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A modified accelerated monotone iterative method for finite difference reaction–diffusion–convection equations☆

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ABSTRACT

This paper is concerned with monotone algorithms for the finite difference solutions of a class of nonlinear reaction–diffusion–convection equations with nonlinear boundary conditions. A modified accelerated monotone iterative method is presented to solve the finite difference systems for both the time-dependent problem and its corresponding steady-state problem. This method leads to a simple and yet efficient linear iterative algorithm. It yields two sequences of iterations that converge monotonically from above and below, respectively, to a unique solution of the system. The monotone property of the iterations gives concurrently improving upper and lower bounds for the solution. It is shown that the rate of convergence for the sum of the two sequences is quadratic. Under an additional requirement, quadratic convergence is attained for one of these two sequences. In contrast with the existing accelerated monotone iterative methods, our new method avoids computing local maxima in the construction of these sequences. An application using a model problem gives numerical results that illustrate the effectiveness of the proposed method.

1. Introduction

Nonlinear reaction–diffusion–convection equations arise in various physical processes. A great deal of work has been devoted to the qualitative analysis of these equations and numerical algorithms for the computation of their solutions; see [1–10]. We seek a simple and yet efficient computational algorithm for computing numerical solutions of the following nonlinear reaction–diffusion–convection problem:

$$\begin{cases} \partial u/\partial t - \nabla\cdot(D\nabla u) + \mathbf{v}\cdot\nabla u = f(x,t,u), & x\in\Omega,\ t>0,\\ \alpha\partial u/\partial\nu + \beta u = g(x,t,u), & x\in\partial\Omega,\ t>0,\\ u(x,0)=\psi(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded connected domain in $\mathbf{R}^p$ with boundary $\partial\Omega$ $(p=1,2,\ldots)$. ∇ is the gradient operator in Ω, $\mathbf{v}\cdot\nabla u = v_1(x,t)\partial u/\partial x_1 + \cdots + v_p(x,t)\partial u/\partial x_p$. $\partial u/\partial\nu$ is the outward normal derivative of u on $\partial\Omega$. The coefficient $D\equiv D(x,t)$ is differentiable in x and continuous in t. The coefficients $\alpha\equiv\alpha(x,t)$, $\beta\equiv\beta(x,t)$ and $\mathbf{v}\equiv(v_1(x,t),\ldots,v_p(x,t))$ are continuous functions of (x,t). It is assumed that $D(x,t)>0$ on $\overline{\Omega}\times[0,+\infty)$, where $\overline{\Omega}=\Omega\cup\partial\Omega$, $\alpha(x,t)\geq 0$ and

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$\beta(x,t) \geq 0$ with $\alpha(x,t)+\beta(x,t) > 0$ on $\partial\Omega \times [0,+\infty)$. The functions $f(x,t,u)$ and $g(x,t,u)$ (which, in general, are nonlinear in u) are continuous in (x,t) and continuously differentiable in u.

The problem (1.1) describes a number of physical models in various fields of applied science such as chemical reaction–diffusion, heat conduction and population dynamics (see [1,2,5]). In these models, the unknown quantity $u(x,t)$ is called the density function (for example, the chemical concentration in chemical reaction–diffusion processes or the temperature distribution in heat conduction problems or the population density in population dynamics), the function $D(x,t)$ is the diffusion coefficient (for example, the thermal diffusivity in heat conduction problems), the term $\nabla\cdot(D\nabla u)$ represents the rate of change due to diffusion, f is the reaction term which depends on the density function u and possibly on (x,t) explicitly, $\mathbf{v}$ is a given convection velocity field which may vary with time t, and it is usually assumed solenoidal. The boundary condition in (1.1) shows that the flux for u across the boundary may depend nonlinearly on u. When the reaction–diffusion–convection process reaches a steady state, the density function u is independent of t. In this case, the problem (1.1) becomes the corresponding steady-state problem:

$$\begin{cases}-\nabla\cdot(D\nabla u)+\mathbf{v}\cdot\nabla u = f(x,u), & x\in\Omega,\\ \alpha\partial u/\partial\nu+\beta u = g(x,u), & x\in\partial\Omega,\end{cases} \tag{1.2}$$

where the coefficients $D\equiv D(x)$, $\alpha\equiv\alpha(x)$, $\beta\equiv\beta(x)$ and $\mathbf{v}\equiv(v_1(x),\ldots,v_p(x))$ are independent of t.

In the study of numerical solutions of (1.1) and (1.2) by the finite difference method, the corresponding discrete problem is usually formulated as a system of nonlinear algebraic equations. For such a system, a major concern is to obtain reliable and efficient computational algorithms for computing its solution. A fruitful approach is the method of upper and lower solutions coupled with its associated monotone iterative technique. By using an upper solution and a lower solution as the initial iterations, this method produces two sequences that converge monotonically from above and below, respectively, to the extremal solutions of the given nonlinear system. Consequently, this leads to a monotone computational algorithm and it has been applied to different nonlinear problems (cf. [4,6–9,11–19]). Most of these monotone iteration processes are of Picard type, Jacobi type, or Gauss–Seidel type. The rate of convergence of these iterations is only of linear order. To increase the rate of convergence while maintaining the monotone property of the iterations, it is well known that an accelerated technique is to adopt the local maximum in the iteration process (see [6,8,18]). In fact, by utilizing the local maximum of the first-order derivative of the nonlinear function, the algorithm in [6] leads to a monotone iterative method for the finite difference solutions of the time-dependent problem (1.1). As shown in [6], a remarkable fact about this method is that the rate of convergence for the sum of the two sequences of iterations is quadratic. (The error metric is the sum of the norm of the error between the mth iteration of the upper solution and the true solution with the norm of the error between the mth iteration of the lower solution and the true solution). Under an additional requirement, the quadratic rate of convergence is attained for one of these two sequences. As a result, it improves the rate of convergence of Picard, Jacobi, and Gauss–Seidel methods. It is known as *Accelerated Monotone Iterative* (AMI) method. A similar AMI method was developed in [8] for the finite difference solutions of the steady-state problem (1.2). It was extended in [18] using less restrictive assumptions so as to be applicable to a larger class of nonlinear functions. In terms of the order of improvement, the AMI method substantially improves Picard, Jacobi, and Gauss–Seidel methods. However, each evaluation of the local maximum in the AMI method causes additional complications in the computations because it may require another algorithm. This is often time consuming especially for complicated nonlinear functions f and g (such as oscillating functions, etc). In this paper, we present a new accelerated technique for constructing a new monotone iterative method that avoids computing any local maximum. It directly constructs monotone sequences, but still maintains the monotone and quadratic convergence of the AMI method.

Specifically, we develop a new accelerated technique for the finite difference solutions of (1.1) and (1.2). This new technique leads to a simple and yet efficient computational algorithm. By the algorithm, one can construct two sequences of iterations from the corresponding linear algebraic system. These two sequences converge monotonically to a unique solution of the nonlinear difference system in exactly the same way as the AMI method. As with the AMI method, the proposed method has quadratic convergence for the sum of the two sequences of iterations. Under an additional requirement, the quadratic rate of convergence is attained for one of these two sequences. But, unlike the AMI method, the construction of the sequences in this method do not involve any local maximum. Thus, the computation is easily carried out and a great deal of computational time is saved. We call such an algorithm as the *Modified Accelerated Monotone Iterative* (MAMI) method.

The outline of the paper is as follows. In Section 2, we formulate a finite difference system for (1.1) and (1.2). The basic idea of the MAMI method for the finite difference solution of (1.1) is discussed in Section 3. Two monotone convergent sequences are constructed. The quadratic convergence for the sum of these two sequences is proved by giving explicitly an estimate of the rate of convergence in the infinity norm. Section 4 is devoted to the finite difference system for (1.2), and a parallel discussion of Section 3 is given. A theoretical comparison between the MAMI and AMI methods is discussed in Section 5. In Section 6, we give an application of the MAMI method to a model problem. We use some numerical results to demonstrate the monotone and rapid convergence of the iterations, and to compare the MAMI method with the AMI method. The final section is for some concluding remarks.

2. The finite difference system

To approximate problem (1.1) by a finite difference system, we let $k_n \equiv t_n - t_{n-1}$ be the time increment, and let $h_\nu (\nu = 1,2,\ldots,p)$ be the spatial increment in the x_ν-coordinate direction. Let $x_i = (x_{i_1},\ldots,x_{i_p})$ be a mesh point in

$\overline{\Omega}$. Denote by Ω_p, $\partial\Omega_p$ and $\overline{\Lambda}_p$ the sets of mesh points in Ω, $\partial\Omega$ and $\overline{\Omega}\times[0,+\infty)$, respectively. For clarity of presentation we use $u_h(x_i,t_n)$ to represent the approximation of u, at any mesh point (x_i,t_n) in $\overline{\Lambda}_p$, and define

$$u_{i,n}=u_h(x_i,t_n),\qquad f_{i,n}(u_{i,n})=f(x_i,t_n,u_{i,n}),\qquad g_{i,n}(u_{i,n})=g(x_i,t_n,u_{i,n}).\tag{2.1}$$

Define also the difference operators

$$\begin{cases}\delta_+^{(\nu)}[u_{i,n}]=u_h(x_i+h_\nu e_\nu,t_n)-u_h(x_i,t_n),\qquad \delta_-^{(\nu)}[u_{i,n}]=u_h(x_i,t_n)-u_h(x_i-h_\nu e_\nu,t_n),\\ \delta^{(\nu)}[u_{i,n}]=(2h_\nu)^{-1}(\delta_+^{(\nu)}[u_{i,n}]+\delta_-^{(\nu)}[u_{i,n}]),\\ \Delta^{(\nu)}[u_{i,n}]=h_\nu^{-2}(D(x_i,t_n)\delta_+^{(\nu)}[u_{i,n}]-D(x_i-h_\nu e_\nu,t_n)\delta_-^{(\nu)}[u_{i,n}]),\quad \nu=1,2,\ldots,p,\\ \mathcal{L}[u_{i,n}]=k_n^{-1}(u_{i,n}-u_{i,n-1})-\displaystyle\sum_{\nu=1}^{p}(\Delta^{(\nu)}[u_{i,n}]-v_\nu(x_i,t_n)\delta^{(\nu)}[u_{i,n}]),\end{cases}\tag{2.2}$$

where e_ν is the unit vector in $\mathbf{R}^p$ with the νth component one and zero elsewhere. For the approximation of the boundary operator in (1.1), we define

$$\mathcal{B}[u_{i,n}]=\alpha(x_i,t_n)|x_i-x_{i^*}|^{-1}(u_{i,n}-u_{i^*,n})+\beta(x_i,t_n)u_{i,n},\quad x_i\in\partial\Omega_p,\tag{2.3}$$

where x_{i^*} is a suitable mesh point in Ω_p and $|x_i-x_{i^*}|$ is the distance between x_i and x_{i^*}. Then we approximate (1.1) by the finite difference system

$$\begin{cases}\mathcal{L}[u_{i,n}]=f_{i,n}(u_{i,n}),\quad x_i\in\Omega_p,\\ \mathcal{B}[u_{i,n}]=g_{i,n}(u_{i,n}),\quad x_i\in\partial\Omega_p,\quad n=1,2,\ldots,\\ u_{i,0}=\psi(x_i),\qquad\quad x_i\in\Omega_p.\end{cases}\tag{2.4}$$

Using the same notations as those in (2.1) and (2.2), but without the index n (all functions are independent of t_n), we obtain the finite difference approximation of the steady-state problem (1.2)

$$\begin{cases}\mathcal{L}_s[u_i]=f_i(u_i),\qquad x_i\in\Omega_p,\\ \mathcal{B}_s[u_i]=g_i(u_i),\qquad x_i\in\partial\Omega_p,\end{cases}\tag{2.5}$$

where

$$\mathcal{L}_s[u_i]=-\sum_{\nu=1}^{p}(\Delta^{(\nu)}[u_i]-v_\nu(x_i)\delta^{(\nu)}[u_i]),\qquad x_i\in\Omega_p,\tag{2.6}$$

and $\mathcal{B}_s$ is defined by (2.3) without the index n and with $\alpha\equiv\alpha(x_i)$, $\beta\equiv\beta(x_i)$ independent of t_n.

To develop numerical methods for the systems (2.4) and (2.5), it is more convenient to express the systems in vector form. Let M be the total number of mesh points in $\Omega_p\cup\partial\Omega_p$ at which the solution $u_{i,n}$ (or u_i) is to be determined. Also, let the mesh points be arranged lexicographically. Corresponding to this arrangement, we define vectors

$$U_n=(u_{1,n},\ldots,u_{M,n})^T,\qquad \Psi=(\psi(x_1),\ldots,\psi(x_M))^T,\tag{2.7}$$

where $(\cdot)^T$ denotes the transpose of a row vector. Then, we may express the finite difference system (2.4) in vector form

$$\begin{cases}(I+k_nA_n)U_n=U_{n-1}+k_nF_n(U_n),\quad n=1,2,\ldots,\\ U_0=\Psi,\end{cases}\tag{2.8}$$

where I is the identity matrix and for each n, A_n is an $M\times M$ matrix associated with the operators $\mathcal{L}$ and $\mathcal{B}$, and the vector $F_n(U_n)$ is given by

$$\begin{aligned}&F_n(U_n)=(f_{1,n}^*(u_{1,n}),f_{2,n}^*(u_{2,n}),\ldots,f_{M,n}^*(u_{M,n}))^T,\\ &f_{i,n}^*(u_{i,n})=f^*(x_i,t_n,u_{i,n}),\\ &f^*(x_i,t_n,u_{i,n})=f(x_i,t_n,u_{i,n})+\theta_{i,n}g(x_i,t_n,u_{i,n}).\end{aligned}\tag{2.9}$$

The coefficients $\theta_{i,n}$ in the definition of $f^*(x_i,t_n,u_{i,n})$ are nonnegative quantities that are nonzero only for mesh points on $\partial\Omega_p$ or possibly neighboring mesh points of $\partial\Omega_p$. Since our main concern is the mathematical structure of the finite difference approximation, detailed formulation of the system (2.8) is omitted (see [6–8,20–22] for some details).

In the same manner, the vector form of system (2.5) is given by

$$AU=F(U),\tag{2.10}$$

where A has the same structure as A_n in (2.8) and

$$\begin{aligned}&U=(u_1,\ldots,u_M)^T,\qquad F(U)=(f_1^*(u_1),\ldots,f_M^*(u_M))^T,\\ &f_i^*(u_i)=f^*(x_i,u_i),\qquad f^*(x_i,u_i)=f(x_i,u_i)+\theta_ig(x_i,u_i)\end{aligned}\tag{2.11}$$

with θ_i having the similar definition as $\theta_{i,n}$ in (2.9).

Motivated by the difference approximations in (2.2), we impose the following basic hypothesis (H) on the matrices A_n and A (see [6–8,16,20,22]).

(H) Let $B = A_n$ or $B = A$. The matrix $B \equiv (b_{i,j})$ is irreducible and

$$b_{i,i} > 0, \qquad b_{i,j} \le 0\ (j \ne i), \qquad \sum_{j=1}^{M} b_{i,j} \ge 0, \quad i, j = 1, 2, \ldots, M. \tag{2.12}$$

It is easy to see from the difference approximations in (2.2) that if the convection term $\mathbf{v} \cdot \nabla u$ does not dominate the diffusion term $\nabla \cdot (D\nabla u)$, then property (2.12) can always be satisfied by the matrices A_n and A. On the other hand, if the convection term dominates the diffusion term, then the matrices A_n and A also possess property (2.12) by either taking spatial increment h_ν suitably small (which depends on the relative magnitude between $\upsilon_\nu(x_i, t_n)$ and $D(x_i, t_n)$) or using an upwind difference approximation instead of the central difference approximation for the convection term without any restriction on the increment h_ν (see [7,22]). The connectedness assumption on Ω ensures that A_n or A is irreducible (see [23]). Therefore, the properties in hypothesis (H) can always be satisfied by the matrices A_n and A.

A direct consequence of hypothesis (H) is that for any nonnegative diagonal matrix $\mathcal{D} \not\equiv 0$, the inverse matrix $(B + \mathcal{D})^{-1}$ exists and is positive (cf. [23–26]). Moreover, the smallest eigenvalue λ^* of B is real and nonnegative. (Here the smallest eigenvalue is in the module sense.) If the strict inequality in the last relation of (2.12) holds for at least one i (such as Dirichlet or Robin boundary condition), then the smallest eigenvalue λ^* is positive (cf. [23,24]). Moreover, for any diagonal matrix $\mathcal{D} = \mathrm{diag}(d_1, \ldots, d_M)$ with $\min_i d_i > -\lambda^*$, the inverse $(B + \mathcal{D})^{-1}$ exists and is nonnegative (see [18]). Otherwise, if the equality in the last relation of (2.12) holds for all i (such as Neumann boundary condition), then the matrix B is singular and its smallest eigenvalue $\lambda^* = 0$ (cf. [23,24]). For convenience, we summarize the above results in the following lemma.

Lemma 2.1. *Let hypothesis (H) hold. Then the smallest eigenvalue λ^* of B is real and nonnegative. Moreover, for any diagonal matrix $\mathcal{D} = \mathrm{diag}(d_1, \ldots, d_M)$, the inverse $(B + \mathcal{D})^{-1}$ exists and is nonnegative if*

$$\begin{cases} \min\limits_i d_i > -\lambda^*, & \text{for } \lambda^* > 0, \\ \min\limits_i d_i \ge 0 \text{ and } \max\limits_i d_i > 0, & \text{for } \lambda^* = 0. \end{cases} \tag{2.13}$$

3. MAMI method for time-dependent system

To obtain a sequence of iterations that converges monotonically to a solution of (2.8), we use the method of upper and lower solutions. The upper and lower solutions of (2.8) are defined as follows.

Definition 3.1. A vector $\widetilde{U}_n$ in $\mathbf{R}^M$ is called an *upper solution* of (2.8) if

$$\begin{cases} (I + k_n A_n)\widetilde{U}_n \ge \widetilde{U}_{n-1} + k_n F_n(\widetilde{U}_n), \quad n = 1, 2, \ldots, \\ \widetilde{U}_0 \ge \Psi. \end{cases} \tag{3.1}$$

Similarly, $\widehat{U}_n$ in $\mathbf{R}^M$ is called a *lower solution* if it satisfies the above inequalities in the reversed order. A pair of upper and lower solutions $\widetilde{U}_n, \widehat{U}_n$ are said to be ordered if $\widetilde{U}_n \ge \widehat{U}_n$.

In the above definition, inequalities between vectors are in the sense of componentwise. It is clear that every solution of (2.8) is an upper solution as well as a lower solution. For any pair of ordered upper and lower solutions $\widetilde{U}_n = (\widetilde{u}_{1,n}, \ldots, \widetilde{u}_{M,n})^T$ and $\widehat{U}_n = (\widehat{u}_{1,n}, \ldots, \widehat{u}_{M,n})^T$, we define the sectors

$$\langle \widehat{U}_n, \widetilde{U}_n \rangle = \{U_n \in \mathbf{R}^M; \widehat{U}_n \le U_n \le \widetilde{U}_n\}, \qquad \langle \widehat{u}_{i,n}, \widetilde{u}_{i,n} \rangle = \{u_{i,n} \in \mathbf{R}; \widehat{u}_{i,n} \le u_{i,n} \le \widetilde{u}_{i,n}\}. \tag{3.2}$$

Let $\widetilde{U}_n$ and $\widehat{U}_n$ be a pair of ordered upper and lower solutions of (2.8). It is known that under hypothesis (H) and a condition on k_n (such as condition (2.19) in [6] or (3.7)), system (2.8) has a unique solution U_n^* in the sector $\langle \widehat{U}_n, \widetilde{U}_n \rangle$ (see [6]). To compute the solution U_n^*, we assume that the function $f^*(x_i, t_n, u_{i,n})$ in (2.9) is given by

$$f^*(x_i, t_n, u_{i,n}) = f^{(1)}(x_i, t_n, u_{i,n}) + f^{(2)}(x_i, t_n, u_{i,n}), \tag{3.3}$$

where $f_u^{(1)}(x_i, t_n, u_{i,n})$ is monotone nondecreasing in $u_{i,n}$ and $f_u^{(2)}(x_i, t_n, u_{i,n})$ is monotone nonincreasing in $u_{i,n}$ for $u_{i,n} \in \langle \widehat{u}_{i,n}, \widetilde{u}_{i,n} \rangle$. (Note: $f_u(x_i, t_n, u_{i,n}) = \frac{\partial f}{\partial u}(x_i, t_n, u_{i,n})$.) This assumption can always be satisfied. In fact, the decomposition (3.3) is trivial if $f_u^*(x_i, t_n, u_{i,n})$ is monotone in $u_{i,n} \in \langle \widehat{u}_{i,n}, \widetilde{u}_{i,n} \rangle$. On the other hand, if $f_u^*(x_i, t_n, u_{i,n})$ is not monotone in $u_{i,n} \in \langle \widehat{u}_{i,n}, \widetilde{u}_{i,n} \rangle$, we can also always find the functions $f^{(1)}$ and $f^{(2)}$ so that the decomposition (3.3) holds (see Remark 3.2 for some details). By using $U_n^{(0)} = \widetilde{U}_n$ and $U_n^{(0)} = \widehat{U}_n$ as initial iterations, we construct the corresponding sequences $\{\overline{U}_n^{(m)}\} = \{(\overline{u}_{1,n}^{(m)}, \ldots, \overline{u}_{M,n}^{(m)})^T\}$ and $\{\underline{U}_n^{(m)}\} = \{(\underline{u}_{1,n}^{(m)}, \ldots, \underline{u}_{M,n}^{(m)})^T\}$, respectively, from the following linear iterative scheme.

Algorithm A1.

$$\begin{cases} P_n^{(m)} U_n^{(m+1)} = U_{n-1}^* + k_n [C_n^{(m)} U_n^{(m)} + F_n(U_n^{(m)})], \quad n = 1, 2, \ldots, \\ U_0^{(m+1)} = \Psi, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} P_n^{(m)} &= I + k_n A_n + k_n C_n^{(m)}, \qquad C_n^{(m)} = \mathrm{diag}(c_{1,n}^{(m)}, c_{2,n}^{(m)}, \ldots, c_{M,n}^{(m)}), \\ c_{i,n}^{(m)} &= -(f_u^{(1)}(x_i, t_n, \underline{u}_{i,n}^{(m)}) + f_u^{(2)}(x_i, t_n, \overline{u}_{i,n}^{(m)})). \end{aligned} \tag{3.5}$$

As compared with the *Accelerated Monotone Iterative* (AMI) method (see [6] or Section 5), a new feature of the above algorithm is the construction of $C_n^{(m)}$. In the above algorithm, the element $c_{i,n}^{(m)}$ of $C_n^{(m)}$ can be directly obtained by the values of the functions $f_u^{(1)}$ and $f_u^{(2)}$ without any other computations such as the evaluation of a local maximum in the AMI method. Therefore, it is much easier to use in practical computations. But, as we shall see later, it still maintains the monotone and quadratic convergence of the AMI method. So we call it the *Modified Accelerated Monotone Iterative* (MAMI) method for (2.8).

3.1. Monotone convergence

It is clear that if the inverse $(P_n^{(m)})^{-1}$ exists, then the sequences $\{\overline{U}_n^{(m)}\}$ and $\{\underline{U}_n^{(m)}\}$ from Algorithm A1 are well defined for an arbitrary C^1-function $F_n(U_n)$. They can be computed by solving a linear algebraic system for each m (with any fixed $n \ge 1$) starting from $n = 1$. To ensure the existence of $(P_n^{(m)})^{-1}$ and the monotone convergence of the sequences, we let

$$\sigma_n = \max_i \{f_u^{(1)}(x_i, t_n, \widetilde{u}_{i,n}) + f_u^{(2)}(x_i, t_n, \widehat{u}_{i,n})\}, \tag{3.6}$$

and impose the following condition on k_n:

$$k_n(\sigma_n - \lambda_n) < 1, \quad n = 1, 2, \ldots, \tag{3.7}$$

where λ_n is the smallest eigenvalue of A_n. It is clear that the condition (3.7) is trivially satisfied for all k_n if $\sigma_n \le \lambda_n$. Hence it is needed only for the case $\sigma_n > \lambda_n$. Since $f_u^{(1)}(x_i, t_n, u_{i,n})$ is monotone nondecreasing in $u_{i,n}$ and $f_u^{(2)}(x_i, t_n, u_{i,n})$ is monotone nonincreasing in $u_{i,n}$ for $u_{i,n} \in \langle \widehat{u}_{i,n}, \widetilde{u}_{i,n} \rangle$, we have from condition (3.7) that

$$1 + k_n c_{i,n}^{(m)} > -k_n \lambda_n \quad \text{whenever } \widehat{u}_{i,n} \le \underline{u}_{i,n}^{(m)} \le \widetilde{u}_{i,n}, \widehat{u}_{i,n} \le \overline{u}_{i,n}^{(m)} \le \widetilde{u}_{i,n}. \tag{3.8}$$

This implies that the diagonal matrix $\frac{1}{k_n} I + C_n^{(m)}$ satisfies condition (2.13) if $\widehat{U}_n \le \underline{U}_n^{(m)} \le \widetilde{U}_n$ and $\widehat{U}_n \le \overline{U}_n^{(m)} \le \widetilde{U}_n$. Therefore by Lemma 2.1, the inverse $(P_n^{(m)})^{-1}$ exists and

$$(P_n^{(m)})^{-1} \ge 0 \quad \text{whenever } \widehat{U}_n \le \underline{U}_n^{(m)} \le \widetilde{U}_n, \widehat{U}_n \le \overline{U}_n^{(m)} \le \widetilde{U}_n. \tag{3.9}$$

Again by the monotone property of $f_u^{(l)}(x_i, t_n, u_{i,n})$ $(l = 1, 2)$ in $u_{i,n}$,

$$-(f_{i,n}^*(u_{i,n}) - f_{i,n}^*(v_{i,n})) \le c_{i,n}^{(m)}(u_{i,n} - v_{i,n}) \quad \text{whenever } \widetilde{u}_{i,n} \ge \overline{u}_{i,n}^{(m)} \ge u_{i,n} \ge v_{i,n} \ge \underline{u}_{i,n}^{(m)} \ge \widehat{u}_{i,n}. \tag{3.10}$$

In vector form, this becomes

$$C_n^{(m)}(U_n - V_n) + F_n(U_n) - F_n(V_n) \ge 0 \quad \text{whenever } \widetilde{U}_n \ge \overline{U}_n^{(m)} \ge U_n \ge V_n \ge \underline{U}_n^{(m)} \ge \widehat{U}_n. \tag{3.11}$$

This property leads to the following well-defined and monotone properties of the sequences from Algorithm A1.

Lemma 3.1. *Let $\widetilde{U}_n$ and $\widehat{U}_n$ be a pair of ordered upper and lower solutions of (2.8) and let f^* be given by (3.3). Also, let hypothesis (H) and condition (3.7) hold. Then, for all $m \ge 1$, the sequences $\{\overline{U}_n^{(m)}\}$ and $\{\underline{U}_n^{(m)}\}$ given by Algorithm A1, with $\overline{U}_n^{(0)} = \widetilde{U}_n$ and $\underline{U}_n^{(0)} = \widehat{U}_n$, are well defined and possess the monotone property*

$$\widehat{U}_n \le \underline{U}_n^{(m-1)} \le \underline{U}_n^{(m)} \le \overline{U}_n^{(m)} \le \overline{U}_n^{(m-1)} \le \widetilde{U}_n, \quad n = 1, 2, \ldots. \tag{3.12}$$

Proof. Let $m = 0$ in (3.4) with any fixed $n = 1, 2, \ldots$. Since $\overline{U}_n^{(0)} = \widetilde{U}_n$ and $\underline{U}_n^{(0)} = \widehat{U}_n$, the right-hand side of (3.4) is known when $m = 0$. By (3.9), the inverse $(P_n^{(0)})^{-1}$ exists and is nonnegative. Hence, the first iterations $\overline{U}_n^{(1)}$ and $\underline{U}_n^{(1)}$ exist uniquely and satisfy

$$P_n^{(0)}(\overline{U}_n^{(1)} - \underline{U}_n^{(1)}) = k_n[C_n^{(0)}(\overline{U}_n^{(0)} - \underline{U}_n^{(0)}) + F_n(\overline{U}_n^{(0)}) - F_n(\underline{U}_n^{(0)})]. \tag{3.13}$$

Since $\overline{U}_n^{(0)} = \widetilde{U}_n \geq \widehat{U}_n = \underline{U}_n^{(0)}$, relation (3.11) implies $P_n^{(0)}(\overline{U}_n^{(1)} - \underline{U}_n^{(1)}) \geq 0$. It follows from the nonnegativity of $(P_n^{(0)})^{-1}$ that $\overline{U}_n^{(1)} \geq \underline{U}_n^{(1)}$. By (3.4), (3.5) and (3.1), we have

$$\begin{aligned} P_n^{(0)}(\widetilde{U}_n - \overline{U}_n^{(1)}) &= (I + k_nA_n + k_nC_n^{(0)})\widetilde{U}_n - [U_{n-1}^* + k_n(C_n^{(0)}\overline{U}_n^{(0)} + F_n(\overline{U}_n^{(0)}))] \\ &= (I + k_nA_n)\widetilde{U}_n - (U_{n-1}^* + k_nF_n(\widetilde{U}_n)) \\ &\geq \widetilde{U}_{n-1} - U_{n-1}^* \geq 0. \end{aligned}$$

In view of $(P_n^{(0)})^{-1} \geq 0$, the above relation ensures $\overline{U}_n^{(1)} \leq \widetilde{U}_n$. Similarly by the property of a lower solution, $\widehat{U}_n \leq \underline{U}_n^{(1)}$. This shows $\widehat{U}_n \leq \underline{U}_n^{(1)} \leq \overline{U}_n^{(1)} \leq \widetilde{U}_n$ which proves (3.12) with $m = 1$.

Assume, by induction, that the conclusion of the lemma holds for some $m \geq 1$. Using relation (3.9), we obtain that the inverse $(P_n^{(m)})^{-1}$ exists and is nonnegative. This ensures that the iterations $\overline{U}_n^{(m+1)}$ and $\underline{U}_n^{(m+1)}$ are well defined. The iteration process (3.4) implies that

$$\begin{aligned} &P_n^{(m)}(\overline{U}_n^{(m+1)} - \underline{U}_n^{(m+1)}) = k_n[C_n^{(m)}(\overline{U}_n^{(m)} - \underline{U}_n^{(m)}) + F_n(\overline{U}_n^{(m)}) - F_n(\underline{U}_n^{(m)})], \\ &P_n^{(m)}(\overline{U}_n^{(m)} - \overline{U}_n^{(m+1)}) = k_n[C_n^{(m-1)}(\overline{U}_n^{(m-1)} - \overline{U}_n^{(m)}) + F_n(\overline{U}_n^{(m-1)}) - F_n(\overline{U}_n^{(m)})], \\ &P_n^{(m)}(\underline{U}_n^{(m+1)} - \underline{U}_n^{(m)}) = k_n[C_n^{(m-1)}(\underline{U}_n^{(m)} - \underline{U}_n^{(m-1)}) + F_n(\underline{U}_n^{(m)}) - F_n(\underline{U}_n^{(m-1)})]. \end{aligned} \tag{3.14}$$

By relation (3.11), the right-hand side of the above relation is nonnegative. So by the nonnegativity of $(P_n^{(m)})^{-1}$, $\underline{U}_n^{(m)} \leq \underline{U}_n^{(m+1)} \leq \overline{U}_n^{(m+1)} \leq \overline{U}_n^{(m)}$. This proves that the conclusion of the lemma is also true for $m + 1$. Finally, the conclusion of the lemma, for all $m \geq 1$, follows from the principle of induction. □

In view of the monotone property (3.12), the limits

$$\lim_{m\to\infty} \overline{U}_n^{(m)} = \overline{U}_n, \qquad \lim_{m\to\infty} \underline{U}_n^{(m)} = \underline{U}_n \tag{3.15}$$

exist and satisfy $\underline{U}_n^{(m)} \leq \underline{U}_n \leq \overline{U}_n \leq \overline{U}_n^{(m)}$ for every m and n. Let

$$c_{i,n}^* = -(f_u^{(1)}(x_i, t_n, \underline{u}_{i,n}) + f_u^{(2)}(x_i, t_n, \overline{u}_{i,n})), \tag{3.16}$$

where $\underline{u}_{i,n}$ and $\overline{u}_{i,n}$ denotes the components of $\underline{U}_n$ and $\overline{U}_n$, respectively. Then the monotone property of $f_u^{(l)}(x_i, t_n, u_{i,n})$ $(l = 1, 2)$ in $u_{i,n}$ implies that $c_{i,n}^{(m-1)} \geq c_{i,n}^{(m)} \geq c_{i,n}^*$ for every i, n and m. So the sequence $\{C_n^{(m)}\}$ converges as $m \to \infty$. Letting $m \to \infty$ in (3.4), we see that the limits $\overline{U}_n$ and $\underline{U}_n$ satisfy

$$\begin{cases} (I + k_nA_n)U_n = U_{n-1}^* + k_nF_n(U_n), \quad n = 1, 2, \ldots, \\ U_0 = \Psi, \end{cases} \tag{3.17}$$

where $U_n = \overline{U}_n$ or $\underline{U}_n$. Using the same argument as that in [6], we can show that $\overline{U}_n = \underline{U}_n$ and is the unique solution U_n^* of (2.8) in $\langle \widehat{U}_n, \widetilde{U}_n \rangle$. This leads to the following conclusion.

Theorem 3.1. *Let conditions in Lemma 3.1 be satisfied. Then system (2.8) has a unique solution U_n^* in $\langle \widehat{U}_n, \widetilde{U}_n \rangle$. In addition, the sequences $\{\overline{U}_n^{(m)}\}$ and $\{\underline{U}_n^{(m)}\}$ from Algorithm A1 with $\overline{U}_n^{(0)} = \widetilde{U}_n$ and $\underline{U}_n^{(0)} = \widehat{U}_n$, converge monotonically to U_n^* and satisfy*

$$\widehat{U}_n \leq \underline{U}_n^{(m-1)} \leq \underline{U}_n^{(m)} \leq U_n^* \leq \overline{U}_n^{(m)} \leq \overline{U}_n^{(m-1)} \leq \widetilde{U}_n, \quad m, n = 1, 2, \ldots. \tag{3.18}$$

3.2. Quadratic convergence

We see from the previous discussions that Algorithm A1 possesses the monotone convergence. Another advantage of this algorithm is its quadratic rate of convergence, under the following Lipschitz condition

$$\begin{aligned} &f_u^{(1)}(x_i, t_n, u_{i,n}) - f_u^{(1)}(x_i, t_n, v_{i,n}) \leq \sigma_n^*(u_{i,n} - v_{i,n}), \\ &f_u^{(2)}(x_i, t_n, u_{i,n}) - f_u^{(2)}(x_i, t_n, v_{i,n}) \geq -\sigma_n^*(u_{i,n} - v_{i,n}) \quad \text{whenever } \widehat{u}_{i,n} \leq v_{i,n} \leq u_{i,n} \leq \widetilde{u}_{i,n}, \end{aligned} \tag{3.19}$$

where σ_n^* is a nonnegative constant. Let σ_n be defined by (3.6). Then, by condition (3.7) and Lemma 2.1, the inverse $((1 - k_n\sigma_n)I + k_nA_n)^{-1}$ exists and is nonnegative. This property leads to the following estimate for the rate of convergence of Algorithm A1.

Theorem 3.2. *Let conditions in Lemma 3.1 and (3.19) be satisfied. Also let U_n^* be the unique solution of (2.8) in $\langle \widehat{U}_n, \widetilde{U}_n \rangle$, and $\{\overline{U}_n^{(m)}\}$ and $\{\underline{U}_n^{(m)}\}$ be the sequences given by Algorithm A1 with $\overline{U}_n^{(0)} = \widetilde{U}_n$ and $\underline{U}_n^{(0)} = \widehat{U}_n$. Then, for all $m, n = 0, 1, 2, \ldots$,*

$$\|\overline{U}_n^{(m+1)} - U_n^*\|_\infty + \|\underline{U}_n^{(m+1)} - U_n^*\|_\infty \le k_n \sigma_n^* \rho_n (\|\overline{U}_n^{(m)} - U_n^*\|_\infty + \|\underline{U}_n^{(m)} - U_n^*\|_\infty)^2, \tag{3.20}$$

where $\rho_n = \|((1 - k_n \sigma_n)I + k_n A_n)^{-1}\|_\infty$.

Proof. We first consider the sequence $\{\overline{U}_n^{(m)}\}$. By (2.8) and Algorithm A1, we have

$$P_n^{(m)}(\overline{U}_n^{(m+1)} - U_n^*) = k_n[C_n^{(m)}(\overline{U}_n^{(m)} - U_n^*) + F_n(\overline{U}_n^{(m)}) - F_n(U_n^*)]. \tag{3.21}$$

An application of the mean-value theorem gives

$$F_n(\overline{U}_n^{(m)}) - F_n(U_n^*) = (F_u)_n(\boldsymbol{\xi}_n^{(m)})(\overline{U}_n^{(m)} - U_n^*), \tag{3.22}$$

where

$$\begin{aligned} &(F_u)_n(\boldsymbol{\xi}_n^{(m)}) = \mathrm{diag}(f_u^*(x_1, t_n, \xi_{1,n}^{(m)}), \ldots, f_u^*(x_M, t_n, \xi_{M,n}^{(m)})), \\ &\boldsymbol{\xi}_n^{(m)} = (\xi_{1,n}^{(m)}, \ldots, \xi_{M,n}^{(m)})^T \in \langle U_n^*, \overline{U}_n^{(m)} \rangle. \end{aligned} \tag{3.23}$$

Using the relation (3.22) in (3.21) yields

$$P_n^{(m)}(\overline{U}_n^{(m+1)} - U_n^*) = k_n(C_n^{(m)} + (F_u)_n(\boldsymbol{\xi}_n^{(m)}))(\overline{U}_n^{(m)} - U_n^*). \tag{3.24}$$

Since by (3.5), (3.3) and (3.19),

$$\begin{aligned} &-(f_u^{(1)}(x_i, t_n, \underline{u}_{i,n}^{(m)}) + f_u^{(2)}(x_i, t_n, \overline{u}_{i,n}^{(m)})) + f_u^*(x_i, t_n, \xi_{i,n}^{(m)}) \\ &\quad = f_u^{(1)}(x_i, t_n, \xi_{i,n}^{(m)}) - f_u^{(1)}(x_i, t_n, \underline{u}_{i,n}^{(m)}) - (f_u^{(2)}(x_i, t_n, \overline{u}_{i,n}^{(m)}) - f_u^{(2)}(x_i, t_n, \xi_{i,n}^{(m)})) \\ &\quad \le \sigma_n^*(\xi_{i,n}^{(m)} - \underline{u}_{i,n}^{(m)}) + \sigma_n^*(\overline{u}_{i,n}^{(m)} - \xi_{i,n}^{(m)}) = \sigma_n^*(\overline{u}_{i,n}^{(m)} - \underline{u}_{i,n}^{(m)}), \end{aligned}$$

we obtain

$$C_n^{(m)} + (F_u)_n(\boldsymbol{\xi}_n^{(m)}) \le (\sigma_n^* \|\overline{U}_n^{(m)} - \underline{U}_n^{(m)}\|_\infty) I. \tag{3.25}$$

It follows from (3.24) that

$$P_n^{(m)}(\overline{U}_n^{(m+1)} - U_n^*) \le k_n \sigma_n^* \|\overline{U}_n^{(m)} - \underline{U}_n^{(m)}\|_\infty (\overline{U}_n^{(m)} - U_n^*). \tag{3.26}$$

This estimate and the nonnegative property of $(P_n^{(m)})^{-1}$ imply that

$$0 \le \overline{U}_n^{(m+1)} - U_n^* \le k_n \sigma_n^* \|\overline{U}_n^{(m)} - \underline{U}_n^{(m)}\|_\infty (P_n^{(m)})^{-1}(\overline{U}_n^{(m)} - U_n^*).$$

This leads to the following estimate in the infinity norm

$$\|\overline{U}_n^{(m+1)} - U_n^*\|_\infty \le k_n \sigma_n^* \|(P_n^{(m)})^{-1}\|_\infty \|\overline{U}_n^{(m)} - \underline{U}_n^{(m)}\|_\infty \|\overline{U}_n^{(m)} - U_n^*\|_\infty. \tag{3.27}$$

Since $((1 - k_n \sigma_n)I + k_n A_n)^{-1} \ge 0$ and $C_n^{(m)} \ge -\sigma_n I$, we get

$$0 \le (P_n^{(m)})^{-1} = (I + k_n A_n + k_n C_n^{(m)})^{-1} \le ((1 - k_n \sigma_n)I + k_n A_n)^{-1}$$

(cf. [23,25,26]). This implies $\|(P_n^{(m)})^{-1}\|_\infty \le \|((1 - k_n \sigma_n)I + k_n A_n)^{-1}\|_\infty = \rho_n$. Therefore by (3.27), we obtain

$$\|\overline{U}_n^{(m+1)} - U_n^*\|_\infty \le k_n \sigma_n^* \rho_n \|\overline{U}_n^{(m)} - \underline{U}_n^{(m)}\|_\infty \|\overline{U}_n^{(m)} - U_n^*\|_\infty. \tag{3.28}$$

Similarly, we have

$$\|\underline{U}_n^{(m+1)} - U_n^*\|_\infty \le k_n \sigma_n^* \rho_n \|\overline{U}_n^{(m)} - \underline{U}_n^{(m)}\|_\infty \|\underline{U}_n^{(m)} - U_n^*\|_\infty. \tag{3.29}$$

Addition of the relations (3.28) and (3.29) yields

$$\|\overline{U}_n^{(m+1)} - U_n^*\|_\infty + \|\underline{U}_n^{(m+1)} - U_n^*\|_\infty \le k_n \sigma_n^* \rho_n \|\overline{U}_n^{(m)} - \underline{U}_n^{(m)}\|_\infty (\|\overline{U}_n^{(m)} - U_n^*\|_\infty + \|\underline{U}_n^{(m)} - U_n^*\|_\infty). \tag{3.30}$$

Then the estimate (3.20) follows immediately from the above relation. □

We end this section with three remarks.

Remark 3.1. Theorem 3.1 shows that Algorithm A1 yields two sequences of iterations that converge monotonically to a unique solution of the nonlinear difference system (2.8). Moreover, Theorem 3.2 gives a quadratic convergence for the sum of the two produced sequences from Algorithm A1. These results imply that Algorithm A1 possesses the same advantages as the AMI method (see [6] or Section 5). It should also be mentioned that the estimate for the rate of convergence in Theorem 3.2 is explicitly given by the infinity norm.

Remark 3.2. The monotone and quadratic convergence of Algorithm A1 is based on the decomposition (3.3). This decomposition includes a wide variety of functions.

(a) If $f_u^*(x_i, t_n, u_{i,n})$ is monotone nonincreasing in $u_{i,n}$, we have the decomposition (3.3) with $f^{(1)}(x_i, t_n, u_{i,n}) \equiv 0$. In this case, the sequence $\{\overline{U}_n^{(m)}\}$ from Algorithm A1 is independent of the sequence $\{\underline{U}_n^{(m)}\}$, and a simple modification of the proof of (3.20) leads to

$$\|\overline{U}_n^{(m+1)} - U_n^*\|_\infty \le k_n \sigma_n^* \rho_n \|\overline{U}_n^{(m)} - U_n^*\|_\infty^2, \tag{3.31}$$

where the constants σ_n^* and ρ_n are the same as those in (3.20). The estimate (3.31) implies that the sequence $\{\overline{U}_n^{(m)}\}$ converges quadratically to the solution U_n^*.

