Presentation

Presented on 2021/6/29. Slides are available here.

QUBO2Ising explanation

If the reader does not render LaTeX, please refer to README.pdf.

QUBO formulation of currency exchange problem

$$x = \mathsf{argmax}_x \left[\sum_{(i,j)\in E} x_{ij} \log c_{ij} - M_1 \sum_{i\in V} \left(\sum_{j,(i,j)\in E} x_{ij} - \sum_{j,(j,i)\in E} x_{ji} \right)^2 - M_2 \sum_{i\in V} \sum_{j,(i,j)\in E} x_{ij} \left(\sum_{j,(i,j)\in E} x_{ij} - 1 \right) \right]$$

is equivalent to

$$x = \underset{x}{\operatorname{argmin}} \left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-log_2(c_{ij}))x_{ij} + M_1 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} x_{ij} - \sum_{j=0}^{n-1} x_{ji} \right)^2 + M_2 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} x_{ij} \right) \left(\sum_{j=0}^{n-1} x_{ij} - 1 \right) \right\}$$

Expand the 1st term:

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-log_2(c_{ij}))x_{ij}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-log_2(c_{ij}))x_{ij}x_{ij}$$

Expand the 2nd term:

$$M_1 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} x_{ij} - \sum_{j=0}^{n-1} x_{ji} \right)^2$$

$$= M_1 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} (x_{ij} - x_{ji}) \right)^2$$

$$= M_1 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} (x_{ij} - x_{ji}) \right) \left(\sum_{k=0}^{n-1} (x_{ik} - x_{ki}) \right)$$

$$= M_1 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} (x_{ij} - x_{ji})(x_{ik} - x_{ki})$$

$$= M_1 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} (x_{ij}x_{ik} - x_{ij}x_{ki} - x_{ji}x_{ik} + x_{ji}x_{ki})$$

Expand the 3rd term:

$$M_2 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} x_{ij} \right) \left(\sum_{j=0}^{n-1} x_{ij} - 1 \right)$$

$$= M_2 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} x_{ij} \right) \left(\sum_{k=0}^{n-1} x_{ik} - 1 \right)$$

$$= M_2 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_{ij}x_{ik} - \sum_{j=0}^{n-1} x_{ij} \right)$$

$$= M_2 \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_{ij}x_{ik} - \sum_{j=0}^{n-1} x_{ij}x_{ij} \right)$$

Change subscripts:

$$x_{ab} \rightarrow x_{a\times n+b}$$

Another formulation in "A Currency Arbitrage Machine based on the Simulated Bifurcation Algorithm for Ultrafast Detection of Optimal Opportunity," 2020

$$x = \underset{x}{\operatorname{argmin}} \left\{ m_c \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-log_2(c_{ij})) x_{ij} + \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} x_{ij} - \sum_{j=0}^{n-1} x_{ji} \right)^2 + \sum_{i=0}^{n-1} \sum_{j \neq j'} \left(x_{ij} x_{ij'} + x_{ji} x_{j'i} \right) + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x_{ij} x_{ji} \right\}$$

Expand the 1st term:

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-log_2(c_{ij})) x_{ij}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-log_2(c_{ij})) x_{ij} x_{ij}$$

Expand the 2nd term:

$$\sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} x_{ij} - \sum_{j=0}^{n-1} x_{ji} \right)^2$$

$$= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} (x_{ij} - x_{ji}) \right)^2$$

$$= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} (x_{ij} - x_{ji}) \right) \left(\sum_{k=0}^{n-1} (x_{ik} - x_{ki}) \right)$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} (x_{ij} - x_{ji})(x_{ik} - x_{ki})$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} (x_{ij} x_{ik} - x_{ij} x_{ki} - x_{ji} x_{ik} + x_{ji} x_{ki})$$

Expand the 3rd term:

$$\sum_{i=0}^{n-1} \sum_{j \neq j'} \left(x_{ij} x_{ij'} + x_{ji} x_{j'i} \right)$$

$$= \frac{1}{2} \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} (x_{ij} x_{ik} + x_{ji} x_{ki}) - \sum_{j=0}^{n-1} (x_{ij} x_{ij} + x_{ji} x_{ji}) \right)$$

Expand the 4th term: already in expanded form

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x_{ij} x_{ji}$$

Change subscripts:

$x_{ab} \rightarrow x_{a \times n + b}$

QUBO

$$Q \in \mathbb{R}^{N\times N}, Q = Q^T, x_i \in \{0,1\}$$
$$f_Q(Q,x) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} Q_{ij}x_ix_j$$
$$x_{opt} = \underset{x}{\operatorname{argmin}}\, f_Q(Q,x)$$

Ising model

$$J \in \mathbb{R}^{N\times N}, J = J^T, h \in \mathbb{R}^N, s_i \in \{-1,1\}$$
$$H(J,h,s) = -\frac{1}{2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} J_{ij}s_is_j + \sum_{i=0}^{N-1} h_is_i$$
$$s_{opt} = \underset{s}{\operatorname{argmin}}\, H(J,h,s)$$

Convert QUBO to Ising model

$$\begin{aligned}
&\text{Let } x_i = \frac{s_i+1}{2}\\
&f_Q(Q,x)\\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} Q_{ij}x_ix_j\\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} Q_{ij}\frac{s_i+1}{2}\frac{s_j+1}{2}\\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}(s_is_j+s_i+s_j+1)\\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}s_is_j + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}s_i + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}s_j + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}\\
&= -\frac{1}{2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{-Q_{ij}}{2}s_is_j + \sum_{i=0}^{N-1} s_i\left(\sum_{j=0}^{N-1}\frac{Q_{ij}}{4}\right) + \sum_{j=0}^{N-1} s_j\left(\sum_{i=0}^{N-1}\frac{Q_{ij}}{4}\right) + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}\\
&= -\frac{1}{2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{-Q_{ij}}{2}s_is_j + \sum_{i=0}^{N-1}\left(\sum_{j=0}^{N-1}\frac{Q_{ij}}{4} + \sum_{j=0}^{N-1}\frac{Q_{ji}}{4}\right)s_i + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}\\
&\text{Let } J_{ij} = \frac{-Q_{ij}}{2}, h_i = \left(\sum_{j=0}^{N-1}\frac{Q_{ij}}{4} + \sum_{j=0}^{N-1}\frac{Q_{ji}}{4}\right), C = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \frac{Q_{ij}}{4}\\
&f_Q(Q,x) = -\frac{1}{2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} J_{ij}s_is_j + \sum_{i=0}^{N-1} h_is_i + C = H(J,h,s) + C
\end{aligned}$$