# LEC 01 : Convex Sets, Functions, & Minimizers

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# Mathematical Preliminaries

- Convex Sets
- Convex Functions
- Minimizers

# Convex Sets

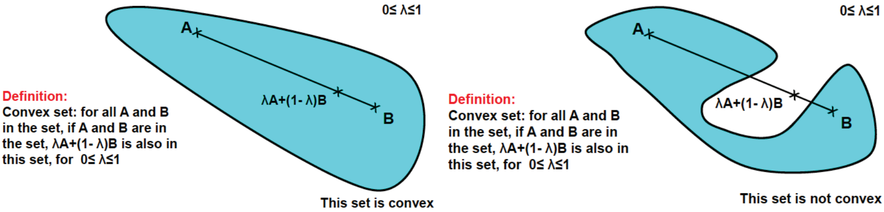
A set $D$ is convex if the line segment connecting any two points in $D$ completely lies in $D$. We formalize this concept into the following definition.

## Definition (Convex Set)

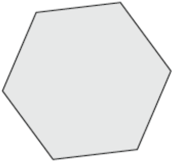
Let $D$ be a subset of $\mathbb{R}^n$. Also, consider scalar parameter $\lambda \in [0, 1]$ and two points $a, b \in D$. The set $D$ is <u>convex</u> if

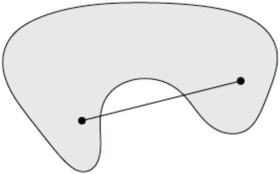
$$\lambda a + (1 - \lambda) b \in D \tag{1}$$

for all points $a, b \in D$.



# Test your knowledge





# Test your knowledge

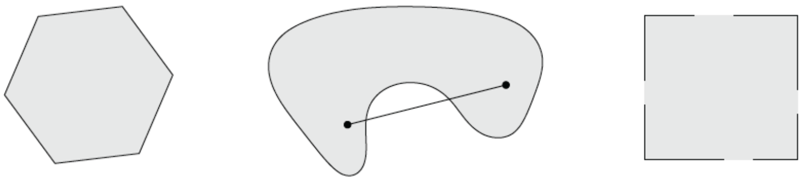


Figure: Some simple convex and nonconvex sets. [Left] The hexagon, which includes its boundary (shown darker), is convex. [Middle] The kidney shaped set is not convex, since the line segment between the two points in the set shown as dots is not contained in the set. [Right] The square contains some boundary points but not others, and is not convex.

# Important Examples of Convex Sets - I

- The empty set , any single point (i.e. a singleton), $\{x_0\}$, and the whole space $\mathbb{R}^n$ are convex.
- Any line in $\mathbb{R}^n$ is convex.
- Any line segment in $\mathbb{R}^n$ is convex.
- A ray, which has the form $\{x_0 + \theta v \mid \theta \geq 0, v \neq 0\}$ is convex.
- A hyperplane, which has the form $\left\{x \mid a^T x = b\right\}$, where $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$.

# Important Examples of Convex Sets - II

- A halfspace, which has the form $\left\{x \mid a^T x \leq b\right\}$, where $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$. See Fig. 2.
- A Euclidean ball in $\mathbb{R}^n$, which is centered at $x_c$ and has radius $r$. Think of the Euclidean ball as a sphere in $n$-dimensions (See Fig. 3.). Mathematically, the Euclidean ball is

$$B(x_c, r) = \{x \mid \ \|x - x_c\|_2 \leq r\} = \left\{x \mid (x - x_c)^T (x - x_c) \leq r^2\right\} \qquad (2)$$

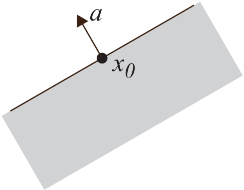



Figure: The shaded set is the halfspace given by $\{x \mid a^T(x - x_0) \leq 0\}$ Vector $a$ points in the outward normal direction of the halfspace.

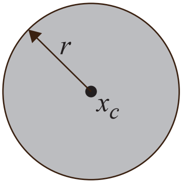



Figure: The Euclidean Ball $B(x_c, r)$ is centered at $x_c$ and has radius $r$. A two-dimensional example is shown.

# Important Examples of Convex Sets - Ellipsoid

An ellipsoid in $\mathbb{R}^n$, which is centered at $x_c$. Think of an ellipsoid as an ellipse in $n$-dimensions. Mathematically,

$$\mathcal{E} = \left\{x \mid (x - x_c)^T A^{-1}(x - x_c) \leq 1\right\} \tag{3}$$

where $A = A^T \succ 0$, i.e. $A$ is symmetric and positive definite. The matrix $A$ encodes how the ellipse extends in each direction of $\mathbb{R}^n$. In particular, the length of the semi-axes for the ellipse are given by $\sqrt{\lambda_i(A)}$, where $\lambda_i$ is the $i$th eigenvalue of $A$. See Fig. on next slide. Another common representation of an ellipsoid, which we use later in this chapter, is:

$$\mathcal{E} = \{x_c + Pu \mid \|u\|_2 \leq 1\} \tag{4}$$

where $P$ is square and positive semi-definite. The semi-axis lengths are given by $\lambda_i(P)$ in this case. If we define $P = A^2$, then this representation is equivalent to (3).

