rate of interest compounded once every 2 years**
1. Period for the equivalent effective rate (in years)	$\frac{1}{4}$ year	2 years
2. Effective rate for the period on line 1	$\frac{1}{4} \times 6\% = 1\frac{1}{2}\%$ ($1\frac{1}{2}\%$ effective per $\frac{1}{4}$-year period)	$2 \times 6\% = 12\%$ (12% effective per 2-year period)
3. Accumulated value of 1 in 1 year	1.015^4	$1.12^{\frac{1}{2}}$

Note that if the nominal rate is compounded quarterly, the period in the above table is $\frac{1}{4}$ year; if the nominal rate is compounded once every 2 years, the period is 2 years; etc.

Also note that the effective rate is determined by multiplying the nominal rate by the period.

For those of you who prefer to work mechanically using symbols, there's nothing wrong with using the relationship between $i^{(m)}$, the nominal rate compounded m times a year, and i, its equivalent

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effective rate, even when m is not an integer. For *any* m, we have:

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i$$

In the above example, we can write the nominal rate of 6% compounded once every 2 years (that is, compounded only a "$\frac{1}{2}$-time" every year) as $i^{(\frac{1}{2})} = 6\%$. Substituting $m = \frac{1}{2}$ in the formula above, we have:

$$\left(1+\frac{.06}{1/2}\right)^{\frac{1}{2}} = 1.12^{\frac{1}{2}} = 1+i\,,$$

as before.

Incidentally, the word "nominal" as used in this context means "in name only" or "so-called," as in the following sentence: "He was the nominal CEO but his sister actually ran the company." We call 6% the "nominal" rate of interest because 6% is not the true, or effective, rate per annum: It is the annual rate in name only.

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Calculator Notes #2: Discount Rates; Nominal Rates

Discount Rates

6. What is the present value of \$1,000 due in $6\frac{1}{4}$ years at an effective rate of discount of 9.27% per annum?

a. Using the regular keys

$PV = 1{,}000(1 - .0927)^{6.25}$

1 [−] .0927 [y^x] 6.25 [×] 1,000 [=]

The answer is \$544.43 to the nearest cent.

Note that we have used a *positive* exponent to find the PV using a discount rate.

b. Using the TVM keys

It's possible to fool the calculator by entering the discount rate 9.27 as a negative interest rate in [I/Y] and 1,000 in [PV] (even though it's actually a FV). We would also enter 6.25 in [N]. If we then press [CPT] [FV], the calculator will compute $1{,}000(1 - .0927)^{6.25}$, or 544.43. (It's also possible to enter −9.27 in [I/Y], 1,000 in [FV] and -6.25 in [N], and then [CPT] [PV].) But if you try to use this trick, you may find that you fool yourself instead of fooling the calculator. Because it's so tricky, we prefer the straightforward approach using the regular keys. But if you would like to be a whiz at using the TVM keys, see pages 5 and 6 of the Study Note for examples involving discount rates.

7. What is the accumulated value of \$1,300 in $10\frac{1}{2}$ years at an effective rate of discount of 5.3286%?

a. Using the regular keys

$AV = 1{,}300(1 - .053286)^{-10.5}$

1 [−] .053286 [y^x] 10.5 [+/−] [×] 1,300 [=]

The answer is \$2,310.18.

Note that we have used a *negative* exponent to find the AV using a discount rate.

b. Using the TVM keys

See the comments in problem 6 about using the TVM keys with discount rates.

Nominal Rates of Interest and Discount

When using a nominal rate, make sure that when you enter the values in the calculator, you use (1) the corresponding effective rate and (2) the number of *interest* periods at that effective rate. Here are some examples:

8. What is the accumulated value of \$50,232 in 17 years at a nominal rate of interest of 13% per annum convertible quarterly?

a. Using the regular keys

.13/4 is the effective rate for a $\frac{1}{4}$-year period and 68 is the number of quarters in 17 years. Thus, the answer is $50{,}232(1 + .13/4)^{68}$. The keystrokes are:

.13 [÷] 4 [+] 1 [y^x] 68 [×] 50,232 [=]

The answer is 442,083.77.

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b. Using the TVM keys

68 [N] 13 [÷] 4 [=] [I/Y] 50,232 [PV] [CPT] [FV]

Again, the answer is $442,083.77 (with a negative sign).

9. Determine the present value of $125,755 due in 18 years at a nominal rate of interest of 12% per annum convertible monthly.

a. Using the regular keys

The effective rate of interest for a one-month period is 1%. There are 216 months in 18 years. Thus, the answer is $125{,}755(1.01)^{-216}$. The keystrokes are:

1.01 [y^x] 216 [+/−] [×] 125,755 [=]

The answer is 14,659.14.

b. Using the TVM keys

216 [N] 1 [I/Y] 125,755 [FV] [CPT] [PV]

The answer is −14,659.14.

10. Find the accumulated value of $32,705 in 10 years at a nominal rate of discount of 5.80% compounded semiannually.

The effective rate of discount for a $\frac{1}{2}$-year period is .029. There are 20 interest periods in 10 years. Thus, the answer is $32{,}705(1 - .029)^{-20}$. The keystrokes are:

1 [−] .029 [y^x] 20 [+/−] [×] 32,705 [=]

The answer is 58,915.50.

11. Find the present value of $82,309 due in 8 years at a nominal rate of discount of 6% per annum compounded monthly.

The effective rate of discount for a one-month period is .005. There are 96 months in 8 years. Thus, the answer is $82{,}309(1 - .005)^{96}$. The keystrokes are:

1 [−] .005 [y^x] 96 [×] 82,309 [=]

The answer is $50,870.16.

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Summary of Concepts and Formulas in Sections 1a(vi) to 1a(ix)

(1) The effective rate of discount in the tth year is based on the amount in the fund at the end of the year:

$$d_t = \frac{a(t) - a(t-1)}{a(t)}$$

(2) For a constant effective rate of interest, it can be shown that:

$$1 - d = v$$

$$i = \frac{d}{1-d}$$

$$d = \frac{i}{1+i} = iv$$

$$i - d = id$$

$$\frac{1}{d} - \frac{1}{i} = 1$$

(3) (a) To accumulate \$1.00 over t years using d:

$$AV = (1+i)^t = v^{-t} = (1-d)^{-t}$$

(b) To compute the PV of \$1.00 due in t years:

$$PV = (1+i)^{-t} = v^t = (1-d)^t$$

(4) The nominal rate of interest compounded (or convertible) m times a year is given the symbol $i^{(m)}$. This means that the effective rate of interest for $\frac{1}{m}$-th of a year is $\frac{i^{(m)}}{m}$.

(5) It follows that:

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i$$

(6) Solving for $i^{(m)}$:

$$i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1]$$

(7) The nominal rate of discount compounded (or convertible) m times a year is given the symbol $d^{(m)}$. This means that $\frac{d^{(m)}}{m}$ is the effective rate of discount for $\frac{1}{m}$-th of a year.

(8) It follows that:

$$\left(1 - \frac{d^{(m)}}{m}\right)^m = 1 - d$$

(9) Solving for $d^{(m)}$:

$$d^{(m)} = m[1 - (1-d)^{\frac{1}{m}}]$$

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Past Exam Questions on Sections 1a(vi) to 1a(ix)

1. Eric deposits X into a savings account at time 0, which pays interest at a nominal rate of i, compounded semiannually. Mike deposits $2X$ into a different savings account at time 0, which pays simple interest at an annual rate of i. Eric and Mike earn the same amount of interest during the last 6 months of the 8th year. Calculate i. [5/03 #12]

   (A) 9.06% (B) 9.26% (C) 9.46% (D) 9.66% (E) 9.86%

2. Jeff deposits 10 into a fund and 20 fifteen years later. Interest is credited at a nominal discount rate of d compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semiannually thereafter. The accumulated balance in the fund at the end of 30 years is 100.

   Calculate d. [5/03 #50]

   (A) 4.33% (B) 4.43% (C) 4.53% (D) 4.63% (E) 4.73%

3. Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits 100 into his bank account, and Robbie deposits 50 into his. Each account earns an annual effective discount rate of d.

   The amount of interest earned in Bruce's account during the 11th year is equal to X. The amount of interest earned in Robbie's account during the 17th year is also equal to X.

   Calculate X. [5/01 #12]

   (A) 28.0 (B) 31.3 (C) 34.6 (D) 36.7 (E) 38.9

4. John invests 1000 in a fund which earns interest during the first year at a nominal rate of K convertible quarterly. During the 2nd year the fund earns interest at a nominal discount rate of K convertible quarterly. At the end of the 2nd year, the fund has accumulated to 1173.54.

   Calculate K. [SOA 5/98 #2]

   (A) 0.064 (B) 0.068 (C) 0.072 (D) 0.076 (E) 0.080

5. Brian and Jennifer each take out a loan of X.

   Jennifer will repay her loan by making one payment of 800 at the end of year 10. Brian will repay his loan by making one payment of 1120 at the end of year 10.

   The nominal semiannual rate being charged to Jennifer is exactly one-half the nominal semiannual rate being charged to Brian.

   Calculate X. [SOA 11/96 #3]

   (A) 562 (B) 565 (C) 568 (D) 571 (E) 574

6. Jennifer deposits 1000 into a bank account. The bank credits interest at a nominal annual rate of i convertible semiannually for the first 7 years and a nominal annual rate of $2i$ convertible quarterly for all years thereafter.

   The accumulated amount in the account at the end of 5 years is X. The accumulated amount in the account at the end of 10.5 years is 1980.

   Calculate X. [SOA 5/95 #1]

   (A) 1200 (B) 1225 (C) 1250 (D) 1275 (E) 1300

7. A deposit of X is made into a fund which pays an annual effective interest rate of 6% for 10 years.

   At the same time, $X/2$ is deposited into another fund which pays an annual effective rate of discount of d for 10 years.

   The amounts of interest earned over the 10 years are equal for both funds.

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Calculate d. [SOA 5/95 #3]

(A) 9.0% (B) 9.5% (C) 10.0% (D) 10.5% (E) 11.0%

8. At time $t = 0$, John deposits 1000 into a fund which credits interest at a nominal interest rate of 10% compounded semiannually. At the same time, he deposits P into a different fund which credits interest at a nominal discount rate of 6% compounded monthly. At time $t = 20$, the amounts in each fund are equal.

   What is the annual effective interest rate earned on the total deposits, $1000 + P$, over the 20-year period? [SOA 11/93 #3]

   (A) 7.44% (B) 7.54% (C) 7.64% (D) 7.74% (E) 7.84%

9. (This problem should appear after Section 2b, where the method of equated time is covered.) Using the method of equated time, a payment of 400 at time $t = 2$ plus a payment of X at time $t = 5$ is equivalent to a payment of $400 + X$ at time $t = 3.125$.

   At an effective annual interest rate of 10%, the above two payments are equivalent to a payment of $400 + X$ at time $t = k$ using the exact method.

   Calculate k. [SOA 5/93 #3]

   (A) 2.90 (B) 2.96 (C) 3.03 (D) 3.14 (E) 3.21

10. On January 1, 1980, Jack deposited 1,000 in Bank X to earn interest at the rate of j per annum compounded semiannually. On January 1, 1985, he transferred his account to Bank Y to earn interest at the rate of k per annum compounded quarterly. On January 1, 1988, the balance at Bank Y is 1,990.76.

    If Jack could have earned interest at the rate of k per annum compounded quarterly from January 1, 1980 through January 1, 1988, his balance would have been 2,203.76.

    Calculate the ratio k/j. [SOA 11/89 #5]

    (A) 1.25 (B) 1.30 (C) 1.35 (D) 1.40 (E) 1.45

11. Investment X for 100,000 is invested at a nominal rate of interest, j, convertible semiannually. After four years, it accumulates to 214,358.88.

    Investment Y for 100,000 is invested at a nominal rate of discount, k, convertible quarterly. After two years, it accumulates to 232,305.73.

    Investment Z for 100,000 is invested at an annual effective rate of interest equal to j in year one and an annual effective rate of discount equal to k in year two.

    Calculate the value of investment Z at the end of two years. [SOA 5/89 #1]

    (A) 168,000 (B) 182,900 (C) 184,425 (D) 200,000 (E) 201,675

12. Which of the following are correct expressions for $d^{(360)}$?

    (i) $1 - v^{\frac{1}{360}}$

    (ii) $360 - 360\left(1 + \frac{i^{(12)}}{12}\right)^{-30}$

    (iii) $\frac{1}{\frac{1}{360} + \frac{1}{i^{(360)}}}$

    [SOA SAMPLE/84 #4]

    (A) None (B) I only (C) II only (D) III only
    (E) The correct answer is not given by (A), (B), (C), or (D).

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Solutions to Past Exam Questions on Sections 1a(vi) to 1a(ix)

1. Note: It is poor practice to use i to represent a nominal rate of interest, since i is defined as the effective rate. However, past exams have used this kind of loose notation on a number of occasions.

   Interest earned in the last 6 months of the 8th year:

   By Eric: (amt. in account after $7\frac{1}{2}$ years) $\times \frac{i}{2}$

   $$= X\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right)$$

   By Mike: constant amt. in each $\frac{1}{2}$year period

   $$= (2X)\left(\frac{i}{2}\right) = Xi$$

   $$X\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right) = Xi$$

   $$\left(1+\frac{i}{2}\right)^{15} = 2$$

   $$i = 2\left(2^{\frac{1}{15}}-1\right) = 9.46\% \qquad \text{ANS. (C)}$$

2. The correct symbol for d is $d^{(4)}$.

   $$10\left(1-\frac{d^{(4)}}{4}\right)^{-40}(1.03)^{40} + 20(1.03)^{30} = 100$$

   $$d^{(4)} = 4\left\{1-\left[10-(2)(1.03)^{30}\right]^{-\frac{1}{40}}(1.03)\right\}$$

   $$= 4.53\% \qquad \text{ANS. (C)}$$

3. We could work with d but it's easier to work with i:

   Bruce: AV in 10 years $= 100(1+i)^{10}$

   Interest in 11th year $= 100(1+i)^{10}(i)$

   Robbie: AV in 16 years $= 50(1+i)^{16}$

   Interest in 17th year $= 50(1+i)^{16}(i)$

   $$100(1+i)^{10}(i) = 50(1+i)^{16}(i)$$

   $$(1+i)^6 = 2, \quad i = 12.2462\%$$

   $$X = (100)(1.122462)^{10}(.122462) = 38.9 \qquad \text{ANS. (E)}$$

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4. $$AV \text{ in 2 years} = 1{,}173.54 = 1{,}000\left(1+\frac{K}{4}\right)^4\left(1-\frac{K}{4}\right)^{-4} = 1{,}000\left(\frac{4+K}{4-K}\right)^4$$

$$\frac{4+k}{4-K} = 1.17354^{1/4}$$

$$= 1.0408$$

Solving for K, we get:

$K = .08$ ANS. **(E)**

5. Let j = Jennifer's effective rate for a half-year period.

Jennifer: $X(1+j)^{20} = 800$

Brian: $X(1+2j)^{20} = 1{,}120$

Dividing: $\left(\frac{1+2j}{1+j}\right)^{20} = 1.4$

$$\frac{1+2j}{1+j} = 1.0169659$$

$$j = .0172588$$

$X = (800)(1.0172588)^{-20} = 568$ ANS. **(C)**

6. $$AV \text{ end of 5 years} = X = 1{,}000\left(1+\frac{i}{2}\right)^{10}$$

$$AV \text{ end of 10.5 years} = 1{,}980 = 1{,}000\left(1+\frac{i}{2}\right)^{14}\left(1+\frac{2i}{4}\right)^{14}$$

We don't have to solve explicitly for i to determine X. From the second equation, we have:

$$1{,}980 = \$1{,}000\left(1+\frac{i}{2}\right)^{28}$$

$$\left(1+\frac{i}{2}\right) = 1.98^{1/28}$$

Substituting for $(1+\frac{i}{2})$ in the first equation:

$X = 1{,}000(1.98)^{10/28} = 1{,}276$ ANS. **(D)**

7. The AV of the deposit of X at the end of 10 years is $1.06^{10}X$. The interest earned over the 10-year period is $1.06^{10}X - X$. Similarly, the interest earned on the deposit of $\frac{X}{2}$ is $\frac{X}{2}(1-d)^{-10} - \frac{X}{2}$. Equating:

$$X\left[1.06^{10}-1\right] = \frac{X}{2}\left[(1-d)^{-10}-1\right]$$

$(1-d)^{-10} = 2.581695,\ d = .0905$ ANS. **(A)**

8. We are given that:

$$1{,}000(1.05)^{40} = P(1-.005)^{-240}$$

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Solving for P, we get $P = 2,114$.

Let i = the effective annual interest rate earned on both deposits, $1,000 + P$, over the 20-year period. Since the AV of the deposit of 1,000 is equal to the AV of the deposit of P, we can write the AV of both deposits as twice the AV of the deposit of 1,000, i.e., as $2,000(1.05)^{40}$:

$$(1,000 + P)(1 + i)^{40} = 2,000(1.05)^{40}$$

Since $P = 2,114$, we have:

$$3,114(1 + i)^{20} = 2,000(1.05)^{40} = 14,080$$

The easiest way to solve for i is by using the TVM keys. (After clearing, enter $N = 20$, $PV = 3,114$ and $FV = -14,080$, then $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$.)

$i = 7.84\%$ ANS. **(E)**

9. The method of equated time uses the weighted average time:

$$\frac{(400)(2) + X(5)}{400 + X} = 3.125$$

$$\therefore\ X = 240$$

Using the exact method:

$$400v^2 + 240v^5 = 640v^k,\ v^k = .749,$$

$k = 3.03$ ANS. **(C)**

10. The second paragraph of the question tells us that:

$$\left(1 + \frac{k}{4}\right)^{32} = 2.20376$$

Using the TVM keys, we find that $\frac{k}{4} = .025$, so $k = .10$.

The first paragraph tells us that:

$$\left(1 + \frac{j}{2}\right)^{10} (1.025)^{12} = 1.99076$$

Using the TVM keys, we find that $\frac{j}{2} = .04$, so $j = .08$.

$\frac{k}{j} = 1.25$ ANS. **(A)**

11.

$$100,000\left(1 + \frac{j}{2}\right)^{8} = 214,358.88,\ j = 20\%$$

$$100,000\left(1 - \frac{k}{4}\right)^{-8} = 232,305.73,\ k = 40\%$$

$$\begin{aligned} AV \text{ in 2 years} &= 100,000(1 + j)(1 - k)^{-1} \\ &= 100,000(1.2)(1 - .4)^{-1} \\ &= 100,000\left(\frac{1.2}{.6}\right) = 200,000 \end{aligned}$$

ANS. **(D)**

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12. (I) $1 - v^{\frac{1}{360}} = \dfrac{d^{(360)}}{360}$

$\therefore\ d^{(360)} = 360\left(1 - v^{\frac{1}{360}}\right)$ (I) is false

(II) $\left(1 - \dfrac{d^{(360)}}{360}\right)^{360} = \left(1 + \dfrac{i^{(12)}}{12}\right)^{-12}$

$d^{(360)} = 360\left[1 - \left(1 + \dfrac{i^{(12)}}{12}\right)^{-\frac{1}{30}}\right]$ (II) is false

(III) The simplest method is to note that

$\dfrac{1}{d^{(p)}} - \dfrac{1}{i^{(p)}} = \dfrac{1}{p}$, so:

$\dfrac{1}{d^{(360)}} - \dfrac{1}{i^{(360)}} = \dfrac{1}{360}$

and $d^{(360)} = \dfrac{1}{\frac{1}{360} + \frac{1}{i^{(360)}}}$ (III) is true ANS. **(D)**

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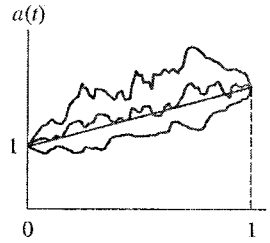
§1b. Why Do We Need a Force of Interest?

Let's begin with an easy concept that you already understand from Section 1a(i): The effective rate of interest is simply the amount of interest earned during a period divided by the amount invested at the beginning of that period.

Consider the following four funds:



The four curves going from time 0 to time 1 show how each of the funds grows during the first year. As you can see, in this example the amount in the fund is the same at the beginning and the end of the year for all four funds. But they certainly grow quite differently at any moment of time during the year.

Question: What can you say about the effective rate of interest for the four funds in this year?

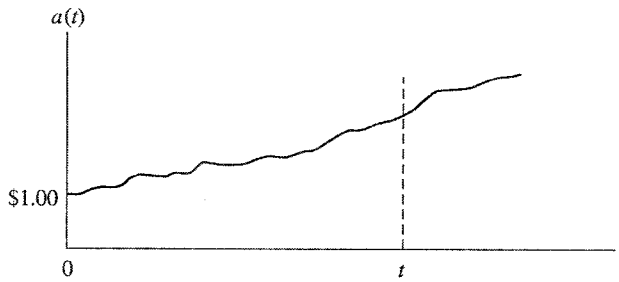
Answer: The effective rate of interest is *the same* for all four funds. You can see that the amount of interest earned in the year is the same for all of the funds (i.e., the increase in each fund over the year is the same). Also, the investment at time 0 for each fund is the same. So by definition, the effective rate of interest is the same for all four.

This shows that the effective rate of interest is all we need to know if we are only interested in how much interest is earned on a fund over the entire year, but gives us absolutely no information about how the fund grows at any moment during the year. In theoretical work, and in practical applications of the theory, we may need to know the rate of growth of a fund at *any moment of time*. That's what the force of interest is.

Please Note! For now, we are considering the general case, where the rate of growth varies with time. In the special case where the rate of growth is constant at every moment (which is the assumption in many of the problems you will see for this exam), things get really simple. This will be covered later.

§1c. Defining the Force of Interest

Suppose $1.00 is invested at time 0 and grows according to the following graph:



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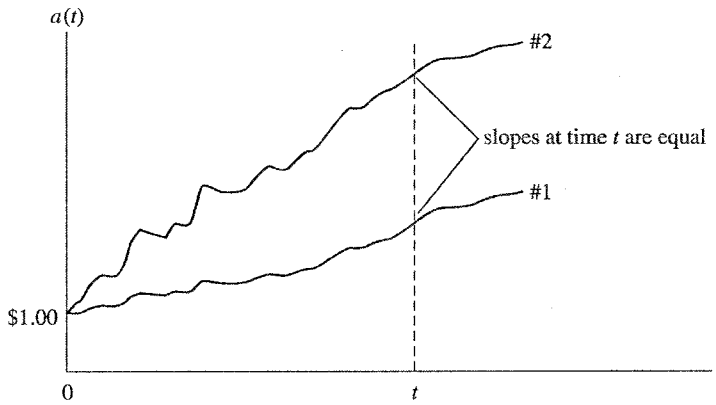
It's plain that this fund is not growing at a constant rate. Suppose we want to measure how fast it's growing at some time t, as indicated on the graph. What would be a reasonable way to define a rate of growth at a point in time?

The first thing that comes to mind is the derivative of $a(t)$. A high derivative—a relatively steep curve—tells us that the fund is increasing by a relatively large amount near time t. A low derivative—a relatively flat curve—tells us that the fund is increasing by a relatively small amount near time t. But is the derivative, in itself, a good measure of the *rate* of growth, as opposed to the *amount* of growth? The answer is "no."

To understand this, let's consider a second fund along with the first:



Suppose the derivatives of the two $a(t)$ functions are the same at time t. Suppose the amount in fund #2 is double the amount in fund #1 at time t. Are the two funds growing at the same rate at that time?

To put the question in practical terms, suppose you are a manager of a fund with a billion dollars of assets at time t. You have to "park" the billion dollars for a very short period of time—say for 30 seconds, or roughly a millionth of a year. Where would you park your money?

Fund #1

Fund #2

It makes no difference

If you said "fund #1," you were right. The reason is that both funds earn about the same *amount* of interest in the thirty-second period but fund #1 earns that amount of interest on only half the investment as fund #2.

A little more detail: The derivative, or slope, is the same for both funds at time t. This means that for a very small change in t, the two funds increase by about the same amount. In other words, both funds earn about the same *amount* of interest in the 30-second period after time t. But fund #1 earns that interest on only half the investment as fund #2 does. So fund #1 is growing at about twice the *rate* of fund #2 at time t.

(If bank #1 offers to pay you $5 in interest in one year on a deposit of $100 and bank #2 also offers to pay $5 in interest but on a $200 deposit, you would have no problem in concluding that bank #1's effective annual rate of growth is twice that of bank #2. The same principle applies to a rate of growth at a *point* in time.)

Our conclusion is that to get a rate of growth that makes sense—one that truly tells us how fast a fund is growing at any point in time—we must divide the derivative, or slope, by the amount in the fund.

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The rate of growth at a point in time is called the "force of interest." If we let δ_t stand for the force of interest at time t, we have:

$$\delta_t = \frac{1}{a(t)} \frac{d}{dt} a(t)$$

Note 1: $a(t)$ (the accumulation function) starts with 1 at time 0. $A(t)$ (the amount function) starts with k at time 0, so $A(t) = ka(t)$. So δ_t can also be defined as:

$$\delta_t = \frac{1}{A(t)} \frac{d}{dt} A(t)$$

since the k's cancel when you substitute $A(t) = ka(t)$.

Note 2: A force of discount could also be defined as minus the derivative of the discount function $(\frac{1}{a(t)})$ divided by the discount function. (The minus sign is used because $(\frac{1}{a(t)})$ is normally a decreasing function of t, so it has a negative derivative. We want the definition to produce a positive rate when a fund is growing.) However, this turns out to be equal to the force of interest. This is not too surprising, since we are measuring rate of growth in the limit, i.e., at a moment of time.

Stepping Stones

EXAMPLE 1

You are given the following facts about a fund:

(i) The amount in the fund at the end of 10 years is \$2,000

(ii) The amount of interest earned in the period from 10 years to 10.001 years is \$.08

Determine the approximate force of interest at time 10.

SOLUTION

The definition of the force of interest is $\delta_t = A'(t)/A(t)$. Since .001 (i.e., 1/1,000th of a year) is a very small increment in the variable t, we can approximate $A'(t)$ at $t = 10$ as follows:

$$A'(t) = \lim[A(t + \Delta t) - A(t)]/\Delta t \quad \text{as } \Delta t \text{ goes to } 0$$

$$A'(10) \approx [A(10 + .001) - A(10)]/.001 = .08/.001 = 80$$

where the symbol $\approx$ means "approximately equal to."

Finally, $\delta_{10} \approx 80/A(10) = 80/2{,}000 = 4\%$.

EXAMPLE 2

You are given that $a'(t) = 3t^2 + 2t + 1$. Determine the force of interest at $t = 5$.

SOLUTION

First, determine $a(t)$ by taking the indefinite integral of $a'(t)$:

$$a(t) = \int (3t^2 + 2t + 1)\, dt = t^3 + t^2 + t + \text{C}$$

where C is the constant of integration. Since $a(0) = 1$ for any accumulation function, C = 1. Thus:

$$a(t) = t^3 + t^2 + t + 1$$

$$\delta_t = a'(t)/a(t) = (3t^2 + 2t + 1)/(t^3 + t^2 + t + 1)$$

$$\delta_5 = 86/156 = 55.13\%$$

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EXAMPLE 3

You are given:

$$A(t) = Kt^2 + Lt + M \quad \text{for } 0 \le t \le 2$$
$$A(0) = 100$$
$$A(1) = 110$$
$$A(2) = 136$$

Determine the force of interest at time $t = \frac{1}{2}$. (Answer to nearest 0.001)

(A) 0.030 (B) 0.049 (C) 0.061 (D) 0.095 (E) 0.097

SOLUTION

$$A(0) = M = 100$$
$$A(1) = 110 = K + L + M = K + L + 100$$
$$A(2) = 136 = 4K + 2L + 100$$

Solving the last two equations, we get $K = 8$, $L = 2$. Thus:

$$A(t) = 8t^2 + 2t + 100$$

By definition:

$$\delta_t = \frac{1}{A(t)}\frac{d}{dt}A(t) = \left.\frac{16t+2}{8t^2+2t+100}\right|_{t=\frac{1}{2}} = \frac{10}{103} = 0.097$$

ANSWER (E)

§ 1d. Finding the Fund in Terms of the Force of Interest

The last section showed that a logical definition of the interest rate at a point in time is the derivative of the fund divided by the fund.

If you know the formula for the fund, it's easy to determine the force of interest. But what if you know the formula for the force of interest and you want to determine the formula for the fund? Can this be done?

The answer is "yes," of course, but it's not quite as easy as going in the other direction. The problem is that we have to solve for $a(t)$, given the definition of δ_t:

$$\delta_t = \frac{1}{a(t)}\frac{d}{dt}a(t)$$

Somehow, we have to "undo" the differentiation (which suggests integration). But this doesn't look easy because of the $\frac{1}{a(t)}$ factor.

The trick is to fondly recall our student days in calculus, when we mastered the art of implicit differentiation. To differentiate the natural log of a function of x, we differentiated the function and divided by the function:

$$\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)$$

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The right-hand side looks very familiar. It is in the same form as the definition of δ_t. This means we can write that definition in terms of the derivative of a logarithm:

$$\delta_t = \frac{1}{a(t)}\frac{d}{dt}a(t) = \frac{d}{dt}\ln a(t)$$

The great advantage of the log form is that we can now undo the differentiation by integrating both sides:

$$\int_0^t \delta_r dr = \int_0^t \frac{d}{dr}\ln a(r)\, dr$$

$$= \ln a(r)|_0^t = \ln a(t) - \ln a(0)$$

$$= \ln a(t) \quad (\text{since } a(0) = 1 \ \ and \ \ ln 1 = 0)$$

The final step is to "de-logarize" the equation by raising e to the power on each side:

$$e^{\int_0^t \delta_r dr} = a(t)$$

This says that if we invest \$1.00 at time 0, the amount in the fund at time t is e to the integral of the force of interest over that time period.

Stepping Stones

EXAMPLE 1

Find δ_t if $A(t) = Ka^t b^{t^2} d^{c^t}$

SOLUTION

Differentiating $A(t)$ is a pain, unless we realize that differentiating the *log* of $A(t)$ is a snap. In other words, use the log form of the definition of δ_t:

$$\delta_t = \frac{d}{dt}\ln A(t) = \frac{d}{dt}(\ln K + t\ln a + t^2 \ln b + c^t \ln d)$$

$$= \ln a + 2t\ln b + c^t \ln c \ln d$$

EXAMPLE 2

You are given $\delta_t = \frac{2}{1+t}$.

A payment of \$300 at the end of 3 years and \$600 at the end of 6 years has the same present value as a payment of \$200 at the end of 2 years and X at the end of 5 years.

Calculate X.

(A) \$306 (B) \$316 (C) \$376 (D) \$456 (E) \$506

SOLUTION

$a(n) = e^{\int_0^n \frac{2}{1+t}dt} = e^{2\ln(1+t)|_0^n} = e^{2\ln(1+n)-0} = e^{\ln(1+n)^2} = (1+n)^2$. To obtain present values, we use the reciprocal of the accumulation function $\frac{1}{a(n)}$ or $[a(n)]^{-1}$ (called the "discount function." See **Section 1a(v)**.)

PV of 1 due in n years $= [a(n)]^{-1} = \frac{1}{(1+n)^2}$

$$\frac{300}{16} + \frac{600}{49} = \frac{200}{9} + \frac{X}{36}$$

X = \$315.82. ANSWER (B)

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EXAMPLE 3

\$100 is invested at a force of interest $\delta_t = 2t/(t^2+1)$. Determine the amount of interest earned in the 5th year.

SOLUTION

$$a(t) = e^{\int_0^t \frac{2r}{r^2+1}\,dr} = e^{[\ln(r^2+1)]_0^t}$$

$$= e^{\ln(t^2+1)-\ln 1} = t^2+1$$

The interest earned in the 5th year $= 100[a(5)-a(4)] = 100(26-17) = 900$.

Trap Alert!

In evaluating $e^{\ln(1+t)^2]_0^n}$, some students are tempted to write this as $(1+t)^2]_0^n = (1+n)^2 - 1$.

But remember, the exponent of e is an integral, which must be evaluated **FIRST**. After you have evaluated the integral, you take e to this power. (Of course, if the power turns out to be the natural log of "something," e raised to that power is equal to the "something.")

§ 1e. The Simplest Case: A Constant Force of Interest

When the force of interest δ_t is constant, we can drop the subscript t and just call it δ. The formula for the accumulation function (see Section 1d) simplifies to:

$$a(t) = e^{\int_0^t \delta_r dr} = e^{\int_0^t \delta dr} = e^{\delta t}$$

This is an exponential. But we know that the accumulation function under a constant effective of rate of interest is also an exponential:

$$a(t) = (1+i)^t$$

at least for integral values of t. Let's assume that this is the accumulation function for all values of t, not just integral values. Then we must conclude that if i is to be equivalent to δ (that is, a fund grows in the same way under either i or δ):

$$e^{\delta t} = (1+i)^t$$

so that $e^{\delta} = 1+i$.

From this we see that:

$$\delta = \ln(1+i)\ .$$

Also, there are relationships among i, δ and other interest measures in terms of series expansions that you should know. This will be covered in greater detail in Section 1f.

Here is another way of looking at δ in the constant case. Consider the formula for $i^{(m)}$. (See Section 1a(vi)):

$$i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1]$$

Now take the limit of $i^{(m)}$ as $m \to \infty$. You can do this by writing the equation as:

$$i^{(m)} = \frac{(1+i)^{\frac{1}{m}} - 1}{1/m}$$

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and noting that the limit of both the numerator and the denominator is 0 as $m \to \infty$. This is the 0/0 case, which can be handled by l'Hospital's rule. Taking the derivative of the numerator and denominator (and remembering that m is the variable), we have

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} (1+i)^{\frac{1}{m}} \frac{-\frac{1}{m^2}\ln(1+i)}{-\frac{1}{m^2}} = \ln(1+i)$$

Note that this is the same result we got a few paragraphs ago when we assumed that the force of interest was a constant δ.

Conclusion: When the accumulation function is $a(t) = (1+i)^t$ (constant effective rate of interest), δ can be described as either:

(1) the force of interest or

(2) the nominal rate of interest compounded an infinite number of times each year (or "compounded continuously"), i.e., $\lim_{m\to\infty} i^{(m)}$. This limit is often written as $i^{(\infty)}$ for short, so we have $\delta = i^{(\infty)}$.

Note: It can also be shown that $\lim_{m\to\infty} d^{(m)} = -\ln(1-d) = \ln(1+i) = \delta$, so δ can also be described as the nominal rate of discount compounded continuously. ($i^{(\infty)} = d^{(\infty)} = \delta$, for short.)

Stepping Stones

EXAMPLE 1

A fund grows at a force of interest of 5% in the first 5 years and 4% in the next 5 years. If \$1,000 is deposited at time 0, how much is in the fund at the end of 7 years?

SOLUTION

$$AV = 1000e^{(5)(.05)}e^{(2)(.04)} = 1000e^{.33} = \$1{,}390.97$$

EXAMPLE 2

What is the equivalent constant force of interest over the first 7 years for the fund in Example 1?

SOLUTION

Let δ = equivalent constant force of interest over the 7-year period. We have:

$$e^{7\delta} = e^{.33}$$
$$7\delta = .33$$
$$\delta = 4.71\% \quad \text{to two decimals}$$

EXAMPLE 3

The force of interest in the first year is 6%. The effective rate of interest in the second year is 5%. The nominal rate of discount compounded semiannually in the third year is 4%. Determine the accumulated value of \$100 at the end of three years.

SOLUTION

$$\begin{aligned} AV &= (100)(e^{.06})(1.05)(1-.04/2)^{-2} \\ &= (100)(1.061837)(1.05)(1.041233) \\ &= \$116.09 \end{aligned}$$

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EXAMPLE 4

A bank offers a nominal annual rate of interest of 6% compounded daily. How much more would be earned on a deposit of \$10,000 in one year if the bank compounded 6% continuously, rather than daily?

SOLUTION

$$\text{Compounded daily:} \quad AV = (10{,}000)(1 + .06/365)^{365} = \$10{,}618.31$$

$$\text{Compounded continuously:} \quad AV = (10{,}000)e^{.06} = \$10{,}618.37$$

Only \$.06 more would be earned in one year on a deposit of \$10,000 if 6% were compounded continuously rather than daily.

The closeness of these two results illustrates the fact that the limit of $(1 + .06/t)^t = e^{.06}$ as t goes to infinity. With an exponent as large as $t = 365$, we would expect the result to be very close to the limit, i.e., we would expect daily compounding to be very close to continuous compounding.

§ 1f. Power Series

Using the relationships between i, d, and δ, any of these rates can be expressed as a power series in any of the others. However, there have been very few (if any) questions on the FM/2 exam in recent years that depend on your knowledge of series expansions. But just in case, let's review a few of these power series.

You probably remember the following series expansions from calculus:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Please be very careful with the second series. Note that the expansion of $\ln(1 + x)$

(i) starts with x, not 1

(ii) has alternating signs

(iii) has denominators of 1, 2, 3, ... for the x, x^2, x^3, ... terms, not 1!, 2!, 3!, ...

In interest theory, the first series can be useful when $x = \delta$, the (constant) force of interest:

$$e^\delta = 1 + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \dots .$$

You will recall from Section 1e that e^δ is the AV of 1 in 1 year, i.e., $e^\delta = 1 + i$. Thus, the series for e^δ gives us i in terms of a power series in δ:

$$i = \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \dots .$$

In the case of the second series at the beginning of this section, putting i for x we get:

$$\ln(1 + i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \dots .$$

Now, $\delta = \ln(1 + i)$. So we have δ in terms of a series expansion in i:

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$$\delta = i - \frac{i^2}{2} + \frac{i^3}{3} - \dots .$$

Trap Alert!

We already cautioned you about the expansion of $\ln(1+x)$. Things get even trickier if the "x" term is negative, as in $\ln(1-d)$. Carefully substitute $(-d)$ for x in the expansion. Do it yourself without looking at the next line:

$$\ln(1-d) = (-d) - \frac{(-d)^2}{2} + \frac{(-d)^3}{3} - \dots = -d - \frac{d^2}{2} - \frac{d^3}{3} - \dots$$

Note that substituting $(-d)$ for x *has the effect of turning all the terms in the expansion negative.*

Furthermore, if we put a negative sign in front of $\ln(1-d)$:

$$-\ln(1-d) = d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots$$

all of the terms in the series are *positive*. This series is useful because $1-d=v$ and $v=(1+i)^{-1}$, so we have $-\ln(1-d) = -\ln v = -\ln(1+i)^{-1} = \ln(1+i) = \delta$, which of course, agrees with a previous result. We have:

$$-\ln(1-d) = \delta = d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots$$

so we have δ in terms of a series expansion in d.

You can also go in the other direction, starting with $e^{-\delta} = (1+i)^{-1} = v = 1-d$, and show that:

$$d = \delta - \frac{\delta^2}{2!} + \frac{\delta^3}{3!} - \dots$$

Stepping Stones

EXAMPLE 1

Express δ as a power series in i and d.

SOLUTION

First, express δ as a power series in i, then as a power series in d. Take one-half of the sum:

$$\delta = \ln(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \dots$$

$$\delta = -\ln(1-d) = d + \frac{d^2}{2} + \frac{d^3}{3} + \dots$$

$$2\delta = i + d + \frac{d^2 - i^2}{2} + \frac{d^3 + i^3}{3} + \dots$$

$$\delta = \frac{d+i}{2} + \frac{d^2 - i^2}{4} + \frac{d^3 + i^3}{6} + \dots$$

EXAMPLE 2

Express:

(i) i in terms of a power series in d, and

(ii) d as a power series in i.

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SOLUTION

(i) $i = d/(1-d) = d(1-d)^{-1} = d(1+d+d^2+\cdots) = d+d^2+d^3+\cdots$

Note that this power series is the sum of an infinite geometric progression with first term d and common ratio d. This sum $= d/(1-d)$ if $-1 < d < 1$.

(ii) $d = i/(1+i) = i(1+i)^{-1} = i(1-i+i^2-\cdots) = i-i^2+i^3-\cdots$

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Calculator Notes #3: Force of Interest

For problems involving the force of interest, you will often need to use the natural log and e^x keys. You have already used the [LN] key in problems 4 and 5 of Calculator Notes #1. The [e^x] key is secondary to the [LN] key.

12. If $\delta = .05$, what is the present value of \$278 due in 6.38 years?

$PV = 278e^{-(.05)(6.38)}$

The keystrokes are:

.05 [+/−] [×] 6.38 [=] [2nd] [e^x] [×] 278 [=]

The answer is \$202.07 to the nearest cent.

13. If $\delta_t = .06t + .01$, what is the accumulated value of \$1,262 in 6 years?

$a(t) = e^{\int_0^t (.06r+.01)dr} = e^{[.03r^2+.01r]_0^t} = e^{.03t^2+.01t}$

$a(6) = 1{,}262e^{1.14}$

The keystrokes are:

1.14 [2nd] [e^x] [×] 1,262 [=]

The answer is \$3,945.98.

14. At a constant force of interest equal to δ, the accumulated value of \$22,537 at the end of 7.828 years is \$31,872. Determine δ.

We have $22{,}537e^{7.828\delta} = 31{,}872$.

$\delta = \frac{\ln \frac{31{,}872}{22{,}537}}{7.828}$

The keystrokes are:

31,872 [÷] 22,537 [=] [LN] [÷] 7.828 [=]

The answer is .0443 or 4.43%.

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Summary of Concepts and Formulas in Sections 1b to 1f

(1) The force of interest at time t is defined as the slope of the accumulation function (i.e., the first derivative) divided by the amount in the fund at time t:

$$\delta_t = \frac{1}{a(t)} \frac{d}{dt} a(t)$$

(2) Solving this equation for $a(t)$:

$$a(t) = e^{\int_0^t \delta_r \, dr}$$

(3) In the simplest case, when δ_t is a constant, $a(t)$ is exponential:

$$a(t) = e^{\delta t}$$

(4) Since $a(t)$ is also an exponential under a constant effective rate i, we have:

$$a(t) = e^{\delta t} = (1+i)^t$$

and $e^{\delta} = 1 + i$ or $\delta = \ln(1+i)$.

(5) It can also be shown that:

$$\lim_{m \to \infty} i^{(m)} = \ln(1+i) = \delta$$

so δ can also be described as the nominal rate of interest compounded continuously.

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Past Exam Questions on Sections 1b to 1f

1. Bruce deposits 100 into a bank account. His account is credited interest at a nominal rate of interest i convertible semiannually. At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at a force of interest of δ. After 7.25 years, the value of each account is 200. Calculate $(i - \delta)$. [5/03 #1]

   (A) 0.12% (B) 0.23% (C) 0.31% (D) 0.39% (E) 0.47%

2. At time $t = 0$, 1 is deposited into each of Fund X and Fund Y. Fund X accumulates at a force of interest $\delta_t = \frac{t^2}{k}$. Fund Y accumulates at a nominal rate of discount of 8% per annum convertible semiannually.

   At time $t = 5$, the accumulated value of Fund X equals the accumulated value of Fund Y.

   Determine k. [5/01 #45]

   (A) 100 (B) 102 (C) 104 (D) 106 (E) 108

3. Tawny makes a deposit into a bank account which credits interest at a nominal interest rate of 10% per annum, convertible semiannually.

   At the same time, Fabio deposits 1000 into a different bank account, which is credited with simple interest.

   At the end of 5 years, the forces of interest on the two accounts are equal, and Fabio's account has accumulated to Z.

   Determine Z. [5/01 #49]

   (A) 1792 (B) 1953 (C) 2092 (D) 2153 (E) 2392

4. At time 0, K is deposited into Fund X, which accumulates at a force of interest $\delta_t = 0.006t^2$. At time m, $2K$ is deposited into Fund Y, which accumulates at an annual effective rate of 10%.

   At time n, where $n > m$, the accumulated value of each fund is $4K$.

   Determine m. [11/00 #53]

   (A) 1.6 (B) 2.4 (C) 3.8 (D) 5.0 (E) 6.2

5. At time 0, deposits of 10,000 are made into each of Fund X and Fund Y. Fund X accumulates at an annual effective interest rate of 5%. Fund Y accumulates at a simple interest rate of 8%.

   At time t, the forces of interest on the two funds are equal.

   At time t, the accumulated value of Fund Y is greater than the accumulated value of Fund X by Z.

   Determine Z. [SAMPLE/00 #23]

   (A) 1625 (B) 1687 (C) 1697 (D) 1711 (E) 1721

6. At time 0, 100 is deposited into Fund X and also into Fund Y. Fund X accumulates at a force of interest $\delta_t = 0.5(1 + t)^{-2}$. Fund Y accumulates at an annual effective interest rate of i.

   At the end of 9 years, the accumulated value of Fund X equals the accumulated value of Fund Y.

   Determine i. [SOA 5/98 #1]

   (A) 4.5% (B) 4.8% (C) 5.1% (D) 5.4% (E) 5.7%

7. You are given:

   (i) Fund X pays interest at the rate of 6% convertible monthly;
   (ii) Fund Y pays interest at a force of interest $\delta_t = 1/(t + 12)$;
   (iii) Ron deposited P into each fund; and

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(iv) at the end of 12 years, the accumulated amount in Fund X was 2000 and the accumulated amount in Fund Y was Z.

Determine Z.

(A) 1950 (B) 1970 (C) 2000 (D) 2030 (E) 2050

8. At a force of interest $\delta_t = \frac{0.1}{1+0.1t}$, $0 \le t \le 14$, the following payments have the same present value:

(i) X at the end of year 5 plus $2X$ at the end of year 10; and

(ii) Y at the end of year 14.

Calculate $\frac{Y}{X}$. [SOA 11/93 #2]

(A) 3.5 (B) 3.8 (C) 4.0 (D) 4.4 (E) 5.0

9. Given a nominal interest rate of 7.5% convertible semiannually, determine the sum of the equivalent:

(i) force of interest; and

(ii) nominal discount rate compounded quarterly.

[SOA 11/92 #1]

(A) 0.1460 (B) 0.1463 (C) 0.1466 (D) 0.1469 (E) 0.1472

10. You are given:

(i) $\delta_t = 1/(2+t)$; and

(ii) the total interest earned during the first n years on an investment of 1 at time $t = 0$ is 8.

