Branch & Bound and Knapsack Lab

Objectives

- Preform the branch and bound algorithm
- Apply branch and bound to the knapsack problem
- Understand the geometry of the branch and bound algorithm

Brief description: In this lab, we will try solving an example of a knapsack problem with the branch-and-bound algorithm. We will also see how adding a cutting plane helps in reducing the computation time and effort of the algorithm. Lastly, we will explore the geometry of the branch and bound algorithm.

In [1]:
```
# imports -- don't forget to run this cell
import pandas as pd
import gilp
from gilp.visualize import feasible_integer_pts
from ortools.linear_solver import pywraplp as OR
```

Part 1: Branch and Bound Algorithm

Recall that the branch and bound algorithm (in addition to the simplex method) allows us to solve integer programs. Before applying the branch and bound algorithm to the knapsack problem, we will begin by reviewing some core ideas. Furthermore, we will identify a helpful property that will make branch and bound terminate quicker later in the lab!

Q1: What are the different ways a node can be fathomed during the branch and bound algorithm? Describe each.

A: We can round the value of the variable up to be greater than or equal to the next integer value. Or, We can round the value of the variable down to be less than or equal to the next integer value.

Q2: Suppose you have a maximization integer program and you solve its linear program relaxation. What does the LP-relaxation optimal value tell you about the IP optimal value? What if it is a minimization problem?

A: If it is a maximization problem, the LP optimal value is the upper bound for our IP optimal value. If it is a minimization problem, the LP optimal value is the lower bound for our IP optimal value.

Q3: Assume you have a maximization integer program with all integral coefficents in the objective function. Now, suppose you are running the branch and bound algorithm and come across a node with an optimal value of 44.5. The current incumbent is 44. Can you fathom this node? Why or why not?

A: Yes, I can. It is because 44 is an integer optimal solution this node can best possibly give in this maximization problem. Since 44 is not greater than the current incumbent 44, the node can be fathomed.

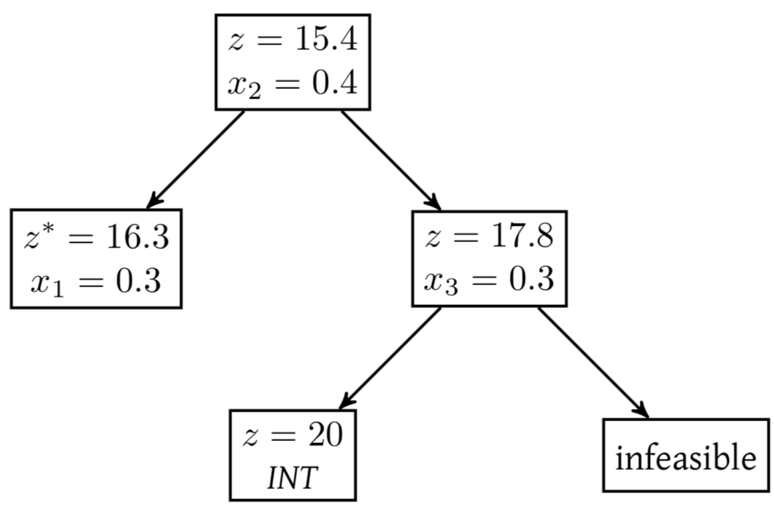
Q4: If the optimal solution to the LP relaxation of the original program is integer, then you have found an optimal solution to your integer program. Explain why this is true.

A: The IP optimal solution can not be better than its corresponding LP optimal solution. So, if the IP optimal solution could be its corresponding LP optimal solution, this optimal solution for LP must also be the optimal solution for the IP relaxation problem.

Q5: If the LP is infeasible, then the IP is infeasible. Explain why this is true.

A: A IP solution is also feasible for a LP. So, if the LP is infeasible, the included IP solution is also infeasible.

The next questions ask about the following branch and bound tree. If the solution was not integral, the fractional x_i that was used to branch is given. If the solution was integral, it is denoted *INT*. In the current iteration of branch and bound, you are looking at the node with the *****.



Q6: Can you determine if the integer program this branch and bound tree is for is a minimization or maximixation problem? If so, which is it?

A: Yes, I can determine that. It is a maximixation problem because in this tree we seek variables that make the objective value greater.

Hint: For **Q7-8**, you can assume integral coefficents in the objective function.

Q7: Is the current node (marked z^*) fathomed? Why or why not? If not, what additional constraints should be imposed for each of the next two nodes?

A: The current node is not fathomed. It's because the bound is better than the value of the best feasible integer solution we get so far.

Q8: Consider the nodes under the current node (where $z = 16.3$). What do you know about the optimal value of these nodes? Why?

A: The optimal value of these nodes must be at most 16.3.

Part 2: The Knapsack Problem

In this lab, you will solve an integer program by branch and bound. The integer program to be solved will be a knapsack problem.

Knapsack Problem: We are given a collection of n items, where each item $i = 1, \dots, n$ has a weight w_i and a value v_i. In addition, there is a given capacity W, and the aim is to select a maximum value subset of items that has a total weight at most W. Note that each item can be brought at most once.

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} v_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i x_i \leq W \\ & 0 \leq x_i \leq 1, \text{integer}, i = 1, \dots, n \end{aligned}$$

Consider the following data which we import from a CSV file:

In [2]:
```
data = pd.read_csv('knapsack_data_1.csv', index_col=0)
data
```

Out[2]:

item	value	weight
1	50	10
2	30	12
3	24	10
4	14	7
5	12	6
6	10	7
7	40	30

and $W = 18$.

Q9: Are there any items we can remove from our input to simplify this problem? Why? If so, replace `index` with the item number that can be removed in the code below. Hint: how many of each item could we possibly take?

A: We can remove item 7 because its own weight already surpasses our weight limit, which is 18. So, we cannot even bring one of item 7.

In [3]:
```
# TODO: replace index
data = data.drop(7)
```

Q10: If we remove item 7 from the knapsack, it does not change the optimal solution to the integer program. Explain why.

A: It's because we will never bring item 7 in any feasible solution. Thus, removing item 7 from the knapsack will not affect any of the feasible solutions to the integer program.

Q11: Consider removing items i such that $w_i > W$ from a knapsack input. How does the LP relaxation's optimal value change?

A: The LP relaxation's optimal value will not change since the removed item is never included in any of the feasible solution.

In **Q10-11**, you should have found that removing these items removes feasible solutions from the linear program but does not change the integer program. This is desirable as the gap between the optimal IP and LP values can become smaller. By adding this step, branch and bound may terminate sooner.

Recall that a branch and bound node can be fathomed if its bound is no better than the value of the best feasible integer solution found thus far. Hence, it helps to have a good feasible integer solution as quickly as possible (so that we stop needless work). To do this, we can first try to construct a good feasible integer solution by a reasonable heuristic algorithm before starting to run the branch and bound procedure.

In designing a heuristic for the knapsack problem, it is helpful to think about the value per unit weight for each item. We compute this value in the table below.

In [4]:
```
data['value per unit weight'] = (data['value'] / data['weight']).round(2)
data
```

Out[4]:

	value	weight	value per unit weight
item			
1	50	10	5.00
2	30	12	2.50
3	24	10	2.40
4	14	7	2.00
5	12	6	2.00
6	10	7	1.43

Q12: Design a reasonable heuristic for the knapsack problem. Note a heuristic aims to find a decent solution to the problem (but is not necessarily optimal).

A: We include all of the items in a list. We find the item of the greatest value per unit weight and add it to our feasible solution. We then eliminate this item from our list. Then, we find the second item of the greatest value per unit weight and do the previous actions and so forth until we cannot add another item that still meet the constraint wi<W.

Q13: Run your heuristic on the data above to compute a good feasible integer solution. Your heuristic should generate a feasible solution with a value of 64 or better. If it does not, try a different heuristic (or talk to your TA!)

A: So, first we add item 1 in our feasible solution. Then, item 4 is the next item we can add while still meet the constraint wi<W. Finally, no other item can be add while still meet the constraint. The value is 50+14=64. The weight is 10+7=17.

We will now use the branch and bound algorithm to solve this knapsack problem! First, let us define a mathematical model for the linear relaxation of the knapsack problem.

Q14: Complete the model below.

In [5]:
```
def Knapsack(table, capacity, integer = False):
    """Model for solving the Knapsack problem.

    Args:
        table (pd.DataFrame): A table indexd by items with a column for value and
        capcity (int): An integer-capacity for the knapsack
        integer (bool): True if the variables should be integer. False otherwise.
    """
    ITEMS = list(table.index)        # set of items
    v = table.to_dict()['value']     # value for each item
    w = table.to_dict()['weight']    # weight for each item
    W = capacity                     # capacity of the knapsack

    # define model
    m = OR.Solver('knapsack', OR.Solver.CBC_MIXED_INTEGER_PROGRAMMING)

    # decision variables
    x = {}
    for i in ITEMS:
        if integer:
            x[i] = m.IntVar(0, 1, 'x_%d' % (i))
        else:
            x[i] = m.NumVar(0, 1, 'x_%d' % (i))

    # define objective function here
    m.Maximize(sum(v[i]*x[i] for i in ITEMS))

    # TODO: Add a constraint that enforces that weight must not exceed capacity
    # recall that we add constraints to the model using m.Add()
    m.Add(sum(w[i]*x[i] for i in ITEMS)<=W)

    return (m, x)  # return the model and the decision variables
```

In [6]:
```
# You do not need to do anything with this cell but make sure you run it!
def solve(m):
    """Used to solve a model m."""
    m.Solve()

    print('Objective =', m.Objective().Value())
    print('iterations :', m.iterations())
    print('branch-and-bound nodes :',m.nodes())

    return ({var.name() : var.solution_value() for var in m.variables()})
```

We can now create a linear relaxation of our knapsack problem. Now, `m` represents our model and `x` represents our decision variables.

In [7]:
```
m, x = Knapsack(data, 18)
```

We can use the next line to solve the model and output the solution

In [8]:
```
solve(m)
```

```
Objective = 70.0
iterations : 0
branch-and-bound nodes : 0
```

Out[8]:
```
{'x_1': 1.0,
 'x_2': 0.6666666666666667,
 'x_3': 0.0,
 'x_4': 0.0,
 'x_5': 0.0,
 'x_6': 0.0}
```

Q15: How does this optimal value compare to the value you found using the heuristic integer solution?

A: This optimal value of 70 is greater than the value I found, which is 64.

Q16: Should this node be fathomed? If not, what variable should be branched on and what additional constraints should be imposed for each of the next two nodes?

A: No, this node cannot be fathomed. The variable x2 is not an integer. So, this variable should be branched on. One of the next nodes has the constraint of x2<=0 while another one of the next node has the constraint of x2>=1.

After constructing the linear relaxation model using `Knapsack(data1, 18)` we can add additional constraints. For example, we can add the constraint $x_2 \leq 0$ and solve it as follows:

In [9]:
```
m, x = Knapsack(data, 18)
m.Add(x[2] <= 0)
solve(m)
```

```
Objective = 69.2
iterations : 0
branch-and-bound nodes : 0
```

Out[9]:
```
{'x_1': 1.0, 'x_2': 0.0, 'x_3': 0.8, 'x_4': 0.0, 'x_5': 0.0, 'x_6': 0.0}
```

NOTE: The line `m, x = Knapsack(data1, 18)` resets the model `m` to the LP relaxation. All constraints from branching have to be added each time.

Q17: Use the following cell to compute the optimal value for the other node you found in **Q16**.

In [10]:
```python
# TODO: Answer Q17
m, x = Knapsack(data, 18)
m.Add(x[2] >= 1)
solve(m)
```

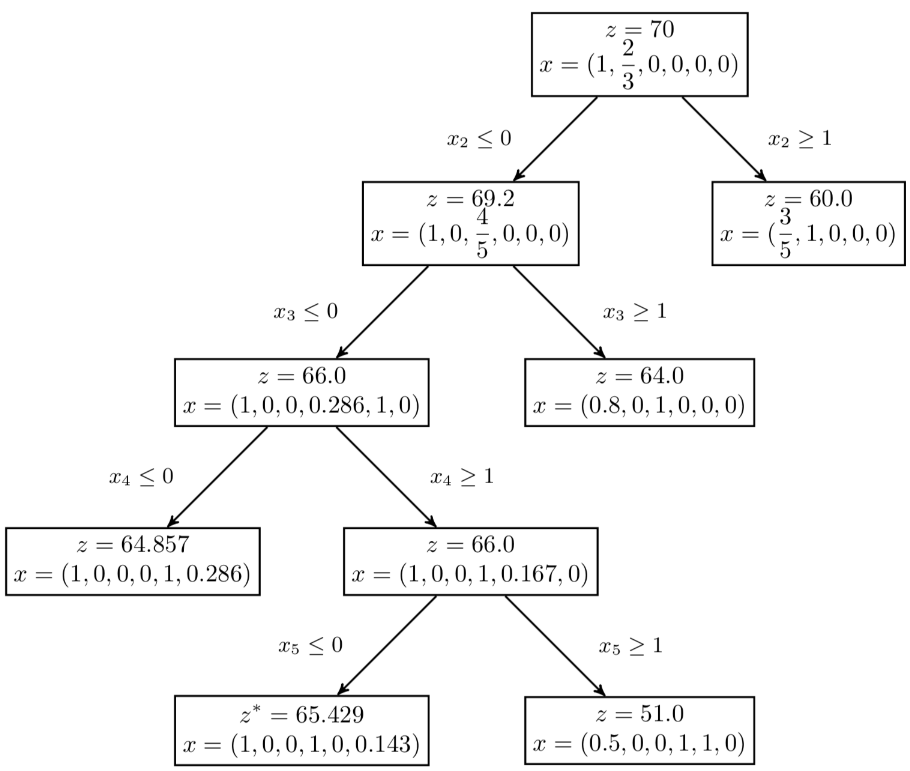
```
Objective = 60.0
iterations : 0
branch-and-bound nodes : 0
```

Out[10]:
```
{'x_1': 0.6000000000000001,
 'x_2': 1.0,
 'x_3': 0.0,
 'x_4': 0.0,
 'x_5': 0.0,
 'x_6': 0.0}
```

Q18: What was the optimal value? Can this node be fathomed? Why? (Hint: In **Q13**, you found a feasible integer solution with value 64.)

A: The optimal value is 69.2. This node cannot be fathomed because the bound, which is 69.2, is larger than our solution of 64 for the integer programming.

If we continue running the branch and bound algorithm, we will eventually reach the branch and bound tree below where the z^* indictes the current node we are looking at.



Q19: The node with $z = 64.857$ was fathomed. Why are we allowed to fathom this node? (Hint: think back to **Q3**)

A: Yes, the node is fathomed because the best integer solution it can give is 64, which is not greater than the optimal value of 64 we found in question 13. So, this node cannot give us any better solution than the current solution.

Q20: Finish running branch and bound to find the optimal integer solution. Use a separate cell for each node you solve and indicate if the node was fathomed with a comment. (Hint: Don't forget to include the constraints further up in the branch and bound tree.)

In [11]:
```
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[2] <= 0)
m.Add(x[3] <= 0)
m.Add(x[4] >= 1)
m.Add(x[5] <= 0)

solve(m)
# fathomed? It is not fathomed.
```

```
Objective = 65.42857142857143
iterations : 0
branch-and-bound nodes : 0
```

Out[11]:
```
{'x_1': 1.0,
 'x_2': 0.0,
 'x_3': 0.0,
 'x_4': 1.0,
 'x_5': 0.0,
 'x_6': 0.14285714285714302}
```

In [12]:
```
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[2] <= 0)
m.Add(x[3] <= 0)
m.Add(x[4] >= 1)
m.Add(x[5] <= 0)
m.Add(x[6] <= 0)
solve(m)
# fathomed? It is fathomed.
```

```
Objective = 64.0
iterations : 0
branch-and-bound nodes : 0
```

Out[12]:
```
{'x_1': 1.0, 'x_2': 0.0, 'x_3': 0.0, 'x_4': 1.0, 'x_5': 0.0, 'x_6': 0.0}
```

In [13]:
```
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[2] <= 0)
m.Add(x[3] <= 0)
m.Add(x[4] >= 1)
m.Add(x[5] <= 0)
m.Add(x[6] >= 1)
solve(m)
# fathomed? It is fathomed.
```

```
Objective = 44.0
iterations : 0
branch-and-bound nodes : 0
```

Out[13]:
```
{'x_1': 0.4, 'x_2': 0.0, 'x_3': 0.0, 'x_4': 1.0, 'x_5': 0.0, 'x_6': 1.0}
```

A: The first node is not fathomed while the second and third nodes are. The optimal integer solution is 64, as shown by the second node.

Q21: How many nodes did you have to explore while running the branch and bound algorithm?

A: I have explored nine nodes before I eventually found the optimal integer solution.

In the next section, we will think about additional constraints we can add to make running branch and bound quicker.

Part 3: Cutting Planes

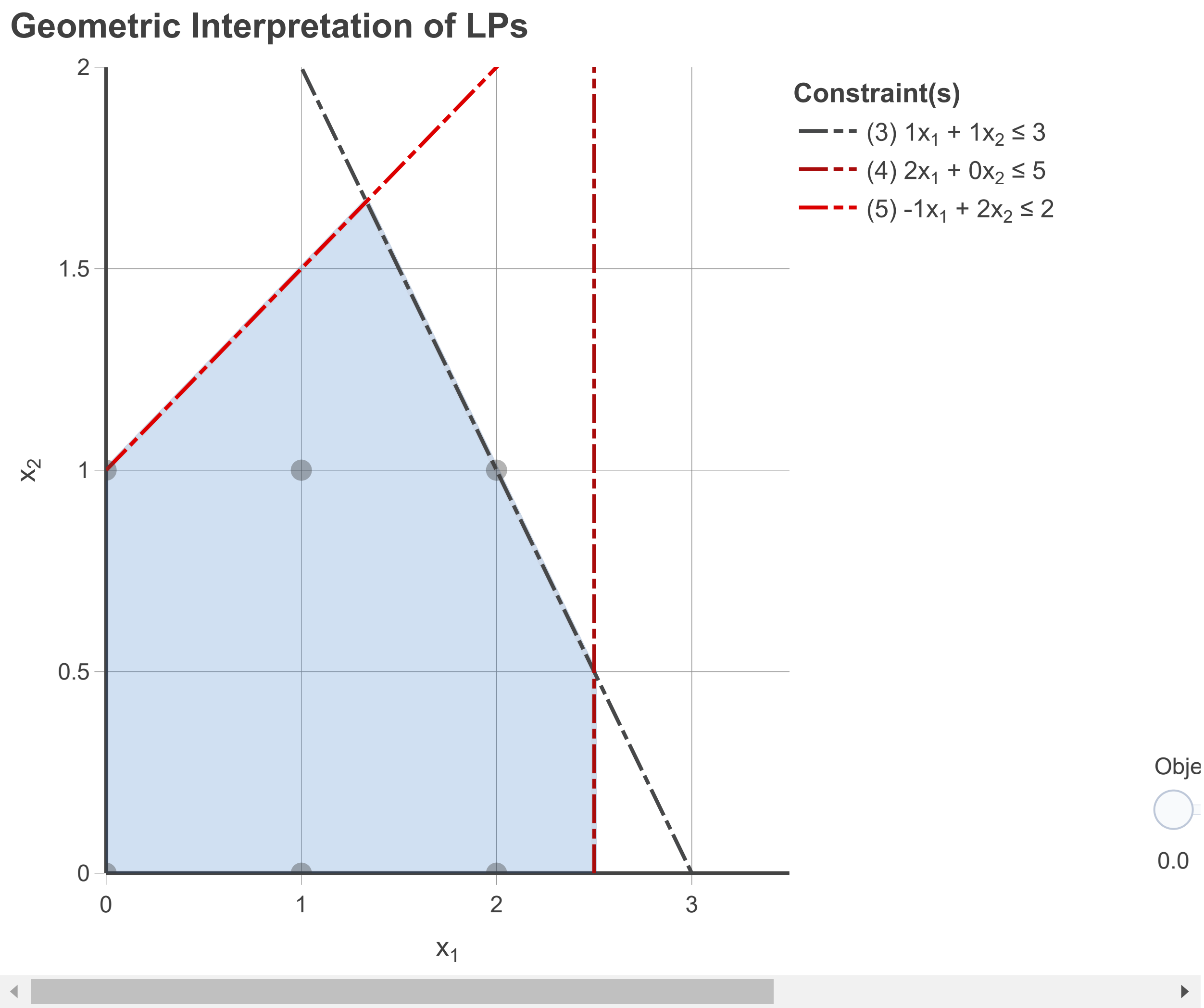
In general, a cutting plane is an additional constraint we can add to an integer program's linear relaxation that removes feasible linear solutions but does not remove any integer feasible solutions. This is very useful when solving integer programs! Recall many of the problems we have learned in class have something we call the "integrality property". This is useful because it allows us to ignore the integrality constraint since we are garunteed to reach an integral solution. By cleverly adding cutting planes, we strive to remove feasible linear solutions (without removing any integer feasible solutions) such that the optimal solution to the linear relaxation is integral!

Conisder an integer program whose linear program releaxation is

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 3 \\ & 2x_1 \leq 5 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We can define this linear program and then visualize its feasible region. The integer points have been highlighted.

In [14]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2]],
             [3,5,2],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```



Q22: List every feasible solution to the integer program.

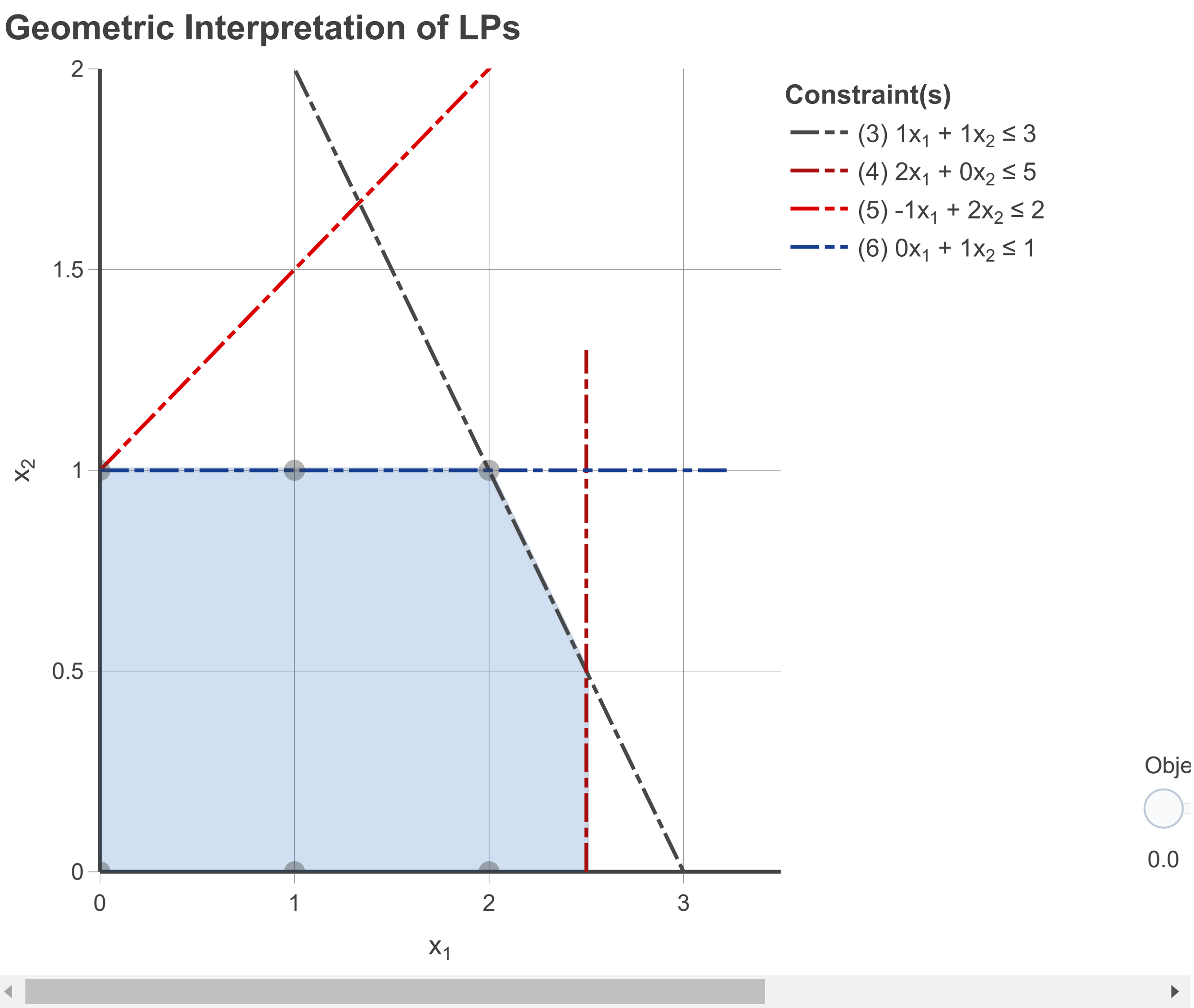
A: The feasible solutions (x,y) are (0,0), (1,0), (2,0), (0,1), (1,1), and (2,1).

Q23: Is the constraint $x_2 \leq 1$ a cutting plane? Why? (Hint: Would any feasible integer points become infeasible? What about feasible linear points?)

A: It is a cutting plane. Every feasible poimts are still feasible while some feasible linear points become infeasible.

Let's add this cutting plane to the LP relaxation!

In [15]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2],[0,1]],
             [3,5,2,1],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```



Q24: Is the constraint $x_1 \leq 3$ a cutting plane? Why?

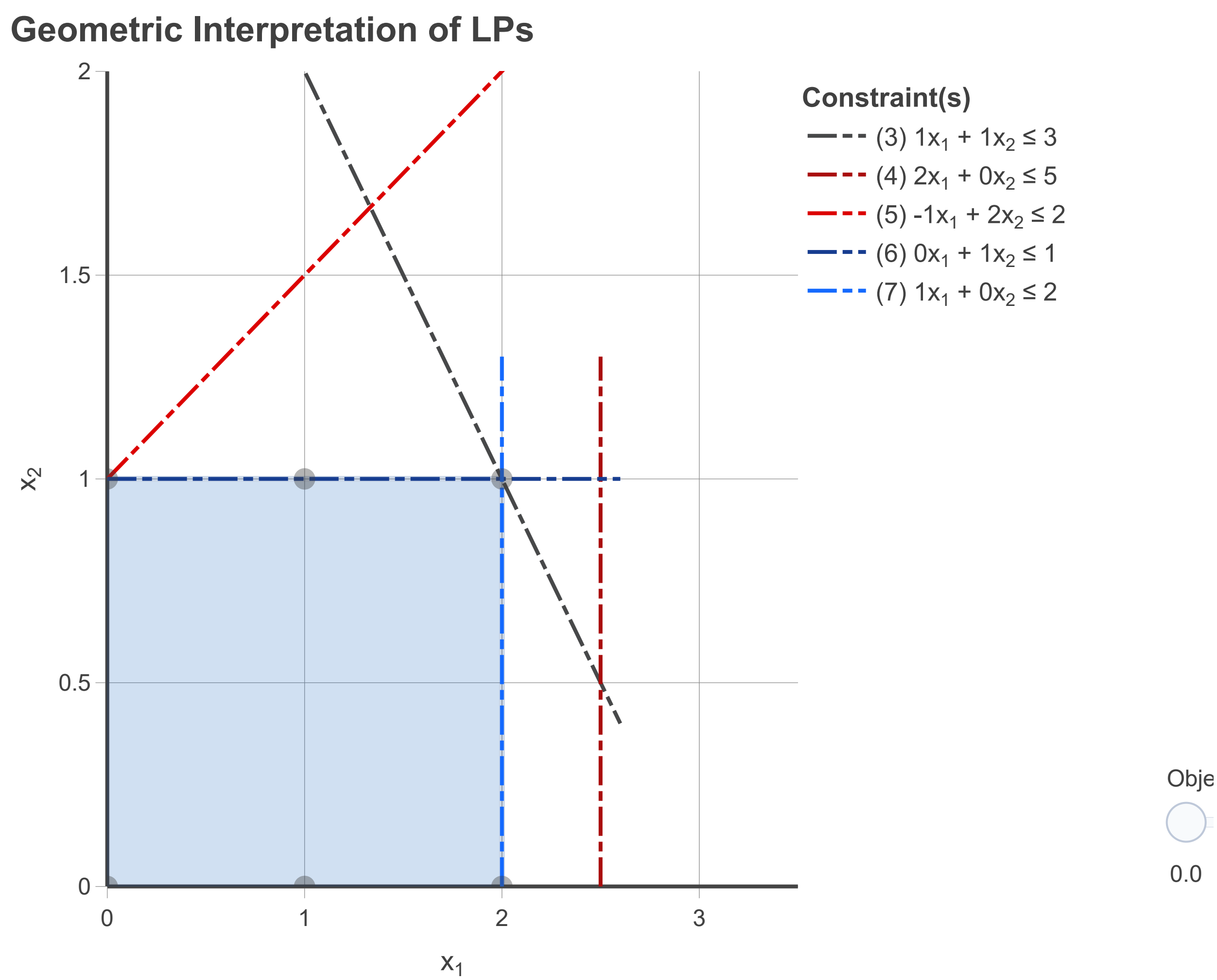
A: No, it is not a cutting plane. Every feasible poimts are still feasible but the feasible linear points before is still feasible after adding the constraint.