# Important Examples of Convex Sets - Ellipsoid

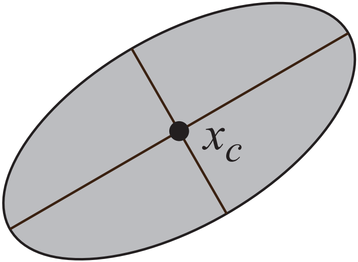



Figure: An ellipsoid $\mathcal{E} = \{x_c + Pu \mid \|u\|_2 \leq 1\}$ in two-dimensions with center $x_c$. The semi-axes have length given by $\lambda_i$ where $\lambda_i =$eig$(P)$.

# Important Examples of Convex Sets - Polyhedron

A <u>polyhedron</u> is defined by the values $x \in \mathbb{R}^n$ that satisfy a finite set of linear inequalities and linear equalities:

$$\mathcal{P} = \left\{ x \mid a_i^T x \le b_i,\ i = 1, \cdots, m,\ \ a_{eq,j}^T x = b_{eq,j},\ j = 1, \cdots, l \right\} \tag{3}$$

See Fig. on next slide. By definition, a polyhedra is the intersection of a finite number of hyperplanes and halfspaces. Thus, all halfspaces, hyperplanes, lines, rays, and line segments are polyhedra. It is convenient to use the compact vector notation

$$\mathcal{P} = \{ x \mid Ax \le b,\ A_{eq}x = b_{eq} \} \tag{4}$$

# Important Examples of Convex Sets - Polyhedron

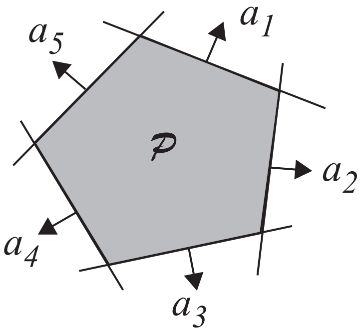



Figure: The polyhedron $\mathcal{P}$ is the intersection of five halfspaces, each with outward pointing normal vector $a_i$.

# Exercise - Square and Disk

Define the square and disk in $\mathbb{R}^2$ respectively as

$$S = \left\{x \in \mathbb{R}^2 \middle| \, 0 \leq x_i \leq 1, i = 1, 2\right\}, \qquad D = \left\{x \in \mathbb{R}^2 \middle| \, \|x\|_2 \leq 1\right\} \tag{3}$$

Are the following statements TRUE or FALSE:

(a) $S \cup D$ is convex, i.e. the union of sets $S$ and $D$ is convex

(b) $S \cap D$ is convex, i.e. the intersection of sets $S$ and $D$ is convex

(c) $S\ D$ is convex, i.e. set difference of $D$ from $S$ is convex

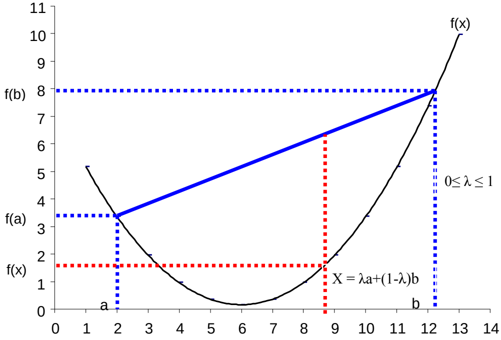
# Convex Functions

## Definition (Convex Function)

Let $D$ be a convex set. Also, consider scalar parameter $\lambda \in [0, 1]$ and two points $a, b \in D$. Then the function $f(x)$ is <u>convex</u> on $D$ if

$$f(x) = f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b) \qquad (4)$$

for all points $a, b \in D$.



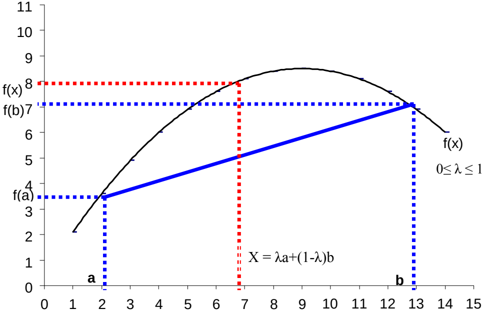
# Concave Functions

## Definition (Concave Function)

Let $D$ be a convex set. Also, consider scalar parameter $\lambda \in [0,1]$ and two points $a, b \in D$. Then the function $f(x)$ is <u>concave</u> on $D$ if

$$f(x) = f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda)f(b) \tag{5}$$

for all points $a, b \in D$.



# Exercises

Which of the following functions are convex, concave, neither, or both, over the set $D = [-10, 10]$? You may use graphical arguments or the definitions to prove your claim.

(a) $f(x) = 0$

(b) $f(x) = x$

(c) $f(x) = x^2$

(d) $f(x) = -x^2$

(e) $f(x) = x^3$

(f) $f(x) = \sin(x)$

(g) $f(x) = e^{-x^2}$

(h) $f(x) = |x|$

# Convex/Concave Function Properties

Consider a function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and compact set $D$.

1. If $f(x)$ is convex on $D$, then $-f(x)$ is concave on $D$.
2. If $f(x)$ is concave on $D$, then $-f(x)$ is convex on $D$.
3. $f(x)$ is a convex function on $D \iff \frac{d^2 f}{dx^2}(x)$ is positive semi-definite $\forall\, x \in D$.
4. $f(x)$ is a concave function on $D \iff \frac{d^2 f}{dx^2}(x)$ is negative semi-definite $\forall\, x \in D$.

# Examples: Scalar Convex/Concave Functions

Consider functions $f(x)$ where $x \in \mathbb{R}$ is scalar.

- *Quadratic.* $\frac{1}{2}ax^2 + bx + c$ is convex on $\mathbb{R}$, for any $a \geq 0$. It is concave on $\mathbb{R}$ for any $a \leq 0$.
- *Exponential.* $e^{ax}$ is convex on $\mathbb{R}$, for any $a \in \mathbb{R}$.
- *Powers.* $x^a$ is convex on the set of all positive $x$, when $a \geq 1$ or $a \leq 0$. It is concave for $0 \leq a \leq 1$.
- *Powers of absolute value.* $|x|^p$, for $p \geq 1$ is convex on $\mathbb{R}$.
- *Logarithm.* $\log x$ is concave on the set of all positive $x$.
- *Negative entropy.* $x \log x$ is convex on the set of all positive $x$.

# Examples: Multivariable Convex/Concave Functions

Consider functions $f(x)$ where $x \in \mathbb{R}^n$ is multivariable.

- *Norms.* Every norm in $\mathbb{R}^n$ is convex.
- *Max function.* $f(x) = \max\{x_1, \cdots, x_n\}$ is convex on $\mathbb{R}^n$.
- *Quadratic-over-linear function.* The function $f(x, y) = x^2/y$ is convex over all positive $x, y$.
- *Log-sum-exp*. The function $f(x) = \log\left(\exp^{x_1} + \cdots + \exp^{x_n}\right)$ is convex on $\mathbb{R}^n$. This function can be interpreted as a differentiable (in fact, analytic) approximation of the max function. Consequently, it is extraordinarily useful for gradient-based algorithms.
- *Geometric mean.* The geometric mean $f(x) = \left(\Pi_{i=1}^n x_i\right)^{1/n}$ is concave for all elements of $x$ positive, i.e. $\{x \in \mathbb{R}^n \mid x_i > 0 \; \forall \; i = 1, \cdots, n\}$.
- *Log determinant.* The function $f(X) = \log\det(X)$ is concave w.r.t. $X$ for all $X$ positive definite. This property is useful in many applications, including optimal experiment design.

# Operations that conserve convexity

- *Linear combinations*. Consider $\alpha_i \geq 0$ for all $i$. A non-negative weighted sum of convex functions is also convex

$$f(x) = \alpha_1 f_1(x) + \cdots + \alpha_m f_m(x) \tag{6}$$

- *Pointwise maximum*. If $f_1(x)$ and $f_2(x)$ are convex functions on $\mathcal{D}$, then their *point-wise maximum* $f$ defined below is convex on $\mathcal{D}$.

$$f(x) = \max\{f_1(x), f_2(x)\} \tag{7}$$

- *Composition with Affine Mapping*. Consider $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with parameters $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$. Define function $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ as

$$g(x) = f(Ax + b) \tag{8}$$

  If $f$ is convex, then $g$ is convex. If $f$ is concave, then $g$ is concave.

# Definition of Minimizers

## Definition (Global Minimizer)

$x^\star \in D$ is a global minimizer of $f(x)$ on $D$ if

$$f(x^\star) \leq f(x), \qquad \forall\, x \in D \tag{9}$$

In words, this means $x^\star$ minimizes $f(x)$ everywhere in $D$. In contrast, we have a local minimizer.

## Definition (Local Minimizer)

$x^\star \in D$ is a local minimizer of $f(x)$ on $D$ if

$$\exists\, \epsilon > 0 \quad \text{s.t.} \quad f(x^\star) \leq f(x), \qquad \forall\, x \in D \cap \{x \in \mathbb{R} \mid \|x - x^\star\| < \epsilon\} \tag{10}$$

In words, this means $x^\star$ minimizes $f(x)$ locally in $D$. That is, there exists some neighborhood whose size is characterized by $\epsilon$ where $x^\star$ minimizes $f(x)$. Examples of global and local minimizers are provided in Fig. on next slide.

# Definition of Minimizers

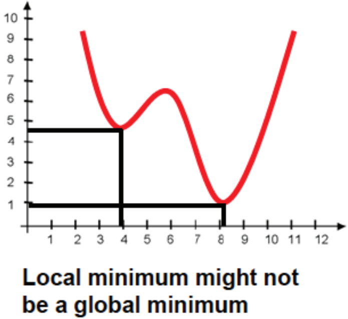




Figure: The LEFT figure contains two local minimizers, but only one global minimizer. The RIGHT figure contains a local minimizer, which is also the global minimizer.