Determine n. [SOA 5/91 #19]

(A) 4 (B) 8 (C) 16 (D) 24 (E) 32

11. A fund earns interest at a force of interest $\delta_t = kt$. A deposit of 100 at time $t = 0$ will grow to 250 at the end of 5 years.

Determine k. [SOA 11/90 #1]

(A) $0.08(\log_e 1.5)$ (B) 0.06 (C) $0.08(\log_e 2.5)$ (D) 0.08 (E) $\log_e 2.5$

12. An investment will triple in 87.88 years at a constant force of interest δ. Another investment will quadruple in t years at a nominal rate of interest numerically equal to δ and convertible once every 4 years.

Calculate t. [SOA 11/90 #16]

(A) 101 years (B) 106 years (C) 109 years (D) 114 years (E) 125 years

13. You are given a loan on which interest is charged over a 4-year period, as follows:

(i) an effective rate of discount of 6% for the first year;

(ii) a nominal rate of discount of 5% compounded every 2 years for the second year;

(iii) a nominal rate of interest of 5% compounded semiannually for the third year; and

(iv) a force of interest of 5% for the fourth year.

Calculate the annual effective rate of interest over the 4-year period. [SOA 5/90 #2]

(A) 0.0500 (B) 0.0525 (C) 0.0550 (D) 0.0575 (E) 0.0600

14. Fund X starts with 1,000 and accumulates with a force of interest $\delta_t = 1/(15-t)$ for $0 \le t < 15$.

Fund Y starts with 1,000 and accumulates with an interest rate of 8% per annum compounded semiannually for the first three years and an effective interest rate of i per annum thereafter.

Fund X equals Fund Y at the end of four years.

Calculate i. [SOA 11/89 #2]

(A) 0.0750 (B) 0.0775 (C) 0.0800 (D) 0.0825 (E) 0.0850

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15. Which of the following are true?
    (i) $\frac{d}{dd}(i) = v^{-2}$
    (ii) $\frac{d}{di}\left(i^{(m)}\right) = v^{-\left(\frac{m-1}{m}\right)}$
    (iii) $\frac{d}{d\delta}(i) = 1 + i$
    [SOA 5/89 #2]

    (A) I and II only (B) I and III only (C) II and III only (D) I, II and III
    (E) The correct answer is not given by (A), (B), (C), or (D).

16. Two funds, X and Y, start with the same amount. You are given:
    (i) Fund X accumulates at a force of interest of 5%.
    (ii) Fund Y accumulates at a rate of interest j, compounded semiannually.
    (iii) At the end of eight years, Fund X is 1.05 times as large as Fund Y.

    Calculate j. [SOA 5/89 #4]

    (A) 0.022 (B) 0.023 (C) 0.042 (D) 0.044 (E) 0.046

17. Fund A accumulates at a force of interest $\frac{0.05}{1+0.05t}$ at time t $(t \geq 0)$.

    Fund B accumulates at a force of interest 0.05.

    You are given:
    (i) The amount in Fund A at time zero is 1,000.
    (ii) The amount in Fund B at time zero is 500.
    (iii) The amount in Fund C at any time t $(t \geq 0)$ is equal to the sum of the amount in Fund A and the amount in Fund B. Fund C accumulates at a force of interest δ_t $(t \geq 0)$.

    Calculate δ_2. [SOA 11/88 #1]

    (A) $\frac{31}{660}$ (B) $\frac{21}{440}$ (C) $\frac{1+e^{0.1}}{22+20e^{0.1}}$ (D) $\frac{2+e^{0.1}}{44+20e^{0.1}}$ (E) $\frac{2+e^{0.1}}{22+20e^{0.1}}$

18. If an investment of 1 will be tripled in 175.778 years at a constant force of interest δ, in how many years will an investment of 1 be doubled at a nominal rate of interest numerically equivalent to δ and convertible once every four years? [CAS 5/88 #3]

    (A) Less than 105
    (B) At least 105, but less than 110
    (C) At least 110, but less than 115
    (D) At least 115, but less than 120
    (E) 120 or more

19. The force of interest at time t is $\frac{t^3}{100}$.

    Calculate $a^{-1}(3)$. [SOA 5/88 #5]

    (A) 0.76 (B) 0.78 (C) 0.80 (D) 0.82 (E) 0.84

20. In Fund A, the accumulated value of 1 at any time $t > 0$ is $1 + t$.

    In Fund B, the accumulated value of 1 at any time $t > 0$ is $1 + t^2$.

    T is the time when the force of interest for Fund A is equal to the force of interest for Fund B.

    Calculate T. [SOA 11/87 #2]

    (A) 0.41 (B) 1.00 (C) 1.41 (D) 2.00 (E) 2.41

21. In Fund X, money accumulates at a force of interest

    $$\delta_t = 0.01t + 0.1 \quad \text{for} \quad 0 \leq t \leq 20.$$

    In Fund Y, money accumulates at an annual effective interest rate i.

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An amount of 1 is invested in each of Fund X and Fund Y for 20 years. The value of Fund X at the end of 20 years is equal to the value of Fund Y at the end of 20 years.

Calculate the value of Fund Y at the end of 1.5 years. [SOA 11/87 #6]

(A) $e^{0.1}$ (B) $e^{0.2}$ (C) $e^{0.3}$ (D) $e^{0.4}$ (E) $e^{0.5}$

22. You are given $\delta_t = \frac{2}{1+t}$.

A payment of 300 at the end of 3 years and 600 at the end of 6 years has the same present value as a payment of 200 at the end of 2 years and X at the end of 5 years.

Calculate X. [SOA 11/87 #12]

(A) 306 (B) 316 (C) 376 (D) 456 (E) 506

23. On January 1, 1986, Sam invests 1000 in a fund for which the force of interest at time t is expressed by $(0.1)(t-1)^2$, where t is the number of years since January 1, 1986.

Calculate the accumulated value of the fund on January 1, 1988. [SOA 5/87 #2]

(A) 1065 (B) 1067 (C) 1069 (D) 1071 (E) 1073

24. It takes 11.553 years for an initial investment to double at a force of interest δ.

How long will it take for an initial investment to triple at a nominal rate of interest numerically equal to δ and convertible twice a year? [CAS 5/86 #2]

(A) Less than 19 years
(B) At least 19 years, but less than 19.5 years
(C) At least 19.5 years, but less than 20 years
(D) At least 20 years, but less than 20.5 years
(E) 20.5 years or more

25. An investment of 1 will double in 27.72 years at a force of interest $= \delta$. An investment of 1 will increase to 7.04 in n years at a nominal rate of interest numerically equal to δ and convertible once every 2 years.

Calculate n. [SOA 5/86 #4]

(A) 78 (B) 79 (C) 80 (D) 81 (E) 82

26. Fund A accumulates at a rate of 12% convertible monthly. Fund B accumulates with a force of interest, $\delta_t = t/6$, for all t. At time $t = 0$, 1 is deposited in each fund. T is the time that the two funds are equal, $T > 0$.

Determine T. [SOA 5/86 #5]

(A) $12 \cdot \ln(1.01)$ (B) $12 \cdot [\ln(1.12) - \ln(1.01)]$ (C) $12 \cdot \ln(1.12)$ (D) $144 \cdot \ln(1.01)$ (E) $144 \cdot \ln(1.12)$

27. Fund A accumulates at a force of interest of $\delta_t = a + bt$. Fund B accumulates at a force of interest of $\delta_t = g + ht$.

You are given:

(i) $a > g > 0$
(ii) $h > b > 0$
(iii) Fund A = Fund B at $t = 0$
(iv) Fund A = Fund B at $t = n$

Determine n. [SOA 5/86 #10]

(A) $\frac{a-g}{h-b}$ (B) $\frac{2(a-g)}{h-b}$ (C) $\frac{h-b}{a-g}$ (D) $\frac{h-b}{2(a-g)}$ (E) $\frac{2(h-b)}{a-g}$

28. Simplify: $\left[\frac{d}{dv}\delta\right] \cdot \left[\frac{d}{di}d\right]$ [SOA 11/85 #11]

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(A) $-v^3$ (B) $-v$ (C) 1 (D) v (E) v^3

29. Two funds are started with equal deposits at time $t = 0$.

Fund A is credited with simple interest at rate i.

Fund B is credited with compound interest at rate i. The force of interest equivalent to i is δ.

Find an expression for the time t, $0 < t < 1$, at which the difference between Fund A and Fund B is greatest. [SOA SAMPLE/84 #6]

(A) $i - \delta$ (B) $\frac{i}{\delta} - 1$ (C) $\log_e\left(\frac{i}{\delta}\right)$ (D) $\delta \log_e\left(\frac{i}{\delta^2}\right)$ (E) $\frac{1}{\delta}\log_e\left(\frac{i}{\delta}\right)$

30. You are given:

(i) $A(t) = Kt^2 + Lt + M$ for $0 \le t \le 2$
(ii) $A(0) = 100$
(iii) $A(1) = 110$
(iv) $A(2) = 136$

Determine the force of interest at time $t = \frac{1}{2}$. [SOA SAMPLE/83 #7]

(A) 0.030 (B) 0.049 (C) 0.061 (D) 0.095 (E) 0.097

31. \$1,000 is deposited into a savings account at time $t = 0$. No other amounts are deposited.

The accumulated amount in the account at any time, t, is given by

$$A(t) = \$1{,}000 \cdot (1 + 2t/35)^2$$

At time t_1 the force of interest equals .04.

What is t_1? [CAS 11/82 #3]

(A) 29.5 (B) 32.5 (C) 35.5 (D) 38.5 (E) 41.5

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Solutions to Past Exam Questions on Sections 1b to 1f

1. Bruce: $\left(1+\frac{i}{2}\right)^{14.5}=2,\ i=9.79\%$

Peter: $e^{7.25\delta}=2,\ \delta=\frac{\ln 2}{7.25}=9.56\%$

$i-\delta=9.79\%-9.56\%=.23\%$ ANS. **(B)**

Note: Some students interpret the statement that "His account is credited with interest at a nominal rate of i convertible semiannually"to mean that Bruce's account earns interest only at the end of each 6-month period. Therefore, they reason that after 7.25 years, Bruce has only $100(1+i/2)^{14}$ in his account. However, it is very important to note that in Exam FM/2, an account is always assumed to grow with interest continuously, no matter how the interest rate is stated (i.e., whether as an annual effective rate, as a nominal rate compounded m-thly, as a force of interest, etc.). If there is an exception to this assumption, it should be clearly stated in the question.

2. $e^{\int_0^5 \frac{t^2}{k}dt}=\left(1-\frac{.08}{2}\right)^{-10}=1.50414$

$e^{\frac{t^3}{3k}\Big]_0^5}=e^{\frac{125}{3k}}=1.50414$

$\frac{125}{3k}=\ln 1.50414=.40822$

$k=102$ ANS. **(B)**

3. Tawny's annual effective rate $i=1.05^2-1=.1025$
$\therefore$ Tawny's constant force of interest $\delta=\ln 1.1025=.0975803$.

Let Fabio's "rate of simple interest" $=i$:

$a(t)=1+it$

$\delta_t=\frac{1}{a(t)}\frac{d}{dt}a(t)=\frac{i}{1+it}$

$\delta_5=\frac{i}{1+5i}=.0975803$

$\therefore\ i=.190550$

AV of Fabio's account in 5 years

$=(1{,}000)(1+5i)=1{,}953$ ANS. **(B)**

4. Fund Y: $2K(1.1)^{n-m}=4K$

$1.1^{n-m}=2$

Using the TVM keys ($\boxed{\text{I/Y}}=10\%$, $\boxed{\text{PV}}=1$, $\boxed{\text{FV}}=-2$, then $\boxed{\text{CPT}}\ \boxed{\text{N}}$), we get:

$n-m=7.27$

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Fund X: $Ke^{\int_0^n .006t^2 dt} = 4K$

$$e^{.002t^3}\Big]_0^n = e^{.002n^3} = 4$$

$$.002n^3 = \ln 4 = 1.38629$$

$$n^3 = 693.147, \quad n = 8.85$$

$m = n - 7.27 = 8.85 - 7.27 = 1.58$ ANS. (A)

5. Fund X: $10{,}000(1.05)^t$

Fund Y: $10{,}000(1 + .08t)$

Force of interest for Fund X $= \ln 1.05$, for all t

Force of interest for Fund Y $= \dfrac{1}{a(t)}\dfrac{d}{dt}a(t)$

$$= \frac{.08}{1 + .08t} = \ln 1.05$$

$\therefore\ t = 7.995934$ (i.e., when the forces of interest are equal)

Fund X: AV in t years $= 10{,}000(1.05)^t$

$= 14{,}772$

Fund Y: AV in t years $= 10{,}000(1 + .08t)$

$= 16{,}397$

$Z = 16{,}397 - 14{,}772 = 1{,}625$ ANS. (A)

6. $e^{\int_0^9 \frac{0.5}{(1+t)^2}dt} = (1+i)^9$

$$e^{-\frac{.5}{1+t}}\Big]_0^9 = e^{.45} = (1+i)^9$$

$$.45 = 9\ln(1+i)$$

$$\ln(1+i) = .05$$

$$1 + i = e^{.05}$$

Enter .05 into the calculator and use secondary key $\boxed{e^x}$ to find that $1 + i = 1.0513$ and $i = 5.1\%$. ANS. (C)

7. $X(1.005)^{144} = 2{,}000,\ X = 975.25$

$$Z = 975.25e^{\int_0^{12} \frac{1}{t+12}dt}$$

$$= 975.25\left(\frac{24}{12}\right) = 1{,}950$$ ANS. (A)

8. Using time 0 as the comparison date:

$$a(t) = e^{\int_0^t \frac{0.1}{1+0.1r}dr}$$

$$= 1 + 0.1t$$

$$\frac{X}{a(5)} + \frac{2X}{a(10)} = \frac{Y}{a(14)}$$

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$$\frac{Y}{X} = \left[\frac{1}{a(5)} + \frac{2}{a(10)}\right] a(14)$$

$$= \left(\frac{1}{1.5} + \frac{2}{2}\right)(2.4) = 4 \qquad \text{ANS. (C)}$$

9. $e^{\delta} = 1.0375^2,\ \delta = .073628$

$$1.0375^2 = \left(1 - \frac{d^{(4)}}{4}\right)^{-4},\ d^{(4)} = .072954$$

$\delta + d^{(4)} = .1466$ ANS. **(C)**

10. $a(t) = e^{\int_0^t \frac{1}{2+r} dr} = e^{\ln(2+r)}\Big]_0^t$

$$= \frac{2+t}{2}$$

Total interest earned on an investment of 1

$$= a(n) - 1 = \frac{2+n}{2} - 1 = 8 \text{ and } n = 16 \qquad \text{ANS. (C)}$$

11. $100e^{\int_0^5 kt\,dt} = 100e^{\frac{kt^2}{2}}\Big]_0^5$

$= 100e^{12.5k} = 250$

$e^{12.5k} = 2.5$

$12.5k = \ln 2.5,\ k = .08 \ln 2.5$ ANS. **(C)**

12. $e^{87.88\delta} = 3,\ 87.88\delta = \ln 3$

$$\delta = \frac{\ln 3}{87.88} = .0125$$

$(1 + 4\delta)^{\frac{t}{4}} = 4 = [1 + 4(.0125)]^{\frac{t}{4}} = 1.05^{\frac{t}{4}}$

$\frac{t}{4} = 28.41,\ t = 113.6$ ANS. **(D)**

13. $(1 - .06)^{-1}[1 - (2)(.05)]^{-0.5}(1.025)^2 e^{.05} = (1 + i)^4$

$(1 + i)^4 = 1.23855, \quad 1 + i = 1.0549$

$i = .055$ ANS. **(C)**

14. $1{,}000(1.04)^6(1 + i) = 1{,}000e^{\int_0^4 \frac{1}{15-t} dt}$

$$= 1{,}000e^{-\ln(15-t)}\Big]_0^4 = 1{,}000\left(\frac{15}{11}\right)$$

$$\therefore\ (1 + i) = \frac{15}{(11)(1.04)^6} = 1.0777 \qquad \text{ANS. (B)}$$

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15. I. $\frac{d}{dd}(i) = \frac{d}{dd}\left(\frac{d}{1-d}\right)$

$$= \frac{(1-d)-(d)(-1)}{(1-d)^2} = \frac{1}{(1-d)^2}$$

$$= \frac{1}{v^2} = v^{-2} \text{ (I is true)}$$

II. $\frac{d}{di}\left(i^{(m)}\right) = \frac{d}{di} m\left[(1+i)^{\frac{1}{m}} - 1\right]$

$$= (m)\left(\frac{1}{m}\right)(1+i)^{\frac{1}{m}-1} = (1+i)^{\frac{1-m}{m}}$$

$$= v^{\frac{m-1}{m}} \text{ (II is false)}$$

III. $\frac{d}{d\delta}(i) = \frac{d}{d\delta}\left(e^{\delta} - 1\right) = e^{\delta} = 1 + i$ (III is true) ANS. (B)

16. AV in 8 years:

Fund X: $e^{(.05)(8)} = e^{0.4}$

Fund Y: $\left(1+\frac{j}{2}\right)^{(2)(8)} = \left(1+\frac{j}{2}\right)^{16}$

$$e^{0.4} = 1.05\left(1+\frac{j}{2}\right)^{16}$$

$$j = 2\left[\left(\frac{e^{0.4}}{1.05}\right)^{\frac{1}{16}} - 1\right] = .044 \qquad \text{ANS. (D)}$$

17. Amount function for Fund C:

$$A(t) = 1{,}000e^{\int_0^t \frac{.05}{1+.05r}dr} + 500e^{.05t}$$

$$= 1{,}000e^{\ln(1+.05r)}\Big]_0^t + 500e^{.05t}$$

$$= 1{,}000(1+.05t) + 500e^{.05t}$$

To determine δ_t for Fund C, we differentiate the amount function and divide by the amount function:

$$\delta_t = \frac{50 + 25e^{.05t}}{1{,}000 + 50t + 500e^{.05t}}$$

$$\delta_2 = \frac{50 + 25e^{0.1}}{1{,}100 + 500e^{0.1}} = \frac{2 + e^{0.1}}{44 + 20e^{0.1}} \qquad \text{ANS. (D)}$$

18. $e^{175.778\delta} = 3$, $\delta = \frac{\ln 3}{175.778} = .00625$

If $i^{(\frac{1}{4})} = .00625$, the effective rate for a 4-year period $= (4)(.00625) = .025$

$$1.025^{\frac{t}{4}} = 2$$

$$t = 4\frac{\ln 2}{\ln 1.025} = 112.3 \qquad \text{ANS. (C)}$$

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19. $a(t) = e^{\int_0^t \frac{r^3}{100} dr} = e^{.0025r^4}\Big]_0^t = e^{.0025t^4}$

$a^{-1}(3) = e^{-0.2025} = .817$ ANS. **(D)**

20. Fund A: $a(t) = 1 + t$

$$\delta_t = \frac{1}{a(t)} \frac{d}{dt}(1+t) = \frac{1}{1+t}$$

Fund B: $a(t) = 1 + t^2$

$$\delta_t = \frac{2t}{1+t^2}$$

$\frac{1}{1+T} = \frac{2T}{1+T^2}$ from which:

$T^2 + 2T - 1 = 0$

$$T = \frac{-2 \pm \sqrt{2^2 - (4)(1)(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$

$= \sqrt{2} - 1$(taking the positive root)

$= 0.41$ ANS. **(A)**

21. Fund X: $e^{\int_0^{20}(.01t+.1)dt} = e^{.01\frac{t^2}{2}+.1t}\Big]_0^{20} = e^4$

Fund Y: $(1+i)^{20}$

$e^4 = (1+i)^{20}$, $(1+i) = e^{.2}$

We want $(1+i)^{1.5} = \left(e^{.2}\right)^{1.5} = e^{.3}$ ANS. **(C)**

22. $a(t) = e^{\int_0^t \frac{2}{1+r} dr} = e^{2\ln(1+r)}\Big]_0^t = (1+t)^2$

The discount factor $a^{-1}(t) = \frac{1}{(1+t)^2}$

$$\frac{300}{(1+3)^2} + \frac{600}{(1+6)^2} = \frac{200}{(1+2)^2} + \frac{X}{(1+5)^2}$$

$$\frac{300}{16} + \frac{600}{49} = \frac{200}{9} + \frac{X}{36}$$

$X = 316$ ANS. **(B)**

23. $A(t) = 1{,}000e^{\int_0^t (.1)(r-1)^2 dr} = 1{,}000e^{\frac{0.1}{3}(r-1)^3}\Big]_0^t$

$= 1{,}000e^{\frac{0.1}{3}(t-1)^3 + \frac{0.1}{3}}$

$A(2) = 1{,}000e^{\frac{0.1}{3}+\frac{0.1}{3}} = 1{,}000e^{\frac{0.2}{3}}$

$= 1{,}000e^{.06\dot{6}} = 1{,}069$ ANS. **(C)**

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24. $e^{11.553\delta} = 2,\ \delta = \dfrac{\ln 2}{11.553} = .06$

$$\left(1 + \frac{.06}{2}\right)^{2t} = 3$$

$$t = \frac{\ln 3}{2 \ln 1.03} = 18.58 \text{ years} \qquad \text{ANS. (A)}$$

25. $e^{27.72\delta} = 2,\ \delta = \frac{\ln 2}{27.72} = .025$

If .025 is the nominal rate convertible once every 2 years, the effective rate for a 2-year period is .05.

$$1.05^{\frac{n}{2}} = 7.04$$

$$n = \frac{2 \ln 7.04}{\ln 1.05} = 80 \qquad \text{ANS. (C)}$$

26. Fund A: 1.01^{12t}

Fund B: $e^{\int_0^t \frac{r}{6} dr} = e^{\frac{t^2}{12}}$

at time T: $1.01^{12T} = e^{\frac{T^2}{12}}$

$$12T \ln 1.01 = \frac{T^2}{12}$$

$$T = 144 \ln 1.01 \qquad \text{ANS. (D)}$$

27. Fund A: $e^{at + b\frac{t^2}{2}}$

Fund B: $e^{gt + h\frac{t^2}{2}}$

$$e^{an + b\frac{n^2}{2}} = e^{gn + h\frac{n^2}{2}}$$

Equating exponents: $an + \frac{bn^2}{2} = gn + \frac{hn^2}{2}$

$$a + \frac{bn}{2} = g + \frac{hn}{2}$$

$$n = \frac{2(a-g)}{h-b} \qquad \text{ANS. (B)}$$

28. $$\left[\frac{d}{dv}\delta\right]\left[\frac{d}{di}d\right] = \left[\frac{d}{dv}\ln v^{-1}\right]\left[\frac{d}{di}\left(\frac{i}{1+i}\right)\right]$$

$$= \left[-\frac{1}{v}\right]\left[\frac{(1+i)-i}{(1+i)^2}\right] = \left(-\frac{1}{v}\right)\left(v^2\right)$$

$$= -v \qquad \text{ANS. (B)}$$

29. Fund A: $1 + it$

Fund B: $(1+i)^t$

Let Δ = difference between Fund A and Fund B. To find t at which Δ is a maximum, differentiate and set equal to 0:

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$$\Delta = 1 + it - (1+i)^t$$

$$\frac{d}{dt}\Delta = i - (1+i)^t \ln(1+i)$$

$$= i - (1+i)^t \delta$$

Let $i - (1+i)^t\delta = 0$

$$(1+i)^t = \frac{i}{\delta}, \quad t\ln(1+i) = \ln\frac{i}{\delta}$$

$$t = \frac{1}{\delta}\ln\frac{i}{\delta} \qquad \text{ANS. (E)}$$

Note: To make sure we have a max, we would take the second derivative.

30. $A(0) = 100 = M$

$$A(1) = K + L + 100 = 110$$

$$A(2) = 4K + 2L + 100 = 136$$

Solving this pair of linear equations, we get $K = 8$, $L = 2$.

$$\therefore\ A(t) = 8t^2 + 2t + 100$$

$$\delta_t = \frac{1}{A(t)}\frac{d}{dt}A(t) = \frac{16t+2}{8t^2+2t+100}$$

$$\delta_{\frac{1}{2}} = \frac{8+2}{2+1+100} = \frac{10}{103} = .097 \qquad \text{ANS. (E)}$$

31. $\delta_t = \dfrac{1}{A(t)}\dfrac{d}{dt}A(t)$

$$= \frac{2000\left(1+\frac{2t}{35}\right)\left(\frac{2}{35}\right)}{1{,}000\left(1+\frac{2t}{35}\right)^2}$$

$$= \frac{(2)\left(\frac{2}{35}\right)}{\left(1+\frac{2t}{35}\right)} = .04$$

Solving for t, we get $t = 32.5$ ANS. **(B)**

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§1g. The Variable Force of Interest Trap

Trap Alert!

Investing at a variable force of interest can be very tricky ***if the investment is not made at time 0.*** Here is an example of a type of question that has appeared on past exams that can easily trap you. Try it now.

Stepping Stones

EXAMPLE 1

The force of interest is $\delta_t = .02t$, where t is the number of years from January 1, 2001. If \$1.00 is invested on January 1, 2003, how much is in the fund on January 1, 2008?

(A) $e^{-.25}$ (B) $e^{.25} - 1$ (C) $e^{.25}$ (D) $e^{.45}$ (E) $e^{.49}$

SOLUTION

The correct answer is (D). If you got (C), you probably fell into the following trap:

> "The money is invested for 5 years. This means that the AV (accumulated value) is $a(5)$, since $a(5)$ is the AV of \$1.00 invested for 5 years. $a(5) = e^{\int_0^5 .02t dt} = e^{[.01t^2]_0^5} = e^{.25}$. ANSWER (C)."

What's wrong with this approach? Remember that t, and therefore δ_t and $a(t)$, are measured from ***January 1, 2001.*** This means that $a(5)$ is the AV on *January 1, 2006* of \$1.00 invested on *January 1, 2001.* It is true that this is a five-year period but it is not the *same* five-year period as from January 1, 2003 ($t = 2$) to January 1, 2008 ($t = 7$), which is the period of the investment in this question.

If the interest rate varies over time, as in this example, we would expect that money will grow differently over different five year periods. If fact, in this case, it is clear that \$1.00 will grow *faster* from $t = 2$ to $t = 7$ than it will from $t = 0$ to $t = 5$, because the force of interest ($.02t$) is increasing with time.

It turns out that it's very easy to get the correct answer. The AV is equal to the accumulation function on the date of the AV divided by the accumulation function on the date of the investment, or $a(7)/a(2)$:

$$a(7) = e^{\int_0^7 .02t dt} = e^{.49}$$

$$a(2) = e^{\int_0^2 .02t dt} = e^{.04}$$

$$\frac{a(7)}{a(2)} = \frac{e^{.49}}{e^{.04}} = e^{.45}$$

ANSWER (D)

An equivalent method is to integrate between the time of the investment and the time at which we want the AV:

$$e^{\int_2^7 .02t dt} = e^{[.01t^2]_2^7} = e^{.49-.04} = e^{.45}$$

An explanation of the above methods appears below.

(If you want the *present value* at a time other than $t = 0$, simply use a negative exponent. For example, the present value at time 2 of \$1.00 due at time 7 at a force of interest δ_t is:

$$PV = e^{-\int_2^7 \delta_t dt}$$

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or $a(2)/a(7)$ or $e^{-.45}$.)

Explanation

The following is an explanation of the above methods:

As in the example, consider an investment of \$1.00 at time 2 which grows to time 7. We can't use the $a(t)$ function directly because it gives us the AV at time t of \$1.00 invested *at time 0*. We do know that \$1.00 invested at time 0 will grow to $a(2)$ at time 2 and that $a(2)$ will continue to grow to $a(7)$ at time 7. (This comes directly from the definition of the accumulation function.)

Since $a(2)$ at time 2 grows to $a(7)$ at time 7, by proportion, \$1.00 at time 2 must grow to $a(7)/a(2)$ at time 7. This verifies the method we used to solve the above example.

(Alternatively, get the PV at time 0 of the \$1.00 investment at time 2. This is $\frac{1}{a(2)}$. In other words, investing $\frac{1}{a(2)}$ at time 0 is equivalent to investing \$1.00 at time 2. Then accumulate $\frac{1}{a(2)}$ to time 7: $\frac{a(7)}{a(2)}$.)

To obtain the integral form, we have:

$$\frac{a(7)}{a(2)} = \frac{e^{\int_0^7 \delta_t dt}}{e^{\int_0^2 \delta_t dt}} = e^{\int_0^7 \delta_t dt - \int_0^2 \delta_t dt}$$

Looking at an integral as an area, the difference between the two integrals in the exponent of e is the difference between the area under δ_t from 0 to 7 and the area under δ_t from 0 to 2. This leaves the area under δ_t from 2 to 7:

$$\frac{a(7)}{a(2)} = e^{\int_2^7 \delta_t dt}$$

This confirms the method in integral form used above to solve this problem.

In general, if you want to accumulate an investment from time t_1 to time t_2, integrate between t_1 and t_2 and take e to that power. Alternatively, $a(t_2)/a(t_1)$ gives the answer in terms of the accumulation function.

EXAMPLE 2

The accumulation function is $a(t) = 1 + .05\ln(t+1)$. X is deposited at time 5. At time 10 there is \$100 in the account. Determine X.

SOLUTION

$$X = 100a(5)/a(10) = 100(1 + .05\ln 6)/(1 + .05\ln 11) = 100(1.089588)/(1.119895)$$
$$= \$97.29$$

EXAMPLE 3

You are given that $\delta_t = 1/(t+1)$. Determine the accumulated value at time 5 of \$1.00 deposited at time 2.

SOLUTION

First we determine $a(t)$:

$$a(t) = e^{\int [1/(1+r)]\,dr}$$

where the limits of integration are from 0 to t. This gives us:

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$$a(t) = t + 1$$
$$AV = a(5)/a(2) = 6/3 = 2$$

A Technical Note on "Simple Interest"

In Section 1a(ii), we defined "simple interest" to mean that the accumulation function is linear:

$$a(t) = 1 + it$$

Let's assume that the linear constant i is equal to .05:

$$a(t) = 1 + .05t$$

Suppose we want to determine the AV at time 7 of \$1.00 invested at time 2. If we follow the methodology that we just used above, the answer would be:

$$AV = \frac{a(7)}{a(2)} = \frac{1 + (.05)(7)}{1 + (.05)(2)} = \frac{1.35}{1.10} = \$1.2273$$

However, please note that "simple interest" is generally understood to mean that the linear function *starts all over again from the date of each deposit or withdrawal.* In other words, if a deposit of \$1.00 is made at time 2, "simple interest" is interpreted to mean that the AV at time 7 is equal to $1 + (.05)(5) = \$1.25$. We realize that this is inconsistent with the treatment of any other accumulation function (e.g., when $\delta(t) = .02t$, as in Example 1 above), but this is how the SOA/CAS expects you to interpret "simple interest" on the exam. See, for example, question 8 following Section 1g, question 2 following Section 2b and question 17 following Section 3c.

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Summary of Concepts and Formulas in Section 1g

(1) If \$1.00 is invested at time t_1 and accumulated to time t_2, and the accumulation function is $a(t)$, then AV of this investment is:

$$AV = \frac{a(t_2)}{a(t_1)}$$

(2) Alternatively, the AV can be expressed as follows:

$$AV = e^{\int_{t_1}^{t_2} \delta_t \, dt}$$

(3) Similarly, to determine the PV at time t_1 of \$1.00 due at time t_2:

$$PV = \frac{a(t_1)}{a(t_2)} = e^{-\int_{t_1}^{t_2} \delta_t \, dt}$$

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Past Exam Questions on Section 1g

1. Ernie makes deposits of 100 at time 0, and X at time 3. The fund grows at a force of interest $\delta_t = \frac{t^2}{100}$, $t > 0$. The amount of interest earned from time 3 to time 6 is X. Calculate X. [11/01 #1]

   (A) 385 (B) 485 (C) 585 (D) 685 (E) 785

2. A bank agrees to lend John 10,000 now and X three years later in exchange for a single repayment of 75,000 at the end of 10 years. The bank charges interest at an annual effective rate of 6% for the first 5 years and at a force of interest $\delta_t = 1/(t+1)$ for $t \geq 5$.

   Determine X. [SOA 5/98 #3]

   (A) 23,500 (B) 24,000 (C) 24,500 (D) 25,000 (E) 25,500

3. On 1/1/97 Kelly deposits X into a bank account. The account is credited with simple interest at the rate of 10% per year.

   On the same date, Tara deposits X into a different bank account. The account is credited interest using a force of interest:

   $$\delta_t = \frac{2t}{(t^2+k)}.$$

   From the end of the 4th year until the end of the 8th year, both accounts earn the same dollar amount of interest.

   Calculate k. [SOA 11/96 #1]

   (A) 96 (B) 104 (C) 112 (D) 120 (E) 128

4. A deposit of 100 is made into a fund at time $t = 0$. The fund pays interest at a nominal rate of discount d compounded quarterly for the first two years.

   Beginning at time $t = 2$, interest is credited at a force of interest $\delta_t = 1/(t+1)$.

   At time $t = 5$, the accumulated value of the fund is 260.

   Calculate d. [SOA 5/95 #2]

   (A) 12.7% (B) 12.9% (C) 13.1% (D) 13.3% (E) 13.5%

5. Peter deposits 400 into a bank account at time $t = 0$. During the first year, the bank credits interest at a nominal rate of 10% compounded semiannually.

   Peter makes an additional deposit of 42 into his bank account at time $t = 1$. During the second year, the bank credits interest at a force of interest $\delta_t = \frac{1}{K+t}$.

   The total amount in Peter's account at time $t = 2$ is 552.

   Calculate K. [SOA 11/93 #1]

   (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

6. At time $t = 0$, Donald puts 1,000 into a fund crediting interest at a nominal rate of i compounded semiannually.

   At time $t = 2$, Lewis puts 1,000 into a different fund crediting interest at a force $\delta_t = \frac{1}{5+t}$ for all t.

   At time $t = 16$, the amounts in each fund will be equal.

   Calculate i. [SOA 5/93 #1]

   (A) 6.9% (B) 7.0% (C) 7.1% (D) 7.2% (E) 7.3%

7. You are given that $\delta_t = \frac{0.2t}{1+0.1t^2}$ for all $t > 0$.

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Determine i_2. [SOA 5/91 #14]

(A) 0.21 (B) 0.24 (C) 0.27 (D) 0.31 (E) 0.36

8. During the first 4 years, interest is credited using a simple interest rate of 5% per year. After 4 years, interest is credited at a force of interest:

$$\delta_t = \frac{0.2}{1+0.2t}, t \geq 4$$

The following are numerically equal:

(i) the current value at time $t = 4$ of payments of 1000 at time $t = 2$ and 400 at time $t = 7$; and
(ii) the present value at time $t = 0$ of a payment of X at time $t = 10$.

Calculate X. [SOA 5/91 #18]

(A) 2800 (B) 2840 (C) 3200 (D) 3660 (E) 4200

9. At time $t = 0$, Billy puts 625 into an account paying 6% simple interest. At the end of year 2, George puts 400 into an account paying interest at a force of interest $\delta_t = \frac{1}{6+t}$ for $t \geq 2$. If both accounts continue to earn interest indefinitely at the levels given above, the amounts in both accounts will be equal at the end of year n.

Calculate n. [SOA 11/90 #10]

(A) 23 (B) 24 (C) 25 (D) 26 (E) 27

10. Jeff puts 100 into a fund that pays an effective annual rate of discount of 20% for the first two years and a force of interest of rate $\delta_t = 2t/(t^2+8)$, $2 \leq t \leq 4$, for the next two years.

At the end of four years, the amount in Jeff's account is the same as what it would have been if he had put 100 into an account paying interest at the nominal rate of i per annum compounded quarterly for four years.

Calculate i. [SOA 11/89 #3]

(A) 0.200 (B) 0.219 (C) 0.240 (D) 0.285 (E) 0.295

11. A fund starts with a zero balance at time zero. The fund accumulates with a varying force of interest

$$\delta_t = \frac{2t}{t^2+1} \quad (t > 0).$$

A deposit of 100,000 is made at time 2.

Calculate the number of years from the time of deposit for the fund to double. [SOA 5/89 #3]

(A) 0.5 (B) 1.0 (C) 1.5 (D) 2.0 (E) 2.5

12. An investment fund accumulates with force of interest

$$\delta_t = \frac{K}{1+(1-t)K}$$

for $0 \leq t \leq 1$.

At time zero, there is 100,000 in the fund. At time one, there is 110,000 in the fund. The only two transactions during the year are a deposit of 15,000 at time 0.25 and a withdrawal of 20,000 at time 0.75.

Calculate K. [SOA 11/88 #18]

(A) 0.047 (B) 0.051 (C) 0.141 (D) 0.150 (E) 0.154

13. A deposit of 1 will accumulate to 2.7183 in 10 years with force of interest

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$$\delta_t = \begin{cases} kt & 0 < t \le 5 \\ \frac{1}{25}kt^2 & 5 < t \le 10 \end{cases}$$

Calculate k. [SOA 11/86 #4]

(A) 0.01 (B) 0.02 (C) 0.03 (D) 0.04 (E) 0.05

14. If $\delta_t = .15\sqrt{t}$ and an amount of 1 is invested at time $t = 1$, what is the accumulated value at time $t = 4$? [CAS 5/84 #2]

(A) $e^{.4}$ (B) $e^{.5}$ (C) $e^{.7}$ (D) $e^{.8}$ (E) $e^{.9}$

15. Jill has \$10,000 to invest at time $t = 0$, and two possible ways to invest it.

Investment A has a force of interest equal to $\frac{.08}{1+.08t}$ at time t.

Investment B provides a 5% effective annual interest rate.

Jill can invest any portion of her principal in either investment A or B, and can transfer any portion of her money between the two investments at any time.

What is the maximum amount Jill can accumulate by time $t = 20$? [CAS 5/84 #4]

(A) Less than \$26,000
(B) At least \$26,000, but less than \$27,000
(C) At least \$27,000, but less than \$28,000
(D) At least \$28,000, but less than \$29,000
(E) \$29,000 or more

16. On July 1, 1984 a person invests 1,000 in a fund for which the force of interest at time t is expressed by the formula:

$$\delta_t = \frac{3 + 2t}{50}$$

where t is the number of years since January 1, 1984.

Determine the accumulated value of the investment on January 1, 1985. [SOA SAMPLE/83 #1]

(A) 1,036 (B) 1,041 (C) 1,045 (D) 1,046 (E) 1,051

17. The force of interest on a savings account at time t is given by

$$\frac{t/25}{\frac{t^2}{50} + 1}$$

At time $t = 0$, \$100 is deposited in the account; at time $t = 10$, an additional \$200 is deposited in the account.

Which of the following is closest to the time t when the balance in the account reaches \$1,500? [CAS 5/83 #13]

(A) 16 (B) 18 (C) 20 (D) 22 (E) 24

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Solutions to Past Exam Questions on Section 1g

1. $a(t) = e^{\int_0^t \frac{r^2}{100} dr} = e^{\frac{r^3}{300}}\Big]_0^t = e^{\frac{t^3}{300}}$

The interest earned from time 3 to 6 is equal to the amount in the fund at time 6 minus the amount in the fund at time 3:

$$100e^{\frac{6^3}{300}} + Xe^{\frac{t^3}{300}}\Big]_3^6 - \left(100e^{\frac{3^3}{300}} + X\right)$$

$$= 100e^{.72} + Xe^{.63} - \left(100e^{.09} + X\right)$$

$$= 205.44 + 1.877610X - (109.42 + X) = X$$

Solving for X: $X = \dfrac{96.02}{.12239} = 784.54$ ANS. (E)

2. Use time 5 (the date the interest rate changes) to evaluate all payments:

$$10{,}000(1.06)^5 + X(1.06)^2 = 75{,}000e^{-\int_5^{10} \frac{dt}{t+1}}$$

$$13{,}382 + 1.1236X = 75{,}000e^{-\ln(t+1)}\Big]_5^{10}$$

$$= 75{,}000e^{-\ln 11 + \ln 6}$$

$$= 75{,}000\left(\frac{6}{11}\right)$$

$X = 24{,}499$ ANS. (C)

3. Kelley: $a(t) = 1 + 0.1t$

Tara: $a(t) = e^{\int_0^t \frac{2r}{r^2+k} dr} = e^{\ln(r^2+k)}\Big]_0^t$

$$= e^{\ln \frac{t^2+k}{k}} = \frac{t^2+k}{k}$$

Since they both deposit the same amount X, X will cancel out.

$$\frac{8^2+k}{k} - \frac{4^2+k}{k} = 1 + (8)(.1) - [1 + (4)(.1)]$$

$$\frac{64+k}{k} - \frac{16+k}{k} = \frac{48}{k} = .4$$

$k = 120$ ANS. (D)

4. $$100\left(1 - \frac{d}{4}\right)^{-8} e^{\int_2^5 \frac{1}{1+t} dt} = 260$$

$$100\left(1 - \frac{d}{4}\right)^{-8} e^{\ln(1+t)}\Big]_2^5 = 100\left(1 - \frac{d}{4}\right)^{-8}(2) = 260$$

$$\left(1 - \frac{d}{4}\right)^{-8} = 1.30, \ d = .129$$ ANS. (B)

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5. $$\left[400(1.05)^2+42\right]e^{\int_1^2 \frac{dt}{K+t}}=552$$

$$483\left(\frac{K+2}{K+1}\right)=552,\ K=6 \qquad \text{ANS. (B)}$$

6. $$1000\left(1+\frac{i}{2}\right)^{32}=1000e^{\int_2^{16}\frac{dt}{5+t}}=3{,}000$$

$$\frac{i}{2}=3.492777\%,\quad i=7.0\% \qquad \text{ANS. (B)}$$

7. $$1+i_2=e^{\int_1^2 \frac{.2t}{1+0.1t^2}dt}=e^{\ln(1+0.1t^2)}\Big]_1^2$$

$$=\frac{1.4}{1.1}=1.2727$$

$$i_2=.2727 \qquad \text{ANS. (C)}$$

8. Note: Occasionally, you may see the term "current value," as in this question. This simply means the value of a series of payments on a specified date, some of which are made before that date and some after that date. Thus, "current value" is the sum of the AV and PV of the payments on the specified date.

$$\text{For } t\geq 4,\ \frac{a(t)}{a(4)}=e^{\int_4^t \frac{0.2}{1+0.2r}dr}$$

$$=e^{\ln(1+0.2r)}\Big]_4^t=\frac{1+0.2t}{1.8}$$

$$\frac{a(4)}{a(t)}=\frac{1.8}{1+0.2t}$$

$$\text{Current value}=1{,}000[1+(2)(.05)]+400\frac{a(4)}{a(7)}$$

$$=1{,}100+(400)\left(\frac{1.8}{2.4}\right)=1{,}400$$

$$PV=X\frac{a(4)}{a(10)}\cdot\frac{1}{1+(4)(.05)}=X\left(\frac{1.8}{3}\right)\left(\frac{1}{1.2}\right)=\frac{X}{2}=1{,}400$$

$$X=2{,}800 \qquad \text{ANS. (A)}$$

9. $$625(1+.06n)=400e^{\int_2^n \frac{dt}{6+t}}$$

$$=400\left(\frac{6+n}{8}\right)=50(6+n)$$

$$1+.06n=.08(6+n),\ n=26 \qquad \text{ANS. (D)}$$

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10.

$$AV = 100(1-0.2)^{-2}e^{\int_2^4 \frac{2t}{t^2+8}dt} = 156.25e^{\ln(t^2+8)}\Big]_2^4$$

$$= 156.25e^{\ln 24 - \ln 12} = (156.25)(2) = \$312.50$$

$$100\left(1+\frac{i}{4}\right)^{16} = \$312.50$$

$$i = 4\left(3.125^{\frac{1}{16}} - 1\right) = .295 \qquad \text{ANS. (E)}$$

11. Let x = number of years from time 0 for the deposit to double. Then the number of years from the date of deposit $= x - 2$.

$$e^{\int_2^x \frac{2t}{t^2+1}dt} = e^{\ln(t^2+1)}\Big]_2^x = e^{\ln\frac{(x^2+1)}{5}}$$

$$\frac{x^2+1}{5} = 2.$$

$$x = 3,\ x-2 = 1 \qquad \text{ANS. (B)}$$

12.

$$100{,}000e^{\int_0^1 \frac{K}{1+(1-t)K}dt}$$

$$= 100{,}000e^{-\ln[1+(1-t)K]}\Big]_0^1 = 100{,}000(1+K)$$

$$\text{Deposit plus int.} = 15{,}000e^{-\ln[1+(1-t)K]}\Big]_{\frac{1}{4}}^1$$

$$= 15{,}000\left(1+\frac{3}{4}K\right)$$

$$\text{Withdrawal plus int.} = 20{,}000\left(1+\frac{1}{4}K\right)$$

$$100{,}000(1+K) + 15{,}000\left(1+\frac{3}{4}K\right) - 20{,}000\left(1+\frac{1}{4}K\right) = 110{,}000$$

$$106{,}250K = 15{,}000,\ K = 0.141 \qquad \text{ANS. (C)}$$

13.

$$e^{\int_0^5 kt\,dt + \int_5^{10} \frac{1}{25}kt^2 dt}$$

$$= e^{\frac{25}{2}k + \left[\frac{1}{25}k\left(\frac{1{,}000}{3} - \frac{125}{3}\right)\right]} = e^{\frac{145}{6}k}$$

$$= 2.7183$$

$$\frac{145}{6}k = \ln 2.7183 = 1,\ k = \frac{6}{145} = 0.041 \qquad \text{ANS. (D)}$$

14.

$$e^{\int_1^4 .15t^{\frac{1}{2}}dt} = e^{.15t^{\frac{3}{2}}(\frac{2}{3})}\Big]_1^4$$

$$= e^{.10t^{\frac{3}{2}}}\Big]_1^4 = e^{.10(4^{\frac{3}{2}}-1)}$$

$$= e^{.10(8-1)} = e^{.7} \qquad \text{ANS. (C)}$$

15. Investment A provides a decreasing force of interest. Investment B provides a constant force of interest $= \ln(1.05) = .048790$. Jill should invest in A until the force of interest decreases to .04879 and then switch to B. To find this point in time:

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$$\frac{.08}{1+.08t} = .04879, \ t = 7.996, \ \text{say } 8$$

$$10{,}000e^{\int_0^8 \frac{.08}{1+.08t}dt}(1.05)^{12}$$

$$= 10{,}000e^{\ln(1+.08t)}\Big]_0^8(1.05)^{12}$$

$$= 10{,}000(1.64)(1.795856) = 29{,}453 \qquad \text{ANS. (E)}$$

16. Since t is the number of years since 1/1/1984, the investment of 1,000 is made at time $t = \frac{1}{2}$.

$$AV = 1{,}000e^{\int_{\frac{1}{2}}^{1} \frac{3+2t}{50}dt} = 1{,}000e^{\frac{3t+t^2}{50}}\Big]_{\frac{1}{2}}^{1}$$

$$= 1{,}000e^{\frac{2.25}{50}} = 1{,}000e^{.045} = 1{,}046 \qquad \text{ANS. (D)}$$

17.

$$a(t) = e^{\int_0^t \frac{r/25}{r^2/50+1}dr} = e^{\ln(\frac{r^2}{50}+1)}\Big]_0^t$$

$$= \frac{t^2}{50} + 1$$

$$1{,}500 = 100a(t) + 200\frac{a(t)}{a(10)}$$

$$= 100\left(\frac{t^2}{50}+1\right) + \frac{200}{3}\left(\frac{t^2}{50}+1\right)$$

$$15 = \frac{5}{3}\left(\frac{t^2}{50}+1\right), \ t^2 = 400, \ t = 20 \qquad \text{ANS. (C)}$$

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§ 1h. Equivalent Rates

§ 1h(i) Constant Rate Case

Two rates are said to be equivalent if both rates produce the same results over the same period of time. (By "same results" we mean that the accumulated value of an investment, or the present value of a future amount, is the same under both rates.)

For example, a nominal rate of interest of 5% compounded semiannually can be said to be equivalent to an effective rate of 5.0625% per annum, since $\left(1+\frac{.05}{2}\right)^2 = 1.050625$.

If a problem asks you to find a rate which is equivalent to another rate, it is generally best to use a one-year period in writing the equations. If you systematically think of all accumulations (or present values) as being over a one-year period, you will be less likely to make a mistake.

Stepping Stones

EXAMPLE 1

What nominal rate of interest compounded semiannually is equivalent to a nominal rate of interest of 3% compounded once every two years?

(A) $2(1.06^{\frac{1}{4}}-1)$ (B) $2(1.03^{\frac{1}{2}}-1)$ (C) $1.06^{\frac{1}{4}}-1$ (D) $1.03^{\frac{1}{2}}-1$ (E) $2(1.03^2-1)$

SOLUTION

Accumulate \$1.00 over one year using each rate and set the two results equal.

If we let $i^{(2)}$ be the unknown nominal rate, the AV of \$1.00 in one year is $\left(1+\frac{i^{(2)}}{2}\right)^2$.

The given nominal rate is 3% compounded once every two years. This means that the *effective* rate for a *two-year* period is 6%. (See Section 1a(ix).) So the AV of \$1.00 over one year is $1.06^{\frac{1}{2}}$. Setting the two results equal:

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = 1.06^{\frac{1}{2}}$$

and solving for $i^{(2)}$, we get $2(1.06^{\frac{1}{4}}-1)$. **ANSWER (A)**

EXAMPLE 2

Determine the nominal annual rate of discount compounded quarterly equivalent to a nominal annual rate of interest of 6% compounded monthly. (Get a numerical result to 4 decimals.)

SOLUTION

$$(1-d^{(4)}/4)^{-4} = (1+.06/12)^{12} = 1.005^{12}$$

$$d^{(4)} = 4(1-1.005^{-3}) = 5.9405\%$$

EXAMPLE 3

Determine the force of interest equivalent to a nominal annual rate of discount of 8% compounded quarterly. (Get a numerical result to 4 decimals.)

$$e^{\delta} = (1-.08/4)^{-4} = .98^{-4}$$

$$\delta = -4\ln .98 = 8.0811\%$$

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§ 1h(ii) Varying Rate Case

Up to now, we have considered equivalent rates in the case where all rates are constant. In the constant case, if two rates are equivalent for, say, a one year period, they are equivalent for any other period, like one month, 5 years, etc. Also, if they are equivalent for, say, 1996, they are equivalent for any other one-year calendar period, like 1999 or 2012.

The same cannot be said when one or more of the rates vary with time. For example, if the force of interest δ_t is $.02t$, a fund grows at a faster and faster rate as time goes on. It is obvious that the equivalent effective rate of interest for the one year period from, say, $t = 0$ to $t = 1$ is less than the equivalent rate from say, $t = 1$ to $t = 2$. Similarly, the equivalent annual effective rate for the period from $t = 0$ to $t = 2$ is in-between the equivalent rates for each of the first two years.

Stepping Stones

EXAMPLE 1

If $\delta_t = .02t$, determine the equivalent annual effective rates for the periods mentioned in the previous paragraph.

SOLUTION

When rates vary, a question would be incomplete if it merely asked what the equivalent rate is. The period ***must*** be specified, since there is a different equivalent rate for each different period.

Let i_1 be the effective rate equivalent to $\delta_t = .02t$ for the year from $t = 0$ to $t = 1$. Accumulating \$1.00 at i_1 over this year gives $(1 + i_1)$. Accumulating at δ_t gives:

$$a(1) = e^{\int_0^1 .02t dt} = e^{.01t^2|_0^1} = e^{.01}$$

Setting the AV's equal:

$$1 + i_1 = e^{.01}$$

$$i_1 = e^{.01} - 1 = 1.005\% \quad \text{(to 3 decimals)}$$

Now let i_2 be the equivalent effective rate for the year from $t = 1$ to $t = 2$. At this rate, the AV of \$1.00 over this year is $(1 + i_2)$. At δ_t, the AV over the year (see Section 1g) is:

$$\frac{a(2)}{a(1)} = e^{\int_1^2 .02t dt} = e^{.04-.01} = e^{.03}$$

$1 + i_2 = e^{.03}$; $i_2 = e^{.03} - 1 = 3.045\%$.

Finally, let the equivalent annual effective rate over the two year period from $t = 0$ to $t = 2$ be j. Over this two-year period, \$1.00 grows to $(1 + j)^2$. Over the same two-year period at δ_t, \$1.00 grows to:

$$a(2) = e^{.01t^2|_0^2} = e^{.04}$$

Thus, $(1 + j)^2 = e^{.04}$ and $j = e^{.02} - 1 = 2.020\%$. Note that j falls between i_1 and i_2, as expected.

EXAMPLE 2

You are given that the accumulation function is $a(t) = .1t^2 + .2t + 1$. Determine the equivalent nominal annual rate of interest compounded semiannually for the period from $t = 0$ to $t = 10$. (Get a numerical result to two decimals.)

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SOLUTION

$$a(10) = 10 + 2 + 1 = 13$$

$$(1 + i^{(2)}/2)^{20} = 13$$

$$i^{(2)} = 2(13^{1/20} - 1) = 27.37\%$$

EXAMPLE 3

You are given that $\delta_t = .03t^2 + .02t$. Determine the equivalent nominal annual rate of discount compounded semiannually for the period from $t = 2$ to $t = 5$. (Get a numerical result to 2 decimals.)

SOLUTION

The AV at time 5 of \$1.00 invested at time 2 is:

$$AV = e^{\int_2^5 (.03t^2 + .02t)\,dt}$$

$$= e^{[.01t^3 + .01t^2]_2^5}$$

$$= e^{1.50 - .12} = e^{1.38} = 3.974902$$

$$\left(1 - \frac{d^{(2)}}{2}\right)^{-6} = 3.974902$$

$$d^{(2)} = 2\left(1 - 3.974902^{-1/6}\right)$$

$$= 41.09\%$$

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Calculator Notes #4: Equivalent Rates

Equivalent rates can be calculated using the regular keys. There is also a special [ICONV] key that can be used to calculate an effective rate that is equivalent to a given nominal rate, or a nominal rate that is equivalent to a given effective rate.

15. What is the effective annual rate of interest equivalent to a nominal rate of interest of 7.5% convertible quarterly?

a. Using the regular keys

In general, $(1+i) = (1+i^{(m)}/m)^m$, so for this problem we have $i = (1+.075/4)^4 - 1$. The keystrokes are:

.075 [÷] 4 [+] 1 [y^x] 4 [−] 1 [=]

The answer is $.0771 = 7.71\%$ to 2 decimals.

b. Using the [ICONV] key

This key is secondary to the "2" key. You can access the registers in [ICONV] by pressing [2nd] [ICONV]. If you now scroll using one of the arrow keys, you will see that there are 3 registers: "NOM," "EFF" and "C/Y." If you enter the nominal rate and the number of interest conversions, you can compute the equivalent effective annual rate. If you enter the effective annual rate and the number of interest conversions, you can compute the equivalent nominal rate.

To determine the effective annual interest rate equivalent to a nominal rate of 7.5% convertible quarterly, we proceed as follows:

[2nd] [ICONV]

"NOM =" will appear, so we enter the nominal rate:

7.5 [ENTER]

Now scroll to "C/Y =." Since compounding is quarterly, enter 4:

4 [ENTER]

Now scroll to "EFF =." To compute the equivalent effective rate, press [CPT]. The answer will appear as "EFF = 7.7136."

To return to the standard calculator mode, press [2nd] [QUIT].

Note: The setting of C/Y when you are in [ICONV] mode has no effect on the setting of C/Y in TVM mode. So if you are keeping P/Y = C/Y = 1 in TVM mode, as we recommended, you don't have to worry that using [ICONV] will mess this up.

16. What is the nominal rate of interest compounded monthly that is equivalent to an effective annual rate of interest of 6.23%?

a. Using the regular keys

In general, $i^{(m)} = m[(1+i)^{1/m} - 1]$. In this problem, we have:

$i^{(12)} = 12(1.0623^{1/12} - 1)$

The keystrokes are:

12 [1/x] [STO] 0 1.0623 [y^x] [RCL] 0 [−] 1 [×] 12 [=]

The answer is .0606, or 6.06% to 2 decimals.

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b. Using the [ICONV] key

[2nd] [ICONV]

"NOM =" will appear. Since we are given the effective rate, scroll to "EFF =." Then press:

6.23 [ENTER]

Scroll to "C/Y =." Then press:

12 [ENTER]

Scroll to "NOM =." Then press [CPT]. The answer will appear as 6.0589%.

17. What is the nominal rate of interest compounded quarterly that is equivalent to a nominal rate of interest of 18% compounded monthly?

a. Using the regular keys

In general, $(1+\frac{i^{(m)}}{m})^m = (1+\frac{i^{(p)}}{p})^p$

In this problem we have:

$(1+i^{(4)}/4)^4 = (1+.18/12)^{12}$

$i^{(4)} = 4(1.015^3 - 1)$

The keystrokes are: 1.015 [y^x] 3 [−] 1 [×] 4 [=]

The answer is 18.27%.

b. Using the [ICONV] key

The idea is to first compute the effective annual rate equivalent to a nominal rate of 18% compounded monthly. We leave that effective rate in the "EFF =" register and compute the equivalent nominal rate compounded quarterly. The keystrokes are as follows:

[2nd] [ICONV]

"NOM =" will appear, so we enter the given nominal rate:

18 [ENTER]

Now scroll to C/Y and enter 12:

12 [ENTER]

Now scroll to "EFF =" and press [CPT]. The equivalent effective annual rate appears as 19.5618%.

Now scroll to C/Y again and enter 4:

4 [ENTER]

Finally, scroll to "NOM =" and press [CPT]. This will compute the equivalent nominal rate compounded quarterly.

The answer is 18.2714%.

18. What effective annual rate of discount is equivalent to a nominal rate of discount of 6% compounded quarterly?

a. Using the regular keys

In general, $1-d = (1-d^{(m)}/m)^m$ and $d = 1-(1-d^{(m)}/m)^m$. In this problem, we have:

$d = 1-(1-.06/4)^4 = 1-.985^4$. The keystrokes are:

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.985 [y^x] 4 [−] 1 [=] [+/−]

The answer is 5.87%.

b. Using the TVM keys

Once again, you can fool the calculator and use the TVM keys to find equivalent discount rates. It's tricky, so we prefer to work from first principles, as in the above. But if you would like to know how to do this, see the material on the bottom of page 7 and most of page 8 of the Study Note.

19. What nominal rate of discount compounded semiannually is equivalent to an effective rate of discount of 8% per annum?

In general, $1 - d = (1 - d^{(m)}/m)^m$, so $d^{(m)} = m[1 - (1 - d)^{1/m}]$. In this problem, we have:

$d^{(2)} = 2[1 - (1 - .08)^{1/2}]$. The keystrokes are:

.92 [y^x] .5 [−] 1 [×] 2 [=] [+/−]

The answer is 8.17%.

20. What nominal rate of interest compounded quarterly is equivalent to a nominal rate of discount of 9% compounded monthly?

In general, $(1 + i^{(m)}/m)^m = (1 - d^{(p)}/p)^{-p} = 1 + i$. In this problem, we have:

$(1 + i^{(4)}/4)^4 = (1 - .09/12)^{-12} = .9925^{-12}$

Thus, $i^{(4)} = 4(.9925^{-3} - 1)$. The keystrokes are:

.9925 [y^x] 3 [+/−] [−] 1 [×] 4 [=]

The answer is 9.14%.

21. The force of interest is $\delta_t = .02t$. Find the equivalent constant nominal rate of discount compounded quarterly over the first 5 years.

We have: $e^{\int_0^5 .02t dt} = (1 - d^{(4)}/4)^{-20}$, where the left-hand side is evaluated as $e^{.25}$. Thus:

$d^{(4)} = 4(1 - e^{-.0125})$

The keystrokes are:

.0125 [+/−] [2nd] [e^x] [−] 1 [×] 4 [=] [+/−]

The answer is .0497 or 4.97%.

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Summary of Concepts and Formulas in Section 1h

(1) Two rates are said to be equivalent if they result in the same AV or PV of the same payment over the same period of time.

(2) The following equations link equivalent rates of interest and discount, and the force of interest:

$$(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^m = \left(1-\frac{d^{(m)}}{m}\right)^{-m} = e^{\delta}$$

(3) If interest rates vary over time, the period over which two rates are to be equivalent must be specified.

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Past Exam Questions on Section 1h

1. A customer is offered an investment where interest is calculated according to the following force of interest:

$$\delta_t = \begin{cases} 0.02t & 0 \le t \le 3 \\ 0.045 & 3 < t \end{cases}$$

The customer invests 1000 at time $t = 0$. What nominal rate of interest, compounded quarterly, is earned over the first four-year period? [5/00 #37]

(A) 3.4% (B) 3.7% (C) 4.0% (D) 4.2% (E) 4.5%

2. Calculate the nominal rate of interest convertible once every four years that is equivalent to a nominal rate of discount convertible quarterly. [SOA 11/88 #2]

(A) $0.25[1 - (1 - 0.25d^{(4)})^{-16}]$
(B) $(1 - 0.25d^{(4)})^{-16} - 1$
(C) $(1 - 0.25d^{(4)})^{-8} - 1$
(D) $0.5[(1 - 0.25d^{(4)})^{-8} - 1]$
(E) $0.25[(1 - 0.25d^{(4)})^{-16} - 1]$

3. You are given $\delta_t = \frac{2}{t-1}$ for $2 \le t \le 10$.

For any one year inverval between n and $n + 1$, with $2 \le n \le 9$, calculate the equivalent $d^{(2)}$. [SOA 5/88 #1]

(A) $\frac{1}{n}$ (B) $\frac{2}{n}$ (C) $\frac{n-1}{n}$ (D) $\frac{n}{n-1}$ (E) $(\frac{n}{n-1})^2$

4. The accumulated value of one at time t (where $0 \le t \le 1$) is given by a second degree polynomial in t. You are given:

(i) The nominal rate of interest convertible semiannually for the first half of the year is 5% per annum.
(ii) The effective rate of interest for the year is 4% per annum.

Calculate $\delta_{3/4}$. [SOA 5/88 #2]

(A) 0.021 (B) 0.023 (C) 0.025 (D) 0.027 (E) 0.029

5. Which simple interest rate over six years is closest to being equivalent to the following:

(i) an effective rate of *discount* of 3% for the first year,
(ii) an effective rate of *discount* of 6% for the second year,
(iii) an effective rate of *discount* of 9% for the third year,
(iv) and an effective rate of *interest* of 5% for the fourth, fifth, and sixth years?

[CAS 5/83 #11]

(A) 6.3% (B) 6.4% (C) 6.5% (D) 6.6% (E) 6.7%

6. What nominal annual discount rate compounded semiannually is equivalent to a nominal rate of interest of 12% compounded monthly? [CAS 11/82 #2]

(A) Less than 11.3%
(B) At least 11.3% but less than 11.5%
(C) At least 11.5% but less than 11.7%
(D) At least 11.7% but less than 11.9%
(E) At least 11.9%

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Solutions to Past Exam Questions on Section 1h

1. $$\left(1+\frac{i^{(4)}}{4}\right)^{16} = e^{\int_0^3 .02t\,dt} \cdot e^{\int_3^4 .045\,dt}$$

$$= e^{.01t^2\big]_0^3 + .045t\big]_3^4} = e^{.09+.045} = e^{.135} = 1.144537$$

$$i^{(4)} = 4\left(1.144537^{\frac{1}{16}} - 1\right) = .0339 \qquad \text{ANS. (A)}$$

2. $$\left(1+4i^{\left(\frac{1}{4}\right)}\right)^{\frac{1}{4}} = \left(1-\frac{d^{(4)}}{4}\right)^{-4}$$

$$i^{\left(\frac{1}{4}\right)} = \frac{1}{4}\left[\left(1-\frac{d^{(4)}}{4}\right)^{-16} - 1\right] \qquad \text{ANS. (E)}$$

3. $$\left(1-\frac{d^{(2)}}{2}\right)^{-2} = e^{\int_n^{n+1} \frac{2}{t-1}dt} = e^{2\ln(t-1)\big]_n^{n+1}}$$

$$= e^{2[\ln n - \ln(n-1)]} = e^{\ln\left(\frac{n}{n-1}\right)^2} = \left(\frac{n}{n-1}\right)^2$$

$$1-\frac{d^{(2)}}{2} = \left(\frac{n}{n-1}\right)^{-1} = \frac{n-1}{n}$$

$$d^{(2)} = 2\left(1-\frac{n-1}{n}\right) = \frac{2}{n} \qquad \text{ANS. (B)}$$

4. Let $a(t) = at^2 + bt + c$

$$a(0) = c = 1$$

$$a\left(\frac{1}{2}\right) = \frac{1}{4}a + \frac{1}{2}b + 1 = 1.025$$

$$a(1) = a + b + 1 = 1.04$$

Solving this pair of linear equations: $a = -.02$, $b = .06$

$$a(t) = -.02t^2 + .06t + 1, \quad \frac{d}{dt}a(t) = -.04t + .06$$

$$\delta_t = \left.\frac{-.04t + .06}{-.02t^2 + .06t + 1}\right|_{t=\frac{3}{4}} = \frac{.03}{1.03375} = .029 \qquad \text{ANS. (E)}$$

5. Let i = required simple interest rate.

$$1 + 6i = (1-.03)^{-1}(1-.06)^{-1}(1-.09)^{-1}(1.05)^3$$

$$= [(.97)(.94)(.91)]^{-1}(1.05)^3$$

$$= (1.205200)(1.157625) = 1.395169$$

$$i = \frac{.395169}{6} = .066 \qquad \text{ANS. (D)}$$

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6. $$\left(1-\frac{d^{(2)}}{2}\right)^{-2}=\left(1+\frac{.12}{12}\right)^{12}=1.01^{12}$$

$$d^{(2)}=2\left(1-1.01^{-6}\right)=11.59\% \qquad \text{ANS. (C)}$$

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2

Practical Applications

§ 2a. Equations of Value, Time Value of Money, and Time Diagrams

Equations of Value and the Time Value of Money

Suppose that deposits made today can earn 5% effective over the next year. Which would you rather have, \$1, 000 today or \$1, 050 in a year?

A basic principle of equivalence that we will use in interest theory is that it doesn't matter to us whether we have \$1, 000 today or \$1, 050 in a year (assuming that the effective rate of interest is 5%). In fact, if we are told that the effective rate is 5% for the next 10 years, we will assume that we would be just as happy with \$1, 000 today as we would be with \$1, 000$(1.05)^t$ in t years, where t is any time from 0 to 10.

This basic principle of equivalence may seem so obvious that it doesn't even have to be stated. But we must agree to it if we want to play the game in accordance with the rules of interest theory.[1]

Another way to say this is that money has a ***time value***: \$1,000 today is worth more than \$1,000 a year from now. So if person A paid person B \$1,000 now in return for a payment from B in a year, A would want that payment to be more than \$1,000. You will often see the term ***time value of money*** in the financial literature.

The principle of equivalence allows us to solve interest problems by setting up ***equations of value*** as of a common ***comparison date***. As an example, let's say that you want to accumulate \$5,000 in two years by making a deposit of X today and another deposit of X in a year. If the effective rate of interest is 6%, determine X.

To solve for X we will equate the value of the deposits and the accumulated value of \$5,000. In order to do this, we must choose a common date, called the comparison date. Let's say we choose time 2. As of that date, the deposits are worth $X(1.06)^2 + X(1.06)$, by the principle of equivalence. The AV we want at that point is \$5,000. Thus, the equation of value as of time 2 is:

$$X(1.06^2 + 1.06) = 5{,}000$$

$$X = \frac{5{,}000}{1.06^2 + 1.06} = \$2{,}289.80$$

Two points should be noted:

[1] There may be some people who would not be just as happy with \$1, 050 a year from now as they would be with \$1, 000 today, even if 5% were a fair rate of interest. (Can you think of any reasons why?) But we will assume that the principle of equivalence applies when we solve problems in this course.

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(i) You ***must*** choose the same date (the comparison date) to evaluate the deposits and the AV. You cannot directly compare X deposited today and X deposited a year from now with \$5,000 received two years from now, unless you determine the equivalent values of these amounts as of the same date.

(ii) ***Any*** date can be used for the comparison date. For example, suppose we had used a ridiculous date like 122 years from now for the comparison date. What would the equation of value be?

$$\text{Answer: } X(1.06)^{122} + X(1.06)^{121} = 5,000(1.06)^{120}$$

This is the same equation of value as before, but with both sides multiplied by 1.06^{120}. Thus, we would get the same solution for X

Certainly, choosing time 122 would unnecessarily complicate the solution. In this example, time 0 or time 2 (and perhaps time 1) are the obvious choices. Choosing a convenient comparison date is something you will learn to do as you get more experience with solving compound interest problems.

Time Diagrams: The Student's Friend

Some people are lucky: They can visualize deposits and withdrawals, payment periods and interest periods, etc., without the aid of a diagram. But for most of us, a time diagram is a very useful aid in solving problems in compound interest.

A time diagram is a statement of a problem in picture form. Once the diagram is drawn, the problem often practically solves itself.

The following examples are basic in nature. If you feel comfortable about setting up time diagrams, or if you don't need them to solve problems, you can skip this section. (But you might want to look at Examples 3 and 4 in any event.)

Stepping Stones

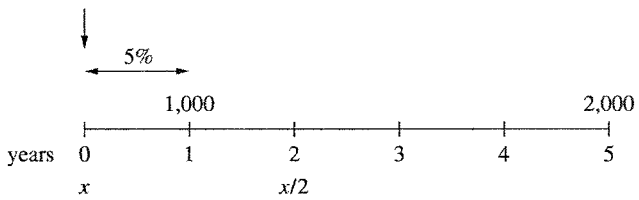
EXAMPLE 1

Draw a time diagram and write an equation of value for the following problem:

A deposit is to be made today and a second deposit, which is one-half the first, is to be made 2 years from now, to provide for withdrawals of \$1,000 one year from now and \$2,000 5 years from now. Interest is at 5% effective. What is the amount of the initial deposit?

SOLUTION

Let x = initial deposit. Then the deposit at time 2 is $x/2$.



Using time 0 as the comparison date, the equation of value is:

$$x + (x/2)v^2 = 1{,}000v + 2{,}000v^5 \quad \text{at } 5\%$$

$$x = \frac{1{,}000v + 2{,}000v^5}{1 + v^2/2} = \$1{,}733.34$$

Note the following points:

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(i) Deposits and withdrawals are placed on opposite sides of the time line (for example, all deposits below the line and all withdrawals above the line, as in the above diagram). This will assure that we get the correct terms on the left and right-hand sides of the equation of value.

(ii) To the left of the time line, the interest period is noted ("years"). In this problem, where the interest period and payment period are the same, it isn't too important but get into the habit of noting it on your diagrams anyway.

(iii) The effective interest rate for one interest period (5%) is marked off, as a reminder of the rate to be used in calculations.

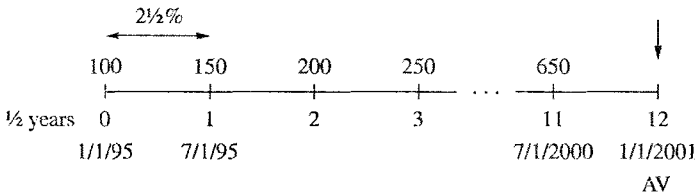
(iv) A vertical arrow is placed at the comparison date chosen; in this solution, time 0 was chosen as the comparison date. Of course, any chosen date would lead to the correct value of x.

EXAMPLE 2

Deposits are made on January 1 and July 1 of every year from 1995 to 2000, inclusive. The initial deposit is \$100 and each subsequent deposit increases by \$50. What is the accumulated value of the deposits on January 1, 2001 at a nominal rate of interest of 5% compounded semiannually?

SOLUTION

In this problem, the effective rate of interest is $2\frac{1}{2}\%$ for a ***half-year*** period and payments are made every ***half-year***, so it's a no-brainer that we should label the diagram in terms of half-year periods.



We have marked off "$\frac{1}{2}$ years" to the left of the time line as a reminder that "time 1" is 6 months from now, "time 2" is a year from now, etc., and that the effective rate of $2\frac{1}{2}\%$ applies to each of these 6-month periods. Also, we have translated the given ***dates*** into ***interest periods***, starting with 1/1/95 as time 0 and ending with 1/1/2001 as time 12 (6 years later).

The equation of value using "brute force" (that is, term-by-term without any fancy annuity symbols) and using time 12 as the comparison date is:

$$100(1.025)^{12} + 150(1.025)^{11} + 200(1.025)^{10} + \cdots + 650(1.025) = AV$$

Of course, the techniques for handling varying annuities that are developed later in this manual would be much more efficient for determining the AV than a term-by-term calculation. But at this point, we are primarily interested in the time diagram as a technique for setting up problems.

EXAMPLE 3

Draw a time diagram and write an equation of value for the following problem:

Find the present value of quarterly payments of \$100 for 10 years, first payment 3 months from now, at a nominal rate of interest of 6% compounded semiannually.

SOLUTION

In this problem, the payment period and the interest period are not the same. The payment period is 3 months and the interest period is 6 months (since the given nominal interest rate implies an effective rate of 3% for a half-year period).

You have a choice:

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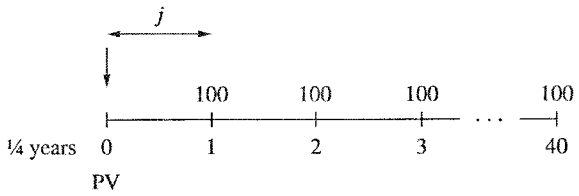
(i) You can draw the diagram in terms of the ***payment period*** (3 months in this case) and determine the equivalent effective rate for this period; or

(ii) You can draw the diagram in terms of the ***interest period*** given in the problem (6 months in this case).

Following the first alternative (the payment period), the diagram would look like this:



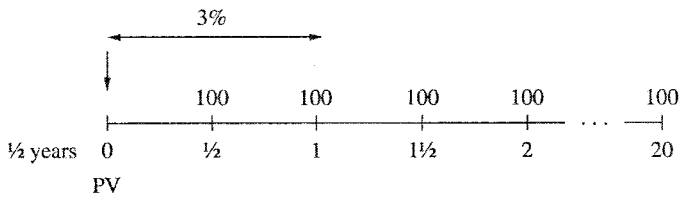
where $j = 1.03^{\frac{1}{2}} - 1 = 1.48892\%$.

(The thinking was: "I want the diagram to be in terms of *payment periods*. This will lead to the simplest form of the equation of value. Since payments are quarterly, the period in the diagram will be quarter-years and the payments will fall at time 1, 2, 3, ... Since there are 40 payments (quarterly payments for 10 years), the last payment is at time 40. The only thing left to do is to find the effective rate of interest for a quarter of a year—call it j—equivalent to the given nominal rate of 6% compounded semiannually. I can do this by accumulating 1 for one year (or for any period of time) at each rate: $1.03^2 = (1+j)^4$, from which $j = 1.03^{\frac{1}{2}} - 1$.")

The equation of value at time 0 is:

$$PV = 100(v + v^2 + \ldots + v^{40}) \text{ at rate } j = 1.03^{\frac{1}{2}} - 1$$

Following the second alternative (the interest period), the diagram looks like this:



(The thinking was: "The effective rate of interest implied by 6% compounded semiannually is 3% for a half-year period, so I will label my diagram in terms of half-year interest periods. The payments are quarterly, or twice each interest period, so I will place the payments at time $\frac{1}{2}$, 1, $1\frac{1}{2}$, etc. This is a 10-year annuity, which consists of 20 interest periods, so the last payment is at time 20.")

The equation of value at time 0 is:

$$PV = 100(v^{\frac{1}{2}} + v^1 + v^{1\frac{1}{2}} + \ldots + v^{20})$$

where $v = 1.03^{-1}$.

Which method is better, using the payment period or using the interest period?

If you only want numerical results, using the payment period is generally the best approach. Interest functions that we will cover later in this manual are at their simplest when the interest period is the same as the payment period. You do have to calculate the effective rate for the payment period and to use it in any equation of value. The fact that this rate will usually not be a round number doesn't matter, since you can handle any rate with the calculator.

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On the other hand, the answers to an exam question may be left in symbolic form, although this has happened very infrequently, if at all, in recent exams. If this is the case, chances are that the examiners will use symbols at the "original" effective rate (3% per half-year in the above problem). In that case, your thinking would have to be in terms of interest periods.

The upshot of this is that you really have to know both approaches and have facility in drawing diagrams and writing equations of value in terms of either the interest period or the payment period. This is covered in greater detail in Sections 4a and 4b of this manual.

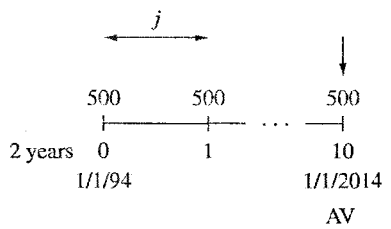
EXAMPLE 4

Draw a time diagram and write an equation of value for the following problem:

Deposits of \$500 are made on January 1 of even years only from 1994 to 2014 inclusive. Find the accumulated value on the date of the last deposit if the nominal rate of interest is 8% compounded quarterly.

SOLUTION

Using the payment period:



where $1+j=1.02^8; j=1.02^8-1=17.1659\%$.

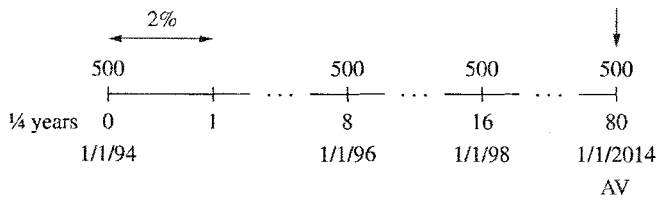
Equation of value:

$$AV = 500[1+(1+j)+\ldots+(1+j)^{10}] \text{ at } j = 17.1659\%$$

(This can be expressed as 500 $s_{\overline{11}|}$ at rate j, using a symbol introduced in the next chapter.)

Note: You can number the time periods in the diagram any way you want to, as long as the payments are spaced correctly. For example, in the above diagram you could have used time 1 for 1/1/94 and time 11 for 1/1/2014. This would have placed the payments at times 1 to 11, inclusive. Verify that if you then choose time 11 (i.e., 1/1/2014) as the comparison date, you would get the same equation of value as the one above.

Using the interest period:



Equation of value:

$$AV = 500(1+1.02^8+1.02^{16}+\cdots+1.02^{80})$$

(In a later chapter, this could be expressed as $500\frac{s_{\overline{88}|}}{s_{\overline{8}|}}$ at $i=2\%$.)

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Note: Once again, you can number the time periods in the diagram any way you want to. For example, in the above diagram, you could have used time 8 for 1/1/94 and time 88 for 1/1/2014. Of course, this would have led to the same equation of value using time 88 as the comparison date.

§ 2b. Unknown Time and Unknown Interest Rate

Unknown Time

Problems involving unknown time can often be solved using the logarithm [LN] key or the [TVM] keys. (See questions 5 and 6 in Calculator Notes #1.)

Stepping Stones

EXAMPLE 1

500 accumulates to 1,500 in t years at an effective annual interest rate of 4%. Determine t.

SOLUTION

$$(500)(1.04^t) = 1{,}500$$
$$1.04^t = 3$$

Using the [LN] key we have $t = \log 3/\log 1.04 = 1.098612/.039221 = 28.01$ years to two decimals.

Using the [TVM] keys, we enter: 4 [I/Y] 500 [PV] 1500 [+/–] [FV] [CPT] [N] for the same result.

EXAMPLE 2

How long will it take money to double at an effective annual rate of interest of 5%?

SOLUTION

$$1.05^t = 2$$

Using either logarithms or the [TVM] keys, you should get $t = 14.2067$ to 4 decimals. Note that this is question 5 in Calculator Notes #1.

In your work or readings, you may come across an approximation for t, sometimes called the Rule of 72: The time it takes money to double at a specified interest rate is equal to 72 divided by the interest rate in percent. At 5%, this results in $72/5 = 14.4$ years. A better approximation is 69.3 divided by the interest rate, plus 0.35. At 5%, this gives 14.21, which is correct to two decimals. (These approximations are based on certain series expansions and the fact that the natural log of 2 is approximately 0.693.)

EXAMPLE 3

Bill will receive two payments of 1,000 each, the first payment in t years and the second in $2t$ years. The present value of these payments is 1,200 at an effective annual rate of 4%. Determine t.

SOLUTION

The equation of value as of time 0 is:

$$1000(v^t + v^{2t}) = 1200 \quad \text{at } 4\%$$
$$v^{2t} + v^t - 1.2 = 0$$

This is a quadratic in v^t. For convenience, we will substitute x for v^t:

$$x^2 + x - 1.2 = 0$$

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Taking the positive root, you should get $x = .704159 = v^t$.

Using either the $\boxed{\text{LN}}$ key or the $\boxed{\text{TVM}}$ keys, you should get $t = 8.943$ years to 3 decimals.

EXAMPLE 4

Fund A grows at a force of interest $\delta_t = 1/(1+t)$. Fund B grows at a constant force of interest of 5%. Equal amounts are invested in each fund at time 0. Determine the time at which the excess of Fund A over Fund B is at a maximum.

SOLUTION

In general, the accumulation function is equal to the base of the natural logarithms, e, raised to the power of the integral of the force of interest. Determine the accumulation functions for Fund A and Fund B, take the difference, take the derivative of this difference, and set it equal to 0 to determine a relative minimum or maximum:

$$\text{Accumulation function for Fund A} = e^{\int_0^t \frac{1}{1+r}dr} = e^{[\ln(1+r)]_0^t} = 1+t$$

$$\text{Accumulation function for Fund B} = e^{.05t}$$

Excess of A over B (call it E(t)):

$$\mathrm{E}(t) = 1 + t - e^{.05t}$$

Take the derivative:

$$\mathrm{E}'(t) = 1 - .05e^{.05t}$$

Setting this equal to 0, we get $e^{.05t} = 20$ and $t = 20 \ln 20 = 59.91$ years to two decimals. (Note that the second derivative of E(t) is negative, which verifies that a relative maximum occurs at 59.91 years.)

Method of Equated Time

A type of problem involving unknown time is one where a series of payments is to be replaced by a single payment equal to the sum of the series. For example, suppose someone is scheduled to make three payments to you, as follows:

Time Due	Payment
1	5
3	1
10	15
Total Payments =	21

Now, suppose that in lieu of the scheduled payments, you and the other party agree that a ***single*** payment will be made to you equal to the ***sum*** of the scheduled payments. In this example, the single payment would be 21.

When should this single payment be made?

Clearly, this would depend on the interest rate we agree on, say, 5%. Let the time of the single payment be t. Then the following equation of value as of time 0 must be satisfied:

$$21v^t = 5v + v^3 + 15v^{10}$$

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You can see that the solution for t involves taking logarithms of both sides, etc. This procedure will give the exact value of t. You can verify that if $i = 5\%$, $t = 7.12$ to 2 decimals.

The "method of equated time" is a method for determining an ***approximate*** value of t using simple arithmetic. The general idea is to find the arithmetic mean of the times at which the payments are due. But it wouldn't be a very good approximation if we just averaged the times at which each payment is due without regard to the ***amount*** due on each date. So to find t by the method of equated time, we compute the ***weighted*** average time. We will use the symbol $\bar{t}$ for the approximation:

$$\bar{t} = \frac{5 \times 1 + 1 \times 3 + 15 \times 10}{21} = \frac{158}{21} = 7.52$$

Note that the time at which each payment is due is weighted by the amount due: 5 is due at time 1 (5×1); 1 is due at time 3 (1×3); 15 is due at time 10 (15×10). Then the sum of these products (158) is divided by the sum of the weights (21) to give the weighted average time of 7.52.

This is certainly an approximation, since we didn't even use the interest rate in calculating it. In fact, the method of equated time gives the same answer regardless of the interest rate.

(Note that in this example, the method of equated time gives a result which is greater than the exact answer at 5%: $7.52 > 7.12$. This is not an accident. It can be shown that the approximation always exceeds the exact answer for a positive interest rate.)

EXAMPLE 5

An annuity provides an infinite series of annual payments of $d, \frac{d^2}{2}, \frac{d^3}{3}, \ldots$, first payment one year from now, where d is the effective rate of discount. In lieu of these payments, a single payment equal to their sum is to be made at time t. Determine t using the method of equated time.

(A) 1 (B) i/δ (C) d/δ (D) $e^{\frac{i}{d}}$ (E) $e^{\frac{i}{d}} - 1$

SOLUTION

This question tests you on a series that we covered in Section 1f, as well as testing you on the method of equated time.

To determine the approximate time using the method of equated time, we calculate the weighted average time. In this problem, there are an infinite number of payments, so the weighted average involves infinite series.

The sum of the products of the weights (payments) and the times is as follows:

$$(d)(1) + (\frac{d^2}{2})(2) + (\frac{d^3}{3})(3) + \cdots$$

$$= d + d^2 + d^3 + \cdots$$

$$= \frac{d}{1-d} \text{ (sum of an infinite geometric progression)}$$

You may recognize this as i or you can substitute $d = iv$ and $1 - d = v$ to obtain i.

To get the weighted average time, we have to divide this result (i) by the sum of the weights (payments), which is:

$$d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots$$

If your memory has held out from Section 1f, you will recognize this as $-\ln(1-d)$, or δ. Therefore, the weighted average time is i/δ. ANS. **(B)**

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EXAMPLE 6

Hannah is scheduled to receive the following payments at the end of the years shown:

End of Year	Scheduled Payment
4	100
10	200
12	300

Hannah agrees to accept a single payment equal to the sum of the scheduled payments at the time determined by the method of equated time. The effective annual interest rate is 6%. Let X be the excess of the present value of Hannah's scheduled payments over the present value of her single payment under the method of equated time. Determine X.

SOLUTION

The time of Hannah's single payment under the method of equated time is:

$$\bar{t} = \frac{(4)(100) + (10)(200) + (12)(300)}{100 + 200 + 300} = \frac{6{,}000}{600} = 10$$

The present value of the single payment is $600v^{10} = 335.04$.

The present value of the scheduled payments is $100v^4 + 200v^{10} + 300v^{12} = 339.98$.

$$X = 339.98 - 335.04 = 4.94$$

Unknown Interest Rate

Problems involving an unknown rate of interest are among the most difficult to solve. This is because the unknown rate may be the root (or roots) of a polynomial (or even of a more complex function).

EXAMPLE 7

At what rate of interest will a payment of 1 now and 2 in one year accumulate to 4 in 2 years?

SOLUTION

The equation of value as of time 2 is:

$$(1+i)^2 + 2(1+i) = 4$$

For convenience, we will let $(1+i) = x$. We have:

$$x^2 + 2x - 4 = 0$$

Taking the positive root:

$$x = \frac{-2 + \sqrt{2^2 - (4)(1)(-4)}}{2}$$

$$= \frac{\sqrt{20} - 2}{2} = \frac{2\sqrt{5} - 2}{2}$$

$$= \sqrt{5} - 1 = 1.236068$$

Thus, $x = 1 + i = 1.236068$ and $i = .236068$, or 23.61%.

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In this case, it was easy to solve for i, since it was the root of a quadratic. But with polynomials of higher degree, or for more complex functions, it could be very difficult to determine a numerical value for i without a financial calculator. (In fact, formulas do not exist for finding the roots of a polynomial of degree 5 or higher.) We might have to resort to an approximation or iterative technique, like interpolation, successive bisection, Newton-Raphson iteration, etc. But as we will see in later sections, a financial calculator such as the BA II Plus can be used to compute an unknown interest rate in many cases.

EXAMPLE 8

At what effective annual rate of interest $i > 0$ is the present value of 100 due at time 5 plus the present value of 200 due at time 15 equal to the present value of 300 due at time 10?

SOLUTION

The equation of value as of time 0 is:

$$100v^5 + 200v^{15} = 300v^{10}$$

Dividing both sides by $100v^5$ and rearranging:

$$2v^{10} - 3v^5 + 1 = 0$$

Let $x = v^5$:

$$2x^2 - 3x + 1 = 0$$

The left-hand side can be factored as follows:

$$(2x - 1)(x - 1) = 0$$

Taking the first root:

$$2x = 1, \quad x = 0.5 = v^5 \text{ or } (1 + i)^5 = 2$$

$$i = 14.87\% \quad \text{to 2 decimals}$$

(The other root is $x = 1 = v^5$, which means that $i = 0$, but we were asked to find $i > 0$.)

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Summary of Concepts and Formulas in Sections 2a and 2b

(1) We say that money has a time value, which means that \$1.00 paid today is equivalent to $(1+i)^t$ paid at time t (assuming that the effective rate is a constant i).

(2) Using the principle of the time value of money, we can set up ***equations of value*** by evaluating all payments as of a common date called the ***comparison date***.

(3) Time diagrams can be very helpful in solving problems. (See examples in Section 2a.)

(4) The "method of equated time" is the weighted average time of a series of scheduled payments. It is a simple approximation to the exact time at which a single payment equal to the sum of the scheduled payments should be made in lieu of the scheduled payments.

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Past Exam Questions on Sections 2a and 2b

1. David can receive one of the following two payment streams:
   (i) 100 at time 0, 200 at time n, and 300 at time $2n$
   (ii) 600 at time 10

   At an annual effective interest rate of i, the present values of the two streams are equal. Given $v^n = 0.75941$, determine i. [11/01 #24]

   (A) 3.5% (B) 4.0% (C) 4.5% (D) 5.0% (E) 5.5%

2. Joe deposits 10 today and another 30 in five years into a fund paying simple interest of 11% per year.

   Tina will make the same two deposits, but the 10 will be deposited n years from today and the 30 will be deposited $2n$ years from today. Tina's deposits earn an annual effective rate of 9.15%.

   At the end of 10 years, the accumulated amount of Tina's deposits equals the accumulated amount of Joe's deposits.

   Calculate n. [5/00 #1]

   (A) 2.0 (B) 2.3 (C) 2.6 (D) 2.9 (E) 3.2

3. An investment of \$1 will double in 20 years at a force of interest δ.

   Determine the number of years required for an investment of \$1 to triple at a nominal rate of interest, convertible 3 times per year, and which is numerically equivalent to δ. [CAS 5/98 #7]

   (A) Less than 31 years
   (B) At least 31 years, but less than 33 years
   (C) At least 33 years, but less than 35 years
   (D) At least 35 years, but less than 37 years
   (E) 37 years or more

4. John is 30 years old. He will receive 2 payments of \$2,500 each. The first payment will be an unknown number of years in the future. The second payment will be five years after the first payment. At an annual effective interest rate of $i = 5\%$, the present value of the two payments is \$2,607.

   Determine at what age John will receive the *second* payment. [CAS 5/94 #1]

   (A) Less than 40
   (B) At least 40 but less than 45
   (C) At least 45 but less than 50
   (D) At least 50 but less than 55
   (E) At least 55

5. Jim borrows \$5,000 from a bank now, an additional \$3,000 one year from now and an additional \$2,000 five years from now. At what point in time, t, would a single payment of \$10,000 be equivalent at a nominal rate of interest of 12% convertible monthly? [CAS 5/91 #4]

   (A) 0.0 years $\leq t <$ 0.9 years
   (B) 0.9 years $\leq t <$ 1.0 years
   (C) 1.0 years $\leq t <$ 1.1 years
   (D) 1.1 years $\leq t <$ 1.2 years
   (E) 1.2 years $\leq t$

6. You are given two loans, with each loan to be repaid by a single payment in the future. Each payment includes both principal and interest.

   The first loan is repaid by a 3,000 payment at the end of four years. The interest is accrued at 10% per annum compounded semiannually.

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The second loan is repaid by a 4,000 payment at the end of five years. The interest is accrued at 8% per annum compounded semiannually.

These two loans are to be consolidated. The consolidated loan is to be repaid by two equal installments of X, with interest at 12% per annum compounded semiannually. The first payment is due immediately and the second payment is due one year from now.

Calculate X. [SOA 11/89 #1]

(A) 2,459 (B) 2,485 (C) 2,504 (D) 2,521 (E) 2,537

7. Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal amount is made.

   Carl withdraws K at the end of each of years 4, 5, 6 and 7. The balance in the account at the end of year 10 is 10,000.

   Calculate K. [SOA 5/89 #5]

   (A) 929 (B) 958 (C) 980 (D) 1,005 (E) 1,031

8. Payments of 300, 500 and 700 are made at the end of years five, six and eight, respectively. Interest is accumulated at an annual effective rate of 4%.

   You are to find the point in time at which a single payment of 1,500 is equivalent to the above series of payments. You are given:

   (i) X is the point in time calculated by the method of equated time.
   (ii) Y is the exact point in time.

   Calculate $X + Y$. [SOA 11/88 #5]

   (A) 13.44 (B) 13.50 (C) 13.55 (D) 14.61 (E) 14.99

9. You are given the following data on three series of payments:

	Payment at end of year			Accumulated value
	6	12	18	at end of year 18
Series A	240	200	300	X
Series B	0	360	700	$X + 100$
Series C	Y	600	0	X

   Assume interest is compounded annually.

   Calculate Y. [SOA 5/88 #4]

   (A) 93 (B) 99 (C) 102 (D) 107 (E) 111

10. The present value of a payment of 1,004 at the end of T months is equal to the present value of the following payments:

At the End of	Amount
1 month	314
18 months	271
24 months	419

   The effective annual interest rate is 5%.

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Calculate T. [SOA 11/87 #8]

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

11. A loan of 1000 is made at an interest rate of 12% compounded quarterly. The loan is to be repaid with three payments: 400 at the end of the first year, 800 at the end of the fifth year, and the balance at the end of the tenth year.

Calculate the amount of the final payment. [SOA 5/87 #1]

(A) 587 (B) 658 (C) 737 (D) 777 (E) 812

12. You are given:

(i) $\delta_t = \frac{2t^3+8t}{t^4+8t^2+16}$, $0 \le t \le 1$.
(ii) i = the effective annual interest rate equivalent to δ_t.
(iii) Fund X accumulates with simple interest at the rate i.
(iv) Fund Y accumulates at δ_t.
(v) An amount of 1 is deposited in each of Fund X and Fund Y at time $t = 0$.

At what time, t, is (Fund X − Fund Y) a maximum? [SOA 5/87 #14]

(A) 0.250 (B) 0.375 (C) 0.500 (D) 0.625 (E) 0.750

13. Fund F accumulates at the rate $\delta_t = \frac{1}{1+t}$. Fund G accumulates at the rate $\delta_t = \frac{4t}{1+2t^2}$.

You are given:

(i) $F(t)$ = Amount in Fund F at time t
(ii) $G(t)$ = Amount in Fund G at time t
(iii) $H(t) = F(t) - G(t)$
(iv) $F(0) = G(0)$
(v) T is the time t when $H(t)$ is a maximum.

Calculate T. [SOA 11/86 #2]

(A) 1/4 (B) 1/2 (C) 3/4 (D) 1 (E) 5/4

14. Jones agrees to pay an amount of $2X$ at the end of 3 years and an amount of X at the end of 6 years. In return he will receive \$2,000 at the end of 4 years and \$3,000 at the end of 8 years.

At an 8% effective annual interest rate, what is the size of Jones' second payment? [CAS 5/86 #4]

(A) Less than \$1,250
(B) At least \$1,250, but less than \$1,300
(C) At least \$1,300, but less than \$1,350
(D) At least \$1,350, but less than \$1,400
(E) \$1,400 or more

15. At a certain interest rate the present value of the following two payment patterns are equal:

(i) 200 at the end of 5 years plus 500 at the end of 10 years
(ii) 400.94 at the end of 5 years

At the same interest rate, 100 invested now plus 120 invested at the end of 5 years will accumulate to P at the end of 10 years.

Calculate P. [SOA 5/86 #1]

(A) 901 (B) 918 (C) 942 (D) 967 (E) 992

16. (This question actually belongs in Section 3 of the manual, although you could answer it by summing a geometric series.)

On January 1, 1985, Marc has the following two options for repaying a loan:

(i) Sixty monthly payments of 100 commencing February 1, 1985.

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(ii) A single payment of 6000 at the end of K months.

Interest is at a nominal annual rate of 12% compounded monthly. The two options have the same present value.

Determine K. [SOA 11/85 #1]

(A) 29.0 (B) 29.5 (C) 30.0 (D) 30.5 (E) 31.0

17. The XYZ Casualty Insurance Company has found that for a particular type of insurance policy it makes the following payments for insurance claims:

(i) on 10% of the policies, XYZ Company pays $1,000 exactly one year after the effective date of the policy;

(ii) on 3% of the policies, XYZ Company pays $10,000 exactly three years after the effective date of the policy;

(iii) on the remaining policies, XYZ Company makes no payment for claims.

In addition to the above payments, XYZ Company pays $20 for the expenses of administering the policy: $10 is paid on the effective date of the policy and the remaining $10 is paid six months after the effective date of the policy.

The annual interest rate is 8%, compounded semiannually.

The premium for this type of insurance policy is due six months after the effective date of the policy.

If the present value of the premium is set equal to the present value of the claim payments and expenses, what is the premium? [CAS 11/82 #5]

(A) Less than $355
(B) At least $355 but less than $380
(C) At least $380 but less than $415
(D) At least $415 but less than $440
(E) At least $440

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Solutions to Past Exam Questions on Sections 2a and 2b

1. $100 + 200v^n + 300v^{2n} = 600v^{10}$

$100 + 200(.75941) + 300(.75941)^2$

$= 100 + 151.882 + 173.011 = 424.893$

$600v^{10} = 424.893,\ v^{10} = .708155,\ i = 3.5\%$ ANS. (A)

2. AV in 10 years:

Joe: $10[1 + (10)(.11)] + 30[1 + (5)(.11)] = 21 + 46.5 = 67.5$

Tina: $10(1.0915)^{10-n} + 30(1.0915)^{10-2n} = 24v^n + 72v^{2n}$

$72v^{2n} + 24v^n - 67.5 = 0$

For simplicity, let $x = v^n$.

$72x^2 + 24x - 67.5 = 0$, or $3x^2 + x - 2.8125 = 0$

$$x = \frac{-1 \pm \sqrt{1 - (4)(3)(-2.8125)}}{6} = .815819 \text{(positive root)}$$

$x = v^n$ at $9.15\% = .815819,\ n = 2.325$ ANS. (B)

3. $e^{20\delta} = 2,\ 20\delta = \ln 2,\ \delta = \dfrac{\ln 2}{20} = .03466$

$$\left(1 + \frac{.03466}{3}\right)^{3x} = 3$$

$3x \ln 1.01155 = \ln 3$

$$x = \frac{1.09861}{(3)(.01148)} = 31.90$$ ANS. (B)

4. Let n = time of first payment.

$2{,}500\left(v^n + v^{n+5}\right) = 2{,}607$

$$v^n\left(1 + v^5\right) = \frac{2{,}607}{2{,}500},\ v^n = \frac{2{,}600}{(1 + v^5)(2{,}500)} = .584684$$

$n = 11,\ \ n + 5 = 16$

John's age at which he receives second payment = 30 + 16 = 46. ANS. (C)

5. $5{,}000 + 3{,}000v^{12} + 2{,}000v^{60} = 10{,}000v^{12t}$ at 1%

$v^{12t} = .876325,\ t = 1.1056$ ANS. (D)

6. $X\left(1 + v_{.06}^2\right) = 3{,}000v_{.05}^8 + 4{,}000v_{.04}^{10}$

$= 3{,}000(.6768) + 4{,}000(.6756) = 4{,}732.80$

$$X = \frac{4{,}732.80}{1 + .8900} = 2{,}504$$ ANS. (C)

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7. With the penalties, the withdrawals are $1.05K$, $1.05K$, K and K at times 4, 5, 6 and 7, respectively. The equation of value is (comparison date time 10):

$$10{,}000(1.04)^{10} = 1.05K\left(1.04^6 + 1.04^5\right) + K\left(1.04^4 + 1.04^3\right) + 10{,}000$$

$$K = \frac{10{,}000\left(1.04^{10} - 1\right)}{1.05\left(1.04^6 + 1.04^5\right) + 1.04^4 + 1.04^3}$$

$$= \frac{4{,}802.44}{4.900794} = 979.93 \qquad \text{ANS. (C)}$$

8. Method of equated time (weighted average time):

$$X = \frac{300(5) + 500(6) + 700(8)}{300 + 500 + 700} = 6.733$$

Exact time Y:

$$1{,}500v^Y = 300v^5 + 500v^6 + 700v^8$$

$$v^Y = .7688, \quad Y = 6.704$$

$$X + Y = 13.437 \qquad \text{ANS. (A)}$$

9.
$$240(1+i)^{12} + 200(1+i)^6 + 300 = X$$

$$360(1+i)^6 + 700 = X + 100$$

$$Y(1+i)^{12} + 600(1+i)^6 = X$$

For simplicity, let $x = (1+i)^6$. Subtracting the second equation from the first:

$$240x^2 - 160x - 300 = 0$$

$$12x^2 - 8x - 15 = 0$$

$$x = \frac{8 \pm \sqrt{64 - (4)(12)(-15)}}{24} = 1.5(\text{positive root})$$

In the second line of this solution, substitute $x = (1+i)^6 = 1.5$:

$$360(1.5) + 700 = X + 100, \quad X = 1{,}140$$

In the third line of this solution, substitute $(1+i)^6 = 1.5$ and $x = 1{,}140$:

$$Y(1.5)^2 + 600(1.5) = 1{,}140, \quad Y = 106.67 \qquad \text{ANS. (D)}$$

10. First determine the equivalent monthly effective rate j:

$$(1+j)^{12} = 1.05, \quad j = 1.05^{\frac{1}{12}} - 1 = .4074\%$$

$$1{,}004v^T = 314v + 271v^{18} + 419v^{24} \text{ at } j$$

$$= 312.73 + 251.88 + 380.05 = 944.66$$

$$v^T = \frac{944.66}{1{,}004} = .940897, \quad T = 14.98 \qquad \text{ANS. (B)}$$

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11. $1{,}000 = 400v^4 + 800v^{20} + Rv^{40}$ at 3%

$$R = \frac{1{,}000 - 400(.8885) - 800(.5537)}{.3066} = 657.66 \qquad \text{ANS. (B)}$$

12. To be more precise, the question should have said that i is the effective annual interest rate equivalent to δ_t *over the first year.*

$$1 + i = e^{\int_0^1 \frac{2t^3+8t}{t^4+8t^2+16}dt} = e^{\frac{1}{2}\ln\left(t^4+8t^2+16\right)\Big]_0^1}$$

$$= \left(\frac{25}{16}\right)^{\frac{1}{2}} = 1.25, \;\; i = .25$$

Let $f(t) =$ Fund X − Fund Y at time t

$$= (1 + .25t) - e^{\int_0^t \frac{2r^3+8r}{r^4+8r^2+16}dr}$$

$$= \frac{4+t}{4} - \left(\frac{t^4+8t^2+16}{16}\right)^{\frac{1}{2}}$$

Note that $\frac{t^4+8t^2+16}{16} = \left(\frac{t^2+4}{4}\right)^2$, so that the second term above is equal to $\frac{t^2+4}{4}$. To determine a relative maximum, take the first derivative and set it equal to 0:

$$f'(t) = \frac{1}{4} - \frac{t}{2} = 0$$

$$t = 0.5 \qquad \text{ANS. (C)}$$

(Check for maximum: 2nd derivative is negative)

13. $$F(t) = e^{\int_0^t \frac{1}{1+r}dr} = e^{\ln(1+r)\big]_0^t} = 1 + t$$

$$G(t) = e^{\int_0^t \frac{4r}{1+2r^2}dr} = e^{\ln(1+2r^2)\big]_0^t} = 1 + 2t^2$$

(Note: We assumed $F(0) = G(0) = 1$, since the initial amount invested in each fund will cancel out anyway.)

$$H(t) = F(t) - G(t) = t - 2t^2$$

To determine a relative maximum, set $H'(t) = 0$:

$$H'(t) = 1 - 4t = 0, \;\; t = \frac{1}{4} \qquad \text{ANS. (A)}$$

(There is a relative maximum at $t = \frac{1}{4}$, since 2nd derivative is negative.)

14. $$X\left(2v^3 + v^6\right) = 2{,}000v^4 + 3{,}000v^8$$

$$X = \frac{1{,}470.06 + 1{,}620.81}{1.587664 + .630170} = \frac{3{,}090.87}{2.217834} = 1{,}393.64 \qquad \text{ANS. (D)}$$

15. Using time 5 as the comparison date:

$$200 + 500v^5 = 400.94, \;\; v^5 = .40188$$

$$P = 100(1+i)^{10} + 120(1+i)^5$$

Note that $(1+i)^5 = \frac{1}{v^5} = \frac{1}{.40188}$ and $(1+i)^{10} = \frac{1}{.40188^2}$.

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$$P = \frac{100}{.40188^2} + \frac{120}{.40188} = 918 \qquad \text{ANS. (B)}$$

16. PV of payments under Option (i) $= 100(v + v^2 + \cdots + v^{60})$

$$= 100v\left(\frac{1 - v^{60}}{1 - v}\right) \quad \text{at } 1\%$$

$$= 4{,}495.5038$$

PV of payment under Option (ii) $= 6{,}000v^K$

$$v^K = \frac{4{,}495.5036}{6{,}000} = .749251$$

$$K = 29.0 \qquad \text{ANS. (A)}$$

Note: in Section 3a, we will see that $v + v^2 + \cdots + v^{60}$ is the PV of an annuity with the symbol $a_{\overline{60}|}$.

17. The effective interest rate is 4% per half year. Set up an equation of value using the date the premium is due as the comparison date (i.e., 6 months after the effective date, or time 1 in terms of interest periods):

$$P = 10(1.04) + 10 + 10\% \times 1{,}000v^1 + 3\% \times 10{,}000v^5$$

$$= 20.40 + 100v^1 + 300v^5$$

$$= 20.40 + 96.15 + 246.58 = 363.13 \qquad \text{ANS. (B)}$$

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3

Annuities

§ 3a. The Geometric Series Trap

Trap Alert!

In this chapter, we begin our study of annuities. An annuity is simply a regular series of payments. For example, let's determine the present value of an annuity of \$1,000 payable at the end of each year for 10 years at 5% effective

$$PV = 1000(v + v^2 + \ldots + v^{10})$$

where $v = 1.05^{-1}$

You can see that the PV is the sum of a geometric progression. (Years ago, a college math prof of the author disdainfully dismissed the subject of compound interest as "glorified geometric progressions." Of course, he never sat for the actuarial exams.)

Because geometric series do play a very important role in this subject, you *must* know how to sum them correctly. Maybe you think you do but our experience has been that many students fall into traps in applying the summation formula. When we ask students to sum series such as those in the following examples, we usually get four or five different answers, most of which are wrong. Try these examples yourself, but don't look at the solutions until you are finished.

Stepping Stones

EXAMPLE 1

Sum the following series:

(a) $v^{10} + v^{15} + \ldots + v^{65} = ?$

(b) $(1+i)^{44} + (1+i)^{40}(1+r)^2 + (1+i)^{36}(1+r)^4 + \ldots + (1+i)^4(1+r)^{20} = ?$

SOLUTION

The answers are $\frac{v^{10}(1-v^{60})}{1-v^5}$ for (a), and $(1+i)^{44}\frac{[1-(1+i)^{-44}(1+r)^{22}]}{1-(1+i)^{-4}(1+r)^2}$ for (b)

If you got different answers, first check whether they are algebraically equivalent to the above answers. Also, in answering (b) you could have used v^4 instead of $(1+i)^{-4}$.

You probably used the following formula for the sum of a geometric progression:

$$S = \frac{a(1-r^n)}{1-r}$$

where a is the first term, r is the common ratio and n is the number of terms. (In other words, this formula gives the sum of the series $a + ar + ar^2 + \ldots + ar^{n-1}$.) Some students remember this

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formula as $\frac{a-ar^n}{1-r}$, where the numerator can be interpreted as the first term minus the (fictitious) $(n+1)^{st}$ term, i.e., the term after the nth or last term, ar^{n-1}.

A trap for some students is that for "n" they use the exponent of the last term in the series, or they make a similar substitution error. The way to avoid such traps is to remember the formula ***in words***, not symbols. You almost can't go wrong if you know the formula in the following way:

$$S = (\text{first term})\,\frac{[1-(\text{ratio})^{\text{no. of terms}}]}{1-\text{ratio}}$$

In series (a) above, the first term is v^{10}, the common ratio is v^5 and the number of terms is 12. (One way to count the number of terms—other than on your fingers—is $\frac{70-10}{5}=12$ where 70 is the exponent of the term ***after*** the actual last term, 10 is the exponent of the first term and 5 is the increment in the exponent from term to term.) If we think of the formula in terms of the ***words***, we have:

$$S = v^{10}\frac{[1-(v^5)^{12}]}{1-v^5} = v^{10}\frac{(1-v^{60})}{1-v^5}$$

For series (b), the first term is $(1+i)^{44}$, the ratio is $(1+i)^{-4}(1+r)^2$ (or $v^4(1+r)^2$) and the number of terms is 11 (which can be counted by using the exponents of the $(1+i)$ terms as $\frac{0-44}{-4}=11$ or by using the exponents of the $(1+r)$ terms as $\frac{22-0}{2}=11$.)

Again, substituting in the formula in terms of the *words*, we have:

$$S = (1+i)^{44}\frac{\{1-[(1+i)^{-4}(1+r)^2]^{11}\}}{1-(1+i)^{-4}(1+r)^2}$$

$$= (1+i)^{44}\frac{[1-(1+i)^{-44}(1+r)^{22}]}{1-(1+i)^{-4}(1+r)^2}$$

which can be further simplifed to:

$$S = \frac{(1+i)^{44}-(1+r)^{22}}{1-(1+i)^{-4}(1+r)^2}$$

The geometric series formula is so simple and yet so treacherous. We urge you to remember it in the plain English version, using words instead of symbols. Practice saying it like a mantra (remember how you used to chant the quadratic formula?) and you will avoid falling into the traps that wait to ensnare you.

EXAMPLE 2

Sum the following series: $v + 1.03v^2 + 1.03^2v^3 + 1.03^3v^4 + \cdots + 1.03^{14}v^{15}$, where v is based on $i = 5.25\%$. (Get a numerical result.)

SOLUTION

Using the formula for the sum of a geometric progression in words, we have:

$$\text{Sum} = \frac{v[1-(103v)^{15}]}{1-1.03v}$$

Substituting $v = 1/1.0525$:

$$\text{Sum} = \frac{1-\left(\frac{1.03}{1.0525}\right)^{15}}{1.0525\left(1-\frac{1.03}{1.0525}\right)} = \frac{1-\left(\frac{1.03}{1.0525}\right)^{15}}{.0225} = 12.30$$

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Note: In Section 4j, we will see that this sum is the PV at 5.25% of a 15-year annuity-immediate with a first payment of 1 and with subsequent payments increasing geometrically at 3% (which could be referred to as an annuity with "inflation protection").

EXAMPLE 3

Find the sum of the series in Example 2 if there are an infinite number of terms, rather than only 15 terms. (Get a numerical result.)

SOLUTION

The sum of an infinite geometric progression is:

$$\text{Sum} = \frac{a}{1-r}$$

which has a limit if $-1 < r < 1$ of:

$$\text{Sum} = \frac{v}{1-1.03v} = \frac{1}{1.0525\left(1-\frac{1.03}{1.0525}\right)} = \frac{1}{.0225} = 44.4\bar{4}$$

EXAMPLE 4

Find the sum of the series: $v + 2v^2 + 3v^3 + 4v^4 + \cdots + 20v^{20}$.

SOLUTION

This is not a geometric series. (Sorry. This was a trick question.) In Section 4k, we will learn how to sum such a series. It represents the PV of an increasing annuity-immediate with payments in arithmetic progression.

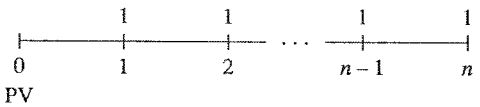
§3b. Annuity-Immediate and Annuity-Due

Almost everyone has had experience with annuities. If you pay rent, insurance premiums, mortgage payments, payments on a car or college loan, etc., the payments are an annuity. If you receive a regular paycheck, you are receiving an annuity. Someday you may receive Social Security benefits or a company pension for life (although traditional pension plans of this kind are on the wane). These, too, are types of annuities.

The stem of the word "annuity" is the Latin annu(us), for "yearly," since the earliest annuities had annual payments. So it's something of an oxymoron to speak of a "monthly annuity," a "quarterly annuity," etc., but that's how the word evolved.

Present Value of an Annuity-Immediate

Let's consider a very simple annuity - a payment of \$1 at the end of each year for n years:



The PV of this annuity is:

$$PV = v + v^2 + \ldots + v^{n-1} + v^n$$

Assuming that you have mastered the summation of geometric series in the last section you should get:

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$$PV = \frac{v(1-v^n)}{1-v}$$

You may recall from Section 1a(vi) that $1 - v = d = iv$, so if we substitute in the denominator, we get:

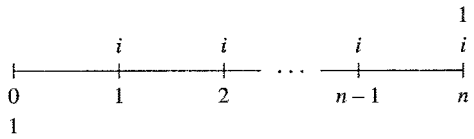
$$PV = \frac{1-v^n}{i}$$

An annuity having the first payment a year from now is called an "***annuity-immediate.***" (It's a little strange to call it "immediate" when the first payment is not immediate.) The PV has a special symbol: $a_{\overline{n}|i}$. This is read "a angle n at i." (If the rate of interest is understood, we can just write $a_{\overline{n}|}$.)

At this point, you should practice getting numerical results. Turn to Calculator Notes #5 following Section 3c and do problems 22 to 26. If you are not using the BA II Plus, use your own calculator and compare your answers.

Here is a simple way to derive the formula for $a_{\overline{n}|}$ without using a summation formula: You deposit \$1.00 in a bank that credits interest at i effective. At the end of a year, you withdraw i, leaving \$1.00 in the account. You do the same thing at the end of years 2, 3 . . . , n, withdrawing i each year. Just after the nth withdrawal, you still have \$1.00 in the account; you withdraw it and close the account.

This transaction can be diagrammed as follows:



If we let X represent the present value of an annuity-immediate of \$1.00 per annum for n years, the present value of the n withdrawals of i equals iX. Thus, we have the following equation of value as of time 0:

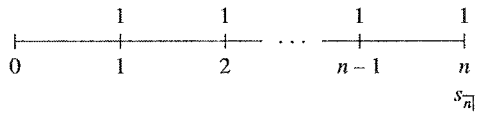
$$1 = iX + v^n$$

$$\text{or } X = \frac{1-v^n}{i}$$

Of course, the way we defined X, it is actually $a_{\overline{n}|}$, and we have derived its formula without making direct use of the summation formula.

Accumulated Value of an Annuity-Immediate

Let's determine the AV of an annuity-immediate of 1 for n periods. (Note that the AV of an annuity-immediate means the AV as of the date of the last payment or deposit.) The AV has the special symbol $s_{\overline{n}|}$.



Accumulating the payments to time n, starting with the last payment, we have:

$$s_{\overline{n}|} = 1 + (1+i) + \ldots + (1+i)^{n-2} + (1+i)^{n-1}$$

(Note carefully that this geometric series starts with 1 and ends with $(1+i)^{n-1}$.)

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Summing the series, we have:

$$s_{\overline{n}|} = \frac{1[1-(1+i)^n]}{1-(1+i)} = \frac{(1+i)^n - 1}{i}$$

Of course, rather than summing a geometric series again, we could have relied on the relationship between $a_{\overline{n}|}$ and $s_{\overline{n}|}$. After all, these symbols represent the value of the ***same*** payments on ***two different dates*** which are n years apart. Thus, we have:

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|} = (1+i)^n \left(\frac{1-v^n}{i}\right) = \frac{(1+i)^n - 1}{i}$$

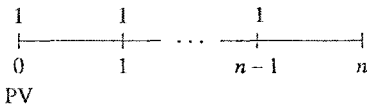
since $(1+i)^n$ and v^n are reciprocals.

(At this point, we caution you that $s_{\overline{n}|}$ is a deceptively simple interest symbol that can easily get you into trouble. More on this in the next section.)

You should now turn to Calculator Notes #5 again and do problems 27 to 29.

Present Value of an Annuity-Due

Suppose we want the PV of an annuity with n payments where the first payment is today, rather than a year from now



$$PV = 1 + v + \ldots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

When the first payment is today, an annuity is called an ***annuity-due***. The PV has the symbol $\ddot{a}_{\overline{n}|}$, which is read "a double-dot angle n."

Mnemonic: Note that the formula for an annuity-*i*mmediate has an i in the denominator:

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

while the formula for an annuity-*d*ue has a d in the denominator:

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

i for *i*mmediate and d for *d*ue.

Relationships Between $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$

There are obvious relationships between the "immediate" and "due" forms of an annuity. The symbols $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ both represent the value of the same payments, but $\ddot{a}_{\overline{n}|}$ is equal to the value of $a_{\overline{n}|}$ a year later. Thus, we have:

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$$

(If you have any doubt about this, ask yourself which you would rather have today: a series of payments that begins today or the ***same*** series of payments beginning a year from now. Clearly, you want the payments to begin as soon as possible, so $\ddot{a}_{\overline{n}|}$ is greater than $a_{\overline{n}|}$; in particular, $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.)

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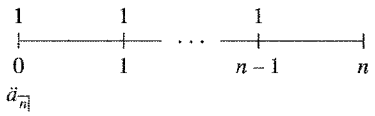
There is another simple relationship between the "immediate" and "due" forms. To derive it, we will play a game that we call "now you see it, now you don't." (We'll play it some more in Section 3d.) Here is the diagram for the annuity due again:



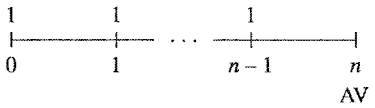
Now, cover up the payment at time 0 (mentally or with your finger). What's left? Answer: An annuity-immediate with $(n-1)$ payments. So we can write the PV of an annuity-due as the payment of \$1.00 at time 0 plus the PV of an annuity-immediate with one less payment:

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

Now turn to Calculator Notes #5 and do problems 30 and 31 for practice in getting numerical results for problems solving $\ddot{a}_{\overline{n}|}$.

Accumulated Value of an Annuity-Due

By the AV of an annuity-due, we mean the AV one period ***after*** the last payment, as in the following diagram:



The symbol for this AV is $\ddot{s}_{\overline{n}|}$. Clearly, the series is:

$$\ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1} + (1+i)^n$$

(Note very carefully that the series begins with $(1+i)$ and ends with $(1+i)^n$, while the series for $s_{\overline{n}|}$ begins with 1 and ends with $(1+i)^{n-1}$.)

Summing the series, we have:

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)[1-(1+i)^n]}{1-(1+i)}$$

If we reverse signs of numerator and denominator and bring $(1+i)$ "downstairs" in the denominator as v, we have:

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{iv} = \frac{(1+i)^n - 1}{d}$$

Earlier, we derived certain relationships between the "immediate" and "due" forms for present value, and we can do the same for accumulated value.

Which is worth more, $\ddot{s}_{\overline{n}|}$ or $s_{\overline{n}|}$? Clearly, you would rather have the AV of a series of deposits ***one year after*** the last deposit than the AV of the same deposits on ***the date of the last deposit***. In particular, $\ddot{s}_{\overline{n}|}$ is equal to the value of $s_{\overline{n}|}$ one year later, i.e.,

$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}$$

Also, we could play "now you see it, now you don't" (or perhaps more accurately, "now you don't see it, now you do"):

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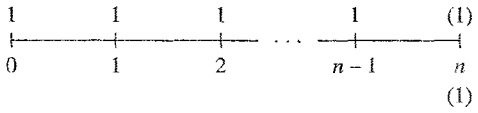

What we have done is to place a fictitious deposit at time n above the time line but to immediately withdraw it at time n below the time line. This leaves the AV of the original annuity exactly the same. Now look at all of the deposits, including the fictitious one. Their AV as of time n is the AV of an annuity-immediate with $(n + 1)$ payments, i.e., $s_{\overline{n+1|}}$. But this AV overstates the true AV by 1, which we must deduct. Thus, we have the following relationship:

$$\ddot{s}_{\overline{n|}} = s_{\overline{n+1|}} - 1.$$

Now turn to Calculator Notes #5 and do problems 32 and 33 for practice in getting numerical results for problems involving $\ddot{s}_{\overline{n|}}$.

Problems Involving Both Present Values and Accumulated Values

Turn to Calculator Notes #5 and do problems 34 to 36 as examples of problems where both the PV and AV of annuities are involved.

Stepping Stones

EXAMPLE 1

Given that $a_{\overline{n|}} = 7.912718$ and $\ddot{a}_{\overline{n|}} = 8.268790$, determine n.

SOLUTION

Since $\ddot{a}_{\overline{n|}} = (1 + i)a_{\overline{n|}}$, $1 + i = \ddot{a}_{\overline{n|}}/a_{\overline{n|}} = 8.268790/7.912718 = 1.045$ and $i = 4.5\%$. We can determine n by using the calculator:

4.5 [I/Y] 7.912718 [PV] 1 [+/-] [PMT] [CPT] [N]

$n = 10$

EXAMPLE 2

Given $a_{\overline{n|}} = 8.50$ and $s_{\overline{n|}} = 16.63$, determine i.

SOLUTION

Since $s_{\overline{n|}} = (1 + i)^n a_{\overline{n|}}$, $(1 + i)^n = s_{\overline{n|}}/a_{\overline{n|}} = 16.63/8.50 = 1.956471$.

Now, $s_{\overline{n|}} = [(1 + i)^n - 1]/i$. Since we have numerical values for $s_{\overline{n|}}$ and $(1 + i)^n$, we can determine i:

$$16.63 = (1.956471 - 1)/i$$

$$i = 5.75\% \quad \text{(to 2 decimals)}$$

§3c. The Great Confusion: Annuity-Immediate and Annuity-Due

Students sometimes get confused by the terms "annuity-immediate" and "annuity-due," and the symbols for the present value and accumulated value of these annuities. In this section, we will consider a few examples that should help clear up this confusion.

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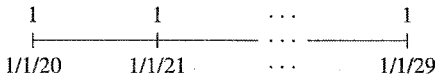
Stepping Stones

EXAMPLE 1

Suppose you are told that an annuity pays \$1.00 on January 1 of each year from 2020 through 2029. Let's draw a time diagram for these 10 payments:



Here's the question: Is this an annuity-immediate or an annuity-due?

Your first impulse would probably be to say that this is an annuity-due. After all, payments are made at the beginning of each year. Isn't that the definition of an annuity-due?

But what if you were asked to determine the present value of this annuity on 1/1/19, one year before the first payment? Wouldn't you say that you were being asked to find the PV of an annuity-immediate, $a_{\overline{10}|}$?

And what if you were asked to determine the present value of the same annuity on 1/1/20, the date of the first payment? Wouldn't you say that you were being asked to find the PV of an annuity-due, $\ddot{a}_{\overline{10}|}$?

It seems as if the same annuity can be an annuity-immediate or an annuity-due, *depending on the valuation date.*

That's exactly right.

Unfortunately, many textbooks define an annuity-immediate and an annuity-due based on whether the payments are made at the "end of each period" or "the beginning of each period." A typical definition might be:

> If payments are made at the end of each period, the annuity is called an "annuity-immediate." If payments are made at the beginning of each period, the annuity is called an "annuity-due."

This definition is deceptive. A more precise definition is:

> An annuity is called an "annuity-immediate" if, in determining its present value, the valuation date is *one period before* the first payment (symbol $a_{\overline{n}|}$). An annuity is called an "annuity-due" if, in determining its present value, the valuation date is *on* the date of the first payment (symbol $\ddot{a}_{\overline{n}|}$).

It may seem as if we are making a big fuss over something that is pretty obvious. But you would be surprised at how often this can lead to confusion, especially when we need to find the *accumulated value* of an annuity.

Let's take an example of an accumulated value.

EXAMPLE 2

\$1.00 is deposited at the beginning of each year for 5 years at i effective. How much is in the account on the date of the last deposit? (Use a symbol.)

Think for a moment and write your answer down.

Hopefully, you wrote $s_{\overline{5}|}$, not $\ddot{s}_{\overline{5}|}$. Remember, you were asked for the AV of this annuity *on the date of the last deposit.* If you drew a time diagram, it might look like this:

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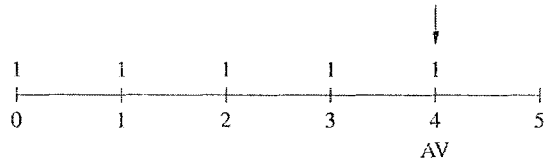


Note that we have placed "AV" at *time 4,* which is the date of the last deposit.

If we write out this AV term by term, starting with the deposit that was just made at time 4, we get:

$$AV = 1 + (1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4$$

Isn't this the definition of $s_{\overline{n}|}$, with $n = 5$, that we just covered in Section 3b?

EXAMPLE 3

\$1.00 is deposited at the beginning of each year for 5 years at i effective. How much is in the account one year after the last deposit? (Use a symbol.)

Clearly, the answer is $\ddot{s}_{\overline{n}|}$. We could use the same diagram as in the previous example, except that we would place "AV" at time 5, one year after the last deposit. Writing out this AV term by term:

$$AV = (1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4 + (1+i)^5$$

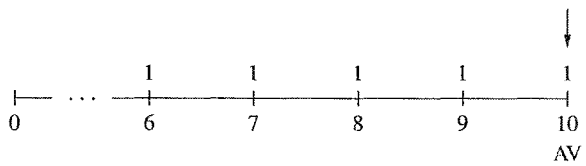
This is the definition of $\ddot{s}_{\overline{n}|}$, with $n = 5$, that we covered in Section 3b.

When it comes to accumulated values, the definition of an annuity-immediate or an annuity-due must take into account the timing of the payments *with respect to the valuation date,* just as in the case of present values. We suggest the following definition:

> An annuity is called an "annuity immediate" if, in determining its accumulated value, the valuation date is *on* the date of the last payment (symbol $s_{\overline{n}|}$). An annuity is called an "annuity-due" if, in determining its accumulated value, the valuation date is *one period after* the date of the last payment (symbol $\ddot{s}_{\overline{n}|}$).

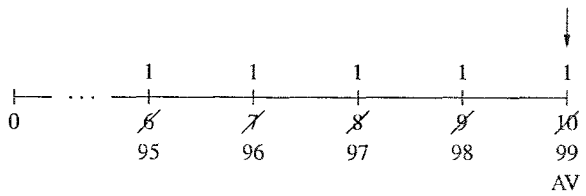
EXAMPLE 4

\$1.00 is deposited at the end of the 6th through 10th years. How much is in the account on the date of the last deposit?



The answer is $s_{\overline{5}|}$. It's certainly not $s_{\overline{10}|}$ or $(s_{\overline{10}|} - s_{\overline{5}|})$.

The time scale can cause confusion. Would it change the answer to this question if we renumbered the dates as follows?



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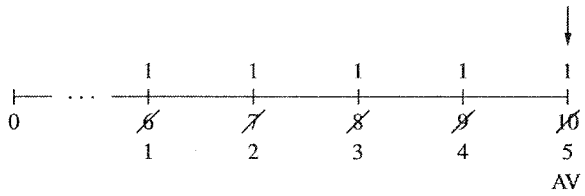
We think you would agree that the answer is not $s_{\overline{99}|}$ or $(s_{\overline{99}|} - s_{\overline{94}|})$. In fact, whatever dates we use, as long as the diagram shows 5 annual payments, with the AV on the date of the last payment, the answer is always $s_{\overline{5}|}$. This includes changing the dates as follows:



In this version of the time diagram, it's clear that the AV is equal to $s_{\overline{5}|}$, just as for the two preceding diagrams. On a term by term basis, we have the following for all three diagrams:

$$AV = 1 + (1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4 = s_{\overline{5}|}$$

The accumulated value of 5 annual deposits of \$1.00 as of the date of the last deposit is always $s_{\overline{5}|}$, whether the payments are labeled as being at time 0 to 4, 1 to 5, 6 to 10, 95 to 99, etc.

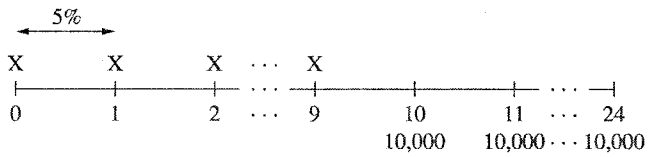
EXAMPLE 5

Joe makes 10 annual deposits of X each into a fund earning 5% effective. These deposits accumulate to an amount that is just sufficient to allow him to withdraw \$10,000 annually for 15 years, with the first withdrawal one year after the last deposit. Set up a time diagram, choose a comparison date, and write an equation of value. Then get a numerical value for X.

SOLUTION

You are not told whether the deposits or withdrawals are made at the beginning or end of each year. However, because you know that the first withdrawal is one year after the last deposit, you have enough information to determine X.

It's up to you as to how you set up the time diagram. Many students would say, "Well, I might as well assume that the deposits start right away, so I'll place X at time 0, 1, 2, . . . , 9." If you take this approach, your time diagram would look like this:



Note that the first withdrawal has been placed one year after the last deposit.

What comparison date would you choose? Many students would think that since the deposits have been placed at the beginning of each year, they should use time 10 as the comparison date. They would then write the following equation of value:

$$X\ddot{s}_{\overline{10}|} = 10{,}000\ddot{a}_{\overline{15}|}$$

There's nothing wrong with this approach from a theoretical point of view. However, we try to avoid using the "double-dot" form of annuities if possible, for the practical reasons given in Calculator Notes #5. (See problems 30 to 33.) Since "double-dots cancel" (this is covered in a later section), we can simply remove them in the above equation. (What this really means is that we cancel d in the denominators of both sides of the equation and replace d by i.) This gives us the following equivalent equation of value:

$$Xs_{\overline{10}|} = 10{,}000a_{\overline{15}|}$$

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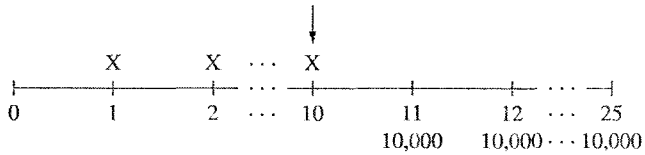
Of course, we could have gotten directly to this equation by using *time 9* as the comparison date, wouldn't you agree?

An even better approach would be to set up the time diagram in the first place as follows:



Using time 10 as the comparison date, we get the above equation of value.

We think that this approach is the best way to avoid making any mistakes. You will learn how to set up your time diagrams in the most convenient way as you get more experience in doing problems.

Using the methods shown in Calculator Notes #5, you should get $X = \$8,252.30$.

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Calculator Notes #5: Annuities

When you do problems involving annuities, it's easier to use the TVM keys than it is to use first principles. You will be using all of the TVM keys, including the [PMT] key. This is the register for the level annuity payment.

If you enter 3 of the 4 variables [N], [I/Y], [PMT] and [PV], you can calculate the unknown 4th variable. You can do the same thing if [FV] is one of the variables, rather than [PV].

Problems 22 to 26 involve $a_{\overline{n}|}$

22. Find the PV of a 10-year annuity-immediate with annual payments of $1,200 at an effective interest rate of 6% per annum.

    The keystrokes are:

    [2nd] [CLR TVM]

    (Just a reminder to always clear the TVM registers at the start of a new problem.)

    10 [N] 6 [I/Y] 1,200 [PMT] [CPT] [PV]

    The answer is −$8,832.10 to 2 decimals.

23. Find the PV of an annuity-immediate of $100 per annum for 15 years at 4% effective.

    15 [N] 4 [I/Y] 100 [PMT] [CPT] [PV]

    The answer is −$1,111.84.

24. Find the PV of an annuity-immediate with payments of $50 every 6 months for 10 years at a nominal rate of interest of 4% compounded semiannually.

    20 [N] 2 [I/Y] 50 [PMT] [CPT] [PV]

    The answer is −$817.57.

25. A 12-year annuity-immediate is purchased for $100,000 at 5.5% effective. What is the level annual payment provided by this annuity?

    12 [N] 5.5 [I/Y] 100,000 [+/−] [PV] [CPT] [PMT]

    The answer is $11,602.92.

26. A loan of $1,000 is to be repaid by equal quarterly installments of X at the end of each quarter over a 3-year period at a nominal rate of interest of 4% compounded quarterly. Determine X.

    12 [N] 1 [I/Y] 1,000 [PV] [CPT] [PMT]

    The answer is −$88.85.

Problems 27 to 29 involve $s_{\overline{n}|}$

27. John deposits $250 at the end of each month for 5 years in an account that credits interests at a nominal rate of 6% per annum compounded monthly. How much is in his account on the date of the last deposit?

    In this problem, one of the four variables is [FV], rather than [PV]. Since we want the accumulated value on the date of the last deposit, the answer in symbolic form is $250s_{\overline{60}|}$ at 0.5%. The keystrokes are:

    60 [N] .5 [I/Y] 250 [+/−] [PMT] [CPT] [FV]

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The answer is \$17,442.51.

28. \$1,200 is deposited in an account on January 1 of each year from 2005 to 2015, inclusive, at 3% effective. How much is in the account on January 1, 2015?

In symbolic form, the answer is $1{,}200s_{\overline{11}|.03}$. (Note that there are 11 deposits from 2005 to 2015, inclusive, and that the question asks for the AV on the date of the last deposit (January 1, 2015). Thus, this is the accumulated value of an annuity-immediate.) The keystrokes are:

11 [N] 3 [I/Y] 1,200 [+/−] [PMT] [CPT] [FV]

The answer is \$15,369.35.

29. What quarterly deposit for 6 years will accumulate to \$22,000 on the date of the last deposit at a nominal rate of interest of 7% compounded quarterly?

In symbolic form, the answer is the solution for X in $Xs_{\overline{24}|1.75\%} = 22{,}000$.

24 [N] 1.75 [I/Y] 22,000 [FV] [CPT] [PMT]

The answer is −\$745.48.

Problems 30 and 31 involve $\ddot{a}_{\overline{n}|}$

30. Find the PV of an annuity-due with payments of \$1,800 every 6 months for 8 years at a nominal rate of interest of 5% per annum compounded semiannually.

This problem asks for the PV of an annuity with payments at the *beginning* of each 6-month period for 16 periods, at an interest rate of 2.5% per period. The answer in symbolic form is $1{,}800\ddot{a}_{\overline{16}|.025}$ or $1{,}800(1.025)a_{\overline{16}|}$ or $1{,}800(1 + a_{\overline{15}|})$.

The calculator has a [BGN] key, secondary to the [PMT] key, which can be used to compute an annuity-due. We will do the problem with and without using the [BGN] key.

a. Without using the [BGN] key

(Make sure that you are in [END] mode. If you are in [BGN] mode, "BGN" will show in the upper right hand corner of the display. To change to [END] mode, see (b) below.)

We will calculate $1{,}800a_{\overline{16}|}$ at 2.5% and then multiply by 1.025. (This is based on the relationship $\ddot{a}_{\overline{16}|} = (1 + i)a_{\overline{16}|}$.)

16 [N] 2.5 [I/Y] 1,800 [PMT] [CPT] [PV] [×] 1.025 [=]

The answer is −\$24,086.48.

b. Using the [BGN] key

Trap Alert!

It is true that you can use the [BGN] key to easily calculate an annuity-due. But we have to warn you that it's very easy to forget that you are in [BGN] mode when you move on to another problem. *The calculator will stay in [BGN] mode even if you clear the TVM keys by pressing* [2nd] [CLR TVM], *and it will remain in this mode until you actively change it!*

How do you avoid this trap? Our own preference is not to use the [BGN] key if we can help it. In problem 30, you could compute an annuity-immediate and multiply by $(1 + i)$, as we did above. You could also compute the PV of an annuity-immediate with one less payment and then add a payment. (This is based on the relationship $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$.)

If you do decide to regularly use the [BGN] key to evaluate an annuity-due, practice a lot, *and make sure that when you move on to another problem, you are in the correct mode for that problem.*

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To put the calculator in [BGN] mode, do the following:

[2nd] [BGN] [2nd] [SET]

If you repeat these 4 keystrokes, the left-hand side of the display will alternate between "END" and "BGN". When it shows "BGN", return to standard calculator mode by pressing [2nd] [QUIT]. You are now in [BGN] mode, and "BGN" will be shown in small font in the upper right-hand corner of the display.

The keystrokes for problem 30 are:

16 [N] 2.5 [I/Y] 1,800 [PMT] [CPT] [PV]

The answer is −\$24,086.48, the same as before. Note that these are the same keystrokes as in (a), except that the last step of multiplying by 1.025 is omitted. Of course, this is because you are in [BGN] mode, which directly calculates an annuity-due.

31. Mary deposits \$15,000 today in a bank crediting interest at a nominal rate of 5% compounded monthly. This deposit is just sufficient to permit her to make monthly withdrawals of X for 6 years, first withdrawal today. Determine X.

X is the solution of $X\ddot{a}_{\overline{72}|5/12\%} = 15{,}000$. (Note that by convention, "monthly withdrawals for 6 years" means 72 withdrawals, not 73, even if the first withdrawal is today.)

a. Without using the [BGN] key

Make sure that you are in [END] mode. The keystrokes are:

72 [N] 5 [÷] 12 [=] [I/Y] 15,000 [PV] [CPT] [PMT]

At this point, you have calculated the withdrawal that Mary could make at the *end* of each month, since you are in [END] mode. If Mary makes each withdrawal one month earlier, you must multiply this amount by v, where $v = (1+i)^{-1}$ and $i = 5/12\%$, the effective monthly rate. The keystrokes are:

.05 [÷] 12 [+] 1 [=] [1/x] [×] [RCL] [PMT] [=]

The answer is −\$240.57.

b. Using the [BGN] key

Make sure you are in [BGN] mode. Use the same keystrokes as in (a), except for the last step of multiplying by v. The answer is −\$240.57, as before.

Problems 32 and 33 involve $\ddot{s}_{\overline{n}|}$

32. Determine the AV of 13 annual deposits of \$1,429 one year after the last deposit, at 2.10% effective.

In symbolic form, the answer is $1{,}429\ddot{s}_{\overline{13}|2.10\%}$ or $1{,}429(1+i)s_{\overline{13}|}$ or $1{,}429(s_{\overline{14}|} - 1)$.

a. Without using the [BGN] key

Make sure that you are in [END] mode. The keystrokes are:

13 [N] 2.10 [I/Y] 1,429 [+/−] [PMT] [CPT] [FV] [×] 1.021 [=]

The answer is \$21,551.03.

b. Using the [BGN] key

Make sure that you are in [BGN] mode. Use the same keystrokes as in (a) but omit the last step of multiplying by 1.021. The answer is the same as before.

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33. Andrea makes deposits of \$100 on the first day of each month in calendar years 2010 through 2015, inclusive, at a nominal rate of 7% per annum convertible monthly. How much is in her account on January 1, 2016?

This is the AV of an annuity-due, since the date of evaluation (January 1, 2016) is one month after the last deposit. There are 72 deposits. (There are 6 years from 2010 through 2015, inclusive.)

a. Without using the [BGN] key

You must be in [END] mode. The keystrokes are:

72 [N] 7 [÷] 12 [=] [I/Y] 100 [+/−] [PMT] [CPT] [FV]

At this point, you have calculated $100s_{\overline{72}|}$, so you must multiply by 1 plus the effective monthly rate:

.07 [÷] 12 [+] 1 [×] [RCL] [FV] [=]

The answer is \$8,968.10.

b. Using the [BGN] key

You must be in [BGN] mode. The keystrokes are the same as in (a), except that you omit the last step of multiplying by $(1 + i)$.

Problems involving both the PV and AV of an annuity

34. Ed makes 15 annual deposits of X, which are just sufficient to allow him to make 8 annual withdrawals of \$1,000, first withdrawal one year after last deposit. Interest is at 5% effective. Determine X.

The most convenient comparison date is the date of the last deposit. X is the solution of $Xs_{\overline{15}|} = 1{,}000a_{\overline{8}|}$ at 5%. First, we calculate [PV], the right-hand side of the equation. Since this is the amount that the 15 deposits of X must accumulate to, we enter it in [FV] and proceed from there. The keystrokes are:

8 [N] 5 [I/Y] 1,000 [PMT] [CPT] [PV] 2nd [CLR TVM] [FV] 15 [N] 5 [I/Y] [CPT] [PMT]

The answer is 299.52.

Note the following:

1. After you clear the TVM registers, the PV (= −6,463.21) is still in the display, so it can be entered in [FV] simply by pressing that key.

2. An alternative is to omit the step 2nd [CLR TVM]. You would then use the keystrokes:

[FV] 15 [N] 0 [PV] [CPT] [PMT]

Again, the answer is 299.52.

Note that if you don't clear the TVM registers after calculating $1{,}000a_{\overline{8}|}$, you don't have to re-enter the 5% interest rate *but you have to zero-out* [PV], *since any values from the previous calculation are still in the registers.* The safest thing to do is to clear the TVM registers, even if you have to re-enter one or more values that don't change. You be the judge.

35. Same as problem 34, except that the interest rate drops to 4% effective immediately after the last deposit.

The keystrokes are exactly the same as in problem 34, except that the interest rate is entered as 4% the first time.

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The answer is 312.01. Note that because a lower interest rate (4%) is earned during the withdrawal period, the deposit X must be larger than it was in problem 34 (312.01 vs. 299.52).

36. Deposits of \$100 are made every month for 5 years into an account crediting interest at a nominal rate of 9% convertible monthly. Starting one month after the last deposit, monthly withdrawals of X are made for 10 years, exhausting the account. Determine X.

The most convenient comparison date is the date of the last deposit. The equation of value is $100s_{\overline{60|}} = Xa_{\overline{120|}}$ at .75%. First, we calculate the left-hand side of the equation as $\boxed{\text{FV}}$. This is entered in $\boxed{\text{PV}}$ for the second part of the calculation. The keystrokes are:

60 $\boxed{\text{N}}$ .75 $\boxed{\text{I/Y}}$ 100 $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$ $\boxed{\text{2nd}}$ [CLR TVM] $\boxed{\text{PV}}$ 120 $\boxed{\text{N}}$ .75 $\boxed{\text{I/Y}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PMT}}$

The answer is 95.54.

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Summary of Concepts and Formulas in Sections 3a to 3c

(1) The best way to remember the formula for the sum of a geometric progression is in words:

$$\text{Sum of geometric progression} = (\text{first term})\frac{[1-(\text{ratio})^{\text{no. of terms}}]}{1-\text{ratio}}$$

(2) (a) PV of an annuity-immediate:

$$\begin{aligned} a_{\overline{n}|} &= v + v^2 + \cdots + v^n \\ &= \frac{1-v^n}{i} \end{aligned}$$

(b) AV on the date of the last payment:

$$\begin{aligned} s_{\overline{n}|} &= 1 + (1+i) + \cdots + (1+i)^{n-1} \\ &= \frac{(1+i)^n - 1}{i} \end{aligned}$$

(3) (a) PV of an annuity-due:

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= \frac{1-v^n}{d} \end{aligned}$$

(b) AV one year after last payment:

$$\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i) + (1+i)^2 + \cdots + (1+i)^n \\ &= \frac{(1+i)^n - 1}{d} \end{aligned}$$

(4) Note the "i" in the denominator for "immediate" and the "d" for "due."

(5) Relationships between annuity-immediate and annuity-due:

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

(6) Avoid the $s_{\overline{n}|}$ trap: The AV *on the date of the last payment* is $s_{\overline{n}|}$ (if there are n payments), regardless of how the time diagram is numbered. The AV *one year after the last payment* is $\ddot{s}_{\overline{n}|}$ (if there are n payments).

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Past Exam Questions on Sections 3a to 3c

1. A man turns 40 today and wishes to provide supplemental retirement income of 3000 at the beginning of each month starting on his 65th birthday. Starting today, he makes monthly contributions of X to a fund for 25 years. The fund earns a nominal rate of 8% compounded monthly. Each 1000 will provide for 9.65 of income at the beginning of each month starting on his 65th birthday until the end of his life. Calculate X. [11/01 #27]

   (A) 324.73 (B) 326.89 (C) 328.12 (D) 355.45 (E) 450.65

2. Susan and Jeff each make deposits of 100 at the end of each year for 40 years.

   Starting at the end of the 41st year, Susan makes annual withdrawals of X for 15 years and Jeff makes annual withdrawals of Y for 15 years. Both funds have a balance of 0 after the last withdrawal.

   Susan's fund earns an annual effective interest rate of 8%. Jeff's fund earns an annual effective interest rate of 10%.

   Calculate $Y - X$. [SAMPLE/00 #27]

   (A) 2792 (B) 2824 (C) 2859 (D) 2893 (E) 2925

3. Jeff and John shared equally in an inheritance.

   Using his inheritance, John immediately bought a 10-year annuity-due with annual payments of 2500 each.

   Jeff put his inheritance in an investment fund earning an annual effective interest rate of 9%. Two years later, Jeff bought a 15-year annuity-immediate with annual payment of Z.

   The present value of both annuities was determined using an annual effective interest rate of 8%.

   Calculate Z. [SOA 11/96 #4]

   (A) 2330 (B) 2470 (C) 2515 (D) 2565 (E) 2715

4. At time $t = 0$, Paul deposits P into a fund crediting interest at an effective annual interest rate of 8%. At the end of each year in years 6 through 20, Paul withdraws an amount sufficient to purchase an annuity-due of 100 per month for 10 years at a nominal interest rate of 12% compounded monthly. Immediately after the withdrawal at the end of year 20, the fund value is zero.

   Calculate P. [SOA 11/93 #4]

   (A) 41,000 (B) 42,000 (C) 43,000 (D) 44,000 (E) 45,000

5. At an annual effective interest rate of 6.3%, an annuity immediate with $4N$ level annual payments of 1,000 has a present value of 14,113.

   Determine the fraction of the total present value represented by the first set of N payments and the third set of N payments combined. [SOA 5/93 #4]

   (A) 57% (B) 60% (C) 63% (D) 66% (E) 69%

6. An investment requires an initial payment of 10,000 and annual payments of 1,000 at the end of each of the first 10 years. Starting at the end of the eleventh year, the investment returns five equal annual payments of X.

   Determine X to yield an annual effective rate of 10% over the 15-year period. [SOA 11/92 #9]

   (A) 10,750 (B) 10,900 (C) 11,050 (D) 11,200 (E) 11,350

7. (This question should actually appear after Section 3i.) You are given an annuity-immediate with 11 annual payments of 100 and a final balloon payment at the end of 12 years. At an annual effective interest rate of 3.5%, the present value at time 0 of all the payments is 1,000.

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Using an annual effective interest rate of 1%, calculate the present value at the beginning of the ninth year of all remaining payments. [SOA 11/89 #6]

(A) 412 (B) 419 (C) 432 (D) 439 (E) 446

8. (This question should actually appear after Section 3i.)

Kimberley is told that she can receive a 250,000 death benefit from her husband's life insurance in annual installments of 25,000 at the beginning of each year for 11 years and a final payment of 16,265 at the beginning of the 12th year.

Alternatively, Kimberley may receive annual installments of 13,000 at the beginning of each year for life, with a certain period of 10 years.

Calculate the present value of a 10-year deferred life annuity-due of one per annum at Kimberley's issue age. [SOA 11/89 #17]

(A) 8.5 (B) 9.0 (C) 9.5 (D) 10.0 (E) 10.5

9. You are given:

(i) X is the current value at the end of year two of a 20-year annuity-due of 1 per annum.
(ii) The annual effective interest rate for year t is $\frac{1}{8+t}$. Calculate X. [SOA 5/89 #6]

(A) $\sum_{t=9}^{28} \frac{10}{t}$ (B) $\sum_{t=9}^{28} \frac{11}{t}$ (C) $\sum_{t=10}^{29} \frac{10}{t}$ (D) $\sum_{t=10}^{29} \frac{11}{t}$ (E) $\sum_{t=11}^{30} \frac{10}{t}$

10. Determine an expression for $\frac{a_{\overline{5}|}}{a_{\overline{6}|}}$. [SOA 11/88 #7]

(A) $\frac{a_{\overline{2}|}+a_{\overline{3}|}}{2a_{\overline{3}|}}$ (B) $\frac{a_{\overline{2}|}+s_{\overline{3}|}}{1+a_{\overline{3}|}+s_{\overline{2}|}}$ (C) $\frac{a_{\overline{2}|}+s_{\overline{3}|}}{a_{\overline{3}|}+s_{\overline{3}|}}$ (D) $\frac{1+a_{\overline{2}|}+s_{\overline{2}|}}{a_{\overline{3}|}+s_{\overline{3}|}}$ (E) $\frac{1+a_{\overline{2}|}+s_{\overline{2}|}}{1+a_{\overline{3}|}+s_{\overline{2}|}}$

11. On January 1, an insurance company has 100,000 which is due to Linden as a life insurance death benefit. He chooses to receive the benefit annually over a period of 15 years, with the first payment immediately. The benefit he receives is based on an effective interest rate of 4% per annum.

The insurance company earns interest at an effective rate of 5% per annum. Every July 1, the company pays 100 in expenses and taxes to maintain the policy.

At the end of nine years, the company has X remaining.

Calculate X. [SOA 11/88 #16]

(A) 46,000 (B) 47,100 (C) 47,700 (D) 52,800 (E) 53,900

12. John took out a 2,000,000 construction loan, disbursed to him in three installments. The first installment of 1,000,000 is disbursed immediately and this is followed by two 500,000 installments at six month intervals.

The interest on the loan is calculated at a rate of 15% convertible semiannually and accumulated to the end of the second year. At that time, the loan and accumulated interest will be replaced by a 30-year mortgage at 12% convertible monthly.

The amount of the monthly mortgage payment for the first five years will be one-half of the payment for the sixth and later years. The first monthly mortgage payment is due exactly two years after the initial disbursement of the construction loan.

Calculate the amount of the 12th mortgage payment. [SOA 5/88 #12]

(A) 13,225 (B) 13,357 (C) 16,787 (D) 16,955 (E) 25,811

13. The proceeds from a life insurance policy are left on deposit, with interest credited at the end of each year. The beneficiary makes withdrawals from the fund at the end of each year t, for $t = 1, 2, \ldots, 10$.

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At the minimum interest rate of 3% guaranteed in the policy, the equal annual withdrawal would be 1,000. However, the insurer credits interest at the rate of 4% for the first four years and 5% for the next six years. The actual amount withdrawn at the end of year t is

$$W_t = \frac{F_t}{\ddot{a}_{\overline{11-t}|.03}}$$

where F_t is the amount of the fund, including interest, prior to the withdrawal.

Calculate W_{10}. [SOA 11/87 #11]

(A) 1,160 (B) 1,167 (C) 1,172 (D) 1,177 (E) 1,183

14. Which of the following are equal to 1?

(i) $\frac{a_{\overline{10}|}(1+is_{\overline{10}|})}{1+\ddot{s}_{\overline{9}|}}$

(ii) $v^{10}\ddot{s}_{\overline{10}|} - a_{\overline{9}|}$

(iii) $(1+i)^{10}a_{\overline{10}|} - \ddot{s}_{\overline{9}|}$

[SOA 5/87 #17]

(A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) The correct answer is not given by (A), (B), (C), or (D).

15. A company has a lease expiring on December 31, 1986. The company is notified that the monthly rent will double as of January 1, 1987. This rate will be good for two years. The company wishes to dampen the effect of the rent increase by paying a higher rent for 2 1/2 years, starting July 1, 1986.

Calculate the percentage increase on July 1, 1986 assuming an interest rate of 12% compounded monthly. [SOA 11/86 #1]

(A) 70% (B) 72% (C) 74% (D) 76% (E) 78%

16. A person age 40 wishes to accumulate a fund for retirement by depositing an amount X at the end of each year into an account paying 4% interest. At age 65, the person will use the entire account balance to purchase a 15-year 5% annuity-immediate with annual payments of $10,000.

Find X. [SOA SAMPLE/84 #1]

(A) $2,490 (B) $2,520 (C) $2,550 (D) $2,580 (E) $2,610

17. At the beginning of each year for ten years $100 is deposited into a savings account.

At a simple annual interest rate of i%, the total amount in the account is $1,275 at the end of ten years.

To the nearest $5, what would be the total amount in the account at the end of ten years if interest had been compounded at an effective annual interest rate of i%. [CAS 11/82 #1]

(A) $1,315 (B) $1,320 (C) $1,325 (D) $1,330 (E) $1,335

18. An annuity provides a payment of $n at the end of each year for n years.

The effective annual interest rate is $1/n$.

What is the present value of the annuity? [CAS 11/82 #7]

(A) $n^2\left[1-\left(\frac{n}{n+1}\right)^n\right]$

(B) $n^2\left[1+\left(\frac{n}{n+1}\right)^n\right]$

(C) $n^2 - n^{n+1}(n+1)^{-n+2}$

(D) $n^2(n+1)^{-n}$

(E) $n^2 - n^{n+1}(n+1)^{-n+1}$

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Solutions to Past Exam Questions on Sections 3a to 3c

1. If you are puzzled by the statement that "Each 1000 will provide for 9.65 of income at the beginning of each month starting on his 65th birthday until the end of his life," you are not alone. Many students don't know what this means. The statement refers to what is known as a ***life annuity***, i.e., an annuity that is paid to someone for as long as he or she lives, rather than for a fixed or certain number of periods. (Obviously, the PV of a life annuity depends on mortality rates as well as the interest rate.)

   Life annuities are covered in later actuarial exams, so it might be considered unfair to include this question in an FM/2 exam. However, there is actually enough information given in the question, if properly interpreted, for an FM/2 student to answer it.

   A life annuity is a product sold by life insurance companies. What the statement means is this: If the man pays the insurance company \$1,000 on his 65th birthday, the company will guarantee to pay him \$9.65 at the beginning of each month for as long as he lives. In this problem, the man wants to receive \$3,000 per month for life starting at age 65. If \$1,000 would guarantee \$9.65 per month, how much does he have to pay the company to guarantee \$3,000 per month?

   One way to look at this is to note that if \$1,000 guarantees \$9.65 per month, then $\frac{\$1,000}{9.65}$ guarantees \$1.00 per month. So to guarantee \$3,000 per month, we multiply this by 3,000:

   Amount to be paid at age 65 to guarantee a monthly life income of \$3,000

   $$= 3{,}000 \times \frac{\$1,000}{9.65}$$

   AV of contributions at age 65 $= X\ddot{s}_{\overline{300}|}$ at $\frac{2}{3}\%$.

   This must equal $(3{,}000)\frac{\$1,000}{9.65}$ to provide a monthly income of 3,000 for life.

   $$X = \frac{(3{,}000)\frac{\$1,000}{9.65}}{\ddot{s}_{\overline{300}|}} = \frac{310{,}880.8}{957.366577} = 324.72 \qquad \text{ANS. (A)}$$

2. The most convenient comparison date is the end of the 40th year.

   Susan: $100s_{\overline{40}|} = Xa_{\overline{15}|}$ at 8%

   $$X = \frac{25{,}905.65}{8.559479} = 3{,}026.55$$

   Jeff: $100s_{\overline{40}|} = Ya_{\overline{15}|}$ at 10%

   $$Y = \frac{44{,}259.26}{7.606080} = 5{,}818.93$$

   $$Y - X = 2{,}792.38 \qquad \text{ANS. (A)}$$

3. The amount of John's inheritance is equal to the PV of the annuity-due that he bought:

   John's inheritance $= 2{,}500\ddot{a}_{\overline{10}|.08}$

   Since Jeff and John shared equally in the inheritance, Jeff's share is also equal to $2{,}500\ddot{a}_{\overline{10}|.08}$.

   We are told that Jeff immediately invests his inheritance for two years at 9% effective, at which point his investment can buy a 15-year annuity-immediate at 8% effective with an annual payment of Z:

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$$2{,}500\ddot{a}_{\overline{10}|.08}(1.09)^2 = Za_{\overline{15}|.08}$$

$$Z = \frac{21{,}525.07}{8.55948} = 2{,}514.76 \qquad \text{ANS. (C)}$$

4. Paul withdraws $100\ddot{a}_{\overline{120}|.01} = 7{,}039.76$ at the end of years 6 to 20. P is the PV of these withdrawals at 8%.

$$P = 7{,}039.76v^5 a_{\overline{15}|} \text{ at } 8\% = (7{,}039.76)(.6806)(8.5595)$$

$$= 41{,}009.68 \qquad \text{ANS. (A)}$$

Note: We will see in Section 3d that the PV at time 0 of payments of 7,039.76 at time 6 through time 20 can also be expressed as $7{,}039.76(a_{\overline{20}|} - a_{\overline{5}|})$. (Imagine that there are 20 payments and then deduct the PV of the first 5 imaginary payments.) Of course, this gives the same numerical result.

5. $14{,}113 = 1{,}000a_{\overline{4N}|}$ at 6.3%, $4N = 36$, $N = 9$. The PV of the first 9 payments is $1{,}000a_{\overline{9}|} = 6{,}713.76$. The PV of the third set of 9 payments is $v^{18}6{,}713.76 = 2{,}235.46$. The total $PV = 14{,}113$. Thus the percentage of the total PV that we want is:

$$\frac{6{,}713.76 + 2{,}235.46}{14{,}113} = .63 \qquad \text{ANS. (C)}$$

6. Use the end of the 10th year as the comparison date. The AV of the payments is:

$$10{,}000(1.1)^{10} + 1{,}000s_{\overline{10}|10\%} = 41{,}874.80$$

This must be equal to the PV of the returns:

$$41{,}874.80 = Xa_{\overline{5}|10\%},\ X = 11{,}046 \qquad \text{ANS. (C)}$$

7. $$1{,}000 = 100a_{\overline{11}|} + Rv^{12} \text{ at } 3.5\%$$

$$R = \frac{1{,}000 - 100(9.002)}{.6618} = 150.80$$

$$PV = 100a_{\overline{3}|.01} + 150.80v^4_{.01} = 100(2.941) + (150.80)(.961) = 439 \qquad \text{ANS. (D)}$$

Notes: (1) The final payment can be computed using the BA II Plus by entering 12 [N] 3.5 [I/Y] 1,000 [PV] 100 [+/–] [PMT] [CPT] [FV]. The result is 50.87, so final payment is $100 + 50.87 = 150.87$. See section 3h for more details.

(2) The question is a little ambiguous. The PV "at the beginning of the ninth year" could be interpreted to include the payment of 100 due at time 8. The official SOA solution excluded this payment.

8. (This question should actually appear after Section 3i.)

The PV of the first annuity that Kimberly could receive must be equal to the death benefit of 250,000:

$$250{,}000 = 25{,}000\ddot{a}_{\overline{11}|} + 16{,}265v^{11}$$

To solve for i using the BA II Plus, put the calculater in [BGN] mode. (This is one of those problems that make it difficult to avoid the [BGN] mode.) Then enter 11 [N] 250,000 [PV] 25,000 [+/–] [PMT] 16,265 [+/–] [FV] [CPT] [I/Y]. (These keystrokes compute the interest rate for which the PV of 11 payments of 25,000 and a final payment of 16,265 at time 11 equals 250,000.) The result is $i = 3.0004\%$.

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Kimberly's alternative is to receive 13,000 at the beginning of each year for 10 years certain, followed by payments of 13,000 for life, i.e., what is called a 10-year deferred life annuity-due. Let X = PV of a 10-year deferred life annuity-due with annual payments of 1. (This is what the question is asking us to find.) Then we have:

$$13{,}000\left(\ddot{a}_{\overline{10}|}+X\right)=250{,}000$$

$$X=\frac{250{,}000}{13{,}000}-8.786=10.5 \qquad \text{ANS. (E)}$$

9. From (ii) we have:

$$i_t=\frac{1}{8+t},\quad 1+i_t=\frac{9+t}{8+t}$$

"The current value at the end of year two" of a 20-year annuity-due is the sum of the AV of the payments at time 0, 1, and 2, and the PV of the payments at time 3, 4, . . . , 19. Since the effective interest rate changes each year, we write the current value as follows:

$$\begin{aligned}
X&=(1+i_1)(1+i_2)+(1+i_2)+1\\
&\quad+\frac{1}{1+i_3}+\frac{1}{(1+i_3)(1+i_4)}+\cdots+\frac{1}{(1+i_3)\ldots(1+i_{19})}\\
&=\left(\frac{10}{9}\right)\left(\frac{11}{10}\right)+\frac{11}{10}+1+\frac{11}{12}+\left(\frac{11}{12}\right)\left(\frac{12}{13}\right)\\
&\quad+\cdots+\left(\frac{11}{12}\right)\left(\frac{12}{13}\right)\cdots\left(\frac{27}{28}\right)\\
&=\frac{11}{9}+\frac{11}{10}+\frac{11}{11}+\frac{11}{12}+\frac{11}{13}+\cdots+\frac{11}{28}\\
&=\sum_{t=9}^{28}\frac{11}{t} \qquad \text{ANS. (B)}
\end{aligned}$$

10. To get the format of the answer choices, express the ratio as follows:

$$\frac{a_{\overline{5}|}}{a_{\overline{6}|}}=\frac{a_{\overline{3}|}+v^3a_{\overline{2}|}}{a_{\overline{3}|}+v^3a_{\overline{3}|}}$$

Multiplying numerator and denominator by $(1+i)^3$:

$$\begin{aligned}
\frac{a_{\overline{5}|}}{a_{\overline{6}|}}&=\frac{(1+i)^3a_{\overline{3}|}+a_{\overline{2}|}}{(1+i)^3a_{\overline{3}|}+a_{\overline{3}|}}\\
&=\frac{s_{\overline{3}|}+a_{\overline{2}|}}{s_{\overline{3}|}+a_{\overline{3}|}} \qquad \text{ANS. (C)}
\end{aligned}$$

11. Annual benefit $=\frac{100{,}000}{\ddot{a}_{\overline{15}|.04}}=\frac{100{,}000}{11.563}=8{,}648.27$

$$\begin{aligned}
X&=100{,}000(1.05)^9-8{,}648.27\ddot{s}_{\overline{9}|.05}-100\ddot{s}_{\overline{9}|.05}v_{.05}^{\frac{1}{2}}\\
&=100{,}000(1.5513)-11.578[8{,}648.27+(100)(.9759)]=53{,}870 \qquad \text{ANS. (E)}
\end{aligned}$$

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12. If x = monthly mortgage payment in the first 5 years, we have (using the end of the 2nd year as the comparison date):

$$x\left(2\ddot{a}_{\overline{360}|.01}-\ddot{a}_{\overline{60}|.01}\right)=1{,}000{,}000(1.075)^4$$

$$+\,500{,}000\left(1.075^3+1.075^2\right)=2{,}534{,}430$$

$$x=\frac{2{,}534{,}430}{2\ddot{a}_{\overline{360}|}-\ddot{a}_{\overline{60}|}}=\frac{2{,}534{,}430}{(2)(98.190514)-45.404589}=16{,}787 \qquad \text{ANS. (C)}$$

13. $F_0=1{,}000a_{\overline{10}|.03}, \quad F_1=1{,}000a_{\overline{10}|.03}(1.04)$

$$W_1=\frac{1{,}000a_{\overline{10}|.03}(1.04)}{\ddot{a}_{\overline{10}|.03}}=1{,}000\left(\frac{1.04}{1.03}\right)$$

$$F_2=\left(F_1-W_1\right)1.04=1{,}000\left(\frac{1.04}{1.03}\right)\left[a_{\overline{10}|}(1.03)-1\right](1.04)$$

$$=1{,}000a_{\overline{9}|.03}\left(\frac{1.04}{1.03}\right)(1.04)$$

$$W_2=\frac{F_2}{\ddot{a}_{\overline{9}|.03}}=1{,}000\left(\frac{1.04}{1.03}\right)^2$$

$$\text{Similarly, } W_{10}=\frac{1{,}000(1.04)^4(1.05)^6}{1.03^{10}}=\frac{1{,}000(1.1699)(1.3401)}{1.3439}$$

$$=1{,}166.59 \qquad \text{ANS. (B)}$$

Note: This is a tough problem under exam conditions. It might be smart to skip it and spend time on questions that you would be more likely to succeed in doing.

14. (i) Numerator: $a_{\overline{10}|}(1+is_{\overline{10}|})=a_{\overline{10}|}[1+(1+i)^{10}-1]=s_{\overline{10}|}$

Denominator: $1+\ddot{s}_{\overline{9}|}=s_{\overline{10}|}$

Thus (i) = 1

(ii) $v^{10}\ddot{s}_{\overline{10}|}-a_{\overline{9}|}=\ddot{a}_{\overline{10}|}-a_{\overline{9}|}=1+a_{\overline{9}|}-a_{\overline{9}|}$

Thus (ii) = 1

(iii) $(1+i)^{10}a_{\overline{10}|}-\ddot{s}_{\overline{9}|}=s_{\overline{10}|}-\ddot{s}_{\overline{9}|}=s_{\overline{10}|}-(s_{\overline{10}|}-1)$

Thus (iii) = 1

15. Let x = monthly rent as originally scheduled for 1986, kx = rent effective 7/1/86 under the revised schedule. Using 7/1/86 as the comparison date:

$$kx\ddot{a}_{\overline{30}|}=x\ddot{a}_{\overline{6}|}+2xv^6\ddot{a}_{\overline{24}|} \text{ at } 1\%$$

$$k=\frac{a_{\overline{6}|}+2v^6a_{\overline{24}|}}{a_{\overline{30}|}}=\frac{5.795+2(.9420)(21.243)}{25.808}$$

$$=1.775, \text{ i.e., increase of } 77.5\% \qquad \text{ANS. (E)}$$

16. $Xs_{\overline{25}|.04}=10{,}000a_{\overline{15}|.05}, \quad X=\frac{103{,}796.58}{41.646}=2{,}492 \qquad$ ANS. (A)

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17. At simple interest rate $i\%$, the AV of the 10 deposits is:

$$100\left[\left(1+\frac{i}{100}\right)+\left(1+\frac{2i}{100}\right)+\cdots+\left(1+\frac{10i}{100}\right)\right]=1{,}275$$

$$100\left(10+\frac{55i}{100}\right)=1{,}275$$

$$i=\frac{275}{55}=5$$

At compound interest effective rate 5%:

$$AV=100\ddot{s}_{\overline{10}|.05}=1{,}320.68 \qquad \text{ANS. } \mathbf{(B)}$$

18. $i=\frac{1}{n}$, $1+i=\frac{n+1}{n}$, $v=\frac{n}{n+1}$

$$PV=na_{\overline{n}|} \text{ at rate } i=\frac{1}{n}$$

$$=n\left(\frac{1-v^n}{i}\right)=n\frac{\left[1-\left(\frac{n}{n+1}\right)^n\right]}{\frac{1}{n}}$$

$$=n^2\left[1-\left(\frac{n}{n+1}\right)^n\right] \qquad \text{ANS. } \mathbf{(A)}$$

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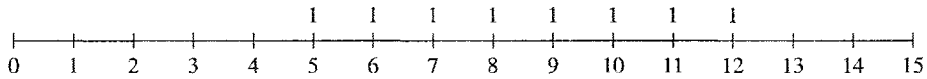
§ 3d. Deferred Annuities (or Playing "Now you see it . . . ")

Consider the following series of payments:



Let's determine the value of these 8 payments as of various dates:

As of times 4, 5, 12, and 13, clearly the values can be expressed using symbols we defined earlier in this section:

$t = 4 \quad a_{\overline{8}|}$

$t = 5 \quad \ddot{a}_{\overline{8}|}$

$t = 12 \quad s_{\overline{8}|}$

$t = 13 \quad \ddot{s}_{\overline{8}|}$

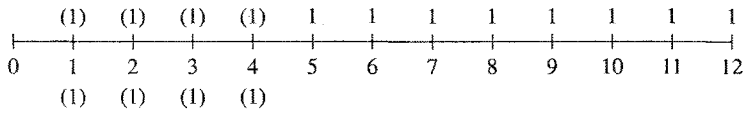
As of $t = 0$, we have a number of choices. We could discount any of the 4 values above back to time 0:

$$PV = v^4 a_{\overline{8}|} = v^5 \ddot{a}_{\overline{8}|} = v^{12} s_{\overline{8}|} = v^{13} \ddot{s}_{\overline{8}|}$$

There is a special symbol for this kind of ***deferred annuity***: ${}_{4|}a_{\overline{8}|}$. We can interpret this symbol as follows: "Go to time 4. Pay what the symbol to the right of the vertical line says—in this case, an annuity-immediate for 8 years. Thus the payments run from $t = 5$ to $t = 12$."

How would you write this PV as a deferred annuity-due? Answer: ${}_{5|}\ddot{a}_{\overline{8}|}$. ("Go to time 5. Start paying what the symbol to the right says," etc.)

We could also determine the present value by playing "Now you see it . . . ." Place ***four*** fictional payments on the diagram and immediately withdraw them:



The PV of the payments above the time line (including the four fictitious ones) is $a_{\overline{12}|}$. But this includes the PV of the fictitious payments, which is $a_{\overline{4}|}$. Thus, the PV of this deferred annuity is $a_{\overline{12}|} - a_{\overline{4}|}$.

The two most common ways to evaluate ${}_{4|}a_{\overline{8}|}$ are:

$${}_{4|}a_{\overline{8}|} = v^4 a_{\overline{8}|} = a_{\overline{12}|} - a_{\overline{4}|}$$

To generalize the symbol for a deferred annuity, consider an m-year deferred n-payment annuity-immediate. (Note that the first payment is at time $m + 1$.) This annuity could also be described as an $(m + 1)$-year deferred n-payment annuity-due. We have:

$${}_{m|}a_{\overline{n}|} = {}_{m+1|}\ddot{a}_{\overline{n}|} = v^m a_{\overline{n}|} = v^{m+1} \ddot{a}_{\overline{n}|}$$
$$= a_{\overline{m+n}|} - a_{\overline{m}|}$$

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Consider the AV of the payments as of time 15. The two most common methods for evaluating this AV are:

$$AV = (1+i)^3 s_{\overline{8}|} = s_{\overline{11}|} - s_{\overline{3}|}$$

where the second method is based on playing "now you see it . . . ."

There is no special symbol for this AV.

Now consider the value of the same payments as of time 7. We could express it as any of the previous values accumulated or discounted to time 6. Perhaps the simplest expression is:

$$s_{\overline{3}|} + a_{\overline{5}|}$$

This value could also be expressed as $(1+i)^3 a_{\overline{8}|}$ and many other forms.

Stepping Stones

EXAMPLE 1

Evaluate the following at $i = 3.25\%$:

$${}_{10|}a_{\overline{20}|} = ?$$

$${}_{20|}\ddot{a}_{\overline{25}|} = ?$$

SOLUTION

For ${}_{10|}a_{\overline{20}|}$, use the "Now you see it . . . " approach: ${}_{10|}a_{\overline{20}|} = a_{\overline{30}|} - a_{\overline{10}|}$. First, determine $a_{\overline{10}|}$, store it, then determine $a_{\overline{30}|}$, recall $a_{\overline{10}|}$ and subtract it:

10 [N] 3.25 [I/Y] 1 [+/-] [PMT] [CPT] [PV] [STO] 0 $\quad (a_{\overline{10}|} = 8.422395)$

We determine $a_{\overline{30}|}$ simply by changing [N] to 30: 30 [N] [CPT] [PV] $\quad (a_{\overline{30}|} = 18.981917)$

Press the minus key, then [RCL] 0 [=]. The answer is 10.559522.

For ${}_{20|}\ddot{a}_{\overline{25}|}$, to avoid using the BGN mode, we can write ${}_{20|}\ddot{a}_{\overline{25}|} = {}_{19}a_{|\overline{25}|} = a_{\overline{44}|} - a_{\overline{19}|}$. Using similar keystrokes as before, you should get 9.224598.

EXAMPLE 2

A deposit of \$1,500 is made on January 1 of every year from 2010 to 2021, inclusive. Determine the accumulated value of these deposits on January 1, 2030 if the effective rate of interest is 1.75% per annum.

SOLUTION

There are 12 deposits (not 11). The comparison date is 9 years after the last deposit.

$$AV = s_{\overline{21}|} - s_{\overline{9}|}$$

Using keystrokes similar to Example 1, but with −1,500 in [PMT] and the [FV] key instead of the [PV] key, you should get $AV = 23{,}189.97$.

EXAMPLE 3

Which of the following is *not* a correct expression for the present value, on January 1, 2018, of 1 paid every January 1 from 2025 to 2035, inclusive?

(A) ${}_{6|}a_{\overline{11}|}$

(B) $\ddot{a}_{\overline{18}|} - \ddot{a}_{\overline{7}|}$

(C) ${}_{7|}\ddot{a}_{\overline{11}|}$

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(D) $a_{\overline{17}|} - a_{\overline{6}|}$

(E) $v^7 a_{\overline{11}|}$

SOLUTION

(A), (B), (C), and (D) are all correct expressions for this PV. (E) is incorrect: It should be $v^6 a_{\overline{11}|}$.

EXAMPLE 4

You are given that:

(i) $a_{\overline{n}|} = 10$

(ii) The present value of an n-year deferred, n-year annuity-immediate is 5.

Determine i.

SOLUTION

From (ii), $v^n a_{\overline{n}|} = 5$. Since $a_{\overline{n}|} = 10$, we have $v^n = 5/10 = 0.5$. Now $a_{\overline{n}|} = (1 - v^n)/i$. Substitute 0.5 for v^n:

$$a_{\overline{n}|} = 10 = (1 - v^n)/i = (1 - 0.5)/i$$

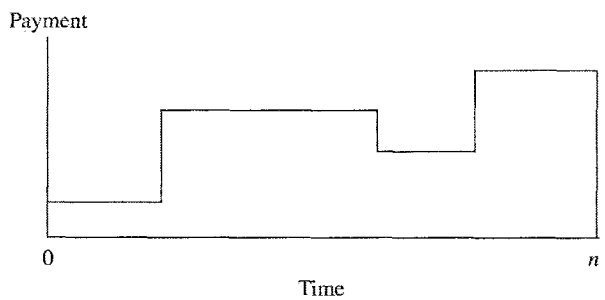
$$10i = 1 - 0.5 = 0.5$$

$$i = 5\%$$

§ 3e. A Short-Cut Method for Annuities with "Block" Payments

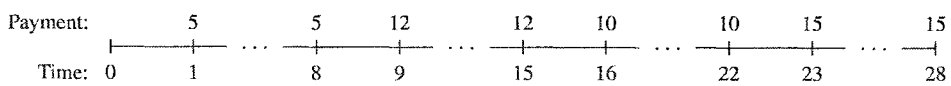
Shortcut Alert!

This section shows how to rapidly write down the value of an annuity with "block" payments; i.e., an annuity which has payments which could be sketched this way:



In other words, payments are level for a period of years, then change to another level for another period of years, etc.

For example, consider the following annuity with block payments: \$5 for the first 8 years, \$12 for the next 7 years, \$10 for the next 7 years and \$15 for the next 6 years:



We will learn to immediately write down the PV (or AV) of an annuity like this directly from the payments.

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The Long Way

Before we learn the shortcut method, let's do it by a more deliberate and slower method. We will write the PV "block-by-block."

PV of first 8 payments:	$5a_{\overline{8}\rceil}$
PV of next 7 payments:	$12(a_{\overline{15}\rceil} - a_{\overline{8}\rceil})$
PV of next 7 payments:	$10(a_{\overline{22}\rceil} - a_{\overline{15}\rceil})$
PV of next 6 payments:	$15(a_{\overline{28}\rceil} - a_{\overline{22}\rceil})$

Adding up all the present values, combining terms and writing in descending order of the periods:

$$PV = 15a_{\overline{28}\rceil} - 5a_{\overline{22}\rceil} + 2a_{\overline{15}\rceil} - 7a_{\overline{8}\rceil}$$

The Short Way

The rules are: (1) Start with the payment furthest from the comparison date; (2) make adjustments (plus or minus) as you move in closer to the comparison date.

(The following description may seem longer than the "long way" but the fact is that once you practice a little, you can look at the block payments and write the PV in simplest form in 10 to 15 seconds.)

We want the PV, so the comparison date is time 0. We start with the furthest payment (\$15 at time 28) and immediately write $15a_{\overline{28}\rceil}$. We move in closer from time 28 toward time 0 until there is a change. This occurs at time 22, when payments *decrease by \$5* (from \$15 to \$10), so we write $-5a_{\overline{22}\rceil}$. We move in closer, see another change at $t = 15$, when payments *increase by \$2* (from \$10 to \$12) so we write $+2a_{\overline{15}\rceil}$. Finally, the last change is at time 8, a *decrease of \$7* (from \$12 to \$5), so we write $-7a_{\overline{8}\rceil}$.

Putting all of this together, we have:

$$PV = 15a_{\overline{28}\rceil} - 5a_{\overline{22}\rceil} + 2a_{\overline{15}\rceil} - 7a_{\overline{8}\rceil}$$

What we have just done is to start with a level annuity of \$15 per year and then make successive adjustments to the \$15 payments to get the payments we want.

Note that this is identical to the answer obtained by "the long way."

Stepping Stones

EXAMPLE 1

Write down directly in simplest annuity form the PV of the following payments:

Time	Payment
1 to 10	\$5
11 to 18	\$8
19 to 23	\$12
24 to 30	\$20

SOLUTION

Starting with the payment of \$20 at time 30 and making the successive adjustments as we move closer in to time 0, we obtain

$$PV = 20a_{\overline{30}\rceil} - 8a_{\overline{23}\rceil} - 4a_{\overline{18}\rceil} - 3a_{\overline{10}\rceil}$$

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Accumulated Value

The AV of annuities with block payments is obtained in much the same way as the PV. For example, consider the annuity just above. The comparison date is time 30 if we want the AV on the date of the last payment, so we start with the *furthest payment* from time 30, which is \$5 at time 1. We immediately write $5s_{\overline{30|}}$. As we move toward the comparison date of time 30, we see that the first change is an *increase of \$3* (from \$5 to \$8) at time 11, so our adjustment term is $+3s_{\overline{20|}}$. (20 is the number of payments that we must increase by \$3, i.e., the payments from time 11 to time 30, inclusive, or 30−10 payments.) The next change is an *increase of \$4* (from \$8 to \$12) at time 19, so the adjustment term is $+4s_{\overline{12|}}$. Finally, payments *increase by \$8* (from \$12 to \$20) at time 24, so the adjustment term is $+8s_{\overline{7|}}$. Putting it all together:

$$AV = 5s_{\overline{30|}} + 3s_{\overline{20|}} + 4s_{\overline{12|}} + 8s_{\overline{7|}}$$

Stepping Stones

EXAMPLE 2

Write the present value at time 0 and the accumulated value on the date of the last payment for the following annuities:

Annuity #1		Annuity #2	
Time	Payment	Time	Payment
1 to 5	5	1 to 10	8
6 to 12	10	11 to 15	0
13 to 18	5	16 to 22	10
19 to 27	10	23 to 25	12

SOLUTION

Annuity #1:

$$PV = 10a_{\overline{27|}} - 5a_{\overline{18|}} + 5a_{\overline{12|}} - 5a_{\overline{5|}}$$

$$AV = 5s_{\overline{27|}} + 5s_{\overline{22|}} - 5s_{\overline{15|}} + 5s_{\overline{9|}}$$

Annuity #2:

$$PV = 12a_{\overline{25|}} - 2a_{\overline{22|}} - 10a_{\overline{15|}} + 8a_{\overline{10|}}$$

$$AV = 8s_{\overline{25|}} - 8s_{\overline{15|}} + 10s_{\overline{10|}} + 2s_{\overline{3|}}$$

EXAMPLE 3

Two annuities have the following payments:

	Payment	
Time	Annuity #1	Annuity #2
1 to 5	5	0
6 to 10	0	5
11 to 15	5	0
16 to 20	0	5

The present value of Annuity #1 is 42.127. The present value of Annuity #2 is 37.692. Determine i.

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SOLUTION

$$\text{PV of \#1} = 5a_{\overline{15}|} - 5a_{\overline{10}|} + 5a_{\overline{5}|} = 42.127$$

$$\text{PV of \#2} = 5a_{\overline{20}|} - 5a_{\overline{15}|} + 5a_{\overline{10}|} - 5a_{\overline{5}|} = 37.692$$

If we add these two PVs, we get:

$$5a_{\overline{20}|} = 79.819 \quad \text{and} \quad a_{\overline{20}|} = 15.9638$$

Using the calculator to determine i, we get $i = 2.25\%$ (to 2 decimals).

EXAMPLE 4

Which of the following is a correct expression for the accumulated value, on the date of the last payment, of 3 paid in years 1 to 7, 8 paid in years 8 to 12 and 5 paid in years 13 to 18?

(A) $3s_{\overline{18}|} + 5s_{\overline{10}|} - 3s_{\overline{5}|}$

(B) $3s_{\overline{7}|}(1+i)^{10} + 8s_{\overline{5}|}(1+i)^5 + 5s_{\overline{5}|}$

(C) $3s_{\overline{18}|} + 8s_{\overline{11}|} + 5s_{\overline{6}|}$

(D) $3s_{\overline{18}|} + 5s_{\overline{11}|} - 3s_{\overline{5}|}$

(E) $3s_{\overline{7}|}(1+i)^{11} + 8s_{\overline{5}|}(1+i)^6 + 5s_{\overline{6}|}$

SOLUTION

Answer (E). Using the "block payment" approach, a correct answer would also be $3s_{\overline{18}|} + 5s_{\overline{11}|} - 3s_{\overline{6}|}$.

§ 3f. Perpetuities

Perpetuity-Immediate

Consider an annuity-immediate with annual payments of \$1.00 where the payments continue forever. Such an annuity is called a ***perpetuity-immediate***. At first glance, it might appear that the PV of this annuity is infinite. But at positive rates of interest, the PV has a limit:

$$\lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty}\left(\frac{1-v^n}{i}\right) = \frac{1}{i}$$

since $v < 1$ and $\lim\limits_{n\to\infty} v^n = 0$ when $i > 0$.

The $\lim\limits_{n\to\infty} a_{\overline{n}|}$ is usually written as $a_{\overline{\infty}|}$.

We could also have determined the limit by summing the infinite geometric progression:

$$a_{\overline{\infty}|} = v + v^2 + \cdots$$

$$= \frac{v}{1-v} = \frac{v}{d} = \frac{v}{iv} = \frac{1}{i}$$

Still another way to determine the limit is to use a verbal explanation ("general reasoning"), as follows: I deposit \$1.00 in a bank crediting interest at i effective. It is clear that I could withdraw i at the end of each year forever and always keep the original deposit intact. Thus, \$1.00 deposited today provides for a perpetuity-immediate with annual payments of i. By proportion, a deposit of $\frac{1}{i}$ provides a perpetuity-immediate with annual payments of \$1.00, i.e.:

$$a_{\overline{\infty}|} = \frac{1}{i}$$

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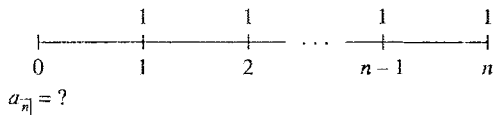
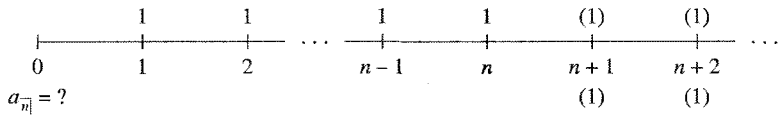
Incidentally, we can use the PV of a perpetuity to derive the formula $a_{\overline{n}|} = \frac{1-v^n}{i}$ by playing "Now you see it . . . ." We would like to determine the PV of the following payments:



Let's put fictitious payments at time $(n+1)$, $(n+2)$, ... forever, and immediately withdraw them:



The payments above the time line (including the fictitious payments) form a perpetuity-immediate, with PV$=\frac{1}{i}$. But the PV overstates $a_{\overline{n}|}$ by the PV of the fictitious payments, which form an n-year deferred perpetuity-immediate. The PV of this deferred perpetuity is $v^n\left(\frac{1}{i}\right)$. Thus we have:

$$a_{\overline{n}|} = \frac{1}{i} - v^n\left(\frac{1}{i}\right)$$
$$= \frac{1-v^n}{i}$$

Perpetuity-Due

A perpetuity-due of \$1.00 per annum begins with a payment of \$1.00 today and continues forever. Thus,

$$\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \lim_{n\to\infty}\left(\frac{1-v^n}{d}\right) = \frac{1}{d}$$

We could also sum the progression:

$$\ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \cdots$$
$$= \frac{1}{1-v} = \frac{1}{d}$$

Another approach is to see that a perpetuity-due provides \$1.00 today plus exactly the same payments as a perpetuity-immediate. We have:

$$\ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|} = 1 + \frac{1}{i} = \frac{1+i}{i} = \frac{1}{d}$$

since $d = \frac{i}{1+i}$

Since $\ddot{a}_{\overline{\infty}|}$ exceeds $a_{\overline{\infty}|}$ by the first payment of \$1.00, we can put it this way:

$$\ddot{a}_{\overline{\infty}|} - a_{\overline{\infty}|} = 1$$
$$\frac{1}{d} - \frac{1}{i} = 1$$

We covered this identity in Section 1a(vii). Now you can give it a verbal explanation.

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Stepping Stones

EXAMPLE 1

The present value of a perpetuity-due with annual payments of 1,000 is 39,610.04. Determine i.

SOLUTION

$$PV = 1{,}000\ddot{a}_{\overline{\infty}|} = 1{,}000/d = 39{,}610.04$$
$$d = .025246$$
$$i = d/(1-d) = 2.59\% \quad \text{(to 2 decimals)}$$

Alternatively:

$$PV = 1{,}000(1 + a_{\overline{\infty}|}) = 1{,}000(1 + 1/i) = 39{,}610.04$$
$$i = 2.59\%$$

EXAMPLE 2

Determine the present value of a 10-year deferred perpetuity-immediate with semiannual payments of 1 at a nominal rate of interest of 1.6% compounded semiannually.

SOLUTION

The effective semiannual rate of interest is 0.8%. You should interpret "a 10-year deferred perpetuity-immediate with semiannual payments" as meaning that the first payment is in $10\frac{1}{2}$ years, or in 21 half-year interest periods.

$$PV = v^{20}/.008, \quad \text{where } v^{20} = 1.008^{-20}$$
$$PV = 106.59$$

EXAMPLE 3

A perpetuity-immediate pays 100 in the first 10 years, 150 in the next 10 years and 200 in all subsequent years. Determine the present value of these payments at $i = 3.85\%$.

SOLUTION

We can use the "block payment" approach with this perpetuity:

$$PV = 200/i - 50a_{\overline{20}|} - 50a_{\overline{10}|}$$

Using the calculator, you should get 4,097.58.

EXAMPLE 4

A perpetuity-immediate of 1 per annum has the same present value as a 10-year annuity-immediate of 4 per annum, at an effective annual rate of interest i. Determine i.

SOLUTION

$$a_{\overline{\infty}|} = 1/i = 4a_{\overline{10}|} = 4(1 - v^{10})/i$$
$$1 = 4 - 4v^{10}$$
$$v^{10} = 0.75$$
$$i = 2.92\% \quad \text{(2 decimals)}$$

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Summary of Concepts and Formulas in Sections 3d to 3f

(1) The PV of an m-year deferred annuity-immediate with n payments is:

$$_{m|}a_{\overline{n|}} = v^m a_{\overline{n|}} = a_{\overline{m+n|}} - a_{\overline{m|}}$$

(2) The symbol for the deferred annuity in (1) can also be written as:

$$_{m+1|}\ddot{a}_{\overline{n|}}$$

(3) The PV of an annuity with "block payments" can be written very quickly by starting with the payments farthest from the comparison date and making adjustments (plus or minus) as you move closer in. (See Section 3e for examples.)

(4) (a) The PV of a perpetuity-immediate is:

$$a_{\overline{\infty|}} = \frac{1}{i}$$

(b) The PV of a perpetuity-due is:

$$\ddot{a}_{\overline{\infty|}} = \frac{1}{d}$$

(c) The PV of a perpetuity-due exceeds the PV of a perpetuity-immediate by the payment of 1 at time 0:

$$\frac{1}{d} - \frac{1}{i} = 1$$

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Past Exam Questions on Sections 3d to 3f

1. To accumulate 8000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first n years and 196 at the end of each of the next $2n$ years.

   The annual effective rate of interest is i. You are given $(1+i)^n = 2.0$.

   Determine i. [11/01 #12]

   (A) 11.25% (B) 11.75% (C) 12.25% (D) 12.75% (E) 13.25%

2. A perpetuity-immediate pays X per year. Brian receives the first n payments, Colleen receives the next n payments, and Jeff receives the remaining payments. Brian's share of the present value of the original perpetuity is 40%, and Jeff's share is K.

   Calculate K. [5/01 #5]

   (A) 24% (B) 28% (C) 32% (D) 36% (E) 40%

3. At an annual effective interest rate of i, $i > 0\%$, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 6, is 32.

   At the same annual effective rate of i, the present value of a perpetuity-immediate paying 1 at the end of each 4-month period is X.

   Calculate X. [5/01 #17]

   (A) 38.8 (B) 39.8 (C) 40.8 (D) 41.8 (E) 42.8

4. The present values of the following three annuities are equal:
   (i) perpetuity-immediate paying 1 each year, calculated at an annual effective interest rate of 7.25%
   (ii) 50-year annuity-immediate paying 1 each year, calculated at an annual effective interest rate of $j\%$
   (iii) n-year annuity-immediate paying 1 each year, calculated at an annual effective interest rate of $j - 1\%$

   Calculate n. [5/01 #50]

   (A) 30 (B) 33 (C) 36 (D) 39 (E) 42

5. Chuck needs to purchase an item in 10 years. The item costs 200 today, but its price inflates at 4% per year.

   To finance the purchase, Chuck deposits 20 into an account at the beginning of each year for 6 years. He deposits an additional X at the beginning of years 4, 5 and 6 to meet his goal.

   The annual effective interest rate is 10%.

   Calculate X. [11/00 #38]

   (A) 7.4 (B) 7.9 (C) 8.4 (D) 8.9 (E) 9.4

6. Jim began saving money for his retirement by making monthly deposits of 200 into a fund earning 6% interest compounded monthly. The first deposit occurred on January 1, 1985.

   Jim became unemployed and missed making deposits 60 through 72. He then continued making monthly deposits of 200.

   How much did Jim accumulate in his fund on December 31, 1999? [5/00 #47]

   (A) 53,572 (B) 53,715 (C) 53,840 (D) 53,966 (E) 54,184

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7. A loan is to be repaid by annual payments continuing forever, the first one due one year after the loan is made. Find the amount of the loan if the payments are 1, 2, 3, 1, 2, 3, . . . assuming an annual effective interest rate of 10%. [CAS 11/99 #5]

(A) Less than 19
(B) At least 19, but less than 20
(C) At least 20, but less than 21
(D) At least 21, but less than 22
(E) At least 22

8. Ms. Smith has two grandchildren, Adam and Evelyn. Adam will be enrolling in college on September 1, 2003, and Evelyn will be enrolling in college on September 1, 2005. Ms. Smith wishes to give both Adam and Evelyn \$1,000 at the beginning of each of their four years of college.

Ms. Smith will fund these payments by making five level annual deposits of P into an account earning an annual effective interest rate of 7%, with the first deposit on September 1, 1998.

Determine the value of P. [CAS 5/99 #5]

(A) Less than \$1,050
(B) At least \$1,050, but less than \$1,150
(C) At least \$1,150, but less than \$1,250
(D) At least \$1,250, but less than \$1,350
(E) At least \$1,350

9. A loan of amount $a_{\overline{10}|}$, made at time $t = 0$, is to be repaid by 10 annual payments of 1, beginning at time $t = 1$ and ending at time $t = 10$. At time $t = 4$, the borrower has financial troubles and can only pay $(1 - v^7)$. If he then returns to his original payment schedule of 1 at times $t = 5$ through $t = 9$, how much will his payment at $t = 10$ need to be in order to pay the loan off in full? [CAS 5/99 #6]

(A) $1 + v$ (B) $1 + v^2$ (C) $v + v^2$ (D) $1 + i$ (E) $1 + i^2$

10. Consider an annuity that pays 1 at the beginning of each year for $k + m$ years.

Which of the following expressions does *not* give the value of this annuity at the end of year k? [CAS 5/98 #10]

(A) $a_{\overline{k+m}|}(1+i)^{k+1}$ (B) $s_{\overline{k+m}|}v^m$ (C) $s_{\overline{k+1}|} + a_{\overline{m-1}|}$ (D) $\ddot{s}_{\overline{k}|} + \ddot{a}_{\overline{m}|}$ (E) $1 + \ddot{s}_{\overline{k}|} + a_{\overline{m-1}|}$

11. Eloise plans to accumulate 100,000 at the end of 42 years. She makes the following deposits:

(i) X at the beginning of years 1–14;
(ii) No deposits at the beginning of years 15–32, and
(iii) Y at the beginning of years 33–42.

The annual effective interest rate is 7%.

$$X - Y = 100.$$

Calculate Y. [SOA 5/98 #7]

(A) 479 (B) 499 (C) 519 (D) 539 (E) 559

12. Jim borrowed 10,000 from Bank X at an annual effective rate of 8%. He agreed to repay the bank with five level annual installments at the end of each year.

At the same time, he also borrowed 15,000 from Bank Y at an annual effective rate of 7.5%. He agreed to repay this loan with five level annual installments at the end of each year.

He lent the 25,000 to Wayne immediately in exchange for four annual level repayments at the end of each year, at an annual effective rate of 8.5%.

Jim can only reinvest the proceeds at an annual effective rate of 6%.

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Immediately after repaying the loans to the banks in full, determine how much Jim has left. [SOA 5/98 #11]

(A) 323 (B) 348 (C) 373 (D) 398 (E) 423

13. Mary deposits 1000 into a fund at the beginning of each year for 10 years. At the end of 15 years, she makes an additional deposit of X.

At the end of 20 years, Mary uses the accumulated balance in the fund to buy a perpetuity-immediate with annual payments of 2000 per year for 10 years, and 1000 per year thereafter.

Interest is credited at an annual effective rate of 5%.

Calculate X. [SOA 5/95 #4]

(A) 4865 (B) 5065 (C) 5265 (D) 5465 (E) 5665

14. Mark receives 500,000 at his retirement. He invests $500{,}000 - X$ in an annual payment 10-year annuity-immediate and X in an annual payment perpetuity-immediate.

His total annual payments received during the first 10 years are twice as large as those received thereafter.

The annual effective rate of interest is 6%.

Calculate X. [SOA 11/93 #5]

(A) 345,835 (B) 346,335 (C) 346,835 (D) 347,335 (E) 348,835

15. Janet buys a \$20,000 car. Prevailing market rates are nominal 8.0% annual interest, convertible monthly. The dealership offers her the choice of a rebate upon purchase of the car for cash, or alternately Janet can make no down payment and 60 monthly payments based on a nominal 2.5% annual interest, convertible monthly. The first payment would be due one month after the purchase of the car. The amount of the rebate is set so that the dealership is indifferent as to whether Janet takes the rebate or finances the car at the offered below market interest rate.

Determine the amount of the rebate. [CAS 5/93 #13]

(A) Less than \$2,450
(B) At least \$2,450 but less than \$2,550
(C) At least \$2,550 but less than \$2,650
(D) At least \$2,650 but less than \$2,750
(E) At least \$2,750

16. Ralph buys a perpetuity-due paying 500 annually. He deposits the payments into a savings account earning interest at an effective annual rate of 10%.

Ten years later, before receiving the eleventh payment, Ralph sells the perpetuity based on an effective annual interest rate of 10%.

Using the proceeds from the sale plus the money in the savings account, Ralph purchases an annuity due paying X per year for 20 years at an effective annual rate of 10%.

Calculate X. [SOA 11/92 #4]

(A) 1145 (B) 1260 (C) 1385 (D) 1525 (E) 1675

17. At a nominal rate of interest i, convertible semiannually, the present value of a series of payments of 1 at the end of every 2 years, forever, is 5.89.

Calculate i. [SOA 5/91 #1]

(A) 6% (B) 7% (C) 8% (D) 9% (E) 10%

18. The following three series of payments have the same present value of P:

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(i) a perpetuity-immediate of 2 per year at an annual effective interest rate of i;
(ii) a 20-year annuity-immediate of X per year at an annual effective interest rate of $2i$; and
(iii) a 20-year annuity-due of $0.96154X$ per year at an annual effective interest rate of $2i$.

Calculate P. [SOA 5/91 #7]

(A) 80 (B) 85 (C) 90 (D) 95 (E) 100

19. Deposits of 1000 are placed into a fund at the beginning of each year for 30 years.

At the end of the 40th year, annual payments commence and continue forever.

Interest is at an effective annual rate of 5%.

Calculate the annual payment. [SOA 5/91 #10]

(A) 5440 (B) 5430 (C) 5420 (D) 5410 (E) 5400

20. The death benefit on a life insurance policy can be paid in any of the following ways, each of which has the same present value as the death benefit:

(i) a perpetuity of 120 at the end of each month;
(ii) 365.47 at the end of each month for n years; and
(iii) a payment of 17866.32 at the end of n years.

Calculate the amount of the death benefit. [SOA 5/91 #17]

(A) 8000 (B) 9000 (C) 10000 (D) 12000 (E) 15000

21. Victor wants to purchase a perpetuity paying 100 per year with the first payment due at the end of year 11. He can purchase it by either:

(i) paying 90 per year at the end of each year for 10 years; or
(ii) paying K per year at the end of each year for the first 5 years and nothing for the next 5 years.

Calculate K. [SOA 11/90 #8]

(A) 150 (B) 160 (C) 170 (D) 175 (E) 180

22. You are given:

(i) the present value of an annuity-due that pays 300 every 6 months during the first 15 years and 200 every 6 months during the second 15 years is 6000;
(ii) the present value of a 15-year deferred annuity-due that pays 350 every 6 months for 15 years is 4000; and
(iii) the present value of an annuity-due that pays 100 every 6 months during the first 15 years and 200 every 6 months during the next 15 years is X.

Determine X. [SOA 11/90 #20]

(A) 3220 (B) 3320 (C) 3420 (D) 3520 (E) 3620

23. An annuity-immediate pays 10 at the ends of years 1 and 2, 9 at the ends of years 3 and 4, etc., with payments decreasing by 1 every second year, until nothing is paid. The effective annual rate of interest is 5%.

Calculate the present value of this annuity-immediate. [SOA 5/90 #8]

(A) 71 (B) 78 (C) 84 (D) 88 (E) 94

24. The present value of a series of payments of 2 at the end of every eight years, forever, is equal to 5.

Calculate the effective rate of interest. [SOA 11/89 #4]

(A) 0.023 (B) 0.033 (C) 0.040 (D) 0.043 (E) 0.052

25. Tom borrows 100 at an annual effective interest rate of 4% and agrees to repay it with 30 annual installments. The amount of each payment in the last 20 years is set at twice that in the first 10 years.

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At the end of 10 years, Tom has the option to repay the entire loan with a final payment X, in addition to the regular payment. This will yield the lender an annual effective rate of 4.5% over the 10-year period.

Calculate X. [SOA 5/89 #12]

(A) 89 (B) 94 (C) 99 (D) 104 (E) 109

26. A perpetuity pays 1 at the end of every year plus an additional 1 at the end of every second year.

The present value of the perpetuity is K for $i > 0$.

Determine K. [SOA 5/86 #9]

(A) $\frac{i+3}{i(i+2)}$ (B) $\frac{i+2}{i(i+1)}$ (C) $\frac{i+1}{i^2}$ (D) $\frac{3}{2i}$ (E) $\frac{i+1}{i(i+2)}$

27. Annuities X and Y provide the following payments:

End of Year	Annuity X	Annuity Y
1–10	1	K
11–20	2	0
21–30	1	K

Annuities X and Y have equal present values at an effective annual interest rate i such that $v^{10} = 1/2$.

Determine K. [SOA SAMPLE/83 #2]

(A) 4/3 (B) 3/2 (C) 5/3 (D) 7/4 (E) 9/5

28. A perpetuity provides the following payments:

(I) Level payments of \$1 at the end of each year for the first $2n$ years.
(II) Level payments of \$3 at the end of each year thereafter.

At the beginning of year 1 the present value of the payments specified in I equals the present value of the payments specified in II.

The effective annual interest rate is i.

What is $(1+i)^n$? [CAS 5/83 #1]

(A) 4/3 (B) 2 (C) 3/2 (D) 3 (E) 4

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Solutions to Past Exam Questions on Sections 3d to 3f

1. Using the "block payments" approach:

$$98s_{\overline{3n}|} + 98s_{\overline{2n}|} = 8{,}000$$

$$\frac{(1+i)^{3n}-1}{i} + \frac{(1+i)^{2n}-1}{i} = 81.63$$

We are given that $(1+i)^n = 2$:

$$\frac{8-1}{i} + \frac{4-1}{i} = 81.63, \quad i = 12.25\% \qquad \text{ANS. (C)}$$

2. PV of the perpetuity $= \dfrac{X}{i}$

Brian's share $= Xa_{\overline{n}|} = .4\dfrac{X}{i}$

Colleen's share $= v^n X a_{\overline{n}|}$

Jeff's share $= v^{2n}\dfrac{X}{i}$

From Brian's share we have:

$$a_{\overline{n}|} = \frac{.4}{i} = \frac{1-v^n}{i}$$

$$\therefore\ v^n = .6,\ v^{2n} = .36$$

Thus Jeff's share $= .36\dfrac{X}{i}$ ANS. (D)

3. 1st perpetuity: $PV = 10(v^6 + v^9 + \cdots)$

$$= \frac{10v^6}{1-v^3} = 32$$

Letting $x = v^3$, this reduces to the quadratic $5x^2 + 16x - 16 = 0$, which has the positive root $x = v^3 = .8$.

2nd perpetuity: $X = v^{\frac{1}{3}} + v^{\frac{2}{3}} + \cdots$

$$= \frac{v^{\frac{1}{3}}}{1-v^{\frac{1}{3}}} = \frac{.8^{\frac{1}{9}}}{1-.8^{\frac{1}{9}}} = 39.8 \qquad \text{ANS. (B)}$$

4. $\dfrac{1}{.0725} = a_{\overline{50}|j} = a_{\overline{n}|j-1\%}$

$13.793103 = a_{\overline{50}|j}$, $\therefore\ j = 7.004382\%$ and $a_{\overline{n}|6.004382\%} = 13.793103$

$n = 30.2$ ANS. (A)

5. The AV of Chuck's deposits at time 6 is $20\ddot{s}_{\overline{6}|} + X\ddot{s}_{\overline{3}|}$. The price of the item 10 years from now is $200(1.04)^{10}$. Using time 10 as the comparison date:

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$$\left(20\ddot{s}_{\overline{6}|} + X\ddot{s}_{\overline{3}|}\right)(1.10)^4 = 200(1.04)^{10}$$

$$169.74 + X(3.641) = 202.20535$$

$$3.641X = 32.46, \quad X = 8.92 \qquad \text{ANS. (D)}$$

6. In solving this problem, we have to carefully count the number of monthly periods using the given dates. One way to do this is as follows.

   Let the first monthly deposit on 1/1/1985 be numbered as being made at $t = 1$. The deposits are made over a 15-year period, from 1985 through 1999 inclusive. (This includes the period of the missed deposits.) Under this numbering scheme, you will find that the date 12/31/1999 is at $t = 181$. (Note that the date 1/1/2000, which is one day later, is also at $t = 181$.) We want the AV at $t = 181$, just before the deposit on 1/1/2000 is made.

   As of t = 181, the first 59 deposits have an AV of $200s_{\overline{59}|}(1.005)^{122}$ and the AV of the 108 deposits made at $t = 73$ through $t = 180$ inclusive is $200\ddot{s}_{\overline{108}|} = 200s_{\overline{108}|}(1.005)$. The sum of these AVs is 53,839.83. ANS. (C)

7. As of the end of the 3rd year, the 3 payments 1, 2, 3 accumulate to $1.1^2 + 2(1.1) + 3 = 6.41$. So a payment of 6.41 at time 3 is equivalent to these 3 payments. Similarly, a payment of 6.41 at times 6, 9, 12, . . . , covers all of the payments of the perpetuity.

$$PV = 6.41\left(v^3 + v^6 + \cdots\right) = 6.41\frac{v^3}{1 - v^3}$$

$$= \frac{(6.41)(.751315)}{1 - .751315} = 19.37 \qquad \text{ANS. (B)}$$

8. The deposits are made on 9/1/98 to 9/1/02, inclusive, so the first withdrawal for Adam will be one year after the last deposit. The first withdrawal for Evelyn will be 2 years after that. Using 9/1/02 as the comparison date:

$$Ps_{\overline{5}|} = 1{,}000a_{\overline{4}|}\left(1 + v^2\right)$$

$$P = \frac{1{,}000a_{\overline{4}|}\left(1 + v^2\right)}{s_{\overline{5}|}} = \frac{(3{,}387.21)(1.873439)}{5.750739}$$

$$= 1{,}103.46 \qquad \text{ANS. (B)}$$

9. The simplest approach is to note that the borrower underpaid by v^7 at time 4. He must pay the *AV* of v^7 at time 10 (6 years later), in addition to the regular payment of 1. Thus, his total payment at time $10 = v^7(1 + i)^6 + 1 = v + 1$ ANS. (A)

10. At time 0, the value of the annuity is $\ddot{a}_{\overline{k+m}|}$. At time k (k years later), its value is $\ddot{a}_{\overline{k+m}|}(1 + i)^k$. Since $\ddot{a}_{\overline{k+m}|} = (1 + i)a_{\overline{k+m}|}$, we see immediately that (A) *does* give the correct value.

    To check (B), we have $a_{\overline{k+m}|}(1 + i)^{k+1} = a_{\overline{k+m}|}(1 + i)^{k+m}(1 + i)^{-(m-1)} = s_{\overline{k+m}|}v^{m-1}$. $\therefore$ (B) is incorrect. There is no need to check the other choices. ANS. (B)

11. Using the end of 42 years as the comparison date:

$$X\ddot{s}_{\overline{14}|}(1.07)^{28} + Y\ddot{s}_{\overline{10}|} = 100{,}000$$

Since $X - Y = 100$, we substitute $X = 100 + Y$:

$$(100 + Y)(160.42997) + Y(14.78360) = 100{,}000$$

$$Y = 479.17 \qquad \text{ANS. (A)}$$

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12. Bank X: $10{,}000 = Xa_{\overline{5}|.08}$, $X = 2{,}504.56$

Bank Y: $15{,}000 = Ya_{\overline{5}|.075}$, $Y = 3{,}707.47$

Total $= 6{,}212.03$

Wayne: $25{,}000 = Wa_{\overline{4}|.085}$, $W = 7{,}632.20$

Difference to invest for 4 years $= 1{,}420.17$

AV of investment at end of 5 years $= 1{,}420.17\ddot{s}_{\overline{4}|.06}$

$= 6{,}585.46$

Jim has $(6{,}585.46 - 6{,}212.03)$ left $= 373$ ANS. **(C)**

13. Use time 20 as the comparison date:

$$1{,}000\ddot{s}_{\overline{10}|}(1.05)^{10} + X(1.05)^5 = \frac{1{,}000}{.05} + 1{,}000a_{\overline{10}|}$$

$X = 4{,}865.13$ ANS. **(A)**

14. Let $2P$ = annual payment during the first 10 years and P = annual payment thereafter. Thus, the perpetuity must provide P annually and the annuity-immediate must provide P for 10 years. Since Mark invests a total of 500,000, we have:

$$500{,}000 = P\left(a_{\overline{10}|} + \frac{1}{i}\right), \quad P = 20{,}810.13$$

Since Mark invests X in the perpetuity, we have:

$$X = \frac{P}{i} = \frac{20{,}810.13}{.06} = 346{,}835.50 \qquad \text{ANS. (C)}$$

15. Let X = amount of rebate. Using the rebate option, Janet would pay $(20{,}000 - X)$ now. Using the low interest rate option, Janet would make monthly payments of $\frac{20{,}000}{a_{\overline{60}|j}}$, where $j = \frac{2.5\%}{12}$.

For the dealership, the PV of these monthly payments at the market rate is $\frac{20{,}000}{a_{\overline{60}|\frac{2.5\%}{12}}}a_{\overline{60}|\frac{8\%}{12}}$. Thus:

$$20{,}000 - X = \frac{20{,}000}{a_{\overline{60}|\frac{2.5\%}{12}}}a_{\overline{60}|\frac{8\%}{12}} = \frac{(20{,}000)(49.318433)}{56.346404}$$

and $X = 2{,}494.56$ ANS. **(B)**

16. Ten years after buying the perpetuity-due, the AV of the first 10 payments is $500\ddot{s}_{\overline{10}|10\%} = 8{,}765.58$. Ralph sells the perpetuity-due at this point for a price of $\frac{500}{d} = \frac{500}{i}(1+i)$ at $10\% = \frac{500}{0.1}(1.1) = 5{,}500$. Thus, the sum of the money in the savings account and the proceeds from the sale is $8{,}765.58 + 5{,}500 = 14{,}265.58$. This amount is used to buy the annuity-due:

$$14{,}265.58 = X\ddot{a}_{\overline{20}|}$$

$X = 1{,}523.30$ ANS. **(D)**

17. $5.89 = \frac{1}{j}$, where j is the effective rate for a *2-year* period. Thus, $j = \frac{1}{5.89} = .169779$. Since $\frac{i}{2}$ is the effective rate for a $\frac{1}{2}$-year period, we have:

$$\left(1 + \frac{i}{2}\right)^2 = (1+j)^{\frac{1}{2}} \text{ and } i = 2\left[(i+j)^{\frac{1}{4}} - 1\right] = 2\left(1.169779^{\frac{1}{4}} - 1\right) = .08 \qquad \text{ANS. (C)}$$

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Note: Using the symbol i for the nominal rate compounded semiannually is poor notation. The symbol should be $i^{(2)}$.

18. Setting (ii) = (iii):

$$Xa_{\overline{20}|2i} = .96154X\ddot{a}_{\overline{20}|2i}$$

$$a_{\overline{20}|2i} = .96154(1+2i)a_{\overline{20}|2i}$$

$$1 = .96154(1+2i),\ \ i = .02$$

From(i):

$$P = \frac{2}{.02} = 100 \qquad \text{ANS. (E)}$$

19. Let X = annual payment commencing at the end of the 40th year. Using time 39 as the comparison date:

$$1{,}000\ddot{s}_{\overline{30}|}(1.05)^9 = \frac{X}{.05},\ \ X = 5{,}411 \qquad \text{ANS. (D)}$$

20. (i): $PV = \dfrac{120}{i}$, where i is the monthly effective rate

(ii): $PV = 365.47a_{\overline{12n}|i}$

Dividing (i) by (ii): $\frac{120}{365.47(1-v^{12n})} = 1$

$$v^{12n} = .671656$$

(iii): $PV = 17{,}866.32v^{12n} = (17{,}866.32)(.671656) = 12{,}000$ ANS. (D)

21. $90s_{\overline{10}|} = \dfrac{100}{i}$; $90\left[(1+i)^{10}-1\right] = 100$,

$$(1+i)^{10} = \frac{19}{9} \text{ and } (1+i)^5 = \frac{\sqrt{19}}{3} = 1.452966$$

$$Ks_{\overline{5}|}(1+i)^5 = \frac{K\left[(1+i)^{10}-(1+i)^5\right]}{i} = \frac{100}{i}$$

$$K = \frac{100}{(1+i)^{10}-(1+i)^5} = \frac{100}{1.452966(1.452966-1)} = 151.94 \qquad \text{ANS. (A)}$$

22. (i): $6{,}000 = 200\ddot{a}_{\overline{60}|} + 100\ddot{a}_{\overline{30}|}$ at effective rate j for $\frac{1}{2}$-year

(ii): $4{,}000 = 350\left(\ddot{a}_{\overline{60}|} - \ddot{a}_{\overline{30}|}\right)$

(iii): $X = 200\ddot{a}_{\overline{60}|} - 100\ddot{a}_{\overline{30}|}$

(i) and (ii) are simultaneous linear equations in $\ddot{a}_{\overline{30}|}$ and $\ddot{a}_{\overline{60}|}$. Solving:

$$\ddot{a}_{\overline{30}|} = 12.380952,\ \ \ddot{a}_{\overline{60}|} = 23.809524,$$

$$X = 200(23.809524) - 100(12.380952)$$

$$= 3{,}523.81 \qquad \text{ANS. (D)}$$

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Note: This problem is actually defective, although we don't believe it was ever designated as such by the SOA.

The examiners wanted you to assume that the same interest rate applied to all 3 statements (i), (ii) and (iii). But this is inconsistent with the given information. Statement (i) is true for an effective semiannual rate of about 4.517%. (You can check this out on the calculator.) Statement (ii) is true for an effective semiannual interest rate of about 2.234%. Thus, we cannot treat these two equations as if they were simultaneous linear equations. And it's meaningless to ask for the PV in (iii), since we don't know what interest rate to use.

23. Using the "block payments" approach:

$$PV = a_{\overline{20|}} + a_{\overline{18|}} + \cdots + a_{\overline{2|}}$$

$$= \frac{10 - \left(v^2 + v^4 + \cdots + v^{20}\right)}{i}$$

The series in parenthesis can be summed (geometric progression), or you may notice that this is the present value of an annuity of 1 payable every other year over 20 years, i.e., $\frac{a_{\overline{20|}}}{s_{\overline{2|}}}$. Either way:

$$PV = \frac{10 - \frac{a_{\overline{20|}}}{s_{\overline{2|}}}}{.05} = 78.42 \qquad \text{ANS. (B)}$$

24.

$$2\left(v^8 + v^{16} + \cdots\right) = 5$$

$$\frac{2v^8}{1 - v^8} = 5, \ v^8 = \frac{5}{7} \text{ and } (1+i)^8 = \frac{7}{5} = 1.4$$

$$\therefore \ i = 1.4^{\frac{1}{8}} - 1 = 4.3\% \qquad \text{ANS. (D)}$$

25. Let R = payment in first 10 years, $2R$ = payment in next 20 years.

$$100 = 2Ra_{\overline{30|}} - Ra_{\overline{10|}} \text{ at } 4\%$$

$$R = \frac{100}{2a_{\overline{30|}} - a_{\overline{10|}}} = \frac{100}{34.5841 - 8.11090} = 3.7774$$

For the lender's yield rate to be 4.5%, we have:

$$100 = 3.7774a_{\overline{10|}.045} + Xv^{10}_{.045}$$

$$X = \left(100 - 3.7774a_{\overline{10|}.045}\right)\left(1.045^{10}\right)$$

$$= 108.88 \qquad \text{ANS. (E)}$$

26.

$$K = \frac{1}{i} + v^2 + v^4 + \cdots$$

$$= \frac{1}{i} + \frac{v^2}{1 - v^2}$$

To get everything in terms of i, multiply the numerator and denominator of the second term by $(1+i)^2$:

$$K = \frac{1}{i} + \frac{1}{(1+i)^2 - 1}$$

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Combine into one fraction:

$$K = \frac{(1+i)^2 - 1 + i}{i[(1+i)^2 - 1]} = \frac{1 + 2i + i^2 - 1 + i}{i(1 + 2i + i^2 - 1)} = \frac{3i + i^2}{i(2i + i^2)}$$

$$= \frac{3+i}{2i+i^2} = \frac{3+i}{i(2+i)} \qquad \text{ANS. (A)}$$

27. Annuity X: $PV = a_{\overline{30}|} + a_{\overline{20}|} - a_{\overline{10}|}$

Annuity Y: $PV = Ka_{\overline{30}|} - Ka_{\overline{20}|} + Ka_{\overline{10}|}$

$$K\left(a_{\overline{30}|} - a_{\overline{20}|} + a_{\overline{10}|}\right) = a_{\overline{30}|} + a_{\overline{20}|} - a_{\overline{10}|}$$

$$K = \frac{a_{\overline{30}|} + a_{\overline{20}|} - a_{\overline{10}|}}{a_{\overline{30}|} - a_{\overline{20}|} + a_{\overline{10}|}}$$

$$= \frac{(1-v^{30}) + (1-v^{20}) - (1-v^{10})}{(1-v^{30}) - (1-v^{20}) + (1-v^{10})} = \frac{1 - \left(\frac{1}{2}\right)^3 + 1 - \left(\frac{1}{2}\right)^2 - \left(1 - \frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^3 - \left[1 - \left(\frac{1}{2}\right)^2\right] + \left(1 - \frac{1}{2}\right)}$$

$$\frac{9/8}{5/8} = \frac{9}{5} \qquad \text{ANS. (E)}$$

28. I. $PV = a_{\overline{2n}|}$

II. $PV = 3v^{2n}\left(\frac{1}{i}\right)$

$$a_{\overline{2n}|} = \frac{3v^{2n}}{i}, \quad \frac{1 - v^{2n}}{i} = \frac{3v^{2n}}{i}, \quad 1 - v^{2n} = 3v^{2n},$$

$$4v^{2n} = 1, \quad v^{2n} = \frac{1}{4}, \quad v^n = \frac{1}{2}$$

$$(1+i)^n = 2 \qquad \text{ANS. (B)}$$

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§ 3g. The $a_{\overline{2n}|}/a_{\overline{n}|}$ Trick (and Variations)

Trick Alert!

One type of problem that has appeared on past exams a number of times can be solved by using the quotient $a_{\overline{2n}|}/a_{\overline{n}|}$, or variations on this theme. The following example illustrates the method:

Stepping Stones

EXAMPLE 1

A "plain vanilla" version of this type of problem is as follows: Given that $a_{\overline{n}|} = 10$ and $a_{\overline{2n}|} = 15$, determine i.

SOLUTION

A few lines further down, we will show that $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$. Using this result, we have:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{15}{10} = 1.5 = 1 + v^n$$

This gives $v^n = 0.5$. All that remains is to plug v^n into the formula for $a_{\overline{n}|}$:

$$a_{\overline{n}|} = \frac{1 - v^n}{i} = \frac{1 - 0.5}{i} = 10$$

Thus, $i = 5\%$.

It is easy to show that $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$. First, a purely algebraic approach:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{\frac{1-v^{2n}}{i}}{\frac{1-v^n}{i}} = \frac{1 - v^{2n}}{1 - v^n}$$

Now, $1 - v^{2n}$ is the difference between two squares, which can be factored into

$(1 + v^n)(1 - v^n)$. Substituting in the above:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{(1 + v^n)(1 - v^n)}{1 - v^n} = 1 + v^n$$

Another way to derive this result is by a general reasoning approach. Think of $a_{\overline{2n}|}$ as consisting of two annuities: (1) an immediate annuity for n years, followed by (2) a deferred annuity for another n years. The relationship is:

$$a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} = a_{\overline{n}|}(1 + v^n)$$

Hence $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$.

It would be a good idea to know both of these approaches. It will help you to remember the above relationship and similar ones.

A couple of additional points:

(1) In Section 4e, we will show that "double dots cancel." This means that the quotient $\ddot{a}_{\overline{2n}|}/\ddot{a}_{\overline{n}|}$ is the same as the quotient $a_{\overline{2n}|}/a_{\overline{n}|}$:

$$\frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}} = \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

(More advanced Chapter 4 annuities with "upper m's" in their symbols also have the same quotient. See Section 4e.)

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(2) Formulas can also be derived for other annuities, for example, $a_{\overline{3n}|}/a_{\overline{n}|}$. Think of $a_{\overline{3n}|}$ as consisting of 3 annuities: an immediate annuity for n years, followed by a deferred annuity for n years, followed by another deferred annuity for n years. You should then be able to visualize the following relationship:

$$a_{\overline{3n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|}$$
$$= a_{\overline{n}|}(1 + v^n + v^{2n})$$

so that $\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$.

EXAMPLE 2

Given that $a_{\overline{10}|} = X$ and $a_{\overline{20}|} = 1.6X$, determine i.

SOLUTION

$$a_{\overline{20}|}/a_{\overline{10}|} = 1.6X/X = 1.6 = 1 + v^{10}$$
$$v^{10} = 0.6$$
$$i = 5.24\% \quad \text{(2 decimals)}$$

EXAMPLE 3

The present value of a 20-year annuity-immediate with annual payments of 5 is equal to the present value of a 20-year annuity-immediate with annual payments of 3 for the first 10 years and 8 for the next 10 years. Determine i.

SOLUTION

We are given that:

$$5a_{\overline{20}|} = 8a_{\overline{20}|} - 5a_{\overline{10}|}$$
$$a_{\overline{20}|}/a_{\overline{10}|} = 5/3 = 1.6\dot{6} = 1 + v^{10}$$
$$v^{10} = 0.6\dot{6}$$
$$i = 4.14\% \quad \text{(2 decimals)}$$

EXAMPLE 4

Given that $a_{\overline{n}|} = 10$ and $a_{\overline{3n}|} = 20$, determine i.

SOLUTION

$$a_{\overline{3n}|}/a_{\overline{n}|} = 20/10 = 2 = 1 + v^n + v^{2n}$$

This is a quadratic in v^n. For simplicity, let $v^n = x$:

$$x^2 + x - 1 = 0$$

The positive root is $x = .618034 = v^n$. Since $a_{\overline{n}|} = (1 - v^n)/i$, we have $10 = (1 - .618034)/i$ and $i = 3.82\%$ (2 decimals).

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Summary of Concepts and Formulas in Section 3g

(1) $$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

(2) This can be derived algebraically or by using the fact that the PV of a $2n$-year annuity is the sum of the PV of an n-year annuity and the PV of an n-year deferred n-year annuity:

$$\begin{aligned} a_{\overline{2n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} \\ &= (1 + v^n) a_{\overline{n}|} \end{aligned}$$

(3) Similar formulas can be derived for $3n$-year annuities, etc. For example:

$$\begin{aligned} a_{\overline{3n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|} \\ &= (1 + v^n + v^{2n}) a_{\overline{n}|} \end{aligned}$$

(4) These formulas can be used to solve for i and n; for example, when numerical values are given for $a_{\overline{n}|}$ and $a_{\overline{2n}|}$.

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Past Exam Questions on Section 3g

1. At an annual effective interest rate of i, $i > 0$, both of the following annuities have a present value of X:
   (i) a 20-year annuity-immediate with annual payments of 55
   (ii) a 30-year annuity-immediate with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years

   Calculate X. [5/03 #33]

   (A) 575 (B) 585 (C) 595 (D) 605 (E) 615

2. Dottie receives payments of X at the end of each year for n years. The present value of her annuity is 493.

   Sam receives payments of $3X$ at the end of each year for $2n$ years. The present value of his annuity is 2748.

   Both present values are calculated at the same annual effective interest rate.

   Determine v^n. [SOA 5/98 #4]

   (A) 0.86 (B) 0.87 (C) 0.88 (D) 0.89 (E) 0.90

3. An annuity pays 2 at the end of each year for 18 years.

   Another annuity pays 2.5 at the end of each year for 9 years.

   At an effective annual interest rate of i, $0 < i < 1$, the present values of both annuities are equal.

   Calculate i. [SOA 11/92 #5]

   (A) 14% (B) 17% (C) 20% (D) 23% (E) 26%

4. You are given:
   (i) $a_{\overline{n}|} = 10.00$; and
   (ii) $a_{\overline{3n}|} = 24.40$.

   Determine $a_{\overline{4n}|}$. [SOA 11/90 #2]

   (A) 28.74 (B) 29.00 (C) 29.26 (D) 29.52 (E) 29.78

5. Eric receives 12000 from a life insurance policy. He uses the fund to purchase two different annuities, each costing 6000.

   The first annuity is a 24-year annuity-immediate paying K per year to himself. The second annuity is an 8-year annuity-immediate paying $2K$ per year to his son.

   Both annuities are based on an annual effective interest rate of i, $i > 0$.

   Determine i. [SOA 11/90 #7]

   (A) 6.0% (B) 6.2% (C) 6.4% (D) 6.6% (E) 6.8%

6. At an effective annual interest rate i, you are given:
   (i) the present value of an annuity-immediate with annual payments of 1 for n years is 40; and
   (ii) the present value of an annuity-immediate with annual payments of 1 for $3n$ years is 70.

   Calculate the accumulated value of an annuity-immediate with annual payments of 1 for $2n$ years. [SOA 5/90 #6]

   (A) 240 (B) 243 (C) 260 (D) 268 (E) 280

7. Samantha receives a 10,000 life insurance benefit. If she uses the proceeds to buy an n-year immediate annuity, the annual payout will be 1,538. If a $2n$-year immediate annuity is purchased, the annual payout will be 1,072. Both calculations are based on an effective annual interest rate of i.

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Calculate i. [SOA 5/88 #16]

(A) 0.087 (B) 0.088 (C) 0.089 (D) 0.090 (E) 0.091

8. At a rate of interest, i, where $i > 0$, a 36-year annuity-immediate with annual payments of \$4 has the same present value as an 18-year annuity-immediate with annual payments of \$5.

   In how many years does money double at rate of interest, i? [SOA SAMPLE/84 #8]

   (A) 9 (B) 12 (C) 18 (D) 27 (E) 36

9. An annuity-due has the following present value and accumulated value:

   $$\ddot{a}_{\overline{n+2|}} = 13.987$$

   $$\ddot{s}_{\overline{n|}} = 51.632$$

   Which of the following is closest to the effective annual rate of interest? [CAS 5/83 #7]

   (A) 5.2% (B) 5.4% (C) 5.6% (D) 5.8% (E) 6.0%

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Solutions to Past Exam Questions on Section 3g

1. $$X = 55a_{\overline{20}|} = 30a_{\overline{10}|} + 60v^{10}a_{\overline{10}|} + 90v^{20}a_{\overline{10}|}$$
$$= a_{\overline{10}|}\left(30 + 60v^{10} + 90v^{20}\right)$$

Dividing by $a_{\overline{10}|}$, we have:

$$55\frac{a_{\overline{20}|}}{a_{\overline{10}|}} = 55\left(1 + v^{10}\right) = 30 + 60v^{10} + 90v^{20}$$
$$90v^{20} + 5v^{10} - 25 = 0$$

This is a quadratic in v^{10}:

$$v^{10} = \frac{-5 + \sqrt{25 - (4)(90)(-25)}}{180} \text{ (positive root)}$$
$$= \frac{-5 + \sqrt{9025}}{180} = \frac{90}{180} = 0.5$$
$$\therefore\ i = 7.177346\%$$
$$X = 55a_{\overline{20}|i} = 574.97 \qquad \text{ANS. (A)}$$

2. Dottie: $Xa_{\overline{n}|} = 493$

Sam: $3Xa_{\overline{2n}|} = 2{,}748$

Dividing: $\dfrac{3Xa_{\overline{2n}|}}{Xa_{\overline{n}|}} = 3\left(1 + v^n\right) = \dfrac{2{,}748}{493}$

$$v^n = \frac{2{,}748}{(3)(493)} - 1 = .858 \qquad \text{ANS. (A)}$$

3. $$2a_{\overline{18}|} = 2.5a_{\overline{9}|}$$
$$\frac{2a_{\overline{18}|}}{a_{\overline{9}|}} = 2\left(1 + v^9\right) = 2.5$$
$$v^9 = \frac{2.5}{2} - 1 = .25,\ \ i = 16.65\% \qquad \text{ANS. (B)}$$

4. Dividing (ii) by (i):

$$\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n} = \frac{24.40}{10} = 2.44$$
$$v^{2n} + v^n - 1.44 = 0,\ \ v^n = \frac{-1 + \sqrt{1 - (4)(-1.44)}}{2} = 0.8$$
$$a_{\overline{4n}|} = a_{\overline{n}|} + v^n a_{\overline{3n}|} = 10 + (0.8)(24.4) = 29.52 \qquad \text{ANS. (D)}$$

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5. $Ka_{\overline{24}|} = 2Ka_{\overline{8}|}$; Dividing:

$$\frac{a_{\overline{24}|}}{a_{\overline{8}|}} = 2 = 1 + v^8 + v^{16}$$

$$v^{16} + v^8 - 1 = 0,\ v^8 = \frac{-1 + \sqrt{1 - (4)(-1)}}{2}$$

$$= \frac{\sqrt{5} - 1}{2} = .618034$$

$i = 6.2\%$ ANS. **(B)**

6. $a_{\overline{n}|} = 40,\ a_{\overline{3n}|} = 70$

$$\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = \frac{7}{4} = 1 + v^n + v^{2n}$$

$$v^{2n} + v^n - \frac{3}{4} = 0,\ v^n = \frac{-1 + \sqrt{1 - (4)\left(-\frac{3}{4}\right)}}{2} = 0.5$$

$$a_{\overline{n}|} = 40 = \frac{1 - v^n}{i} = \frac{1 - 0.5}{i},\ i = \frac{0.5}{40} = .0125$$

$$s_{\overline{2n}|} = \frac{(1+i)^{2n} - 1}{i} = \frac{4 - 1}{.0125} = 240$$ ANS. **(A)**

7. $1{,}538a_{\overline{n}|} = 1{,}072a_{\overline{2n}|}$

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n = \frac{1{,}538}{1{,}072} = 1.434701$$

and $v^n = .434701$. Substituting v^n:

$$10{,}000 = 1{,}538a_{\overline{n}|} = \frac{1{,}538(1 - .434701)}{i},$$

$$i = \frac{(1{,}538)(.565299)}{10{,}000} = .0870$$ ANS. **(A)**

8. $4a_{\overline{36}|} = 5a_{\overline{18}|}$

$$\frac{a_{\overline{36}|}}{a_{\overline{18}|}} = 1 + v^{18} = \frac{5}{4},\ v^{18} = \frac{1}{4},$$

$(1+i)^{18} = 4,\ (1+i)^9 = 2$ ANS. **(A)**

9. $\ddot{a}_{\overline{n+2}|} = 1 + a_{\overline{n+1}|} = 13.987,\ a_{\overline{n+1}|} = 12.987$

$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1 = 51.632,\ s_{\overline{n+1}|} = 52.632$

$$\frac{s_{\overline{n+1}|}}{a_{\overline{n+1}|}} = \frac{52.632}{12.987} = 4.05267 = (1+i)^{n+1}$$

$$s_{\overline{n+1}|} = \frac{(1+i)^{n+1} - 1}{i} = \frac{4.05267 - 1}{i} = 52.632$$

$$i = \frac{3.05267}{52.632} = .058$$ ANS. **(D)**

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§ 3h. Balloon Payments and Drop Payments

What's wrong with the following question:

A loan of \$10,000 is to be repaid at 5% effective by payments of \$1,000 at the end of each year until the loan is repaid. How many payments are required?

We think you would agree that it is very, very unlikely that an ***integral*** number of payments of \$1,000 would just happen to have a PV of \$10,000 at 5% effective. So the question is incomplete because it really doesn't tell us how to finish paying off the loan.

Let's solve for the term of the annuity satisfying the following equation of value:

$$10{,}000 = 1{,}000a_{\overline{n}|.05}$$

Using the BA II Plus we enter 5 [I/Y] 10,000 [PV] 1,000 [+/-] [PMT] [CPT] [N]. You should get $N =$ 14.2067.

Since $14 < n < 15$, we know that 14 payments of \$1,000 would not repay the loan but 15 payments of \$1,000 would overpay the loan. So in order to exactly repay the loan, we have to do something special between time 14 and 15 that doesn't follow the regular schedule of \$1,000 payments.

At least three alternatives come to mind:

(1) A final payment ***larger*** than \$1,000 could be made at ***time 14***. This is called a ***balloon payment***.

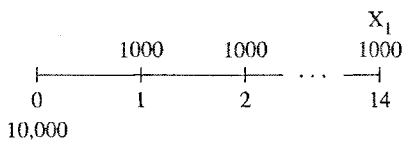
(2) A final payment ***smaller*** than \$1,000 could be made at ***time 15***. This is called a ***drop payment***.

(3) A final payment ***smaller*** than \$1,000 could be made at ***time n*** that is the solution to the equation of value, i.e., at n=14.2067. (This is about $2\frac{1}{2}$ months after time 14.)

So in order to make the original question unambiguous, it would have to specify exactly how to handle the final payment. For example, if the question intended that a balloon payment be made, it could have said:

A loan of \$10,000 is to be repaid at 5% effective by payments of \$1,000 at the end of each year for as long as possible. How many regular payments are required if an additional smaller payment is to be made on the date of the last regular payment? What is the amount of the additional payment?

First, we determine that $14 < n < 15$ as before. Thus, there are 14 regular payments of \$1,000. The question asks for the additonal amount (call it X_1) at time 14 that will repay the loan. The time diagram is:



The equation of value is:

$$10{,}000 = 1{,}000a_{\overline{14}|} + X_1 v^{14}$$

X_1 can be calculated very easily, especially if you haven't cleared the TVM registers yet. (If you have cleared them, you would have to re-enter 5 [I/Y] 10,000 [PV] 1,000 [+/-] [PMT].) All you have to do is:

14 [N] [CPT] [FV]

The answer is −200.68. This shows that a payment of 200.68 must be made at time 14 to repay the loan, in addition to the regular payments of 1,000 at the end of each year for 14 years. (You told

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the calculator that you received the loan amount of 10,000 (*a cash flow in*) at time 0 and then made payments (*cash flows out*) of 1,000 at the end of each year for 14 years. You then computed $\boxed{\text{FV}}$ = −200.68. The fact that $\boxed{\text{FV}}$ is *negative* shows that 200.68 is an additional *cash flow out* that the borrower must pay to satisfy the loan.)

For the drop payment X_2 at time 15, the equation of value is:

$$10{,}000 = 1{,}000a_{\overline{14}|} + X_2 v^{15}$$

To solve for X_2, use these keystrokes:

15 $\boxed{\text{N}}$ 5 $\boxed{\text{I/Y}}$ 10,000 $\boxed{\text{PV}}$ 1,000 $\boxed{\text{+/-}}$ $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$

The answer is 789.28. This shows that if we had made 15 payments of 1,000, we would have overpaid the loan by 798.28 at time 15. (The *positive* $\boxed{\text{FV}}$ of 789.28 means that it is a *cash flow in*, i.e., it should be paid back to the borrower at time 15.) So the borrower should pay only 1,000 − 789.28 = 210.72 at time 15, which is the amount of the drop payment.

Note the relationship between X_1 and X_2. X_2 should be X_1 with one year's interest:

$$X_2 = 1.05X_1 = (1.05)(200.68) = 210.71$$

(There is a one cent rounding difference from the previous result of $X_2 = 210.72$.)

The third alternative is to make a final payment X_3 at time 14.2067. It is unlikely that this would be done in practice because it would be inconvenient to make the final payment on a completely different month/day than that of all of the other payments.

One way to determine X_3 is to compute X_1 or X_2 and accumulate or discount to time 14.2067. For example, we have already calculated X_1 as \$200.68, so we have:

$$X_3 = 1.05^{0.2067}X_1 = \$202.71$$

(It can be shown that, in general, $X_3 = Xs_{\overline{k}|}$, where X is the *regular* payment (\$1,000 in this example) and k is the fraction of the year (0.2067 in this example)).

Stepping Stones

EXAMPLE 1

Colin makes 10 annual deposits of 1 into an account earning 5% effective. One year after the last deposit he begins making annual withdrawals of 1. How many full withdrawals of 1 can Colin make?

SOLUTION

We determine the accumulated value of the 10 deposits on the date of the last deposit as follows:

10 $\boxed{\text{N}}$ 5 $\boxed{\text{I/Y}}$ 1 $\boxed{\text{+/-}}$ $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$

The result is 12.577893. We simply enter this in $\boxed{\text{PV}}$, enter 0 in $\boxed{\text{FV}}$ and then $\boxed{\text{CPT}}$ $\boxed{\text{N}}$. The result is $N = 20.3\ldots$, which shows that Colin can make 20 full withdrawals of 1.

What we just did was to set $s_{\overline{10}|}$ equal to $a_{\overline{n}|}$ and then compute n.

EXAMPLE 2

In Example 1, what is the final drop withdrawal that Colin can make at the end of 21 years that would exhaust the account?

SOLUTION

Enter 21 in $\boxed{\text{N}}$ and leave the other registers as they are. Then $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$. You should get .677714 (positive). Since $\boxed{\text{PMT}}$ has been entered as a negative (i.e., as −1), this shows that a 21st withdrawal

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of 1 overdraws the account by .677714. Thus, the final drop withdrawal that Colin can make is $1 - .677714 = .322286$.

EXAMPLE 3

In Example 1, suppose Colin withdrew the entire account balance at the end of 20 years (a balloon withdrawal). What would the amount of this withdrawal be?

SOLUTION

If you had already done Example 2, you would simply discount the drop withdrawal by one year to time 20: $.322286/1.05 = .306939$, and then add this to the regular withdrawal of 1 at time 20, for a total balloon withdrawal of 1.306939.

If you were determining the balloon withdrawal directly, you would first determine that it would be made at time 20 (see the solution to Example 1). Then enter 20 in [N] and [CPT] [FV]. The result is $-.306939$. Thus, the balloon withdrawal at time 20 is the regular withdrawal of 1 plus the additional withdrawal of .306939.

EXAMPLE 4

In Example 1, suppose Colin's account earns 5% up to the date of the last deposit, but the account earns only 4% thereafter. How many full withdrawals of 1 can Colin make?

SOLUTION

The steps are similar to the solution to Example 1, except that after we enter the value of $s_{\overline{10}|}$ at 5% in [PV], we enter 4 in [I/Y], 0 in [FV] and then [CPT] [N]. The result is 17.8 . . ., which shows that Colin could make 17 full withdrawals of 1. (Since interest is at only 4% during the withdrawal period, we would expect that Colin could make fewer than 20 withdrawals, as in Example 1.)

§ 3i. What If the Rate Is Unknown?

If we know the term, the PV and the amount of the payment of an annuity, we can use our calculator to directly compute the unknown interest rate.

Stepping Stones

(See Sections 3c to 3g for additional examples of an unknown interest rate.)

EXAMPLE 1

The present value of an annuity-immediate with semiannual payments of \$1,300 for 9 years is \$15,400. Determine the nominal annual rate of interest compounded semiannually and the effective annual rate of interest.

SOLUTION

To compute the effective *semiannual* rate, the keystrokes on the BA II Plus are:

18 [N] 15,400 [PV] 1,300 [+/−] [PMT] [CPT] [I/Y]

The answer is 4.8303%. The nominal annual rate compounded semiannually is two times this rate, or 9.6605%. The equivalent effective annual rate is computed as follows:

[RCL] [I/Y] [÷] 100 [+] 1 [y^x] 2 [−] 1 [=]

The answer is 9.89%. (This could also be computed using the [ICONV] registers.)

Without a calculator, we would have to resort to an approximate or iterative technique, such as Newton-Raphson. See the comments at the end of Section 2b.

Sometimes an unknown rate of interest can be determined by simple algebraic techniques. See, for example, Section 3g.

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EXAMPLE 2

Grady pays \$1,000 for a 12-year annuity-immediate and receives annual payments of \$100 plus an additional payment of \$50 at the end of 12 years. What is Grady's effective annual rate of return?

SOLUTION

Use the calculator as follows:

12 [N] 1,000 [PV] 100 [+/-] [PMT] 50 [+/-] [FV] [CPT] [I/Y]

The result is 3.49% (2 decimals).

EXAMPLE 3

In Example 2, suppose Grady receives the additional payment of \$50 at the end of 13 years, rather than 12 years. What is his effective annual rate of return?

SOLUTION

In order to set up the calculator to register 12 payments of \$100 and a final payment of \$50 at time 13, we do the following:

13 [N] 1,000 [PV] 100 [+/-] [PMT] 50 [FV] [CPT] [I/Y]

Note that this tells the calculator that there are 13 payments of \$100 (registered as negatives) and a final payment of \$50 (registered as a positive) at time 13, for a net payment of \$50 at time 13. (Of course, you could have reversed the signs of all of these entries and you would get the same result for the interest rate.)

The answer is 3.47%, only slightly lower than Example 2, which is not surprising.

EXAMPLE 4

Tiffany buys an annuity-immediate for \$100,000. The annuity provides for 20 equal annual payments based on an annual effective rate of interest of 3.5%. Just after receiving the 8th payment, she decides to sell the remaining payments to Leo for \$70,000. What effective annual rate of interest did she earn on the entire transaction?

SOLUTION

First, we must determine the annual payment provided by this annuity. We use the calculator as follows:

20 [N] 3.5 [I/Y] 100,000 [PV] [CPT] [PMT]

We find that Tiffany's annual payment is \$7,036.11. Thus, after selling the annuity to Leo, Tiffany has received 8 payments of \$7,036.11 and a final payment at time 8 of \$70,000. Her effective annual rate of interest on this transaction is the solution of the following equation of the value for i:

$$100{,}000 = 7{,}036.11 a_{\overline{8}|} + 70{,}000 v^8$$

We can use the calculator to solve for i by leaving everything in the registers except for [N] and [FV]:

8 [N] 70,000 [+/-] [FV] [CPT] [I/Y]

The answer is 3.75% (2 decimals).

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Summary of Concepts and Formulas in Sections 3h and 3i

(1) There are at least three possibilities for repaying a loan by level payments, followed by an unequal final payment:

  (a) the final payment is larger than the level payments (a ***balloon payment***).
  (b) the final payment is smaller than the level payments (a ***drop payment***).
  (c) the final payment is smaller than the level payments and is made on a date that is between regular payment dates. (This is not often done in practice.)

(2) Section 3h shows how to compute the number of level payments and the final payment in each of these cases.

(3) If we know the PV of an annuity, and the number and amount of the payments, we can solve for the interest rate using the calculator.

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Past Exam Questions on Sections 3h and 3i

1. Peter borrows \$5,000 from Kevin for a term of 5 years. Peter agrees to pay interest at the end of each year at an effective annual interest rate of 8% and to repay the entire \$5,000 as a lump sum at the end of 5 years. Immediately after the third payment, Kevin sells his right to future payments to Martha at a price that will yield Martha an effective annual rate of return of 5%. Determine Kevin's overall yield rate. [CAS 5/99 #9]

   (A) $< 8.6\%$ (B) 8.6%/9.0% (C) 9.0%/9.4% (D) 9.4%/9.8% (E) $\geq 9.8\%$

2. John's estate is to be divided into three equal parts and invested, to be paid out as follows:
   (i) John's two children will each receive their share in 20 level annual payments, beginning one year after John's death.
   (ii) Charity Q will receive its share as equal annual payments in perpetuity beginning 21 years after John's death.

   Q's annual payment is twice the annual payment for one child.

   Determine the effective interest rate at which the estate is invested. [CAS 11/93 #9]

   (A) Less than 5%
   (B) At least 5% but less than 5.25%
   (C) At least 5.25% but less than 5.50%
   (D) At least 5.50% but less than 5.75%
   (E) At least 5.75%

3. Bob purchased Amy's engagement ring on January 1, 1992 with a \$10,000 loan. His loan carries an interest rate of 21% per year convertible monthly. He pays \$600 per month starting February 1, 1992, plus an additional \$1600 on June 1, 1992. His last payment will be a partial payment.

   Determine when he will make the last full payment of \$600. [CAS 11/92 #10]

   (A) April 1, 1993 (B) May 1, 1993 (C) June 1, 1993 (D) July 1, 1993 (E) August 1, 1993

4. On 1/1/87, Cathy borrows \$1,000 from ABS Finance Company and agrees to pay it back in five annual installments at an annual effective rate of 11%. The first payment is due one year after the loan is issued. On 1/1/89 Cathy pays off the balance of the loan plus a \$50 prepayment penalty. Find ABC's effective overall yield rate i. [CAS 5/91 #20]
   (A) $i < 12.8\%$
   (B) $12.8\% \leq i < 13.0\%$
   (C) $13.0\% \leq i < 13.2\%$
   (D) $13.2\% \leq i < 13.4\%$
   (E) $13.4\% \leq i$

5. A renter with \$1,104 has a one year lease. The landlord is willing to accept two payment options:
   (i) \$1,104 now; or
   (ii) \$100 paid at the beginning of each month for twelve months.

   What monthly interest rate would be required for the two options to be equivalent? [CAS 5/90 #20]

   (A) less than .012
   (B) at least .012 but less than .013
   (C) at least .013 but less than .014
   (D) at least .014 but less than .015
   (E) at least .015

6. Using an annual effective interest rate $j > 0$, you are given:
   (i) The present value of 2 at the end of each year for $2n$ years, plus an additional 1 at the end of each of the first n years, is 36.

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(ii) The present value of an n-year deferred annuity-immediate paying 2 per year for n years is 6.

Calculate j. [SOA 11/89 #7]

(A) 0.03 (B) 0.04 (C) 0.05 (D) 0.06 (E) 0.07

7. You are given an n-year annuity-due of one per year plus a final payment at time $n+k-1$ $(0 < k < 1)$. The present value of the payments can be simplified to:

$$\frac{1-v^{n+k}}{d}.$$

Calculate the final payment. [SOA 5/88 #6]

(A) 1 (B) k (C) $1-k$ (D) $\frac{(1+i)^k-1}{i}$ (E) $\frac{(1+i)^k-1}{d}$

8. A sum, P, is used to buy a deferred perpetuity-due of 1 payable annually. The effective annual rate of interest is i, $i > 0$.

Calculate the deferred period. (logs are to the base e) [SOA 11/86 #7]

(A) $\log\left(\frac{P}{d}\right)$ (B) $1-\frac{\log(iP)}{\delta}$ (C) $-\frac{\log(iP)}{\delta}$ (D) $1+\frac{\log(dP)}{\delta}$ (E) $\frac{\log(dP)}{\delta}$

9. A 100,000 loan is to be repaid by 30 equal payments at the end of each year. The outstanding balance is amortized at 4%.

In addition to the annual payments, the borrower must pay an origination fee at the time the loan is made. This fee is 2% of the loan but does not reduce the loan balance.

When the second payment is due, the borrower pays the remaining loan balance.

Determine the yield to the lender considering the origination fee and the early pay-off of the loan. [SOA 5/86 #7]

(A) 4.9% (B) 5.0% (C) 5.1% (D) 5.2% (E) 5.3%

10. You are given the following series of payments:

(i) 100 at time $t = 1, 3, 5, \ldots, 19$

(ii) 200 at time $t = 2, 4, 6, \ldots, 20$

An actuary is asked to determine the time, t^*, such that the present value of the series of payments is equal to the present value of a single payment of 3000 made at time t^*.

Derive an exact expression for t^* for $i > 0$. (logs are to the base e) [SOA 11/85 #9]

(A) $-\frac{1}{\delta}\log\left[\frac{(1+2v)\ddot{a}_{\overline{20}|}}{30a_{\overline{2}|}}\right]$

(B) $-\frac{1}{\delta}\log\left[\frac{(1+2v)\ddot{a}_{\overline{20}|}}{30\ddot{a}_{\overline{2}|}}\right]$

(C) $-\frac{1}{\delta}\log\left[\frac{(v+2v^2)\ddot{a}_{\overline{20}|}}{30a_{\overline{2}|}}\right]$

(D) $-\frac{1}{\delta}\log\left[\frac{(v+2v^2)a_{\overline{20}|}}{30\ddot{a}_{\overline{2}|}}\right]$

(E) $-\frac{1}{\delta}\log\left[\frac{(v+2v^2)a_{\overline{20}|}}{30a_{\overline{2}|}}\right]$

11. Two perpetuities have the same annual effective interest rate. Perpetuity A pays \$8 at the end of each year for the first 20 years and then \$4 at the end of each year thereafter. Perpetuity B is a perpetuity due which has a level annual payment of \$6.

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At time $t = 0$, the present value of perpetuity A is equal to that of perpetuity B. What is the effective annual interest rate i? [CAS 5/85 #6]

(A) Less than 3.6%
(B) At least 3.6%, but less than 3.8%
(C) At least 3.8%, but less than 4.0%
(D) At least 4.0%, but less than 4.2%
(E) 4.2% or more

12. A \$74.93 loan is to be repaid in two \$50 installments. The effective annual rate of interest on the loan is 3%.

The first installment is due in six years.

When is the second installment due? [CAS 5/84 #12]

(A) 14 years (B) 15 years (C) 16 years (D) 17 years (E) 18 years

13. A person deposits 100 at the beginning of each year for 20 years. Simple interest at an annual rate of i is credited to each deposit from the date of deposit to the end of the twenty year period. The total amount thus accumulated is 2,840.

If instead, compound interest had been credited at an effective annual rate of i, what would the accumulated value of these deposits have been at the end of 20 years? [SOA SAMPLE/83 #12]

(A) 2,980 (B) 3,100 (C) 3,200 (D) 3,310 (E) 3,470

14. Smith deposits \$100 at the beginning of each year in Savings Account A and \$100 at the beginning of each year in Savings Account B. At the end of N years the balance in each account is \$2,350.

The effective annual rate of interest on Savings Account A is 3.5%.

The effective annual rate of interest on Savings Account B is i% for the first $N - 6$ years and 5% for the last six years.

Which of the following is closest to i? [CAS 5/83 #8]

(A) 1.50% (B) 1.75% (C) 2.00% (D) 2.25% (E) 2.50%

15. A loan of \$12,000 is to be repaid within one year with level monthly payments, due at the beginning of each month.

The 12 payments equal \$1,000 each. A finance charge of \$632 is also due with the first payment.

Which of the following is closest to the effective annual interest rate on the loan? [CAS 5/83 #14]

(A) 12.7% (B) 12.9% (C) 13.1% (D) 13.3% (E) 13.5%

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Solutions to Past Exam Questions on Sections 3h and 3i

1. Kevin lends \$5,000 in return for the following payments: 8% × \$5,000 = \$400 at the end of years 1 to 3 plus a lump sum X from Martha, where $X = 400a_{\overline{2}|} + 5{,}000v^2$ at 5% = 743.76 + 4,535.15 = 5,278.91. (X could be determined in one operation on the BA II Plus by entering 2 [N] 5 [I/Y] 400 [+/–] [PMT] 5,000 [+/–] [FV] [CPT] [PV].) Thus, for Kevin we have:

$$5{,}000 = 400a_{\overline{3}|} + 5{,}278.91v^3$$

This can be solved for i on the BA II Plus (while 5,278.91 is still in the display) by entering [2nd] [CLR TVM] [FV] 3 [N] 5,000 [+/–] [PV] 400 [PMT] [CPT] [I/Y]. The result is $i = 9.69\%$ ANS. (D)

2. Let the total value of the estate = 1

Payment to each child for 20 years $= \dfrac{1}{3a_{\overline{20}|}}$

Payment to charity in perpetuity, starting in 21 years $= \dfrac{i}{3v^{20}}$

We are given that $\frac{i}{3v^{20}} = 2\left(\frac{1}{3a_{\overline{20}|}}\right) = \frac{2i}{3(1-v^{20})}$

Cancelling i and taking reciprocals:

$$3v^{20} = \frac{3}{2}\left(1 - v^{20}\right) \text{ or } v^{20} = \frac{1}{3}$$

Solving for i: $i = 5.65\%$. ANS. (D)

3. Let n = number of months of \$600 payments.

$$10{,}000 = 600a_{\overline{n}|} + 1{,}600v^5 \text{ at } \frac{21}{12}\% = 1.75\% \text{ per month}$$

$$a_{\overline{n}|} = \frac{10{,}000 - 1{,}600v^5}{600} = 14.221567$$

Solve for n on the BA II Plus by entering 1.75 [I/Y] 14.2216 [PV] 1 [+/–] [PMT] [CPT] [N], resulting in $n = 16.50$. Thus, the last full payment of \$600 will be made at the end of 16 months (and the drop payment will be made at the end of 17 months). 16 months after 1/1/92 is 5/1/93. ANS. (B)

4. Cathy makes the following payments:

$$\frac{1{,}000}{a_{\overline{5}|11\%}} = \$270.57 \text{ at times 1 and 2}$$

$$270.57a_{\overline{3}|11\%} = \$661.20 \text{ at time 2}$$

(i.e., the PV of the 3 remaining payments)

\$50 at time 2 (pre-payment penalty)

Thus, ABC's equation of value is:

$$1{,}000 = 270.57a_{\overline{2}|i} + (661.20 + 50)v_i^2$$

This can be solved for i on the BA II Plus by entering 2 [N] 1,000 [PV] 270.57 [+/–] [PMT] 711.20 [+/–] [FV] [CPT] [I/Y]. The answer is 13.53%. ANS. (E)

5. $1{,}104 = 100\ddot{a}_{\overline{12}|j}$

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If we want to avoid the [BGN] mode on the BA II Plus calculator, we can substitute $1 + a_{\overline{11}|}$ for $\ddot{a}_{\overline{12}|}$, giving: $1{,}004 = 100a_{\overline{11}|j}$. Enter 11 [N] 1,004 [PV] 100 [+/–] [PMT] [CPT] [I/Y]. The answer is 1.55%. ANS. (E)

6. $$2a_{\overline{2n}|} + a_{\overline{n}|} = 36$$

$$2\left(a_{\overline{2n}|} - a_{\overline{n}|}\right) = 6$$

Solving this pair of simultaneous equations, we get $a_{\overline{n}|} = 10$, $a_{\overline{2n}|} = 13$

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n = \frac{13}{10}, \quad v^n = 0.3$$

$$a_{\overline{n}|} = \frac{1 - v^n}{j} = \frac{1 - 0.3}{j} = 10,$$

$$j = \frac{0.7}{10} = .07 \qquad \text{ANS. (E)}$$

7. The payments are 1 at the beginning of each year for n years and a final payment (call it X) a fraction of a year after the last payment of 1, i.e., at time $n - 1 + k$. (This is an annuity-due, so the last payment of 1 is at time $n - 1$.) Thus:

$$\frac{1 - v^{n+k}}{d} = \ddot{a}_{\overline{n}|} + Xv^{n+k-1}$$

$$X = \frac{\left[1 - v^{n+k} - (1 - v^n)\right]}{d}(1+i)^{n+k-1}$$

$$= \frac{v\left[(1+i)^k - 1\right]}{d} = \frac{(1+i)^k - 1}{i} \qquad \text{ANS. (D)}$$

8. $$P = v^n\left(\frac{1}{d}\right)$$

$$\ln P = n \ln v - \ln d$$

$$n = \frac{\ln(dP)}{\ln v} = -\frac{\ln(dP)}{\delta}$$

Since this answer does not appear, substitute $d = \frac{i}{1+i}$:

$$n = -\frac{\ln\left[\left(\frac{i}{1+i}\right)P\right]}{\delta} = -\frac{[\ln(iP) - \ln(1+i)]}{\delta}$$

$$= -\frac{[\ln(iP) - \delta]}{\delta} = 1 - \frac{\ln(iP)}{\delta} \qquad \text{ANS. (B)}$$

9. The scheduled payments are $\frac{100{,}000}{a_{\overline{30}|4\%}} = 5{,}783.01$ at the end of each year for 30 years and $2\% \times 100{,}000 = 2{,}000$ on the date of the loan. The borrower makes 2 payments of 5,783.01 and a payment of 5,783.01 $a_{\overline{28}|.04} = 96{,}362.66$ (the outstanding balance) at time 2. Thus, the lender's equation of value is:

$$100{,}000 = 2{,}000 + 5{,}783.01a_{\overline{2}|j} + 96{,}362.66v_j^2$$

This can be solved for j on the BA II Plus by entering 2 [N] 98,000 [PV] 5,783.01 [+/–] [PMT] 96,362.66 [+/–] [FV] [CPT] [I/Y]. The answer is 5.0864%. ANS. (C)

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10. The PV of the payments is equal to

$$100\left(v+v^3+\cdots+v^{19}\right)+200\left(v^2+v^4+\cdots+v^{20}\right)$$
$$=\frac{100v\left(1-v^{20}\right)+200v^2\left(1-v^{20}\right)}{1-v^2}$$
$$=\frac{100va_{\overline{20|}}+200v^2a_{\overline{20|}}}{a_{\overline{2|}}}$$

This is equal to $3{,}000v^{t*}$, so taking natural logs, we get:

$$t^*=-\frac{1}{\delta}\ln\left[\frac{\left(v+2v^2\right)a_{\overline{20|}}}{30a_{\overline{2|}}}\right]$$

since $\ln v=-\delta$ ANS. (E)

11. A: $PV=\dfrac{4}{i}+4a_{\overline{20|}}$

B: $PV=\dfrac{6}{d}=\dfrac{6(1+i)}{i}$

$$\frac{4}{i}+\frac{4\left(1-v^{20}\right)}{i}=\frac{6(1+i)}{i}$$
$$4+4-4v^{20}=6+6i$$
$$2v^{20}+3i-1=0$$

Trial and error shows that $i>4.2\%$ ANS. (E)

12. $74.93=50v^6+50v^t$ at 3%

$$v^t=\frac{74.93-50v^6}{50}=.661116,\ t=14$$ ANS. (A)

13. $100[(1+20i)+(1+19i)+\cdots+(1+i)]=2{,}840$

$$20+\frac{(20)(21)}{2}i=28.4,\ i=.04$$

At $i=.04$ compounded:

$100\ddot{s}_{\overline{20|}.04}=3{,}096.92$ ANS. (B)

14. A: $100\ddot{s}_{\overline{N|}.035}=2{,}350,\ N=17$

B: $100\ddot{s}_{\overline{11|}i}(1.05)^6+100\ddot{s}_{\overline{6|}.05}=2{,}350$

$$\ddot{s}_{\overline{11|}i}=\frac{\left(2{,}350-100\ddot{s}_{\overline{6|}.05}\right)}{100}v_{.05}^6=12.206585$$

$i=1.725\%$ ANS. (B)

15. The payments on the \$12,000 loan are \$1,632 immediately and \$1,000 at the end of each of the next 11 months:

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$12{,}000 = 1{,}632 + 1{,}000a_{\overline{11}|j}$, where j = monthly effective rate

$a_{\overline{11}|j} = 10.368, \quad j = .999386\%$

$i = (1+j)^{12} - 1 = 12.67\%$ ANS. **(A)**

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§ 3j. What If the Rate Varies?

When the effective rate of interest varies over the term of an investment, things can get a little tricky. We have to be very careful about how a question describes the rates.

Consider the following two examples, which are worded similarly but have different answers.

Stepping Stones

EXAMPLE 1

Annual deposits of \$100 are made at the beginning of each year for 10 years. Find the accumulated value at the end of 10 years if the effective rate of interest is 8% for the first 6 years and 7% for the last 4 years.

SOLUTION

The first 6 deposits have an AV at time 6 of $100\ddot{s}_{\overline{6}|8\%} = \$792.28$. At time 10, the AV is $792.28(1.07)^4 = \$1{,}038.52$. The last 4 deposits have an AV at time 10 of $100\ddot{s}_{\overline{4}|7\%} = \$475.07$. The total $AV = \$1{,}513.59$.

Notes:

(1) The total AV could also be written as $100s_{\overline{7}|8\%}(1.07)^4 + 100\ddot{s}_{\overline{3}|7\%}$.

(2) We can't play "Now you see it . . . " with the first 6 payments, i.e., their AV at time 10 *cannot* be expressed as $100(\ddot{s}_{\overline{10}|7\%} - \ddot{s}_{\overline{4}|7\%})$. This expression, which is equivalent to $100\ddot{s}_{\overline{6}|7\%}(1.07)^4$, implies that the deposits earn 7% during the first 6 years, when they actually earn 8%.

EXAMPLE 2

Annual deposits of \$100 are made at the beginning of each year for 10 years. Find the accumulated value at the end of 10 years if the first 6 deposits are invested at 8% effective and the last 4 deposits are invested at 7% effective.

SOLUTION

The wording of Example 2 tells us that the first 6 deposits can earn 8% for the *entire* 10-year period (not 7% in the last 4 years). Thus, the AV should be higher than the AV in Example 1. The AV of all the deposits at time 10 is:

$$100\ddot{s}_{\overline{6}|8\%}(1.08)^4 + 100\ddot{s}_{\overline{4}|7\%} = \$1{,}077.89 + \$475.07 = \$1{,}552.96$$

(The first term could also be expressed as $100(\ddot{s}_{\overline{10}|8\%} - \ddot{s}_{\overline{4}|8\%})$).

As expected, this AV is higher than the AV in Example 1.

Another type of varying interest is where the rate varies continuously, e.g., the force of interest is a function of t. Section 4g will cover examples of this.

EXAMPLE 3

Determine the present value of a 15-year annuity-immediate with payments of \$100 for the first 10 years and \$200 for the next 5 years, given that the effective annual rate of interest is 6% for the first 10 years and 3% thereafter.

SOLUTION

$$PV = 100a_{\overline{10}|.06} + 200v_{.06}^{10}a_{\overline{5}|.03}$$

Using your calculator, you should get \$736.01 for the first term and \$511.46 for the second, for a total PV of \$1,247.47.

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EXAMPLE 4

Determine the accumulated value of the annuity in Example 3 on the date of the last payment.

SOLUTION

$$AV = 100s_{\overline{10}|.06}(1.03)^5 + 200s_{\overline{5}|.03}$$

The first term is \$1,528.02 and the second term is \$1,061.83, for a total of \$2,589.85.

Note: The answers to Examples 3 and 4 are consistent, i.e., the present value, accumulated with the appropriate interest rates, is equal to the accumulated value: $1{,}247.47(1.06)^{10}(1.03)^5 = 2{,}589.85$.

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Summary of Concepts and Formulas in Sections 3j

(1) Two examples of varying interest rates:

(a) Four annual deposits of \$1.00, followed by 6 annual deposits of \$1.00, are invested in a fund that credits interest at 5% effective up to the date of the 4th deposit, and 4% effective thereafter. Find the AV on the date of the last deposit.

$$AV = s_{\overline{4}|.05}(1.04)^6 + s_{\overline{6}|.04}$$

(b) Four annual deposits of \$1.00 are invested in a fund that credits interest at 5% effective, followed by 6 annual deposits that are invested in another fund at 4% effective. Find the AV on the date of the last deposit.

$$AV = s_{\overline{4}|.05}(1.05)^6 + s_{\overline{6}|.04}$$

(2) For the above two examples, find the PV of the deposits one year before the first deposit.

(a) $PV = a_{\overline{4}|.05} + v^4_{.05}\, a_{\overline{6}|.04}$

(b) $PV = a_{\overline{4}|.05} + v^4_{.04}\, a_{\overline{6}|.04}$

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Past Exam Questions on Section 3j

1. A loan of 10,000 is to be amortized in 10 annual payments beginning 6 months after the date of the loan. The first payment, X, is half as large as the other payments. Interest is calculated at an annual effective rate of 5% for the first 4.5 years and 3% thereafter. Determine X. [SAMPLE/00 #28]

   (A) 640 (B) 648 (C) 656 (D) 664 (E) 672

2. Jeff deposits 100 at the end of each year for 13 years into Fund X. Antoinette deposits 100 at the end of each year for 13 years into Fund Y.

   Fund X earns an annual effective rate of 15% for the first 5 years and an annual effective rate of 6% thereafter. Fund Y earns an annual effective rate of i.

   At the end of 13 years, the accumulated value of Fund X equals the accumulated value of Fund Y.

   Calculate i. [SOA 5/98 #5]

   (A) 6.4% (B) 6.7% (C) 7.0% (D) 7.4% (E) 7.8%

3. Sam deposits 1000 every year on his birthday into a retirement fund earning an annual effective rate of 15%. The first deposit is made on his 39th birthday and the last deposit is made on his 60th birthday.

   Immediately after the last deposit, the accumulated value of the fund is transferred to a fund earning an annual effective rate of j.

   Five years later, a 35-year monthly annuity-due paying 1000 each month is purchased with the funds.

   The purchase price of the annuity was determined using a nominal rate of interest convertible monthly at 6%.

   Calculate j. [SOA 11/96 #5]

   (A) 4.6% (B) 5.1% (C) 5.6% (D) 6.1% (E) 6.6%

4. John deposits 100 at the end of each year for 20 years into a fund earning an annual effective interest rate of 7%.

   Mary makes 20 deposits into a fund at the end of each year for 20 years. The first 10 deposits are 100 each, while the last 10 deposits are $100 + X$ each. The fund earns an annual effective interest rate of 8% during the first 10 years and 6% annual effective interest thereafter.

   At the end of 20 years, the amount in John's fund equals the amount in Mary's fund.

   Calculate X. [SOA 5/93 #5]

   (A) 8 (B) 10 (C) 12 (D) 14 (E) 16

5. You are given $\delta_t = \frac{2}{10+t}$, $t \geq 0$.

   Calculate $a_{\overline{4}|}$. [SOA 5/87 #3]

   (A) 2.34 (B) 2.62 (C) 2.85 (D) 3.01 (E) 3.23

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Solutions to Past Exam Questions on Section 3j

1. $$10{,}000 = \left[2Xa_{\overline{5}|5\%} - Xv\right]\left(1.05^{\frac{1}{2}}\right) + 2Xa_{\overline{5}|.03}v_{.05}^{4.5}$$
$$= X[(8.6590 - .9524)(1.0247) + 7.3539]$$
$$= 15.2509X$$
$$X = 655.70 \qquad \text{ANS. (C)}$$

2. Fund X: $AV = 100s_{\overline{5}|15\%}(1.06)^8 + 100s_{\overline{8}|.06}$

Fund Y: $AV = 100s_{\overline{13}|i}$

$s_{\overline{13}|i} = (6.7424)(1.5938) + 9.8975 = 20.6435$

$i = 7.4\% \qquad$ ANS. (D)

3. $$1{,}000s_{\overline{22}|15\%}(1+j)^5 = 1{,}000\ddot{a}_{\overline{420}|0.5\%}$$
$$137.63164(1+j)^5 = 176.25713$$
$$j = 5.07\% \qquad \text{ANS. (B)}$$

4. John's $AV = 100s_{\overline{20}|7\%}$

Mary's $AV = 100s_{\overline{10}|8\%}(1.06)^{10} + (100 + X)s_{\overline{10}|6\%}$

Equating and solving for X:

$$X = \frac{100\left[s_{\overline{20}|7\%} - s_{\overline{10}|8\%}(1.06)^{10} - s_{\overline{10}|6\%}\right]}{s_{\overline{10}|6\%}}$$
$$= 100\frac{[40.9955 - (14.4866)(1.7908) - 13.1808]}{13.1808}$$
$$= 14.20 \qquad \text{ANS. (D)}$$

5. $$a(t) = e^{\int_0^t \frac{2}{10+r}dr} = e^{2\ln(10+r)\big]_0^t}$$
$$= e^{\ln\left(\frac{10+t}{10}\right)^2} = \left(\frac{10+t}{10}\right)^2$$
$$a_{\overline{4}|} = \frac{1}{a(1)} + \frac{1}{a(2)} + \frac{1}{a(3)} + \frac{1}{a(4)}$$
$$= \left(\frac{10}{11}\right)^2 + \left(\frac{10}{12}\right)^2 + \left(\frac{10}{13}\right)^2 + \left(\frac{10}{14}\right)^2 = 2.62 \qquad \text{ANS. (B)}$$

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4

Complex Annuities

§ 4a. Annuities with "Off-Payments"—Part I

This section and the next one are companion sections that show you how to handle annuities with "off-payments". By this we mean payments which are less frequent or more frequent than the interest period.

Suppose the effective rate of interest is given as 5% per annum. The following are examples of "off-payment" annuities:

(1) An annuity with payments of 1 at the end of each 5-year period over 40 years. (Payments are ***less frequent*** than the interest period of one year.)

(2) An annuity with payments of $\frac{1}{12}$ at the end of each month for 10 years. (Payments are ***more frequent*** than the interest period of one year.)

The payments in (2) are sometimes described as "1 per annum payable in monthly installments", which means monthly payments of 1/12 each.

We have divided the subject into two sections because there are two general approaches for handling these types of annuities:

(1) We could use interest functions at the equivalent effective rate of interest for the ***payment period***. (In example (1) above, we would use the equivalent effective rate for a ***5-year*** period. In example (2), we would use the equivalent effective rate for a ***one-month*** period.)

(2) We could use interest functions at the effective rate of interest ***given in the problem***. (In the above examples, this means interest functions at 5% per annum.)

In this section, we will cover the first approach. We will see that if you want numerical results, the first approach is the easiest. Since most exam questions have numerical answers, this is the approach you will use most of the time. However, occasionally an exam question may have symbolic answers. In this case, the symbols are normally shown at the effective rate given in the question (5% in the above example). So you would need to know the second approach as well. (In the published exams under the 2000 and later syllabi, very few questions had symbolic answers).

Stepping Stones

EXAMPLE 1

Take as an example the first annuity described above, namely, payments of 1 at the end of every 5 years over 40 years at 5% effective. Find the present value of this annuity in terms of interest functions at the effective rate for a 5-year period.

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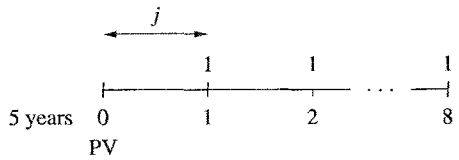
SOLUTION

Since the payments are at 5-year intervals, we determine the effective rate for a 5-year period. If we call this rate j, we have:

$$1+j=1.05^5$$

$$j=1.05^5-1=27.6282\%$$

The time diagram, in terms of 5-year periods, is:



The PV is given by an annuity-immediate for 8 periods at rate j:

$$PV = a_{\overline{8}|} \text{ at rate } j = 27.6282\%$$

$$= 3.1054$$

EXAMPLE 2

Find the PV of an annuity-immediate with annual payments of \$1,500 for 8 years at a nominal rate of 4% compounded quarterly.

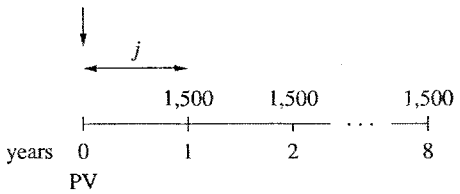
SOLUTION

Since payments are annual, determine the equivalent annual effective rate j as follows:

$$1+j=1.01^4$$

$$j=1.01^4-1=4.0604\%$$

The time diagram, in terms of annual interest periods, looks like this:



$$PV = 1500\, a_{\overline{8}|} \text{ at rate } j = 4.0604\%$$

$$= \$10,073.98$$

EXAMPLE 3

Find the PV of \$500 paid at the beginning of every 3 years for 15 years at 3% effective per annum.

SOLUTION

If j is the effective rate for a 3-year period, we have:

$$1+j=1.03^3$$

$$j=1.03^3-1=9.2727\%$$

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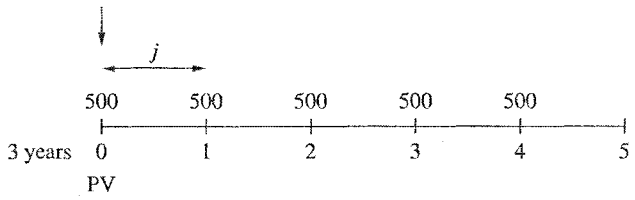
The time diagram in terms of 3-year periods is as follows:



$$PV = 500\,\ddot{a}_{\overline{5}|} \text{ at } j = 9.2727\%$$
$$= \$2{,}110.21$$

EXAMPLE 4

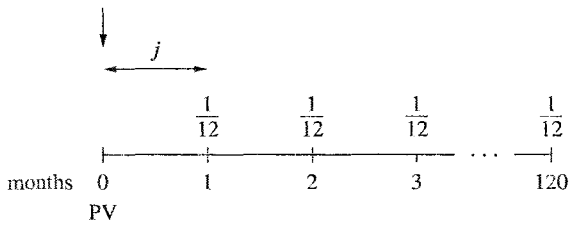
Find the PV of an annuity of $\frac{1}{12}$ paid at the end of each month for 10 years at an effective rate of 5% per annum.

SOLUTION

If j is the equivalent monthly effective rate of interest, we have

$$(1+j)^{12} = 1.05$$
$$j = 1.05^{\frac{1}{12}} - 1 = 0.4074\%$$

The time diagram in terms of monthly periods is as follows:



$$PV = \frac{1}{12}\,a_{\overline{120}|} \text{ at } j = 0.4074\%$$
$$= 7.8971$$

§ 4b. Annuities with "Off Payments"—Part II

In this section, we will take the second approach referred to in Section 4a, i.e., we will use the effective interest rate *given in the problem.*

§ 4b(i) Payments Less Frequent Than the Interest Period ("Fission" Method)

Stepping Stones

EXAMPLE 1

Determine the PV of the annuity with payments of 1 at the end of each 5-year period over 40 years at an annual effective rate of 5%, in terms of interest functions at 5%.

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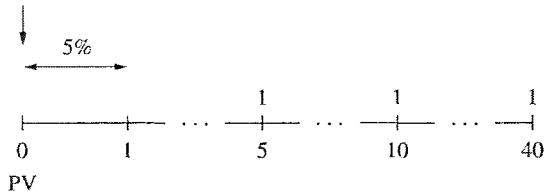
SOLUTION

The time diagram for example (1) above, in terms of one-year periods, is as follows:



We can use a "brute force" approach to express the PV of the annuity at 5% by discounting each payment:

$$PV = v^5 + v^{10} + \ldots + v^{40} \text{ at } 5\%$$

Summing the geometric progression (and recalling Section 3a. to avoid any traps), we get:

$$PV = \frac{v^5[1-(v^5)^8]}{1-v^5} = \frac{v^5(1-v^{40})}{1-v^5}$$

We can simplify this a bit by multiplying numerator and denominator by $(1+i)^5$:

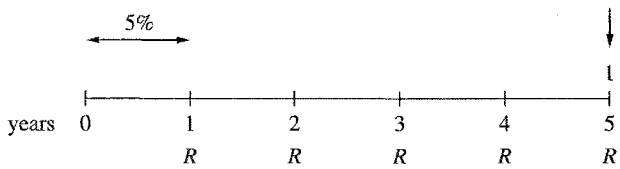
$$PV = \frac{1-v^{40}}{(1+i)^5-1}$$

Examine this result to see if anything looks familiar. Did you notice that $1-v^{40}$ is the numerator of $a_{\overline{40|}}$ and $(1+i)^5-1$ is the numerator of $s_{\overline{5|}}$? This gives us:

$$PV = \frac{1-v^{40}}{(1+i)^5-1} = \left(\frac{1}{s_{\overline{5|}}}\right) a_{\overline{40|}} = \frac{17.1591}{5.5256} = 3.1054$$

Note that this is the same answer we got for Example 1 in Section 4a where we used the equivalent rate j for the 5-year payment period. It appears that the given annuity (paid once every 5 years over 40 years) is equivalent in value to an annuity paid ***annually*** for 40 years. When we get such a neat result, we can be almost certain that there is a "general reasoning" explanation for it: an explanation that relies more on a verbal argument than it does on algebra.

Here is the explanation: We can change the given annuity (which has payments every 5 years) into an equivalent annuity with annual payments. Consider the first payment of 1 at time 5. What ***annual*** payment for 5 years would you be willing to accept in lieu of a single payment of 1 at the end of 5 years? If we call the equivalent annual payment R, the time diagram looks like this:



A convenient comparison date is time 5. The equation of value is:

$$R\, s_{\overline{5|}} = 1$$

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from which $R = \frac{1}{s_{\overline{5}|}}$. We are saying that 5 annual payments of $\frac{1}{s_{\overline{5}|}}$ would make us just as happy as a single payment of 1 at the end of 5 years. This is clearly true, because if we deposited $\frac{1}{s_{\overline{5}|}}$ for 5 years at 5%, the deposits would accumulate to 1.

If you carefully consider the ***second*** payment of 1 at time 10 under the original annuity, you should be able to verify that the same payments $R = \frac{1}{s_{\overline{5}|}}$ paid at time 6 to 10 are equivalent to the payment of 1 at time 10. And so on for the payments of 1 at times 15, 20, ... In fact, $\frac{1}{s_{\overline{5}|}}$ paid annually over the entire 40 year period is equivalent to 1 paid once every 5 years.

So the general reasoning explanation leads to the same result:

$$PV = \left(\frac{1}{s_{\overline{5}|}}\right) a_{\overline{40}|}$$

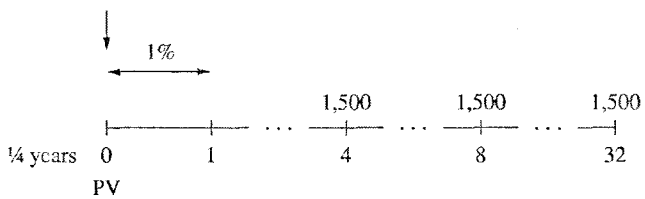
We can call this process of splitting up the actual payments into more frequent payments "fission". We strongly suggest that you practice visualizing the fission process. In the long run, you will be able to handle these types of annuities faster and more accurately by general reasoning than you could by using algebra.

EXAMPLE 2

Try visualizing the fission process in the following problem: Find the PV of an annuity-immediate of \$1,500 a year for 8 years at a nominal rate of interest of 4% compounded quarterly, in terms of interest functions at 1%.

SOLUTION

The time diagram (drawn in terms of $\frac{1}{4}$-year interest periods) is as follows:



The algebraic (geometric progression) approach gives:

$$\begin{aligned} PV &= 1{,}500(v^4 + v^8 + \ldots + v^{32}) \text{ at } 1\% \\ &= 1500v^4 \frac{(1 - v^{32})}{1 - v^4} = 1500 \left(\frac{1 - v^{32}}{(1+i)^4 - 1}\right) \\ &= 1500 \frac{a_{\overline{32}|}}{s_{\overline{4}|}} \text{ at } 1\% = 1500 \frac{27.269589}{4.060401} = \$10{,}073.98 \end{aligned}$$

This is the same answer as for Example 2 in Section 4a.

The fission approach says that 4 quarterly payments of $\frac{1500}{s_{\overline{4}|}}$ at 1% are equivalent to a single payment of \$1,500 at the end of the year. So the given annuity of \$1,500 payable annually is equivalent in value to an annuity of $\frac{1500}{s_{\overline{4}|}}$ payable quarterly:

$$PV = \frac{1{,}500}{s_{\overline{4}|}} a_{\overline{32}|} \quad \text{at } 1\%$$

which is the same result as under the algebraic approach.

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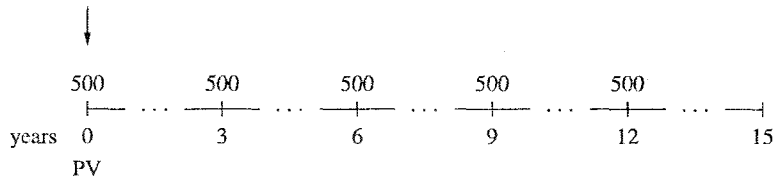
EXAMPLE 3

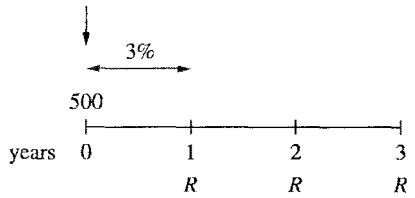
Find the PV of an annuity with payments of 500 at the beginning of every 3 years for 15 years at 3% effective per annum, in terms of interest functions at 3%.

SOLUTION



There are various ways to write the answer. If you got $500\ddot{a}_{\overline{15}|}/\ddot{a}_{\overline{3}|}$, you're right but note that "double-dots cancel" (Section 4e), so this reduces to $500\ a_{\overline{15}|}/a_{\overline{3}|}$. This is the "preferred" form of the answer; usually, symbolic answers will use the "immediate" form in preference to the "due" form whenever possible. (This goes back to "the old days" before calculators, when people used interest tables that showed values for the "immediate" form.)

We will explain why the PV is $500\ a_{\overline{15}|}/a_{\overline{3}|}$. Consider the first payment of \$500 at time 0. Let's use fission and split it into 3 annual payments:



Note that we have placed the equivalent annual payment R at the ***end*** of each year. We choose to do this because we want the answer to be expressed in terms of annuities-immediate. It wouldn't be incorrect to place R at the ***beginning*** of each year but we would end up with "double-dots" in the answer.

Using time 0 as the comparison date, we have

$$500 = R\ a_{\overline{3}|}\,, \qquad R = \frac{500}{a_{\overline{3}|}}$$

Continuing the fission process with each of the \$500 payments, you will find that an annuity with payments of $500/a_{\overline{3}|}$ at the end of each year for 15 years is equivalent in value to the given annuity. The present value of such an annual payment annuity is $\left(\frac{500}{a_{\overline{3}|}}\right) a_{\overline{15}|} = \$2{,}110.21$. This is the same answer we got for Example 3 in Section 4a.

§ 4b(ii) Payments More Frequent Than the Interest Period ("Fusion" Method)

Stepping Stones

EXAMPLE 1

Express the present value of an annuity-immediate with monthly payments of $\frac{1}{12}$ for 10 years at 5% effective per annum in terms of annuity functions at 5%.

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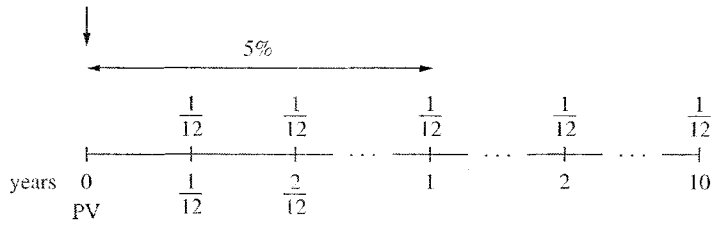
SOLUTION

The following is the time diagram:



$$PV = \frac{1}{12}(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \ldots + v^{\frac{120}{12}})$$

The standard symbol for the PV is $a_{\overline{10|}}^{(12)}$ (\$1 per annum payable in monthly installments of $\frac{1}{12}$ each for 10 years). It can be shown that by summing the geometric progression, the following formula can be derived:

$$a_{\overline{10|}}^{(12)} = \frac{1}{12}(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \ldots + v^{\frac{120}{12}}) = \frac{1 - v^{10}}{i^{(12)}} = 7.897133$$

(Recall that $i^{(12)} = 12[(1+i)^{\frac{1}{12}} - 1]$)

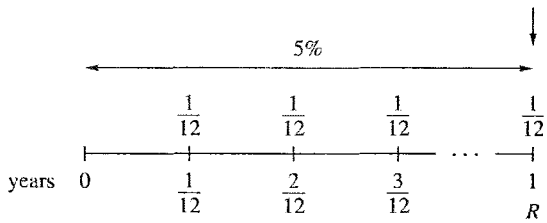
Once again, this is the same result as in the corresponding example (Example 4) in Section 4a.

Since $a_{\overline{10|}} = \frac{1-v^{10}}{i}$, the following relationship between $a_{\overline{10|}}$ and $a_{\overline{10|}}^{(12)}$ can be stated:

$$a_{\overline{10|}}^{(12)} = \frac{i}{i^{(12)}} a_{\overline{10|}}$$

In this form, we can see that the given annuity (with monthly payments of $\frac{1}{12}$) is equivalent in value to an annuity with ***annual*** payments of $\frac{i}{i^{(12)}}$. This can be explained by general reasoning:

Consider the payments in the first year:



The "R" at time 1 is the accumulated value of the 12 monthly payments of $\frac{1}{12}$. It represents the single annual payment you would be just as happy with as you would be with the 12 monthly payments of $\frac{1}{12}$.

$$R = \frac{1}{12}[1 + (1+i)^{\frac{1}{12}} + \ldots + (1+i)^{\frac{11}{12}}]$$

$$= \frac{1}{12} \cdot \frac{1 - [(1+i)^{\frac{1}{12}}]^{12}}{1 - (1+i)^{\frac{1}{12}}} = \frac{i}{12[(1+i)^{\frac{1}{12}} - 1]}$$

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Since $i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1]$ (Section 1a(vii)), we can express this result as:

$$R = \frac{i}{i^{(12)}}$$

We can call this process of accumulating the monthly payments into one equivalent annual payment "fusion". You should be able to verify that the same $R = i/i^{(12)}$ paid at the end of the second year is equivalent to the monthly payments of $\frac{1}{12}$ in the second year. And so on for years 3, 4, . . . , 10. This verifies by general reasoning that an annuity with annual payments of $i/i^{(12)}$ is equivalent to an annuity with monthly payments of $\frac{1}{12}$.

One important note about symbols: The fused amount $i/i^{(12)}$ can be written another way. Since $i/i^{(12)}$ is an accumulated value, it can be represented by an "s" symbol. Since the accumulation is over one interest period, $n = 1$ and the symbol (so far) is $s_{\overline{1}|}$. Since payments are monthly, we modify the symbol as follows: $s_{\overline{1}|}^{(12)}$. And since the total payment in an interest period is 1, the coefficient of the symbol is 1. Thus, $i/i^{(12)} = s_{\overline{1}|}^{(12)}$.

We can verify this in the other direction, that is, starting with $s_{\overline{1}|}^{(12)}$. (See Section 4f.) All "s" functions have $(1+i)^n - 1$ as numerators, so the numerator of $s_{\overline{1}|}^{(12)}$ is $(1+i)^1 - 1 = i$. The denominator is told by the symbol; in this case, the "upper 12" and the fact that this is an annuity-immediate tell us that the denominator is $i^{(12)}$. Therefore, $s_{\overline{1}|}^{(12)} = i/i^{(12)}$.

So in general, the PV of an n-year annuity-immediate of 1 per annum payable in m-thly installments is:

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|} = s_{\overline{1}|}^{(m)} a_{\overline{n}|}$$

Similarly, the AV is:

$$s_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} s_{\overline{n}|} = s_{\overline{1}|}^{(m)} s_{\overline{n}|}$$

EXAMPLE 2

A 10-year annuity-immediate provides quarterly payments of \$1,000 at 4% effective per annum. What is the present value of this annuity, using a symbol based on a 4% effective rate of interest?

SOLUTION

The total payment in a one-year interest period is \$4,000, paid in quarterly installments. The present value can be expressed as:

$$PV = 4{,}000 a_{\overline{10}|}^{(4)} \quad \text{at } 4\%$$

Annuity-Due

It doesn't take much imagination to guess that the annuity-due formulas are the same as the annuity-immediate formulas except that $d^{(m)}$ is in the denominator instead of $i^{(m)}$. This, indeed, turns out to be true:

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}} = \frac{i}{d^{(m)}} a_{\overline{n}|} = \ddot{s}_{\overline{1}|}^{(m)} a_{\overline{n}|}$$

and similarly for $\ddot{s}_{\overline{n}|}^{(m)}$.

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EXAMPLE 3

A 5-year annuity-due provides annual payments of \$15,000, payable in semiannual installments, at an effective annual interest rate of 3.5%. What is the present value of this annuity, using a symbol based on a 3.5% effective rate of interest?

SOLUTION

The total payment in a one-year period is \$15,000, paid in semiannual installments. The present value can be expressed as:

$$PV = 15{,}000 a_{\overline{5}|}^{(2)} \quad \text{at } 3.5\%$$

Perpetuities

If the m-thly payments continue forever, we have a perpetuity. Consider the perpetuity-immediate:

$$\lim_{n\to\infty} a_{\overline{n}|}^{(m)} = a_{\overline{\infty}|}^{(m)} = \lim_{n\to\infty}\left(\frac{1-v^n}{i^{(m)}}\right) = \frac{1}{i^{(m)}}$$

Similarly, for the perpetuity-due, we find that:

$$\ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}}$$

Question: What's the only difference between the perpetuity-due and the perpetuity-immediate?

Answer: The only difference is the first payment of $\frac{1}{m}$ at time 0 included in the perpetuity-due. All of the subsequent payments are included in both perpetuities. Thus, we can state the following identity:

$$\ddot{a}_{\overline{\infty}|}^{(m)} - a_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \frac{1}{m}$$

This is an identity that's good to know. You won't forget it if you remember it as the difference between the "due" and "immediate" forms of the m-thly perpetuities.

EXAMPLE 4

A perpetuity-immediate provides monthly payments of \$500 at an effective annual rate of interest of 6%. What is the present value of this perpetuity in terms of a symbol based on a 6% effective rate of interest?

SOLUTION

The total payment in a one-year interest period is \$6,000, paid in monthly installments. The present value can be expressed as:

$$PV = 6{,}000/i^{(12)} \quad \text{at } 6\%$$

EXAMPLE 5

Get a numerical result for the present value in Example 4, based on the symbol at 6% effective.

SOLUTION

$$i^{(12)} = 12(1.06^{1/12} - 1) = .058411$$

$$PV = 6{,}000/i^{(12)} = \$102{,}721$$

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§4c. Avoiding the m^{thly} Annuity Trap

Trap Alert!

Let's say that we are using the approach in Section 4b for evaluating annuities payable more frequently than the interest period. In other words, we are using symbols such as $a_{\overline{n}|}^{(m)}$ at the given effective rate of interest. In this case, there is one trap that almost everyone (including the author) falls into from time to time. It's just a careless mistake that's very easy to make—and very easy to avoid if you follow the suggestions below.

Stepping Stones

EXAMPLE 1

An annuity will pay \$100 at the end of every month for 10 years at an effective annual rate of interest of 5%. Express the present value of the payments in terms of an annuity symbol at 5%.

(No choices—you're on your own.)

SOLUTION

The correct answer is $1{,}200\, a_{\overline{10}|}^{(12)}$. The trap is to use a coefficient of 100 (the amount of the payment) rather than the correct coefficient of \$1,200 (the sum of the payments in a one-year interest period).

If you go back to Section 4b to see how $a_{\overline{n}|}^{(m)}$ was defined, you will find that it is the present value of 1 per interest period ***paid in m-thly installments***. In other words, the actual payment is $1/m$, paid at the end of $\frac{1}{m}, \frac{2}{m}, \ldots,$ years for n years. Thus, the coefficient of the $a_{\overline{n}|}^{(m)}$ symbol is ***not*** the actual payment $\frac{1}{m}$ but the ***sum of the payments in one interest period*** (i.e., 1).

So to avoid the trap, remember the following rule:

> The coefficient of the $a_{\overline{n}|}^{(m)}$ symbol is the ***sum of the payments*** in each interest period.

In the example, the total payment in each (annual) interest period is \$1,200 (\$100 for 12 months). Thus, the PV is $1200\, a_{\overline{10}|}^{(12)}$.

After you write your answer, as a final check divide the coefficient by the frequency m. This should give the actual payment amount. For example, if you thought the answer to the example was $1200\, a_{\overline{10}|}^{(12)}$, the final check would be to divide 1,200 (the coefficient) by 12 (the frequency m). The result is 100, which is the correct amount of the payment. Had you written $100\, a_{\overline{10}|}^{(12)}$ in error and applied this check, you would have gotten 100/12, which is not the actual payment. This would tell you that you made a mistake.

(You could also multiply the frequency by the number of interest periods: $12 \times 10 = 120$, which should be the total number of payments.)

EXAMPLE 2

An 8-year annuity-immediate provides quarterly payments of \$18,000 at a nominal annual rate of interest of 6%, compounded semiannually. What is the present value of this annuity, in terms of a symbol based on an effective rate of interest of 3% for a six-month period?

SOLUTION

This is a little tricky.

The effective rate of interest implied by a nominal annual rate of 6% compounded semiannually is 3% for a 6-month period.

There are two payments each 6-month interest period (the payments are quarterly), so $m = 2$.

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There are 16 6-month interest periods in the 8-year term of the annuity, so $n = 16$.

The total payment in each 6-month interest period is \$36,000 (2 quarterly payments of \$18,000 each).

Putting this all together, we have:

$$PV = 36{,}000 a_{\overline{16|}}^{(2)} \quad \text{at } 3\%$$

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Summary of Concepts and Formulas in Sections 4a to 4c

(1) There are two general approaches for evaluating annuities with "off-payments," i.e., annuities with payments that are more or less frequent than the interest period:

(a) Determine the interest rate for the payment period that is equivalent to the interest rate given in the problem. This simplifies the problem to a Section 3 type of annuity, where the payment period and the interest period are the same.

(b) Stay with the original interest rate given in the problem. Determine the payment in each interest period that is equivalent to the original series of payments (using "fission" or "fusion").

(2) Approach (1)(a) is the most convenient approach for numerical work, so you will be using it most of the time. Occasionally, if the answer choices are left in symbolic form, you might have to use approach (1)(b).

(3) (a) Under approach (1)(b), if payments are made every k interest periods (k an integer) over n interest periods (n a multiple of k), the PV of an annuity-immediate (first payment at time k) is $\frac{a_{\overline{n}|}}{s_{\overline{k}|}}$ and the PV of an annuity-due (first payment at time 0) is $\frac{a_{\overline{n}|}}{a_{\overline{k}|}}$.

(b) Under approach (1)(b), if payments are made m times each interest period over n interest periods, the PV of an annuity-immediate with payments of $1/m$ (first payment at time $1/m$) is $a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}}$ and the PV of an annuity-due (first payment at time 0) is $\ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}}$.

(4) Avoid the m^{thly} annuity trap. For example, the PV of a 10-year annuity-immediate with monthly payments of \$1.00 is $12a_{\overline{10}|}^{(12)}$, not $a_{\overline{10}|}^{(12)}$.

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Past Exam Questions on Sections 4a to 4c

1. Kathryn deposits 100 into an account at the beginning of each 4-year period for 40 years. The account credits interest at an annual effective interest rate of i. The accumulated amount in the account at the end of 40 years is X, which is 5 times the accumulated amount in the account at the end of 20 years. Calculate X. [5/03 #8]

   (A) 4695 (B) 5070 (C) 5445 (D) 5820 (E) 6195

2. Jerry will make deposits of 450 at the end of each quarter for 10 years. At the end of 15 years, Jerry will use the fund to make annual payments of Y at the beginning of each year for 4 years, after which the fund is exhausted. The annual effective rate of interest is 7%. Determine Y. [11/00 #22]

   (A) 9573 (B) 9673 (C) 9773 (D) 9873 (E) 9973

3. A perpetuity paying 1 at the beginning of each 6-month period has a present value of 20. A second perpetuity pays X at the beginning of every 2 years. Assuming the same annual effective interest rate, the two present values are equal. Determine X. [5/00 #14]

   (A) 3.5 (B) 3.6 (C) 3.7 (D) 3.8 (E) 3.9

4. Raj deposits 100 into a fund at the end of each 2-year period for 20 years. The fund pays interest at an annual effective rate of i. The total amount of interest earned by the fund during the 19th and 20th years is 250. Calculate the accumulated amount in Raj's account at the end of year 20. [SOA 5/98 #12]

   (A) 1925 (B) 1950 (C) 1975 (D) 2000 (E) 2025

5. The present value of a perpetuity is \$9.25. The perpetuity pays \$1 at the end of every 2 years (that is to say, one payment every other year), with the first payment due immediately. Cheryl will make 3 payments of \$300 each. The first payment is due one year from now, with successive payments due every third year thereafter. Determine the present value of Cheryl's payments at the same annual effective interest rate used to determine the present value of the perpetuity described above. [CAS 11/97 #2]

   (A) Less than \$725
   (B) At least \$725, but less than \$740
   (C) At least \$740, but less than \$755
   (D) At least \$755, but less than \$770
   (E) \$770 or more

6. Mr. Lewis borrows \$10,000 on January 1, 1996. He begins to repay the loan as follows:
   January 1, 1997 \$1,000
   January 1, 1998 \$1,000
   January 1, 1999 \$1,000
   On July 1, 1999, he makes an additional payment of \$5,000. Mr. Lewis makes no further payments. You may assume $d^{(4)} = 0.08$. Determine how much Mr. Lewis owes on this loan as of January 1, 2005. [CAS 11/96 #21]

   (A) Less than \$7,000
   (B) At least \$7,000, but less than \$7,250
   (C) At least \$7,250, but less than \$7,500
   (D) At least \$7,500, but less than \$7,750
   (E) \$7,750 or more

7. The present value of a 5-year annuity immediate, with payments of \$1,000 each 6 times per year, is \$20,930. Determine the force of interest. [CAS 5/95 #1]

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(A) Less than 14.65%
(B) At least 14.65%, but less than 14.75%
(C) At least 14.75%, but less than 14.85%
(D) At least 14.85%, but less than 14.95%
(E) At least 14.95%

8. Trish had reimbursable health care expenses of \$100 per week for the first 50 weeks of the year. Trish sent in claim forms for all of her reimbursable expenses in bulk at the end of the 50th week and received payment at the end of the 51st week, which she deposited in her account on the same day that she received it. Assume the annual effective rate of interest is 10.95%. Determine the difference between the amount actually accumulated in her savings account at year end and the amount that could have been accumulated if she had sent in her claim forms at the end of each week and received payment at the end of the following week. [CAS 5/95 #7]

(A) Less than \$230
(B) At least \$230, but less than \$240
(C) At least \$240, but less than \$250
(D) At least \$250, but less than \$260
(E) At least \$260

9. Norma receives \$500,000 from a life insurance policy with which she purchases an annuity-certain. The annuity will pay 10 equal annual installments, with the first payment made immediately. On the day she receives her fourth payment she is offered, in lieu of the future annual payments, a new payment scheme:

(i) 4 annual payments of \$40,000, beginning in one year, followed by a monthly perpetuity of X
(ii) The first monthly perpetuity payment would occur one month after the fourth annual payment of \$40,000.

The effective annual rate of interest is 9% for the entire time period. Determine the value of X [CAS 5/95 #11]

(A) Less than \$1,500
(B) At least \$1,500, but less than \$1,700
(C) At least \$1,700, but less than \$1,900
(D) At least \$1,900, but less than \$2,100
(E) At least \$2,100

10. Harriet wishes to accumulate \$60,000 in a fund at the end of 25 years. She plans to deposit \$80 into the fund at the end of each of the first 120 months. She then plans to deposit \$80 + x into the fund at the end of each of the last 180 months. Assume the fund earns interest at an annual effective rate of 3.66%. Determine x. [CAS 5/95 #15]

(A) Less than \$86
(B) At least \$86, but less than \$88
(C) At least \$88, but less than \$90
(D) At least \$90, but less than \$92
(E) At least \$92

11. Betty received \$500,000 from a life insurance policy to be distributed to her as an annuity certain in 10 equal annual installments with the first payment made immediately. On the day she receives her 3rd payment, she is offered a monthly perpetuity of X in lieu of the future annual payments. The first payment will be made in exactly one month. The effective annual rate of interest is 7%. Determine the value X. [CAS 5/94 #6]

(A) Less than \$1,800
(B) At least \$1,800, but less than \$1,900
(C) At least \$1,900, but less than \$2,000

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(D) At least \$2,000, but less than \$2,100
(E) At least \$2,100

12. The University of the State of Turmoil wishes to invest \$100,000 in an interest bearing account. Beginning 6 years after the deposit, scholarship payments of \$10,000 per year are to be made in perpetuity. Determine the minimum effective annual rate of interest that the University must earn on its investments in order to fund this perpetuity as planned. [CAS 5/94 #9]

(A) Less than 7.04%
(B) At least 7.04% but less than 7.08%
(C) At least 7.08% but less than 7.12%
(D) At least 7.12% but less than 7.16%
(E) At least 7.16%

13. Jim and Sue are planning to retire on January 1, 1995. Their goal is to have enough money in savings to be able to withdraw \$3,000 per month beginning one month after retirement and continuing for 25 years after retirement. They earn an annual effective rate of interest of 10% on their account. Determine the minimum amount needed in their savings account on January 1, 1995, to accomplish their goal. [CAS 11/93 #4]

(A) Less than \$325,000
(B) At least \$325,000, but less than \$335,000
(C) At least \$335,000, but less than \$345,000
(D) At least \$345,000, but less than \$355,000
(E) At least \$355,000

14. Kendall borrows \$100,000 on January 1, 1993 to be repaid in 12 annual installments at an effective annual rate of interest of 8%. The first payment is due on January 1, 1994. Instead of annual payments she decides to make monthly payments equal to one-twelfth the annual payment beginning on February 1, 1993. Determine how many months will be needed to pay off the loan. [CAS 5/93 #4]

(A) 136 (B) 137 (C) 138 (D) 139 (E) 140

15. The present value of a perpetuity of 6,500 paid at the end of each year plus the present value of a perpetuity of 8,500 paid at the end of every 5 years is equal to the present value of an annuity of k paid at the end of each year for 25 years. Interest is 6% convertible quarterly. Calculate k. [SOA 5/93 #7]

(A) 10,340 (B) 11,340 (C) 12,340 (D) 19,370 (E) 19,560

16. Tom has served in the Navy for 21 years and plans to retire sometime in the future. Every three years he has received a re-enlistment bonus and deposited a certain dollar amount, X, in an account for when he retires. After making his current deposit, the value of his account is \$180,176. Assuming an annual effective interest rate of 9%, what were his tri-annual deposits? [CAS 5/91 #8]

(A) $X < \$10{,}400$ (B) $\$10{,}400 \le X < \$10{,}425$ (C) $\$10{,}425 \le X < \$10{,}450$ (D) $\$10{,}450 \le X < \$10{,}475$ (E) $\$10{,}475 \le X$

17. John wins \$1,000,000 in a lottery and will be paid 20 equal annual installments of \$50,000 with the first payment due today. A bank offers to exchange John's winnings for a perpetuity of \$X per month with the first payment due today. Find the value closest to \$X assuming a 10% effective rate of interest. [CAS 5/90 #6]

(A) \$3,330 (B) \$3,360 (C) \$3,550 (D) \$3,700 (E) \$3,730

18. Jim buys a perpetuity of 100 per year, with the first payment 1 year from now. The price for the perpetuity is 975.61, based on a nominal yield of i compounded semiannually. Immediately after

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the second payment is received, the perpetuity is sold for 1642.04, to earn for the buyer a nominal yield of j compounded semiannually. Calculate i–j. [SOA 5/90 #3]

(A) 0.020 (B) 0.030 (C) 0.035 (D) 0.040 (E) 0.042

19. A 20-year annuity pays 100 every other year beginning at the end of the second year, with additional payments of 300 each at the end of years 3, 9 and 15. The effective annual interest rate is 4%. Calculate the present value of the annuity. [SOA 5/90 #7]

(A) 1310 (B) 1340 (C) 1370 (D) 1400 (E) 1430

20. You are given:

(i) the present value of a $6n$-year annuity-immediate of 1 at the end of every year is 9.996;
(ii) the present value of a $6n$-year annuity-immediate of 1 at the end of every second year is 4.760; and
(iii) the present value of a $6n$-year annuity-immediate of 1 at the end of every third year is X;

Calculate X. [SOA 5/90 #9]

(A) 2.87 (B) 3.02 (C) 3.17 (D) 3.32 (E) 4.17

21. The proceeds of a 10,000 death benefit are left on deposit with an insurance company for seven years at an annual effective interest rate of 5%. The balance at the end of seven years is paid to the beneficiary in 120 equal monthly payments of X, with the first payment made immediately. During the payout period, interest is credited at an annual effective interest rate of 3%. Calculate X. [SOA 11/89 #16]

(A) 117 (B) 118 (C) 129 (D) 135 (E) 158

22. You are given a perpetuity, with annual payments as follows:

(i) Payments of 1 at the end of the first year and every three years thereafter.
(ii) Payments of 2 at the end of the second year and every three years thereafter.
(iii) Payments of 3 at the end of the third year and every three years thereafter.

The interest rate is 5% convertible semiannually. Calculate the present value of this perpetuity. [SOA 5/89 #8]

(A) 24 (B) 29 (C) 34 (D) 39 (E) 47

23. Which of the following statements are true?

(1) $s_{\overline{n}|} - a_{\overline{n}|} = i\ a_{\overline{n}|}\ s_{\overline{n}|}$
(2) $\ddot{s}^{(m)}_{\overline{n}|i} - s^{(m)}_{\overline{n}|i} = \frac{i^{(m)}}{m}\ s^{(m)}_{\overline{n}|i}$
(3) ${}_{1/4|}\ddot{a}^{(2)}_{\overline{n}|i} + a^{(2)}_{\overline{n}|i} = a^{(4)}_{\overline{n}|i}$

[CAS 5/87 #5]

(A) (1) (B) (3) (C) (1), (2) (D) (2) ,(3) (E) (1), (2), (3)

24. You currently have \$1000 in an account which pays a nominal rate of interest of 8% compounded quarterly. You plan to deposit \$200 every two months with the first deposit one month from now. What will be the value of the account one month after the eighteenth deposit? [CAS 5/87 #10]

(A) Less than \$5,280
(B) At least \$5,280, but less than \$5,300 (C) At least \$5,300, but less than \$5,320 (D) At least \$5,320, but less than \$5,340 (E) \$5,340 or more

25. A car dealer offers to sell a car for 10,000. The current market loan rate is a nominal rate of interest of 12% per annum, compounded monthly. As an inducement, the dealer offers 100% financing at an effective annual interest rate of 5%. The loan is to be repaid in equal installments at the end of each month over a four-year period. Calculate the cost to the dealer of this inducement. [SOA 5/87 #4]

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(A) 700 (B) 900 (C) 1100 (D) 1300 (E) 1500

26. Determine the present value of 1 payable at the end of years 7, 11, 15, 19, 23 and 27. [SOA 5/87 #7]

(A) $\frac{a_{\overline{28}|}-a_{\overline{4}|}}{a_{\overline{4}|}}$ (B) $\frac{a_{\overline{28}|}-a_{\overline{4}|}}{s_{\overline{4}|}}$ (C) $\frac{a_{\overline{28}|}-a_{\overline{4}|}}{s_{\overline{3}|}+d}$ (D) $\frac{a_{\overline{28}|}-a_{\overline{4}|}}{s_{\overline{3}|}-a_{\overline{1}|}}$ (E) $\frac{a_{\overline{28}|}-a_{\overline{4}|}}{s_{\overline{3}|}+a_{\overline{1}|}}$

27. Janis needs an amount on January 1, 2025 to provide for a lump sum of 50,000 and a 15-year annuity-due with semiannual payments of K. The amount will be accumulated by 25 annual deposits of K beginning on January 1, 2000. The deposits accumulate at a nominal rate of 4% compounded semiannually. The annuity payout is based on a nominal rate of 3% compounded semiannually. Determine an expression for K. [SOA 5/87 #9]

(A) $\frac{50,000}{\frac{s_{\overline{50}|}^{0.02}}{a_{\overline{2}|}^{0.02}}-\ddot{a}_{\overline{30}|}^{0.015}}$

(B) $\frac{50,000}{\frac{s_{\overline{50}|}^{0.02}}{s_{\overline{2}|}^{0.02}}-\ddot{a}_{\overline{30}|}^{0.015}}$

(C) $\frac{50,000}{\frac{s_{\overline{50}|}^{0.02}}{s_{\overline{2}|}^{0.02}}-\frac{a_{\overline{30}|}^{0.015}}{a_{\overline{2}|}^{0.015}}}$

(D) $\frac{50,000}{\frac{s_{\overline{50}|}^{0.02}}{a_{\overline{2}|}^{0.02}}-\frac{a_{\overline{30}|}^{0.015}}{s_{\overline{2}|}^{0.015}}}$

(E) $\frac{50,000}{\frac{s_{\overline{50}|}^{0.02}}{a_{\overline{2}|}^{0.02}}-a_{\overline{30}|}^{0.015}}$

28. As settlement of a 100,000 death benefit, a beneficiary elected to take an annuity-immediate payable monthly for 25 years. The monthly payment was calculated using an effective annual interest rate of 3%. After making payments for 10 years, the insurance company decides to increase the monthly payments for the remaining 15 years by changing the effective annual interest rate to 5%. Calculate the increase in the monthly payment. [SOA 5/87 #15]

(A) 60 (B) 62 (C) 64 (D) 66 (E) 68

29. You anticipate having to pay $30,000 per year for your child's college education starting 10 years from now. You plan to finance four years of college by making quarterly deposits in a savings account starting now. The final deposit is made three months prior to the first college payment, for a total of 40 deposits. Each annual college payment is made in full at the beginning of the school year. If the savings account earns 8% per annum convertible quarterly, what should your quarterly deposit be? [CAS 5/86 #6]

(A) Less than $1,700
(B) At least $1,700, but less than $1,710
(C) At least $1,710, but less than $1,720
(D) At least $1,720, but less than $1,730
(E) $1,730 or more

30. You have an annuity which pays $1,200 every two years. The first payment is two years from now and the last payment is ten years from now. You can trade that annuity for another annuity of equivalent present value, which pays $180 per quarter starting today. The interest rate for both annuities is 4% per annum convertible quarterly. If you took the second annuity, how many quarterly payments would you receive? The last payment may be less than $180 but not more than $180. [CAS 5/86 #8]

(A) Fewer than 30 (B) 30 (C) 31 (D) 32 (E) More than 32

31. Jones has been depositing $150 in a savings account every three months for the past 3 years. This account paid 4% convertible quarterly. Jones has just made the last deposit. Jones is buying a car for $8,000. He is taking out a car loan. He will use the accumulated value of his savings account

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as the downpayment on the car. The loan is at 8% interest convertible quarterly and has a term of 4 years. Find the size of Jones' *monthly* car loan payment. [CAS 5/86 #9]

(A) Less than \$146
(B) At least \$146, but less than \$147
(C) At least \$147, but less than \$148
(D) At least \$148, but less than \$149
(E) \$149 or more.

32. Which of the following is equal to $a_{\overline{5}|} + a_{\overline{10}|} + a_{\overline{15}|} + \cdots + a_{\overline{100}|}$? [CAS 5/85 #5]

(A) $\frac{1}{i\,a_{\overline{5}|}}\,[20a_{\overline{5}|} - a_{\overline{100}|}]$

(B) $\frac{1}{i\,s_{\overline{5}|}}\,[20s_{\overline{5}|} - a_{\overline{100}|}]$

(C) $\frac{1}{i\,s_{\overline{5}|}}\,[20 - a_{\overline{100}|}]$

(D) $\frac{20}{i} - \frac{a_{\overline{100}|}}{s_{\overline{5}|}}$

(E) $\frac{20 - a_{\overline{100}|}}{s_{\overline{5}|}}$

33. A loan of \$100,000 is to be repaid by 20 equal quarterly payments including principal and interest at a nominal interest rate of 6% per year compounded semiannually. The first payment is at the end of the first quarter. What is the size of each payment? [CAS 5/85 #8]

(A) Less than \$5,790
(B) At least \$5,790, but less than \$5,800
(C) At least \$5,800, but less than \$5,810
(D) At least \$5,810, but less than \$5,820
(E) \$5,820 or more

34. A perpetuity of \$1 each year, with the first payment due immediately, has a present value of \$25 at an annual effective rate of i%. The owner exchanges it for another perpetuity with the first payment due immediately and subsequent payments due at two year intervals. What should the payment of the second perpetuity be, in order to keep the same interest rate, i%, and the same present value? [CAS 5/84 #1]

(A) Less than \$1.90
(B) At least \$1.90, but less than \$1.94
(C) At least \$1.94, but less than \$1.98
(D) At least \$1.98, but less than \$2.02
(E) \$2.02 or more

35. A college education fund is to be accumulated by twenty level semiannual deposits, the first due on January 1, 1986. The fund is to provide sixteen quarterly withdrawals of \$1000 each, the first due on October 1, 1995. Interest is compounded semiannually at an annual rate of 10%. What is the amount of each deposit? [CAS 5/84 #5]

(A) Less than \$375
(B) At least \$375, but less than \$380
(C) At least \$380, but less than \$385
(D) At least \$385, but less than \$390
(E) \$390 or more

36. Carl deposits \$200 at the beginning of each month in a savings account that provides an effective annual interest rate of 4%. At the end of each month he withdraws \$100. What is the balance in the account at the end of three years, i.e., following the 36th withdrawal? [CAS 5/84 #15]

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(A) Less than \$3832
(B) At least \$3832, but less than \$3835
(C) At least \$3835, but less than \$3838
(D) At least \$3838, but less than \$3841
(E) \$3841 or more

37. A series of nineteen payments of \$1,900 each is to be made with the first payment one year hence and with each subsequent payment two years after the preceding one. The annual effective interest rate is $3\frac{1}{2}\%$. Find the present value of the series of payments. [SOA SAMPLE/84 #7]

(A) \$18,900 (B) \$19,500 (C) \$20,100 (D) \$20,800 (E) \$21,600

38. A payment of \$100 is made at the end of each two months for a period of six years. The nominal annual rate of interest is 3%, convertible every 8 months. Find the present value of this series of payments. [SOA SAMPLE/84 #13]

(A) \$3,150 (B) \$3,210 (C) \$3,250 (D) \$3,290 (E) \$3,330

39. A perpetuity paying \$50 on the last day of each year was purchased on January 1, 1928. On January 1, 1978, the perpetuity was exchanged for a 15-year annuity-due with semiannual payments of amount X. The interest rate is 6%, convertible monthly. Find X. [SOA SAMPLE/84 #16]

(A) \$10,000 $\dfrac{a_{\overline{6}|.005}}{a_{\overline{180}|.005}\, a_{\overline{12}|.005}}$

(B) \$10,000 $\dfrac{a_{\overline{6}|.005}}{a_{\overline{180}|.005}\, \ddot{a}_{\overline{12}|.005}}$

(C) \$10,000 $\dfrac{a_{\overline{6}|.005}}{a_{\overline{180}|.005}\, s_{\overline{12}|.005}}$

(D) \$10,000 $\dfrac{\ddot{a}_{\overline{6}|.005}}{A_{\overline{180}|.005}\, a_{\overline{12}|.005}}$

(E) \$10,000 $\dfrac{\ddot{a}_{\overline{6}|.005}}{a_{\overline{180}|.005}\, s_{\overline{12}|.005}}$

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Solutions to Past Exam Questions on Sections 4a to 4c

1. End of 40 years: $AV = 100\dfrac{s_{\overline{40|}}}{a_{\overline{4|}}} = X$

End of 20 years: $AV = 100\dfrac{s_{\overline{20|}}}{a_{\overline{4|}}}$

$100\dfrac{s_{\overline{40|}}}{a_{\overline{4|}}} = 500\dfrac{s_{\overline{20|}}}{a_{\overline{4|}}}$

$(1+i)^{20} + 1 = 5$

(Note: $\dfrac{s_{\overline{2n|}}}{s_{\overline{n|}}} = (1+i)^n + 1$)

$(1+i)^{20} = 4,\ \ i = 7.1773\%$

$X = 100\dfrac{s_{\overline{40|}}}{a_{\overline{4|}}} = \dfrac{20{,}899.09}{3.3737} = 6{,}195$ ANS. (E)

Note: This problem could also be solved by using the 4-year payment period at an effective rate $j = (1+i)^4 - 1$ per period.

2. The AV of the deposits of 450 at the end of 10 years (40 quarters) = $450s_{\overline{40|}j} = 25{,}513.23$, where $j = 1.07^{\frac{1}{4}} - 1 = 1.7059\%$.

The AV 5 years later (i.e., on the date of the 1st withdrawal of Y) is $25{,}513.23(1.07)^5 = 35{,}783.63$. This is equal to the PV of the withdrawals.

$Y\ddot{a}_{\overline{4|}.07} = 35{,}783.63,\ Y = 9{,}873$ ANS. (D)

3. Perpetuity #1: $\dfrac{2}{d^{(2)}} = 20$

Perpetuity #2: $PV = X(1 + v^2 + v^4 + \cdots)$

$= \dfrac{X}{1-v^2} = 20$

and $X = 20\left(1 - v^2\right)$

From #1: $d^{(2)} = 2\left[1 - (1-d)^{\frac{1}{2}}\right]$

$= 2\left[1 - v^{\frac{1}{2}}\right]$, so $\dfrac{2}{d^{(2)}}$

$= \dfrac{1}{1 - v^{\frac{1}{2}}} = 20$

$1 - v^{\frac{1}{2}} = .05,\ \ v^{\frac{1}{2}} = .95,\ \ v = .9025$

Since $X = 20\left(1 - v^2\right)$ (see above),

$X = 20\left(1 - .9025^2\right) = 3.71$ ANS. (C)

4. Let j = effective rate for a 2-year period. The interest earned during the two-year period from the end of the 18th to the end of the 20th year is equal to j times the amount in the fund at the end of

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the 18th year:

$$j \cdot 100 s_{\overline{9}|j} = 100\left[(1+j)^9 - 1\right] = 250$$

$$(1+j)^9 = 3.5,\ j = 14.9349\%$$

$$AV \text{ at end of 20 years} = 100 s_{\overline{10}|j}$$

$$= 2{,}023.93 \qquad \text{ANS. (E)}$$

5. Let j = effective rate for a 2-year period. Then the PV of the perpetuity due =

$$1 + \frac{1}{j} = 9.25,\ j = \frac{1}{8.25} = .1212\ldots$$

$$(1+i) = (1+j)^{\frac{1}{2}} = (1.1212\ldots)^{\frac{1}{2}} = 1.0589,$$

where i is the effective annual rate

$$v = \frac{1}{1+i} = .9444.$$

PV of Cheryl's payments $= 300\left(v + v^4 + v^7\right) = 300(.9444 + .7955 + .6700) = 723$
ANS. (A)

6. (This question could also be included in Chapter 6.)

The amount he owes is the AV of the 10,000 loan minus the AV of the payments, i.e.,

$$10{,}000(1+i)^9 - 1{,}000\left(s_{\overline{9}|} - s_{\overline{6}|}\right) - 5{,}000(1+i)^{5.5},$$

where i is the annual effective rate.

$$d^{(4)} = 4\left[1 - (1-d)^{\frac{1}{4}}\right] = .08$$

$$\therefore\ (1-d)^{\frac{1}{4}} = .98 = v^{\frac{1}{4}},\ v = .9224$$

and $i = 8.4166\%$

$\therefore$ amount owed $= 20{,}694.80 - 5{,}293.42 - 7{,}798.24 = 7{,}603.14$ ANS. (D)

7. Let j = effective rate for $\left(\frac{1}{6}\right)$th of a year.

$$1{,}000 a_{\overline{30}|j} = 20{,}930$$

$$j = .025,\ 1 + i = (1+j)^6 = 1.1597$$

$$\delta = \ln(1+i) = \ln 1.1597 = 14.816\% \qquad \text{ANS. (C)}$$

8. If j is the effective rate for a one-week period, $(1+j)^{52} = 1+i$ and $(1+j) = 1.1095^{\frac{1}{52}} = 1.002000257$, $j = .2000257\%$. The actual amount in Trish's account at the end of the year (one week after she received her 5,000 reimbursement) is $5{,}000(1+j) = 5{,}010$. The amount that could have been accumulated is $100\ddot{s}_{\overline{50}|}$. (She would have received her first reimbursement of 100 at the end of week 2 and her 50th at the end of week 51. At the end of week 52, the AV is $100\ddot{s}_{\overline{50}|}$.) This $AV = 5{,}263.57$, so the difference is $5{,}263.57 - 5{,}010.00 = 253.57$ ANS. (D)

9. The original annual payment R is determined by $R\ddot{a}_{\overline{10}|.09} = 500{,}000$. $\therefore$ $R = 71{,}477.11$. The remaining value of the annuity just after the 4th payment is $71{,}477.11 a_{\overline{6}|.09} = 320{,}640.45$. The PV of

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the payments under the new scheme is $40{,}000a_{\overline{4}|} + \frac{Xv^4}{j}$, where $j = 1.09^{\frac{1}{12}} - 1 = .7207\%$. $\therefore$ $PV = 129{,}588.80 + 98.292415X = 320{,}640.45$ and $X = 1{,}943.71$ ANS. **(D)**

10. The monthly effective rate $j = 1.0366^{\frac{1}{12}} - 1 = .3\%$

$$80s_{\overline{300}|j} + xs_{\overline{180}|j} = 60{,}000$$

$$x = \frac{60{,}000 - 80s_{\overline{300}|j}}{s_{\overline{180}|j}} = \frac{60{,}000 - 38{,}834}{238.2067} = 88.86 \qquad \text{ANS. (C)}$$

11. If the original annual payment is R, $R\ddot{a}_{\overline{10}|.07} = 500{,}000$ and $R = 66{,}531.54$.

The present value of the annuity just after the 3rd payment = $66{,}531.54a_{\overline{7}|} = 358{,}557.74$

To compute the PV of the perpetuity, first determine the equivalent monthly effective rate j:

$$(1+j)^{12} = 1.07$$

$$j = .005654$$

$$\text{PV of perpetuity} = X/j = X/.005654$$

We set this equal to 358,557.74 and solve for X:

$X = 2{,}027.34$ ANS. **(D)**

12. Using time 5 as a comparison date:

$$100{,}000(1+i)^5 = \frac{10{,}000}{i}$$

$$(1+i)^5 i = 0.1$$

Trying $i = 7.04\%, 7.08\%, 7.12\%, \ldots$, we find that $7.08\% < i < 7.12\%$ ANS. **(C)**

13. The monthly effective rate $j = 1.1^{\frac{1}{12}} - 1 = .7974\%$, $3{,}000a_{\overline{300}|j} = 341{,}493$ ANS. **(C)**

14. If R = annual payment, $Ra_{\overline{12}|.08} = 100{,}000$ and $R = 13{,}269.50$. Monthly payment $= \frac{R}{12} = 1{,}105.79$. $1{,}105.79a_{\overline{n}|j} = 100{,}000$, where $j = 1.08^{\frac{1}{12}} - 1 = .6434\%$.

$n = 135.95$ ANS. **(A)**

15.
$$\frac{6{,}500}{i} + \frac{8{,}500v^5}{1-v^5} = ka_{\overline{25}|i}$$

at $i = 1.015^4 - 1 = 6.1364\%$

$k = 10{,}336$ ANS. **(A)**

16. The effective rate for a 3-year period $= j = 1.09^3 - 1 = 29.5029\%$.

$Xs_{\overline{7}|j} = 180{,}176$, $X = 10{,}405$ ANS. **(B)**

17. PV of John's prize $= 50{,}000\ddot{a}_{\overline{20}|.10} = 468{,}246$. The effective monthly rate $j = 1.10^{\frac{1}{12}} - 1 = .0080$.

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