(b) If $f_u^*(x_i, t_n, u_{i,n})$ is monotone nondecreasing in $u_{i,n}$, we have the decomposition (3.3) with $f^{(2)}(x_i, t_n, u_{i,n}) \equiv 0$. In this case, the result in (a) holds true for the sequence $\{\underline{U}_n^{(m)}\}$. In particular, the estimate (3.31) remains to be true for the sequence $\{\underline{U}_n^{(m)}\}$. Thus, the sequence $\{\underline{U}_n^{(m)}\}$ converges quadratically to the solution U_n^*.

(c) If $f_u^*(x_i, t_n, u_{i,n})$ is not monotone in $u_{i,n}$, we can always find a decomposition (3.3) so that the conditions of Theorems 3.1 and 3.2 are fulfilled. For example, if there exists a nonnegative constant M_n such that

$$f_u^*(x_i, t_n, u_{i,n}) - f_u^*(x_i, t_n, v_{i,n}) \ge -M_n(u_{i,n} - v_{i,n}) \quad \text{whenever } \widehat{u}_{i,n} \le v_{i,n} \le u_{i,n} \le \widetilde{u}_{i,n}, \tag{3.32}$$

we have the decomposition (3.3) with

$$f^{(1)}(x_i, t_n, u_{i,n}) = f^*(x_i, t_n, u_{i,n}) + M_n u_{i,n}^2/2, \qquad f^{(2)}(x_i, t_n, u_{i,n}) = -M_n u_{i,n}^2/2.$$

Also, if there exists a nonnegative constant M_n such that

$$f_u^*(x_i, t_n, u_{i,n}) - f_u^*(x_i, t_n, v_{i,n}) \le M_n(u_{i,n} - v_{i,n}) \qquad \text{whenever } \widehat{u}_{i,n} \le v_{i,n} \le u_{i,n} \le \widetilde{u}_{i,n}, \tag{3.33}$$

we can choose $f^{(1)}$ and $f^{(2)}$ in (3.3) as

$$f^{(1)}(x_i, t_n, u_{i,n}) = M_n u_{i,n}^2/2, \qquad f^{(2)}(x_i, t_n, u_{i,n}) = f^*(x_i, t_n, u_{i,n}) - M_n u_{i,n}^2/2.$$

Remark 3.3. The use of the diagonal matrix $C_n^{(m)} = \mathrm{diag}(c_{1,n}^{(m)}, c_{2,n}^{(m)}, \ldots, c_{M,n}^{(m)})$ in Algorithm A1 is mainly for ensuring that

$$0 \le c_{i,n}^{(m)} + f_u^*(x_i, t_n, \xi_{i,n}^{(m)}) \le \sigma_n^*(\overline{u}_{i,n}^{(m)} - \underline{u}_{i,n}^{(m)}) \quad \text{whenever } \underline{u}_{i,n}^{(m)} \le \xi_{i,n}^{(m)} \le \overline{u}_{i,n}^{(m)}. \tag{3.34}$$

This can be seen from (3.10) and (3.25). The above estimate (3.34) is key for us to prove the monotone property (3.18) and the quadratic rate of convergence given in (3.20). This estimate is clearly satisfied if $c_{i,n}^{(m)}$ is replaced by the local maximum $\max_s\{-f_u^*(x_i, t_n, s); \underline{u}_{i,n}^{(m)} \le s \le \overline{u}_{i,n}^{(m)}\}$. In this case, Algorithm A1 becomes the AMI method given in [6]. However, our choice of $c_{i,n}^{(m)}$ here avoids computing the local maximum.

4. MAMI method for steady-state system

The MAMI method for time-dependent system (2.8) can be extended to the corresponding steady-state system (2.10). For this system, upper and lower solutions are defined as follows.

Definition 4.1. Two vectors $\widetilde{U}$ and $\widehat{U}$ in $\mathbf{R}^M$ are called a pair of *ordered upper and lower solutions* of (2.10) if $\widetilde{U} \ge \widehat{U}$ and

$$A\widetilde{U} \ge F(\widetilde{U}), \qquad A\widehat{U} \le F(\widehat{U}). \tag{4.1}$$

For a pair of ordered upper and lower solutions $\widetilde{U} = (\widetilde{u}_1, \ldots, \widetilde{u}_M)^T$ and $\widehat{U} = (\widehat{u}_1, \ldots, \widehat{u}_M)^T$, we define the sectors

$$\langle \widehat{U}, \widetilde{U} \rangle = \{U \in \mathbf{R}^M; \widehat{U} \le U \le \widetilde{U}\}, \qquad \langle \widehat{u}_i, \widetilde{u}_i \rangle = \{u_i \in \mathbf{R}; \widehat{u}_i \le u_i \le \widetilde{u}_i\}. \tag{4.2}$$

It is easily shown (e.g., see [4,18]) that, under hypothesis (H), system (2.10) has a maximal solution $\overline{U}$ and a minimal solution $\underline{U}$ in the sector $\langle \widehat{U}, \widetilde{U} \rangle$. Here the maximal and minimal property of $\overline{U}$ and $\underline{U}$ means that if U is any solution of (2.10) in $\langle \widehat{U}, \widetilde{U} \rangle$ then $\underline{U} \le U \le \overline{U}$. To compute $\overline{U}$ and $\underline{U}$, we utilize the technique developed in Section 3.

Assume that the function $f^*(x_i, u_i)$ in (2.11) is given by

$$f^*(x_i, u_i) = f^{(1)}(x_i, u_i) + f^{(2)}(x_i, u_i), \tag{4.3}$$

where $f_u^{(1)}(x_i, u_i)$ is monotone nondecreasing in u_i and $f_u^{(2)}(x_i, u_i)$ is monotone nonincreasing in u_i for $u_i \in \langle \widehat{u}_i, \widetilde{u}_i \rangle$. This assumption can always be satisfied (see Remark 3.2 for some details). Let λ_0 be the smallest eigenvalue of A. Starting from $U^{(0)} = \widetilde{U}$ and $U^{(0)} = \widehat{U}$, we compute the corresponding sequences $\{\overline{U}^{(m)}\} = \{(\overline{u}_1^{(m)}, \overline{u}_2^{(m)}, \ldots, \overline{u}_M^{(m)})^T\}$ and $\{\underline{U}^{(m)}\} = \{(\underline{u}_1^{(m)}, \underline{u}_2^{(m)}, \ldots, \underline{u}_M^{(m)})^T\}$, respectively, from the following linear iterative scheme (called the *Modified Accelerated Monotone Iterative* (MAMI) method).

Algorithm A2.

$$(A + \Gamma^{(m)})U^{(m+1)} = \Gamma^{(m)}U^{(m)} + F(U^{(m)}), \tag{4.4}$$

where

$$\Gamma^{(m)} = \mathrm{diag}(\gamma_1^{(m)}, \gamma_2^{(m)}, \ldots, \gamma_M^{(m)}), \qquad \gamma_i^{(m)} = \begin{cases} c_i^{(m)}, & \text{if } c_i^{(m)} > -\lambda_0, \\ \delta, & \text{if } c_i^{(m)} \le -\lambda_0, \end{cases} \tag{4.5}$$

with $c_i^{(m)} = -(f_u^{(1)}(x_i, \underline{u}_i^{(m)}) + f_u^{(2)}(x_i, \overline{u}_i^{(m)}))$ and δ being any positive constant.

The choice of $\gamma_i^{(m)}$ in (4.5) ensures that the inverse $(A+\Gamma^{(m)})^{-1}$ exists and is nonnegative (see Lemma 2.1). It also implies that

$$\Gamma^{(m)}(U - V) + F(U) - F(V) \ge 0 \quad \text{whenever } \widetilde{U} \ge \overline{U}^{(m)} \ge U \ge V \ge \underline{U}^{(m)} \ge \widehat{U}. \tag{4.6}$$

The same reasoning as for the time-dependent system leads to the following analogous result as that in Lemma 3.1.

Lemma 4.1. *Let $\widetilde{U}$ and $\widehat{U}$ be a pair of ordered upper and lower solutions of (2.10) and let f^* be given by (4.3). Also let hypothesis (H) hold. Then, for all $m \ge 1$, the sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$ given by Algorithm A2 with $\overline{U}^{(0)} = \widetilde{U}$ and $\underline{U}^{(0)} = \widehat{U}$ are well defined and possess the monotone property*

$$\widehat{U} \le \underline{U}^{(m-1)} \le \underline{U}^{(m)} \le \overline{U}^{(m)} \le \overline{U}^{(m-1)} \le \widetilde{U}. \tag{4.7}$$

Define

$$\overline{\sigma} = \max_i (f_u^{(1)}(x_i, \widetilde{u}_i) + f_u^{(2)}(x_i, \widehat{u}_i)), \qquad \underline{\sigma} = \min_i (f_u^{(1)}(x_i, \widetilde{u}_i) + f_u^{(2)}(x_i, \widehat{u}_i)). \tag{4.8}$$

An immediate consequence of Lemma 4.1 is as follows.

Theorem 4.1. *Let the conditions in Lemma 4.1 be satisfied. Then the sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$ given by Algorithm A2 with $\overline{U}^{(0)} = \widetilde{U}$ and $\underline{U}^{(0)} = \widehat{U}$ converge monotonically from above and below, respectively, to the maximal solution $\overline{U}$ and the minimal solution $\underline{U}$ of (2.10) in $\langle \widehat{U}, \widetilde{U} \rangle$. Moreover,*

$$\widehat{U} \le \underline{U}^{(m-1)} \le \underline{U}^{(m)} \le \underline{U} \le \overline{U} \le \overline{U}^{(m)} \le \overline{U}^{(m-1)} \le \widetilde{U}, \quad m = 1, 2, \ldots. \tag{4.9}$$

If, in addition, we have

$$\begin{cases} \overline{\sigma} < \lambda_0, & \text{for } \lambda_0 > 0, \\ \overline{\sigma} \le 0 \text{ and } \underline{\sigma} < 0, & \text{for } \lambda_0 = 0, \end{cases} \tag{4.10}$$

then $\overline{U} = \underline{U}$ and is the unique solution of (2.10) in $\langle \widehat{U}, \widetilde{U} \rangle$.

Proof. In view of the monotone property (4.7), the limits

$$\lim_{m\to\infty} \overline{U}^{(m)} = \overline{U}, \qquad \lim_{m\to\infty} \underline{U}^{(m)} = \underline{U}$$

exist and satisfy relation (4.9). Again by the monotone property (4.7), the element $\gamma_i^{(m)}$ of $\Gamma^{(m)}$ is monotone nonincreasing in m (possibly for $m \ge m_0$, where m_0 is a positive integer). Moreover, the sequence $\{\gamma_i^{(m)}\}$ is bounded from below by $\gamma_i = -(f_u^{(1)}(x_i, \underline{u}_i) + f_u^{(2)}(x_i, \overline{u}_i))$, where $\overline{u}_i$ and $\underline{u}_i$ are the respective components of $\overline{U}$ and $\underline{U}$. This implies that the sequence $\{\Gamma^{(m)}\}$ converges as $m \to \infty$. Letting $m \to \infty$ in (4.4), we see that $\overline{U}$ and $\underline{U}$ are solutions of (2.10) in $\langle \widehat{U}, \widetilde{U} \rangle$.

Let U be any solution of (2.10) in $\langle \widehat{U}, \widetilde{U} \rangle$. Then, by an induction using the nonnegative property of $(A + \Gamma^{(m)})^{-1}$ and the relation (4.6), we can easily show

$$\underline{U}^{(m)} \le U \le \overline{U}^{(m)}, \quad m = 0, 1, 2, \ldots. \tag{4.11}$$

This implies $\underline{U} \le U \le \overline{U}$ and so the maximal and minimal property of $\overline{U}$ and $\underline{U}$.

To prove $\overline{U} = \underline{U}$, under the present condition (4.10), we observe from the mean-value theorem that

$$(A - C)(\overline{U} - \underline{U}) = 0, \qquad C = \mathrm{diag}(f_u^*(x_1, \xi_1), \ldots, f_u^*(x_M, \xi_M)), \tag{4.12}$$

where $\xi_i \in \langle \underline{u}_i, \overline{u}_i \rangle$. The condition (4.10) ensures that the matrix $-C$ satisfies condition (2.13) of $\mathcal{D}$ in Lemma 2.1. So by this lemma, the inverse $(A - C)^{-1}$ exists. Hence, $\overline{U} = \underline{U}$. □

To show the quadratic convergence for the sum of sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$, we assume that the functions $f^{(l)}(x, u)$ $(l = 1, 2)$ in (4.3) satisfy the following Lipschitz condition

$$\begin{aligned} &f_u^{(1)}(x_i, u_i) - f_u^{(1)}(x_i, v_i) \le \sigma^*(u_i - v_i), \\ &f_u^{(2)}(x_i, u_i) - f_u^{(2)}(x_i, v_i) \ge -\sigma^*(u_i - v_i) \quad \text{whenever } \widehat{u}_i \le v_i \le u_i \le \widetilde{u}_i, \end{aligned} \tag{4.13}$$

where σ^* is a nonnegative constant. If $\overline{\sigma} < \lambda_0$, where $\overline{\sigma}$ is defined by (4.8), we have from Lemma 2.1 that the inverse $(A - \overline{\sigma} I)^{-1}$ exists and is nonnegative. Moreover, we have from Theorem 4.1 that the system (2.10) has a unique solution U^* in $\langle \widehat{U}, \widetilde{U} \rangle$. In analogy to Theorem 3.2, we have the following theorem for the rate of convergence of Algorithm A2.

Theorem 4.2. *Let the conditions in Lemma 4.1 and (4.13) be satisfied, and let $\overline{\sigma} < \lambda_0$. Also let U^* be the unique solution of (2.10) in $\langle \widehat{U}, \widetilde{U} \rangle$, and $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$ be the sequences from Algorithm A2 with $\overline{U}^{(0)} = \widetilde{U}$ and $\underline{U}^{(0)} = \widehat{U}$. Then, for all $m = 0, 1, 2, \ldots$,*

$$\|\overline{U}^{(m+1)} - U^*\|_\infty + \|\underline{U}^{(m+1)} - U^*\|_\infty \le \sigma^* \rho (\|\overline{U}^{(m)} - U^*\|_\infty + \|\underline{U}^{(m)} - U^*\|_\infty)^2, \tag{4.14}$$

where $\rho = \|(A - \overline{\sigma} I)^{-1}\|_\infty$.

Proof. By (2.10) and Algorithm A2, we obtain

$$(A + \Gamma^{(m)})(\overline{U}^{(m+1)} - U^*) = \Gamma^{(m)}(\overline{U}^{(m)} - U^*) + F(\overline{U}^{(m)}) - F(U^*). \tag{4.15}$$

The condition $\overline{\sigma} < \lambda_0$ ensures $c_i^{(m)} > -\lambda_0$. This implies $\Gamma^{(m)} = \mathrm{diag}(c_1^{(m)}, \ldots, c_M^{(m)})$. It follows from the argument in the proof of Theorem 3.2 that

$$0 \le \overline{U}^{(m+1)} - U^* \le \sigma^* \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty (A + \Gamma^{(m)})^{-1} (\overline{U}^{(m)} - U^*). \tag{4.16}$$

This leads to

$$\|\overline{U}^{(m+1)} - U^*\|_\infty \le \sigma^* \|(A + \Gamma^{(m)})^{-1}\|_\infty \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty \|\overline{U}^{(m)} - U^*\|_\infty. \tag{4.17}$$

Since the inverse $(A - \overline{\sigma} I)^{-1} \ge 0$ and $\Gamma^{(m)} \ge -\overline{\sigma} I$, we have $0 \le (A + \Gamma^{(m)})^{-1} \le (A - \overline{\sigma} I)^{-1}$. Thus, by (4.17), we obtain

$$\|\overline{U}^{(m+1)} - U^*\|_\infty \le \sigma^* \rho \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty \|\overline{U}^{(m)} - U^*\|_\infty. \tag{4.18}$$

Similarly,

$$\|\underline{U}^{(m+1)} - U^*\|_\infty \le \sigma^* \rho \|\overline{U}^{(m)} - \underline{U}^{(m)}\|_\infty \|\underline{U}^{(m)} - U^*\|_\infty. \tag{4.19}$$

The estimate (4.14) follows immediately from the addition of the above two relations (4.18) and (4.19). □

Remarks 3.1–3.3 are also valid for Algorithm A2.

5. Comparison of MAMI and AMI methods

In this section, we compare the monotone sequence of the MAMI method with that of the AMI method given in [6,18]. The only difference between the MAMI method and the AMI method is the construction of the matrix $C_n^{(m)}$ in (3.4) or $\Gamma^{(m)}$ in (4.4) (see Remark 3.3). In the AMI method, these two matrices are defined by a local maximum. Specifically, the AMI method for the time-dependent system (2.8) is given by (3.4), where the matrix $C_n^{(m)} = \mathrm{diag}(c_{1,n}^{(m)}, c_{2,n}^{(m)}, \ldots, c_{M,n}^{(m)})$ is defined by

$$c_{i,n}^{(m)} = \max_s \left\{ -f_u^*(x_i, t_n, s);\ \underline{u}_{i,n}^{(m)} \le s \le \overline{u}_{i,n}^{(m)} \right\}, \tag{5.1}$$

and the AMI method for the steady-state system (2.10) is given by (4.4), where the matrix $\Gamma^{(m)} = \mathrm{diag}(\gamma_1^{(m)}, \gamma_2^{(m)}, \ldots, \gamma_M^{(m)})$ is defined by

$$\gamma_i^{(m)} = \begin{cases} c_i^{(m)}, & \text{if } c_i^{(m)} > -\lambda_0, \\ \delta, & \text{if } c_i^{(m)} \le -\lambda_0, \end{cases} \qquad c_i^{(m)} = \max_s \left\{ -f_u^*(x_i, s);\ \underline{u}_i^{(m)} \le s \le \overline{u}_i^{(m)} \right\} \tag{5.2}$$

with δ being any positive constant.

Let $\widetilde{U}_n$ and $\widehat{U}_n$ be a pair of ordered upper and lower solutions of (2.8). It is well known that under hypothesis (H) and condition (3.7), the sequences $\{(\overline{U}_n^{(m)})_{\mathrm{A}}\}$ and $\{(\underline{U}_n^{(m)})_{\mathrm{A}}\}$ from the AMI method for (2.8) with $(\overline{U}_n^{(0)})_{\mathrm{A}} = \widetilde{U}_n$ and $(\underline{U}_n^{(0)})_{\mathrm{A}} = \widehat{U}_n$ are well defined. Also, they possess the monotone convergence described in Theorem 3.1 (see [6]). To establish a comparison result for the sequences in the MAMI and AMI methods, we denote by $\{(\overline{U}_n^{(m)})_{\mathrm{M}}\}$ and $\{(\underline{U}_n^{(m)})_{\mathrm{M}}\}$ the sequences from the MAMI method for (2.8) with $(\overline{U}_n^{(0)})_{\mathrm{M}} = \widetilde{U}_n$ and $(\underline{U}_n^{(0)})_{\mathrm{M}} = \widehat{U}_n$. Then, we have the following comparison result.

Theorem 5.1. *Let conditions in Lemma 3.1 be satisfied. Then*

$$(\overline{U}_n^{(m)})_{\mathrm{A}} \le (\overline{U}_n^{(m)})_{\mathrm{M}}, \qquad (\underline{U}_n^{(m)})_{\mathrm{M}} \le (\underline{U}_n^{(m)})_{\mathrm{A}}, \quad m, n = 1, 2, \ldots. \tag{5.3}$$

Proof. Denote the matrix $C_n^{(m)}$ in the MAMI method and the AMI method by $(C_n^{(m)})_{\mathrm{M}}$ and $(C_n^{(m)})_{\mathrm{A}}$, respectively. Let $(P_n^{(m)})_{\mathrm{J}} = I + k_n A_n + k_n (C_n^{(m)})_{\mathrm{J}} (J = M, A)$. We see from the AMI method that the equations for $\{(\overline{U}_n^{(m)})_{\mathrm{A}}\}$ and $\{(\underline{U}_n^{(m)})_{\mathrm{A}}\}$ are, respectively, equivalent to

$$\begin{aligned}(P_n^{(m)})_{\mathrm{M}}(\overline{U}_n^{(m+1)})_{\mathrm{A}} &= U_{n-1}^* + k_n((C_n^{(m)})_{\mathrm{M}} - (C_n^{(m)})_{\mathrm{A}})((\overline{U}_n^{(m+1)})_{\mathrm{A}} - (\overline{U}_n^{(m)})_{\mathrm{A}}) \\ &\quad + k_n((C_n^{(m)})_{\mathrm{M}}(\overline{U}_n^{(m)})_{\mathrm{A}} + F_n((\overline{U}_n^{(m)})_{\mathrm{A}}))\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}(P_n^{(m)})_{\mathrm{M}}(\underline{U}_n^{(m+1)})_{\mathrm{A}} &= U_{n-1}^* + k_n((C_n^{(m)})_{\mathrm{M}} - (C_n^{(m)})_{\mathrm{A}})((\underline{U}_n^{(m+1)})_{\mathrm{A}} - (\underline{U}_n^{(m)})_{\mathrm{A}}) \\ &\quad + k_n((C_n^{(m)})_{\mathrm{M}}(\underline{U}_n^{(m)})_{\mathrm{A}} + F_n((\underline{U}_n^{(m)})_{\mathrm{A}})).\end{aligned} \tag{5.5}$$

Let $\overline{W}_n^{(m)} = (\overline{U}_n^{(m)})_{\mathrm{M}} - (\overline{U}_n^{(m)})_{\mathrm{A}}$ and $\underline{W}_n^{(m)} = (\underline{U}_n^{(m)})_{\mathrm{A}} - (\underline{U}_n^{(m)})_{\mathrm{M}}$. We have from (3.4), (5.4) and (5.5) that

$$\begin{aligned}(P_n^{(m)})_{\mathrm{M}}\overline{W}_n^{(m+1)} &= k_n((C_n^{(m)})_{\mathrm{M}}\overline{W}_n^{(m)} + F_n((\overline{U}_n^{(m)})_{\mathrm{M}}) - F_n((\overline{U}_n^{(m)})_{\mathrm{A}})) \\ &\quad - k_n((C_n^{(m)})_{\mathrm{M}} - (C_n^{(m)})_{\mathrm{A}})((\overline{U}_n^{(m+1)})_{\mathrm{A}} - (\overline{U}_n^{(m)})_{\mathrm{A}})\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}(P_n^{(m)})_{\mathrm{M}}\underline{W}_n^{(m+1)} &= k_n((C_n^{(m)})_{\mathrm{M}}\underline{W}_n^{(m)} + F_n((\underline{U}_n^{(m)})_{\mathrm{A}}) - F_n((\underline{U}_n^{(m)})_{\mathrm{M}})) \\ &\quad + k_n((C_n^{(m)})_{\mathrm{M}} - (C_n^{(m)})_{\mathrm{A}})((\underline{U}_n^{(m+1)})_{\mathrm{A}} - (\underline{U}_n^{(m)})_{\mathrm{A}}).\end{aligned} \tag{5.7}$$

Consider the case $m = 0$. Since $f_u^{(1)}(x_i, t_n, \widehat{u}_{i,n}) + f_u^{(2)}(x_i, t_n, \widetilde{u}_{i,n}) \le f_u^*(x_i, t_n, s)$ for all $s \in \langle \widehat{u}_{i,n}, \widetilde{u}_{i,n} \rangle$, we have $(C_n^{(0)})_{\mathrm{M}} \ge (C_n^{(0)})_{\mathrm{A}}$ for all $n = 1, 2, \ldots$. By this relation and the monotone properties $(\overline{U}_n^{(1)})_{\mathrm{A}} \le (\overline{U}_n^{(0)})_{\mathrm{A}}$ and $(\underline{U}_n^{(1)})_{\mathrm{A}} \ge (\underline{U}_n^{(0)})_{\mathrm{A}}$, the relations (5.6) and (5.7) with $m = 0$ imply that

$$(P_n^{(0)})_{\mathrm{M}}\overline{W}_n^{(1)} \ge 0, \qquad (P_n^{(0)})_{\mathrm{M}}\underline{W}_n^{(1)} \ge 0.$$

It follows from the nonnegative property of $(P_n^{(0)})_{\mathrm{M}}^{-1}$ that $\overline{W}_n^{(1)} \ge 0$ and $\underline{W}_n^{(1)} \ge 0$. That is, $(\overline{U}_n^{(1)})_{\mathrm{A}} \le (\overline{U}_n^{(1)})_{\mathrm{M}}$ and $(\underline{U}_n^{(1)})_{\mathrm{M}} \le (\underline{U}_n^{(1)})_{\mathrm{A}}$ for $n = 1, 2, \ldots$.

Assume, by induction, that $(\overline{U}_n^{(m)})_{\mathrm{A}} \le (\overline{U}_n^{(m)})_{\mathrm{M}}$ and $(\underline{U}_n^{(m)})_{\mathrm{M}} \le (\underline{U}_n^{(m)})_{\mathrm{A}}$ for some $m \ge 1$. Then $f_u^{(1)}(x_i, t_n, (\underline{u}_{i,n}^{(m)})_{\mathrm{M}}) + f_u^{(2)}(x_i, t_n, (\overline{u}_{i,n}^{(m)})_{\mathrm{M}}) \le f_u^*(x_i, t_n, s)$ for all $s \in \langle (\underline{u}_{i,n}^{(m)})_{\mathrm{A}}, (\overline{u}_{i,n}^{(m)})_{\mathrm{A}} \rangle$. This implies that $(C_n^{(m)})_{\mathrm{M}} \ge (C_n^{(m)})_{\mathrm{A}}$ for all $n = 1, 2, \ldots$. Therefore, by (3.11), (5.6) and (5.7), we have

$$(P_n^{(m)})_{\mathrm{M}}\overline{W}_n^{(m+1)} \ge 0, \qquad (P_n^{(m)})_{\mathrm{M}}\underline{W}_n^{(m+1)} \ge 0.$$

This leads to $\overline{W}_n^{(m+1)} \ge 0$ and $\underline{W}_n^{(m+1)} \ge 0$ which proves

$$(\overline{U}_n^{(m+1)})_{\mathrm{A}} \le (\overline{U}_n^{(m+1)})_{\mathrm{M}}, \qquad (\underline{U}_n^{(m+1)})_{\mathrm{M}} \le (\underline{U}_n^{(m+1)})_{\mathrm{A}}, \quad n = 1, 2, \ldots.$$

The monotone property (5.3) follows from the principle of induction. □

We next consider the steady-state system (2.10). Let $\{\overline{U}_{\mathrm{A}}^{(m)}\}$ and $\{\underline{U}_{\mathrm{A}}^{(m)}\}$ be the sequences from the AMI method for (2.10) with $\overline{U}_{\mathrm{A}}^{(0)} = \widetilde{U}$ and $\underline{U}_{\mathrm{A}}^{(0)} = \widehat{U}$, where $\widetilde{U}$ and $\widehat{U}$ are a pair of ordered upper and lower solutions of (2.10). Then we have from [18] that under hypothesis (H) the sequences $\{\overline{U}_{\mathrm{A}}^{(m)}\}$ and $\{\underline{U}_{\mathrm{A}}^{(m)}\}$ converge monotonically from above and below, respectively, to the maximal solution $\overline{U}$ and the minimal solution $\underline{U}$ of (2.10) in $\langle \widehat{U}, \widetilde{U} \rangle$. Denote by $\{\overline{U}_{\mathrm{M}}^{(m)}\}$ and $\{\underline{U}_{\mathrm{M}}^{(m)}\}$ the sequences from the MAMI method for (2.10) with $\overline{U}_{\mathrm{M}}^{(0)} = \widetilde{U}$ and $\underline{U}_{\mathrm{M}}^{(0)} = \widehat{U}$. The following theorem gives a comparison result for these sequences.

Theorem 5.2. *Let the conditions in Lemma 4.1 be satisfied. Then*

$$\overline{U}_{\mathrm{A}}^{(m)} \le \overline{U}_{\mathrm{M}}^{(m)}, \qquad \underline{U}_{\mathrm{M}}^{(m)} \le \underline{U}_{\mathrm{A}}^{(m)}, \quad m = 1, 2, \ldots. \tag{5.8}$$

Proof. The proof follows from a similar argument as that in the proof of Theorem 5.1. □

The comparison results in Theorems 5.1 and 5.2 show that with the same initial iterations (which are a pair of ordered upper and lower solutions) the sequences from the AMI method may converge faster than the sequences from the MAMI method in terms of the number of iterations. This property holds true for both the time-dependent system (2.8) and the steady-state system (2.10). However, the AMI method involves a local maximum in each iteration (see (5.1) and (5.2)). By itself, this often requires an algorithm when the function f^* is much complicated (such as oscillating functions). So the overall computational cost may be expensive. In contrast, the MAMI method is very simple to apply in computations. A great deal of computational time can be saved especially for complicated function f^*. On the other hand, the MAMI method preserves the same monotone and quadratic convergence as the AMI method (as shown in Theorems 3.1 and 3.2 or 4.1 and 4.2). The above theoretical comparison results, as well as the numerical results in the next section, demonstrate that the MAMI method is preferable for numerical computations especially for complicated function f^*.

6. An application and numerical results

In this section, we apply the MAMI method to a model problem. We use some numerical results to demonstrate monotone and rapid convergence and to compare the MAMI and AMI methods.

It is seen from the previous sections that in order to implement the MAMI method it is necessary to find a pair of ordered upper and lower solutions of (2.8) and (2.10). Discussions for the construction of such a pair can be found in [4,6,7] for time-dependent systems and in [4,8,18] for steady-state systems. Our next example also illustrates one technique for constructing upper and lower solutions of the systems (2.8) and (2.10).

Let

$$f(x, u) = a\sin(\omega\pi u) + q(x), \quad x \in \Omega, \tag{6.1}$$

where a and ω are positive constants and $q(x)$ is a nonnegative continuous function in Ω. We consider the following problem

$$\begin{cases} \partial u/\partial t - D\nabla^2 u = f(x, u), & x \in \Omega, t > 0, \\ \alpha\partial u/\partial \nu + \beta u = \phi(x), & x \in \partial\Omega, t > 0, \\ u(x, 0) = \psi(x), & x \in \Omega, \end{cases} \tag{6.2}$$

where D is a positive diffusion constant, α and β are nonnegative constants with $\alpha + \beta > 0$, and the functions $\phi(x)$ and $\psi(x)$ are nonnegative. For this problem, the finite difference approximations (2.8) and (2.10) are reduced, respectively, to

$$\begin{cases} (I + k_n A)U_n = U_{n-1} + k_n(F(U_n) + G), \quad n = 1, 2, \ldots, \\ U_0 = \Psi, \end{cases} \tag{6.3}$$

and

$$AU = F(U) + G, \tag{6.4}$$

where

$$F(U) = (f(x_1, u_1), \ldots, f(x_M, u_M))^T, \tag{6.5}$$

and $G \ge 0$ is an M-dimensional vector associated with the boundary function $\phi(x)$.

To construct a pair of ordered upper and lower solutions, we assume that the boundary condition in (6.2) is of Dirichlet or Robin type. Let W be the positive solution of the following linear system

$$AW = Q, \tag{6.6}$$

where $Q \ge (a + \|q\|_\infty + \|G\|_\infty, \ldots, a + \|q\|_\infty + \|G\|_\infty)^T$ is sufficiently large so that $W \ge \Psi$. It is easy to verify that $\widetilde{U} = W$ and $\widehat{U} = 0$ are a pair of ordered upper and lower solutions of (6.4). Moreover, they are also ordered upper and lower solutions of the time-dependent system (6.3) since $W \ge \Psi \ge 0$. The above construction of upper and lower solutions will be used for our numerical computations.

Since $f_u(x, u) = a\omega\pi\cos(\omega\pi u)$, a simple calculation shows that

$$-a\omega^2\pi^2(u - v) \le f_u(x, u) - f_u(x, v) \le a\omega^2\pi^2(u - v) \quad \text{whenever } v \le u. \tag{6.7}$$

Therefore, the function $f(x, u)$ can be decomposed into the form (3.3) with

$$f^{(1)}(x, u) = f(x, u) + a\omega^2\pi^2u^2/2, \qquad f^{(2)}(x, u) = -a\omega^2\pi^2u^2/2 \tag{6.8}$$

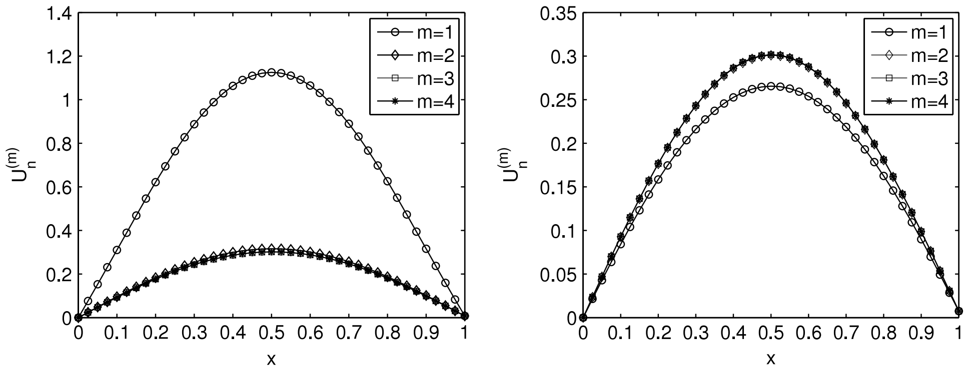

Fig. 6.1. The monotone convergence of the sequences $\{\overline{U}_n^{(m)}\}$ and $\{\underline{U}_n^{(m)}\}$ at $(x_i, 0.2, 1)$ [left: $\{\overline{U}_n^{(m)}\}$, right: $\{\underline{U}_n^{(m)}\}$].

or

$$f^{(1)}(x,u) = a\omega^2\pi^2u^2/2, \qquad f^{(2)}(x,u) = f(x,u) - a\omega^2\pi^2u^2/2. \tag{6.9}$$

To give some numerical results, we consider problem (6.2) in the square domain $\Omega = \{(x,y); 0 < x < 1, 0 < y < 1\}$ under the non-homogeneous Dirichlet boundary condition

$$u(x,y,t) = \phi(x,y), \quad (x,y) \in \partial\Omega,\ t > 0,$$

and take an equal mesh size in each of the space and time directions, i.e., $h_x = h_y = h$ and $k_n = k$. In this case, the matrix A is an $M_0^2 \times M_0^2$ matrix where $M_0 = 1/h - 1$. It is given by $A = (D/h^2)\text{tridiag}(C, A_0, C)$ with the $M_0 \times M_0$ diagonal matrix C and tridiagonal matrix A_0

$$C = \text{diag}(-1, \ldots, -1), \qquad A_0 = \text{tridiag}(-1, 4, -1).$$

In the block form, the vector G is defined by

$$G = (D/h^2)(G_1, \ldots, G_{M_0})^T,$$

where G_i are the M_0-dimensional vectors of the form

$$\begin{cases} G_1 = (\phi(x_1,0) + \phi(0,y_1), \phi(x_2,0), \ldots, \phi(x_{M_0-1},0), \phi(x_{M_0},0) + \phi(1,y_1))^T, \\ G_i = (\phi(0,y_i), 0, \ldots, 0, \phi(1,y_i))^T \quad (i = 2,3,\ldots,M_0-1), \\ G_{M_0} = (\phi(x_1,1) + \phi(0,y_{M_0}), \phi(x_2,1), \ldots, \phi(x_{M_0-1},1), \phi(x_{M_0},1) + \phi(1,y_{M_0}))^T. \end{cases}$$

It is easy to see that A satisfies the conditions in hypothesis (H). Choose the physical parameters $D = 1$, $\omega = 20$, $a = (\omega\pi)^{-2}$ and $q(x,y) = 10\sin(\pi x)\sin(\pi y)$. The boundary function $\phi(x,y)$ and initial function $\psi(x,y)$ are given as

$$\phi(x,y) = \psi(x,y) = \sin(\pi x/5)\sin(\pi y/5)/10.$$

Let W be the solution of (6.6) with $Q = (150, 150, \ldots, 150)^T$. We have from the construction of the upper and lower solutions that the pair $\widetilde{U} = W$ and $\widehat{U} = 0$ are ordered upper and lower solutions of (6.3) as well as (6.4). Using this pair as the initial iterations, we compute the corresponding sequences from Algorithms A1 and A2 where the decomposition (3.3) is given by (6.8). All computations are carried out by using MATLAB on a Pentium-4 computer with 2 G memory. In the iteration process, the termination criterion of iterations is given by

$$\|\overline{W}^{(m)} - \underline{W}^{(m)}\|_\infty < 10^{-10}, \tag{6.10}$$

where $(\overline{W}^{(m)}, \underline{W}^{(m)})$ represents the mth time-dependent iteration $(\overline{U}_n^{(m)}, \underline{U}_n^{(m)})$ or the mth steady-state iteration $(\overline{U}^{(m)}, \underline{U}^{(m)})$.

6.1. Monotone convergence

Let $h = k = 1/40$. Using $\overline{U}_n^{(0)} = W$ and $\underline{U}_n^{(0)} = 0$, we compute the corresponding sequences $\{\overline{U}_n^{(m)}\}$ and $\{\underline{U}_n^{(m)}\}$ from Algorithm A1 for (6.3). In all the numerical computations, the basic feature of monotone convergence of the sequences was observed. In Fig. 6.1, we present some numerical results of the sequences $\{\overline{U}_n^{(m)}\}$ and $\{\underline{U}_n^{(m)}\}$ at $(y_j, t_n) = (0.2, 1)$ and various x_i. As expected from our theoretical analysis in Theorem 3.1, the monotone convergence of the sequences holds at every mesh point (x_i, y_j, t_n).

Table 6.1
Numerical solution U_n^* of (6.3) at $t_n = 1$ by Algorithm A1.

x_i	y_j				
	0.1	0.2	0.3	0.4	0.5
0.1	0.0487624037	0.0927866375	0.1277992603	0.1504083112	0.1584368605
0.2	0.0927866375	0.1765611912	0.2431941076	0.2862338603	0.3015395872
0.3	0.1277992603	0.2431941076	0.3349948789	0.3943217972	0.4154752784
0.4	0.1504083112	0.2862338603	0.3943217972	0.4642328527	0.4892665050
0.5	0.1584368605	0.3015395872	0.4154752784	0.4892665050	0.5158681062

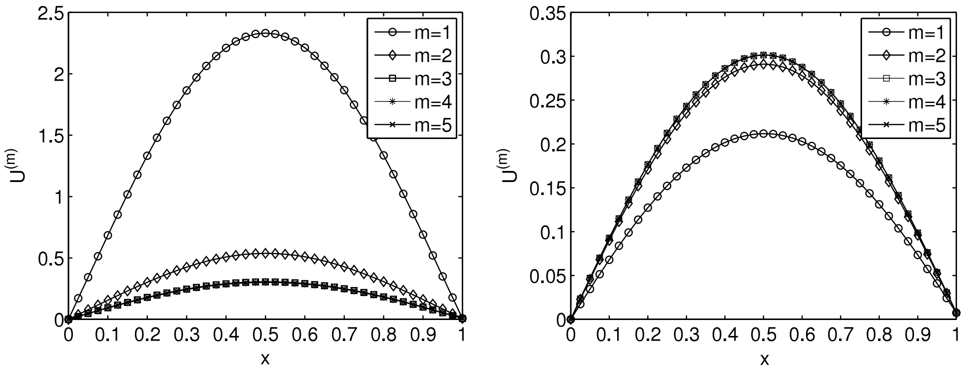

Fig. 6.2. The monotone convergence of the sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$ at $(x_i, 0.2)$ [left: $\{\overline{U}^{(m)}\}$, right: $\{\underline{U}^{(m)}\}$].

Table 6.2
Numerical solution U^* of (6.4) by Algorithm A2.

x_i	y_j				
	0.1	0.2	0.3	0.4	0.5
0.1	0.0487624090	0.0927866475	0.1277992741	0.1504083274	0.1584368774
0.2	0.0927866475	0.1765612102	0.2431941337	0.2862338910	0.3015396195
0.3	0.1277992741	0.2431941337	0.3349949148	0.3943218394	0.4154753228
0.4	0.1504083274	0.2862338910	0.3943218394	0.4642329023	0.4892665572
0.5	0.1584368774	0.3015396195	0.4154753228	0.4892665572	0.5158681611

Since the convergence of our iterations is quadratic, we obtained the unique solution U_n^* of (6.3) in $\langle 0, W\rangle$ only after four iterations as shown in Fig. 6.1. More numerical results of U_n^* at $t_n = 1$ are explicitly presented in Table 6.1.

We next compute the sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$ from Algorithm A2 for (6.4) with $\overline{U}^{(0)} = W$ and $\underline{U}^{(0)} = 0$. Since the smallest eigenvalue λ_0 of A is about $2\pi^2$ and the maximum $\overline{\sigma}$ defined by (4.8) is approximately 11 for this example, the condition $\overline{\sigma} < \lambda_0$ is satisfied. Let $h = 1/40$. Some numerical results of the sequences $\{\overline{U}^{(m)}\}$ and $\{\underline{U}^{(m)}\}$ at $y_j = 0.2$ and various x_i are plotted in Fig. 6.2. It is seen that the sequences possess the monotone convergence described by Theorem 4.1. In only five iterations, they converge to the unique solution U^* of (6.4) in $\langle 0, W\rangle$. More numerical results of U^* are listed in Table 6.2.

6.2. Comparison of MAMI and AMI methods

In order to further demonstrate the effectiveness of the MAMI method, we now compare it with the AMI method in terms of the number of iterations and the corresponding CPU time (in seconds). We use MATLAB function "fminbnd" to compute the local maxima in (5.1) and (5.2) when the AMI method is applied.

For the time-dependent system (6.3), the corresponding number of iterations and the CPU time (in seconds) at $t_n = 1$ are given in Table 6.3, where $k = h$ and the termination criterion is still determined by (6.10). For the same mesh size h, the number of iterations for the MAMI and AMI methods is about the same, but the MAMI method needs less computational time to converge. When $h = 1/60$, for example, the number of iterations for the MAMI method is four and the computational cost is 18.1273 CPU seconds. In contrast, the AMI method takes three iterations to converge within 294.3894 CPU seconds. Similar comparisons can be made with other data. Thus, in terms of the computational time, the MAMI method converges faster than the AMI method.

In Table 6.4, we give the number of iterations and the CPU time (in seconds) required for the MAMI and AMI methods for the steady-state system (6.4). Clearly, similar comparison results as in Table 6.3 can be obtained from the data in Table 6.4.

Table 6.3
Number of iterations (I) and CPU seconds (T) for (6.3) at $t_n = 1$.

h	MAMI method		AMI method	
	I	T	I	T
1/20	5	0.9828	3	10.5768
1/40	4	5.0076	3	86.8614
1/60	4	18.1273	3	294.3894
1/80	4	47.7519	3	698.8221
1/100	4	94.9266	3	1773.7448

Table 6.4
Number of iterations (I) and CPU seconds (T) for (6.4).

h	MAMI method		AMI method	
	I	T	I	T
1/120	5	1.8720	3	20.9821
1/140	5	2.7144	3	28.6262
1/160	5	3.7128	3	37.6430
1/180	5	4.9764	3	47.8143
1/200	5	6.5052	3	59.7172

7. Concluding remarks

In this paper, we developed a new monotone iterative method, called the *Modified Accelerated Monotone Iterative* (MAMI) method. It can be used to solve finite difference systems of a class of nonlinear reaction–diffusion–convection equations with nonlinear boundary conditions, including time-dependent systems and its corresponding steady-state systems. This method preserves the same monotone and quadratic convergence as the known *Accelerated Monotone Iterative* (AMI) method. It improves the AMI method in the sense that the construction of sequences of iterations avoids computing local maxima in each iteration and the computation is easily carried out. The numerical result presented coincides with the analysis. The MAMI method is shown to be preferable for numerical computations especially for complicated nonlinear functions.

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