CHAPTER 3: STATE ESTIMATION

1 Overview

State estimation is the process of determining the internal state of an energy system, from measurements of the input/output data. The algorithm is typically computer implemented, and is fundamental to many analysis, monitoring, and energy management tasks.

Knowledge of the energy system's state is necessary to solve many energy systems and control problems. A naive engineer might propose to place sensors everywhere, and on everything. However, in most applications it is not practical to coat the entire system with sensors. This could be expensive, difficult to manage, compromising of the original design, etc. Consequently, we ask the question, "Is it possible to determine the internal state of an energy system from a limited number of measurements?" In many situations, the answer is a resounding yes. This chapter provides you with the essential tools to complete this task.

Consider the block diagram in Fig.1. The top block represents a physical energy system (e.g. wind farm, building environment, or power system distribution network). This system receives inputs $u(t)$. Moreover, we may place sensors on a limited number of locations within the system to measure relevant variables (e.g. wind turbine rotor speed, room temperature, substation voltages). The measurements provide signals represented by $y(t)$. However, we seek to monitor the internal state $x(t)$. This state is not directly measurable, for a plethora of possible practical reasons. The state estimator places a mathematical model (implemented on a computer) in parallel with the physical energy system. This mathematical model is fed the same input data $u(t)$, and provides a prediction of "estimate" of the internal state, $\hat{x}(t)$. Although we formulate the mathematical model to accurately capture the energy system dynamics to the best of our ability, it will contain errors. Consequently, we can compare the model output $\hat{y}(t)$ with the true measured output $y(t)$. The error between these two signals can be properly injected into the mathematical model to "correct" for the model error. Put simply, this algorithm fuses models and measurements to obtain the best estimates. Consequently, the technique is robust to both model inaccuracies and measurement noise. This is an extremely powerful framework, and is without a doubt one of the greatest practical accomplishments in control theory.

1.1 Chapter Organization

In this chapter, we shall (i) explain conditions under which *it is possible* to estimate the internal state from measurements, (ii) provide a procedure to design a simple state estimator (called the Luenberger observer), (iii) describe a way to generate estimates "optimally" via the celebrated Kalman filter, and (iv) introduce its nonlinear variant called the Extended Kalman filter. The remainder of this chapter is organized as follows:

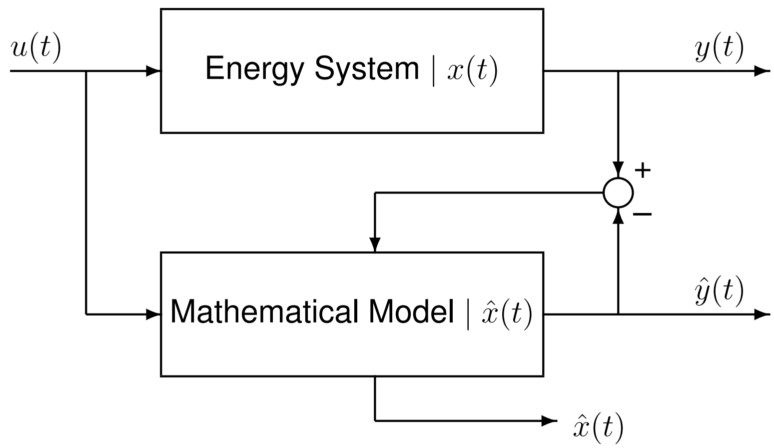

Figure 1: State estimation deals with inferring $x(t)$ from measurements of u, y.

1. Open-Loop Observers and Observability
2. Luenberger Observer
3. Kalman Filter (KF)
4. Extended Kalman Filter (EKF)

2 Open Loop Observers and Observability

In this section we motivate the use of feedback to correct the mathematical model. We also provide a checkable condition to determine if it is possible to estimate internal states from measurements.

2.1 Open Loop Observer

Consider the so-called "open loop" observer in Fig.2. This algorithm places a mathematical model running in software in parallel with the physical energy system. The measured input data is fed into the mathematical model, which generates its own internal state estimates. To make this example concrete, suppose the true energy system evolves according to linear time-invariant (LTI) dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0. \tag{1}$$

Moreover, suppose the mathematical model is an identical copy of the energy system dynamics (i.e. perfect modeling):

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \qquad \hat{x}(0) = \hat{x}_0. \tag{2}$$

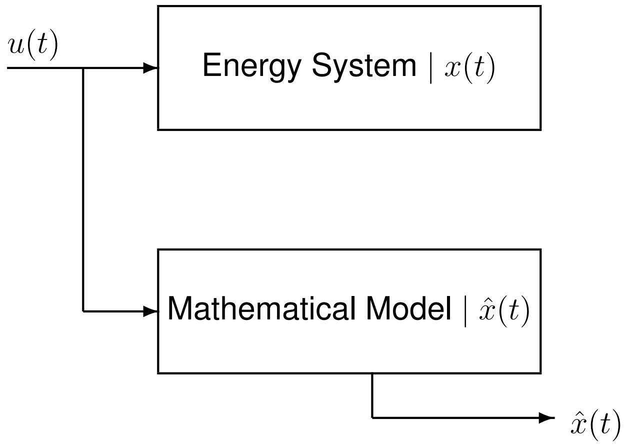

Figure 2: Block diagram of open loop observer.

Consider the "error state" $\tilde{x}(t) = x(t) - \hat{x}(t)$. Clearly, the goal is to drive the error towards zero as we collect data. One can derive the error dynamics as

$$\begin{aligned}
\dot{\tilde{x}}(t) &= \dot{x} - \dot{\hat{x}}, \\
&= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t), \\
&= A(x(t) - \hat{x}(t)), \\
\dot{\tilde{x}}(t) &= A\tilde{x}(t), \qquad\qquad \tilde{x}(0) = x_0 - \hat{x}_0 = \tilde{x}_0
\end{aligned} \tag{3}$$

From Chapter 1, we can conclude the LTI estimation error system is:

- **Marginally stable** or **stable in the sense of Lyapunov** if the real part of the eigenvalues of A are less than or equal to zero, i.e. Re$[\lambda_i] \leq 0$, for all i, where λ_i are the eigenvalues of A.
- **Asymptotically stable** if the real part of the eigenvalues of A are strictly less than zero, i.e. Re$[\lambda_i] < 0$, for all i, where λ_i are the eigenvalues of A.

Consequently, if the true energy system is asymptotically stable and our mathematical model is perfect, then the estimates $\hat{x}(t) \to x(t)$ as $t \to \infty$. There are three questionable issues with this open-loop observer:

1. Energy systems are not always asymptotically stable.
2. Our mathematical models are never perfect.
3. Even under these idealized conditions, the speed at which the estimates converge is only as fast as the energy system itself. That is, the error system and physical energy system are characterized by the same eigenvalues. In practice, we seek to have our estimates converge faster than the energy system evolves.

Feedback, in the form of output error injection shown in Fig. 1, alleviates all three of these issues.

2.2 Observability

Next we discuss the concept of observability. Analogous to identifiability in Chapter 2, this concept helps us determine if it is possible to estimate the internal states from measured input/output data. Also like before, we seek an easily checkable condition that allows us to determine observability a priori. Satisfaction of this condition is a required assumption to design an state estimation algorithm.

Consider the $n-$dimensional, $p-$input, $q-$output LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{4}$$

$$y(t) = Cx(t) + Du(t). \tag{5}$$

where A, B, C, D are, respectively $n \times n$, $n \times p$, $q \times n$, $q \times p$ constant matrices. We now introduce the formal definition of observability

Definition 2.1 (Observability)**.** *The LTI system (4)-(5) is said to be* <u>*observable*</u> *if for any unknown initial state x_0, there exists a finite time $t_1 > 0$ such that the knowledge of the input $u(t)$ and output vector $y(t)$ over $[0, t_1]$ suffices to uniquely determine the initial state x_0. Otherwise the LTI system is said to be unobservable.*

Less formally, this means that from the system's input/output data it is possible to determine the behavior of the entire system. If a system is unobservable, this means it is impossible to determine at least some internal states from the input/output data.

Although this definition provides concreteness to the notion of observability, it does not provide an easily checkable condition. We provide a test now, in the form of a theorem without proof.

Theorem 2.1 (Observability Test)**.** *Consider the LTI system (4)-(5). Define the so-called "observability matrix" $\mathcal{O} \in \mathbb{R}^{nq \times n}$ as*

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{6}$$

Then the LTI system (4)-(5) is observable if $\mathcal{O}$ has full column rank, that is rank$(\mathcal{O}) = n$. Under this condition, we say the pair (A,C) is observable.

Remark 2.1. The observability matrix $\mathcal{O}$ can be generated in Matlab with the `obsv` command. Note that this test can be performed immediately after formulating a mathematical model for the energy system. Moreover, it is independent of the input/output data. That is, the input/output data values have no impact on the observability property. It is strictly a system property for LTI systems.