Q25: Can you provide another cutting plane? If so, what is it?

A: The constraint $x1≤2$ is a cutting plane because every feasible poimts before are still feasible while some feasible linear points before become infeasible.

Let's look at the feasible region after adding the cutting plane from **Q23** and one of the possible answers from **Q25**. Notice the optimal solution to the LP relaxation is now integral!

In [16]:
```
lp = gilp.LP([[1,1],[2,0],[-1,2],[0,1],[1,0]],
             [3,5,2,1,2],
             [2,1])
fig = gilp.lp_visual(lp)
fig.set_axis_limits([3.5,2])
fig.add_trace(feasible_integer_pts(lp, fig))
fig
```



Let's try applying what we know about cutting planes to the knapsack problem! Again, recall our input was $W = 18$ and:

In [17]:
```
data
```

Out[17]:

	value	weight	value per unit weight
item			
1	50	10	5.00
2	30	12	2.50
3	24	10	2.40
4	14	7	2.00
5	12	6	2.00
6	10	7	1.43

Q26: Look at items 1, 2, and 3. How many of these items can we take simultaneously? Can you write a new constraint to capture this? If so, please provide it.

A: We can only take one simultaneously. The new constraint is x1+x2+x3<=1.

Q27: Is the constraint you found in **Q26** a cutting plane? If so, provide a feasible solution to the linear program relaxation that is no longer feasible (i.e. a point the constraint *cuts off*).

A: Yes, it is a cutting plane. An example is x1=0.5, x2=0.5, and x3=0.

Q28: Provide another cutting plane involving items 4,5 and 6 for this integer program. Explain how you derived it.

A: Another cutting plane is x4+x5+x6<=2. It is because the sume of weight of either two of the three items is less than W=18 but the sum ofthe weight of all three items is greater than W=18.

Q29: Add the cutting planes from **Q26** and **Q28** to the model and solve it. You should get a solution in which we take items 1 and 4 and $\frac{1}{6}$ of item 5 with an objective value of 66.

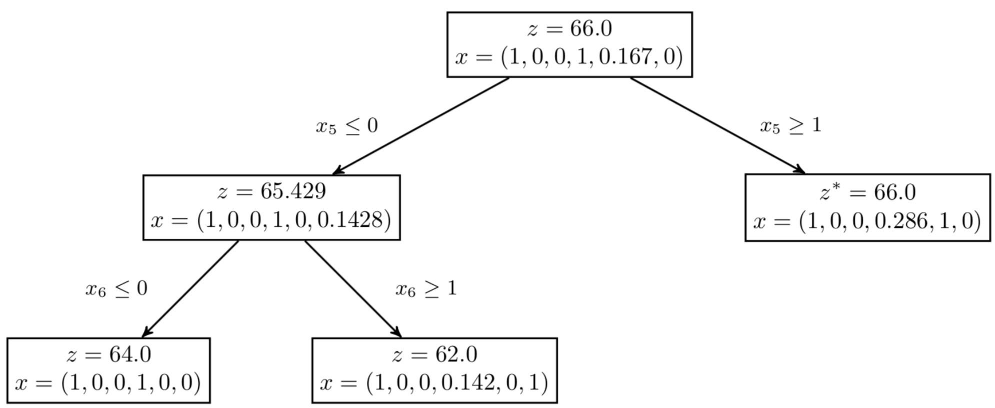
In [18]:
```
m, x = Knapsack(data, 18)
# TODO: Add cutting planes here
m.Add(x[1]+x[2]+x[3] <= 1)
m.Add(x[4]+x[5]+x[6] <= 2)
solve(m)
```

```
Objective = 66.0
iterations : 0
branch-and-bound nodes : 0
```

Out[18]:
```
{'x_1': 1.0,
 'x_2': 0.0,
 'x_3': 0.0,
 'x_4': 1.0,
 'x_5': 0.16666666666666696,
 'x_6': 0.0}
```

Let's take a moment to pause and reflect on what we are doing. Recall from **Q9-11** that we dropped item 7 becuase its weight was greater than the capcity of the knapsack. Essentially we added the constraint $x_7 \leq 0$. This constraint was a cutting plane! It eliminated some linear feasible solutions but no integer ones. By adding these two new cutting planes, we can get branch and bound to terminate earlier yet again! So far, we have generated cutting planes by inspection. However, there are more algorithmic ways to identify them (which we will ignore for now).

If we continue running the branch and bound algorithm, we will eventually reach the branch and bound tree below where the z^* indictes the current node we are looking at.



NOTE: Do not forget about the feasible integer solution our heuristic gave us with value 64.

Q30 Finish running branch and bound to find the optimal integer solution. Use a separate cell for each node you solve and indicate if the node was fathomed with a comment. Hint: Don't forget the cutting plane constraints should be included in every node of the branch and bound tree.

In [19]:
```
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[1]+x[2]+x[3] <= 1)
m.Add(x[4]+x[5]+x[6] <= 2)
m.Add(x[5] >= 1)

solve(m)
# fathomed? It is not fathomed.
```

```
Objective = 66.0
iterations : 0
branch-and-bound nodes : 0
```

Out[19]:
```
{'x_1': 1.0,
 'x_2': 0.0,
 'x_3': 0.0,
 'x_4': 0.28571428571428586,
 'x_5': 1.0,
 'x_6': 0.0}
```

In [20]:
```
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[1]+x[2]+x[3] <= 1)
m.Add(x[4]+x[5]+x[6] <= 2)
m.Add(x[5] >= 1)
m.Add(x[4] >= 1)

solve(m)
# fathomed? It is fathomed.
```

```
Objective = 51.0
iterations : 0
branch-and-bound nodes : 0
```

Out[20]:
```
{'x_1': 0.5, 'x_2': 0.0, 'x_3': 0.0, 'x_4': 1.0, 'x_5': 1.0, 'x_6': 0.0}
```

In [21]:
```
# Template
m, x = Knapsack(data, 18)
# Add constraints here
m.Add(x[1]+x[2]+x[3] <= 1)
m.Add(x[4]+x[5]+x[6] <= 2)
m.Add(x[5] >= 1)
m.Add(x[4] <= 0)

solve(m)
# fathomed? It is fathomed.
```

```
Objective = 64.85714285714286
iterations : 0
branch-and-bound nodes : 0
```

Out[21]:
```
{'x_1': 1.0,
 'x_2': 0.0,
 'x_3': 0.0,
 'x_4': 0.0,
 'x_5': 1.0,
 'x_6': 0.28571428571428586}
```

A: The first node is not fathomed whiler the other two nodes are. The optimal integer solution has the value of 64.

Q31: Did you find the same optimal solution? How many nodes did you explore? How did this compare to the number you explored previously?

A: I find the same optimal solution with the same value of 64. I explored 5 nodes, which is less then the number I explored previously, which is 9.

Part 4: Geometry of Branch and Bound

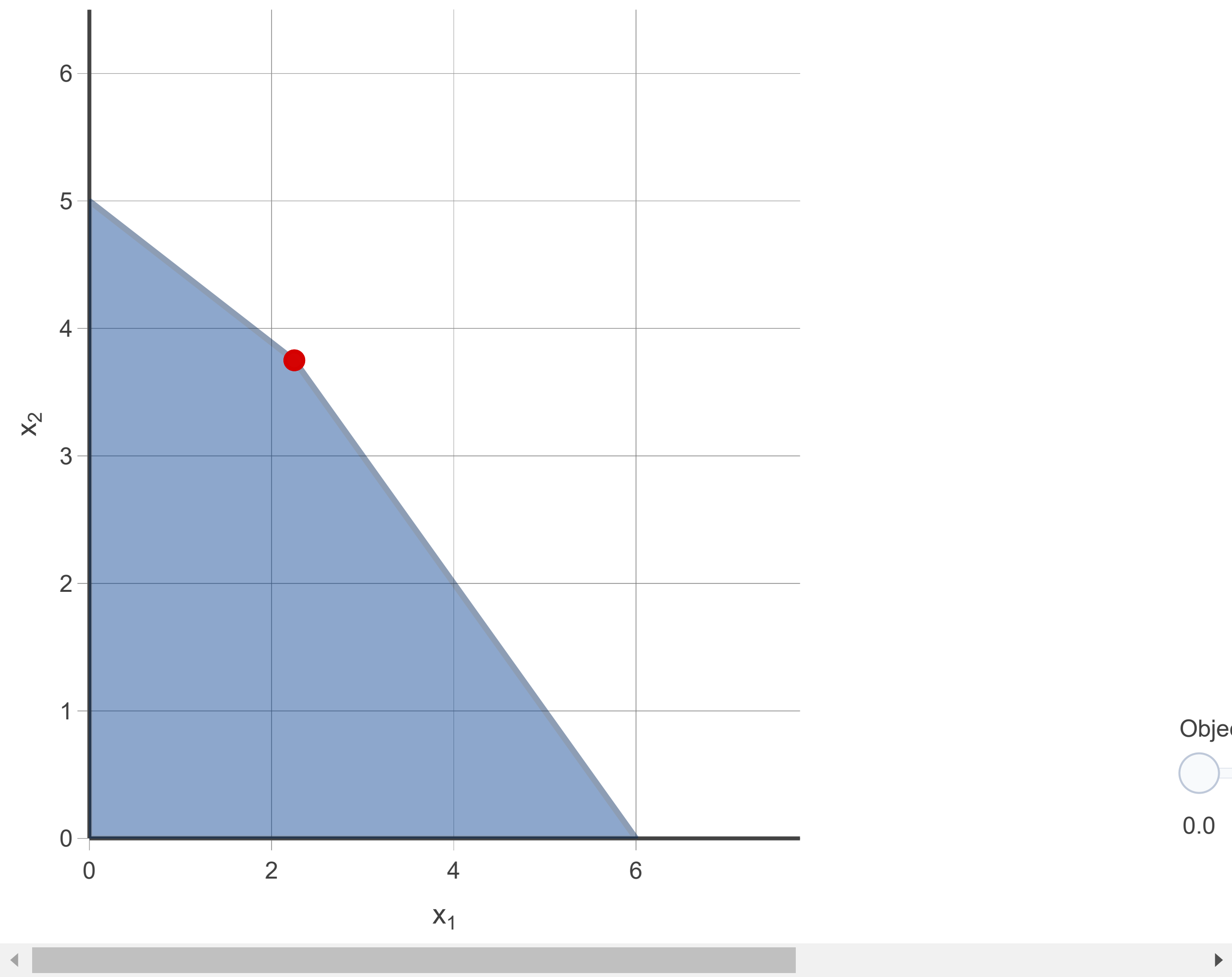
Previously, we used the `gilp` package to viusualize the simplex algorithm but it also has the functionality to visualize branch and bound. We will give a quick overview of the tool. Similar to `lp_visual` and `simplex_visual`, the function `bnb_visual` takes an `LP` and returns a visualization. It is assumed that every decision variable is constrained to be integer. Unlike previous visualizations, `bnb_visual` returns a series of figures for each node of the branch and bound tree. Let's look at a small 2D example:

$$\begin{aligned} \max \quad & 5x_1 + 8x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_1, x_2 \geq 0, \quad \text{integral} \end{aligned}$$

In [22]:
```
nodes = gilp.bnb_visual(gilp.examples.STANDARD_2D_IP)
```

In [23]:
```
nodes[0].show()
```

Geometric Interpretation of LPs



Run the cells above to generate a figure for each node and view the first node. At first, you will see the LP relaxation on the left and the root of the branch and bound tree on the right. The simplex path and isoprofit slider are also present.

Q32: Recall the root of a branch and bound tree is the unaltered LP relaxation. What is the optimal solution? (Hint: Use the objective slider and hover over extreme points).

A: The optimal solution is (2.25, 3.75) with the objective value of 41.25.

Q33: Assume that we always choose the variable with the minimum index to branch on if there are multiple options. Write down (in full) each of the LPs we get after branching off the root node.

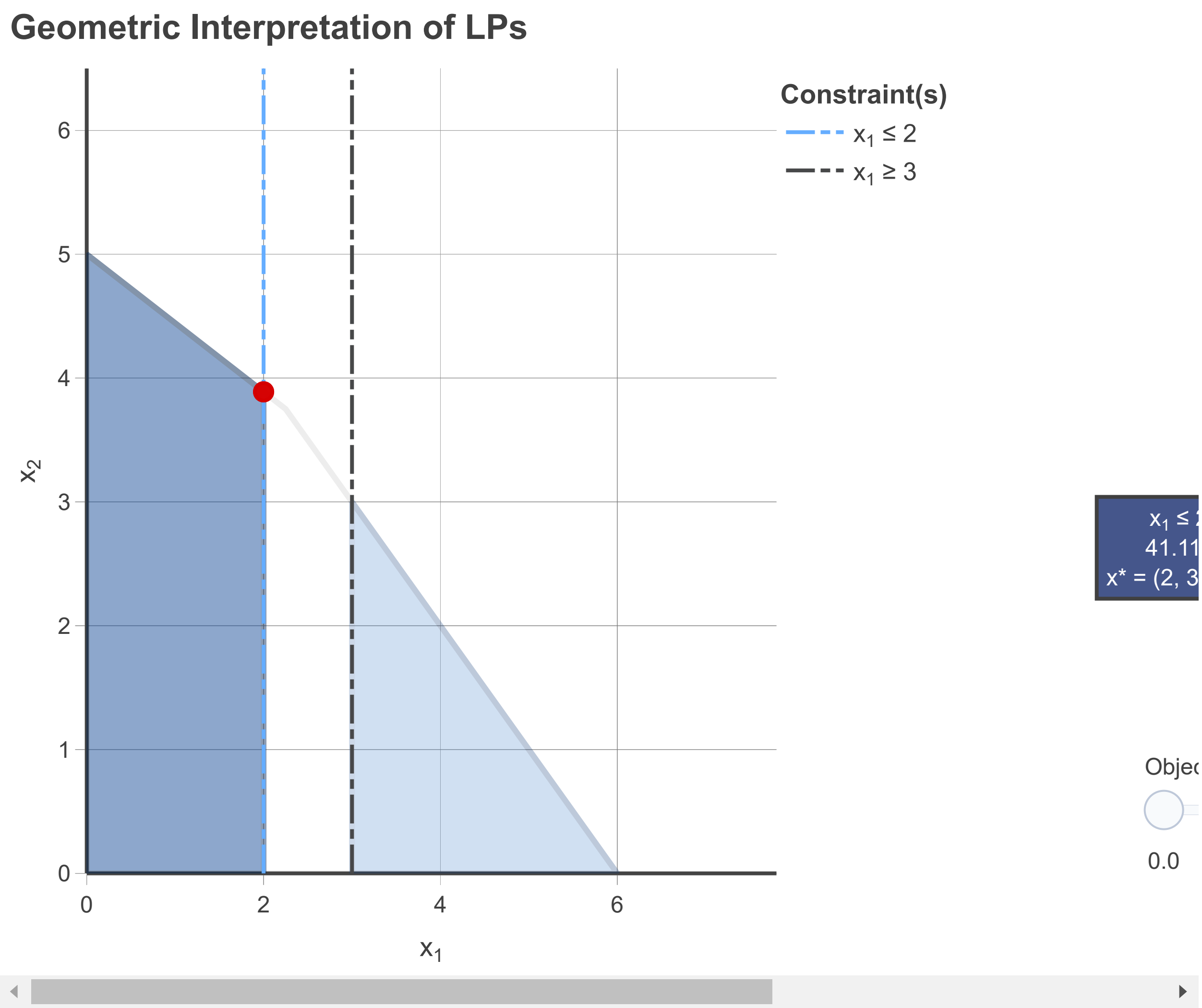
A: As we branching on, the first node has the constraint x1<=2, the objective value of 41.111, and the solution of x=(2,3839). The second node have the constraint x1>=3, but we won't go down to this node because what the question says.

Q34: Draw the feasible region to each of the LPs from **Q33** on the same picture.

A: The feasible region would be divided into two separated region. One of the region has the constraint x1<=2 while another one of the region has the constraint x1>=3.

Run the following cell to see if the picture you drew in **Q34** was correct.

In [24]:
```
nodes[1].show()
```



The outline of the original LP relaxation is still shown on the left. Now that we have eliminated some of the fractional feasible solutions, we now have 2 feasible regions to consider. The darker one is the feasible region associated with the current node which is also shaded darker in the branch and bound tree. The unexplored nodes in the branch and bound tree are not shaded in.

Q35: Which feasible solutions to the LP relaxation are removed by this branch?

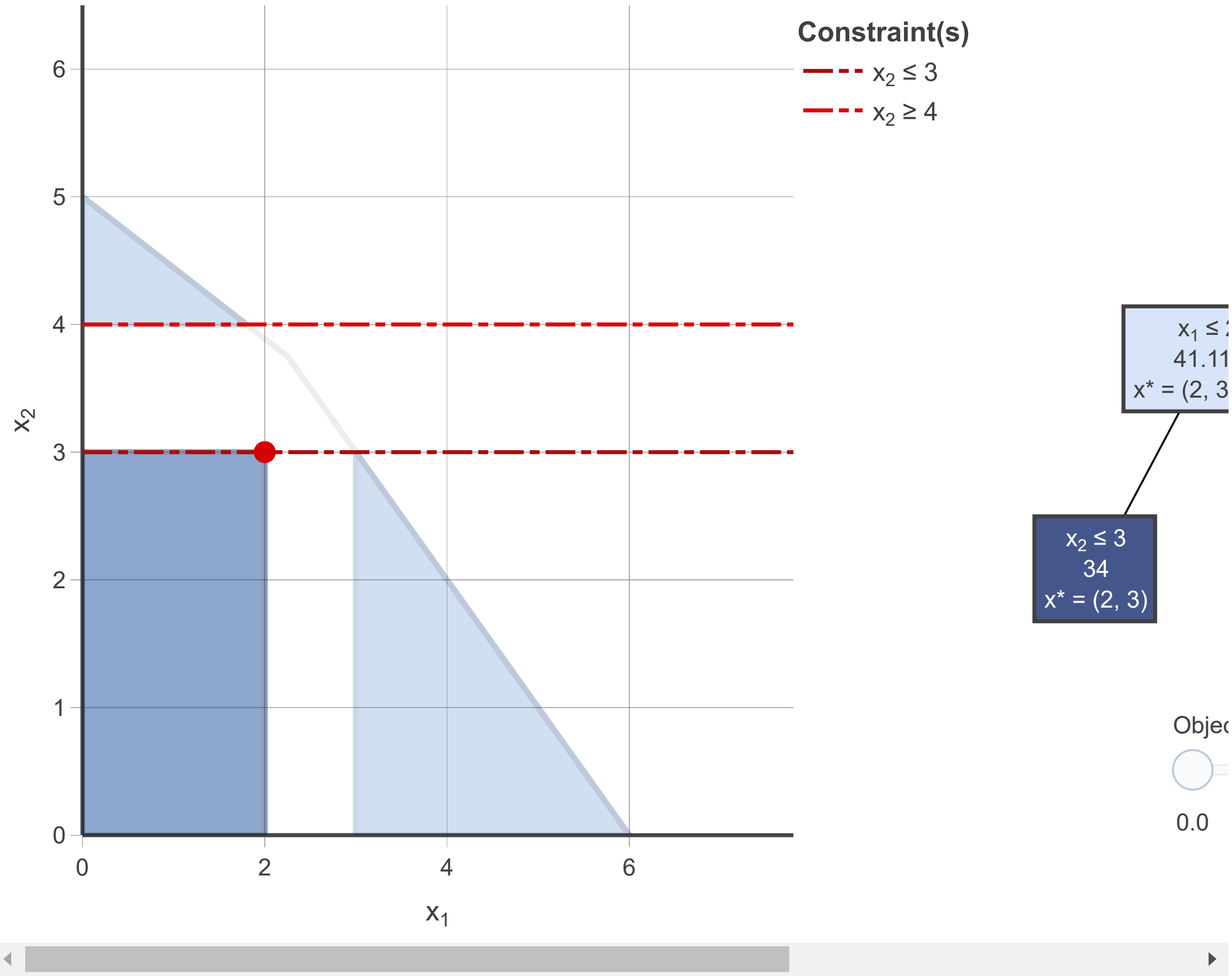
A: Those feasible solutions that have x1>=2 is removed.

Q36: At the current (dark) node, what constraints will we add? How many feasible regions will the original LP relaxation be broken into?

A: We will add the constraint x2<= 3 in one node and the constraint x2>= 4 in another node. The original LP relaxation will be broken into 3 parts.

In [25]:
```
nodes[2].show()
```

Geometric Interpretation of LPs



Q37: What is the optimal solution at the current (dark) node? Do we have to further explore this branch? Explain.

A: The optimal solution is (x,y)=(2,3), which has the value of 34. We don't need to further explore the branch because every variable is a integer now.

Q38: Recall shaded nodes have been explored and the node shaded darker (and feasible region shaded darker) correspond to the current node and its feasible region. Nodes not shaded have not been explored. How many nodes have not yet been explored?

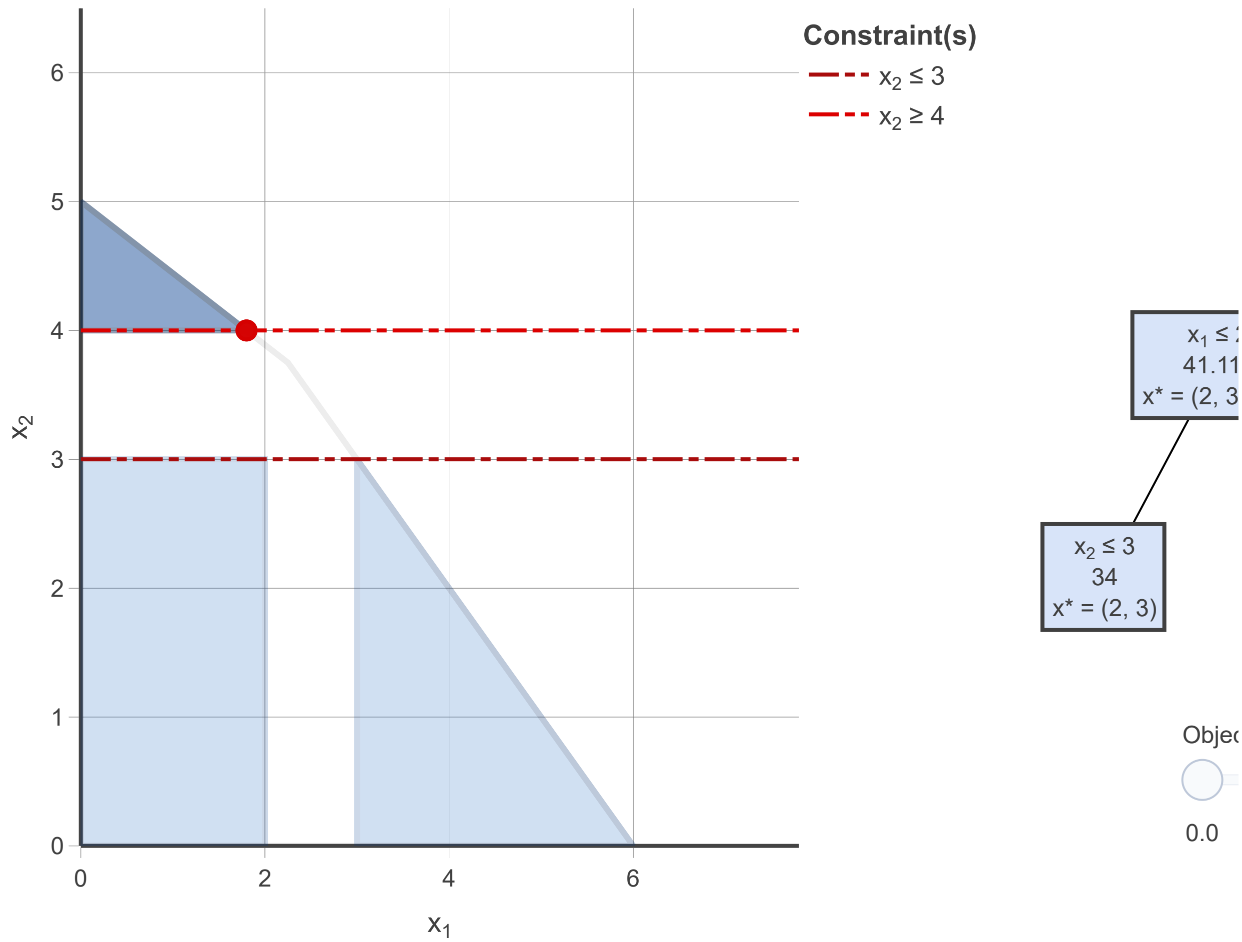
A: There are two nodes have not yet been explored.

Q39: How many nodes have a degree of one in the branch and bound tree? (That is, they are only connected to one edge). These nodes are called leaf nodes. What is the relationship between the leaf nodes and the remaining feasible region?

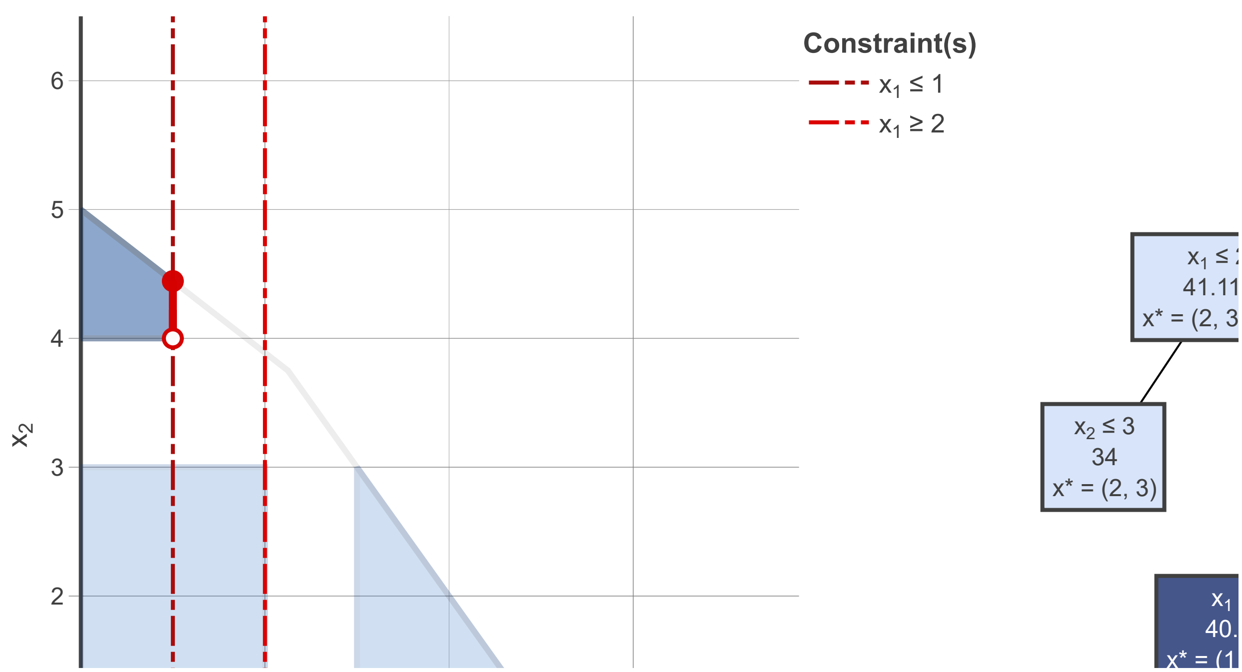
A: There are three nodes of this type. The three of these kind of nodes are the three separated feasible region respectively.

