# Branch & Bound and Knapsack Lab

**Objectives**

- Preform the branch and bound algorithm
- Apply branch and bound to the knapsack problem
- Understand the geometry of the branch and bound algorithm

**Brief description:** In this lab, we will try solving an example of a knapsack problem with the branch-and-bound algorithm. We will also see how adding a cutting plane helps in reducing the computation time and effort of the algorithm. Lastly, we will explore the geometry of the branch and bound algorithm.

In [1]:
```
# imports -- don't forget to run this cell
import pandas as pd
import gilp
from gilp.visualize import feasible_integer_pts
from ortools.linear_solver import pywraplp as OR
```

## Part 1: Branch and Bound Algorithm

Recall that the branch and bound algorithm (in addition to the simplex method) allows us to solve integer programs. Before applying the branch and bound algorithm to the knapsack problem, we will begin by reviewing some core ideas. Furthermore, we will identify a helpful property that will make branch and bound terminate quicker later in the lab!

**Q1:** What are the different ways a node can be fathomed during the branch and bound algorithm? Describe each.

**A:** If we have found the most optimal solution given the LP solution, we can fathom the node. We can also do so if the restrictions on the latest node is greater than another node's solution.

**Q2:** Suppose you have a maximization integer program and you solve its linear program relaxation. What does the LP-relaxation optimal value tell you about the IP optimal value? What if it is a minimization problem?

**A:** If it is a maximization problem, the LP relaxation shows the upper bound to the solution. If it is a minimisation problem, it shows a lower bound.

**Q3:** Assume you have a maximization integer program with all integral coefficents in the objective function. Now, suppose you are running the branch and bound algorithm and come across a node with an optimal value of 44.5. The current incumbent is 44. Can you fathom this node? Why or why not?

**A:** Yes, because 44 is the largest integer possible with an upper bound of 44.5.

**Q4:** If the optimal solution to the LP relaxation of the original program is integer, then you have found an optimal solution to your integer program. Explain why this is true.

**A:** The bounds of the LP relaxation are inclusive of the LP solution, so if the solution is an integer it can be counted in the set of possible solutions, and is the bound so it is the largest/smallest possible solution.

**Q5:** If the LP is infeasible, then the IP is infeasible. Explain why this is true.

**A:** If there were a feasible IP, then that would be the solution to the LP problem. Since there is no LP solution, there cannot be an IP one either.

The next questions ask about the following branch and bound tree. If the solution was not integral, the fractional $x_i$ that was used to branch is given. If the solution was integral, it is denoted *INT*. In the current iteration of branch and bound, you are looking at the node with the *****.

**Q6:** Can you determine if the integer program this branch and bound tree is for is a minimization or maximixation problem? If so, which is it?

**A:** We can tell that this is a minimisation problem, as the solution of the first node is the smallest of all other solutions and is thus a lower bound.

Hint: For **Q7-8**, you can assume integral coefficents in the objective function.

**Q7:** Is the current node (marked $z^*$) fathomed? Why or why not? If not, what additional constraints should be imposed for each of the next two nodes?

**A:** No, as the solution is smaller than the integer solution found. This means that one of this node's subnodes might have a more optimal solution.

**Q8:** Consider the nodes under the current node (where $z = 16.3$). What do you know about the optimal value of these nodes? Why?

**A:** I know that the optimal value of these nodes is greater than or equal to 16.3, as the current node provides a lower bound for its 'children' nodes.

# Part 2: The Knapsack Problem

In this lab, you will solve an integer program by branch and bound. The integer program to be solved will be a knapsack problem.

**Knapsack Problem:** We are given a collection of $n$ items, where each item $i = 1, \ldots, n$ has a weight $w_i$ and a value $v_i$. In addition, there is a given capacity $W$, and the aim is to select a maximum value subset of items that has a total weight at most $W$. Note that each item can be brought at most once.
max

Consider the following data which we import from a CSV file:

In [2]:
```
data = pd.read_csv('knapsack_data_1.csv', index_col=0)
data
```

Out[2]:

| item | value | weight |
|---|---|---|
| **1** | 50 | 10 |
| **2** | 30 | 12 |
| **3** | 24 | 10 |
| **4** | 14 | 7 |
| **5** | 12 | 6 |
| **6** | 10 | 7 |
| **7** | 40 | 30 |

and W = 18.

**Q9:** Are there any items we can remove from our input to simplify this problem? Why? If so, replace `index` with the item number that can be removed in the code below. Hint: how many of each item could we possibly take?

**A:** We can remove weight 7, as it's weight is already greater than the limit.

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In [4]:
```python
# TODO: replace index
data = data.drop(7)
```

**Q10:** If we remove item 7 from the knapsack, it does not change the optimal solution to the integer program. Explain why.

**A:** 7 would never be feasible to include in the situation, as its weight of 30 exceeds the given capacity of 18.

**Q11:** Consider removing items i such that w_i > W from a knapsack input. How does the LP relaxation's optimal value change?

**A:** It is highly unlikely to change as the LP relaxation's optimal value would only have a fraction of this weight's value. It would be impossible for it to be included in the IP solution.

In **Q10-11**, you should have found that removing these items removes feasible solutions from the linear program but does not change the integer program. This is desirable as the gap between the optimal IP and LP values can become smaller. By adding this step, branch and bound may terminate sooner.

Recall that a branch and bound node can be fathomed if its bound is no better than the value of the best feasible integer solution found thus far. Hence, it helps to have a good feasible integer solution as quickly as possible (so that we stop needless work). To do this, we can first try to construct a good feasible integer solution by a reasonable heuristic algorithm before starting to run the branch and bound procedure.

In designing a heuristic for the knapsack problem, it is helpful to think about the value per unit weight for each item. We compute this value in the table below.

In [5]:
```python
data['value per unit weight'] = (data['value'] / data['weight']).round(2)
data
```

Out[5]:

| item | value | weight | value per unit weight |
|---|---|---|---|
| 1 | 50 | 10 | 5.00 |
| 2 | 30 | 12 | 2.50 |
| 3 | 24 | 10 | 2.40 |
| 4 | 14 | 7 | 2.00 |
| 5 | 12 | 6 | 2.00 |
| 6 | 10 | 7 | 1.43 |

**Q12:** Design a reasonable heuristic for the knapsack problem. Note a heuristic aims to find a decent solution to the problem (but is not necessarily optimal).

**A:** Choose items starting from most value per unit weight, going through the list and skipping items that would exceed the weight limit until no more can be added.

**Q13:** Run your heuristic on the data above to compute a good feasible integer solution. Your heuristic should generate a feasible solution with a value of 64 or better. If it does not, try a different heuristic (or talk to your TA!)

**A:** Items chosen: 1, 4. Total value = 50 + 14 = 64.

We will now use the branch and bound algorithm to solve this knapsack problem! First, let us define a mathematical model for the linear relaxation of the knapsack problem.

**Q14:** Complete the model below.

In [11]:
```python
def Knapsack(table, capacity, integer = False):
    """Model for solving the Knapsack problem.

    Args:
        table (pd.DataFrame): A table indexd by items with a column for value and weight
        capcity (int): An integer-capacity for the knapsack
        integer (bool): True if the variables should be integer. False otherwise.
    """
    ITEMS = list(table.index)        # set of items
    v = table.to_dict()['value']     # value for each item
    w = table.to_dict()['weight']    # weight for each item
    W = capacity                     # capacity of the knapsack

    # define model
    m = OR.Solver('knapsack', OR.Solver.CBC_MIXED_INTEGER_PROGRAMMING)

    # decision variables
    x = {}
    for i in ITEMS:
        if integer:
            x[i] = m.IntVar(0, 1, 'x_%d' % (i))
        else:
            x[i] = m.NumVar(0, 1, 'x_%d' % (i))

    # define objective function here
    m.Maximize(sum(v[i]*x[i] for i in ITEMS))

    # TODO: Add a constraint that enforces that weight must not exceed capacity
    # recall that we add constraints to the model using m.Add()
    m.Add(sum((w[i]*x[i]) for i in ITEMS) <= W)

    return (m, x)  # return the model and the decision variables
```

In [12]:
```python
# You do not need to do anything with this cell but make sure you run it!
def solve(m):
    """Used to solve a model m."""
    m.Solve()

    print('Objective =', m.Objective().Value())
    print('iterations :', m.iterations())
    print('branch-and-bound nodes :',m.nodes())

    return ({var.name() : var.solution_value() for var in m.variables()})
```

We can now create a linear relaxation of our knapsack problem. Now, `m` represents our model and `x` represents our decision variables.

In [13]:
```python
m, x = Knapsack(data, 18)
```

We can use the next line to solve the model and output the solution

In [14]:
```
solve(m)
```

```
Objective = 70.0
iterations : 0
branch-and-bound nodes : 0
```

Out[14]:
```
{'x_1': 1.0,
 'x_2': 0.6666666666666667,
 'x_3': 0.0,
 'x_4': 0.0,
 'x_5': 0.0,
 'x_6': 0.0}
```

**Q15:** How does this optimal value compare to the value you found using the heuristic integer solution?

**A:** This optimal value is greater than mine.

**Q16:** Should this node be fathomed? If not, what variable should be branched on and what additional constraints should be imposed for each of the next two nodes?

**A:** No, as $x_2$ has a non-integer value. Variable $x_2$ should be branched on with constraints $x_2 \leq 0$ and $x_2 \geq 1$

After constructing the linear relaxation model using `Knapsack(data1, 18)` we can add additional constraints. For example, we can add the constraint $x_2 \leq 0$ and solve it as follows:

In [15]:
```
m, x = Knapsack(data, 18)
m.Add(x[2] <= 0)
solve(m)
```

```
Objective = 69.2
iterations : 0
branch-and-bound nodes : 0
```

Out[15]:
```
{'x_1': 1.0, 'x_2': 0.0, 'x_3': 0.8, 'x_4': 0.0, 'x_5': 0.0, 'x_6': 0.0}
```

**NOTE:** The line `m, x = Knapsack(data1, 18)` resets the model `m` to the LP relaxation. All constraints from branching have to be added each time.

**Q17:** Use the following cell to compute the optimal value for the other node you found in **Q16**.

In [16]:
```
m, x = Knapsack(data, 18)
m.Add(x[2] >= 1)
solve(m)
```

```
Objective = 60.0
iterations : 0
branch-and-bound nodes : 0
```

Out[16]:
```
{'x_1': 0.6000000000000001,
 'x_2': 1.0,
 'x_3': 0.0,
 'x_4': 0.0,
 'x_5': 0.0,
 'x_6': 0.0}
```

**Q18:** What was the optimal value? Can this node be fathomed? Why? (Hint: In **Q13**, you found a feasible integer solution with value 64.)

**A:** The optimal value was 60, and since I have found a feasible integer solution with greater value this node can be fathomed.

If we continue running the branch and bound algorithm, we will eventually reach the branch and bound tree below where the z^* indictes the current node we are looking at.

**Q19:** The node with $z = 64.857$ was fathomed. Why are we allowed to fathom this node? (Hint: think back to **Q3**)

**A:** The maximum integer value possible beyond this node is 64, and we have already found a solution with a value of 64 or greater, so we need not go further.

**Q20:** Finish running branch and bound to find the optimal integer solution. Use a separate cell for each node you solve and indicate if the node was fathomed with a comment. (Hint: Don't forget to include the constraints further up in the branch and bound tree.)

In [17]:
```
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[2] <= 0)
m.Add(x[3] <= 0)
m.Add(x[4] >= 1)
m.Add(x[5] <= 0)
m.Add(x[6] <= 0)

solve(m)
# fathomed? yes
```

```
Objective = 64.0
iterations : 0
branch-and-bound nodes : 0
```

Out[17]:
```
{'x_1': 1.0, 'x_2': 0.0, 'x_3': 0.0, 'x_4': 1.0, 'x_5': 0.0, 'x_6': 0.0}
```

In [18]:
```
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[2] <= 0)
m.Add(x[3] <= 0)
m.Add(x[4] >= 1)
m.Add(x[5] <= 0)
m.Add(x[6] >= 1)

solve(m)
# fathomed? yes
```

```
Objective = 44.0
iterations : 0
branch-and-bound nodes : 0
```

Out[18]:
```
{'x_1': 0.4, 'x_2': 0.0, 'x_3': 0.0, 'x_4': 1.0, 'x_5': 0.0, 'x_6': 1.0}
```

In [ ]:
```

```

**A:** The optimal solution is where x_1 and x_4 equal 1 and the rest have a value of 0 with optimal value 64.

**Q21:** How many nodes did you have to explore while running the branch and bound algorithm?

**A:** I had to run a total of 11 nodes to find the optimal solution.

In the next section, we will think about additional constraints we can add to make running branch and bound quicker.

# Part 3: Cutting Planes

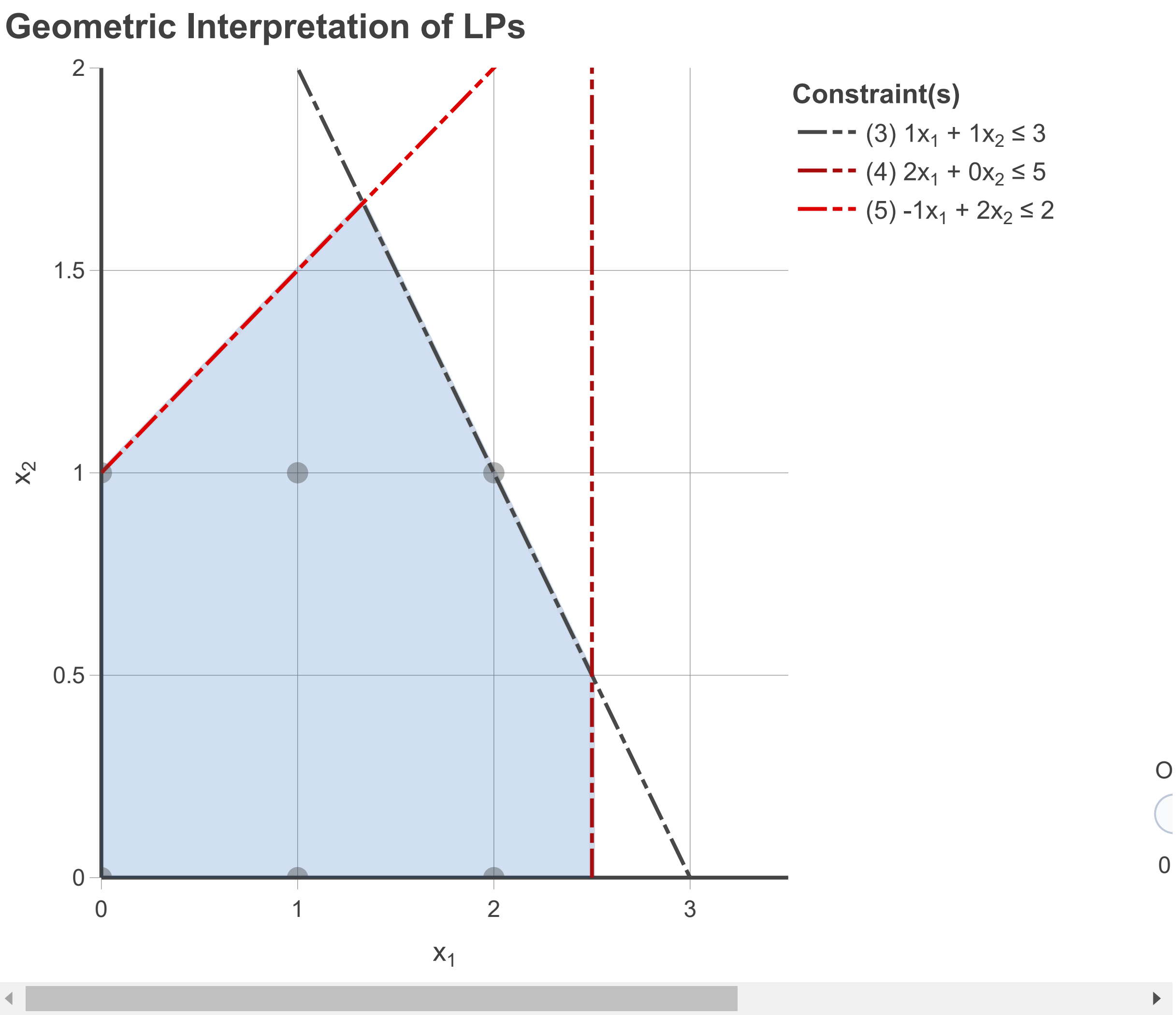
In general, a cutting plane is an additional constraint we can add to an integer program's linear relaxation that removes feasible linear solutions but does not remove any integer feasible solutions. This is very useful when solving integer programs! Recall many of the problems we have learned in class have something we call the "integrality property". This is useful because it allows us to ignore the integrality constraint since we are garunteed to reach an integral solution. By cleverly adding cutting planes, we strive to remove feasible linear solutions (without removing any integer feasible solutions) such that the optimal solution to the linear relaxation is integral!

Conisder an integer program whose linear program releaxation is

$$\begin{align*} \max \quad & 2x_1+x_2\\ \text{s.t.} \quad & x_1 + x_2 \leq 3 \\ & 2x_1 \leq 5 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{align*}$$

We can define this linear program and then visualize its feasible region. The integer points have been highlighted.

In [19]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2]],
             [3,5,2],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```




**Q22:** List every feasible solution to the integer program.

**A:** (0,0), (0,1), (1,0), (1,1), (2,0), (2,1)

**Q23:** Is the constraint x_2 \leq 1 a cutting plane? Why? (Hint: Would any feasible integer points become infeasible? What about feasible linear points?)

**A:** Yes, as there is no feasible solution where x_2 is greater than 1.

Let's add this cutting plane to the LP relaxation!

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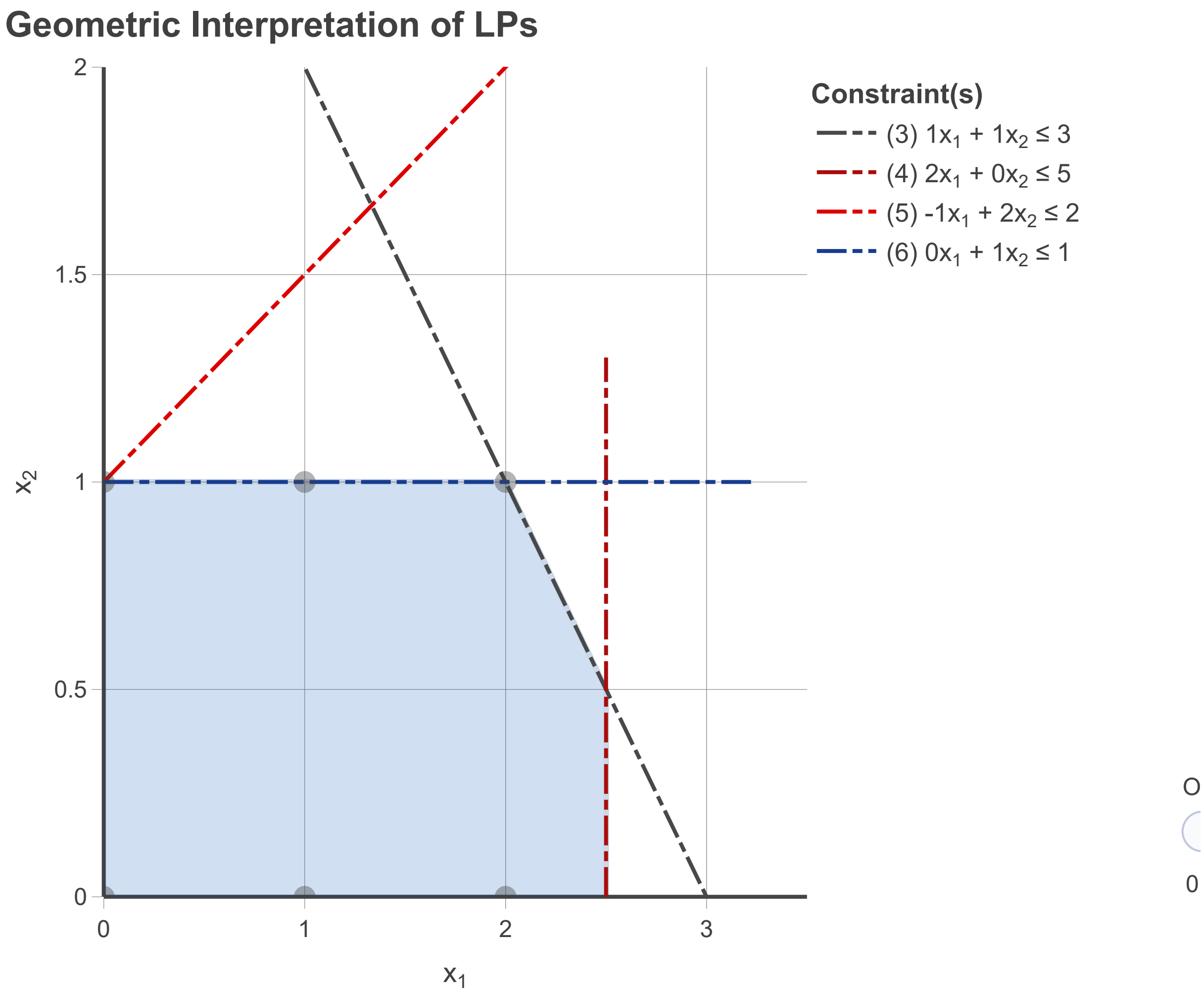
In [20]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2],[0,1]],
             [3,5,2,1],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```




**Q24:** Is the constraint $x_1 \leq 3$ a cutting plane? Why?

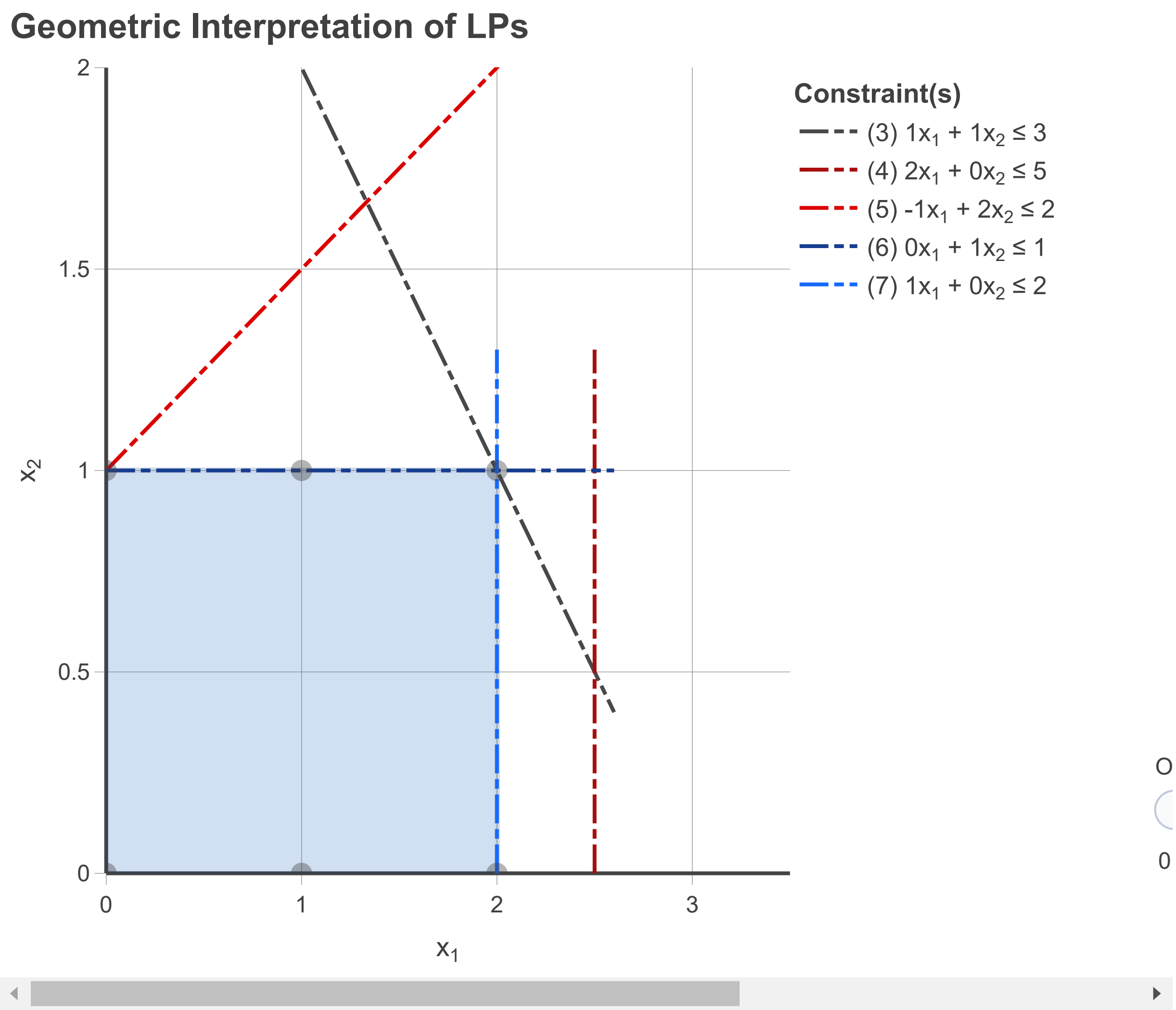
**A:** No, as it is a redundant constraint and no LP optimum solutions are cut.

**Q25:** Can you provide another cutting plane? If so, what is it?

**A:** $x_1 \leq 2$

Let's look at the feasible region after adding the cutting plane from **Q23** and one of the possible answers from **Q25**. Notice the optimal solution to the LP relaxation is now integral!

In [21]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2],[0,1],[1,0]],
             [3,5,2,1,2],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```




Let's try applying what we know about cutting planes to the knapsack problem! Again, recall our input was W = 18 and:

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In [22]:
```
data
```

Out[22]:

| item | value | weight | value per unit weight |
|---|---|---|---|
| 1 | 50 | 10 | 5.00 |
| 2 | 30 | 12 | 2.50 |
| 3 | 24 | 10 | 2.40 |
| 4 | 14 | 7 | 2.00 |
| 5 | 12 | 6 | 2.00 |
| 6 | 10 | 7 | 1.43 |

**Q26:** Look at items 1, 2, and 3. How many of these items can we take simultaneously? Can you write a new constraint to capture this? If so, please provide it.

**A:** We can take one of these items simultaneously, so we know that $x_1 + x_2 + x_3 \leq 1$

**Q27:** Is the constraint you found in **Q26** a cutting plane? If so, provide a feasible solution to the linear program relaxation that is no longer feasible (i.e. a point the constraint *cuts off*).

**A:** This is a cutting plane, as one of the no longer feasible solutions is $x_1 = 1, x_2 = 2/3$, while the remaining x equal 0

**Q28:** Provide another cutting plane involving items 4,5 and 6 for this integer program. Explain how you derived it.

**A:** At maximum, two of the three items may be chosen, so we know that $x_4 + x_5 + x_6 \leq 2$

**Q29:** Add the cutting planes from **Q26** and **Q28** to the model and solve it. You should get a solution in which we take items 1 and 4 and $\frac{1}{6}$ of item 5 with an objective value of 66.

In [23]:
```
m, x = Knapsack(data, 18)
# TODO: Add cutting planes here
m.Add(x[1]+x[2]+x[3] <= 1)
m.Add(x[4]+x[5]+x[6] <= 2)

solve(m)
```

```
Objective = 66.0
iterations : 0
branch-and-bound nodes : 0
```

Out[23]:
```
{'x_1': 1.0,
 'x_2': 0.0,
 'x_3': 0.0,
 'x_4': 1.0,
 'x_5': 0.16666666666666696,
 'x_6': 0.0}
```

Let's take a moment to pause and reflect on what we are doing. Recall from **Q9-11** that we dropped item 7 becuase its weight was greater than the capcity of the knapsack. Essentially we added the constraint $x_7 \leq 0$. This constraint was a cutting plane! It eliminated some linear feasible solutions but no integer ones. By adding these two new cutting planes, we can get branch and bound to terminate earlier yet again! So far, we have generated cutting planes by inspection. However, there are more algorithmic ways to identify them (which we will ignore for now).

If we continue running the branch and bound algorithm, we will eventually reach the branch and bound tree below where the $z^*$ indictes the current node we are looking at.

**NOTE:** Do not forget about the feasible integer solution our heuristic gave us with value 64.

**Q30** Finish running branch and bound to find the optimal integer solution. Use a separate cell for each node you solve and indicate if the node was fathomed with a comment. Hint: Don't forget the cutting plane constraints should be included in every node of the branch and bound tree.

In [24]:
```python
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[1]+x[2]+x[3] <= 1)
m.Add(x[4]+x[5]+x[6] <= 2)
m.Add(x[5] >= 1)
m.Add(x[4] <= 0)

solve(m)
# fathomed? Yes
```

```
Objective = 64.85714285714286
iterations : 0
branch-and-bound nodes : 0
```

Out[24]:
```
{'x_1': 1.0,
 'x_2': 0.0,
 'x_3': 0.0,
 'x_4': 0.0,
 'x_5': 1.0,
 'x_6': 0.28571428571428586}
```

In [25]:
```python
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[1]+x[2]+x[3] <= 1)
m.Add(x[4]+x[5]+x[6] <= 2)
m.Add(x[5] >= 1)
m.Add(x[4] >= 1)

solve(m)
# fathomed? Yes
```

```
Objective = 51.0
iterations : 0
branch-and-bound nodes : 0
```

Out[25]:
```
{'x_1': 0.5, 'x_2': 0.0, 'x_3': 0.0, 'x_4': 1.0, 'x_5': 1.0, 'x_6': 0.0}
```

In [ ]:
```python

```

**A:** Both of these nodes are fathomed, as the first one can have a maximum IP solution of 64 which has already been found, and the secomd has an IP solution which is less than 64.

**Q31:** Did you find the same optimal solution? How many nodes did you explore? How did this compare to the number you explored previously?

**A:** The optimal solution is still 64. This time, I only explored 7 nodes, which is less than the previous 11 nodes.

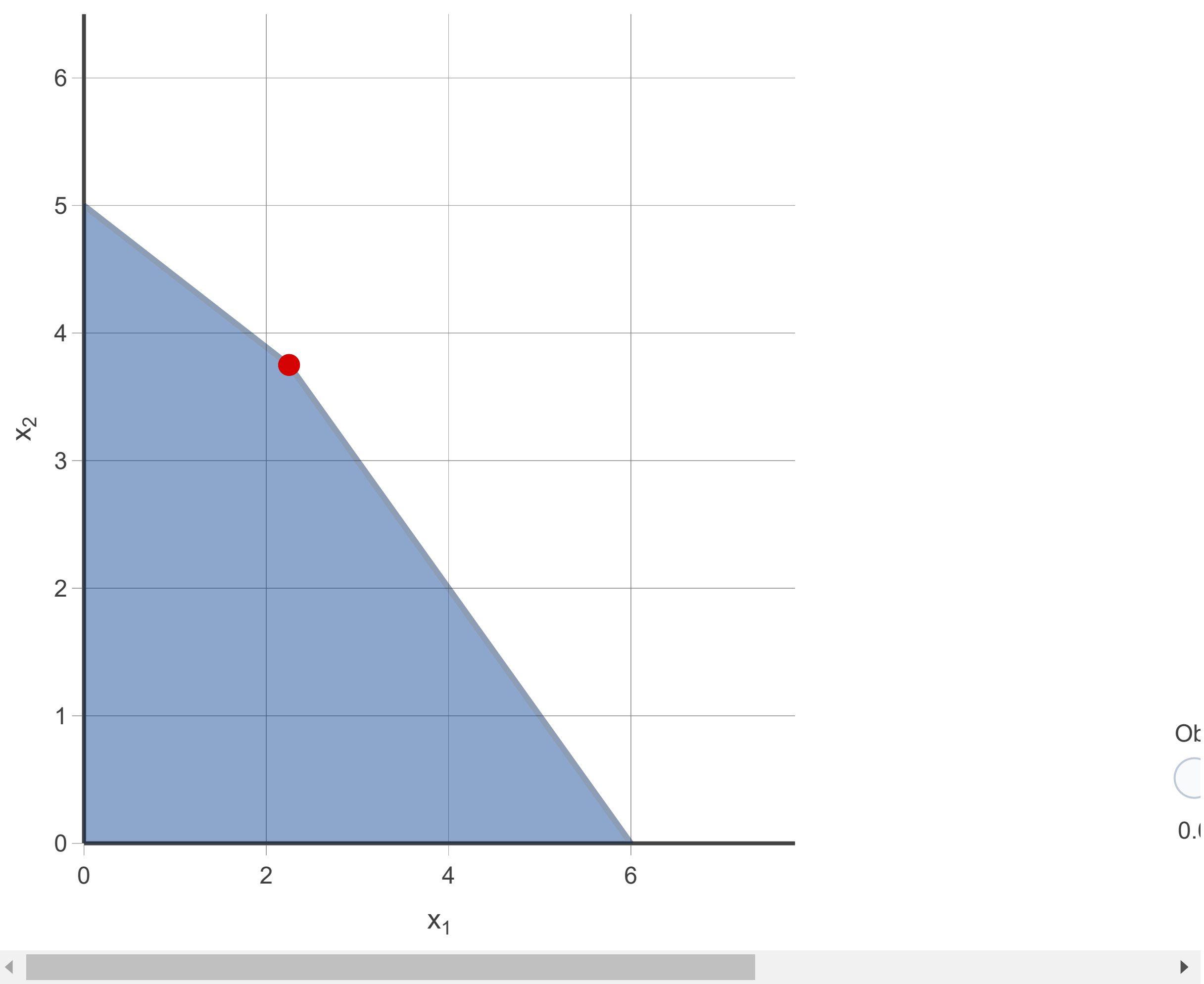
# Part 4: Geometry of Branch and Bound

Previously, we used the `gilp` package to viusualize the simplex algorithm but it also has the functionality to visualize branch and bound. We will give a quick overview of the tool. Similar to `lp_visual` and `simplex_visual`, the function `bnb_visual` takes an `LP` and returns a visualization. It is assumed that every decision variable is constrained to be integer. Unlike previous visualizations, `bnb_visual` returns a series of figures for each node of the branch and bound tree. Let's look at a small 2D example: \begin{align*} \max \quad & 5x_1+ 8x_2\\ \text{s.t.} \quad & x_1 + x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_1, x_2 \geq 0, \quad \text{integral} \end{align*}

In [26]:
```
nodes = gilp.bnb_visual(gilp.examples.STANDARD_2D_IP)
```

In [27]:
```
nodes[0].show()
```

**Geometric Interpretation of LPs**




Run the cells above to generate a figure for each node and view the first node. At first, you will see the LP relaxation on the left and the root of the branch and bound tree on the right. The simplex path and isoprofit slider are also present.

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**Q32:** Recall the root of a branch and bound tree is the unaltered LP relaxation. What is the optimal solution? (Hint: Use the objective slider and hover over extreme points).

**A:** The optimal IP solution is (5,0), with a value of 40.

**Q33:** Assume that we always choose the variable with the minimum index to branch on if there are multiple options. Write down (in full) each of the LPs we get after branching off the root node.

**A:** I would say the two LP's would be:

1. $\max 5x_1+ 8x_2\\ x_1 + x_2 \leq 6 \\ 5x_1 + 9x_2 \leq 45 \\ x_1 \leq 2 \\ x_1, x_2 \geq 0, \quad \text{integral}$

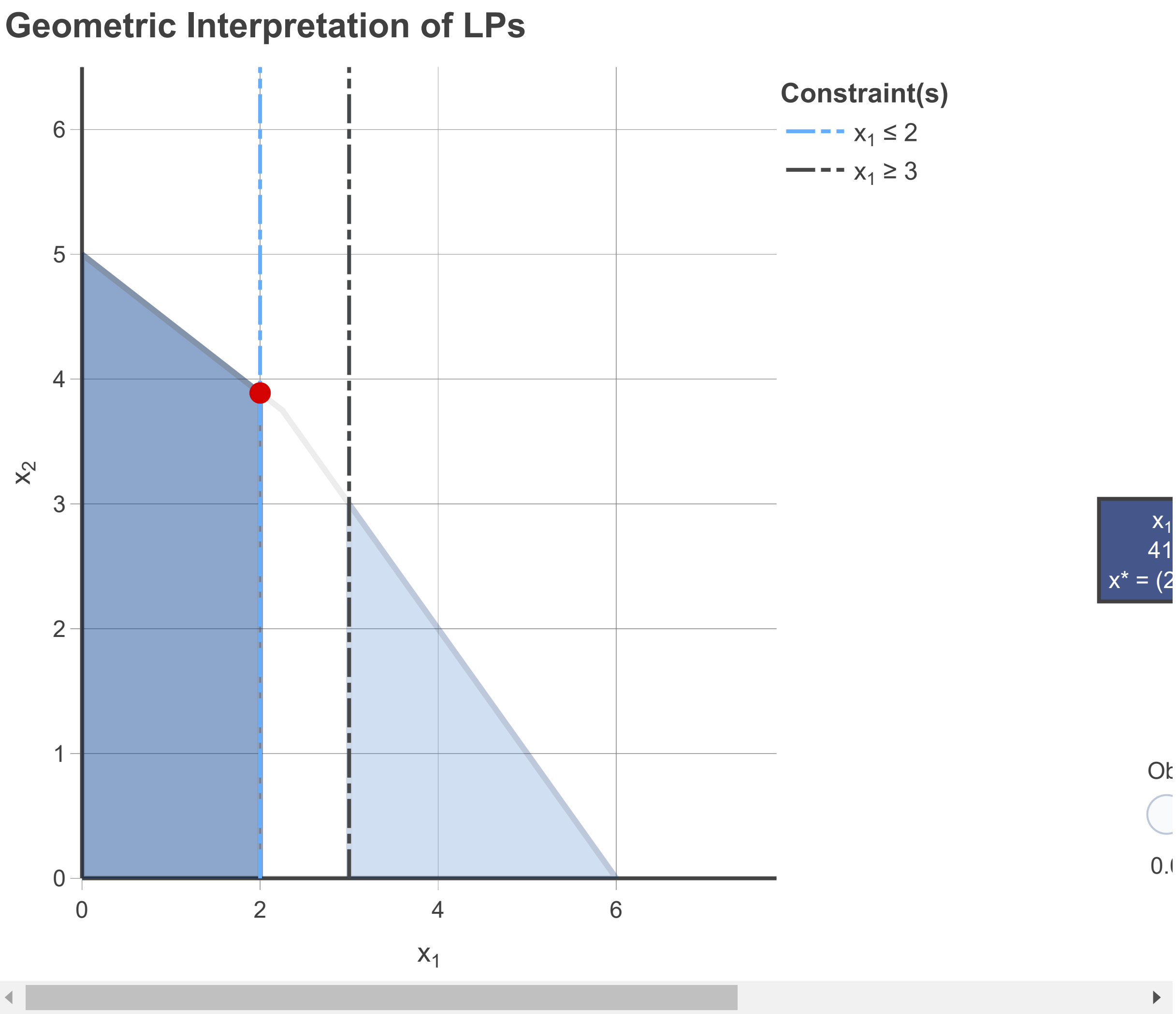
2. $\max 5x_1+ 8x_2\\ x_1 + x_2 \leq 6 \\ 5x_1 + 9x_2 \leq 45 \\ x_1 \geq 3 \\ x_1, x_2 \geq 0, \quad \text{integral}$

**Q34:** Draw the feasible region to each of the LPs from **Q33** on the same picture.

**A:** There would be two vertical inequalities, one of $x_1 \leq 2$ and the other of $x_1 \geq 3$

Run the following cell to see if the picture you drew in **Q34** was correct.

In [28]:
```
nodes[1].show()
```




The outline of the original LP relaxation is still shown on the left. Now that we have eliminated some of the fractional feasible solutions, we now have 2 feasible regions to consider. The darker one is the feasible region associated with the current node which is also shaded darker in the branch and bound tree. The unexplored nodes in the branch and bound tree are not shaded in.

**Q35:** Which feasible solutions to the LP relaxation are removed by this branch?

**A:** All feasible solutions where 2 < x_1 < 3 are removed, including the previous optimal solution.

**Q36:** At the current (dark) node, what constraints will we add? How many feasible regions will the original LP relaxation be broken into?

**A:** We will add x_1 \leq 3 and x_1 \geq 4, which will break up the original relaxation into 3 parts.

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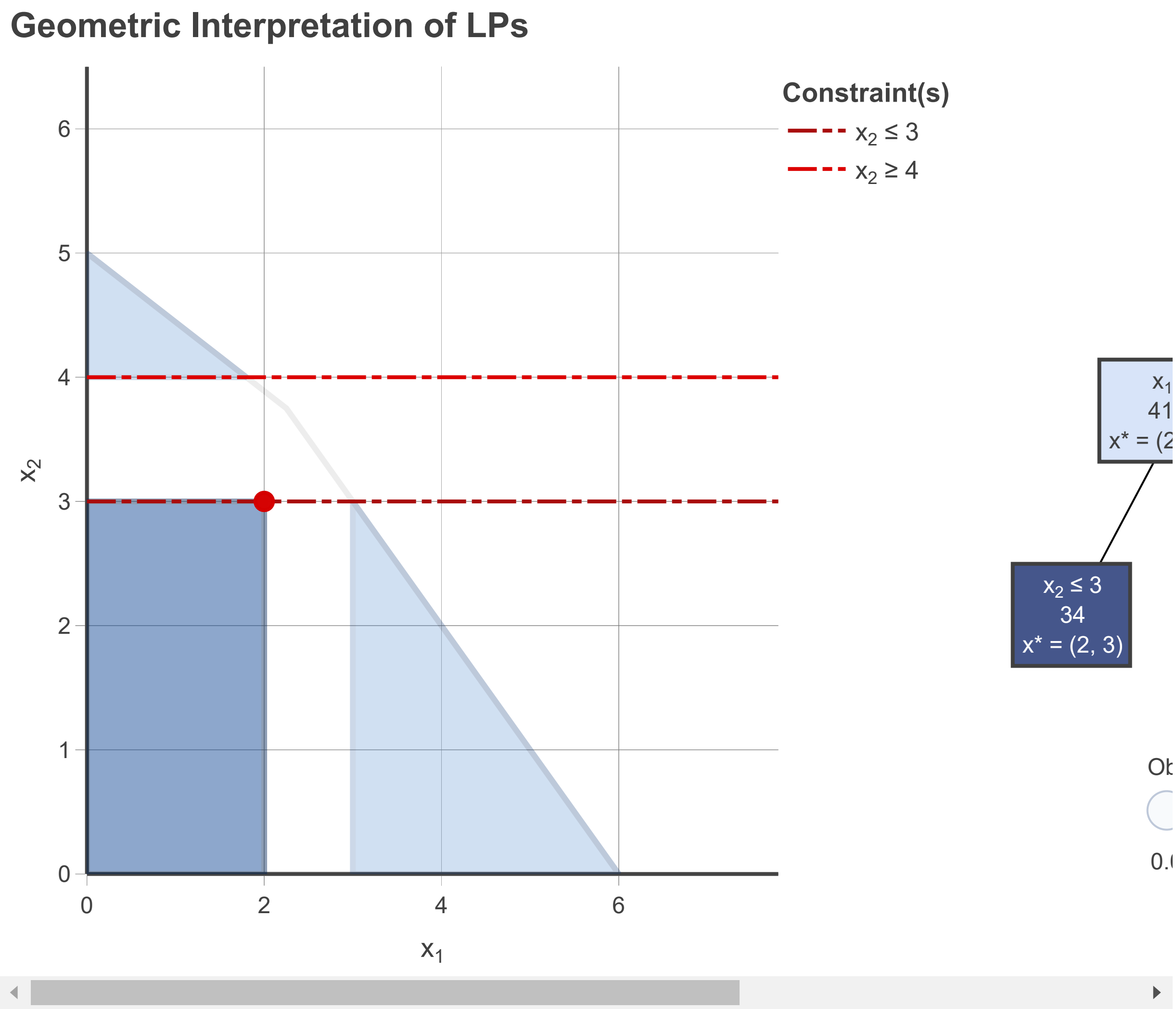
In [29]:
```
nodes[2].show()
```




**Q37:** What is the optimal solution at the current (dark) node? Do we have to further explore this branch? Explain.

**A:** The optimal solution is an integer, 34, so we no longer have to explore this branch.

**Q38:** Recall shaded nodes have been explored and the node shaded darker (and feasible region shaded darker) correspond to the current node and its feasible region. Nodes not shaded have not been explored. How many nodes have not yet been explored?

**A:** Two nodes have not yet been explored.

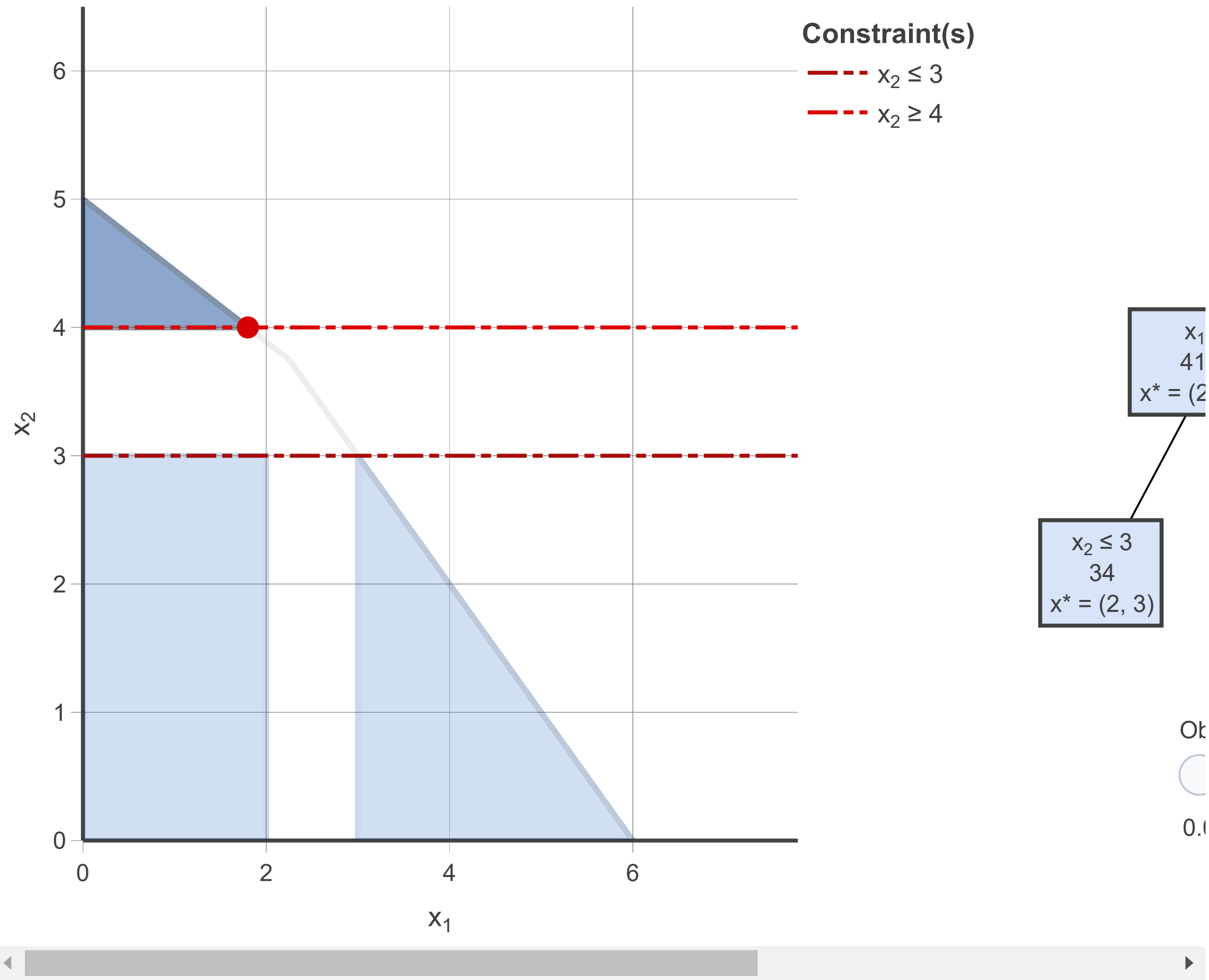
**Q39:** How many nodes have a degree of one in the branch and bound tree? (That is, they are only connected to one edge). These nodes are called leaf nodes. What is the relationship between the leaf nodes and the remaining feasible region?

**A:** There are three leaf nodes, each of which correspond to a remaining feasible region.

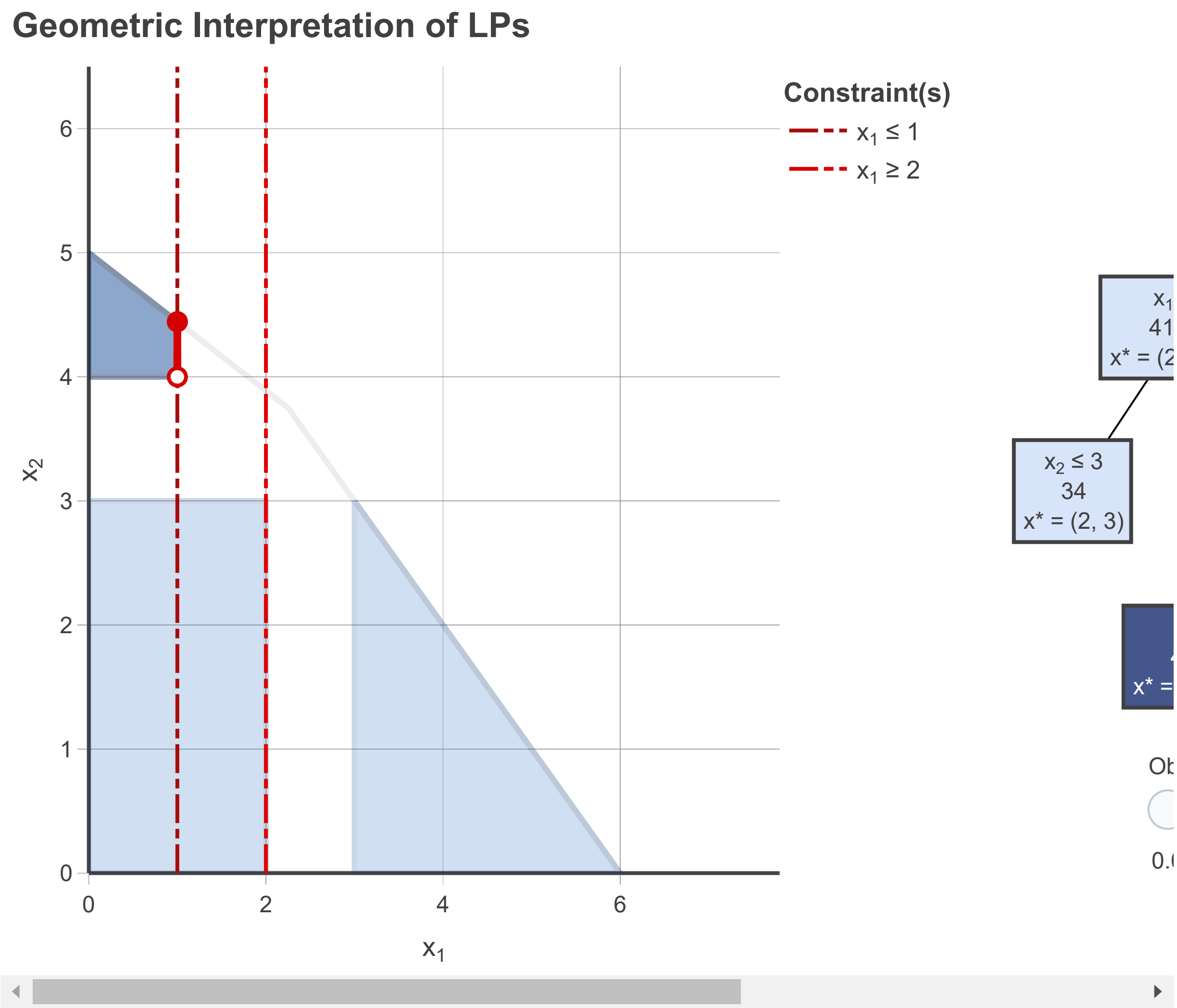
In [30]:
```
# Show the next two iterations of the branch and bound algorithm
nodes[3].show()
nodes[4].show()
```

Processing math: 16%

## Geometric Interpretation of LPs




Processing math: 16%




**Q40:** At the current (dark) node, we added the constraint x_1 \leq 1. Why were the fractional solutions 1 < x_1 < 2 not eliminated for x_2 <= 3?

**A:** The optimal value found within x_2 \leq 3 was an integer, so we did not have to further restrict the area.

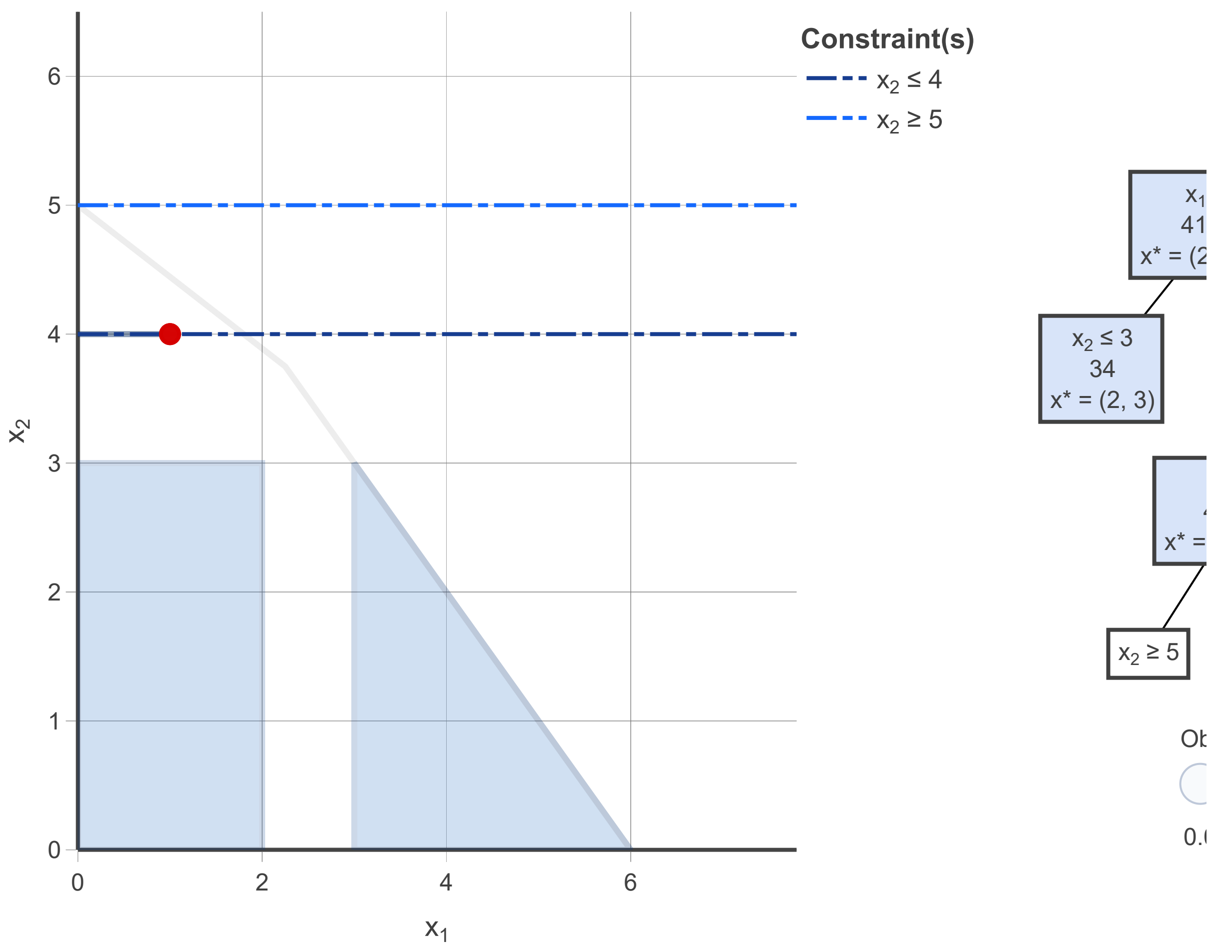
Processing math: 16%

In [31]:

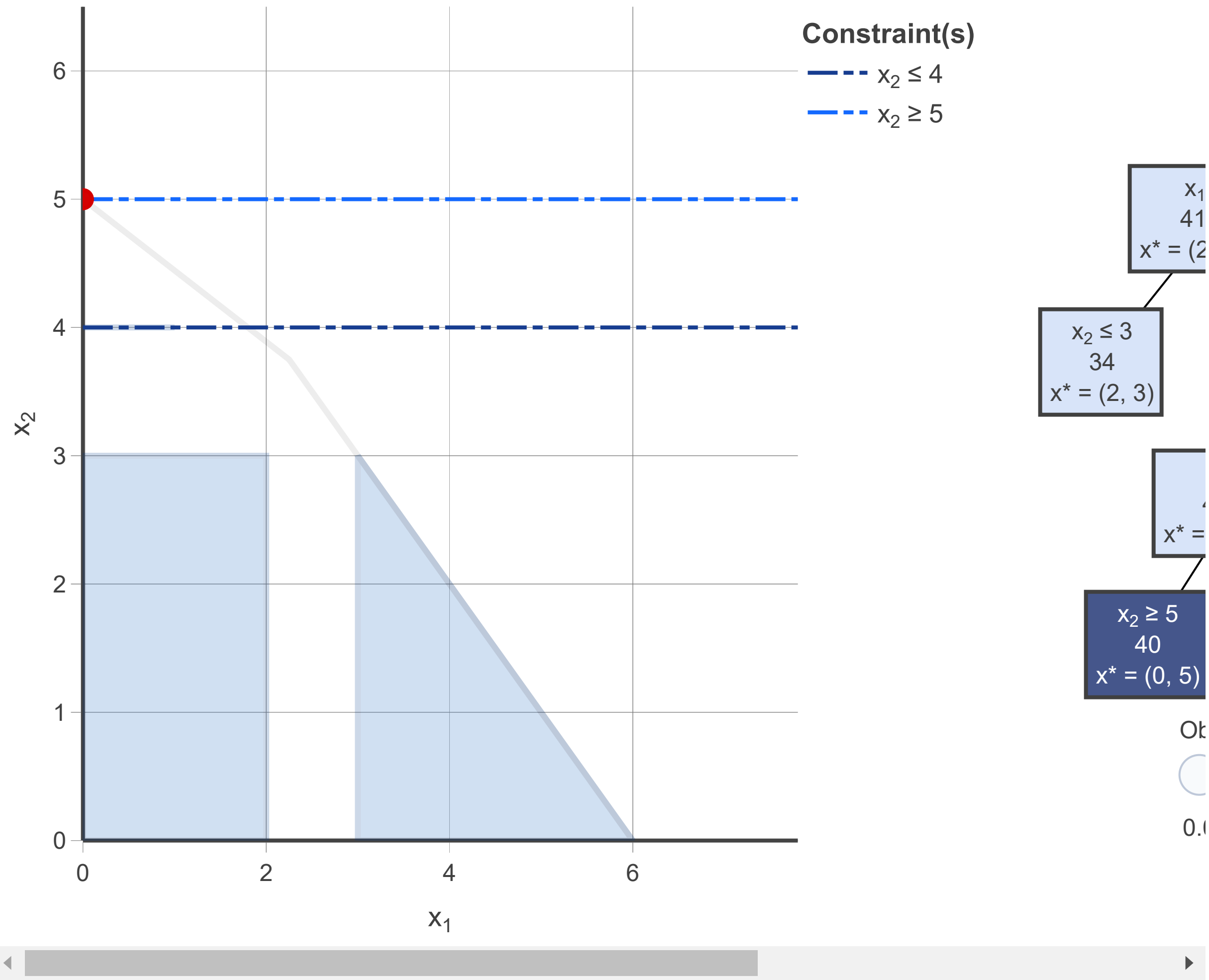
```
# Show the next three iterations of the branch and bound algorithm
nodes[5].show()
nodes[6].show()
nodes[7].show()
```

Processing math: 16%

## Geometric Interpretation of LPs

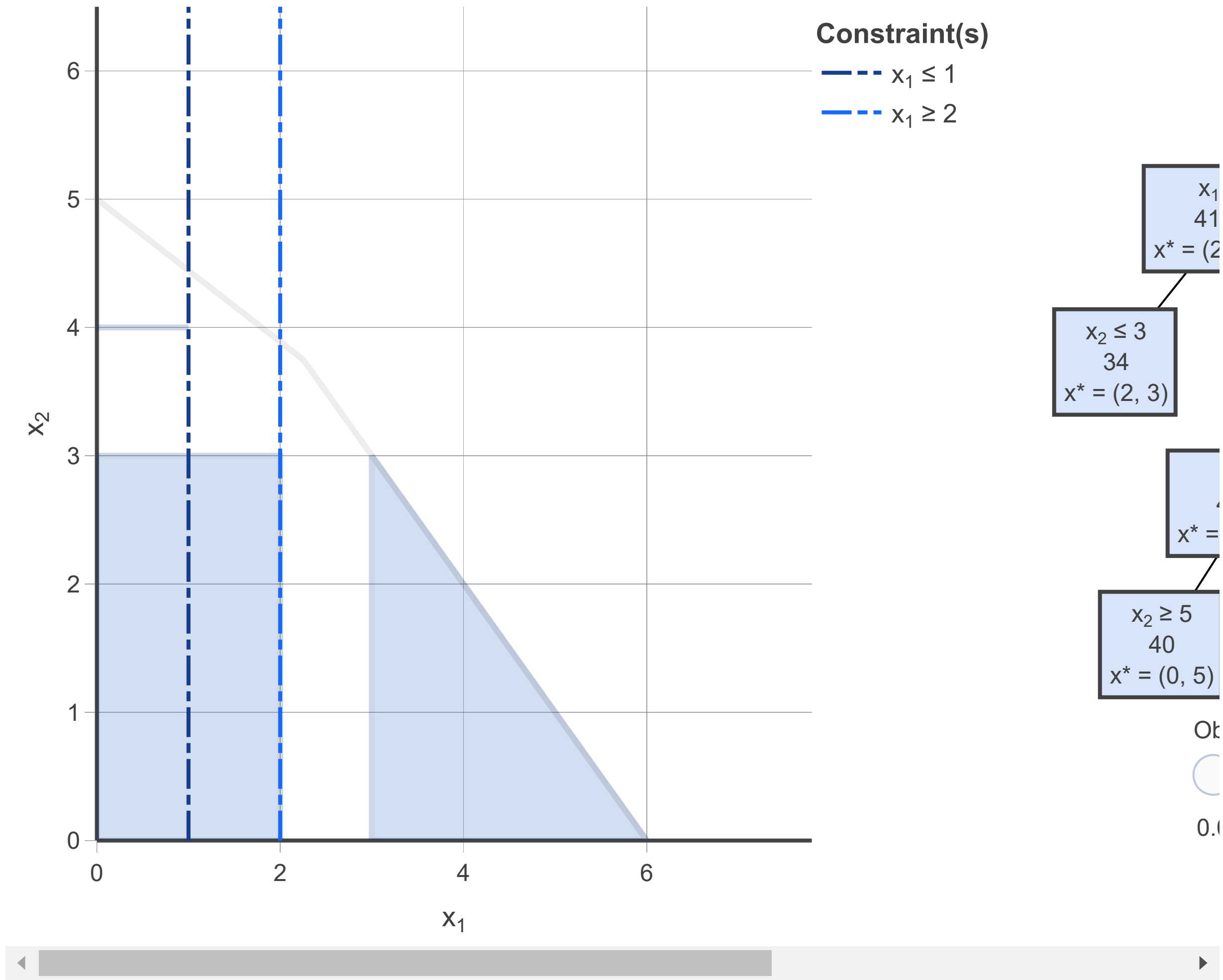


## Geometric Interpretation of LPs




Processing math: 16%

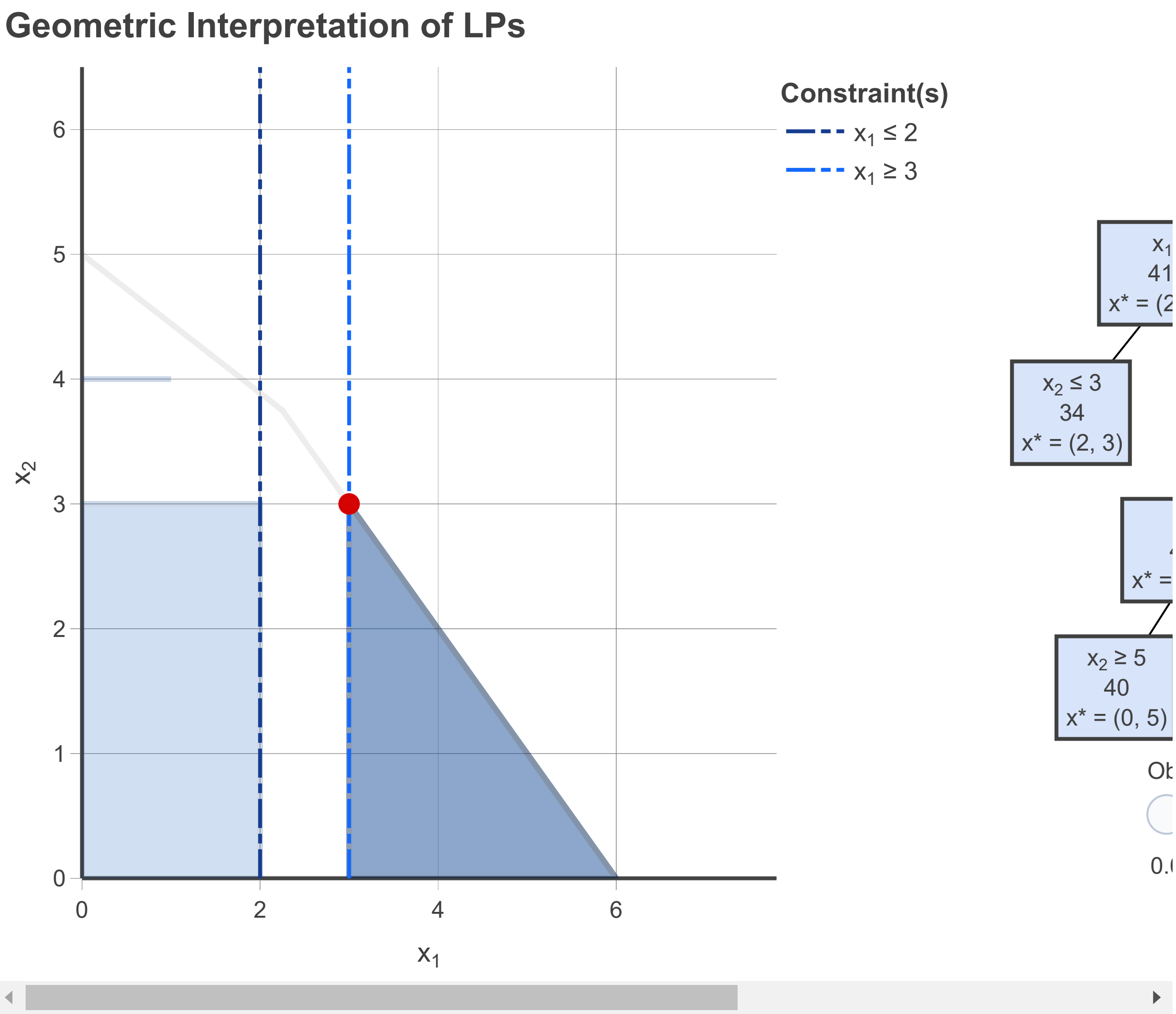
## Geometric Interpretation of LPs




**Q41:** What constraints are enforced at the current (dark) node? Why are there no feasible solutions at this node?

**A:** The constrants enforced are $x_1 \leq 2$, $x_2 \geq 4$, and $x_1 \geq 2$. The only possible $x_1$ value is 2, which isnt possible when $x_2 \geq 4$.

In [32]:
```
nodes[8].show()
```




**Q42:** Are we done? If so, what nodes are fathomed and what is the optimal solution? Explain.

**A:** We are done, as we have found integer or infeasible solutions at every leaf node.

Let's look at branch and bound visualization for an integer program with 3 decision variables!

In [33]:
```
nodes = gilp.bnb_visual(gilp.examples.VARIED_BRANCHING_3D_IP)
```

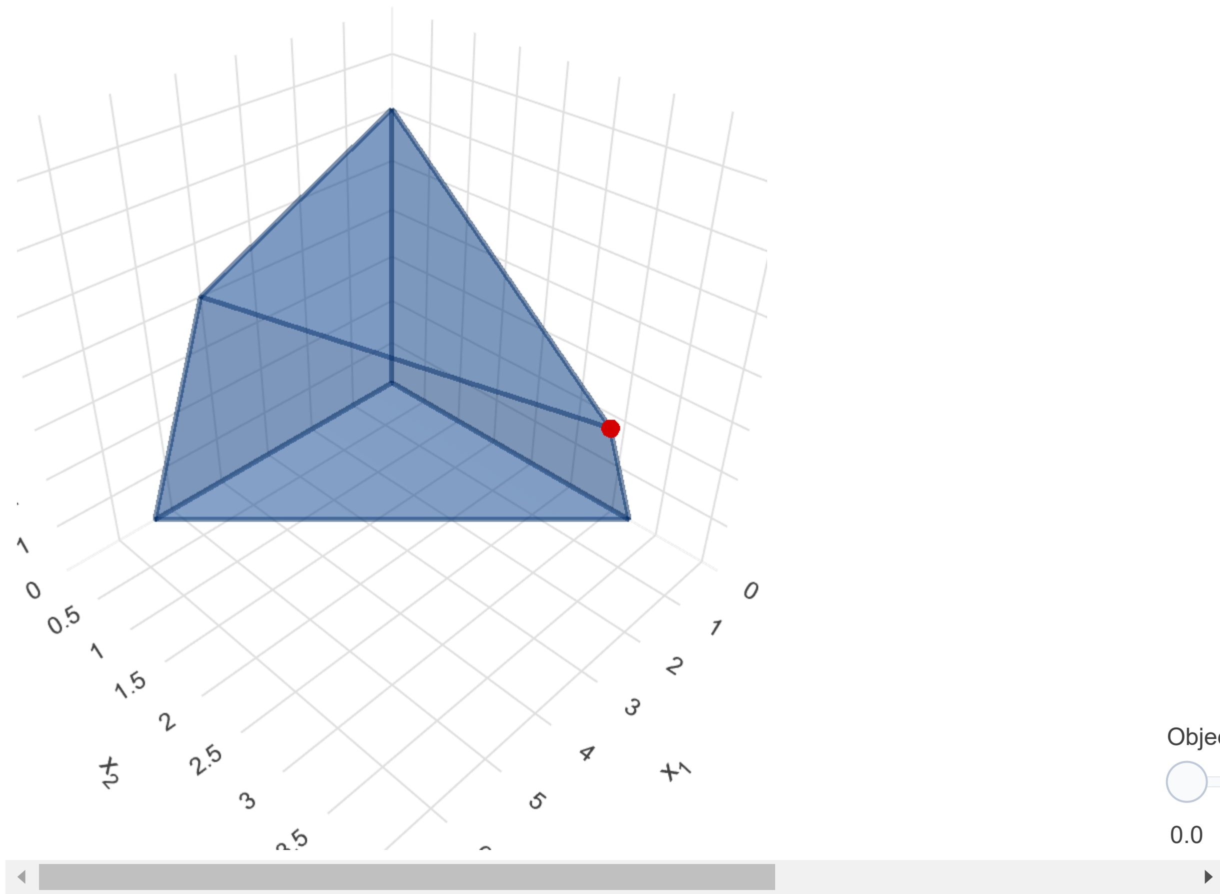
Processing math: 16%

In [34]:

```
# Look at the first 3 iterations
nodes[0].show()
nodes[1].show()
nodes[2].show()
```

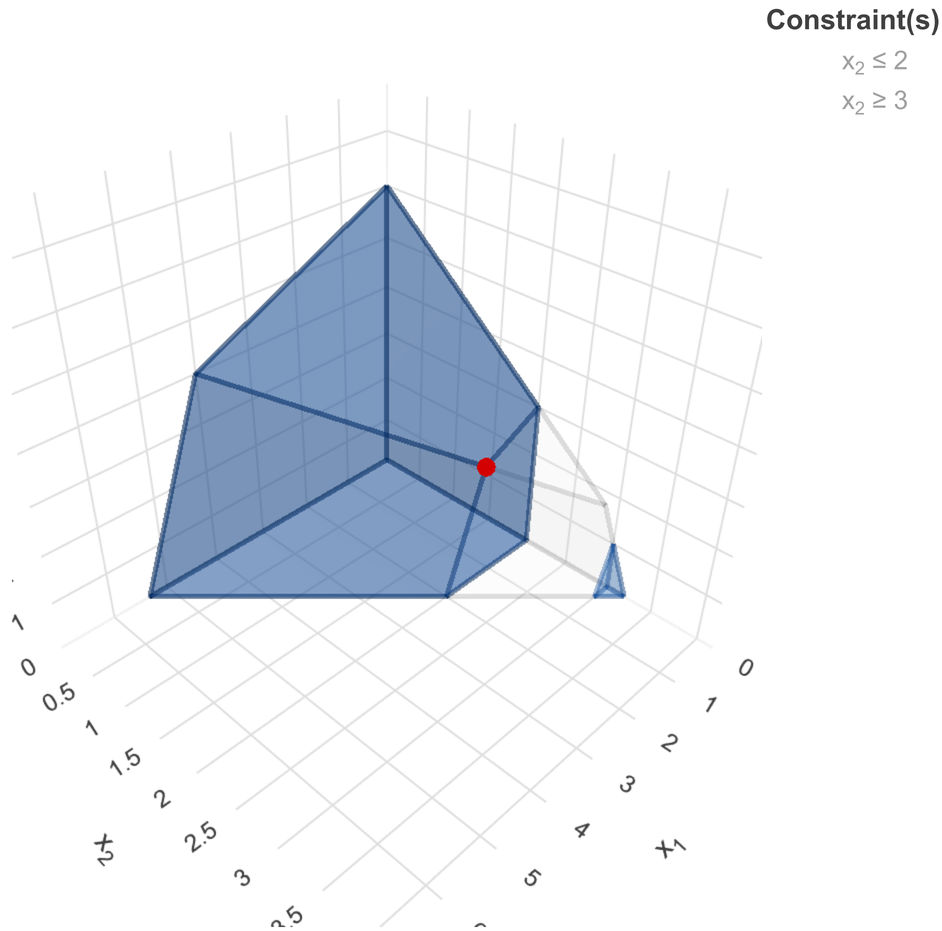
Processing math: 16%

## Geometric Interpretation of LPs




Processing math: 16%

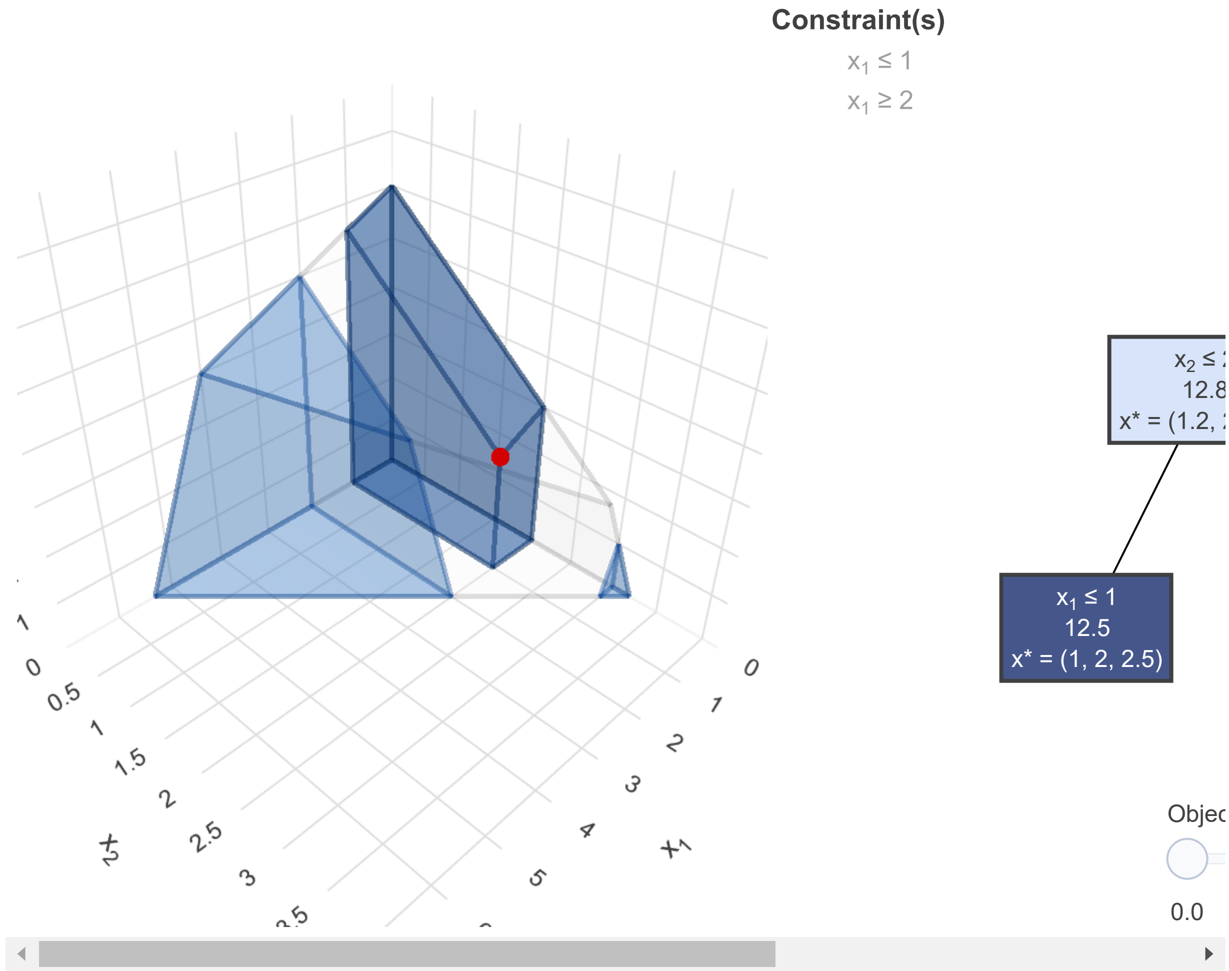
## Geometric Interpretation of LPs





Processing math: 16%

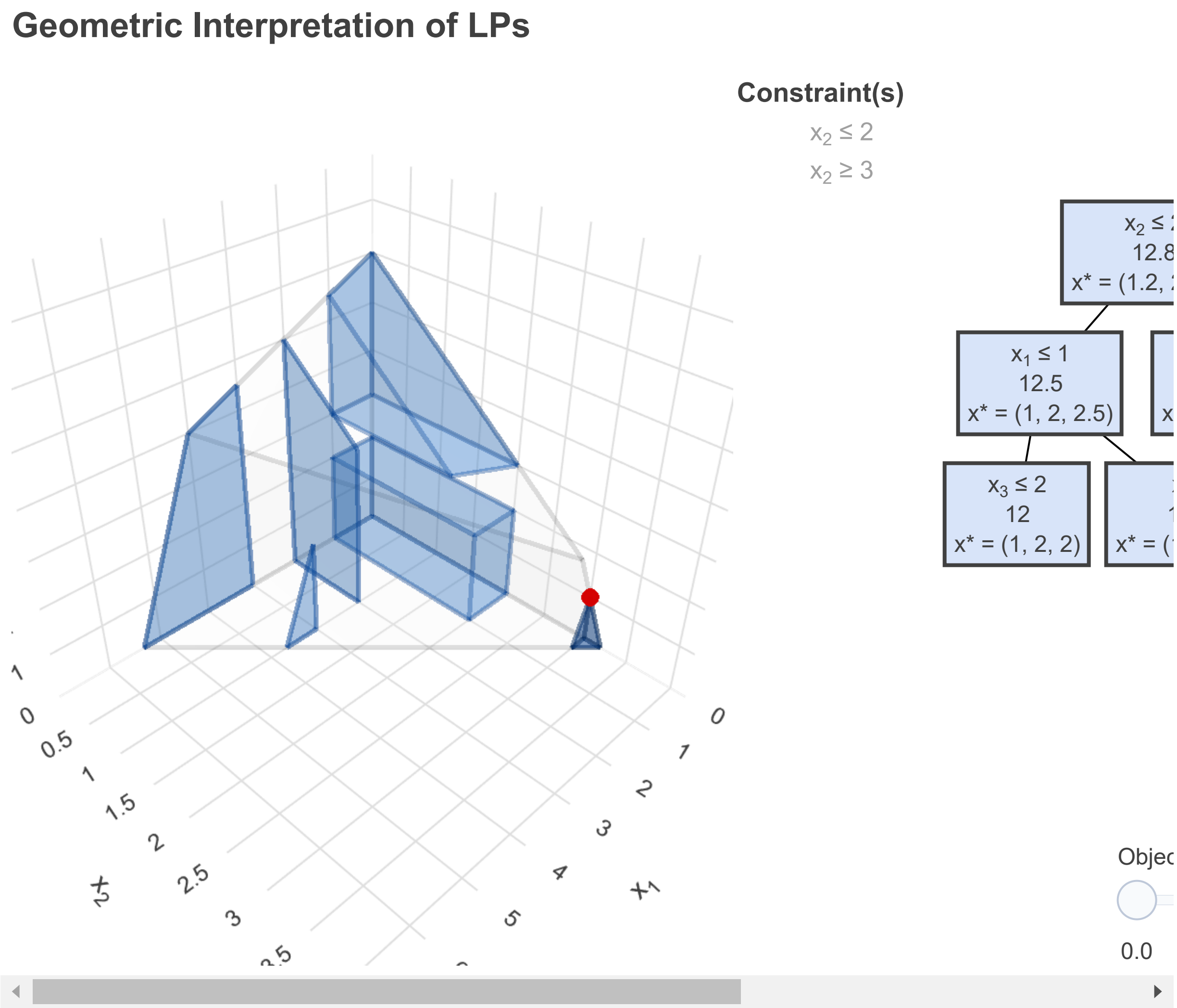
## Geometric Interpretation of LPs




Let's fast-forward to the final iteration of the branch and bound algorithm.

Processing math: 16%

```
In [35]: nodes[-1].show()
```




**Q43:** Consider the feasible region that looks like a rectangular box with one corner point at the origin. What node does it correspond to in the tree? What is the optimal solution at that node?

**A:** It corresponds to the leftmost node in the fourth row, with constraint x_3 \leq 2 The optimal solution at that node is (1,1,2) with value 12.

**Q44:** How many branch and bound nodes did we explore? What was the optimal solution? How many branch and bound nodes would we have explored if we knew the value of the optimal solution before starting branch and bound?

**A:** We would only have explored three nodes, as the first node on the second row is already less than the optimal solution.

## Bonus: Branch and Bound for Knapsack

Consider the following example:

| item | value | weight |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 9 | 3 |
| 3 | 6 | 2 |

The linear program formulation will be:

$$\begin{align*} \max \quad & 2x_1+9x_2+6x_3\\ \text{s.t.} \quad & 1x_1+3x_2+2x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0, \quad \text{integer} \end{align*}$$

In gilp, we can define this lp as follows:

In [ ]:
```
lp = gilp.LP([[1,3,2]],
             [10],
             [2,9,6])

for fig in gilp.bnb_visual(lp):
    fig.show()
```