In [ ]:
```
lp = gilp.examples.ALL_INTEGER_3D_LP # get LP example
gilp.lp_visual(lp).show() # visualize it
```

The 3D feasible region is shown on the left. Hold and drag the mouse to examine it from different angles. Next, click on a constraint to un-mute it. Each constraint is a gray plane in 3D space. Un-mute the constraints one by one to see how they define the 3D feasible region. Move the objective slider to see the isoprofit planes. The isoprofit plane is light gray and the intersection with the feasible region is shown in red. Like the 2D visualization, you can hover over corner points to see information about that point.

Q5: Use the objective slider to solve this LP graphically. Give an optimal solution and objective value. (Hint: The objective slider shows the isoprofit plane for some objective value in light gray and the intersection with the feasible region in red.)

A:

When it comes to LPs with 4 or more decision variables, our graphical approaches fail. We need to find a different way to solve linear programs of this size.

Part II: The Simplex Algorithm for Solving LPs

Dictionary Form LP

First, let's answer some guiding questions that will help to motivate the simplex algorithm.

Q6: Does there exist a unique way to write any given inequality constraint? If so, explain why each constraint can only be written one way. Otherwise, give 2 ways of writing the same inequality constraint.

A: There is not necessarily a unique way to write any given inequality. For example, the inequalities $x_1 - x_2 \leq 5$ and $x_1 - 3 \leq x_2 + 2$ are the same inequality.

Q7: Consider the following two constraints: $2x_1 + 1x_2 \leq 20$ and $2x_1 + 1x_2 + x_3 = 20$ where all x are nonnegative. Are these the same constraint? Why? (This question is tricky!)

A: The variable x_3, stands in for the $\leq$ sign in the first inequality, as the fact that all x are nonnegative ensures that $2x_1 + 1x_2 \leq 20$ in the second equation, which is identical to the first inequality.

Q8: Based on your answers to **Q6** and **Q7**, do you think there exists a unique way to write any given LP?

A: No. There is always another way to write any given LP.

You should have found that there are many ways to write some LP. This begs a new question: are some ways of writing an LP harder or easier to solve than others? Consider the following LP:

$$\begin{aligned} \max \quad & 56 - 2x_3 - 1x_4 \\ \text{s.t.} \quad & x_1 = 4 - 1x_3 + 1x_4 \\ & x_2 = 12 + 1x_3 - 2x_4 \\ & x_5 = 3 + 1x_3 - 1x_4 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Q9: Just by looking at this LP, can you give an optimal solution and its objective value. If so, explain what property of the LP allows you to do this. (Hint: Look at the objective function)

A: The coefficients of all x in the objective function are nonpositive, so the maximum would require the x with non-zero coefficients in the objective to equal 0. The optimal solution is (4,12,0,0,3), with objective value 56.

The LP above is the same as `ALL_INTEGER_2D_LP` just rewritten in a different way! This rewitten form (which we found is easier to solve) was found using the simplex algorithm. At its core, the simplex algorithm strategically rewrites an LP until it is in a form that is "easy" to solve.

The simplex algorithm relies on an LP being in **dictionary form**. Recall the following properties of an LP in dictionary form:

- All constraints are equality constraints
- All variables are constrained to be nonnegative
- Each variable only appears on the left-hand side (LHS) or the right-hand side (RHS) of the constraints (not both)
- Each constraint has a unique variable on the LHS
- The objective function is in terms of the variables that appear on the RHS of the constraints only.
- All constants on the RHS of the constraints are nonnegative

Q10: Rewrite the example LP `ALL_INTEGER_2D_LP` in dictionary form. Show your steps!

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + 1x_2 \leq 20 \\ & 1x_1 + 1x_2 \leq 16 \\ & 1x_1 + 0x_2 \leq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

A: $2x_1 + 1x_2 \leq 20$ becomes $2x_1 + 1x_2 + 1x_3 = 20$

$1x_1 + 1x_2 \leq 16$ becomes $1x_1 + 1x_2 + 1x_4 = 16$

$1x_1 + 0x_2 \leq 7$ becomes $1x_1 + 0x_2 + 1x_5 = 7$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Dictionary: $max \quad 5x_1 + 3x_2$

$$\begin{aligned} & x_3 = 20 - 2x_1 - x_2 \\ & x_4 = 16 - x_1 - x_2 \\ & x_5 = 7 - x_1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Most Limiting Constraint

Once our LP is in dictionary form, we can run the simplex algorithm! In every iteration of the simplex algorithm, we will take an LP in dictionary form and strategically rewrite it in a new dictionary form. Note: it is important to realize that rewriting the LP **does not** change the LP's feasible region. Let us examine an iteration of simplex on a new LP.

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 \\ \text{s.t.} \quad & 1x_1 + 0x_2 \leq 4 \\ & 0x_1 + 1x_2 \leq 6 \\ & 2x_1 + 1x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Q11: Is this LP in dictionary form? If not, rewrite this LP in dictionary form.

A: This is not in dictionary form. The dictionary form is:

$$
\begin{aligned}
&max \quad 5x_1 + 3x_2 \\
&x_3 = 4 - x_1 \\
&x_4 = 6 - x_2 \\
&x_5 = 9 - 2x_1 - x_2 \\
&x_6 = 15 - 3x_1 - 2x_2 \\
&x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

Q12: Recall from **Q9** how you found a feasible solution (which we argued to be optimal) just by looking at the LP. Using this same stratagy, look at the LP above and give a feasible solution and its objective value for this LP. Describe how you found this feasible solution. Is it optimal? Why?

A: A feasible solution would be (0,0,4,6,9,15). This is not optimal as the solution is 0. In order for us to find an optimal solution, the coefficients of all x in the max expression must be either 0 or negative.

From **Q12** we see that every dictionary form LP has a corresponding feasible solution. Furthermore, there are positive coefficents in the objective function. Hence, we can increase the objective value by increasing the corresponding variable. In our example, both x_1 and x_2 have positive coefficents in the objective function. Let us choose to increase x_1.

Q13: What do we have to be careful about when increasing x_1?

A: We have to make sure that we don't increase the value of x_1 to the point of it invalidating the given restrictions.

Q14: After choosing a variable to increase, we must determine the most limiting constraint. Let us look at the first constraint $x_3 = 4 - 1x_1 - 0x_2$. How much can x_1 increase? (Hint: what does a dictionary form LP require about the constant on the RHS of constraints?)

A: x_1 can have a maximum value of 4.

Q15: Like in **Q14**, determine how much each constraint limits the increase in x_1 and identify the most limiting constraint.

A: The limits on x_1 are: 4, No limits, 4.5, and 5. The most limiting constraint is $x_1 = 4$

If we increase x_1 to 4, note that x_3 will become zero. Earlier, we identified that each dictionary form has a corresponding feasible solution acheived by setting variables on the RHS (and in the objective function) to zero. Hence, since x_3 will become zero, we want to rewrite our LP such that x_3 appears on the RHS. Furthermore, since x_1 is no longer zero, it should now appear on the LHS.

Q16: Rewrite the most limiting constraint $x_3 = 4 - 1x_1 - 0x_2$ such that x_1 appears on the left and x_3 appears on the right.

A: $x_1 = 4 - 0x_2 - 1x_3$

Q17: Using substitution, rewrite the LP such that x_3 appears on the RHS and x_1 appears on the LHS. (Hint: Don't forget the rule about which variables can appear in the objective function)

A: $max \quad 5(4 - x_3) + 3x_2 = 20 - 5x_3 + 3x_2$

$$
\begin{aligned}
x_1 &= 4 - x_3 \\
x_4 &= 6 - x_2 \\
x_5 &= 9 - 2(4 - x_3) - x_2 = 1 - 2x_3 - x_2 \\
x_6 &= 15 - 3(4 - x_3) - 2x_2 = 3 + 3x_3 - 2x_2 \\
&x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

Q18: We have now completed an iteration of simplex! What is the corresponding feasible solution of the new LP?

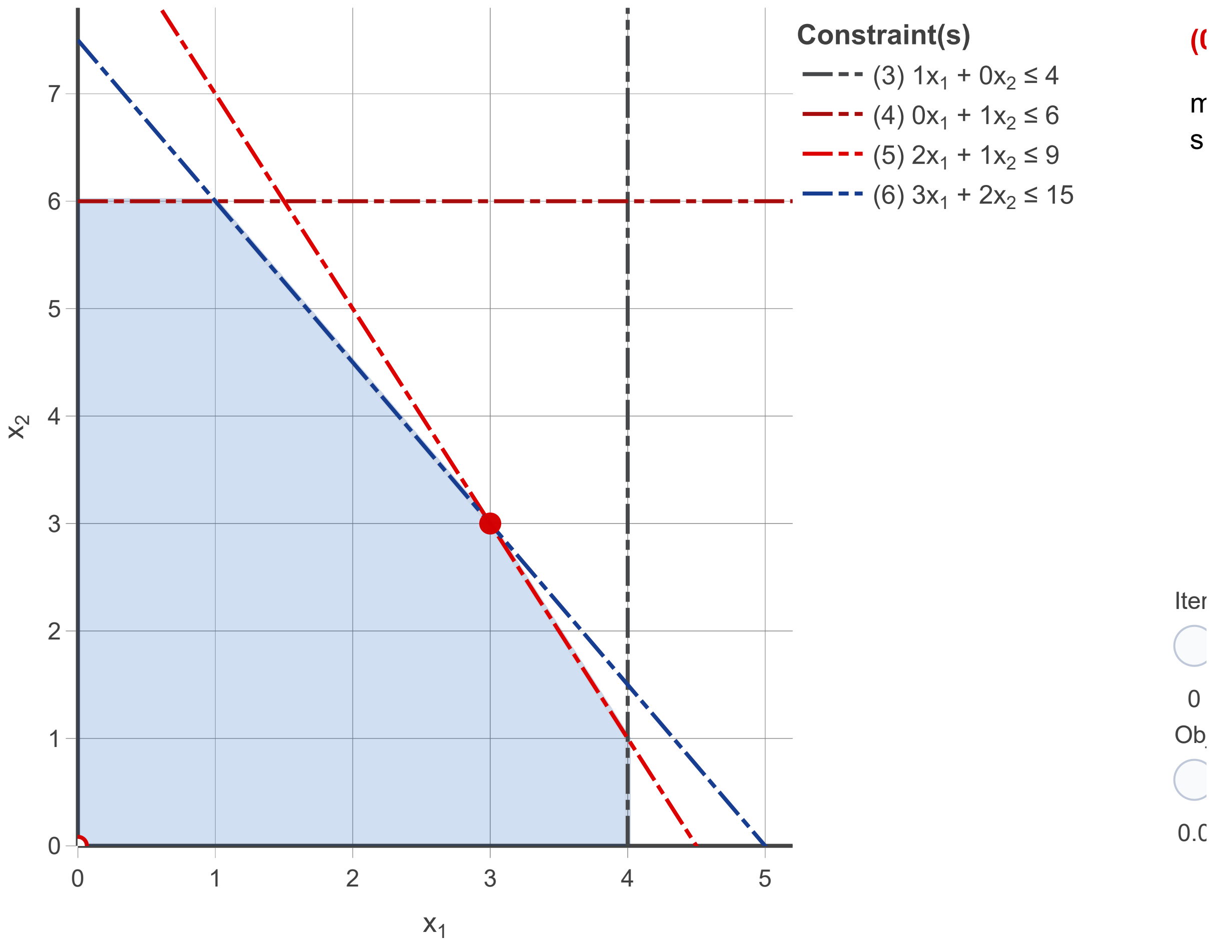
A: The corresponding solution would have $x_2, x_3 = 0$, which is (4,0,0,6,1,3), with an objective value of 20.

Now that we have seen an iteration of simplex algebraically, let's use GILP to visualize it! The LP example we have been using is called `LIMITING_CONSTRAINT_2D_LP` . To visualize simplex, we must import a function called `simplex_visual()` .

In [5]:

```
lp = gilp.examples.LIMITING_CONSTRAINT_2D_LP # get the LP example
gilp.simplex_visual(lp, initial_solution=np.array([[0],[0]])).show() # show th
e simplex visualization
```

Geometric Interpretation of LPs



This visualization is much the same as the previous one but we now have an addtional slider which allows you to toggle through iterations of simplex. Furthermore, the corresponding dictionary at every iteration of simplex is shown in the top right. If you toggle between two iterations, you can see the dictionary form for both the previous and next LP at the same time.

Q19: Starting from point (0,0), by how much can you increase x_1 before the point is no longer feasible? Which constraint do you *hit* first? Does this match what you found algebraically?

A: You can increase x_1 by 4 before hitting constraint 3, which matches what I found algebraically.

Q20: Which variable will be the next increasing variable and why? (Hint: Look at the dictionary form LP at iteration 1)

A: x_2 will be increasing next, as it is the only x with a positive coefficient in the max expression.

Q21: Visually, which constraint do you think is the most limiting constraint? How much can x_2 increase? Give the corresponding feasible solution and its objective value of the next dictionary form LP. (Hint: hover over the feasible points to see information about them.)

A: Visually, the x_5 seems the most limiting as it is the first to intersect with the path of x_2 increasing. The corresponding feasible solution would be (4,1,0,5,0,1), with objective value 23.

$max \quad 20 - 5x_3 + 3(1 - 2x_3 - x_5) = 23 + x_3 - 3x_5$

$x_1 = 4 - x_3$

$x_2 = 1 + 2x_3 - x_5$

$x_4 = 6 - (1 + 2x_3 - x_5) = 5 - 2x_3 + x_5$

$x_6 = 3 + 3x_3 - 2(1 + 2x_3 - x_5) = 1 - x_3 + 2x_5$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Q22: Move the slider to see the next iteration of simplex. Was your guess from **Q21** correct? If not, describe how your guess was wrong.

A: My guess was correct.

Q23: Look at the dictionary form LP after the second iteration of simplex. What is the increasing variable? Identify the most limiting constraint graphically and algebraically. Show your work and verify they are the same constraint. In addition, give the next feasible solution and its objective value.

A: The increasing variable is now x_3, with limiting constraint x_6. The next solution is (3,3,1,5,0,0), with objective value 24. $max \quad 23 + (1 - x_6 + 2x_5) - 3x_5 = 24 - x_6 - x_5$

$x_1 = 4 - (1 - x_6 + 2x_5) = 3 + x_6 - 2x_5$

$x_2 = 1 + 2(1 - x_6 + 2x_5) - x_5 = 3 - 2x_6 + 4x_5$

$x_3 = 1 - x_6 + 2x_5$

$x_4 = 5 - 2(1 - x_6 + 2x_5) + x_5 = 3 + 2x_6 - 3x_5$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Q24: Is the new feasible solution you found in **Q23** optimal? (Hint: Look at the dictionary form LP)

A: Yes, as the coefficients of all x in the max expression are either 0 or negative.

Q25: In **Q21** and **Q23**, how did you determine the most limiting constraint graphically?

A: I traveled along the previous constraining line, and the first intersection is the most limiting constraint.

(BONUS): In 2D, we can increase a variable until we hit a 2D line representing the most limiting constraint. What would be the analagous situation in 3D?

A: The analogous situation in 3D would be to travel until we hit a 3D plane representing the most limiting constraint.

Part III: Geometrical Interpretation of the Dictionary

We have seen how the simplex algorithm transforms an LP from one dictionary form to another. Each dictionary form has a corresponding dictionary defined by the variables on the LHS of the constraints. Furthermore, each dictionary form has a corresponding feasible solution obtained by setting all non-dictionary variables to 0 and the dictionary variables to the constants on the RHS. In this section, we will explore the geometric interpretation of a dictionary.

In [ ]:
```
lp = gilp.examples.ALL_INTEGER_2D_LP # get LP example
gilp.simplex_visual(lp, initial_solution=np.array([[0],[0]])).show() # visualize it
```

Recall, we can hover over the corner points of the feasible region. **BFS** indicates the feasible solution corresponding to that point. For example, (7,0,6,9,0) means $x_1 = 7, x_2 = 0, x_3 = 6, x_4 = 9$, and $x_5 = 0$. **B** gives the indices of the variables "being defined" in that dictionary – that is, the variables that are on the LHS of the constraints. For simplicity, we will just say these variables are *in the dictionary*. For example, if **B** = $(1, 3, 4)$, then x_1, x_3, and x_4 are in the dictionary. Lastly, the objective value at that point is given.

Q26: Hover over the point (7,6) where $x_1 = 7$ and $x_2 = 6$. What is the feasible solution at that point ?

A:

We have a notion of *slack* for an inequality constraint. Consider the constraint $x_1 \geq 0$. A feasible solution where $x_1 = 7$ has a slack of 7 in this constraint. Consider the constraint $2x_1 + 1x_2 \leq 20$. The feasible solution with $x_1 = 7$ and $x_2 = 6$ has a slack of 0 in this constraint.

Q27: What is the slack in constraint $1x_1 + 1x_2 \leq 16$ when $x_1 = 7$ and $x_2 = 6$?

A:

Q28: Look at the constraint $2x_1 + 1x_2 \leq 20$. After rewriting in dictionary form, the constraint is $x_3 = 20 - 2x_1 - 1x_2$. What does x_3 represent?

A: