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Mathematical analysis and numerical methods for a PDE model of a stock loan pricing problem[☆]

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ABSTRACT

In this paper the mathematical analysis of a model for pricing stock loan contracts, when the accumulative dividend yield associated to the stock is returned by the lender to the borrower on redemption, is carried out. More precisely, the model is formulated in terms of an obstacle problem associated to a Kolmogorov equation and the existence and uniqueness in the set of solutions with polynomial growth are obtained. Also some regularity properties of the solution are analyzed. Next, for the numerical solution of the problem the combination of Crank–Nicolson Lagrange–Galerkin with the augmented Lagrangian active set method is described. Finally, some numerical examples illustrate the theoretical properties of the optimal redeeming boundary previously stated in the literature.

1. Introduction

A stock loan is a contract between a lender (for example, a bank) and a borrower (for example, a client of the bank). The borrower owns a share of a stock which acts as the collateral of the loan obtained from the lender. At any time before or at loan maturity, the borrower may recover the stock by repaying the lender the principal and the fixed interest rate associated to the loan. Otherwise, the borrower can surrender the stock instead of paying the loan. The product is a way of financing in which stocks are employed as the only guarantee for the loan, this secured feature being one advantage with respect to traditional loans. The stock loan price must be here understood as the fair price that the lender should charge to the borrower and this is the target of the pricing problem here addressed. On the other hand, as the borrower has the option to redeem, for him/her the question about the optimal redeeming strategy arises.

An important feature from the financial and mathematical points of view is the contract specifications concerning the destination of the dividends associated to the stock: they can be either gained by the lender or by the borrower and in both cases, also either before or on redemption. The first attempt of a quantitative analysis to price a stock loan contract appears in [24], where the case in which the dividends of the stock are collected by the lender until redemption and not credited to the borrower. In this case the pricing problem is similar to the American call option problem with time dependent strike. Moreover, in [24] the authors deduce a pricing formula when the maturity of the loan is infinite in analogy to the American perpetual option with a time varying exercise value. Finally, the paper indicates different interesting open problems and some of them are treated in [8]. Thus, in [8] the different PDE based pricing models for the finite maturity case subjected

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to different possibilities of dividend yield distribution are presented; the mathematical analysis mainly focuses on the properties of the redeeming boundary, which is the unknown free boundary that separates the redemption region from the no redemption one, thus characterizing the optimal redemption policy to be followed by the borrower. More precisely, the first three situations analyzed in [8] correspond to the cases of dividends gained by the lender before redemption, reinvested dividends returned to the borrower on redemption and dividends always delivered to the borrower, and all lead to one-dimensional Black–Scholes variational inequalities. From the mathematical point of view, the most complex case arises when the accumulative dividends yield is returned to the borrower on redemption. In this fourth case, the introduction of a path dependent variable allows to pose an obstacle problem associated to an ultraparabolic PDE of Kolmogorov type, as in the case of Asian options with continuous arithmetic averaging. For this case, in [8] the existence of a redeeming boundary and their properties are analyzed. We acknowledge the recent paper [25], that analyzes the stock loan pricing problem under a pair of regime switching examples: a single regime jump and a two-state Markov change, although the policy in the contract about the stock dividends is not treated. We also notice the recent paper [23] that considers a stochastic volatility model for the asset without dividend yield.

In the present paper, for the first time the mathematical analysis of the PDE model for the stock loan pricing problem is addressed in the case when the accumulative dividend yield is returned to the borrower on redemption. Furthermore, the optimal regularity of the solution in anisotropic Sobolev spaces is analyzed. For this purpose, the techniques developed in [18,16] to study obstacle problems associated to hypoelliptic equations of Kolmogorov type are applied.

Secondly, as the analytical solution cannot be obtained, we propose a numerical method to approximate it. More precisely, first the unbounded domain is truncated to a large enough computational bounded domain with appropriate boundary conditions. Next, taking into account that the Kolmogorov equation is strongly convection-dominated, we propose the characteristics method to discretize the material derivative associated to the time derivative and first order spatial derivatives terms. The classical method of characteristics of first order has been introduced in [21] and first applied for the resolution of financial problems in [22] for vanilla options and for pricing Asian options in [9]. More recently, the higher order Crank–Nicolson Lagrange–Galerkin method has been analyzed in [3,4] for a general (possibly degenerated) convection–diffusion–reaction equation and applied to pricing problems of Asian options with continuous arithmetic averaging in [5]. Additionally to the discretization method, in order to deal with the nonlinearity associated to the obstacle condition (free boundary problem), the augmented Lagrangian active set method proposed in [15] is used. For Asian options, this method has been compared with an alternative duality method in [6]. For the discretization in the asset and accumulative dividend variables, a piecewise quadratic finite elements method is considered, so that the joint time and spatial discretization falls in the frame of the so called Lagrange–Galerkin methods. In order to validate the performance of the proposed numerical techniques, we verify all qualitative properties theoretically proved in [8] about the redemption region and the optimal redeeming boundary.

The paper is organized as follows. In Section 2 some notations and the mathematical model are stated. Moreover, the theoretical properties of the redemption boundary are recalled. In Section 3 the existence and uniqueness of solution is obtained. Section 4 is devoted to the analysis of the regularity of the solution in anisotropic Sobolev spaces. Numerical techniques to solve the problem are described in Section 5. In Section 6 some numerical results illustrate the theoretical results.

2. Formulation of the pricing problem

We assume that the risk neutral price of the stock evolves according to the classical geometric Brownian motion dynamics

$$dS_t = (r-\delta)S_t dt + \sigma S_t dW_t, \tag{2.1}$$

where W is a standard real Brownian motion on a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\right)$. The parameters r, δ and σ denote the risk-free interest rate, the dividend yield and the volatility respectively. Hereafter we assume that the dividend yield is a positive constant, that is

$$\delta > 0. \tag{2.2}$$

As in previous papers related to stock loans, we consider the classical Black–Scholes dynamics for simplicity and we remark that our theoretical and numerical results can be extended to more general diffusion models (e.g. local volatility models).

As indicated in the introduction, we consider a stock loan in which the borrower receives a loan from the lender who in turn receives the stock as collateral. The loan contract provides that the accumulative dividends of the stock will be returned to the borrower on redemption. Redemption can take place at any time before or at the loan maturity and in case of no redemption the lender maintains the stock. The parameters of the stock loan contract are the principal value K, the (continuously compounded) interest rate γ of the loan and the maturity T.

Let us assume that the initial date of the loan is $t = 0$: then the intrinsic value of the stock loan is given by

$$\left(S_t - Ke^{\gamma t} + I_t\right)^+, \qquad t \in [0, T], \tag{2.3}$$

where the auxiliary path dependent process

$$I_t = \delta \int_0^t e^{r(t-u)} S_u du, \qquad t \in [0, T], \tag{2.4}$$

represents the value of the accumulative dividends. In differential form, we have

$$dI_t = (rI_t + \delta S_t)dt. \tag{2.5}$$

By classical arguments (see, for instance, Chap. 11 in [19]), the unique price of the stock loan which avoids the introduction of arbitrage opportunities is given by $V_t = V(t, S_t, I_t)$ where V is the solution to the following free-boundary problem

$$\begin{cases} \max\{\mathcal{L}V, \Psi - V\} = 0 & \text{in } [0,T) \times \mathbb{R}^2_+, \\ V(T,S,I) = \Psi(T,S,I) & (S,I) \in \mathbb{R}^2_+. \end{cases} \tag{2.6}$$

In (2.6), $\mathcal{L}$ is the Kolmogorov operator related to the processes (S_t, I_t) in (2.1)–(2.5), that is

$$\mathcal{L}V = \frac{\sigma^2 S^2}{2}\partial_{SS}V + (r-\delta)S\partial_S V + (rI + \delta S)\partial_I V + \partial_t V - rV, \tag{2.7}$$

and

$$\Psi(t,S,I) = \left(S - Ke^{\gamma t} + I\right)^+ \tag{2.8}$$

is the payoff/obstacle function.

The existence of solutions to (2.6) is a delicate matter. Indeed, on the one hand it is well-known that obstacle problems do not generally admit classical (smooth) solutions; on the other hand, $\mathcal{L}$ is not a uniformly parabolic operator and the classical PDE theory of generalized solutions does not apply to problem (2.6). We emphasize that, differently from the standard Black&Scholes case, (2.6) is a two-dimensional time-dependent problem and does not admit dimension reduction; indeed, the solution V is a function of the time variable t and the spatial variables S and I. Operator $\mathcal{L}$ *is not uniformly parabolic* because only the first order derivative with respect to I appears; in other terms, we have two spatial variables but only one source of diffusion (i.e. one Brownian motion). We mention that similar operators to $\mathcal{L}$ were recently studied in [1,10,12,16,18,20] in connection with the analysis of American Asian options. Taking into account certain analogies between stock loans and Asian options, in the Section 3 (cf. Theorem 3.5) we give some results on the existence and optimal regularity of solutions; moreover, we prove the following stochastic representation formula for the solution to (2.6):

$$V(t,S,I) = \sup_{\tau \in \mathcal{T}_{t,T}} E\left[e^{-r(\tau-t)}\left(S_\tau - Ke^{\gamma\tau} + I_\tau\right)^+ \mid S_t = S,\ I_t = I\right]. \tag{2.9}$$

In (2.9), $\mathcal{T}_{t,T}$ denotes the set of all $(\mathcal{F}_t)$-stopping times with values in $[t, T]$.

We remark explicitly that the first equation in (2.6) can be decomposed in the following system on $(0,T) \times \mathbb{R}^2_+$

$$\begin{cases} \mathcal{L}V \le 0, \\ V \ge \Psi, \\ \mathcal{L}V \cdot (V - \Psi) = 0, \end{cases} \tag{2.10}$$

where the third equation is known as *complementarity condition*. Then, as for most contracts with the early exercise feature, the domain of the solution V can be divided in two regions:

(i) the redemption region

$$R_0 = \{(t,S,I) \in [0,T) \times \mathbb{R}^2_+ \mid V(t,S,I) = \Psi(t,S,I)\}; \tag{2.11}$$

(ii) the no-redemption region

$$R_+ = \{(t,S,I) \in [0,T) \times \mathbb{R}^2_+ \mid V(t,S,I) > \Psi(t,S,I)\}. \tag{2.12}$$

The surface separating the two regions is the so-called *optimal redeeming boundary* which identifies the critical price of the stock at which it is worth redeeming the loan.

By assuming the existence and exploiting the regularity of the solution V, in [8] some qualitative properties of the optimal redeeming boundary have been proved. Specifically, early redemption never happens for $r > \gamma$; while when $r = \gamma$ it is optimal to hold the loan until maturity, even if in some occasion early redemption may be optimal as well. Furthermore, for $r < \gamma$ the redemption region is non-empty. These results are summarized in the following proposition proved in [8]:

Proposition 2.1. *Assume that $\delta > 0$. We have:*

(i) *if $r > \gamma$ then $R_0 = \emptyset$;*
(ii) *if $r = \gamma$ then it is optimal to hold the stock loan before maturity;*

(iii) *if* $r < \gamma$ *and we put*

$$O = \{(t, I) \in [0, T) \times \mathbb{R}_+ \mid I \geq Ke^{\gamma t}\}$$

then

$$\{(t, S, I) \in [0, T) \times \mathbb{R}_+^2 \mid (t, I) \in O\} \subset R_0. \tag{2.13}$$

Moreover the optimal redeeming boundary is a graph, that is there exists a function $S^* : O \longrightarrow \mathbb{R}_+$ *such that*

$$R_0 = \left\{(t, S, I) \mid S \geq S^*(t, I)\right\}.$$

The function S^* *is monotonically decreasing in* t *and* I*, with*

$$\lim_{t \to T} S^*(t, I) = e^{\gamma T} K - I. \tag{2.14}$$

The above result is based on the existence and other properties of the solution V, which we examine in detail in this paper. In particular, we put some emphasis on the regularity properties of generalized solutions because those properties also give some hint for the efficient numerical solution of (2.6): specifically, we will show that the numerical schemes can take advantage of the degenerate structure of $\mathcal{L}$ as a strongly convection-dominated operator.

3. Existence and uniqueness of solutions

In order to study the existence and uniqueness of solution we first introduce a suitable functional setting. So, we denote by

$$Y = (r - \delta) S \partial_S + (rI + \delta S)\partial_I + \partial_t \tag{3.15}$$

the first order part of $\mathcal{L}$.

For any domain $\Omega \subset \mathbb{R}^3$ and $p \geq 1$, we define the anisotropic Sobolev spaces

$$\mathcal{S}^p(\Omega) = \left\{U \in L^p(\Omega) \mid \partial_S U,\ \partial_{SS} U,\ YU \in L^p(\Omega)\right\} \tag{3.16}$$

endowed with the semi-norm

$$\|U\|_{\mathcal{S}^p} = \|U\|_{L^p} + \|\partial_S U\|_{L^p} + \|\partial_{SS} U\|_{L^p} + \|YU\|_{L^p}.$$

If $U \in \mathcal{S}^p(H)$ for any compact subset $H \subseteq \Omega$, then we write $U \in \mathcal{S}^p_{\text{loc}}(\Omega)$. Next, we introduce the notion of *strong solution* of the free boundary problem (2.6).

Definition 3.1 (*Strong Solution*)**.** A strong solution to problem (2.6) is a function $V \in \mathcal{S}^1_{\text{loc}} \cap C((0, T] \times \mathbb{R}_+^2)$ which satisfies the differential inequality a.e. in $(0, T) \times \mathbb{R}_+^2$ and the final condition in the pointwise sense.

Although the goal of this section is the proof of the existence of a strong solution to problem (2.6), in the sense of Definition 3.1, as an intermediate result we first construct a supersolution.

Definition 3.2. A function $\bar{V} \in C^2([0, T) \times \mathbb{R}_+^2) \cap C([0, T] \times \mathbb{R}_+^2)$ such that

$$\mathcal{L}\bar{V} \leq 0 \quad \text{and} \quad \bar{V} \geq \Psi \text{ in } (0, T) \times \mathbb{R}_+^2, \tag{3.17}$$

is called a supersolution to problem (2.6).

As the following lemma shows, it is not difficult to give the explicit expression of a supersolution to (2.6) with Ψ as in (2.8).

Lemma 3.3. *For any* β *and* q *suitably large constants, the function*

$$\bar{V}(t, S, I) = \beta e^{-qt} \sqrt{S^2 + I^2} \tag{3.18}$$

is a super-solution to problem (2.6).

Proof. We have

$$\mathcal{L}\bar{V}(t, S, I) = \frac{\beta e^{-qt}}{2\left(I^2 + S^2\right)^{3/2}} W(S, I)$$

where

$$W(S, I) = -2\left(S^2 + I^2\right)\left(q\left(S^2 + I^2\right) + \delta S\left(S - I\right)\right) + \sigma^2 S^2 I^2.$$

Therefore $\mathcal{L}\bar{V} \le 0$ if and only if $W(S, I) \le 0$. By using repeatedly the elementary inequality

$$SI \le \frac{S^2 + I^2}{2},$$

we have

$$\begin{aligned} W(S, I) &\le \frac{S^2 + I^2}{2} \left(\left(-4q + \sigma^2\right) \left(S^2 + I^2\right) - 4\delta S\,(S - I) \right) \\ &\le \frac{S^2 + I^2}{2} \left(S^2 \left(-4q - 2\delta + \sigma^2\right) + I^2 \left(-4q + 2\delta + \sigma^2\right) \right). \end{aligned}$$

Thus $W(S, I) \le 0$ if q is positive and suitably large. Once q is fixed, it is clear that there exists $\beta > 0$ such that

$$\bar{V}(t, S, I) \ge \Psi(t, S, I), \qquad (t, S, I) \in (0, T) \times \mathbb{R}^2_+,$$

and therefore $\bar{V}$ is a supersolution. □

Now we prove the main result of this section. Generally speaking, we study problem (2.6) in the framework of hypoelliptic equations of Kolmogorov type. The obstacle problem for a general class of degenerate parabolic operators including (2.7) was first studied for the free boundary problem for arithmetic Asian options in [16].

As it appears in next theorem, we first define the concept of functions with polynomial growth.

Definition 3.4. A function $f : [0, T] \times \mathbb{R}^2_+ \to \mathbb{R}$ has polynomial growth if

$$|f(t, S, I)| \le C\left(1 + S^p + I^p\right), \quad (t, S, I) \in [0, T] \times \mathbb{R}^2_+,$$

for some positive constants C, p.

Theorem 3.5. *There exists a strong solution V of problem (2.6) with Ψ as in (2.8): we have that $V \in \mathcal{S}^p_{loc}\left((0, T] \times \mathbb{R}^2_+\right)$ for any $p \ge 1$ and*

$$V \le \bar{V} \tag{3.19}$$

where $\bar{V}$ is the supersolution in (3.18). Moreover, V is the unique solution with polynomial growth of problem (2.6), which solves the optimal stopping problem (2.9).

Proof. Following [16], let $D_\rho(x_1, x_2)$ denote the Euclidean ball centered at $(x_1, x_2) \in \mathbb{R}^2$, with radius ρ. We consider the sequence of domains $O_n = D_n\left(n + \frac{1}{n}, 0\right) \cap D_n\left(0, n + \frac{1}{n}\right)$ covering $\mathbb{R}^2_+$. For any $n \in \mathbb{N}$, the cylinder $H_n = (0, T) \times O_n$ is a $\mathcal{L}$*-regular domain* in the sense that the Cauchy–Dirichlet problem for $\mathcal{L}$ is well-posed because it is possible to find a barrier function (cf. Remark 3.1 in [12]) at any point of the "parabolic" boundary

$$\partial_P H_n := \partial H_n \setminus (\{0\} \times O_n).$$

In particular, since $\mathcal{L}$ satisfies condition (4.23) on any H_n, then by Theorem 3.1 in [12], for any $n \in \mathbb{N}$, problem

$$\begin{cases} \max\{\mathcal{L}U - f, \Psi - U\} = 0 & \text{in } H_n, \\ U|_{\partial_P H_n} = \Psi \end{cases} \tag{3.20}$$

has a strong solution $U \in \mathcal{S}^p_{\mathrm{loc}}(H_n) \cap C(H_n \cup \partial_P H_n)$. Moreover, for every $p \ge 1$ and $H \subset\subset H_n$ there exists a positive constant C, only depending on $H, H_n, p, \|\Psi\|_{L^\infty(H_n)}$ such that

$$\|U_n\|_{\mathcal{S}^p(H)} \le C. \tag{3.21}$$

Next we consider a sequence of cut-off functions $\chi_n \in C_0^\infty(\mathbb{R}^2_+)$, such that $\chi_n = 1$ in O_{n-1}, $\chi_n = 0$ in $\mathbb{R}^2_+ \setminus O_n$ and $0 \le \chi_n \le 1$ in $O_n \setminus O_{n-1}$. We set

$$\Psi_n(t, S, I) = \chi_n(S, I)\Psi(t, S, I) + (1 - \chi_n(S, I))\bar{V}(t, S, I),$$

where $\bar{V}$ is the supersolution in (3.18), and we denote by V_n the strong solution to (3.20) with $\Psi = \Psi_n$. By the comparison principle we have $\Psi \le V_{n+1} \le V_n \le \bar{V}$. Therefore, by (3.21), for every compact set H and $n \in \mathbb{N}$ such that $H_n \supset H$ we have

$$\|V_n\|_{\mathcal{S}^p(H)} \le C, \qquad p \ge 1,$$

for some constant C depending on H and p but not on n. Then we can pass to the limit as $n \to \infty$, on compact subsets of $(0, T) \times \mathbb{R}^2_+$, to get a strong solution of $\max\{\mathcal{L}V - f, \Psi - V\} = 0$ in the space $\mathcal{S}^p_{\mathrm{loc}}$. A standard argument based on barrier functions shows that $V(t, \cdot)$ is continuous up to $t = T$ and attains the final datum. Finally, the uniqueness and the Feynman–Kac representation of strong solutions is a consequence of the local summability properties of the transition density of the process (cf. [16], Theorem 1-(ii)) and it can be proved as in [18], Theorem 4.3. □

4. Anisotropic regularity of solutions

In the previous section we have proven the existence and uniqueness of solution to problem (2.6) in the anisotropic Sobolev spaces $\mathcal{S}^p$ defined in (3.16). It is interesting to notice that the regularity in $\mathcal{S}^p$ is optimal for this kind of problems and gives a clear picture of the peculiar properties of the solution. Other notions of generalized solutions (for instance, in the viscosity or variational sense) can be considered as well, albeit the stochastic representation (2.9) entails uniqueness among different solutions.

In this section we analyze the regularity properties of V and, in particular, the anisotropic Hölder continuity of V is compared with the classical Euclidean regularity.

For greater convenience, we put $x = (S, I)$ and, using the matrix notation, we rewrite the vector field Y in (3.15) as

$$Y = \langle Bx, \nabla_x\rangle + \partial_t$$

where B is the convection matrix

$$B = \begin{pmatrix} r-\delta & 0 \\ \delta & r \end{pmatrix}$$

and ∇_x is the gradient in the variables x. It is possible to introduce a functional setting, induced by the convection field Y, which is natural for the study of the interior regularity of strong solutions. Let us first consider an operator in $\mathbb{R}^3$ of the form

$$\bar{\mathcal{L}} = \bar{a}(t,x)\partial_{x_1x_1} + Y, \tag{4.22}$$

with Y as in (3.15). It is known (cf. [11], Theorem 1.4) that under the assumption (2.2) (i.e. $\delta > 0$) and if the coefficient $\bar{a}$ is a smooth function such that

$$\frac{1}{\mu} \le \bar{a} \le \mu \quad \text{on } \mathbb{R}^3, \tag{4.23}$$

where μ is a positive constant, then $\bar{\mathcal{L}}$ has a fundamental solution which can be globally estimated by Gaussian functions from above and below. Moreover, if the coefficient $\bar{a}$ in (4.22) is constant, then the operator $\bar{\mathcal{L}}$ is invariant[1] w.r.t. the left translations in the group law

$$(\tau,\xi) * (t,x) = (\tau + t, x + e^{tB}\xi), \tag{4.24}$$

where the exponential matrix of B is equal to

$$\exp(tB) = \exp(t(r-\delta))\begin{pmatrix} 1 & 0 \\ \exp(t\delta)-1 & \exp(t\delta) \end{pmatrix}.$$

Since the function $\bar{a}(t,S,I) = \frac{\sigma^2 S^2}{2}$ verifies the non-degeneracy condition (4.23) on any compact subset of $\mathbb{R}\times\mathbb{R}^2_+$, then the pricing operator $\mathcal{L}$ is *locally* of the form (4.22). Consequently, it is natural to characterize the interior regularity of solutions to $\mathcal{L}$ in terms of the group law (4.24). Indeed, the following embedding theorem holds (cf. [12]).

Theorem 4.1 (*Embedding Theorem*)**.** *Let O, Ω be bounded domains of $\mathbb{R}^3$ such that $O \subset\subset \Omega$ and $p > 6$. There exists a positive constant c, only dependent on B, Ω, O and p, such that*

$$\|U\|_{C_B^{1,\alpha}(O)} \le c\|U\|_{\mathcal{S}^p(\Omega)}, \qquad \alpha = 1 - \frac{6}{p}, \tag{4.25}$$

for any $u \in \mathcal{S}^p(\Omega)$. In (4.25), $C_B^{1,\alpha}$ is the anisotropic Hölder space defined by the following norms[2]:

$$\|U\|_{C_B^{0,\alpha}(\Omega)} = \sup_{\Omega}|U| + \sup_{\substack{(t,x),(\tau,\xi)\in\Omega \\ (t,x)\neq(\tau,\xi)}} \frac{|U(t,x)-U(\tau,\xi)|}{\|(\tau,\xi)^{-1}*(t,x)\|_B^{\alpha}},$$

$$\|U\|_{C_B^{1,\alpha}(\Omega)} = \|U\|_{C_B^{0,\alpha}(\Omega)} + \|\partial_{x_1}U\|_{C_B^{0,\alpha}(\Omega)} + \sup_{\substack{(t,x),(\tau,\xi)\in\Omega \\ (t,x)\neq(\tau,\xi)}} \frac{|U(t,x)-U(\tau,\xi)-(x_1-\xi_1)\partial_{x_1}U(\tau,\xi)|}{\|(\tau,\xi)^{-1}*(t,x)\|_B^{1+\alpha}},$$

where $\|\cdot\|_B$ is the anisotropic norm in $\mathbb{R}^3$ defined by

$$\|(t,x_1,x_2)\|_B = |t|^{\frac{1}{2}} + |x_1| + |x_2|^{\frac{1}{3}}.$$

[1] $\bar{\mathcal{L}}$ is left-$*$-invariant if

$$\bar{\mathcal{L}}U((\tau,\xi)*(t,x)) = \left(\bar{\mathcal{L}}U\right)((\tau,\xi)*(t,x)).$$

[2] We adopt the notation $x = (S, I)$ and $\xi = (S', I')$.

As a consequence of Theorem 4.1, the strong solutions to problem (2.6) belong locally to the space $C_B^{1,\alpha}$ for any $\alpha < 1$. Actually, according to the recent results in [13], the solutions to (2.6) belong to the class $\mathcal{S}^{\infty}_{\mathrm{loc}}$ and this regularity is optimal.

Now we briefly compare the intrinsic notion of $C_B^{1,\alpha}$-regularity with the more familiar regularity in the standard Euclidean sense. First notice that, for any *bounded* domain Ω, there exists a positive constant c_Ω such that

$$\begin{aligned}\|(\tau,\xi)^{-1} * (t,x)\|_B &= \|(t-\tau, x-\xi) + (0, (\mathrm{Id}_2 - e^{(t-\tau)B})\xi)\|_B \\ &\le c_\Omega |(t-\tau, x-\xi)|^{\frac{1}{3}}, \quad (\tau,\xi),(t,x) \in \Omega,\end{aligned}$$

where Id_2 is the identity matrix in $\mathbb{R}^2$. It immediately follows that

$$C_B^{0,\alpha}(\Omega) \subseteq C^{0,\frac{\alpha}{3}}(\Omega)$$

where $C^{0,\alpha}$ denotes the standard Euclidean Hölder space.

Remark 4.2 (*Euclidean Regularity*). If $U \in C_B^{1,\alpha}(\Omega)$ then $U, \partial_S U \in C^{0,\frac{\alpha}{3}}(\Omega)$ and also

$$|U((t,x) * (\tau,0)) - U(t,x)| = |U((t+\tau, e^{\tau B}x)) - U(t,x)| \le c_\Omega |\tau|^{\frac{1+\alpha}{2}}. \tag{4.26}$$

Estimate (4.26) is equivalent to the Hölder regularity of order $\frac{1+\alpha}{2}$ *along the integral curves of* Y. As a matter of fact, if we identify Y with the vector field $Y(t,x) = (1, Bx)$, then $\gamma(\tau) := (t+\tau, e^{\tau B}x)$ is the integral curve of Y starting from (t,x), that is the solution of the problem

$$\begin{cases}\dot\gamma(\tau) = Y(\gamma(\tau)),\\ \gamma(0) = (t,x).\end{cases}$$

Notice that the $C_B^{1,\alpha}$-regularity of U does not imply the existence of the Euclidean derivative $\partial_I U$: roughly speaking, since ∂_I is obtained by commuting ∂_S and Y

$$[\partial_S, Y] = \partial_S Y - Y\partial_S = (r-\delta)\partial_S + \delta\partial_I,$$

then intrinsically it has to be considered a third order derivative.

Keeping in mind the above remarks, in the numerical solution of problem (2.6) we adopt the natural approach of using a semi-Lagrangian method for time discretization, that mainly consists of a finite differences scheme along the integral curves of the convective part Y of the equation.

5. Numerical methods

In order to enumerate the numerical techniques, the main difficulties and the way to overcome them numerically are briefly outlined. First, a localization technique is used to cope with the initial formulation in an unbounded domain. Also, as the diffusive term is strongly degenerated, the PDE can be understood as an example of extreme convective dominated case, so that we propose a Crank–Nicolson characteristics time discretization scheme combined with a piecewise quadratic Lagrange finite element method. For the inequality constraints associated to the early redemption opportunity, we propose a mixed formulation and the use of an augmented Lagrangian active set technique.

5.1. Divergence form and localization in a bounded domain

Taking into account that we apply finite elements methods based on variational formulation, we first rewrite the PDE in (2.6) in divergence form. For simplicity, we introduce the new time variable $\tau = T - t$ and pose the equivalent problem:

$$\overline{\mathcal{L}}[V] \ge 0 \quad \text{in } (0,T)\times\mathbb{R}^2_+, \tag{5.27}$$

$$V \ge \overline{\Lambda} \quad \text{in } (0,T)\times\mathbb{R}^2_+, \tag{5.28}$$

$$\overline{\mathcal{L}}[V]\cdot(V-\overline{\Lambda}) = 0 \quad \text{in } (0,T)\times\mathbb{R}^2_+, \tag{5.29}$$

$$V(0,S,I) = \overline{\Lambda}(0,S,I) \quad \text{in } \mathbb{R}^2_+, \tag{5.30}$$

where the new operator and obstacle are respectively given by

$$\overline{\mathcal{L}}[V] = \partial_\tau V + \vec{v}\cdot\nabla V - \mathrm{div}(A\nabla V) + rV, \tag{5.31}$$

$$\overline{\Lambda}(\tau,S,I) = \Psi(T-\tau,S,I), \tag{5.32}$$

with

$$A(S,I)=\begin{pmatrix}\frac{1}{2}\sigma^2S^2 & 0\\ 0 & 0\end{pmatrix},\tag{5.33}$$

$$\vec{v}(S,I)=\begin{pmatrix}(\sigma^2-r+\delta)S\\ -(\delta S+rI)\end{pmatrix}.\tag{5.34}$$

As in most problems arising in finance, the numerical solution with finite differences, finite volumes or finite elements requires the approximation of the original problem in an unbounded domain by another one posed in a bounded computational domain. This technique is known as localization procedure, that has to be performed so that the truncation by the bounded domain and the associated boundary conditions do not affect the solution in the region of financial interest. For the classical problem of European vanilla options and Dirichlet boundary conditions, a rigorous analysis has been carried out in [14]. In general, the required boundary conditions at the new boundaries of the bounded domain are obtained with financial and/or mathematical arguments.

For the localization purpose, let us consider both S^∞ and I^∞ large enough real numbers suitably chosen and let the bounded domain be $\Omega=(0,S^\infty)\times(0,I^\infty)$, with Lipschitz boundary Γ, such that $\Gamma=\Gamma_1^+\bigcup\Gamma_2^+\bigcup\Gamma_1^-\bigcup\Gamma_2^-$, where $\Gamma_1^-=\Gamma\cap\{S=0\}$, $\Gamma_2^-=\Gamma\cap\{I=0\}$, $\Gamma_1^+=\Gamma\cap\{S=S^\infty\}$, $\Gamma_2^+=\Gamma\cap\{I=I^\infty\}$.

Then, problem (5.27)–(5.30) is replaced by the following one:

Find $V:[0,T]\times\Omega\to\mathbb{R}$ such that

$$\overline{\mathcal{L}}[V]\geq 0\quad\text{in }(0,T)\times\Omega,\tag{5.35}$$

$$V\geq\overline{\Lambda}\quad\text{in }(0,T)\times\Omega,\tag{5.36}$$

$$\overline{\mathcal{L}}[V]\cdot(V-\overline{\Lambda})=0\quad\text{in }(0,T)\times\Omega,\tag{5.37}$$

$$V(0,S,I)=\overline{\Lambda}(0,S,I)\quad\text{in }\Omega,\tag{5.38}$$

We note that in a certain abuse of notation we maintain the use of V also for the solution in the new time variable.

Next, by applying the theory of second order partial differential equations with nonnegative characteristics that can be found in [17] and taking into account the expression of the matrix A and the vector $\vec{v}$, only boundary conditions at Γ_1^+ and Γ_2^+ are required.

More precisely, following the ideas in [17], for simplicity let us introduce the notation

$$x_1=S,\qquad x_2=I.\tag{5.39}$$

Then, the operator associated to the Cauchy problem can be written in the form:

$$\mathcal{L}^*=\sum_{i,j=1}^{2}a_{ij}^*\frac{\partial^2}{\partial x_i x_j}+\sum_{j=1}^{2}b_j^*\frac{\partial}{\partial x_j}+l^*+\frac{\partial}{\partial t},\tag{5.40}$$

where the involved data are defined as follows

$$A^*(x_1,x_2)=(a_{ij}^*)=\begin{pmatrix}\dfrac{\sigma^2x_1^2}{2} & 0\\ 0 & 0\end{pmatrix},\tag{5.41}$$

$$v^*(x_1,x_2)=(b_j^*)=\begin{pmatrix}(r-\delta)x_1\\ \delta x_1+rx_2\end{pmatrix},\tag{5.42}$$

$$l^*(x_1,x_2)=-r.\tag{5.43}$$

Thus, in terms of the inwards normal vector to the boundary of Ω, $\vec{m}=(m_1,m_2)$, we introduce the following subsets of Γ:

$$\Sigma^1=\left\{(x_1,x_2)\in\Gamma,\ \sum_{i,j=1}^{2}a_{ij}^*m_im_j>0\right\},\tag{5.44}$$

$$\Sigma^2=\left\{(x_1,x_2)\in\Gamma-\Sigma_1,\ \sum_{i=1}^{2}\left(b_i^*-\sum_{j=1}^{2}\frac{\partial a_{ij}^*}{\partial x_j}\right)m_i<0\right\}.\tag{5.45}$$

As indicated in [17], the boundary conditions at $\Sigma_1\cup\Sigma_2$ for the initial boundary value problem associated to (5.40) are required. So, considering each boundary of Ω, we get:

- On boundary $\Gamma_1^+ : x_1 = x_1^\infty, 0 \le x_2 \le x_2^\infty, \vec{m} = (-1, 0)$

$$\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = \frac{\sigma^2 x_1^2}{2} > 0.$$

- On boundary $\Gamma_2^+ : 0 \le x_1 \le x_1^\infty, x_2 = x_2^\infty, \vec{m} = (0, -1)$

$$\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0$$

$$\sum_{i=1}^{2} \left(b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = -(\delta x_1 + r x_2^\infty) < 0.$$

- On boundary $\Gamma_1^- : x_1 = 0, 0 \le x_2 \le x_2^\infty, \vec{m} = (1, 0)$

$$\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0$$

$$\sum_{i=1}^{2} \left(b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = (-\sigma^2 + r - \delta) x_1 = 0.$$

- On boundary $\Gamma_2^- : 0 \le x_1 \le x_1^\infty, x_2 = 0, \vec{m} = (0, 1)$

$$\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0$$

$$\sum_{i=1}^{2} \left(b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = \delta x_1 > 0.$$

Therefore, we obtain that $\Sigma_1 = \Gamma_1^+$ and $\Sigma_2 = \Gamma_2^+$, so that $\Sigma_1 \cup \Sigma_2 = \Gamma_1^+ \cup \Gamma_2^+$.
Next, we propose the following non-homogeneous Neumann conditions:

$$\frac{\partial V}{\partial S}(t, S, I) = g_1(t, S, I) \quad \text{on } [0, T] \times \Gamma_1^+, \tag{5.46}$$

$$\frac{\partial V}{\partial I}(t, S, I) = g_2(t, S, I) \quad \text{on } [0, T] \times \Gamma_2^+, \tag{5.47}$$

the functions g_1 and g_2 being defined by

$$g_1(t, S, I) = \frac{\partial \overline{\Lambda}}{\partial S}(0, S, I) = 1, \quad (t, S, I) \in [0, T] \times \Gamma_1^+, \tag{5.48}$$

$$g_2(t, S, I) = \frac{\partial \overline{\Lambda}}{\partial I}(0, S, I) = 1, \quad (t, S, I) \in [0, T] \times \Gamma_2^+, \tag{5.49}$$

which are derived from the exercise value function $\overline{\Lambda}$, provided that we choose the bounded domain satisfying the condition

$$\min(S_\infty, I_\infty) > K \exp(\gamma T), \tag{5.50}$$

that guarantees the inequality

$$S + I - K \exp(\gamma T) > 0, \quad \forall (S, I) \in \Gamma_1^+ \cup \Gamma_2^+. \tag{5.51}$$

Notice that condition (5.50) is satisfied by the data in the forthcoming test examples.

Moreover, we propose a mixed formulation to deal with obstacle problem by introducing the multiplier $P : [0, T] \times \Omega \longrightarrow \mathbb{R}$, so that we can replace Eqs. (5.35)–(5.37) by the equation

$$V_\tau - \operatorname{div}(A \nabla V) + \vec{v} \cdot \nabla V + rV + P = 0 \quad \text{in } (0, T) \times \Omega, \tag{5.52}$$

and the complementarity conditions

$$V \ge \overline{\Lambda}, \; P \le 0, \quad (V - \overline{\Lambda}) \cdot P = 0 \quad \text{in } (0, T) \times \Omega. \tag{5.53}$$

This kind of mixed formulations have been previously used in early exercise Asian options with arithmetic averaging in [5] or in pension plans with early retirement opportunity pricing problems in [7], for example. In practice, we will apply the mixed formulation (5.52)–(5.53) to the fully discretized problem.

5.2. Discretization in time

Very often, in differential equations for pricing financial products, the diffusive term is quite small relative to the convective one for some regions of the domain or due to the presence of particular values of the involved parameters. This is specially reinforced in the case of the equations here considered for the stock loans pricing, due to the fact that there is no diffusion in one of the spatial dimensions. In such circumstances numerical schemes present difficulties.

A relatively large variety of ideas and approaches have been proposed in widely different contexts to solve these difficulties and the characteristics method for time discretization constitutes a possible up-winding scheme that leads to symmetric and stable approximations, reducing temporal errors and allowing for large time steps without loss of accuracy.

In order to cope with the extremely convection dominated feature that appears in the Kolmogorov equation associated to the stock loan model, we use the Crank–Nicolson Lagrange–Galerkin method to approximate the material derivative

$$\frac{D}{D\tau} = \partial_\tau + \vec{v} \cdot \nabla. \tag{5.54}$$

For this purpose, we define the characteristics curve through the point (S, I) at time $\bar{\tau}$, $X_e(x, \bar{\tau}; \tau)$, which solves the following final value problem:

$$\partial_\tau X_e((S, I), \bar{\tau}; \tau) = \vec{v}(X_e((S, I), \bar{\tau}; \tau)), \qquad X_e((S, I), \bar{\tau}; \bar{\tau}) = (S, I). \tag{5.55}$$

The final value problem (5.55) can be exactly solved, so that depending on the parameter values, we obtain:

- If $\sigma^2 - r + \delta = 0$

$$X_e^1((S, I), \bar{\tau}; \tau) = S$$
$$X_e^2((S, I), \bar{\tau}; \tau) = -\frac{\delta}{r} S + \exp(r(\bar{\tau} - \tau)) \left(I + \frac{S\delta}{r} \right)$$

- If $\sigma^2 - r + \delta \neq 0$

$$X_e^1((S, I), \bar{\tau}; \tau) = S \exp(-(\sigma^2 - r + \delta)(\bar{\tau} - \tau))$$
$$X_e^2((S, I), \bar{\tau}; \tau) = \frac{-\delta S \exp(-(\sigma^2 - r + \delta)(\bar{\tau} - \tau))}{\sigma^2 + \delta} + \exp(r(\bar{\tau} - \tau)) \left(I + \frac{S\delta}{\sigma^2 + \delta} \right)$$

Next, in order to describe the time discretization taking into account previous computations, for $N > 0$ let us consider the time step $\Delta\tau = \frac{T}{N}$ and the time mesh points $\tau^n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. Then, at time $\tau^{n+\frac{1}{2}}$ the material derivative approximation by characteristics method is given by:

$$\frac{DV}{D\tau} \approx \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau},$$

where $X_e^n(S, I) := X_e(S, I, \tau^{n+1}; \tau^n)$, the components of which are given by

- If $\sigma^2 - r + \delta = 0$

$$X_e^{n,1}(S, I) = S, \qquad X_e^{n,2}(S, I) = -\frac{\delta}{r} S + \exp(r\Delta\tau) \left(I + \frac{S\delta}{r} \right)$$

- If $\sigma^2 - r + \delta \neq 0$

$$X_e^{n,1}(S, I) = S \exp(-(\sigma^2 - r + \delta)\Delta\tau),$$
$$X_e^{n,2}(S, I) = \frac{-\delta S \exp(-(\sigma^2 - r + \delta)\Delta\tau)}{\sigma^2 + \delta} + \exp(r\Delta\tau) \left(I + \tfrac{S\delta}{\sigma^2 + \delta} \right).$$

The velocity field $\vec{v}$ is shown in Fig. 1 for the conditions $\sigma^2 - \delta + r > 0$ (left) and $\sigma^2 - \delta + r > 0$ (right).

Remark 5.1. Note that the velocity field at the boundary Γ_2^+ points towards the interior of the domain if $\sigma^2 - r + \delta \geq 0$ (see Fig. 1 left). Also, if the quantity $\sigma^2 - r + \delta < 0$ then the velocity field at the boundaries Γ_2^+ and Γ_1^+ points towards the interior of the domain (see Fig. 1 right). So, even for small enough time steps, the point $X_e^n(S, I)$ may not belong to the domain and some approximations will be used. More precisely, if the point $X_e^n(S, I)$ is located outside the domain, we use a suitable Taylor approximation at the corresponding boundary, taking in account the functions appearing in the Neumann boundary conditions (5.46) and (5.47).

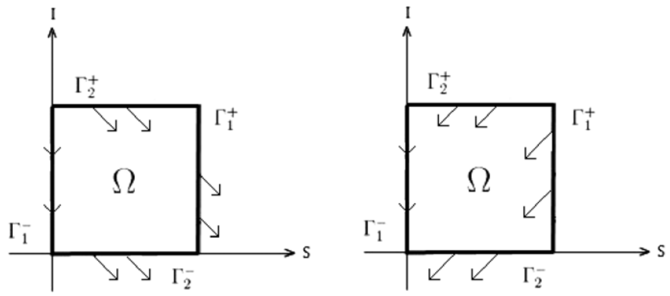

Fig. 1. Velocity field in the domain Ω for $\sigma^2 - \delta + r > 0$ (left) and $\sigma^2 - \delta + r < 0$ (right).

Next, if we consider a Crank–Nicolson scheme around the particular point $\left(X_e((S, I), \tau^{n+1}; \tau), \tau\right)$ with $\tau = \tau^{n+\frac{1}{2}}$ for $n = 0, \ldots, N-1$, then the time discretized PDE operator can be written as follows:

$$\overline{\mathcal{L}}[V]\left(X_e((S,I),\tau^{n+1};\tau^{n+\frac{1}{2}}),\tau^{n+\frac{1}{2}}\right) \approx \frac{V^{n+1}(S,I) - V^n(X_e^n(S,I))}{\Delta\tau} - \frac{1}{2}\mathrm{div}(A\nabla V^{n+1})(S,I) - \frac{1}{2}\mathrm{div}(A\nabla V^n)(X_e^n(S,I)) + \frac{1}{2}(rV^{n+1}(S,I)) + \frac{1}{2}(rV^n(X_e^n(S,I))). \tag{5.56}$$

For simplicity, let us introduce the notation $\left(\overline{\mathcal{L}}[V]\right)^{n+\frac{1}{2}}$:

$$\left(\overline{\mathcal{L}}[V]\right)^{n+\frac{1}{2}}(S,I) = \overline{\mathcal{L}}[V]\left(X_e((S,I),\tau^{n+1};\tau^{n+\frac{1}{2}}),\tau^{n+\frac{1}{2}}\right). \tag{5.57}$$

In order to state the weak formulation for the semi-discretized problem, by multiplying the terms in (5.56) by $\psi \in H^1(\Omega)$ and integrating in Ω, we have:

$$\left(\left(\overline{\mathcal{L}}[V]\right)^{n+\frac{1}{2}}, \psi\right) \approx \int_\Omega \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau}\psi\, dSdI - \frac{1}{2}\int_\Omega \mathrm{div}(A\nabla V^{n+1})\psi\, dSdI - \frac{1}{2}\int_\Omega \left(\mathrm{div}(A\nabla V^n)\right)\circ X_e^n \psi\, dSdI + \frac{1}{2}\int_\Omega rV^{n+1}\psi\, dSdI + \frac{1}{2}\int_\Omega (rV^n)\circ X_e^n\psi\, dSdI. \tag{5.58}$$

Next, applying Lemma 3.1 in [5] to the third term on the right hand side in (5.58) and the usual Green's formula to the second term, expression (5.58) is equivalent to:

$$\begin{aligned}\left(\left(\overline{\mathcal{L}}[V]\right)^{n+\frac{1}{2}}, \psi\right) \approx{}& \int_\Omega \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau}\psi\, dSdI + \frac{1}{2}\int_\Omega A\nabla V^{n+1}\nabla\psi\, dSdI \\ &+ \frac{1}{2}\int_\Omega (F_e^n)^{-1}(A\nabla V^n)\circ X_e^n\nabla\psi\, dSdI + \frac{1}{2}\int_\Omega (\mathrm{div}(F_e^n)^{-t}(A\nabla V^n))\circ X_e^n\psi\, dSdI \\ &+ \frac{1}{2}\int_\Omega rV^{n+1}\psi\, dSdI - \frac{1}{2}\int_\Gamma \vec{n}\cdot A\nabla V^{n+1}\psi\, dA + \frac{1}{2}\int_\Omega (rV^n)\circ X_e^n\psi\, dSdI \\ &- \frac{1}{2}\int_\Gamma \left((F_e^n)^{-t}\vec{n}\cdot(A\nabla V^n)\right)\circ X_e^n\psi\, dA,\end{aligned} \tag{5.59}$$

where notation dA is used for the integration measure in Γ.

Notice that the tensor $(F_e^n)^{-t}(S,I) = (\nabla X_e(S,I,\tau_{n+1};\tau_n))^{-t}$ can be easily computed and takes the form

$$(F_e^n)^{-t} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22}\end{pmatrix},$$

where the tensor components are actually independent of S and I. More precisely, by taking into account the different cases depending on the value of $\sigma^2 - r + \delta$, we have:

- If $\sigma^2 - r + \delta = 0$:

$$b_{11} = \exp(r\Delta\tau), \qquad b_{22} = 1, \qquad b_{12} = \frac{\delta}{r}(1 - \exp(r\Delta\tau)).$$

- If $\sigma^2 - r + \delta \neq 0$:

$$b_{11} = \exp(r\Delta\tau),$$
$$b_{22} = \exp(-(\sigma^2 - r + \delta)\Delta\tau),$$
$$b_{12} = \frac{\delta \exp(-(\sigma^2 - r + \delta)\Delta\tau)}{\sigma^2 + \delta} - \frac{\delta \exp(r\Delta\tau)}{\sigma^2 + \delta}.$$

Next, let us precise the boundary integrals appearing in formulation (5.59). First, notice that we have $\vec{n} \cdot A\nabla V^{n+1} = 0$ on $\Gamma_1^- \cup \Gamma_2^-$ and we can use the Neumann boundary conditions on $\Gamma_1^+ \cup \Gamma_2^+$. Therefore, in the first boundary integral on the right hand side of Eq. (5.59) we can introduce the function

$$\bar{g}^{n+1} = \begin{cases} a_{11}\, g_1^{n+1} & \text{on } \Gamma_1^+ \\ a_{11}\, g_2^{n+1} & \text{on } \Gamma_2^+. \end{cases} \tag{5.60}$$

Moreover, for the second integral, we have

$$\int_\Gamma \left((F_e^n)^{-t}\vec{n} \cdot (A\nabla V^n)\right) \circ X_e^n \psi \, dA = \int_\Gamma g^n \psi \, dA, \tag{5.61}$$

where $g^n : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is given by

$$g^n(S, I) = \begin{cases} 0 & \text{on } \Gamma_1^- \\ -((F_e^n)^{-t})_{11} a_{11}(X_e^n(S, I)) \dfrac{\partial V}{\partial I}(X_e^n(S, I)) & \text{on } \Gamma_1^+ \\ -\dfrac{1}{2}((F_e^n)^{-t})_{12} a_{11}(X_e^n(S, I)) \dfrac{\partial V^n}{\partial I}(X_e^n(S, I)) & \text{on } \Gamma_2^- \\ -((F_e^n)^{-t})_{12} a_{11}(X_e^n(S, I)) \dfrac{\partial V}{\partial I}(X_e^n(S, I)) & \text{on } \Gamma_2^+. \end{cases}$$

Therefore, expression (5.59) becomes

$$\begin{aligned} \left(\left(\bar{\mathcal{L}}[V]\right)^{n+\frac{1}{2}}, \psi\right) \approx{} & \int_\Omega \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau} \psi \, dSdI + \frac{1}{2}\int_\Omega A\nabla V^{n+1} \nabla\psi \, dSdI \\ & + \frac{1}{2}\int_\Omega \left((F_e^n)^{-1}(A\nabla V^n)\right) \circ X_e^n \nabla\psi \, dSdI + \frac{1}{2}\int_\Omega rV^{n+1}\psi \, dSdI \\ & + \frac{1}{2}\int_\Omega \left(\operatorname{div}(F_e^n)^{-t}(A\nabla V^n)\right) \circ X_e^n \psi \, dSdI - \frac{1}{2}\int_\Gamma g^n(S, I)\psi \, dA \\ & + \frac{1}{2}\int_\Omega (rV^n) \circ X_e^n \psi \, dSdI - \frac{1}{2}\int_{\Gamma_1^+ \cup \Gamma_2^+} \bar{g}^{n+1}\psi \, dA \end{aligned} \tag{5.62}$$

for all $\psi \in H^1(\Omega)$.

5.3. Finite elements discretization

In order to obtain the fully discretized problem, we combine the previously described time discretization with a finite elements based spatial discretization. For this purpose, we consider a family of quadrangular meshes $\{\tau_h\}$ of the domain Ω. Associated to the mesh $\{\tau_h\}$, let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ denotes the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. Now, let us define the finite elements space $\mathcal{V}_h$:

$$\mathcal{V}_h = \{\varphi_h \in \mathcal{C}^0(\bar{\Omega}) : \varphi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\},$$

where $\mathcal{C}^0(\bar{\Omega})$ denotes the space of continuous functions on $\bar{\Omega}$.

Therefore, if $V_h \in \mathcal{V}_h$ denotes the finite element approximation of $V \in H^1(\Omega)$, then the spatial discretization of (5.62) can be written in the form

$$\begin{aligned} \left(\left(\bar{\mathcal{L}}[V_h]\right)^{n+\frac{1}{2}}, \psi_h\right) \approx{} & \int_\Omega \frac{V_h^{n+1} - V_h^n \circ X_e^n}{\Delta\tau} \psi_h \, dSdI + \frac{1}{2}\int_\Omega A\nabla V_h^{n+1} \nabla\psi_h \, dSdI \\ & + \frac{1}{2}\int_\Omega \left((F_e^n)^{-1}(A\nabla V_h^n)\right) \circ X_e^n \nabla\psi_h \, dSdI + \frac{1}{2}\int_\Omega rV_h^{n+1}\psi_h dSdI \\ & + \frac{1}{2}\int_\Omega \left(\operatorname{div}(F_e^n)^{-t}(A\nabla V_h^n)\right) \circ X_e^n \psi_h \, dSdI - \frac{1}{2}\int_\Gamma g_h^n(S, I)\psi_h \, dA \\ & + \frac{1}{2}\int_\Omega (rV_h^n) \circ X_e^n \psi_h dSdI - \frac{1}{2}\int_{\Gamma_1^+ \cup \Gamma_2^+} \bar{g}^{n+1}\psi_h \, dA \end{aligned} \tag{5.63}$$

for all $\psi_h \in \mathcal{V}_h$.

5.4. Mixed formulation and an augmented Lagrangian active set method

Once the previous discretizations have been applied, we are led to the following fully discretized complementarity problem at each time step n:

$$M_h \mathbf{V}_h^n \geq b_h^{n-1},\ \mathbf{V}_h^n \geq \overline{\boldsymbol{\Lambda}}_h, \quad \left(M_h \mathbf{V}_h^n - b_h^{n-1}\right) \cdot \left(\mathbf{V}_h^n - \overline{\boldsymbol{\Lambda}}_h\right) = 0, \tag{5.64}$$

where M_h denotes the matrix that is obtained after the discretization with the finite element space $\mathcal{V}_h$, $\mathbf{V}_h^n$ denotes de vector containing the values of the solution at the nodes of the finite element mesh and $\overline{\boldsymbol{\Lambda}}_h$ is the vector of the node values of the function $\overline{\Lambda}$. So, the corresponding mixed formulation of the complementarity problem (5.64) can be written in the form

$$M_h \mathbf{V}_h^n + \mathbf{P}_h^n = b_h^{n-1}, \tag{5.65}$$

jointly with the complementarity conditions

$$\mathbf{V}_h^n \geq \overline{\boldsymbol{\Lambda}}_h,\ \mathbf{P}_h^n \leq \mathbf{0}, \qquad \mathbf{P}_h^n \cdot \left(\mathbf{V}_h^n - \overline{\boldsymbol{\Lambda}}_h\right) = 0, \tag{5.66}$$

where $\mathbf{P}_h^n$ denotes the vector containing the nodal values of the multiplier.

The basic iteration of the augmented Lagrangian active set algorithm has been introduced in [15] and mainly consists of two steps. In the first one the domain is decomposed into active and inactive parts (depending on whether the constraints are active or not), and in the second step, a reduced linear system associated to the inactive part is solved. Although the algorithm can be used in bilateral problems (in case of upper and lower constraints), we use the algorithm for unilateral problems, which are based on the augmented Lagrangian formulation. The method has been already successfully used when pricing early exercise Asian option with continuous arithmetic average [6] and pension plans [7]. We address the reader to both papers for further details on the algorithm.

6. Numerical results

After verifying the performance of the numerical methods with some academic tests with analytical solution, we consider the real test proposed in [8] in which the authors apply the shooting grid method introduced in [2], in turn based on binomial trees techniques. Our approach can be classified into the fixed domain methods for free boundary problems. More precisely, the financial data appearing in [2] are the following:

$$\sigma = 0.4, \qquad r = 0.05, \qquad \delta = 0.03, \qquad \gamma = 0.09, \qquad K = 0.7, \qquad T = 3. \tag{6.67}$$

We notice that for the previous data set, the relation $r < \gamma$ holds, so that Proposition 2.1 states the existence of a redeeming boundary and that the redemption region always contains a specific known region. The numerical solution confirms those results.

After using different meshes, time discretization steps and parameters of the numerical method, we show the results obtained for the localization parameters $S^\infty = 3K$ and $L^\infty = 3K$, a quadrangular finite elements mesh with 4096 elements and 16641 nodes, and the time step $\Delta\tau = 0.001$. Notice that the particular choice of the bounded domain guarantees that condition (5.50) is satisfied.

Fig. 2 shows the computed optimal redeeming boundary at the times to maturity $\tau = T - t = 0,\ 1$, and 3, which coincide with those presented in [8] by using different scales in the axes. The boundaries have been computed by taking into account that the multiplier passes from zero in the continuation region to a negative value in the redemption region. The redemption region is located above the redeeming boundary curve and condition (2.13) is numerically satisfied. On the other hand, also the limit property (2.14) is clearly illustrated by Fig. 2.

Fig. 3 shows the computed stock loan value for $r < \gamma$ with the data in (6.67) at $t = 0$ which qualitatively resembles the kind of results obtained for Asian options with early exercise opportunity (see [5,6], for example).

In addition to the example in [8], also the tests corresponding to cases with $r = \gamma$ and $r \geq \gamma$ have been performed: also in these cases the numerical results agree with the theoretical results of Proposition 2.1. The following experiments have been performed with the same parameters of the numerical methods as in previous case.

In the case $r = \gamma$ the following financial data set has been chosen:

$$\sigma = 0.4, \qquad r = 0.09, \qquad \delta = 0.03, \qquad \gamma = 0.09, \qquad K = 0.7, \qquad T = 3. \tag{6.68}$$

For these data, Fig. 4 shows the computed stock loan prices for $t = 0$. Taking into account that the no redemption region corresponds to the points where the multiplier is identically zero, we notice that in this case the multiplier vanishes in the whole domain, thus confirming the theoretical property proved for this case in Proposition 2.1.

For the case $r > \gamma$ the following financial data set has been taken:

$$\sigma = 0.4, \qquad r = 0.13, \qquad \delta = 0.03, \qquad \gamma = 0.09, \qquad K = 0.7, \qquad T = 3. \tag{6.69}$$

For these data, Fig. 5 shows the computed stock loan prices for $t = 0$. Again the numerical solution confirms the theoretical properties proved for this case in Proposition 2.1.

Finally, in order to better compare the stock loan prices in the three cases, Figs. 6 and 7 show the difference of these prices in the cases $r = \gamma$ (Fig. 4) and $r > \gamma$ (Fig. 5) with respect to the case $r < \gamma$ (Fig. 3), respectively.

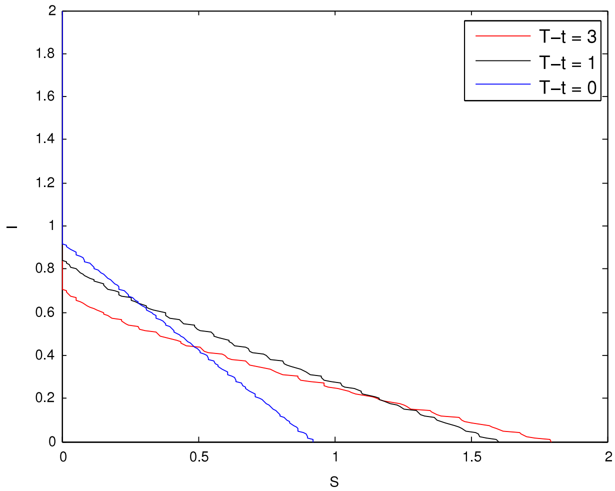

Fig. 2. Optimal redeeming boundary for $\sigma = 0.4$, $r = 0.05$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

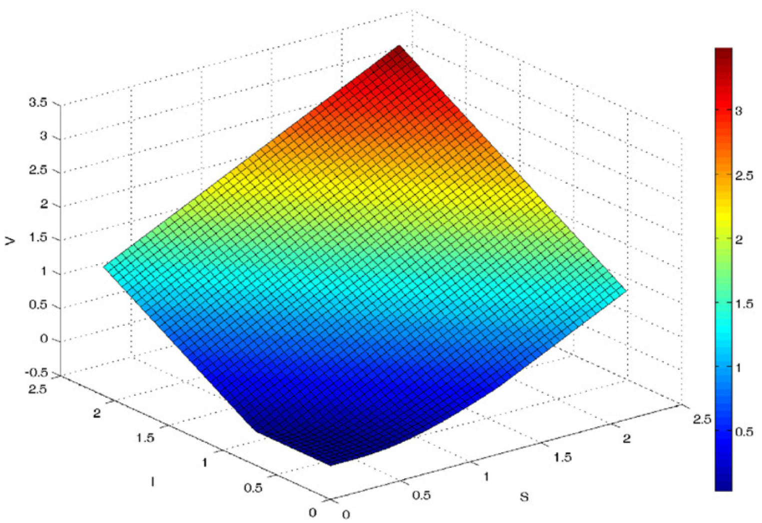

Fig. 3. Stock loan price at $t = 0$ for the data $\sigma = 0.4$, $r = 0.05$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

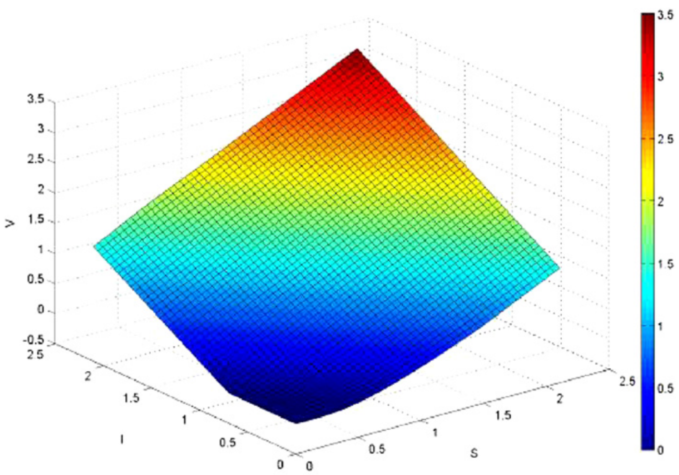

Fig. 4. Stock loan price at $t = 0$ for the data $\sigma = 0.4$, $r = 0.09$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

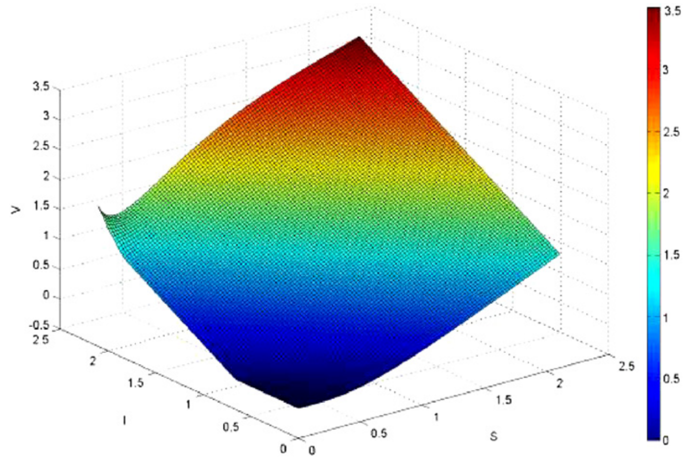

Fig. 5. Stock loan price at $t = 0$ for the data $\sigma = 0.4$, $r = 0.13$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

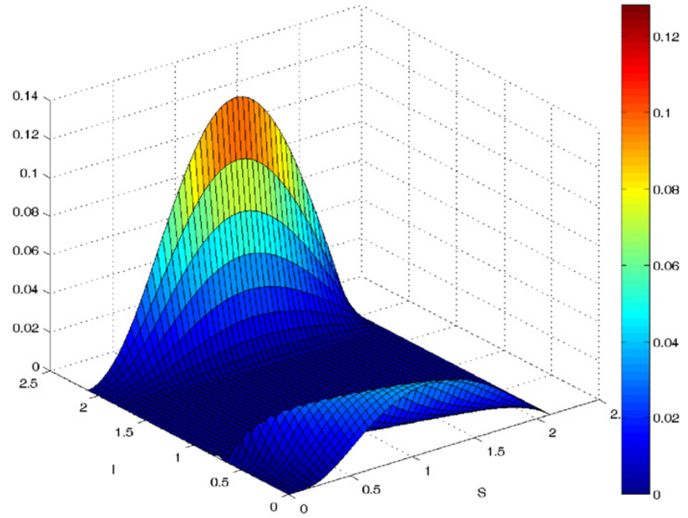

Fig. 6. Difference between stock loan prices in Figs. 4 and 3.

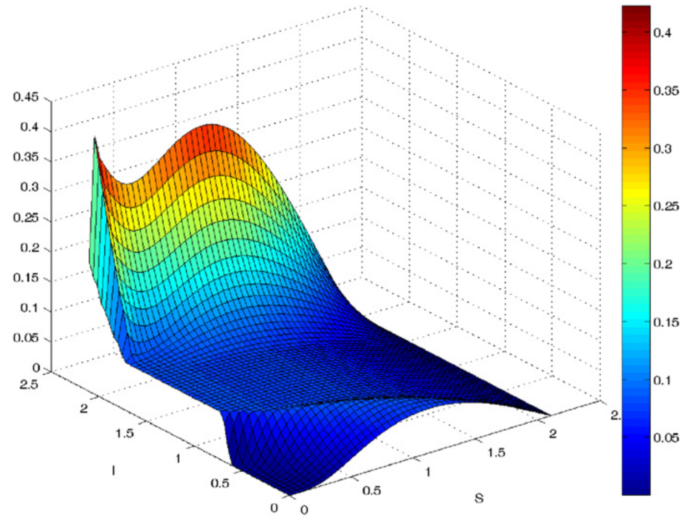

Fig. 7. Difference between stock loan prices in Figs. 5 and 3.

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