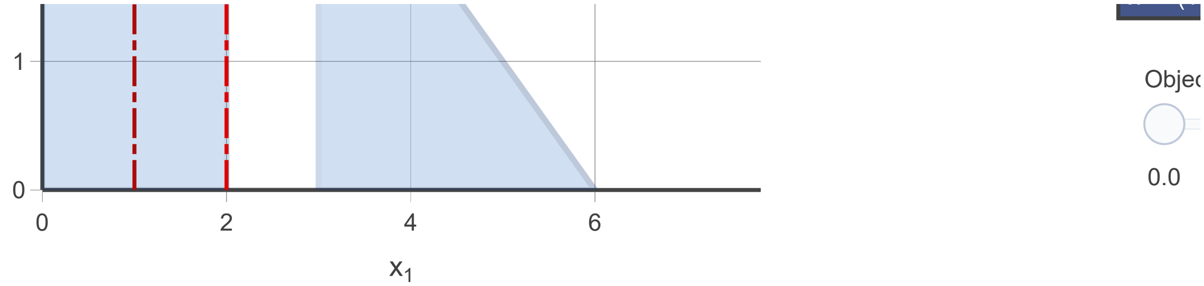
In [26]:
```
# Show the next two iterations of the branch and bound algorithm
nodes[3].show()
nodes[4].show()
```

Geometric Interpretation of LPs



Geometric Interpretation of LPs



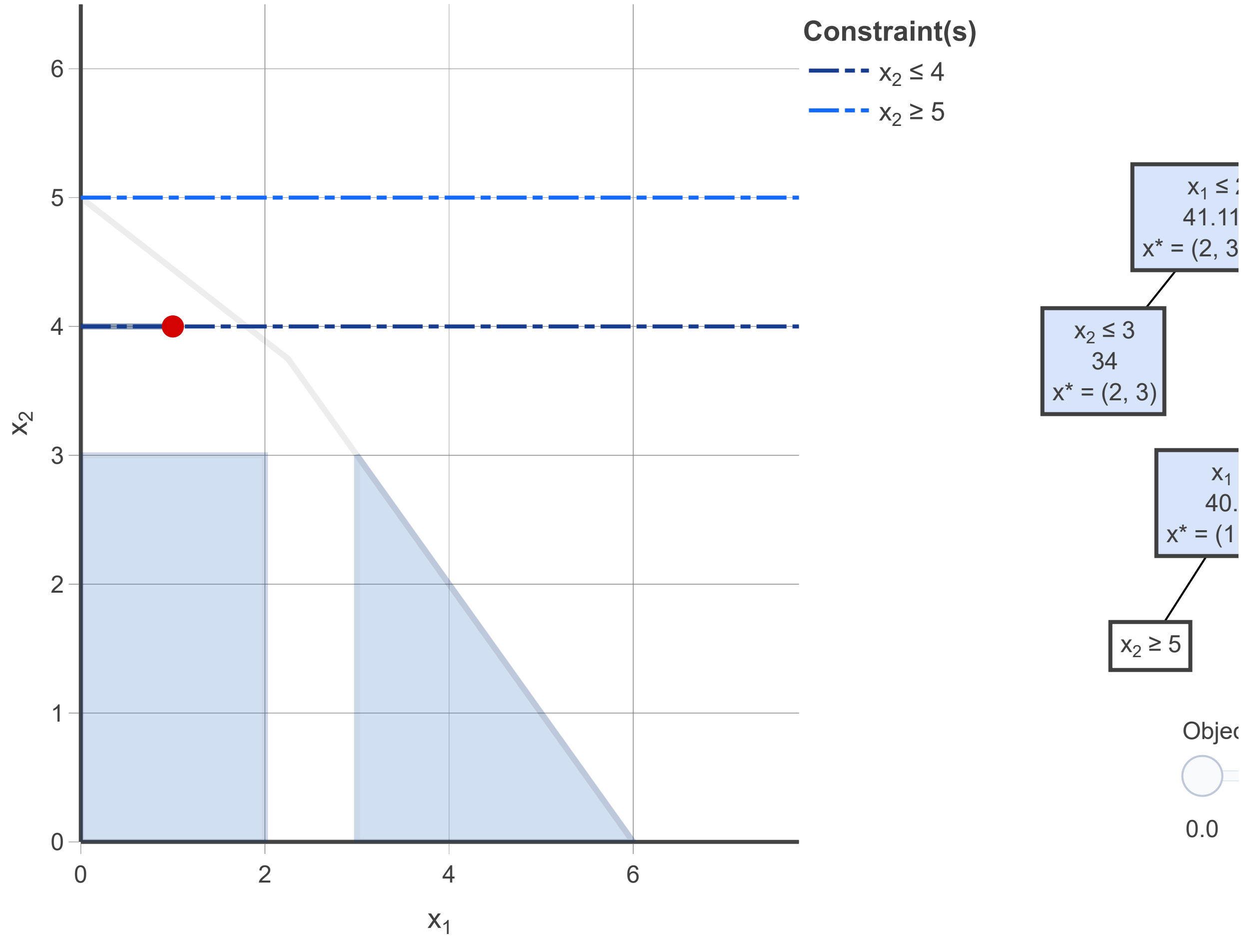


Q40: At the current (dark) node, we added the constraint $x_1 \leq 1$. Why were the fractional solutions $1 < x_1 < 2$ not eliminated for $x_2 <= 3$?

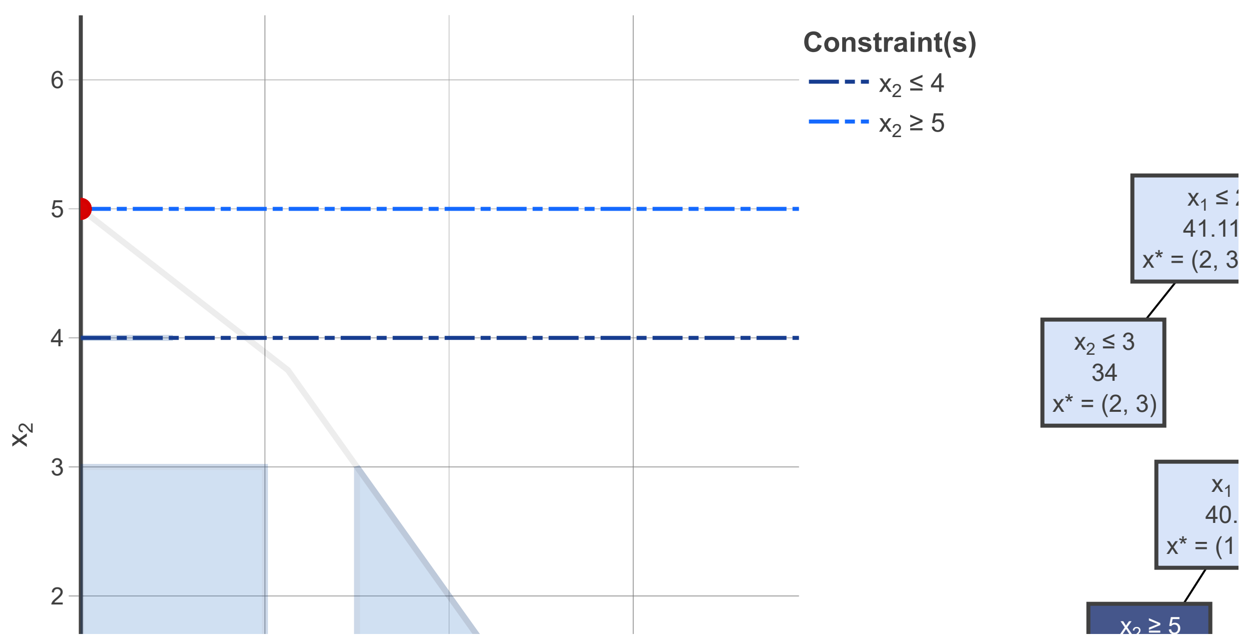
A: It is because the constraint $x_1 \leq 1$ is only connected to the node that don't have the constraint $x_2 <= 3$. So, the constraint $x_1 \leq 1$ does not apply to the node that have the constraint $x_2 <= 3$.

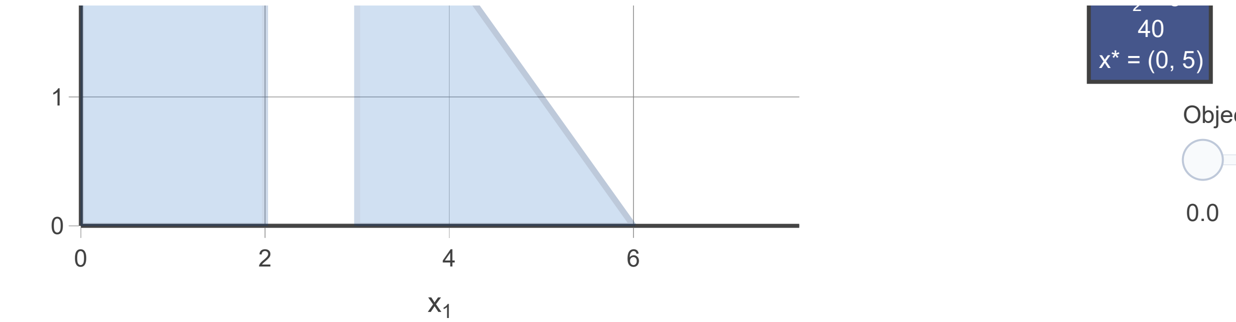
In [27]:
```
# Show the next three iterations of the branch and bound algorithm
nodes[5].show()
nodes[6].show()
nodes[7].show()
```

Geometric Interpretation of LPs

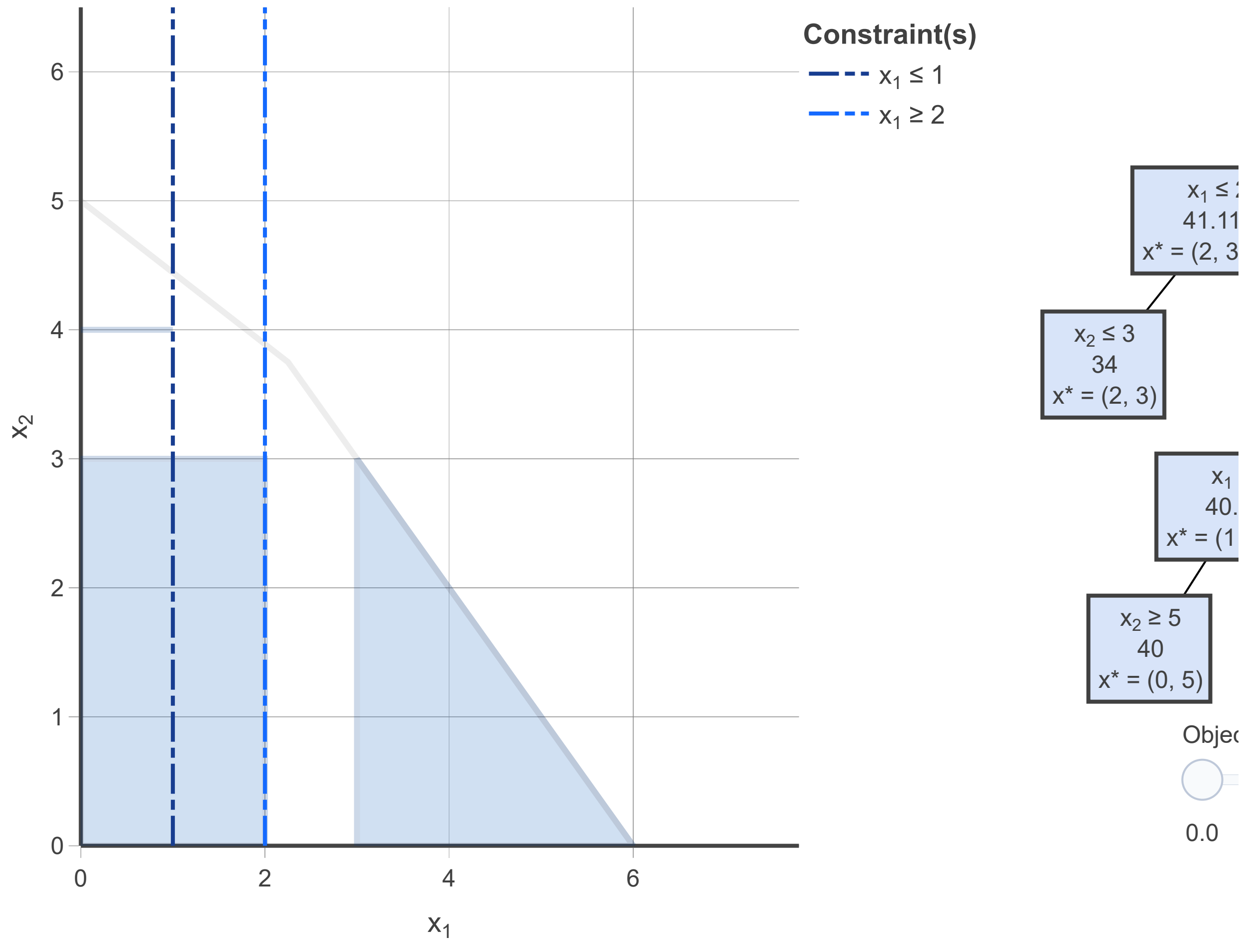


Geometric Interpretation of LPs





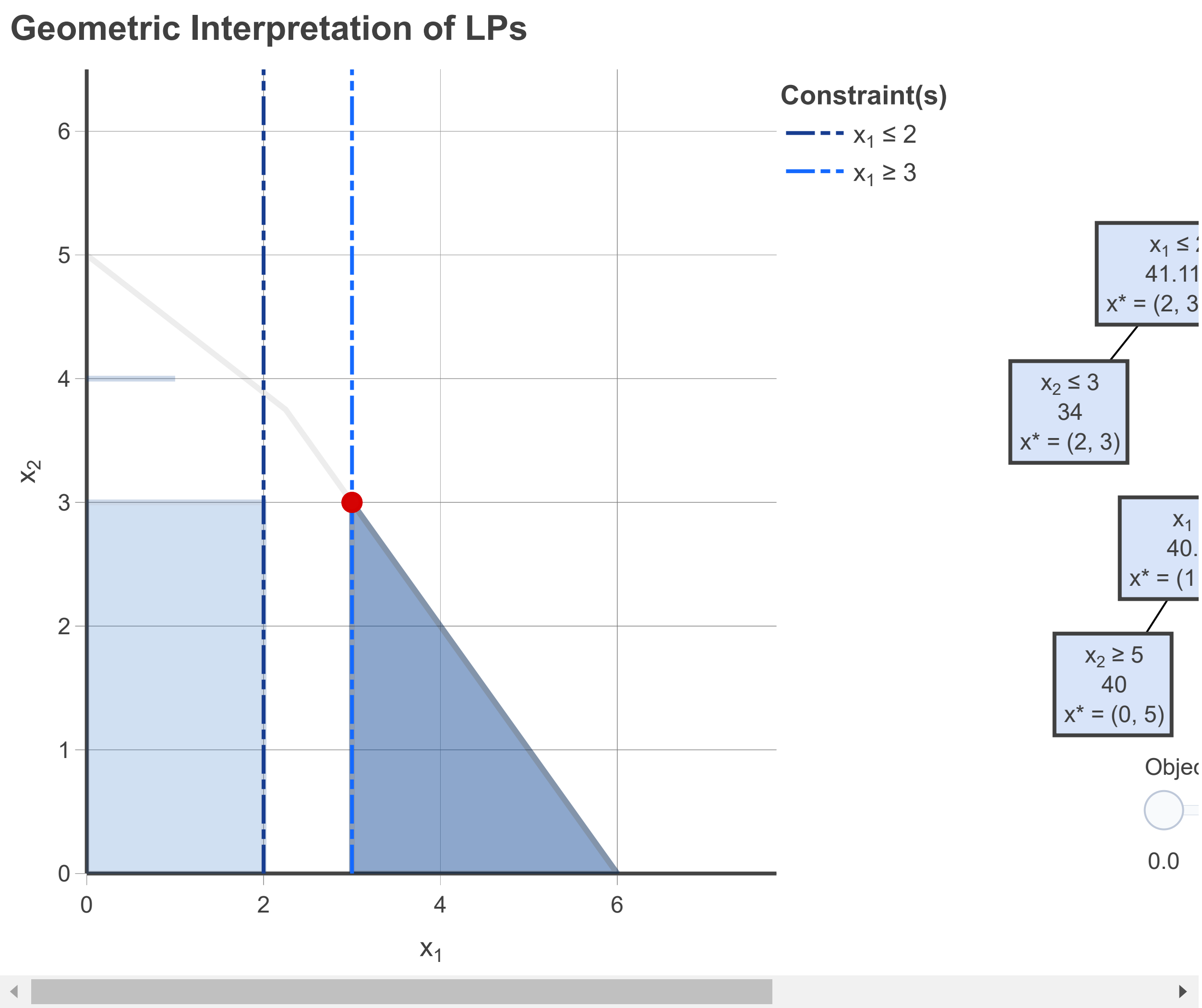
Geometric Interpretation of LPs



Q41: What constraints are enforced at the current (dark) node? Why are there no feasible solutions at this node?

A: The constraints are x1<=2, x2>=4, x1>=2. There are no feasible solutions because there is no region that has both x1>=2 and x1<=2.

In [28]:
```
nodes[8].show()
```



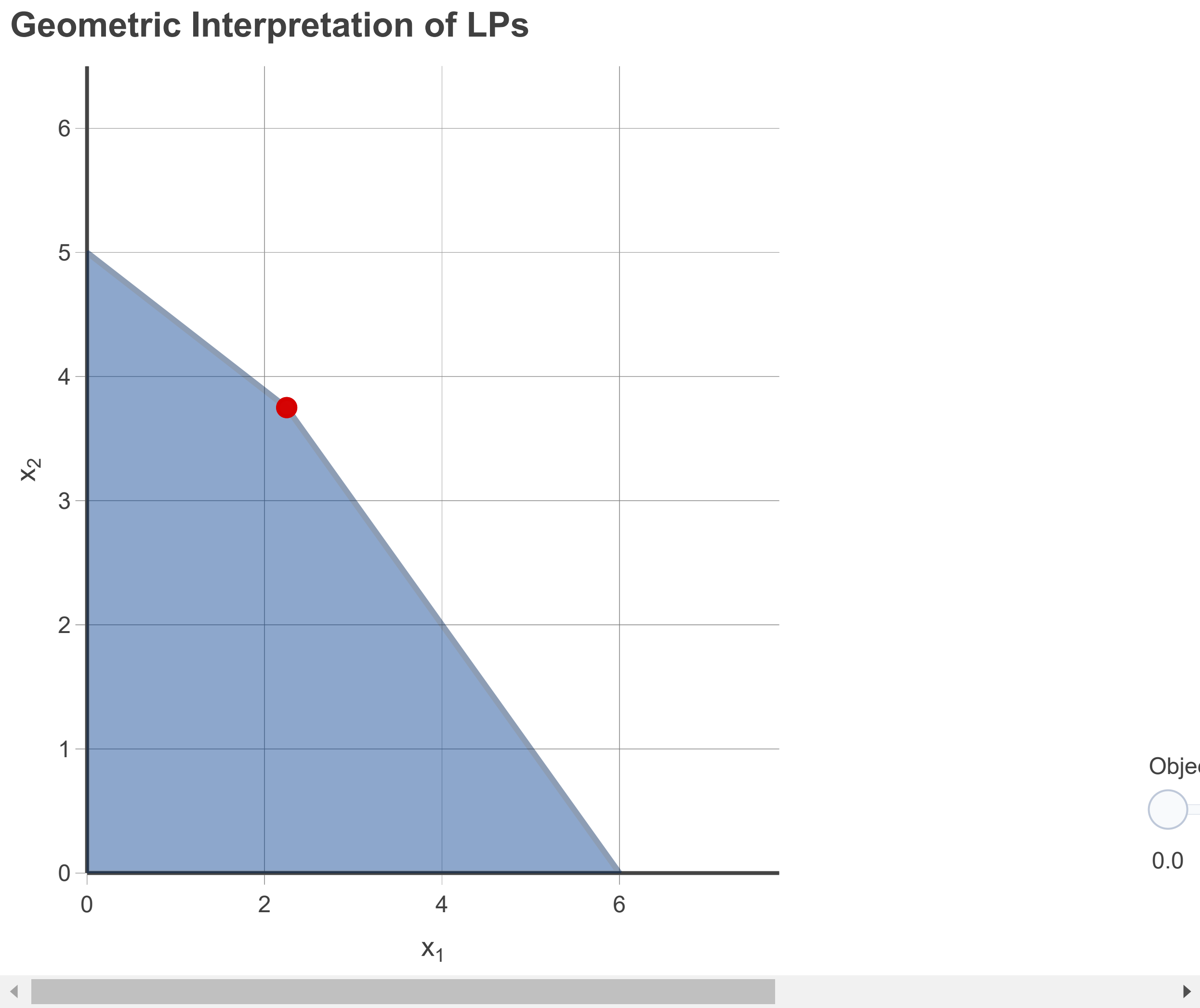
Q42: Are we done? If so, what nodes are fathomed and what is the optimal solution? Explain.

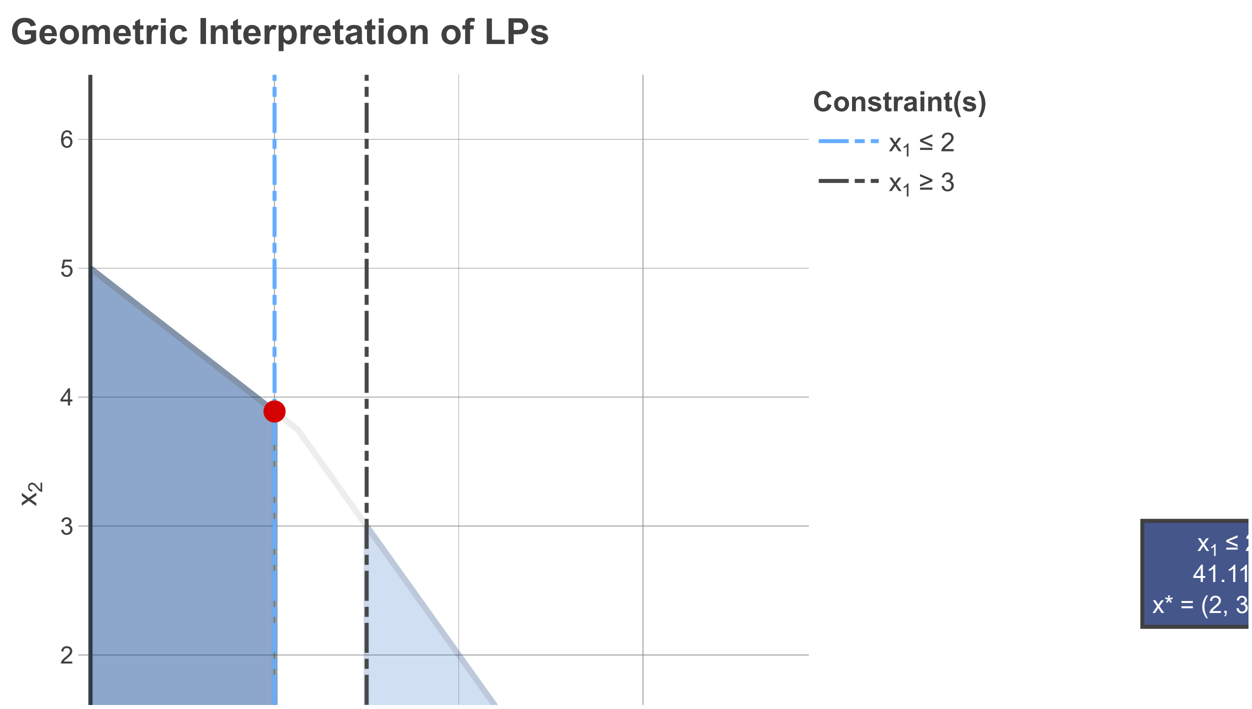
A: Yes, we are done. The nodes with a value of 34,39,40.556,37,and 40 are fathomed. The optimal solution is (0,5) with the value of 40.

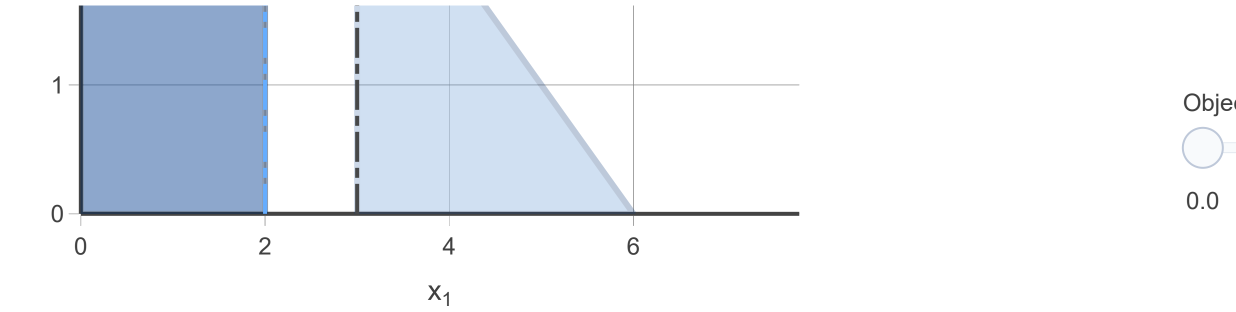
Let's look at branch and bound visualization for an integer program with 3 decision variables!

In [ ]:
```
nodes = gilp.bnb_visual(gilp.examples.VARIED_BRANCHING_3D_IP)
```

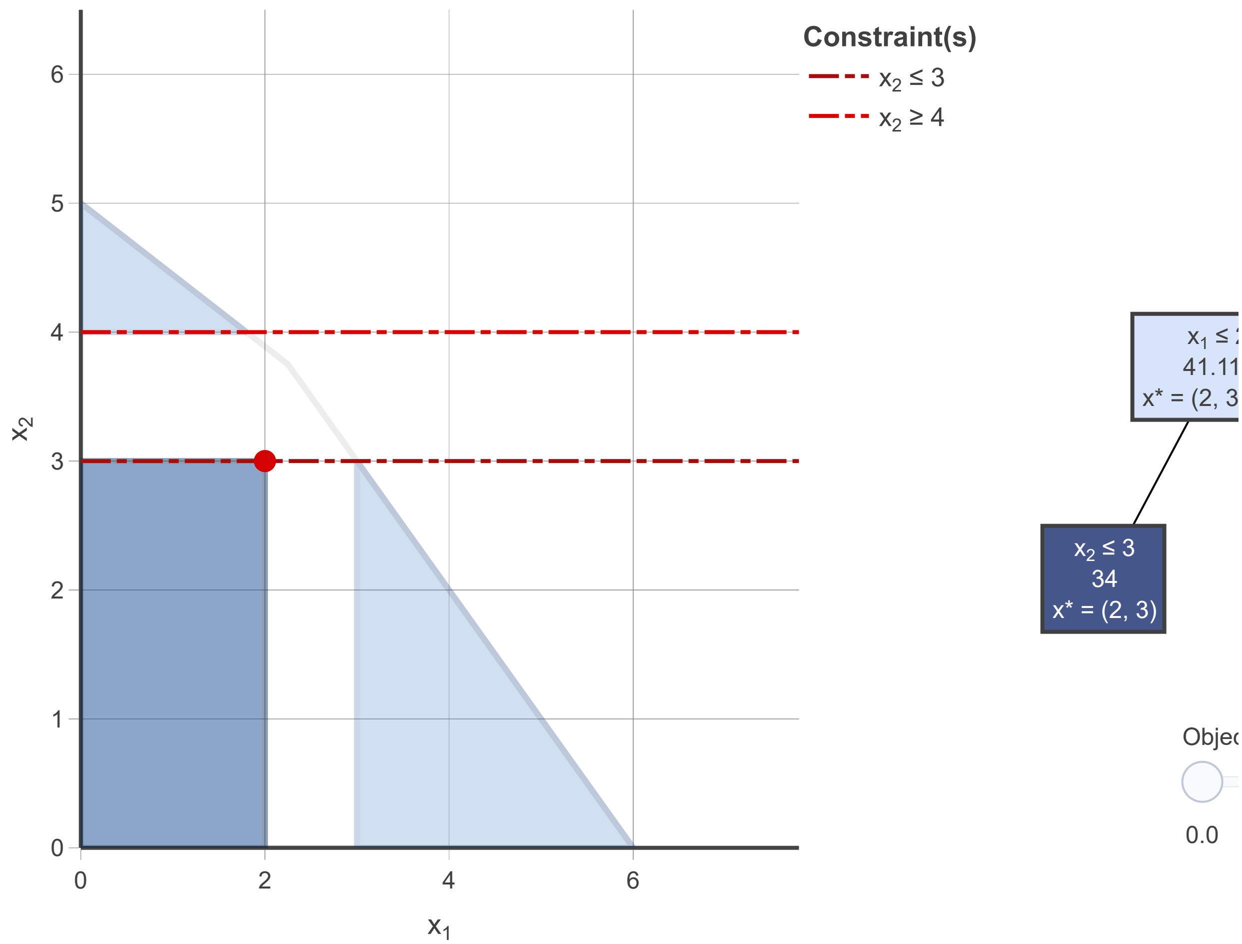
In [30]:
```
# Look at the first 3 iterations
nodes[0].show()
nodes[1].show()
nodes[2].show()
```





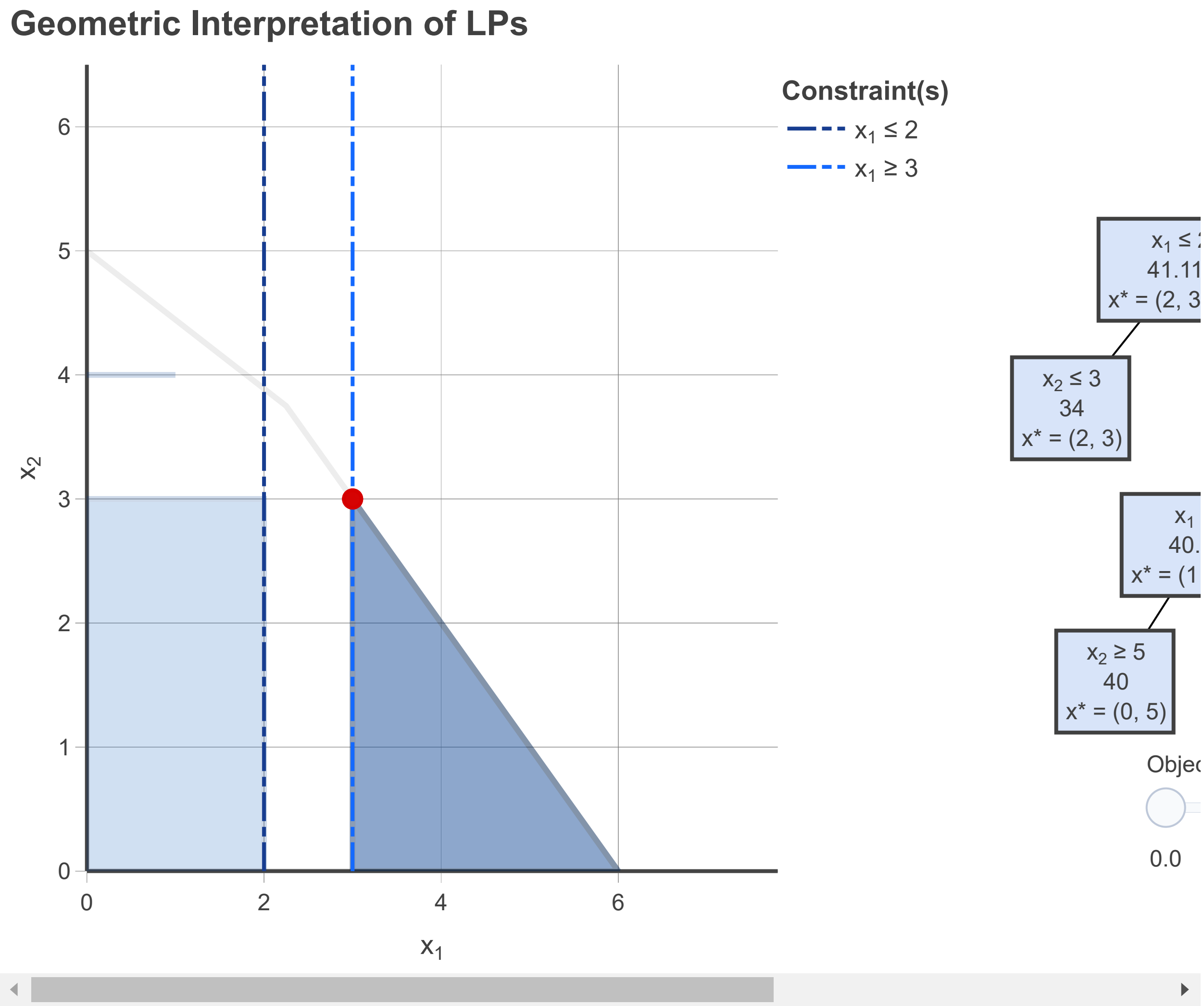


Geometric Interpretation of LPs



Let's fast-forward to the final iteration of the branch and bound algorithm.

In [31]:
```
nodes[-1].show()
```



Q43: Consider the feasible region that looks like a rectangular box with one corner point at the origin. What node does it correspond to in the tree? What is the optimal solution at that node?

A: It corresponds to the node that hvae the constraints x1<=2 and x2<=3. The optimal solution at that node is (x1,x2)=(2,3).

Q44: How many branch and bound nodes did we explore? What was the optimal solution? How many branch and bound nodes would we have explored if we knew the value of the optimal solution before starting branch and bound?

A: We have explored 9 nodes and the optimal solution is (x1,x2)=(0,5) with the value of 40. In that case, we would have only 8 nodes and 7 branches in the tree.

Bonus: Branch and Bound for Knapsack

Consider the following example:

item	value	weight

item	value	weight
1	2	1
2	9	3
3	6	2

The linear program formulation will be:

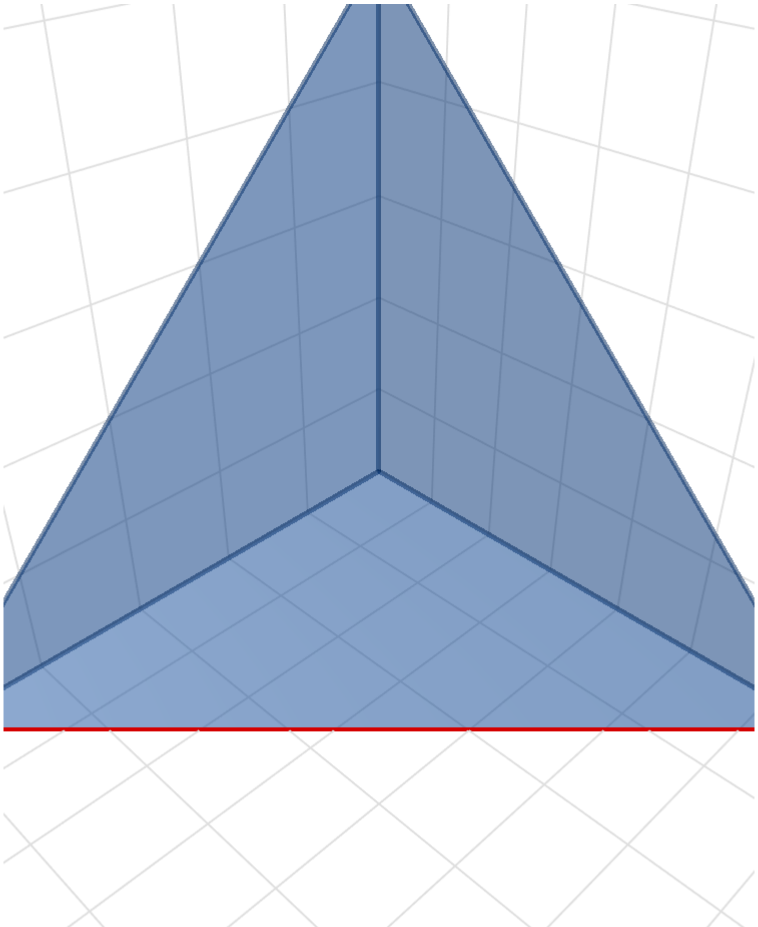
$$\begin{array}{rl} \max & 2x_1 + 9x_2 + 6x_3 \\ \text{s.t.} & 1x_1 + 3x_2 + 2x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0, \quad \text{integer} \end{array}$$

In gilp, we can define this lp as follows:

In [32]:
```
lp = gilp.LP([[1,3,2]],
             [10],
             [2,9,6])

for fig in gilp.bnb_visual(lp):
    fig.show()
```

Geometric Interpretation of LPs



Objectiv

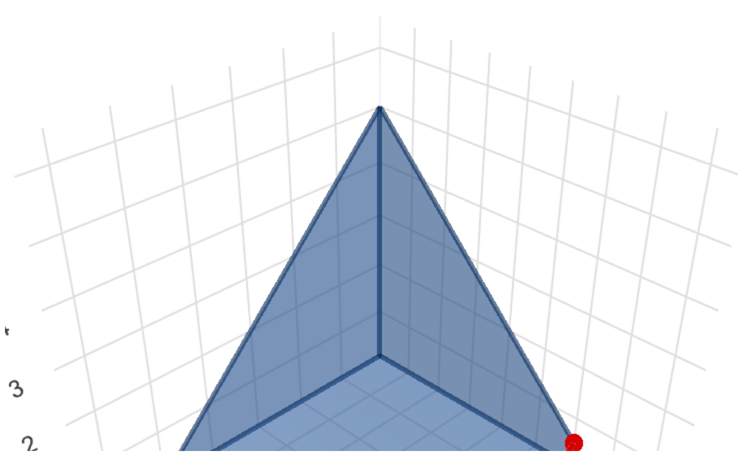
0.0

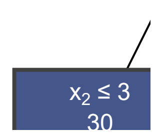
Geometric Interpretation of LPs

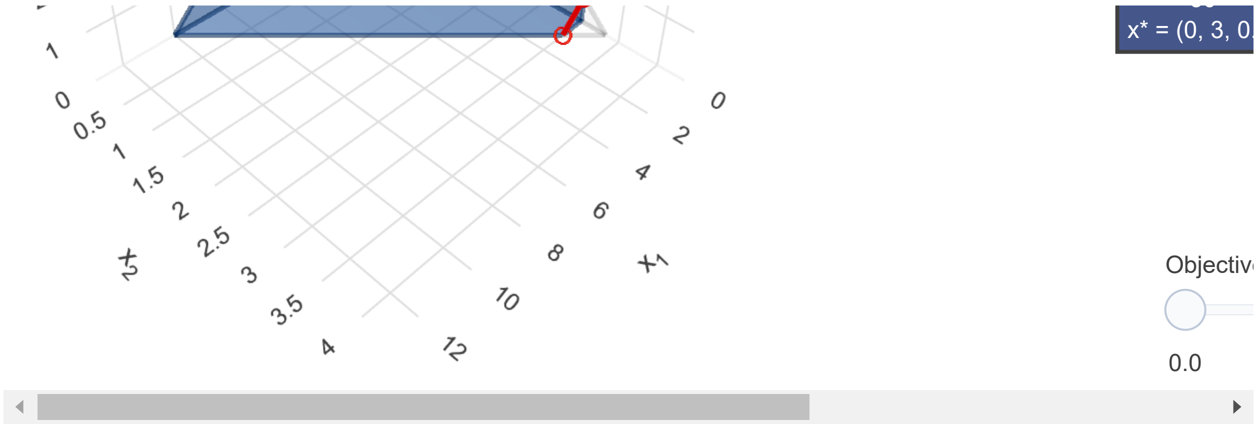
Constraint(s)

$x_2 \leq 3$

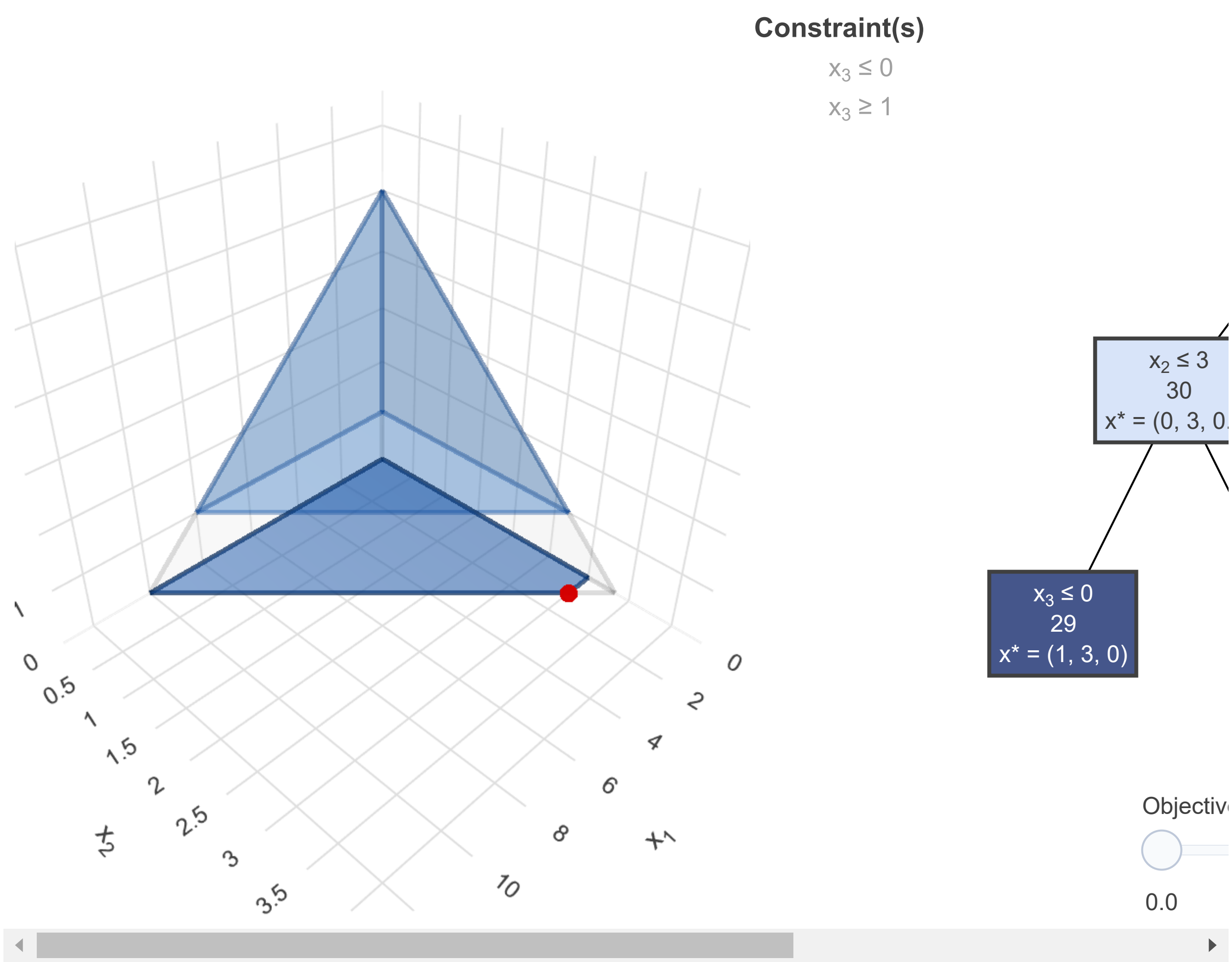
$x_2 \geq 4$



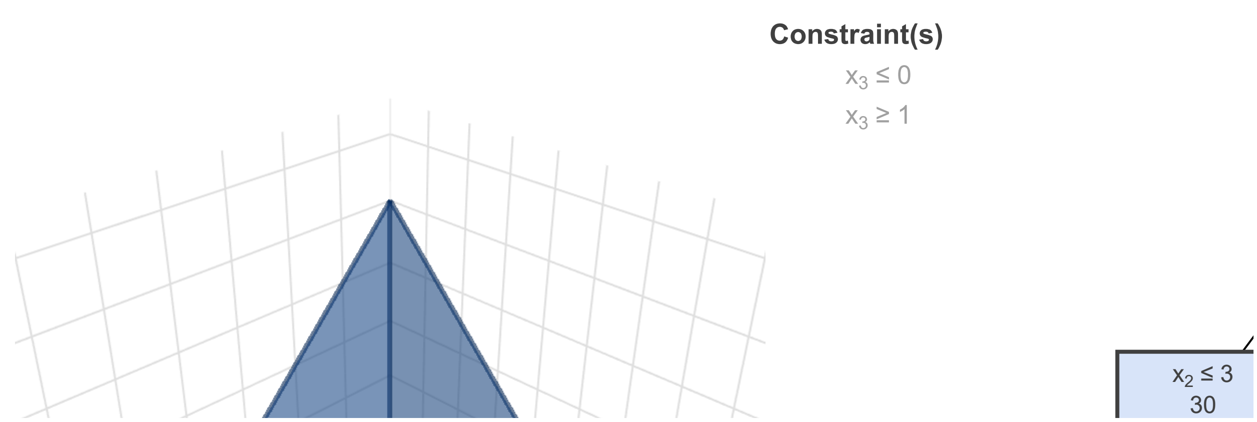


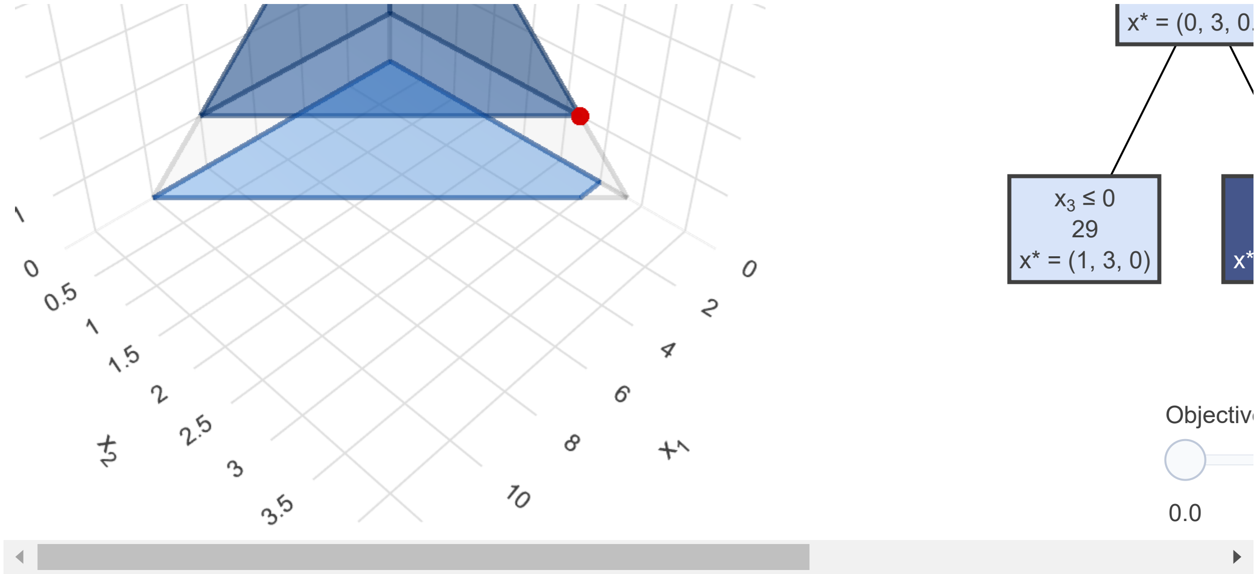


Geometric Interpretation of LPs

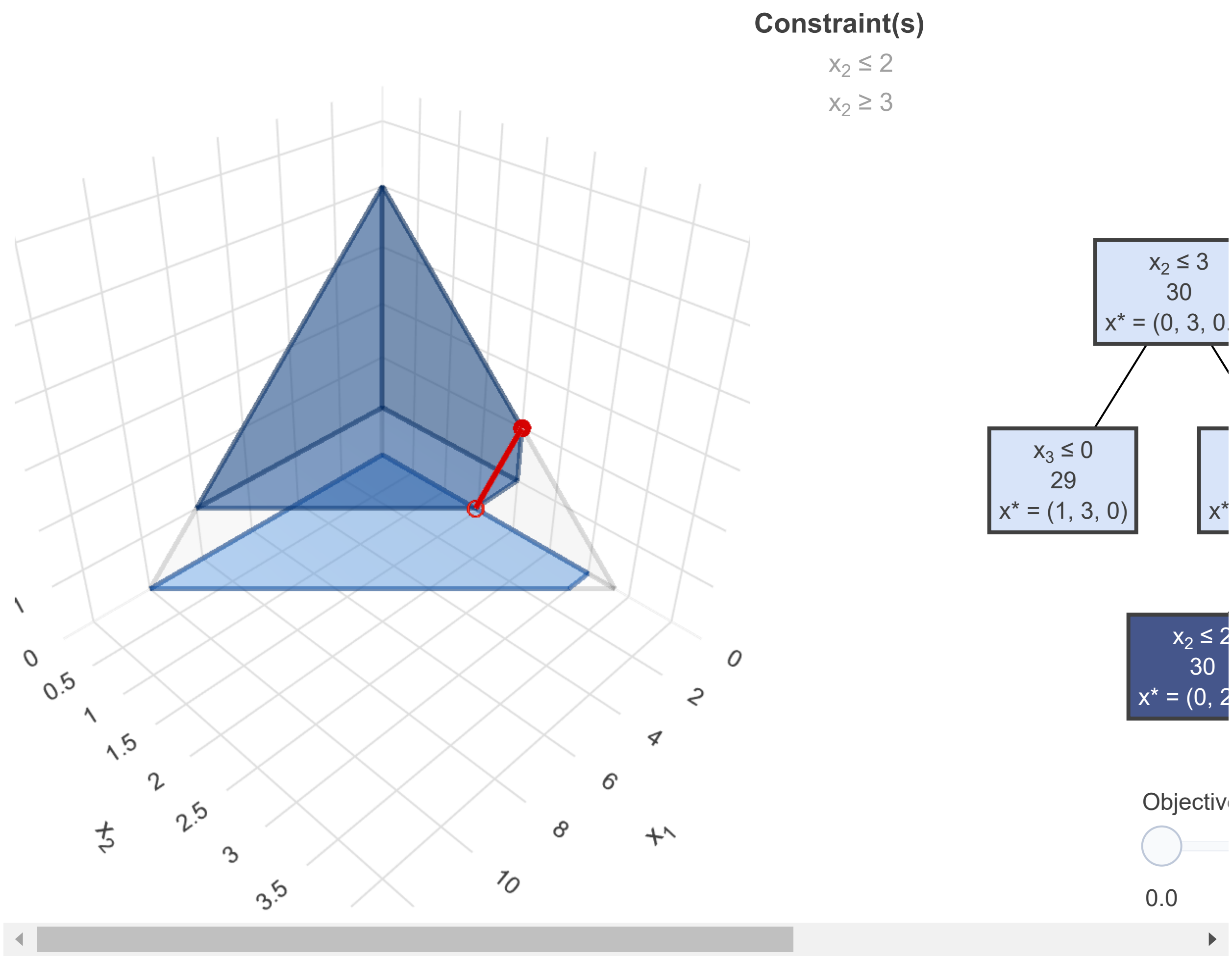


Geometric Interpretation of LPs

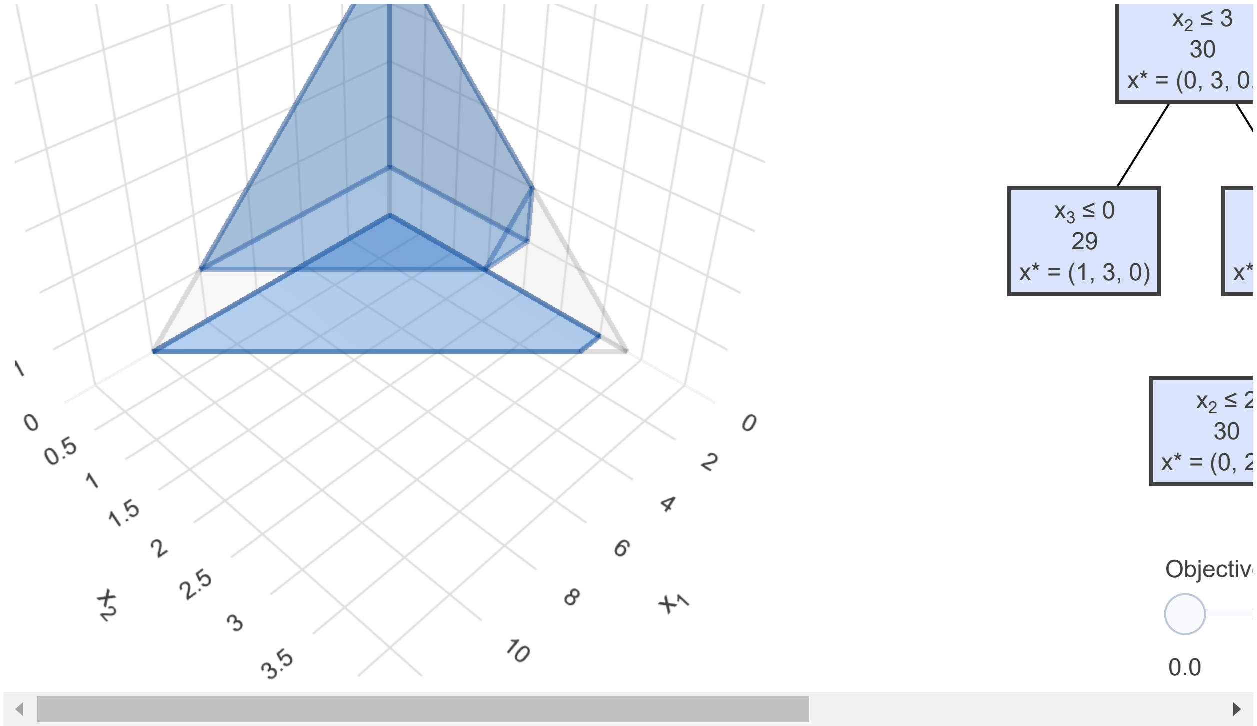




Geometric Interpretation of LPs



Geometric Interpretation of LPs



Geometric Interpretation of LPs