Remark 2.2. In fact, several equivalent tests exist to determine observability. These additional tests are provided in the Appendix.

Example 2.1 (Unobservable Tank System)**.** This example demonstrates an unobservable system. Although this example appears synthetic, it captures the essence of why some formulations of the battery state-of-charge estimation problem are not solvable due to model unobservability.

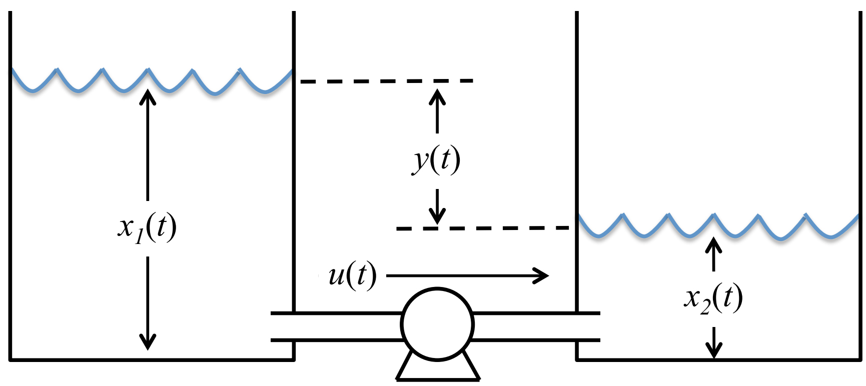

Figure 3: Two tank system that demonstrates an unobservable system.

Consider two tanks connected by a pipe in Fig. 3 This pipe contains a pump, which can transfer fluid from one tank to the other. Each tank serves as a "reservoir," where we define the states $x_1(t), x_2(t)$ as the respective fluid levels in each tank. Moreover, assign variable $u(t)$ as the fluid flow rate from tank 1 to tank 2. Finally, assume we can only measure the relative tank levels, yet seek to know their absolute values. That is, we measure $y(t) = x_1(t) - x_2(t)$ yet seek to monitor $x_1(t), x_2(t)$ individually. It is straight forward to utilize the concepts from Chapter 1 to formulate the following mathematical model

$$\dot{x}_1(t) = -u(t), \tag{7}$$

$$\dot{x}_2(t) = u(t), \tag{8}$$

$$y(t) = x_1(t) - x_2(t) \tag{9}$$

which can be arranged into matrix state-space form as

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t), \tag{10}$$

$$y(t) \quad = \quad \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{11}$$

where matrices A, B, C, D take the form

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \qquad D = 0 \tag{12}$$

The observability matrix $\mathcal{O}$ is given by

$$\mathcal{O} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{13}$$

Note that the two columns are linear multiples of each other (by a factor of -1). Consequently rank$(\mathcal{O}) = 1$, meaning the pair (A,C) fails the observability test. Consequently, it is impossible to determine the individual tank levels $x_1(t), x_2(t)$ by observing only the pump flow rate $u(t)$ and the level difference $y(t)$.

Exercise 1. *Determine if the following state-space systems are observable*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

where

(a) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

(c) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

3 Luenberger Observer

Motivated by the need for measured feedback and armed with the observability concept, we next discuss a simple observer design. This design is based on eigenvalue assignment (a.k.a. pole placement), and is known as the Luenberger observer.

Consider an energy system modeled by the $n-$dimensional, $p-$input, $q-$output LTI system, as before

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, && (14) \\ y(t) &= Cx(t) + Du(t). && (15) \end{aligned}$$

Similar to the open loop observer, the Luenberger observer will employ a carbon copy of the energy system dynamics with one notable difference,

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L\left[y(t) - \hat{y}(t)\right], \qquad \hat{x}(0) = \hat{x}_0, && (16) \\ \hat{y}(t) &= C\hat{x}(t) + Du(t). && (17) \end{aligned}$$

The term $L\left[y(t) - \hat{y}(t)\right]$ injects the error between measurements and model predictions, scaled by a user-selectable "observer gain" vector $L \in \mathbb{R}^{n\times q}$. This concept is logically named "output error injection," and is fundamental to state estimation design. To understand this state estimation algorithm, we consider the error dynamics $\tilde{x}(t) = x(t) - \hat{x}(t)$, which evolve according to

$$\begin{aligned} \dot{\tilde{x}}(t) &= \dot{x} - \dot{\hat{x}}, \\ &= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) - L\left[y(t) - \hat{y}(t)\right], \\ &= A(x(t) - \hat{x}(t)) - L\left[Cx(t) + Du(t) - C\hat{x}(t) - Du(t)\right], \\ &= A(x(t) - \hat{x}(t)) - LC\left(x(t) - \hat{x}(t)\right), \\ \dot{\tilde{x}}(t) &= (A - LC)\tilde{x}(t), \qquad \tilde{x}(0) = x_0 - \hat{x}_0 = \tilde{x}_0 && (18) \end{aligned}$$

From Chapter 1, we can conclude the estimation error system is asymptotically stable if we select L such that the eigenvalues of $(A - LC)$ have negative real parts. That is, we can assign the eigenvalues (i.e. speed) of the error system by selecting L appropriately. Mathematically $\tilde{x}(t) \to 0$ or $\hat{x}(t) \to x(t)$ as $t \to \infty$, where the speed of convergence is characterized by the eigenvalues of $(A - LC)$. This result is summarized by the following theorem, which requires the observability concept.

Theorem 3.1. *(Luenberger Observer) If the pair (A,C) of the LTI system (14)-(15) is observable, then the eigenvalues of $(A - LC)$ can be placed arbitrarily.*

Remark 3.1. The Matlab command `place` provides the observer gain L, given matrix pair (A,C) and the desired eigenvalues for error system matrix $(A - LC)$.

```
>> L = place(A',C', [lam1, lam2, ..., lamN])';
```

where `[lam1, lam2, ..., lamN]` are the desired eigenvalues of $(A - LC)$. Note the transpose operator `'` used throughout this syntax. Read the Matlab documentation for more detail.

The Python Control Systems library "python-control" (https://github.com/python-control/python-control) contains a function `acker`, which provides the observer gain L, given matrix pair (A,C) and the desired eigenvalues

```
>> L = acker(A.T,C.T, [lam1, lam2, ..., lamN]).T
```

Note the transpose method `.T` used throughout this syntax. Read the Python Control Systems documentation for more detail.

Remark 3.2. A general rule-of-thumb is that the observer eigenvalues should be placed 2-10 times faster than the slowest stable eigenvalue of the energy system itself. Let $\lambda_A^\star$ represent the slowest stable eigenvalue of the energy system, i.e. $\text{Re}[\lambda_A^\star] = \max_i \{\text{Re}[\lambda_{A,i}] \mid \lambda_{A,i} \in \text{eig}(A), \text{Re}[\lambda_{A,i}] < 0\}$.

$$-\infty < 10 \cdot \text{Re}\left[\lambda_A^\star\right] \leq \text{Re}\left[\lambda_{A-LC,i}\right] \leq 2 \cdot \text{Re}\left[\lambda_A^\star\right] < 0, \qquad \forall\, i \tag{19}$$

where $\lambda_{A-LC,i}$ are the eigenvalues of $A - LC$. This rule-of-thumb achieves fast convergence of state estimates, without being hypersensitive to measurement noise, as explained next.

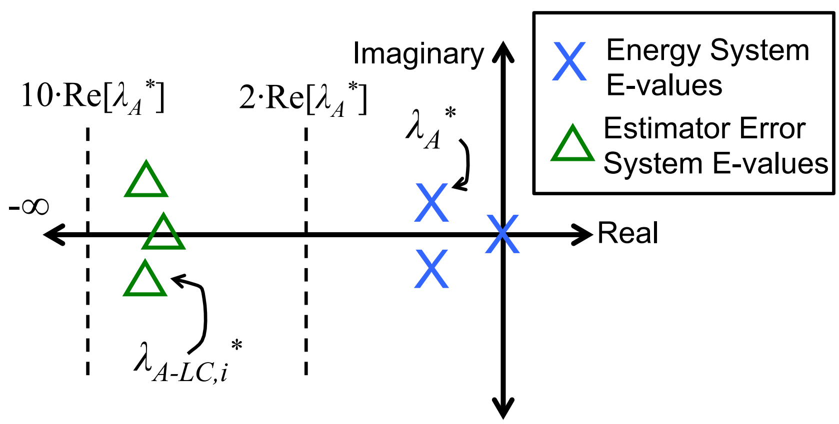

Figure 4: Eigenvalues of energy system and Luenberger estimation plotted on complex plane. Placing the estimator eigenvalues closer to negative infinity increases sensitivity to measurement noise. Ideally, one seeks to balance sensitivity to measurement data and model uncertainty. The Kalman filter accomplishes this goal.

Remark 3.3. Given that the eigenvalues of $(A - LC)$ can be assigned arbitrarily when (A,C) is observable, one might logically argue: "Why not pick $\lambda_i = -10^{99}$?", as shown in Fig. 4. That is, select all eigenvalues for $(A - LC)$ arbitrarily close to negative infinity. Sensor noise is the reason against this decision. Suppose the energy system has dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{20}$$
$$y(t) = Cx(t) + Du(t) + n(t). \tag{21}$$

where $n(t)$ represents measurement noise that exists within any practical sensor. Using this noisy LTI model, it is easy to verify the estimation error dynamics are given by

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x} - Ln(t) \tag{22}$$

If the eigenvalues are selected $\lambda_i = -10^{99}$, then it will render a very large observer gain L. This gain will amplify the sensor noise, and pollute our sensor estimates. Consequently, one must design L such that they appropriate balance convergence speed and robustness to noise. Section 4 describes an optimal way to balance this tradeoff, known as the Kalman Filter.

Example 3.1 (Building Zone Temperature Estimation)**.** Consider a building with three thermal zones, depicted in Fig. 5. Zones 1 and 3 have central heating vents. We measure when the heater is on and off, i.e. $s(t) \in \{0, 1\}$. Zones 1 and 3 also share large walls with the external environment whose temperature $T_\infty(t)$ is measured. Zone 2 is a narrow hallway, and contains a temperature sensor. Our goal is to estimate temperature in all zones via a Luenberger observer, by combining a thermal dynamics model with measurements $T_2(t), T_\infty(t), s(t)$.

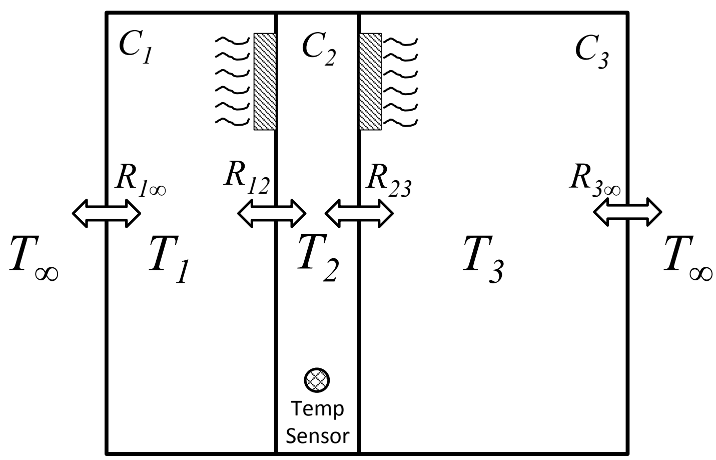

Figure 5: Building with three thermal zones. Our goal is to estimate $T_1(t), T_2(t), T_3(t)$, given measurements of T_2, environmental temperature $T_\infty(t)$, and heater on/off state $s(t)$.

First, we derive the mathematical model for the temperature dynamics using the First Law of Thermodynamics:

$$C_1\dot{T}_1(t) = \frac{1}{R_{1\infty}}\left[T_\infty(t) - T_1(t)\right] + \frac{1}{R_{12}}\left[T_2(t) - T_1(t)\right] + \frac{1}{2}Ps(t) \tag{23}$$

$$C_2\dot{T}_2(t) = \frac{1}{R_{12}}\left[T_1(t) - T_2(t)\right] + \frac{1}{R_{23}}\left[T_3(t) - T_2(t)\right] \tag{24}$$

$$C_3\dot{T}_3(t) = \frac{1}{R_{23}}\left[T_2(t) - T_3(t)\right] + \frac{1}{R_{3\infty}}\left[T_\infty(t) - T_3(t)\right] + \frac{1}{2}Ps(t) \tag{25}$$

This can be written in state-space form: $\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t) + Du(t)$ as follows

$$\frac{d}{dt}\begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_{1\infty}} + \frac{1}{R_{12}}\right) & \frac{1}{C_1 R_{12}} & 0 \\ \frac{1}{C_2 R_{12}} & -\frac{1}{C_2}\left(\frac{1}{R_{12}} + \frac{1}{R_{23}}\right) & \frac{1}{C_2 R_{23}} \\ 0 & \frac{1}{C_3 R_{23}} & -\frac{1}{C_3}\left(\frac{1}{R_{23}} + \frac{1}{R_{3\infty}}\right) \end{bmatrix}\begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1 R_{1\infty}} & \frac{P}{2C_1} \\ 0 & 0 \\ \frac{1}{C_3 R_{3\infty}} & \frac{P}{2C_3} \end{bmatrix}\begin{bmatrix} T_\infty(t) \\ s(t) \end{bmatrix} \tag{26}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} T_\infty(t) \\ s(t) \end{bmatrix} \tag{27}$$

where $x(t) = [T_1(t), T_2(t), T_3(t)]^T$ and $u(t) = [T_\infty(t), s(t)]$. Note we have added the output/measurement equation $y(t) = T_2(t)$. *As an exercise, derive (26)-(27) for yourself.* From this point forward, we consider parameter values given in Table 1.

Table 1: Parameters for Building Thermal Estimation Example

Symbol	Description	Value	Units
$R_{1\infty}$	Thermal resistance btw zone 1 and outside	5	[deg C/kW]
R_{12}	Thermal resistance btw zones 1 and 2	3	[deg C/kW]
R_{23}	Thermal resistance btw zones 2 and 3	3	[deg C/kW]
$R_{3\infty}$	Thermal resistance btw zone 3 and outside	5	[deg C/kW]
C_1	Thermal capacitance of zone 1	24	[kWh/deg C]
C_2	Thermal capacitance of zone 2	15	[kWh/deg C]
C_3	Thermal capacitance of zone 3	48	[kWh/deg C]
P	Thermal power of boiler	8	[kW]

Observability Analysis: First we assess observability. Is it possible to estimates states $T_1(t), T_3(t)$ from measurements of $T_2(t)$? We compute the observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0.0222 & -0.0444 & 0.0222 \\ -0.0015 & 0.0024 & -0.0012 \end{bmatrix} \tag{28}$$

In this case, the observability matrix is full rank, i.e. $rank(\mathcal{O}) = 3$. By Theorem 2.1, the pair (A, C) is observable. Consequently, we can indeed estimate states $T_1(t), T_3(t)$ from measurements of $T_2(t)$. *As an exercise, re-compute the observability matrix and its rank with $C_3 = 24$ kWh/deg C. Is the resulting (A, C) observable? Use the thermal dynamics and mathematics to explain why not.*

Model Simulation & Analysis: To verify the model produces reasonable temperature predictions, we implement (26)-(27) with parameters in Table 1. Figure 6 visualizes the simulations, with ambient temperature T_∞ given by the black dashed line and heater on/off signal $s(t)$ shaded in yellow/white. The initial conditions are: $T(0) = [21, 17, 25]^T$ deg C. From these simulations, we

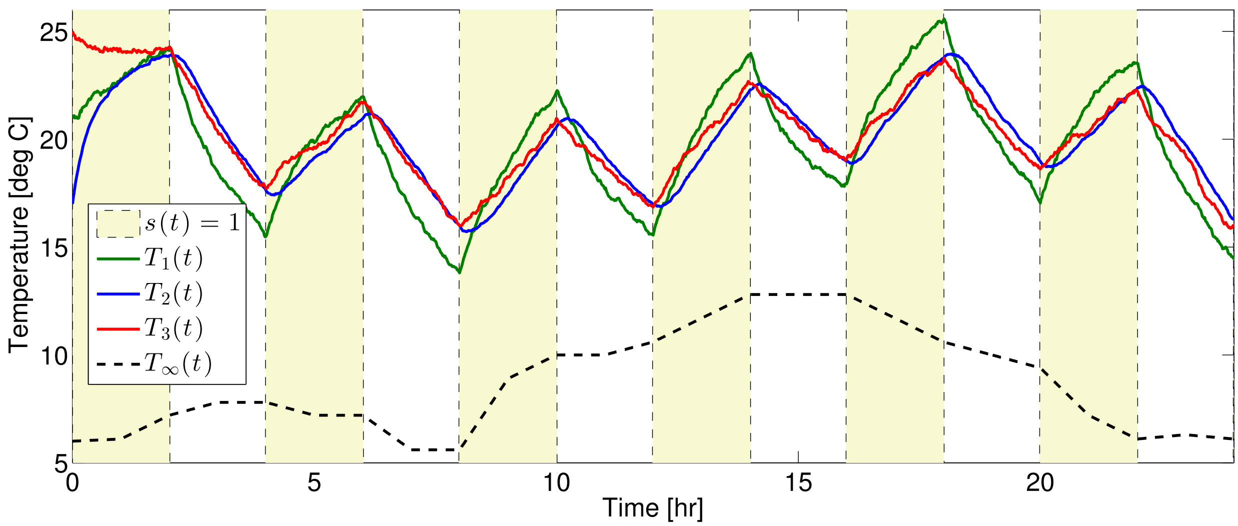

Figure 6: Simulation of 3-zone building temperature dynamics using parameters in Table 1. The ambient temperature T_∞ is given by the black dashed line, the heater on/off signal $s(t)$ is shaded in yellow/white.

observe that T_1 tends to change rapidly, which makes physical sense because it's thermal capacitance is smaller than zone 3 and it is exposed to the environment. In contrast T_3 evolves more slowly, due to larger thermal capacitance. During the heating phase $s(t) = 1$, zone 2 tends to increase its temperature slower than zones 1 and 3. This makes sense, as the vents are in zones 1 and 3. Zone 2 must rely on heat transfer with zones 1 and 3 to heat up.

Next we examine the eigenvalues of the system: eig$(A) = \{-0.0568, -0.0167, -0.0044\}$. All eigenvalues have negative real parts, implying the system is asymptotically stable. This is intuitive, as we expect the temperatures will not explode toward infinity but will eventually equilibrate with the ambient temperature T_∞ if no heating is applied. These eigenvalues are important for the Luenberger observer design, described after the open loop observer.

Open-Loop Observer: We consider an observer of the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \qquad \hat{x}(0) = \hat{x}_0, \tag{29}$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t). \tag{30}$$

This yields estimation error dynamics

$$\dot{\tilde{x}}(t) = A\tilde{x}(t), \qquad \tilde{x}(0) = x_0 - \hat{x}_0 = \tilde{x}_0 \tag{31}$$

As previously discussed, eig$(A) = \{-0.0568, -0.0167, -0.0044\}$. This implies the open loop estimation error dynamics are asymptotically stable. That is, this open loop observer will converge to the true values. Figure 7 provides results from implementing the open loop observer, with incorrect initial state estimates of $\hat{T}(0) = [17, 17, 17]^T$ deg C. Observe that the states estimates converge to

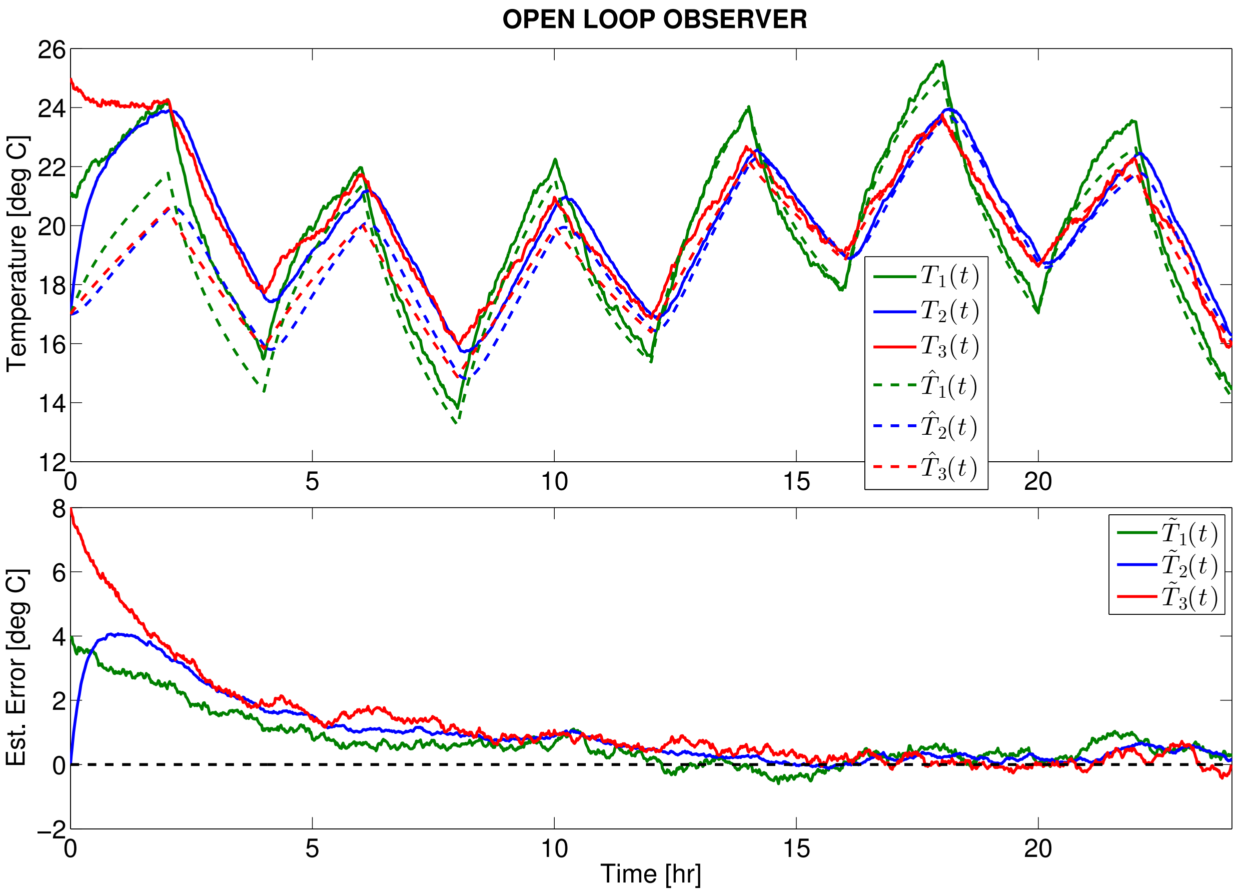

Figure 7: Open loop observer simulation results. [TOP] True and estimated states. [BOTTOM] State estimation error. The estimates converge, but at a rate dictated by the open-loop thermal dynamics.

the true values: $\left[\hat{T}_1(t), \hat{T}_2(t), \hat{T}_3(t)\right]^T \rightarrow [T_1(t), T_2(t), T_3(t)]^T$ as $t \rightarrow \infty$. In other words, the state estimation error goes to zero: $\left[\tilde{T}_1(t), \tilde{T}_2(t), \tilde{T}_3(t)\right]^T \rightarrow [0, 0, 0]^T$ as $t \rightarrow \infty$. The convergence speed is dictated by the eigenvalues of A, which are determined by the buildings thermal dynamics. Ideally, we want a user-selectable parameter that controls the convergence rates, thus enabling convergence faster than 10+ hours. The open-loop observer converges, but does not feature a user-selectable parameter to manage convergence. For this, we turn to the Luenberger observer.

Luenberger Observer: We consider an observer of the form:

$$\dot{\hat{x}}(t) \quad = \quad A\hat{x}(t) + Bu(t) + L\left[y(t) - \hat{y}(t)\right], \qquad \hat{x}(0) = \hat{x}_0, \tag{32}$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t). \tag{33}$$

This yields estimation error dynamics

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t), \qquad \tilde{x}(0) = x_0 - \hat{x}_0 = \tilde{x}_0 \tag{34}$$

where $\tilde{x}(t) = x(t) - \hat{x}(t)$ is the estimation error. Since the pair (A, C) is observable, Theorem 3.1 says we can arbitrarily place the eigenvalues of the error system matrix $(A-LC)$. Recall eig$(A) = \{-0.0568, -0.0167, -0.0044\}$. Remark 3.2 suggest a general rule-of-thumb: The eigenvalues of $(A - LC)$ should be placed 2-10 times faster than the eigenvalues of A. Following this suggestion we place eig$(A - LC) = 5 \times$ eig$(A) = \{-0.2838, -0.0833, -0.0218\}$. Using the `place` command in Matlab, this yields observer gain $L = [0.0444, 0.3111, 0.8556]^T$. We now implement (32)-(33) and simulate with incorrect initial state estimates $\hat{T}(0) = [17, 17, 17]^T$ deg C, shown in Fig. 8. In

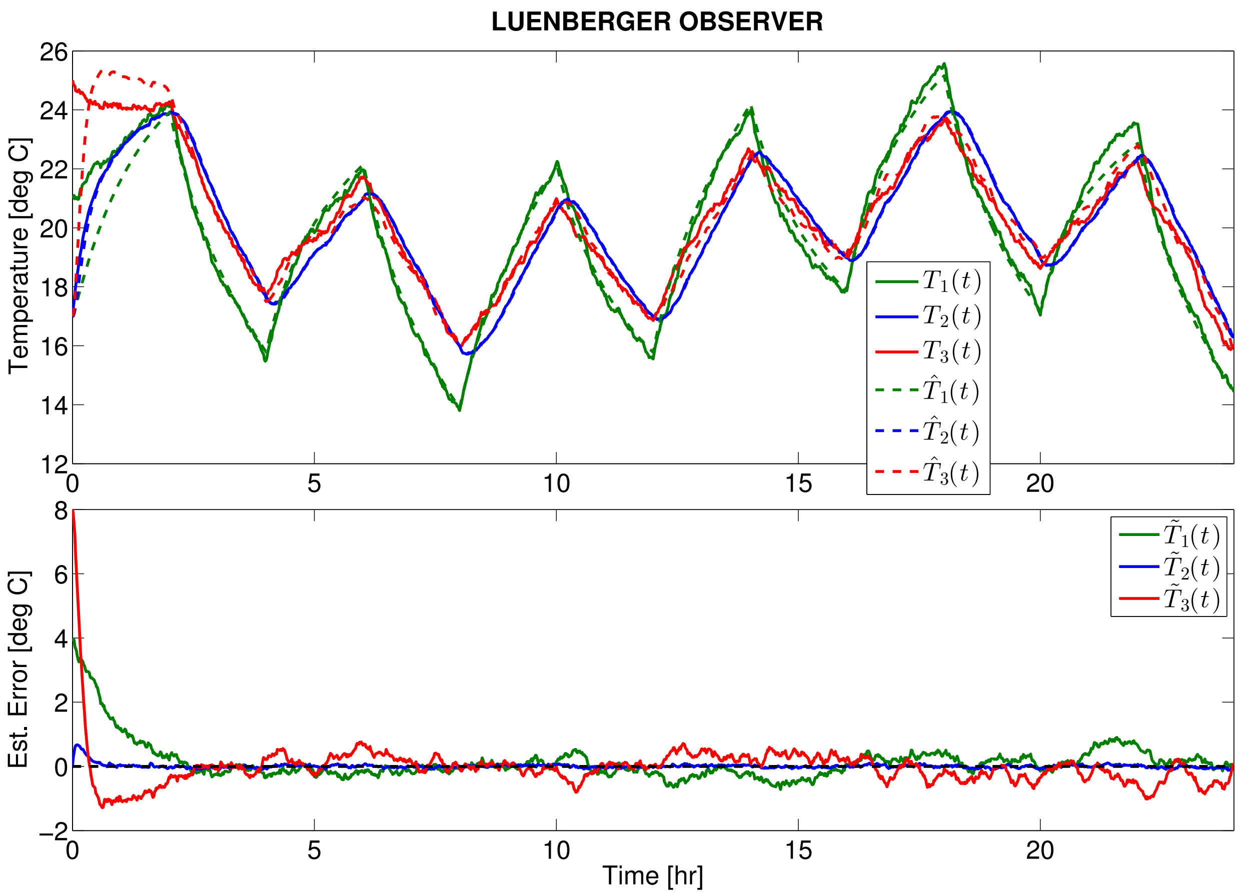

Figure 8: Luenberger observer simulation results. [TOP] True and estimated states. [BOTTOM] State estimation error. The estimates converge at a user-selected rate, dictated by eig$(A - LC)$.

this case the state estimates converge within 2-3 hours. This can be refined further by selecting different values for the eigenvalues of $(A - LC)$. *As an exercise, implement the Luenberger observer and test different eigenvalue placements for* $(A - LC)$.

The Luenberger observer provides a straight-forward method for estimating states. However, placing the eigenvalues of estimation error system matrix $(A - LC)$ is an abstract design methodology. That is, it requires some understanding of matrix theory (computing eigenvalues) and dynamic systems (relating eigenvalue placement to convergence). The Kalman Filter (KF), described next, is an optimization based approach. As we shall see, tuning the KF involves balancing trust between model and sensor accuracy.

4 Kalman Filter

The Kalman Filter (KF), also known as the linear quadratic estimator, is one of the most important developments in systems and control technology. The KF has numerous applications, especially within energy systems. It is named after Rudolf E. Kalman, a Hungarian-American engineer and mathematical scientist. The KF recursively operates on streams of noisy data and a possibly inaccurate model to produce statistically optimal estimates of the underlying system state. Put simply, the KF optimally balances trust in your model and trust in your data to generate state estimates.

Consider the LTI energy system model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t), \quad x(0) = x_0, \quad x, w \in \mathbb{R}^n, u \in \mathbb{R}^p && (35) \\ y_m(t) &= Cx(t) + Du(t) + n(t), \qquad\qquad\qquad y_m, n \in \mathbb{R}^q && (36) \end{aligned}$$

where w and n are "noise" terms that represent model inaccuracy (process noise) and sensor noise. The KF assumes these terms are generated from stationary, zero mean, and Gaussian white noise processes. Moreover, $w(t)$ and $n(t)$ have covariances W and N, respectively. The definition of a covariance implies W, N are positive semi-definite matrices. We shall assume, in addition, that N is positive definite, i.e. $N \succ 0$. This assumption implies that the noise affects all the measured outputs of the system; i.e. there are no "clean measurements."

Assume that the initial state x_0 is also generated from a Gaussian distribution, with mean and covariance

$$\begin{aligned} \overline{x}_0 &= \mathbb{E}\{x_0\}, && (37) \\ \Sigma_0 &= \mathbb{E}\left\{(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T\right\} && (38) \end{aligned}$$

where $\mathbb{E}\{\cdot\}$ represents the expected value of $\{\cdot\}$. Finally, also assume that w, n, and x_0 are mutually uncorrelated:

$$\begin{aligned}\mathbb{E}\left\{w(t)n^T(\tau)\right\} &= 0, \quad \forall\, t, \tau, &(39)\\ \mathbb{E}\left\{x_0 w^T(t)\right\} &= 0, \quad \forall\, t, &(40)\\ \mathbb{E}\left\{x_0 n^T(t)\right\} &= 0, \quad \forall\, t. &(41)\end{aligned}$$

Let $\hat{x}(t)$ denote an estimate of the system state at time $t \geq 0$. Suppose that we know the input and (measured) output of the system for times between 0 and t: $u(\tau), y_m(\tau), \forall \tau \in (0, t)$. Then we seek to use this information to recursively construct a state estimate $\hat{x}(t)$ that minimizes the mean square estimation error $\mathbb{E}\left\{\tilde{x}^T(t)\tilde{x}(t)\right\}$, where $\tilde{x} = x - \hat{x}$.

The following algorithm generates an estimate $\hat{x}$ that minimizes the mean square error. It is generated from a dynamical system with the structure of an observer with time-varying gain $L(t)$.

Theorem 4.1 (Kalman Filter)**.** *Consider the linear system (35)-(36) together with the observer*

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(t)(y_m - \hat{y}), \qquad \hat{x}(0) = \overline{x}_0, &(42)\\ \hat{y} &= C\hat{x} + Du &(43)\end{aligned}$$

Suppose that

$$L(t) = \Sigma(t) C^T N^{-1}, \quad \forall\, t > 0, \tag{44}$$

where $\Sigma(t)$ is the solution to the matrix differential equation

$$\dot{\Sigma}(t) = \Sigma(t)A^T + A\Sigma(t) + W - \Sigma(t)C^T N^{-1} C\Sigma(t), \qquad \Sigma(0) = \Sigma_0 \tag{45}$$

which is known as the "Riccati differential equation". Then the following properties hold:

(i) $\hat{x}(t)$ *minimizes* $\mathbb{E}\left\{\tilde{x}^T(t)\tilde{x}(t)\right\}$*,*

(ii) $\mathbb{E}\{\tilde{x}(t)\} = 0$ *and* $\mathbb{E}\left\{\tilde{x}(t)\tilde{x}^T(t)\right\} = \Sigma(t)$*,*

(iii) $\mathbb{E}\left\{\tilde{x}^T(t)\tilde{x}(t)\right\} = \text{trace}\ \Sigma(t)$

(iv) $\Sigma(t) \geq 0$ *exists and is finite for all* $t > 0$*.*

The system (42) is known as the time-varying optimal estimator, or Kalman filter. The four properties in the theorem have the following respective implications.

(i) The state of this estimator minimizes the mean square estimation error, as desired.

(ii) The estimation error has zero mean, and a covariance given by the solution to the Riccati equation (45). In other words, solving (45) is not just superfluous math, but provides a confidence interval for the generate estimate $\hat{x}(t)$.

(iii) The mean square value of the estimation error is given by the trace of the covariance matrix.

(iv) The covariance matrix exists and is finite.

Remark 4.1. Suppose we have been estimating the state for an arbitrarily long time. Under appropriate conditions, the solution of the Riccati equation (45) will converge to a constant value, and the observer gain (44) will thus become constant. This solution to this problem is known as the time-invariant optimal estimator. We do not cover this algorithm here. Please refer to Section 4.5 of [1] for more details.

Next we demonstrate the Kalman filter on the building zone temperature estimation example.

Example 4.1 (Building Zone Temperature Estimation - revisited)**.** Recall the 3-zone building temperature estimation problem from Example 3.1, Fig. 5, and (26)-(27). We consider an observer of the form:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(t)(y_m - \hat{y}), \qquad \hat{x}(0) = \overline{x}_0, \tag{46}$$

$$\hat{y} = C\hat{x} + Du, \tag{47}$$

where,

$$L(t) = \Sigma(t)C^T N^{-1}, \tag{48}$$

$$\dot{\Sigma}(t) = \Sigma(t)A^T + A\Sigma(t) + W - \Sigma(t)C^T N^{-1} C\Sigma(t), \qquad \Sigma(0) = \Sigma_0. \tag{49}$$

Like the open loop and Luenberger observers, we consider the initial state estimate $\hat{x}(0) = [17, 17, 17]^T$ deg C. The remaining decisions involve selecting initial state covariance $\Sigma_0 \in \mathbb{R}^{3\times 3}$, process noise covariance $W \in \mathbb{R}^{3\times 3}$, and measurement noise covariance $N \in \mathbb{R}^{1\times 1}$. Given sufficient measurement data of the true state and noise-free output, one could fit multivariate Gaussian distributions to empirically determine the covariances Σ_0, W, N. However, this is almost never the case in practice. The entire motivation for state observers was an inability to measure all states, and all sensors have some degree of noise. Are we stuck? In practice, a control engineer treats Σ_0, W, N as tuning gains. That is, they provide control knobs that balance trust in the model versus trust in the measured data. For simplicity, these matrices are almost always defined as diagonal matrices.

In this example, we consider $\Sigma_0 = 10 \cdot \mathbb{I}_{3\times 3}$, $W = \texttt{diag}([0.05, 0.02, 0.05])$, $N = 0.001$. The results are provided in Fig. 9. With these parameters, the state estimates converge within 1 hour. An additional benefit of the KF is confidence intervals. That is, the KF does not just estimate states like the Luenberger observer. It provides a Gaussian distribution for the state estimates, whose mean is given by $\hat{x}(t)$ and covariance given by $\Sigma(t)$. Consequently, the quantity $\hat{x}_i(t) \pm \sqrt{\Sigma_{ii}(t)}$ provides a ± 1 standard deviation confidence interval around the state estimate $\hat{x}_i(t)$. Neat! This is beneficial for a variety of reasons. For example, one might seek to optimally control $s(t)$ to maintain temperature within a comfortable interval $T_{\min} \leq T(t) \leq T_{\max}$. Since the model and measurements are inaccurate, we may pursue a "robust" strategy that constrains $T_{\min} \leq \hat{T}_i(t) \pm \sqrt{\Sigma_{ii}(t)} \leq T_{\max}$.

As an exercise, implement the KF and explore different combinations of Σ_0, W, N. *What effects do you see? What happens if you keep* $\Sigma_0 = 10 \cdot \mathbb{I}_{3\times 3}$, *and simultaneously enhance* W, N *one order of magnitude, i.e.* $W = 10\times \; \mathtt{diag}([0.05, 0.02, 0.05])$, $N = 10 \times 0.001$

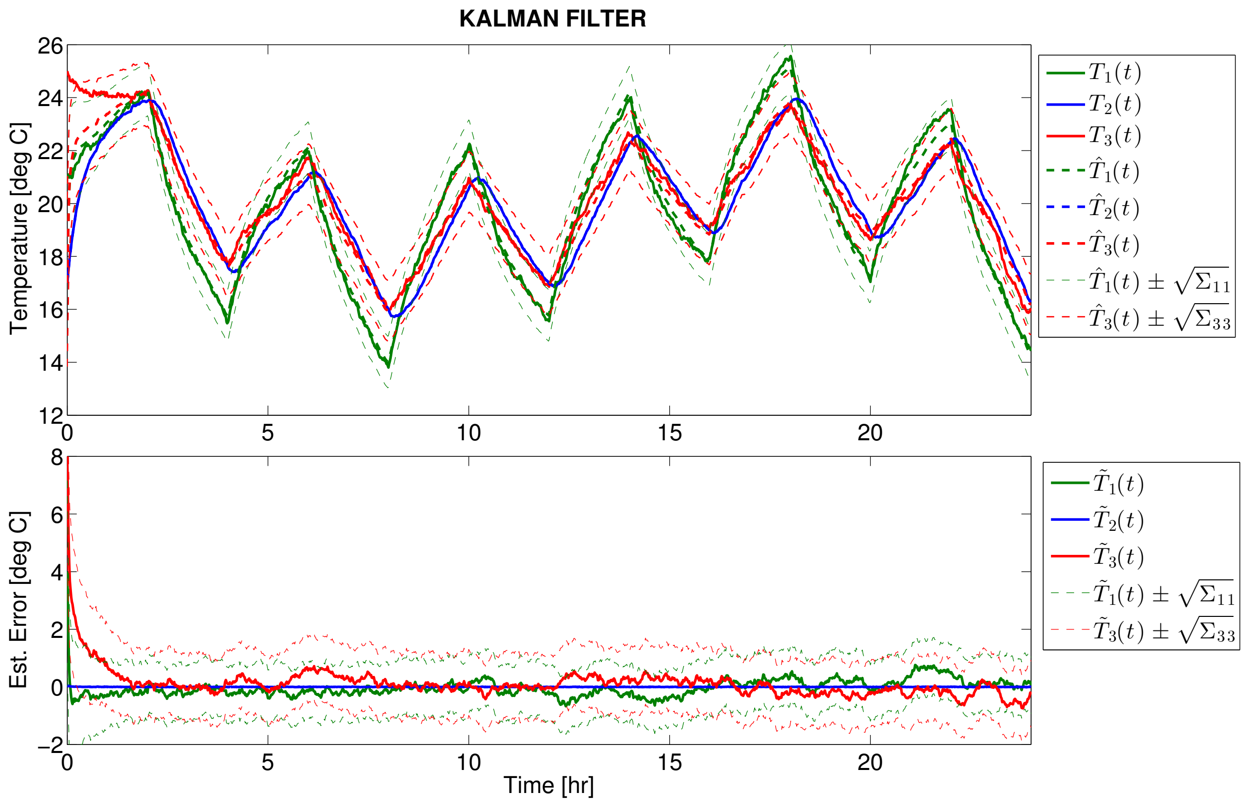

Figure 9: Kalman Filter simulation results. [TOP] True and estimated states. [BOTTOM] State estimation error. In addition to minimizing the mean square estimation error, the KF provides confidence intervals with covariance matrix $\Sigma(t)$.

5 Extended Kalman Filter

The Extended Kalman Filter (EKF) is a nonlinear version of the Kalman Filter. The primary drawback of the Kalman Filter is that it assumes a linear dynamic system model. Unfortunately, most engineering systems are nonlinear, so some attempt was immediately made to apply this filtering method to nonlinear systems. Interestingly, most of this work was done at nearby NASA Ames. The key idea behind the EKF is to linearize the model equations about the current estimate.

Consider the nonlinear dynamical energy system model

$$\dot{x} = f(x,u) + w, \quad x(0) = x_0, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^p \tag{50}$$

$$y_m = h(x,u) + n, \qquad y_m \in \mathbb{R}^q \tag{51}$$

As in the KF, we assume $w(t), n(t)$ are stationary, zero mean, and Gaussian white noise processes. Moreover, they have covariances W and N, respectively, where we assume $N \succ 0$. As before, we also assume the initial state x_0 is also generated from a Gaussian distribution, with mean and covariance

$$\overline{x}_0 = \mathbb{E}\{x_0\}, \tag{52}$$

$$\Sigma_0 = \mathbb{E}\left\{(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T\right\} \tag{53}$$

Finally, also assume that w, n, and x_0 are mutually uncorrelated:

$$\mathbb{E}\left\{w(t)n^T(\tau)\right\} = 0, \quad \forall\, t, \tau, \tag{54}$$

$$\mathbb{E}\left\{x_0 w^T(t)\right\} = 0, \quad \forall\, t, \tag{55}$$

$$\mathbb{E}\left\{x_0 n^T(t)\right\} = 0, \quad \forall\, t. \tag{56}$$

Under identical assumptions as the KF, but with a nonlinear model, we are now positioned to state the EKF algorithm.

Algorithm 5.1 (Extended Kalman Filter)**.** *Define the following Jacobians with respect to the state variable x, evaluated at the current estimate $x = \hat{x}$ and control input u.*

$$F(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \qquad H(t) = \frac{\partial h}{\partial x}(\hat{x}(t), u(t)), \tag{57}$$

Consider the nonlinear system (50)-(51) together with the observer

$$\dot{\hat{x}} = f(\hat{x}, u) + L(t)(y_m - h(\hat{x}, u)), \qquad \hat{x}(0) = \overline{x}_0. \tag{58}$$

Suppose that

$$L(t) = \Sigma(t) H^T(t) N^{-1}, \quad \forall\, t > 0, \tag{59}$$

where $\Sigma(t)$ is the solution to the matrix differential equation

$$\dot{\Sigma}(t) = \Sigma(t) F(t)^T + F(t)\Sigma(t) + W - \Sigma(t) H^T(t) N^{-1} H(t) \Sigma(t), \qquad \Sigma(0) = \Sigma_0 \tag{60}$$

which is known as the "Riccati differential equation".

Remark 5.1. Unlike its linear counterpart, the extended Kalman filter in general is NOT an optimal estimator. Of course, if the differential equation and measurement functions (50)-(51) are linear, then the EKF becomes identical to the KF. Moreover, if the estimate's initial condition is far away from the true value, or the model is inaccurate, then estimates may diverge due to linearization. Consequently, none of the properties of Theorem 4.1 are guaranteed for the EKF. That said, the EKF often works well in practice and is the de-facto estimation technique for nonlinear systems.

Example 5.1 (Building Zone Temperature Estimation w/ Nonlinear Sensors)**.** Recall the 3-zone building temperature estimation problem from Example 3.1, Fig. 5, and (26). In this example we consider a nonlinear sensor. Specifically, suppose we measure T_2 using a thermistor. A thermistor is quite simply a temperature-sensitive resistor. If we measure the resistance of the thermistor (using a Wheatstone bridge for example), then we can map this resistance to a temperature. The relationship between temperature and resistance for a typical negative temperature coefficient (NTC) thermistor is shown in Fig. 10. Clearly, the relationship is nonlinear. We can fit an

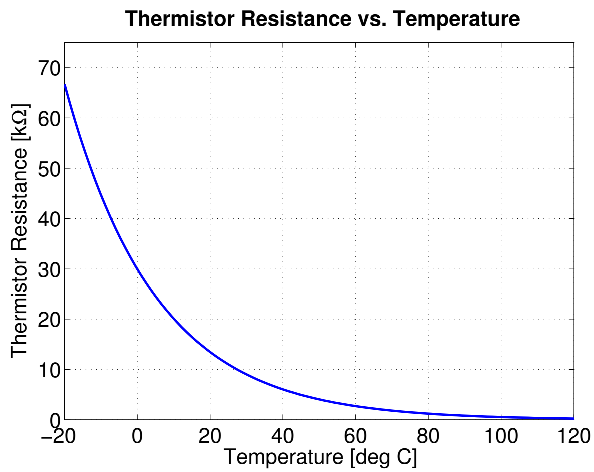

Figure 10: Resistance versus temperature for a typical NTC thermistor.

exponential curve, thereby providing the following nonlinear output equation

$$y_m(t) = \exp\left[-0.04 \cdot T_2(t) + 3.4\right] + n(t) \tag{61}$$

where $n(t)$ is noise due to sensor inaccuracy.

We consider a nonlinear EKF observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(t)(y_m(t) - \hat{y}), \qquad \hat{x}(0) = \overline{x}_0, &(62)\\ \hat{y}(t) &= \exp\left[-0.04 \cdot \hat{T}_2(t) + 3.4\right] &(63)\end{aligned}$$

where A, B are from (26) and the time-varying observer gain

$$L(t) = \Sigma(t)H^T(t)N^{-1}, \quad \forall\, t > 0, \tag{64}$$

where $\Sigma(t)$ is the solution to the matrix differential equation

$$\dot{\Sigma}(t) = \Sigma(t)F(t)^T + F(t)\Sigma(t) + W - \Sigma(t)H^T(t)N^{-1}H(t)\Sigma(t), \qquad \Sigma(0) = \Sigma_0 \tag{65}$$

All that remains is to compute the Jacobians $F(t)$ and $H(t)$. From (57) we have

$$F(t) = A, \tag{66}$$
$$H(t) = \left[0, -0.04e^{-0.04\hat{T}_2(t)+3.4}, 0\right]. \tag{67}$$

Next we implement the EKF with the same parameters from Example 4.1, except using initial condition $\hat{x}(0) = [17, 100, 17]^T$ deg C. In this case, we challenge the EKF by initializing T_2 at 100 deg C - far away from the true value of 17 deg C. As seen from Fig. 10, a linear approximation around 100 deg C is notably different than a linear approximation around 17 deg C. We also set $N = 1$ to reflect the added uncertainty in this nonlinear sensor. The EKF results are provided in Fig. 11. After a large initial transient, the state estimates quickly converge to their true values - de-

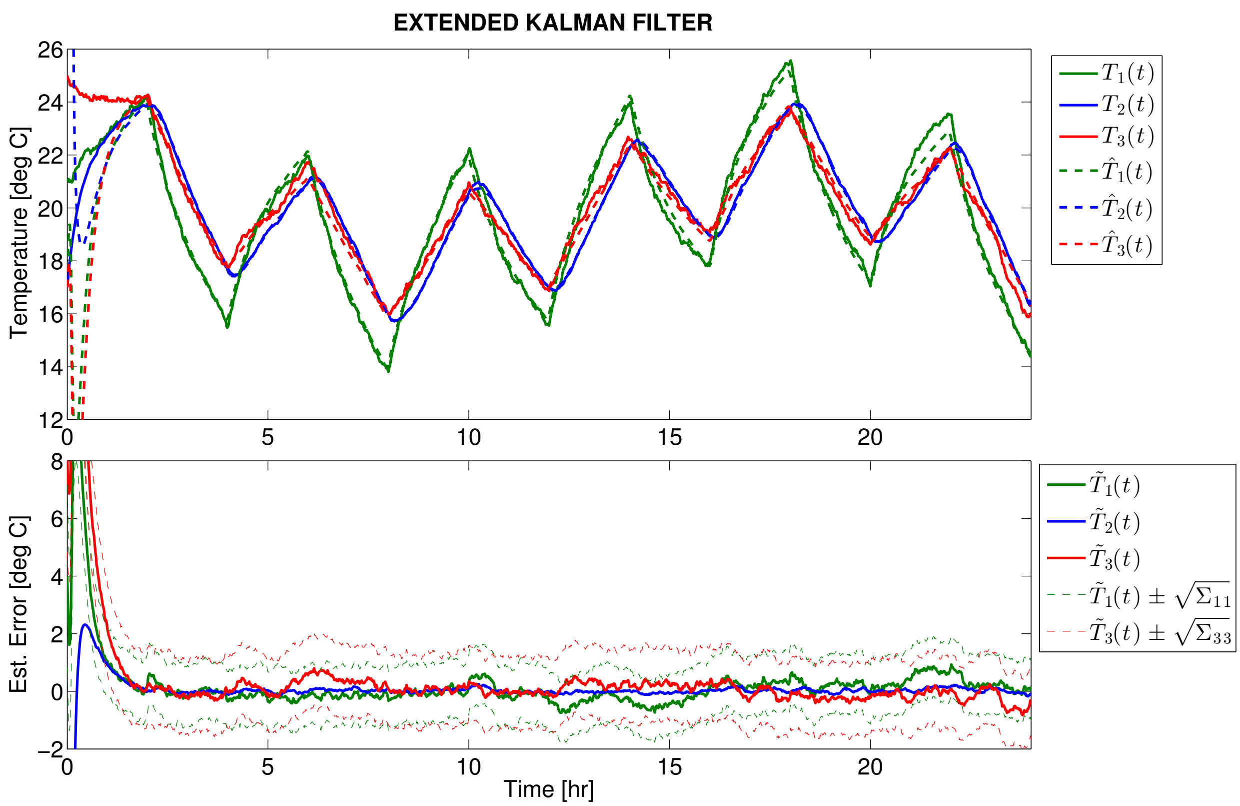

Figure 11: Extended Kalman Filter simulation results. [TOP] True and estimated states. [BOTTOM] State estimation error. The initial state estimate is $\hat{x}(0) = [17, 100, 17]^T$ deg C.

spite a nonlinear sensor equation and grossly incorrect initial conditions. Of course, these results can be further tuned by adjusting Σ_0, W, N. Although the EKF lacks the theoretical properties of the KF, it performs well in practice - as shown here.

Remark 5.2. Of course, state estimation is not limited to temperature monitoring in buildings. We simply use this energy system as a pedagogical case study.

6 Combined State and Parameter Estimation

At this stage in the course, a logical question is: "Can one *simultaneously* estimate the model parameter and states, by combining concepts in CH2 and CH3?" The answer is yes. Next, we present one very simple concept for combined state & parameter estimation, called "joint state and parameter estimation". This is one of many concepts, and therefore this section is NOT a comprehensive resource for state and parameter estimation.

Consider the parameterized LTI energy system model:

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) + w(t), \quad x(0) = x_0, \quad x, w \in \mathbb{R}^n, u \in \mathbb{R}^p && (68) \\ y_m(t) &= C(\theta)x(t) + D(\theta)u(t) + n(t), \qquad y_m, n \in \mathbb{R}^q && (69) \end{aligned}$$

where the key detail is that system matrices A, B, C, D depend on unknown parameter vector $\theta \in \mathbb{R}^{n_\theta}$. The main idea is to augment the state vector by considering dynamics for parameter vector θ as follows:

$$\dot{\theta} = 0 + w_\theta(t), \quad \theta(0) = \theta_0 \tag{70}$$

Then we can define an augmented state vector $x_a = \left[x^T(t), \theta^T(t)\right]^T$, resulting in nonlinear system:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} &= \begin{bmatrix} A(\theta)x(t) + B(\theta)u(t) \\ 0 \end{bmatrix} + \begin{bmatrix} w(t) \\ w_\theta(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \theta(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} && (71) \\ y_m(t) &= C(\theta)x(t) + D(\theta)u(t) + n(t) && (72) \end{aligned}$$

This now formulates a nonlinear system, in which nonlinear state estimation techniques (e.g. the EKF) can be applied to jointly estimated $x(t), \theta(t)$. As mentioned in Section 5, however, properties such as estimation error stability or mean square optimality are generally not guaranteed.

7 Notes

State estimation is a rich and deep area, in both theory and applications. In this chapter, we attempt to provide students with an introduction, several popular tools, and motivation for further study. Several textbooks provide valuable expositions on this topic, including Chen [2], Stengel [1], and Simon [3]. Readers may also be interested in state estimation within the context of power systems [4, 5] or battery systems [6].

A plethora of state estimation techniques exist for nonlinear systems, beyond EKFs. These include unscented Kalman Filters (UKF), ensemble Kalman filters (EnKF), particle filters (PF), sliding mode estimators, and moving horizon estimation (MHE). Each technique offers solutions that trade off some degree of model knowledge, computational power, and algorithmic properties.

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8 Appendix

8.1 Observability Test

The following theorem provides several tests for determining observability of the pair (A,C)

Theorem 8.1 (Observability Tests)**.** The following statements are equivalent:

1. The $n-$dimensional pair (A,C) is observable.
2. The $n \times n$ matrix called the observability Grammian

$$W_O(t) = \int_0^t e^{A^T \tau} C^T C e^{A \tau} d\tau \tag{73}$$

   is nonsingular for any $t > 0$.

3. The $nq \times n$ observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{74}$$

   has rank n (full column rank).

4. The $(n+q) \times n$ matrix
$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \tag{75}$$
has full column rank for all eigenvalue λ of A.

5. If, in additional, all eigenvalues of A have negative real parts, then the unique solution of the Lyapunov equation
$$A^T W_O + W_O A = -C^T C \tag{76}$$
is positive definite. The solution is called the observability Grammain and can be expressed as
$$W_O = \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau \tag{77}$$

8.2 Discrete-time Kalman Filter

In practice, data is time-sampled. Consequently, one might ask if a Kalman filter exists that is native to discrete-time. The answer is yes, and we summarize it here. Consider an energy system with LTI difference equation dynamics given by:

$$x_{k+1} = Ax_k + Bu_k + w_k, \qquad x, w \in \mathbb{R}^n, u \in \mathbb{R}^p \tag{78}$$
$$y_{m,k} = Cx_k + Du_k + n_k, \qquad y_m, n \in \mathbb{R}^q \tag{79}$$

The notation $(\cdot)_k$ refers to the variable $(\cdot)$ at time index k. For example, $x_k = x(k\Delta t)$ where Δt is the sampling time. Terms w and n represent model inaccuracy (process noise) and sensor noise. The KF assumes these terms are generated from stationary, zero mean, and Gaussian white noise processes. Moreover, $w(t)$ and $n(t)$ have covariances $W \succeq 0$ and $N \succ 0$, respectively.

Assume that the initial state x_0 is also generated from a Gaussian distribution, with mean and covariance

$$\overline{x}_0 = \mathbb{E}\{x_0\}, \tag{80}$$
$$\Sigma_0 = \mathbb{E}\left\{(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T\right\} \tag{81}$$

Finally, also assume that w, n, and x_0 are mutually uncorrelated:

$$\mathbb{E}\left\{w(k)n^T(j)\right\} = 0, \quad \forall\, k, j, \tag{82}$$
$$\mathbb{E}\left\{x_0 w^T(k)\right\} = 0, \quad \forall\, k, \tag{83}$$
$$\mathbb{E}\left\{x_0 n^T(k)\right\} = 0, \quad \forall\, k. \tag{84}$$

Let $\hat{x}_k$ denote an estimate of the system state at time index $k \geq 0$. Suppose that we know the input and measured output of the system for time indices between 0 and k: $u_j, y_{m,j}, \forall j \in \{0, 1, \cdots, k\}$. Then we seek to use this information to recursively construct a state estimate $\hat{x}_k$ that minimizes the mean square estimation error $\mathbb{E}\left\{\tilde{x}_k^T \tilde{x}_k\right\}$, where $\tilde{x}_k = x_k - \hat{x}_k$.

The discrete-time KF generates an estimate $\hat{x}_k$ that minimizes the mean square error. Namely, it will produce estimate x_k as a Gaussian random variable with mean $\hat{x}_k$ and covariance Σ_k. The discrete-time KF consists of a prediction step and an update/correction step. To provide clarity, it is helpful to expand the k notation to distinguish between the state estimates produced before and after the correction step. Therefore, at each time step k, the predicted (a priori) state estimate is denoted as $\hat{x}_{k|k-1}$ and represents the mean estimate of x_k given measurements $y_0, \cdots, y_{k-1}$. The corrected (a posterior) state estimate is denoted as $\hat{x}_{k|k}$ and represents the mean estimate of x_k given measurements $y_0, \cdots, y_k$. To reiterate, in this course the uncorrected (a priori) predictions are denoted by subscripts $k|k-1$ or $k+1|k$ whereas the corrected predictions (a posterior) are denoted by subscripts $k|k$, $k-1|k-1$, or $k+1|k+1$.

The KF prediction step is given by

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1}, \tag{85}$$

$$\Sigma_{k|k-1} = AQ_{k-1|k-1}A^T + W \tag{86}$$

The update/correction step is

$$\hat{y}_k = C\hat{x}_{k|k-1} + Du_k, \tag{87}$$

$$L_k = \Sigma_{k|k-1}C^T \left[C\Sigma_{k|k-1}C^T + N\right]^{-1}, \tag{88}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k \left[y_k - \hat{y}_k\right], \tag{89}$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - L_k \left[C\Sigma_{k|k-1}C^T + N\right] L_k^T \tag{90}$$

Theorem 8.2 (Discrete-time Kalman Filter)**.** *Consider the discrete-time linear system (78)-(79) together with the aforementioned noise assumptions and discrete-time KF equations in (85)-(90). Then the following properties hold:*

(i) $\hat{x}_{k|k}$ *minimizes* $\mathbb{E}\left\{\tilde{x}_k^T\tilde{x}_k\right\}$ *where* $\tilde{x}_k = x_k - \hat{x}_{k|k}$*,*

(ii) $\mathbb{E}\{\tilde{x}_k\} = 0$ *and* $\mathbb{E}\left\{\tilde{x}_k\tilde{x}_k^T\right\} = \Sigma_{k|k}$*,*

(iii) $\mathbb{E}\left\{\tilde{x}_k^T\tilde{x}_k\right\} = \text{trace } \Sigma_{k|k}$

(iv) $\Sigma_{k|k} \geq 0$ *exists and is finite for all* $k > 0$*.*

The system (85)-(90) is known as the discrete-time Kalman filter. The four properties in the theorem have the following respective implications.

(i) The a posterior state $\hat{x}_{k|k}$ of this estimator minimizes the mean square estimation error, as desired.

(ii) The estimation error has zero mean, and a covariance given by (90). In other words, computing (90) is not just superfluous math. It provides a confidence interval for the estimate $\hat{x}_{k|k}$ Similarly, we can define $\Sigma_{y,k} = C\Sigma_{k|k-1}C^T + N$ which represents the estimation covariance of measurement $\hat{y}_k$.

(iii) The mean square value of the estimation error is given by the trace of the covariance matrix.

(iv) The covariance matrix exists and is finite.

For more detail, the textbook by Stengel [1] provides an excellent exposition of optimal state estimation and filtering.