Branch & Bound and Knapsack Lab

Objectives

- Preform the branch and bound algorithm
- Apply branch and bound to the knapsack problem
- Understand the geometry of the branch and bound algorithm

Brief description: In this lab, we will try solving an example of a knapsack problem with the branch-and-bound algorithm. We will also see how adding a cutting plane helps in reducing the computation algorithm. Lastly, we will explore the geometry of the branch and bound algorithm.

```
In [2]:  # imports -- don't forget to run this cell
         import pandas as pd
         import gilp
         from gilp.visualize import feasible_integer_pts
         from ortools.linear_solver import pywraplp as OR
```

Part 1: Branch and Bound Algorithm

Recall that the branch and bound algorithm (in addition to the simplex method) allows us to solve integer programs. Before applying the branch and bound algorithm to the knapsack problem, we will be some core ideas. Furthermore, we will identify a helpful property that will make branch and bound terminate quicker later in the lab!

Q1: What are the different ways a node can be fathomed during the branch and bound algorithm? Describe each.

A: A node can be fathomed when the value is higher than the upper bound or lower than the lower bound.

Q2: Suppose you have a maximization integer program and you solve its linear program relaxation. What does the LP-relaxation optimal value tell you about the IP optimal value? What if it is a minimizat

A: For maximization problem, LP relaxation optimal value is the upper bound to the IP optimal value (unless the LP relaxation optimal value is an integer, in that case that is the optimal solution). For mini optimal would be the lower bound for IP optimal.

Q3: Assume you have a maximization integer program with all integral coeffects in the objective function. Now, suppose you are running the branch and bound algorithm and come across a node with 44.5. The current incumbent is 44. Can you fathom this node? Why or why not?

**A:** We can fathom because it is higher than the optimal value of 44 (IP).

Q4: If the optimal solution to the LP relaxation of the original program is integer, then you have found an optimal solution to your integer program. Explain why this is true.

A: LP relaxation solution provides an upper bound to the IP solution, so if that's an integer, it'd mean we found an optimal solution to our integer problem.

Q5: If the LP is infeasible, then the IP is infeasible. Explain why this is true.

A: When the found value is greater than the upper bound or lower than the lower bound. All the IP solutions are feasible for LP so it can't be the other way around.

The next questions ask about the following branch and bound tree. If the solution was not integral, the fractional x_i that was used to branch is given. If the solution was integral, it is denoted *INT*. In the o branch and bound, you are looking at the node with the *****.
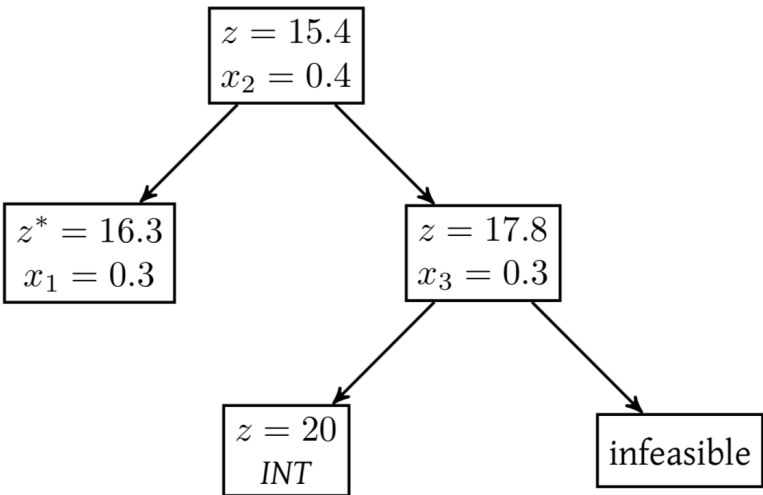
Q6: Can you determine if the integer program this branch and bound tree is for is a minimization or maximixation problem? If so, which is it?

A: This integer program is for a minimization problem. IP >= LP so we're trying to get closer to the LP.

Hint: For **Q7-8**, you can assume integral coeffects in the objective function.

Q7: Is the current node (marked z^*) fathomed? Why or why not? If not, what additional constraints should be imposed for each of the next two nodes?

A: The current node is not fathomed. Greater than or equal to 17, less than or equal to 16.

Q8: Consider the nodes under the current node (where $z = 16.3$). What do you know about the optimal value of these nodes? Why?

A: The optimal value of these nodes must be greater than or equal to 16.3 since that sets a lower bound.

Part 2: The Knapsack Problem

In this lab, you will solve an integer program by branch and bound. The integer program to be solved will be a knapsack problem.

Knapsack Problem: We are given a collection of n items, where each item $i = 1, \ldots, n$ has a weight w_i and a value v_i. In addition, there is a given capacity W, and the aim is to select a maximum valu that has a total weight at most W. Note that each item can be brought at most once.

$$\max \quad \sum_{i=1}^{n} v_i x_i$$
$$\text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \le W$$
$$0 \le x_i \le 1, \text{integer}, i = 1, \ldots, n$$

Consider the following data which we import from a CSV file:

```
In [3]: data = pd.read_csv('knapsack_data_1.csv', index_col=0)
        data
```

Out[3]:

item	value	weight
1	50	10
2	30	12
3	24	10
4	14	7
5	12	6
6	10	7
7	40	30

and $W = 18$.

Q9: Are there any items we can remove from our input to simplify this problem? Why? If so, replace `index` with the item number that can be removed in the code below. Hint: how many of each item c
take?

A: We can remove the column for item 7 to simplify the problem because W=18 and there's no way item 7 can be included with a weight of 30.

```
In [7]: #TODO: replace index
        data = data.drop(7)
```

Q10: If we remove item 7 from the knapsack, it does not change the optimal solution to the integer program. Explain why.

A: W=18 and there's no way item 7 can be included since it has a weight of 30.

Q11: Consider removing items i such that $w_i > W$ from a knapsack input. How does the LP relaxation's optimal value change?

A: The LP relaxation's optimal value would not change after all.

In **Q10-11**, you should have found that removing these items removes feasible solutions from the linear program but does not change the integer program. This is desirable as the gap between the optim
can become smaller. By adding this step, branch and bound may terminate sooner.

Recall that a branch and bound node can be fathomed if its bound is no better than the value of the best feasible integer solution found thus far. Hence, it helps to have a good feasible integer solution a
(so that we stop needless work). To do this, we can first try to construct a good feasible integer solution by a reasonable heuristic algorithm before starting to run the branch and bound procedure.

In designing a heuristic for the knapsack problem, it is helpful to think about the value per unit weight for each item. We compute this value in the table below.

```
In [8]: data['value per unit weight'] = (data['value'] / data['weight']).round(2)
        data
```

Out[8]:

item	value	weight	value per unit weight
1	50	10	5.00
2	30	12	2.50
3	24	10	2.40
4	14	7	2.00
5	12	6	2.00
6	10	7	1.43

Q12: Design a reasonable heuristic for the knapsack problem. Note a heuristic aims to find a decent solution to the problem (but is not necessarily optimal).

A: item 1, item 4.

Q13: Run your heuristic on the data above to compute a good feasible integer solution. Your heuristic should generate a feasible solution with a value of 64 or better. If it does not, try a different heuristic

A: The value = how good the values are. The value = 50+14 = 64.

We will now use the branch and bound algorithm to solve this knapsack problem! First, let us define a mathematical model for the linear relaxation of the knapsack problem.

Q14: Complete the model below.

```
In [43]: def Knapsack(table, capacity, integer = False):
             """Model for solving the Knapsack problem.

             Args:
                 table (pd.DataFrame): A table indexd by items with a column for value and weight
                 capcity (int): An integer-capacity for the knapsack
                 integer (bool): True if the variables should be integer. False otherwise.
             """
             ITEMS = list(table.index)      # set of items
             v = table.to_dict()['value']   # value for each item
             w = table.to_dict()['weight']  # weight for each item
             W = capacity                   # capacity of the knapsack

             # define model
             m = OR.Solver('knapsack', OR.Solver.CBC_MIXED_INTEGER_PROGRAMMING)

             # decision variables
             x = {}
             for i in ITEMS:
                 if integer:
                     x[i] = m.IntVar(0, 1, 'x_%d' % (i))
                 else:
                     x[i] = m.NumVar(0, 1, 'x_%d' % (i))

             # define objective function here
             m.Maximize(sum(v[i]*x[i] for i in ITEMS))

             # TODO: Add a constraint that enforces that weight must not exceed capacity
             # recall that we add constraints to the model using m.Add()
             m.Add(sum(w[i]*x[i] for i in ITEMS) <= W)

             return (m, x)  # return the model and the decision variables
```

```
In [44]:  # You do not need to do anything with this cell but make sure you run it!
          def solve(m):
              """Used to solve a model m."""
              m.Solve()

              print('Objective =', m.Objective().Value())
              print('iterations :', m.iterations())
              print('branch-and-bound nodes :',m.nodes())

              return ({var.name() : var.solution_value() for var in m.variables()})
```

We can now create a linear relaxation of our knapsack problem. Now, `m` represents our model and `x` represents our decision variables.

```
In [45]:  m, x = Knapsack(data, 18)
```

We can use the next line to solve the model and output the solution

```
In [46]:  solve(m)
```

```
Objective = 70.0
iterations : 0
branch-and-bound nodes : 0
```

```
Out[46]:  {'x_1': 1.0,
           'x_2': 0.6666666666666667,
           'x_3': 0.0,
           'x_4': 0.0,
           'x_5': 0.0,
           'x_6': 0.0}
```

Q15: How does this optimal value compare to the value you found using the heuristic integer solution?

A: This one is better.

Q16: Should this node be fathomed? If not, what variable should be branched on and what additional constraints should be imposed for each of the next two nodes?

A: The node should not be fathomed because we can branch out from x2 since it's a non-integer.

After constructing the linear relaxation model using `Knapsack(data1, 18)` we can add additional constraints. For example, we can add the constraint $x_2 \leq 0$ and solve it as follows:

```
In [47]:  m, x = Knapsack(data, 18)
          m.Add(x[2] <= 0)
          solve(m)
```

```
Objective = 69.2
iterations : 0
branch-and-bound nodes : 0
```

```
Out[47]:  {'x_1': 1.0, 'x_2': 0.0, 'x_3': 0.8, 'x_4': 0.0, 'x_5': 0.0, 'x_6': 0.0}
```

NOTE: The line `m, x = Knapsack(data1, 18)` resets the model `m` to the LP relaxation. All constraints from branching have to be added each time.

Q17: Use the following cell to compute the optimal value for the other node you found in **Q16**.

```
In [50]:  m, x = Knapsack(data, 18)
          m.Add(x[2] >= 1)
          solve(m)
```

```
Objective = 60.0
iterations : 0
branch-and-bound nodes : 0
```

```
Out[50]:  {'x_1': 0.6000000000000001,
           'x_2': 1.0,
           'x_3': 0.0,
           'x_4': 0.0,
           'x_5': 0.0,
           'x_6': 0.0}
```

Q18: What was the optimal value? Can this node be fathomed? Why? (Hint: In **Q13**, you found a feasible integer solution with value 64.)

A: Our optimal value is 60 and this node can be vathmoed becuase we have a better feasible value of 64 from Q13.

If we continue running the branch and bound algorithm, we will eventually reach the branch and bound tree below where the z^* indictes the current node we are looking at.
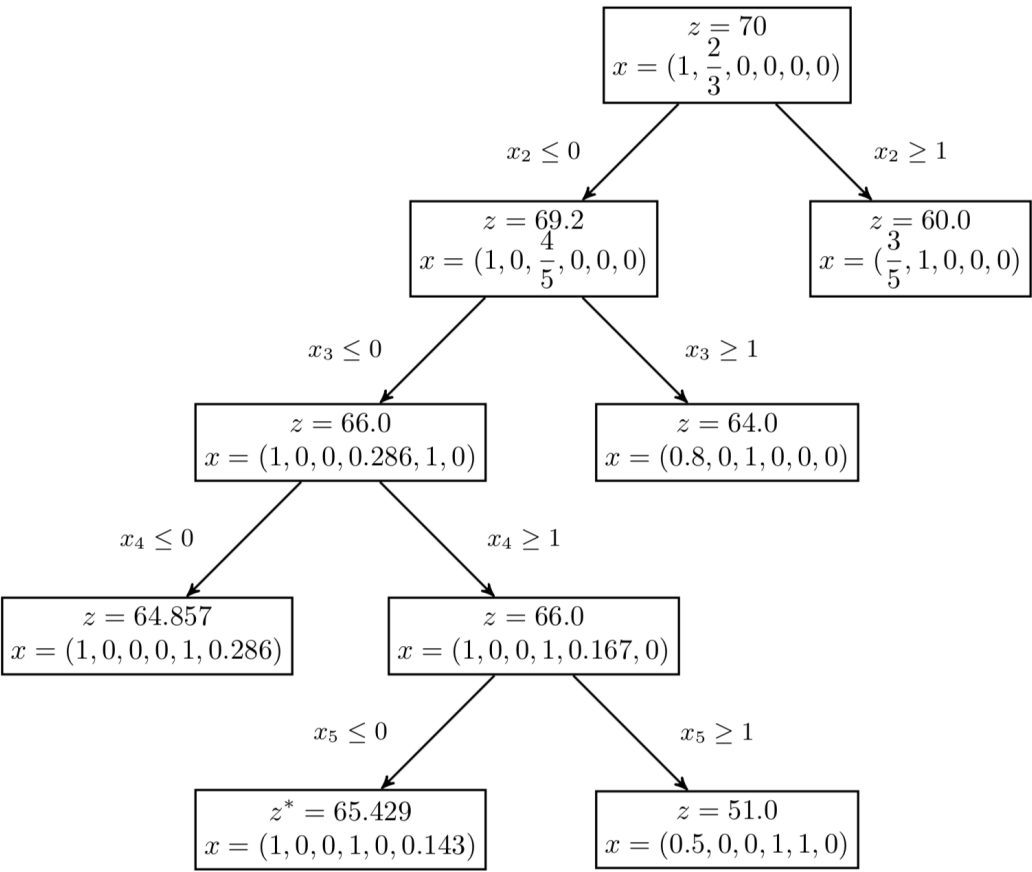
Q19: The node with $z = 64.857$ was fathomed. Why are we allowed to fathom this node? (Hint: think back to **Q3**)

A: LOwer bound is 64 and since it's 64.857, we can fathom this node.

Q20: Finish running branch and bound to find the optimal integer solution. Use a separate cell for each node you solve and indicate if the node was fathomed with a comment. (Hint: Don't forget to inclu[...]
further up in the branch and bound tree.)

```
In [54]:  # Template
          m, x = Knapsack(data, 18)
          # Add constraints here
          m, x = Knapsack(data, 18)
          # Add constraints here
          m.Add(x[2] <= 0)
          m.Add(x[4] >= 1)
          m.Add(x[5] <= 0)
          m.Add(x[6] <= 0)

          solve(m)
          # fathomed?
```

```
Objective = 66.4
iterations : 0
branch-and-bound nodes : 0
```

```
Out[54]: {'x_1': 1.0,
          'x_2': 0.0,
          'x_3': 0.10000000000000009,
          'x_4': 1.0,
          'x_5': 0.0,
          'x_6': 0.0}
```

```
In [ ]:
```

```
In [ ]:
```

A: When added x6 >= 1, objective value: 44 (items, 1, 3, 4, 6). x6 <= 0, obj value: 66.4 (items 1, 3, 4, 6)

Q21: How many nodes did you have to explore while running the branch and bound algorithm?

A: 4 nodes.

In the next section, we will think about additional constraints we can add to make running branch and bound quicker.

Part 3: Cutting Planes

In general, a cutting plane is an additional constraint we can add to an integer program's linear relaxation that removes feasible linear solutions but does not remove any integer feasible solutions. This is useful for solving integer programs! Recall many of the problems we have learned in class have something we call the "integrality property". This is useful because it allows us to ignore the integrality constraint since to reach an integral solution. By cleverly adding cutting planes, we strive to remove feasible linear solutions (without removing any integer feasible solutions) such that the optimal solution to the linear rel
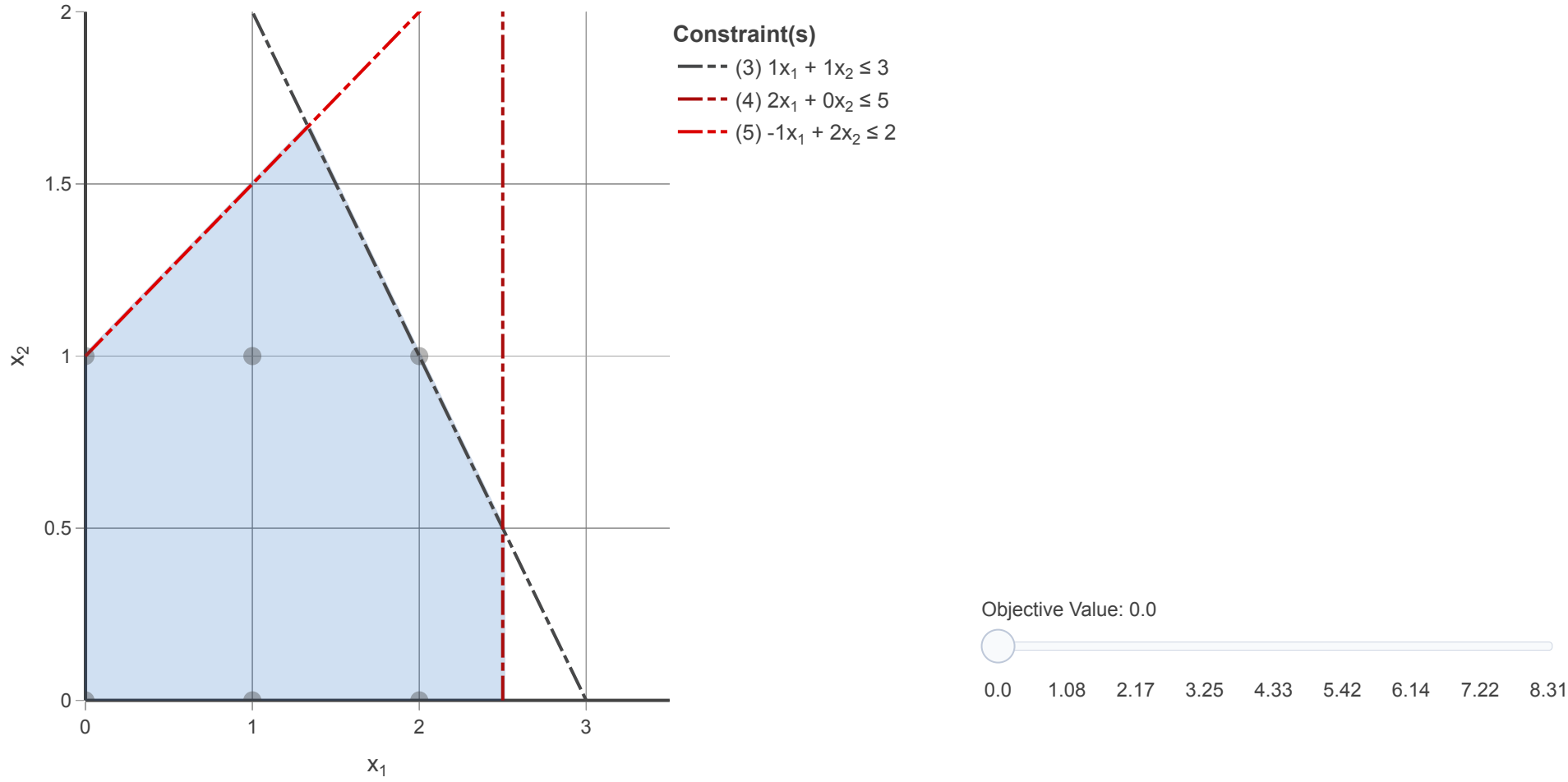
Conisder an integer program whose linear program releaxation is

$$
\begin{aligned}
\max \quad & 2x_1 + x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 3 \\
& 2x_1 \leq 5 \\
& -x_1 + 2x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

We can define this linear program and then visualize its feasible region. The integer points have been highlighted.

```
In [55]:  lp = gilp.LP([[1,1],[2,0],[-1,2]],
                       [3,5,2],
                       [2,1])
          fig = gilp.lp_visual(lp)
          fig.set_axis_limits([3.5,2])
          fig.add_trace(feasible_integer_pts(lp, fig))
          fig
```

Geometric Interpretation of LPs

Constraint(s)
- (3) $1x_1 + 1x_2 \leq 3$
- (4) $2x_1 + 0x_2 \leq 5$
- (5) $-1x_1 + 2x_2 \leq 2$

Objective Value: 0.0

0.0   1.08   2.17   3.25   4.33   5.42   6.14   7.22   8.31

Q22: List every feasible solution to the integer program.

**A:** (0,0), (0,1), (1,0), (2,0), (1,1), (2,1)

Q23: Is the constraint $x_2 \leq 1$ a cutting plane? Why? (Hint: Would any feasible integer points become infeasible? What about feasible linear points?)

A: x2 is a cutting plane because it cuts the little triangle on the top. It limits the LP feasible region.

Let's add this cutting plane to the LP relaxation!

In [57]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2],[0,1]],
             [3,5,2,1],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```

Geometric Interpretation of LPs
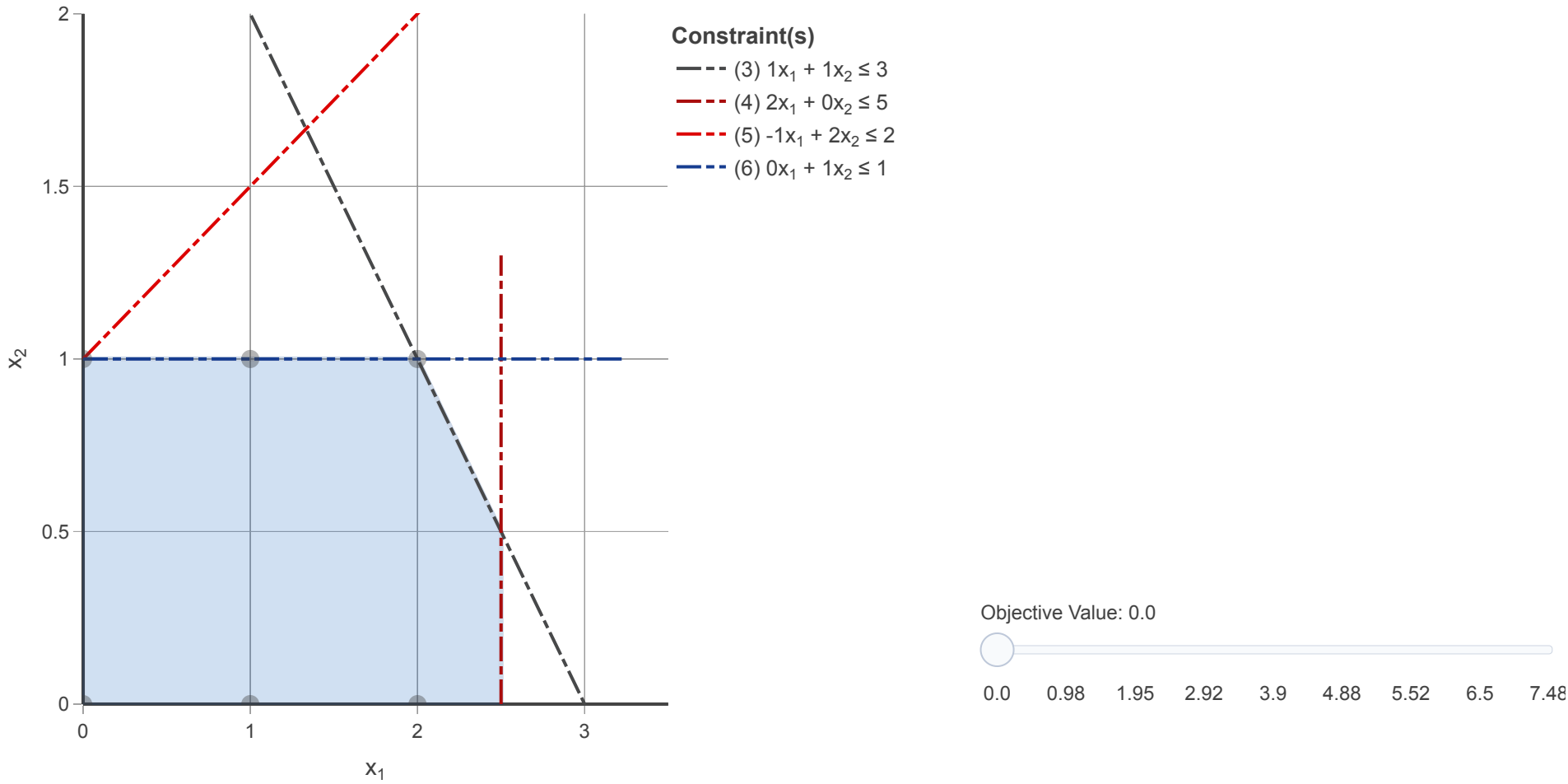
Q24: Is the constraint $x_1 \leq 3$ a cutting plane? Why?

A: The constraint is not a cutting plane because we are rejecting any solutions there.

Q25: Can you provide another cutting plane? If so, what is it?

A: x1 <= 2

Let's look at the feasible region after adding the cutting plane from **Q23** and one of the possible answers from **Q25**. Notice the optimal solution to the LP relaxation is now integral!

In [58]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2],[0,1],[1,0]],
             [3,5,2,1,2],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```

Geometric Interpretation of LPs
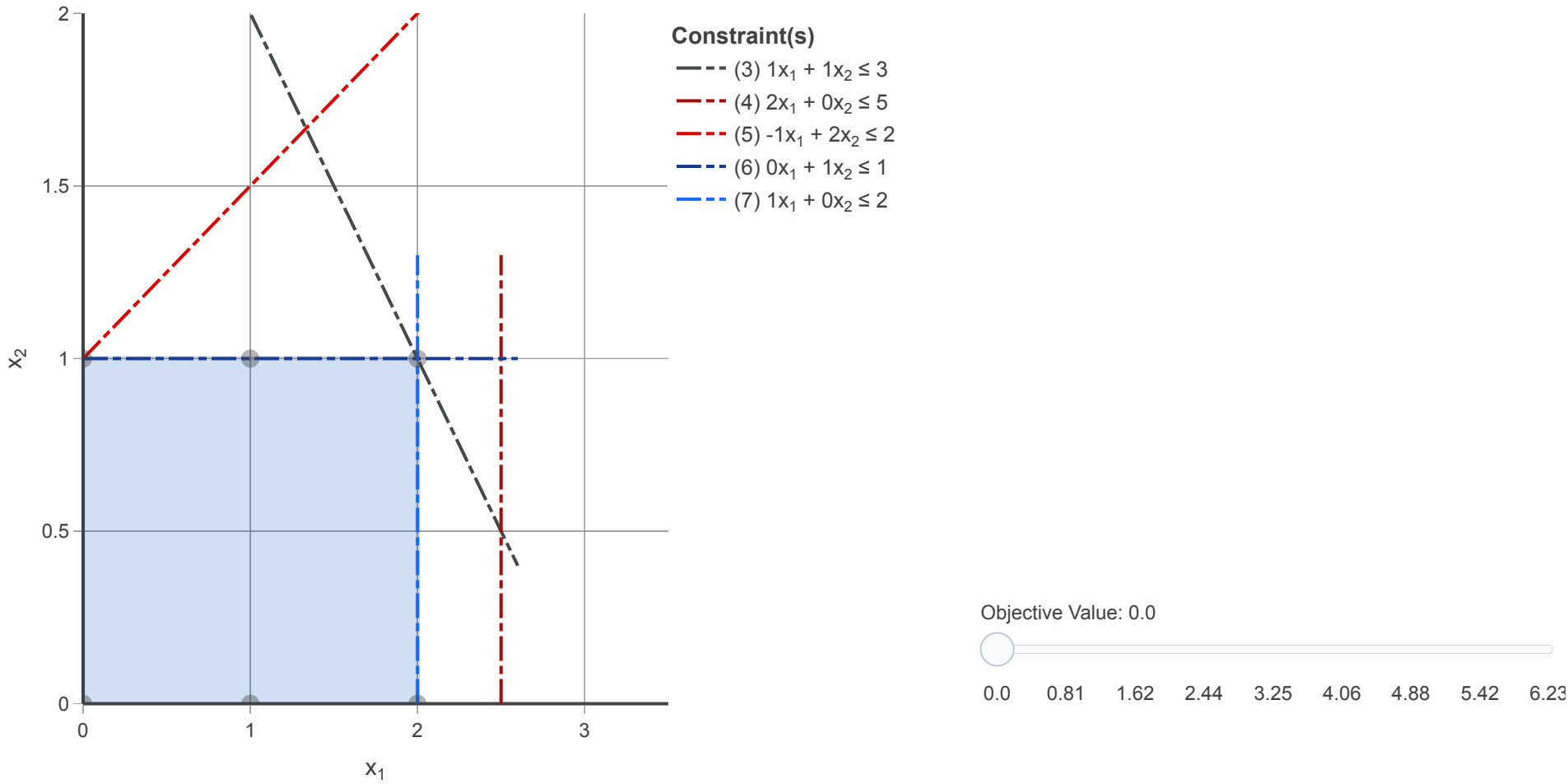
Let's try applying what we know about cutting planes to the knapsack problem! Again, recall our input was $W = 18$ and:

In [59]:
```
data
```

Out[59]:

item	value	weight	value per unit weight
1	50	10	5.00
2	30	12	2.50
3	24	10	2.40
4	14	7	2.00
5	12	6	2.00
6	10	7	1.43

Q26: Look at items 1, 2, and 3. How many of these items can we take simultaneously? Can you write a new constraint to capture this? If so, please provide it.

A: We can only take one of each item simultanouesly. x(1) + x(2) + x(3) <= 1

Q27: Is the constraint you found in **Q26** a cutting plane? If so, provide a feasible solution to the linear program relaxation that is no longer feasible (i.e. a point the constraint *cuts off*).

A: The constraint we found in Q26 is a cutting plane. A feasible solution (1,0.5,0,0,0,0)

Q28: Provide another cutting plane involving items 4,5 and 6 for this integer program. Explain how you derived it.

A: x(4) + x(5) + x(6) <= 2. We could put 2 in each.

Q29: Add the cutting planes from **Q26** and **Q28** to the model and solve it. You should get a solution in which we take items 1 and 4 and $\frac{1}{6}$ of item 5 with an objective value of 66.

```
In [60]: m, x = Knapsack(data, 18)
         # TODO: Add cutting planes here
         m.Add(x[1] + x[2] + x[3] <= 1)
         m.Add(x[4] + x[5] + x[6] <= 2)


         solve(m)
```
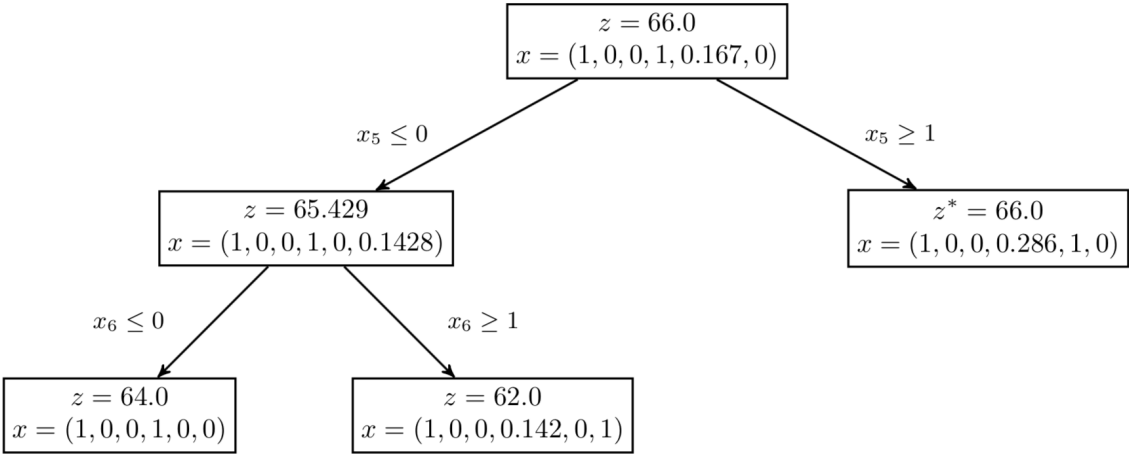
```
Objective = 66.0
iterations : 0
branch-and-bound nodes : 0
```

```
Out[60]: {'x_1': 1.0,
          'x_2': 0.0,
          'x_3': 0.0,
          'x_4': 1.0,
          'x_5': 0.16666666666666696,
          'x_6': 0.0}
```

Let's take a moment to pause and reflect on what we are doing. Recall from **Q9-11** that we dropped item 7 becuase its weight was greater than the capacity of the knapsack. Essentially we added the co
This constraint was a cutting plane! It eliminated some linear feasible solutions but no integer ones. By adding these two new cutting planes, we can get branch and bound to terminate earlier yet again!
generated cutting planes by inspection. However, there are more algorithmic ways to identify them (which we will ignore for now).

If we continue running the branch and bound algorithm, we will eventually reach the branch and bound tree below where the z^* indictes the current node we are looking at.

NOTE: Do not forget about the feasible integer solution our heuristic gave us with value 64.

Q30 Finish running branch and bound to find the optimal integer solution. Use a separate cell for each node you solve and indicate if the node was fathomed with a comment. Hint: Don't forget the cuttir
should be included in every node of the branch and bound tree.

```
In [68]: # Template
         m, x = Knapsack(data, 18)
         m.Add(x[1] + x[2] + x[3] <= 1)
         m.Add(x[4] + x[5] + x[6] <= 2)
         m.Add(x[5] >= 1)

         solve(m)
         # fathomed? no
```

```
Objective = 66.0
iterations : 0
branch-and-bound nodes : 0
```

```
Out[68]: {'x_1': 1.0,
          'x_2': 0.0,
          'x_3': 0.0,
          'x_4': 0.28571428571428586,
          'x_5': 1.0,
          'x_6': 0.0}
```

```
In [72]: m, x = Knapsack(data, 18)
         m.Add(x[1] + x[2] + x[3] <= 1)
         m.Add(x[4] + x[5] + x[6] <= 2)
         m.Add(x[5] >= 1)
         m.Add(x[4] <= 1)
         solve(m)
```

```
Objective = 66.0
iterations : 0
branch-and-bound nodes : 0
```

```
Out[72]: {'x_1': 1.0,
          'x_2': 0.0,
          'x_3': 0.0,
          'x_4': 0.28571428571428586,
          'x_5': 1.0,
          'x_6': 0.0}
```

```
In [77]: m, x = Knapsack(data, 18)
         m.Add(x[1] + x[2] + x[3] <= 1)
         m.Add(x[4] + x[5] + x[6] <= 2)
         m.Add(x[5] >= 1)
         m.Add(x[4] <= 0)
         m.Add(x[2] <= 1)
         m.Add(x[6] <= 1)
         solve(m)
```

```
Objective = 64.85714285714286
iterations : 0
branch-and-bound nodes : 0
```

```
Out[77]: {'x_1': 1.0,
          'x_2': 0.0,
          'x_3': 0.0,
          'x_4': 0.0,
          'x_5': 1.0,
          'x_6': 0.28571428571428586}
```

A: We fathom both nodes.

Q31: Did you find the same optimal solution? How many nodes did you explore? How did this compare to the number you explored previously?

A: 64 is the best solution. We explored 4 nodes. We explored more nodes than the previous branch and bound.

Part 4: Geometry of Branch and Bound

Previously, we used the `gilp` package to viusualize the simplex algorithm but it also has the functionality to visualize branch and bound. We will give a quick overview of the tool. Similar to `lp_visua`
`simplex_visual`, the function `bnb_visual` takes an `LP` and returns a visualization. It is assumed that every decision variable is constrained to be integer. Unlike previous visualizations, `bnb_vis`
of figures for each node of the branch and bound tree. Let's look at a small 2D example:

$$\begin{aligned}
\max \quad & 5x_1 + 8x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 6 \\
& 5x_1 + 9x_2 \leq 45 \\
& x_1, x_2 \geq 0, \quad \text{integral}
\end{aligned}$$

In [78]: `nodes = gilp.bnb_visual(gilp.examples.STANDARD_2D_IP)`

In [79]: `nodes[0].show()`

Geometric Interpretation of LPs
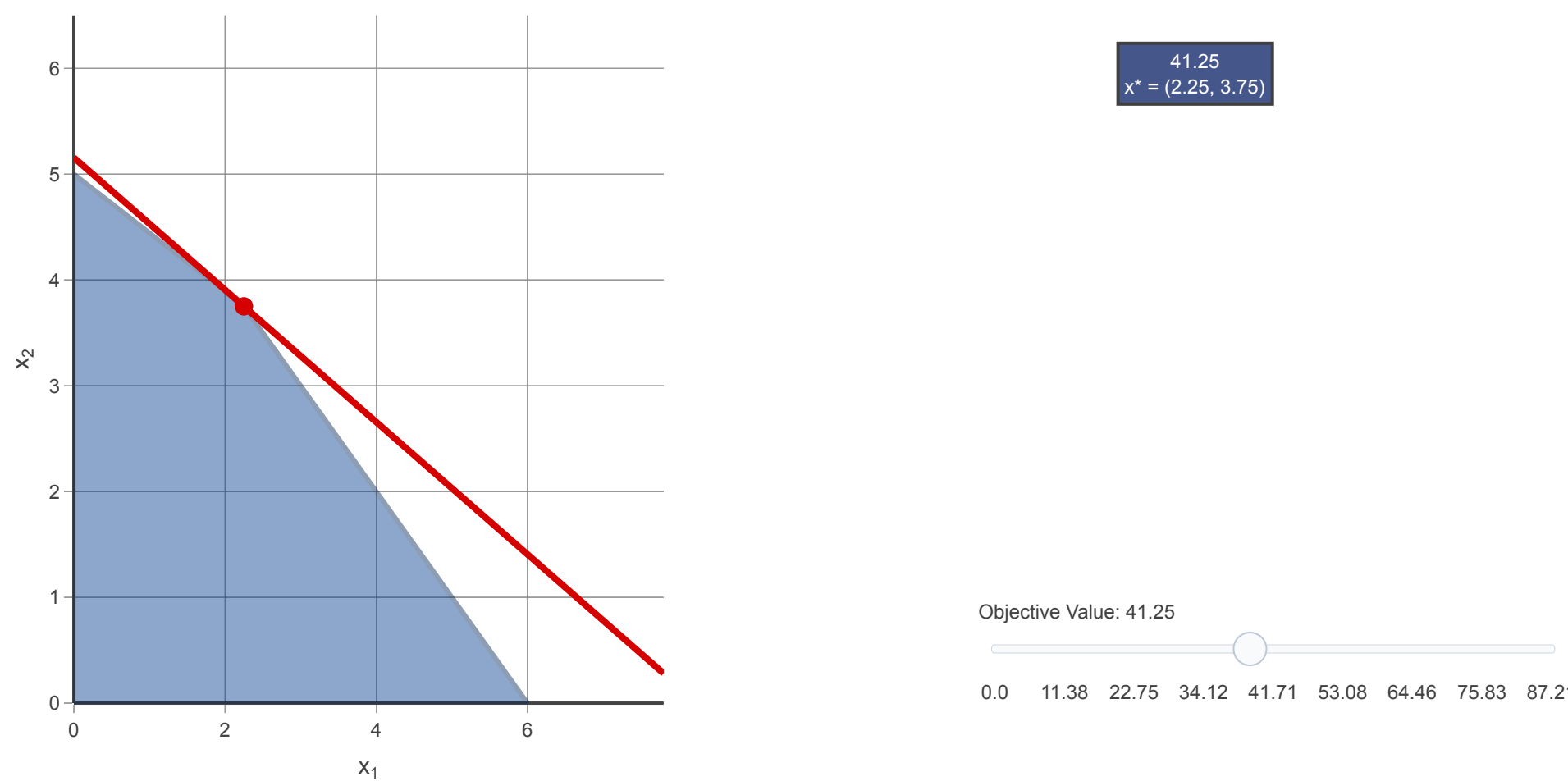
Run the cells above to generate a figure for each node and view the first node. At first, you will see the LP relaxation on the left and the root of the branch and bound tree on the right. The simplex path a also present.

Q32: Recall the root of a branch and bound tree is the unaltered LP relaxation. What is the optimal solution? (Hint: Use the objective slider and hover over extreme points).

A: The optimal value is 41.25. The objective solution is (2.25, 3.75).

Q33: Assume that we always choose the variable with the minimum index to branch on if there are multiple options. Write down (in full) each of the LPs we get after branching off the root node.

A: Shown in the picture.

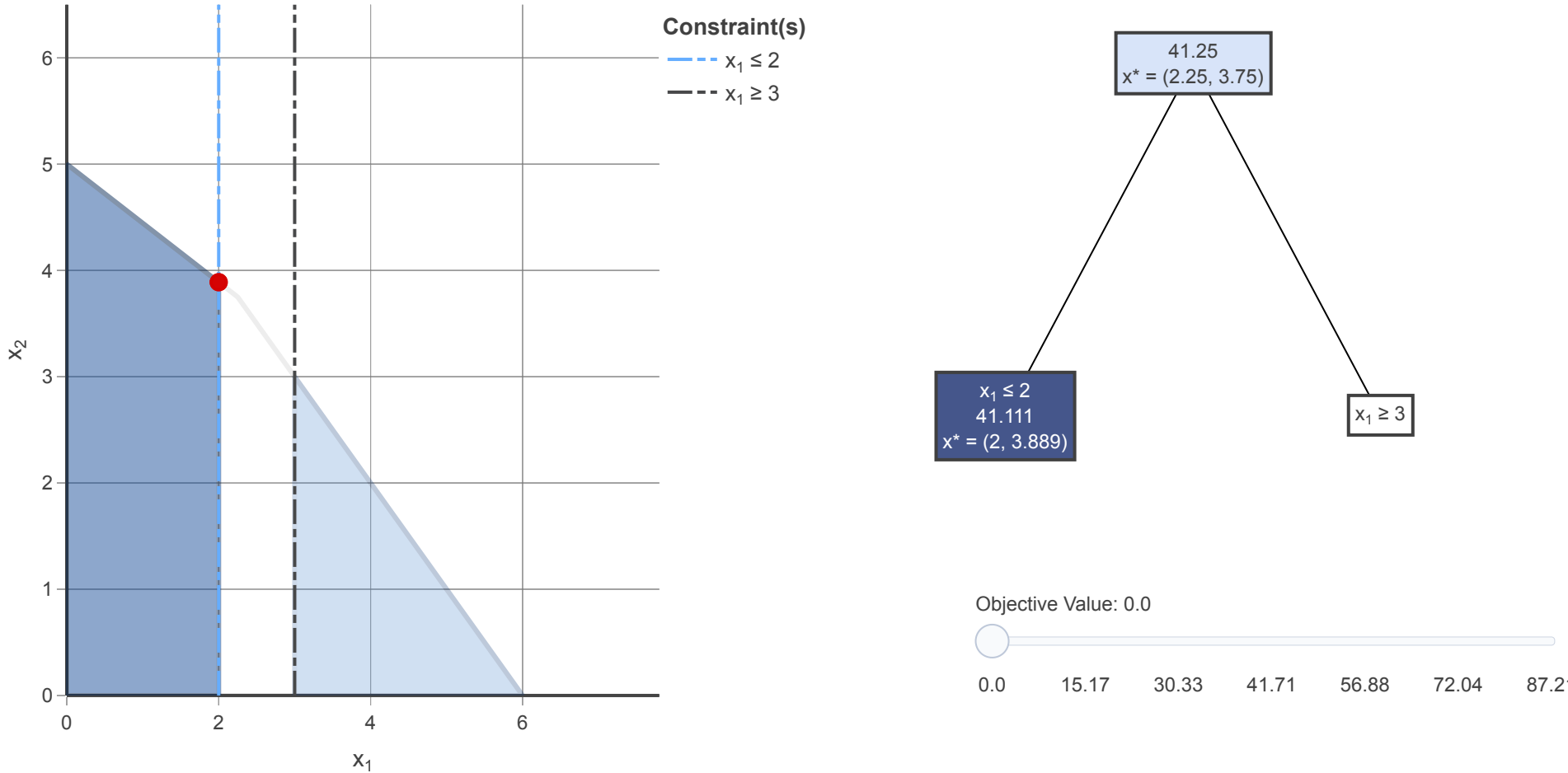**Q34:** Draw the feasible region to each of the LPs from **Q33** on the same picture.

A:

$$\max \quad 5x_1 + 8x_2$$

$$s.t \quad x_1 + x_2 \le 6$$

$$5x_1 + 9x_2 \le 45$$

$$x_1, x_2 \ge 0, \text{ integral}$$

$x_1 \le 2$

$x_1 \ge 3$

$x_2 \le 3$

$x_2 \ge 4$

x_2

5

4

3

$(2.25, 3.75)$

2   3   4   6

x_1
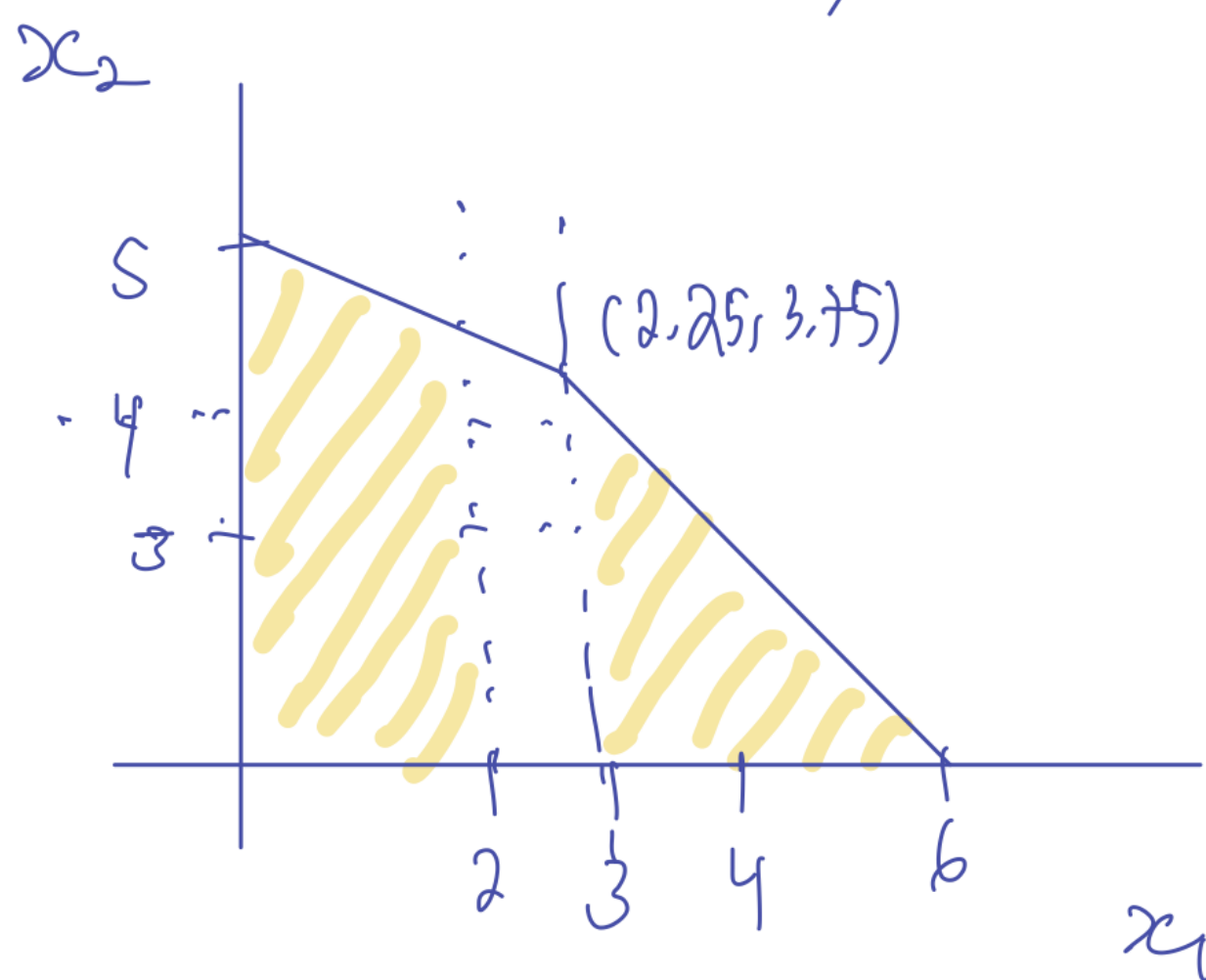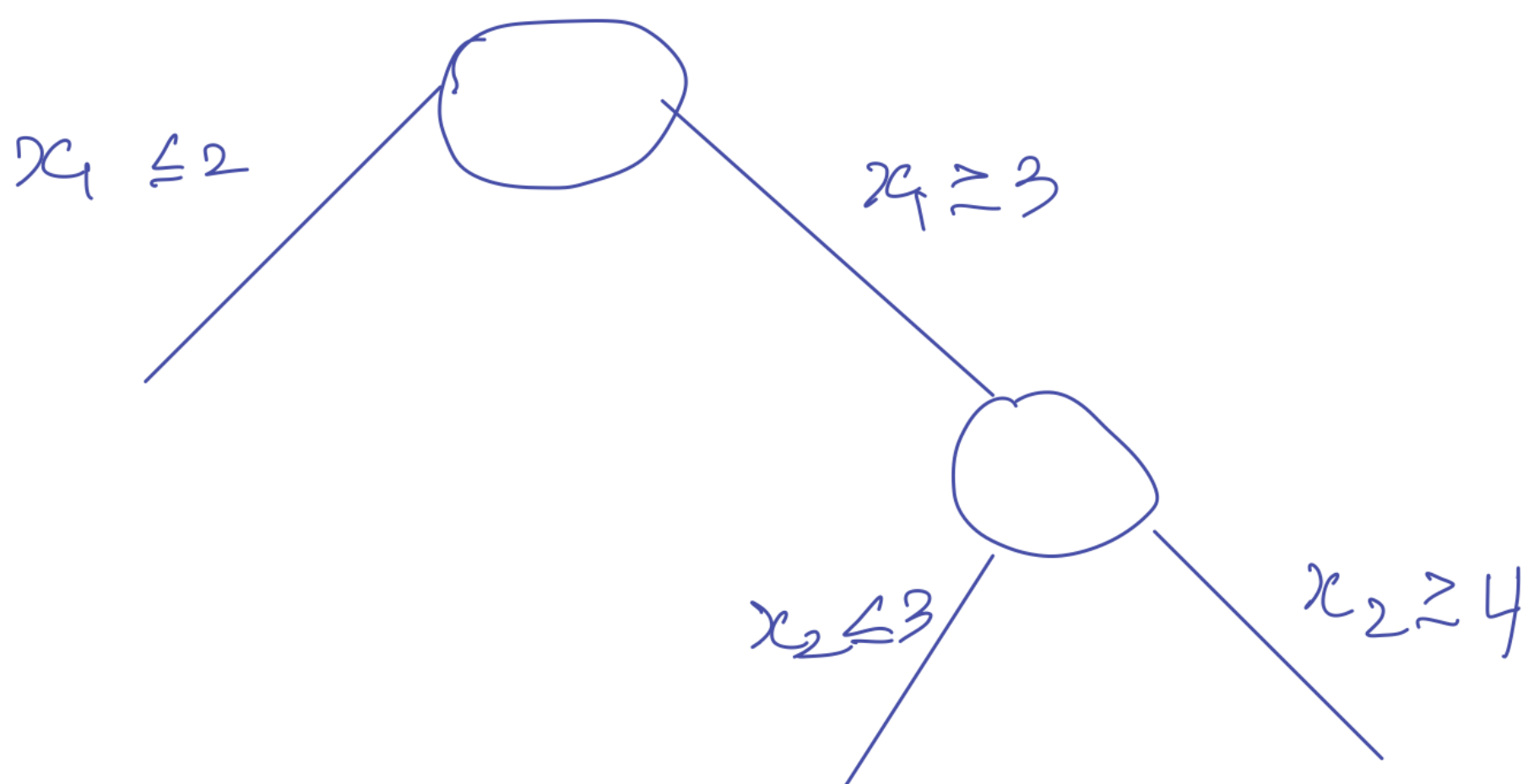
Run the following cell to see if the picture you drew in **Q34** was correct.

In [81]: `nodes[1].show()`

Geometric Interpretation of LPs

The outline of the original LP relaxation is still shown on the left. Now that we have eliminated some of the fractional feasible solutions, we now have 2 feasible regions to consider. The darker one is the f associated with the current node which is also shaded darker in the branch and bound tree. The unexplored nodes in the branch and bound tree are not shaded in.

Q35: Which feasible solutions to the LP relaxation are removed by this branch?

A: Solutions that have x1 between 2 and 3.

Q36: At the current (dark) node, what constraints will we add? How many feasible regions will the original LP relaxation be broken into?

A: x2 >= 4, x2 <= 3, 3 regions

In [82]: `nodes[2].show()`

Geometric Interpretation of LPs
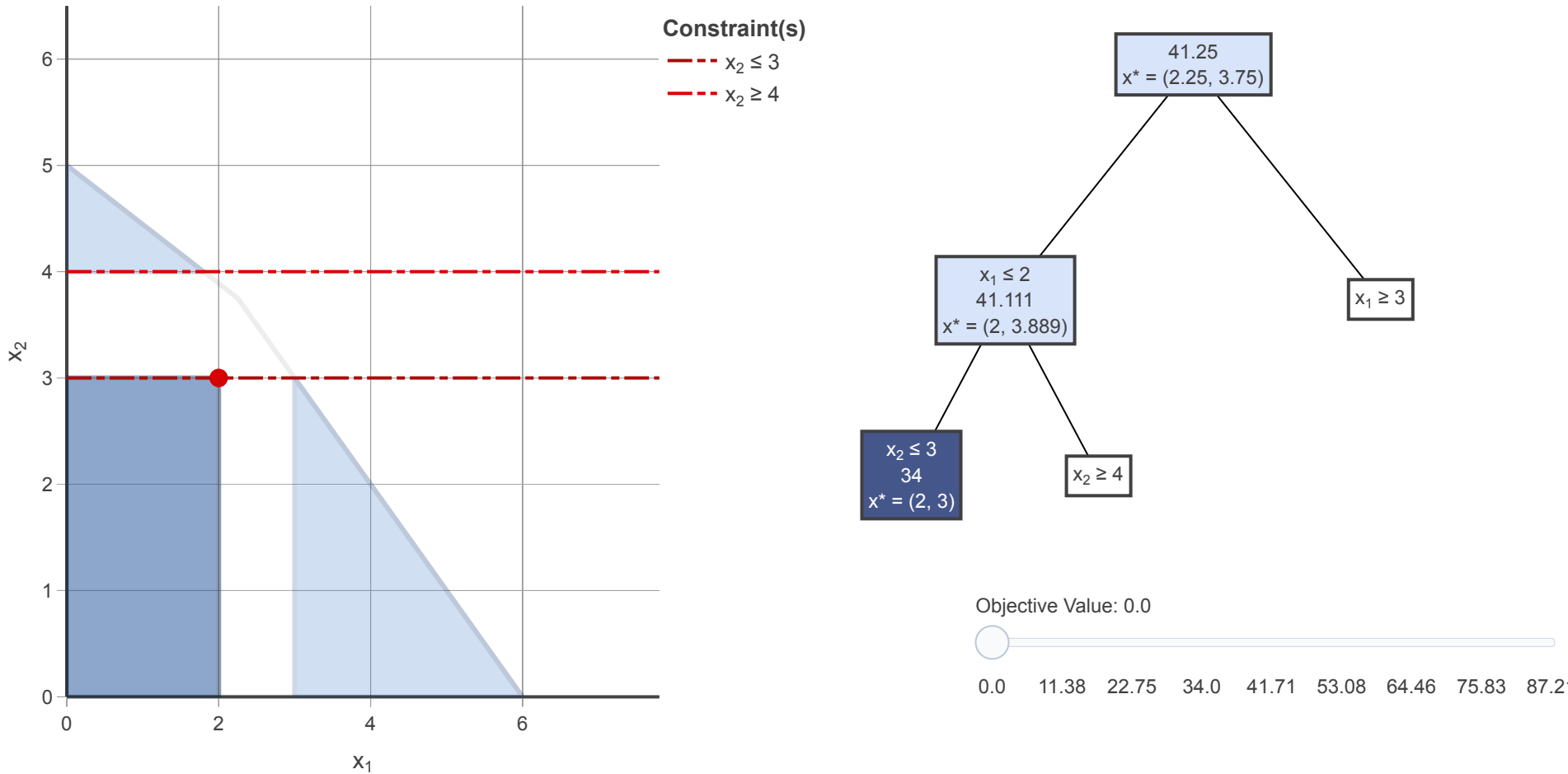
Q37: What is the optimal solution at the current (dark) node? Do we have to further explore this branch? Explain.

A: The optimal solution ins (2,3) with the objective value of 34.

Q38: Recall shaded nodes have been explored and the node shaded darker (and feasible region shaded darker) correspond to the current node and its feasible region. Nodes not shaded have not been nodes have not yet been explored?

A: 2 nodes (Becuase we broke it up into 2).

Q39: How many nodes have a degree of one in the branch and bound tree? (That is, they are only connected to one edge). These nodes are called leaf nodes. What is the relationship between the leaf n remaining feasible region?
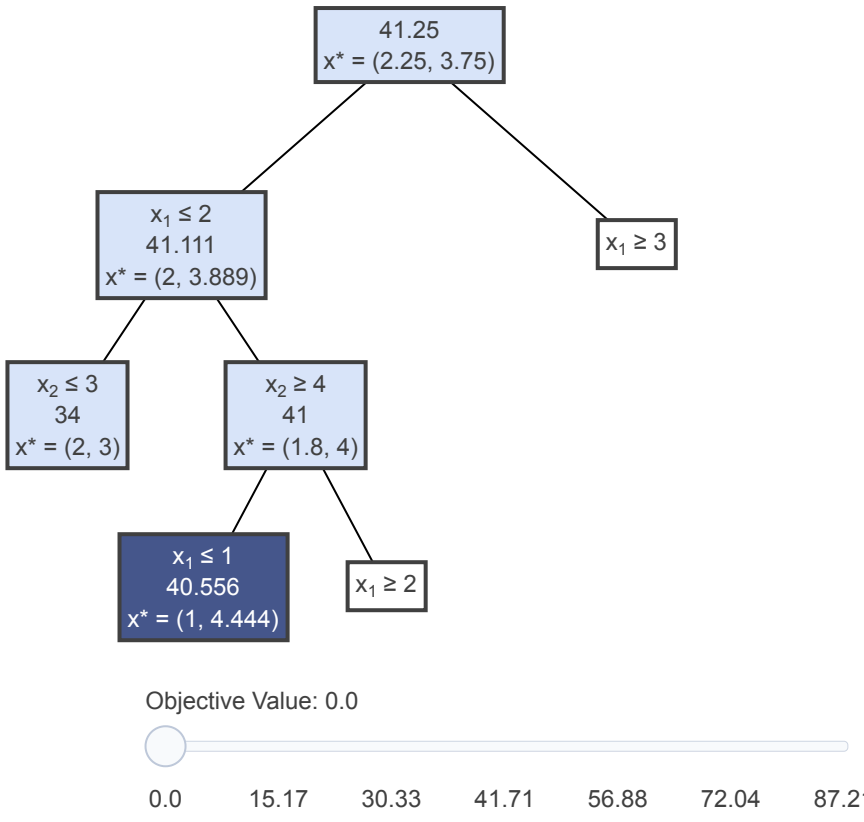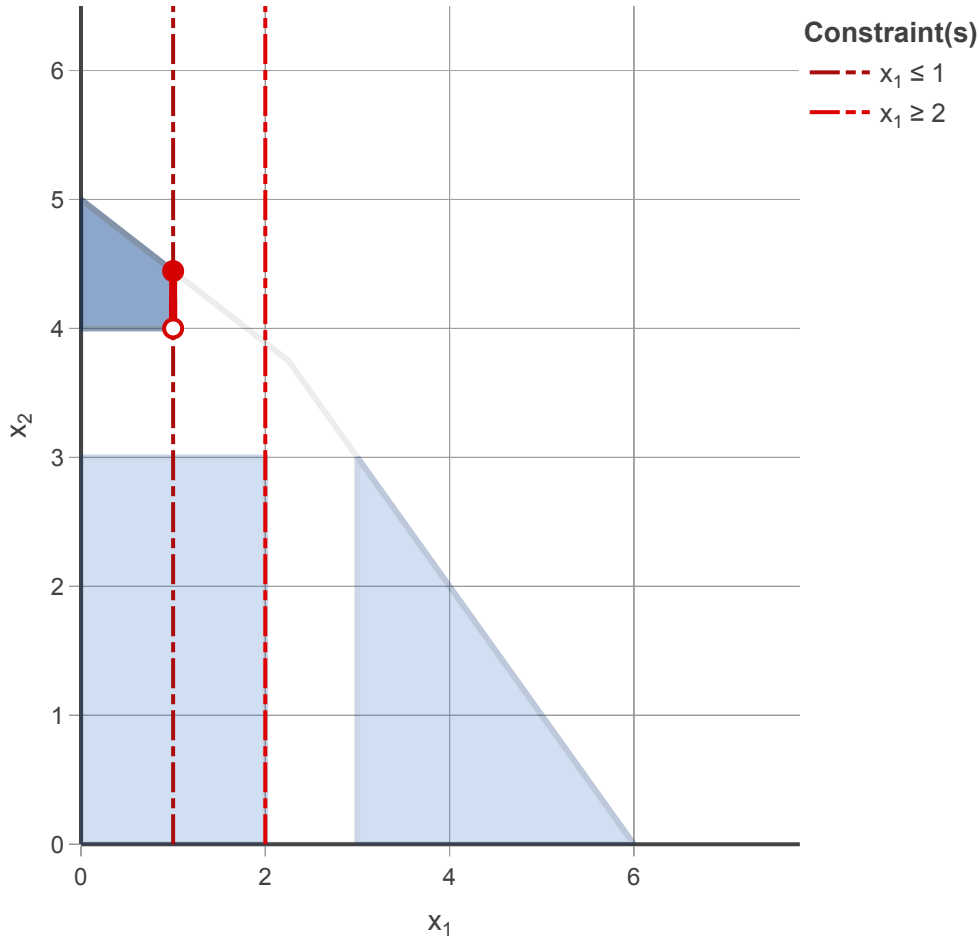
A: 3 nodes. Every region has one leaf node.

In [83]:   `# Show the next two iterations of the branch and bound algorithm`
`nodes[3].show()`
`nodes[4].show()`

Geometric Interpretation of LPs
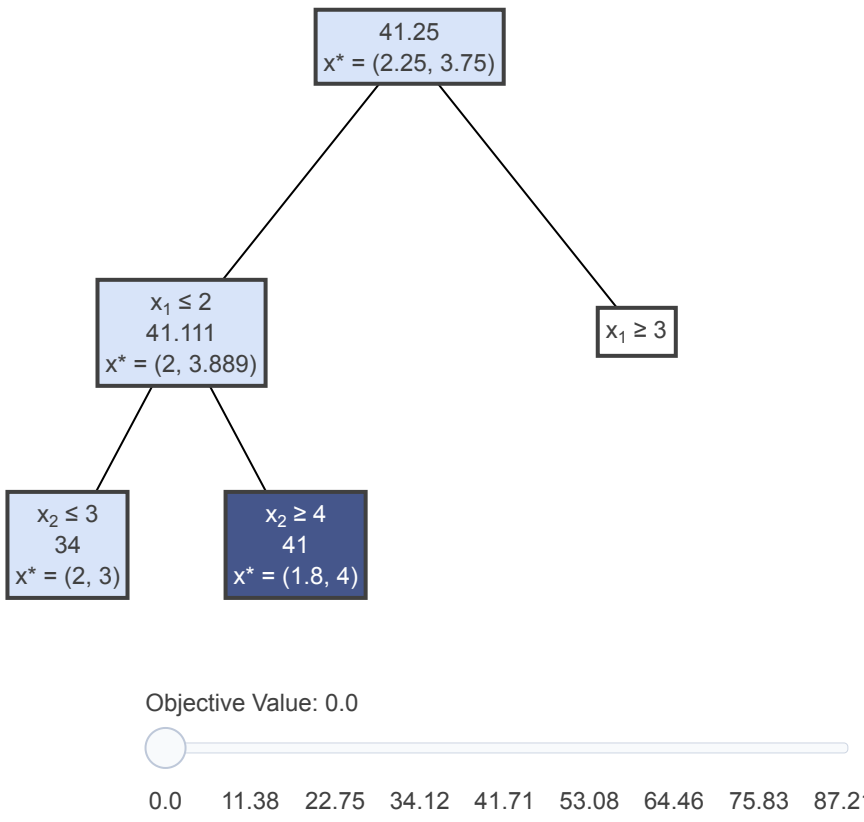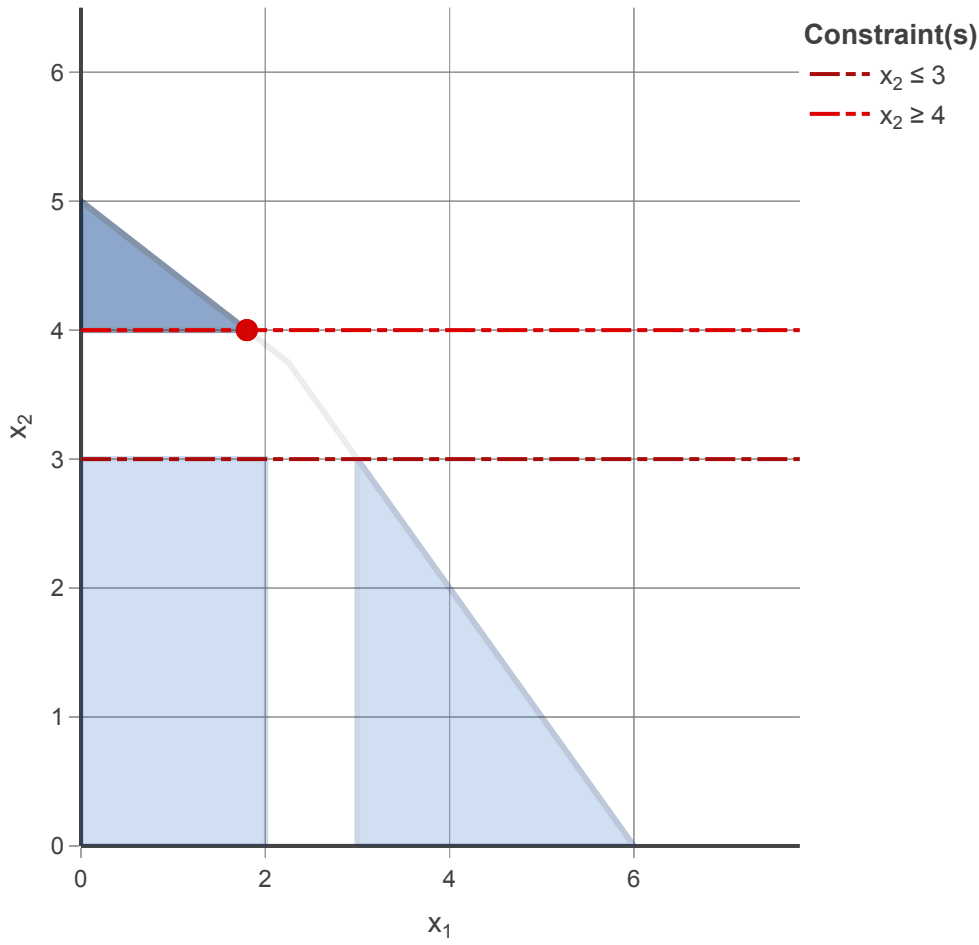
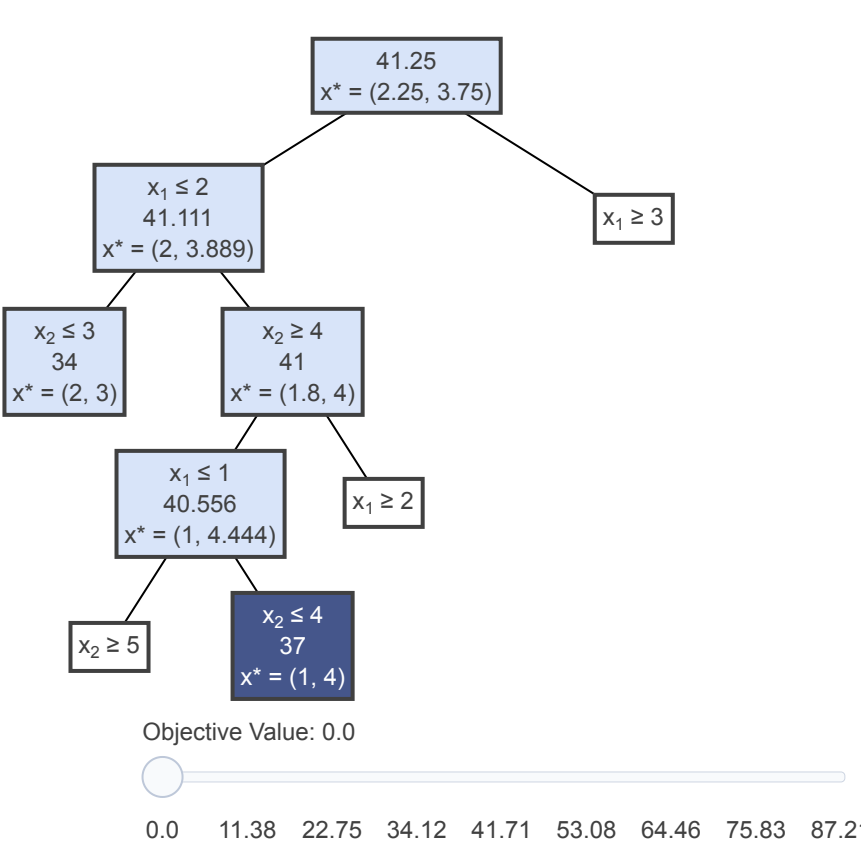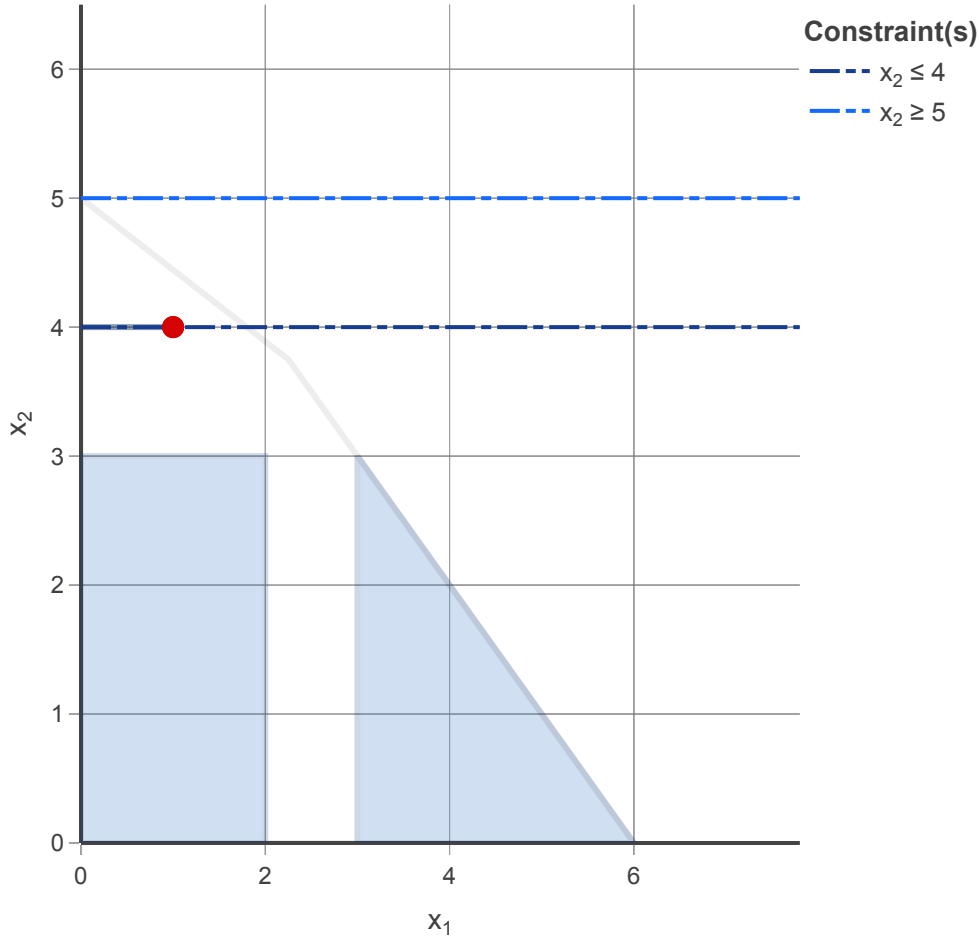### Geometric Interpretation of LPs

Q40: At the current (dark) node, we added the constraint $x_1 \leq 1$. Why were the fractional solutions $1 < x_1 < 2$ not eliminated for $x_2 <= 3$?
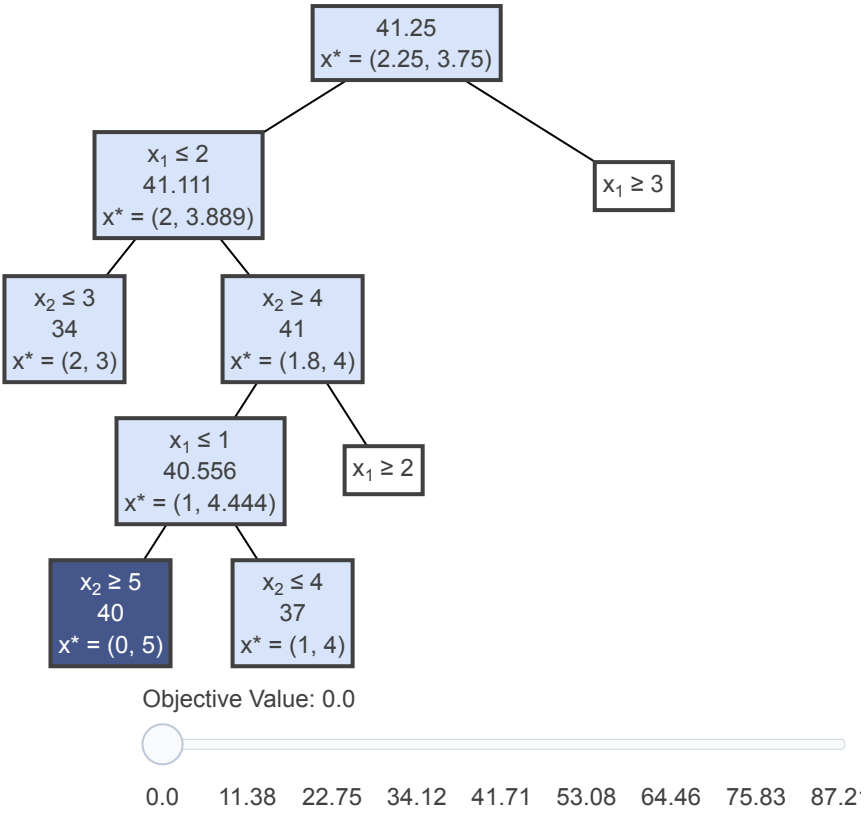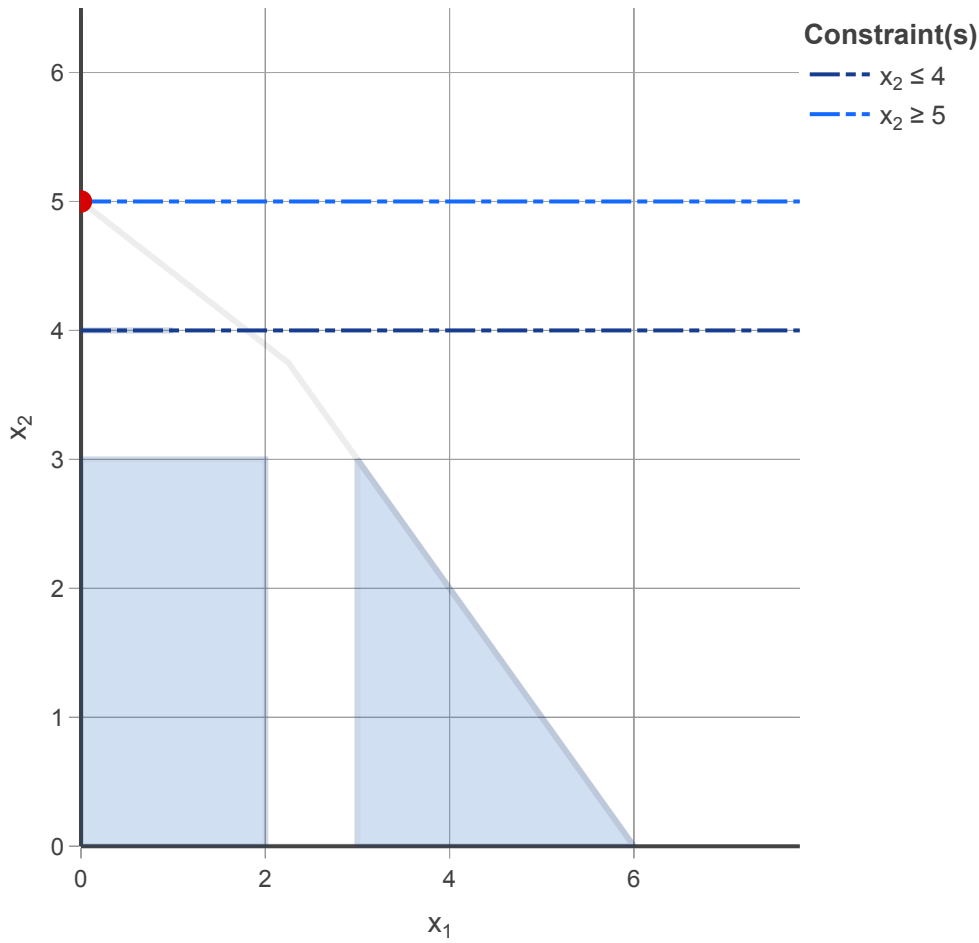
A: The fractional solutions were not eliminated in that area because we didn't follow that branch.

In [84]: `# Show the next three iterations of the branch and bound algorithm`
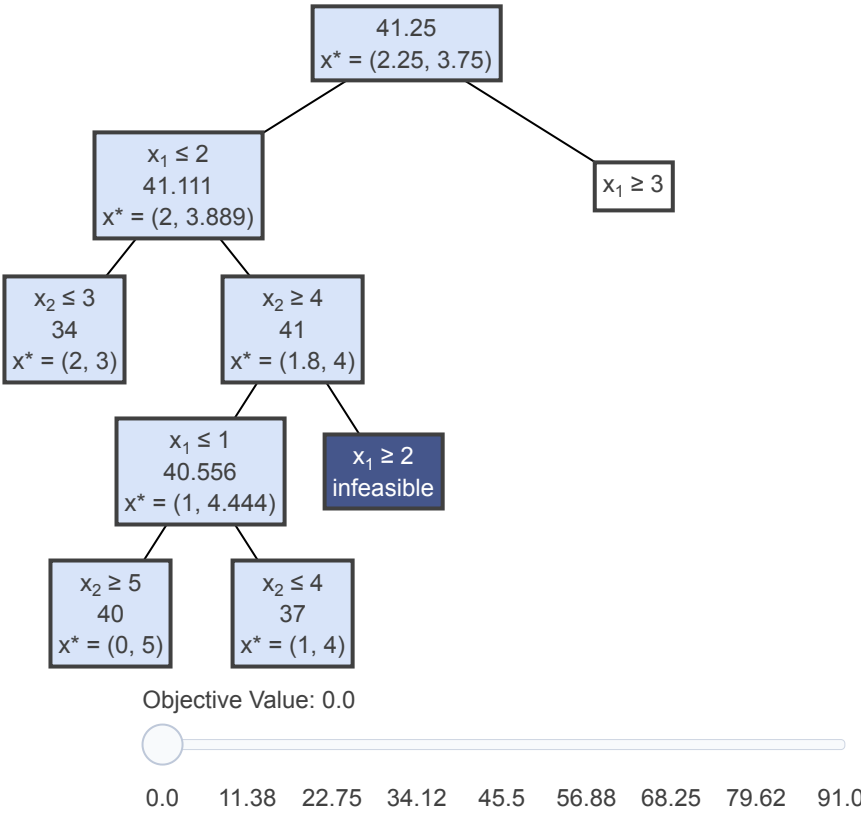`nodes[5].show()`
`nodes[6].show()`
`nodes[7].show()`

Geometric Interpretation of LPs

Geometric Interpretation of LPs


Geometric Interpretation of LPs
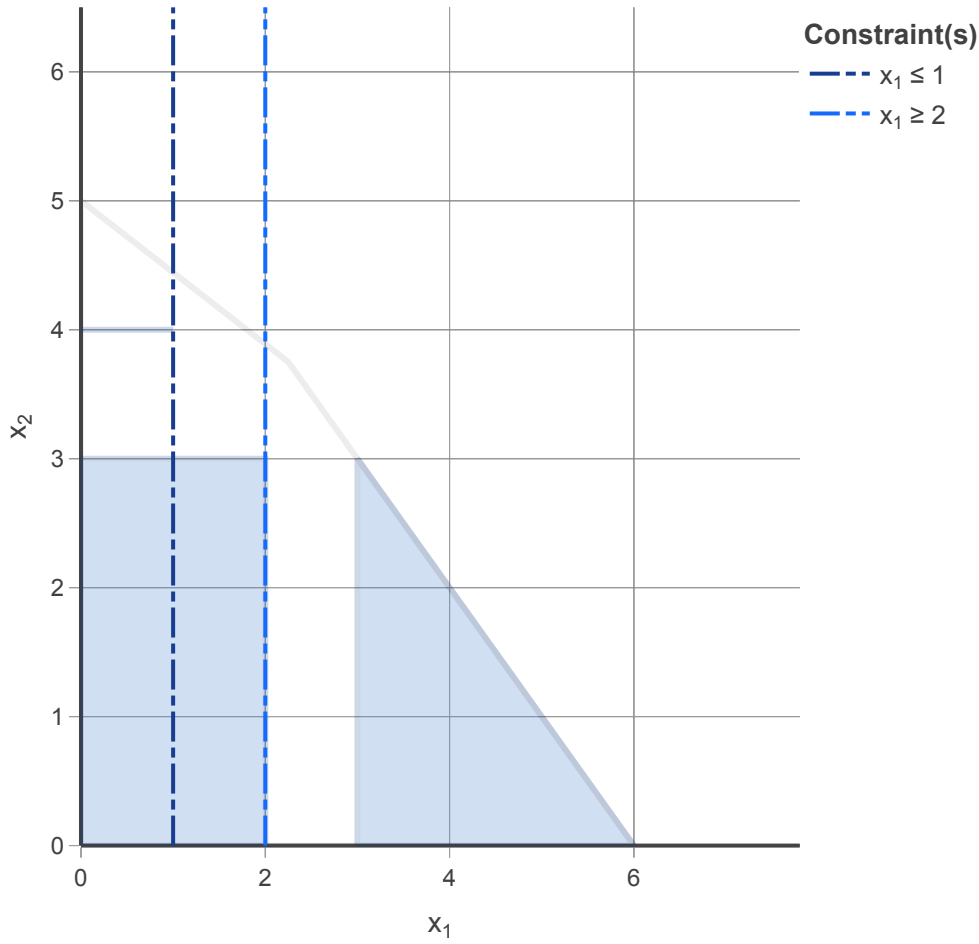
Q41: What constraints are enforced at the current (dark) node? Why are there no feasible solutions at this node?

A: The constraints enforced are x1 <= 2, x2 >=4, and x1 >=2. There are no feasible solutions at this node because x1 <=2 and x1 >=2 are contradictory. There are no regions that satisfy this constraint.

In [85]: `nodes[8].show()`

Geometric Interpretation of LPs
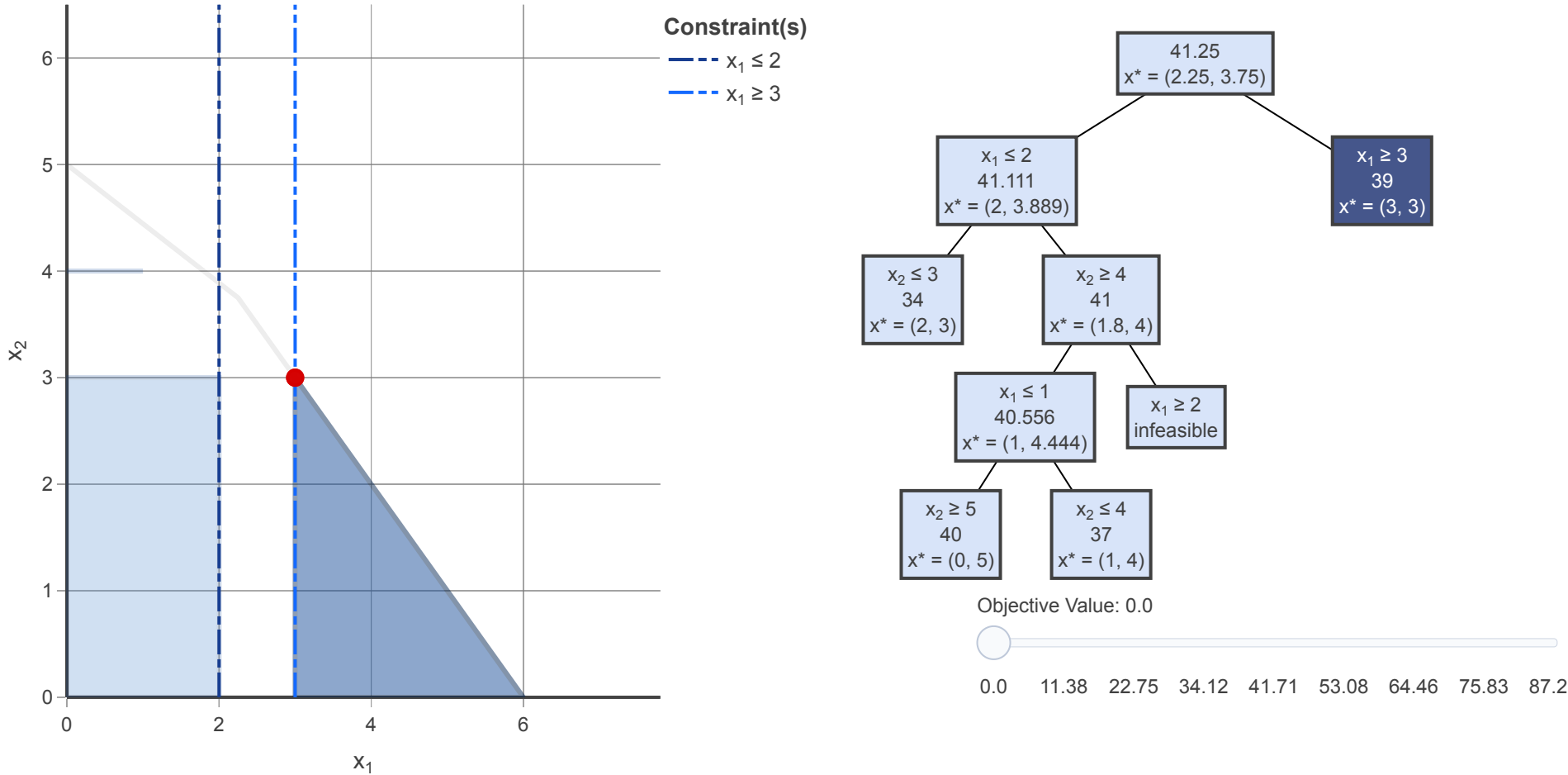
Q42: Are we done? If so, what nodes are fathomed and what is the optimal solution? Explain.

A: The optimal solution is (3,3) with the value of 39. All the leaf nodes that aren't integer solutions are fathomed.

Let's look at branch and bound visualization for an integer program with 3 decision variables!

In [86]: `nodes = gilp.bnb_visual(gilp.examples.VARIED_BRANCHING_3D_IP)`

In [87]: *# Look at the first 3 iterations*
nodes[0].show()
nodes[1].show()
nodes[2].show()

Geometric Interpretation of LPs
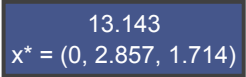
Geometric Interpretation of LPs

Geometric Interpretation of LPs

Let's fast-forward to the final iteration of the branch and bound algorithm.

In [87]: *# Look at the first 3 iterations*
nodes[0].show()
nodes[1].show()
nodes[2].show()

In [88]: `nodes[-1].show()`

Geometric Interpretation of LPs

Constraint(s)
$x_2 \leq 2$
$x_2 \geq 3$


```
                          13.143
                     x* = (0, 2.857, 1.714)
          ┌─────────────────┴──────────────────┐
     x₂ ≤ 2                                  x₂ ≥ 3
     12.8                                      13
  x* = (1.2, 2, 2.4)                      x* = (0, 3, 1)
  ┌────────┴────────┐
 x₁ ≤ 1           x₁ ≥ 2
 12.5             12.571
x* = (1, 2, 2.5)  x* = (2, 1.429, 2.857)
  │         ┌────────┼──────────┐
 x₃ ≤ 2   x₃ ≥ 3   x₂ ≤ 1      x₂ ≥ 2
 12       11.667   12.4        12
x*=(1,2,2) x*=(1,1.667,3) x*=(2.6,1,3.2) x*=(2,2,0)
            ┌────────┴────────┐
          x₁ ≤ 2            x₁ ≥ 3
          11.5             12.286
         x*=(2,1,3.5)    x*=(3,0.714,3.429)
                          ┌────────┴────────┐
                        x₂ ≤ 0            x₂ ≥ 1
                        12                12
                     x*=(4,-0,4)       x*=(3,1,2)
```
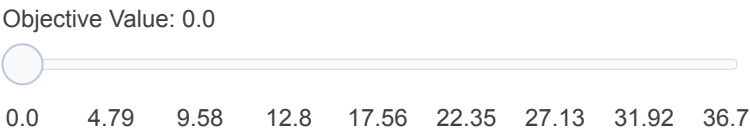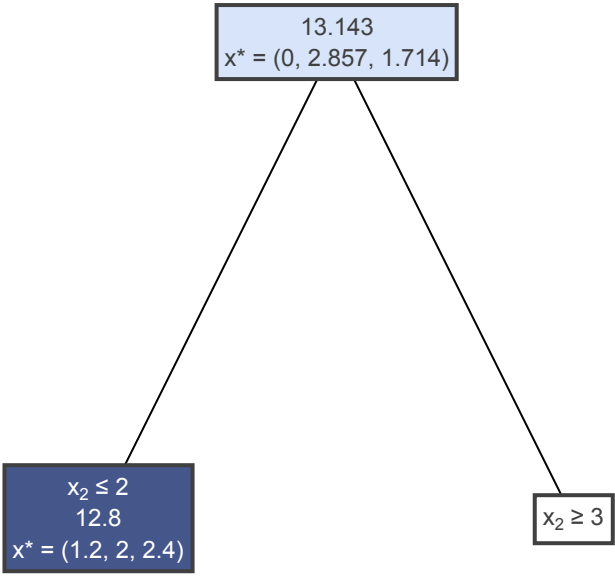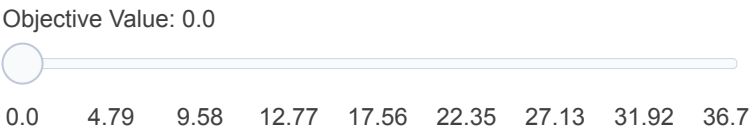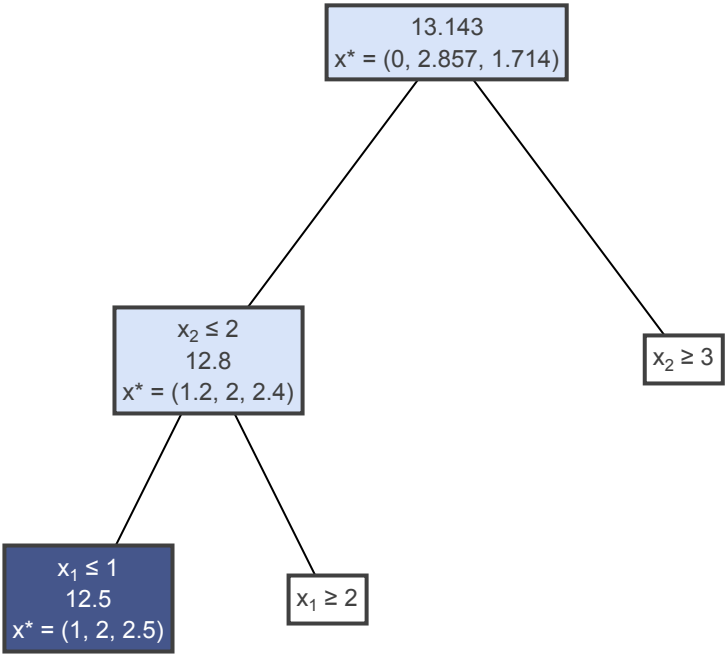
Objective Value: 0.0

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0.0 4.79 9.58 13.0 17.56 22.35 27.13 31.92 36.7
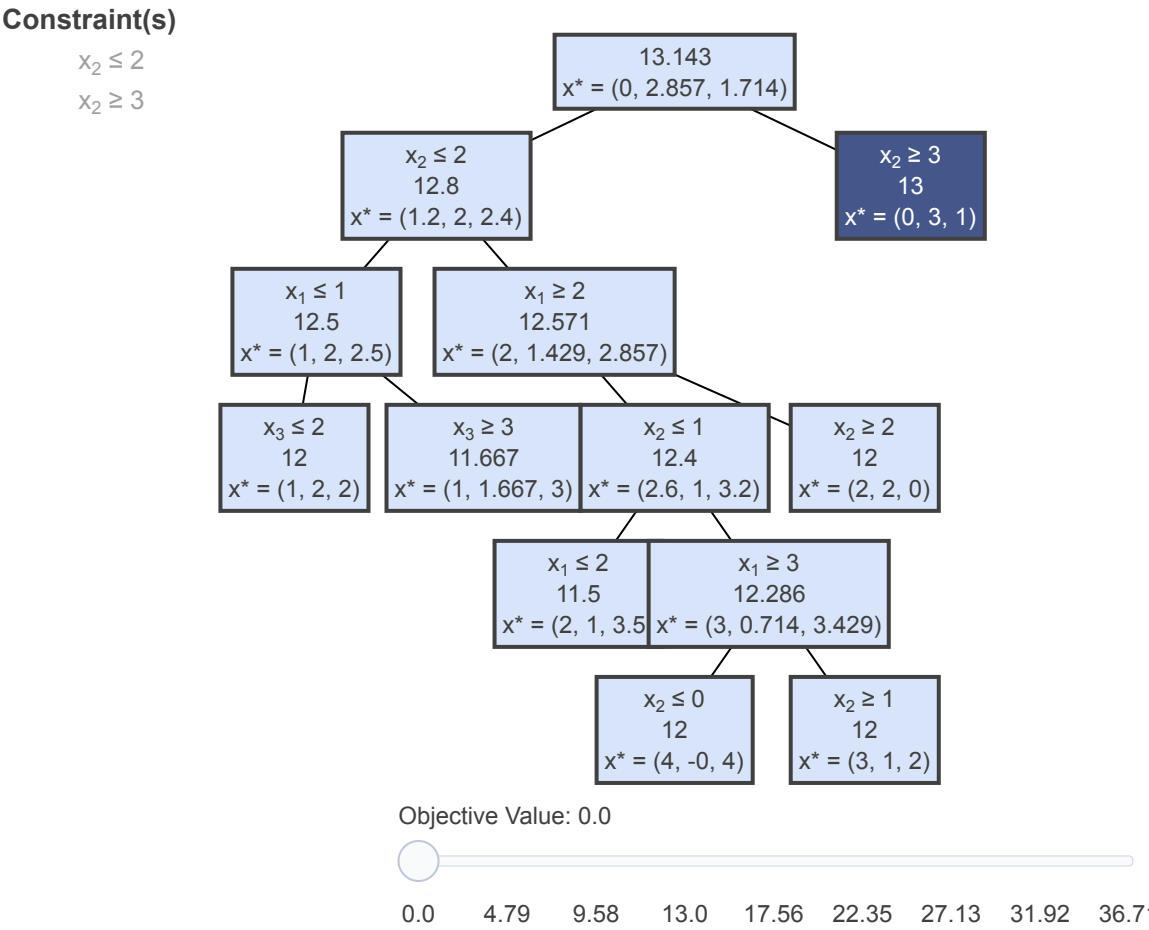
Q43: Consider the feasible region that looks like a rectangular box with one corner point at the origin. What node does it correspond to in the tree? What is the optimal solution at that node?

A: x3 <= 2. The optimal solution is (1,2,2) with the optimal value of 12.

Q44: How many branch and bound nodes did we explore? What was the optimal solution? How many branch and bound nodes would we have explored if we knew the value of the optimal solution before and bound?

A: We explored 13 nodes. The optimal solution is (0,3,1) and the optimal value is 13. We would have to explore 2 nodes (the root node and the second node that gives you the answer).

Bonus: Branch and Bound for Knapsack

Consider the following example:

item	value	weight
1	2	1
2	9	3
3	6	2

The linear program formulation will be:

$$\begin{aligned} \max \quad & 2x_1 + 9x_2 + 6x_3 \\ \text{s.t.} \quad & 1x_1 + 3x_2 + 2x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0, \quad \text{integer} \end{aligned}$$

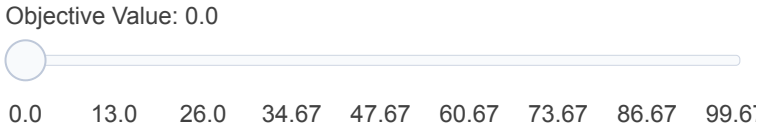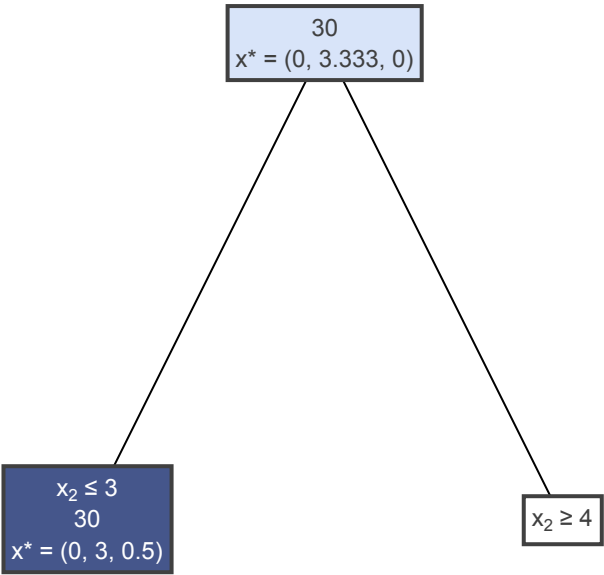In gilp, we can define this lp as follows:

In [89]:
```python
lp = gilp.LP([[1,3,2]],
             [10],
             [2,9,6])

for fig in gilp.bnb_visual(lp):
    fig.show()
```

Geometric Interpretation of LPs

30
x* = (0, 3.333, 0)

Objective Value: 0.0

0.0   13.0   26.0   34.67   47.67   60.67   73.67   86.67   99.6

Geometric Interpretation of LPs

Constraint(s)
$x_2 \leq 3$
$x_2 \geq 4$

30
x* = (0, 3.333, 0)

$x_2 \leq 3$
30
x* = (0, 3, 0.5)

$x_2 \geq 4$

Objective Value: 0.0

0.0   13.0   26.0   34.67   47.67   60.67   73.67   86.67   99.6

Geometric Interpretation of LPs

Constraint(s)
$x_3 \leq 0$
$x_3 \geq 1$

30
x* = (0, 3.333, 0)

$x_2 \leq 3$
30
x* = (0, 3, 0.5)

$x_2 \geq 4$

$x_3 \leq 0$
29
x* = (1, 3, 0)

$x_3 \geq 1$

Objective Value: 0.0

0.0   13.0   26.0   34.67   47.67   60.67   73.67   86.67   99.6

Geometric Interpretation of LPs

Constraint(s)
$x_3 \leq 0$
$x_3 \geq 1$

30
x* = (0, 3.333, 0)

$x_2 \leq 3$
30
x* = (0, 3, 0.5)

$x_2 \geq 4$

$x_3 \leq 0$
29
x* = (1, 3, 0)

$x_3 \geq 1$
30
x* = (0, 2.667, 1)

Objective Value: 0.0

0.0   13.0   26.0   34.67   47.67   60.67   73.67   86.67   99.6

Geometric Interpretation of LPs

Constraint(s)
$x_2 \leq 2$
$x_2 \geq 3$

```
                          30
                     x* = (0, 3.333, 0)

           x₂ ≤ 3
            30
       x* = (0, 3, 0.5)                    x₂ ≥ 4

   x₃ ≤ 0            x₃ ≥ 1
    29                30
x* = (1, 3, 0)    x* = (0, 2.667, 1)

              x₂ ≤ 2
               30              x₂ ≥ 3
           x* = (0, 2, 2)
```

Objective Value: 0.0

○————————————————————————————————————

0.0   13.0   26.0   34.67   47.67   60.67   73.67   86.67   99.6ⁿ

Geometric Interpretation of LPs

Constraint(s)
$x_2 \leq 2$
$x_2 \geq 3$

```
                          30
                     x* = (0, 3.333, 0)

           x₂ ≤ 3
            30
       x* = (0, 3, 0.5)                    x₂ ≥ 4

   x₃ ≤ 0            x₃ ≥ 1
    29                30
x* = (1, 3, 0)    x* = (0, 2.667, 1)

              x₂ ≤ 2          x₂ ≥ 3
               30            infeasible
           x* = (0, 2, 2)
```

Objective Value: 0.0

○————————————————————————————————————

0.0   13.0   26.0   39.0   52.0   65.0   78.0   91.0   104.0

Geometric Interpretation of LPs

Constraint(s)
$x_2 \leq 3$
$x_2 \geq 4$

```
                          30
                     x* = (0, 3.333, 0)

           x₂ ≤ 3
            30                        x₂ ≥ 4
       x* = (0, 3, 0.5)              infeasible

   x₃ ≤ 0            x₃ ≥ 1
    29                30
x* = (1, 3, 0)    x* = (0, 2.667, 1)

              x₂ ≤ 2          x₂ ≥ 3
               30            infeasible
           x* = (0, 2, 2)
```

Objective Value: 0.0

○————————————————————————————————————

0.0   13.0   26.0   39.0   52.0   65.0   78.0   91.0   104.0

In [ ]: