

The 3D feasible region is shown on the left. Hold and drag the mouse to examine it from different angles. Next, click on a constraint to un-mute it. Each constraint is a gray plane in 3D space. Un-mute the constraints one by one to see how they define the 3D feasible region. Move the objective slider to see the isoprofit planes. The isoprofit plane is light gray and the intersection with the feasible region is shown in red. Like the 2D visualization, you can hover over corner points to see information about that point.

Q5: Use the objective slider to solve this LP graphically. Give an optimal solution and objective value. (Hint: The objective slider shows the isoprofit plane for some objective value in light gray and the intersection with the feasible region in red.)

A:

When it comes to LPs with 4 or more decision variables, our graphical approaches fail. We need to find a different way to solve linear programs of this size.

Part II: The Simplex Algorithm for Solving LPs

Dictionary Form LP

First, let's answer some guiding questions that will help to motivate the simplex algorithm.

Q6: Does there exist a unique way to write any given inequality constraint? If so, explain why each constraint can only be written one way. Otherwise, give 2 ways of writing the same inequality constraint.

A: There is no unique way to write any given inequality. For example $x+y \leq 0$ can also be written as $y \leq -x$.

Q7: Consider the following two constraints: $2x_1 + 1x_2 \leq 20$ and $2x_1 + 1x_2 + x_3 = 20$ where all x are nonnegative. Are these the same constraint? Why? (This question is tricky!)

A: These two are the same constraint because the minimum value that x_1 , x_2 , x_3 can individually be is 0. If x_3 is increased, in order for the equality to stand, $x_1+x_2 \leq 20$. Therefore they are the same constraint.

Q8: Based on your answers to **Q6** and **Q7**, do you think there exists a unique way to write any given LP?

A: No, I don't think there exists a unique way to write any given LP.

You should have found that there are many ways to write some LP. This begs a new question: are some ways of writing an LP harder or easier to solve than others? Consider the following LP:

$$\begin{aligned}
 \max \quad & 56 - 2x_3 - 1x_4 \\
 \text{s.t.} \quad & x_1 = 4 - 1x_3 + 1x_4 \\
 & x_2 = 12 + 1x_3 - 2x_4 \\
 & x_5 = 3 + 1x_3 - 1x_4 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

Q9: Just by looking at this LP, can you give an optimal solution and its objective value. If so, explain what property of the LP allows you to do this. (Hint: Look at the objective function)

A: Just by looking at this LP you can give the optimal solution and its objective value. The LP gives us that all the x values are nonnegative. Therefore the minimum value for any x is 0. The x values in the max function are both subtracted, therefore by being nonzero values, the maximum value of the function is less than 56. Plugging 0 into the x values of the max function, you can get an maximum value of 56.

The LP above is the same as `ALL_INTEGER_2D_LP` just rewritten in a different way! This rewritten form (which we found is easier to solve) was found using the simplex algorithm. At its core, the simplex algorithm strategically rewrites an LP until it is in a form that is "easy" to solve.

The simplex algorithm relies on an LP being in **dictionary form**. Recall the following properties of an LP in dictionary form:

- All constraints are equality constraints
- All variables are constrained to be nonnegative
- Each variable only appears on the left-hand side (LHS) or the right-hand side (RHS) of the constraints (not both)
- Each constraint has a unique variable on the LHS
- The objective function is in terms of the variables that appear on the RHS of the constraints only.
- All constants on the RHS of the constraints are nonnegative

Q10: Rewrite the example LP `ALL_INTEGER_2D_LP` in dictionary form. Show your steps!

$$\begin{aligned}
 \max \quad & 5x_1 + 3x_2 \\
 \text{s.t.} \quad & 2x_1 + 1x_2 \leq 20 \\
 & 1x_1 + 1x_2 \leq 16 \\
 & 1x_1 + 0x_2 \leq 7 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{A:} \quad & \max 5x_1 + 3x_2 \\
 & x_3 = 20 - 2x_1 - 1x_2 \\
 & x_4 = 16 - 1x_1 - 1x_2 \\
 & x_5 = 7 - 1x_1 - 0x_2 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

Most Limiting Constraint

Once our LP is in dictionary form, we can run the simplex algorithm! In every iteration of the simplex algorithm, we will take an LP in dictionary form and strategically rewrite it in a new

dictionary form. Note: it is important to realize that rewriting the LP **does not** change the LP's feasible region. Let us examine an iteration of simplex on a new LP.

$$\begin{array}{ll}\max & 5x_1 + 3x_2 \\ \text{s.t.} & 1x_1 + 0x_2 \leq 4 \\ & 0x_1 + 1x_2 \leq 6 \\ & 2x_1 + 1x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 15 \\ & x_1, x_2 \geq 0\end{array}$$

Q11: Is this LP in dictionary form? If not, rewrite this LP in dictionary form.

A: This LP is not in dictionary form $\max 3x_1 + 3x_2$

$$x_3 = 4 - 1x_1 - 0x_2$$

$$x_4 = 6 - 0x_1 - 1x_2$$

$$x_5 = 9 - 2x_1 - 1x_2$$

$$x_6 = 15 - 3x_1 - 2x_2$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Q12: Recall from **Q9** how you found a feasible solution (which we argued to be optimal) just by looking at the LP. Using this same strategy, look at the LP above and give a feasible solution and its objective value for this LP. Describe how you found this feasible solution. Is it optimal? Why?

A: Because x_1 and x_2 both have positive coefficients in the max function, we can start by maximizing x_2 because it has a higher constraint than x_1 . So we set $x_2 = 6$. Plugging $x_2 = 6$ into the third constraint we get $x_1 = 1.5$. Plugging $x_2 = 6$ into the second to last constraint, we have $3x_1 + 2(6) \leq 15$ so $x_1 = 1$. Because $1 < 1.5$, $x_1 = 1$. Therefore a feasible solution is $(1, 6)$ and the objective value is 22. This is an optimal solution because we have maximized the x values as much as possible given the constraints.

From **Q12** we see that every dictionary form LP has a corresponding feasible solution. Furthermore, there are positive coefficients in the objective function. Hence, we can increase the objective value by increasing the corresponding variable. In our example, both x_1 and x_2 have positive coefficients in the objective function. Let us choose to increase x_1 .

Q13: What do we have to be careful about when increasing x_1 ?

A: We would have to be careful that the value of x_1 does not exceed 4.

Q14: After choosing a variable to increase, we must determine the most limiting constraint. Let us look at the first constraint $x_3 = 4 - 1x_1 - 0x_2$. How much can x_1 increase? (Hint: what does a dictionary form LP require about the constant on the RHS of constraints?)

A: x_1 can increase by 4.

Q15: Like in **Q14**, determine how much each constraint limits the increase in x_1 and identify the most limiting constraint.

A: x_3 is the most limiting constraint is the first one where x_1 can only increase by at most 4.

If we increase x_1 to 4, note that x_3 will become zero. Earlier, we identified that each dictionary

form has a corresponding feasible solution achieved by setting variables on the RHS (and in the objective function) to zero. Hence, since x_3 will become zero, we want to rewrite our LP such that x_3 appears on the RHS. Furthermore, since x_1 is no longer zero, it should now appear on the LHS.

Q16: Rewrite the most limiting constraint $x_3 = 4 - 1x_1 - 0x_2$ such that x_1 appears on the left and x_3 appears on the right.

A: $x_1 = 4 - 0x_2 - x_3$

Q17: Using substitution, rewrite the LP such that x_3 appears on the RHS and x_1 appears on the LHS. (Hint: Don't forget the rule about which variables can appear in the objective function)

A: $\max 5(4 - 0x_2 - x_3) + 3x_2$

$x_1 = 4 - 0x_2 - x_3$

$x_4 = 6 - 0(4 - 0x_2 - x_3) - 1x_2$

$x_5 = 9 - 2(4 - 0x_2 - x_3) - 1x_2$

$x_6 = 15 - 3(4 - 0x_2 - x_3) - 2x_2$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

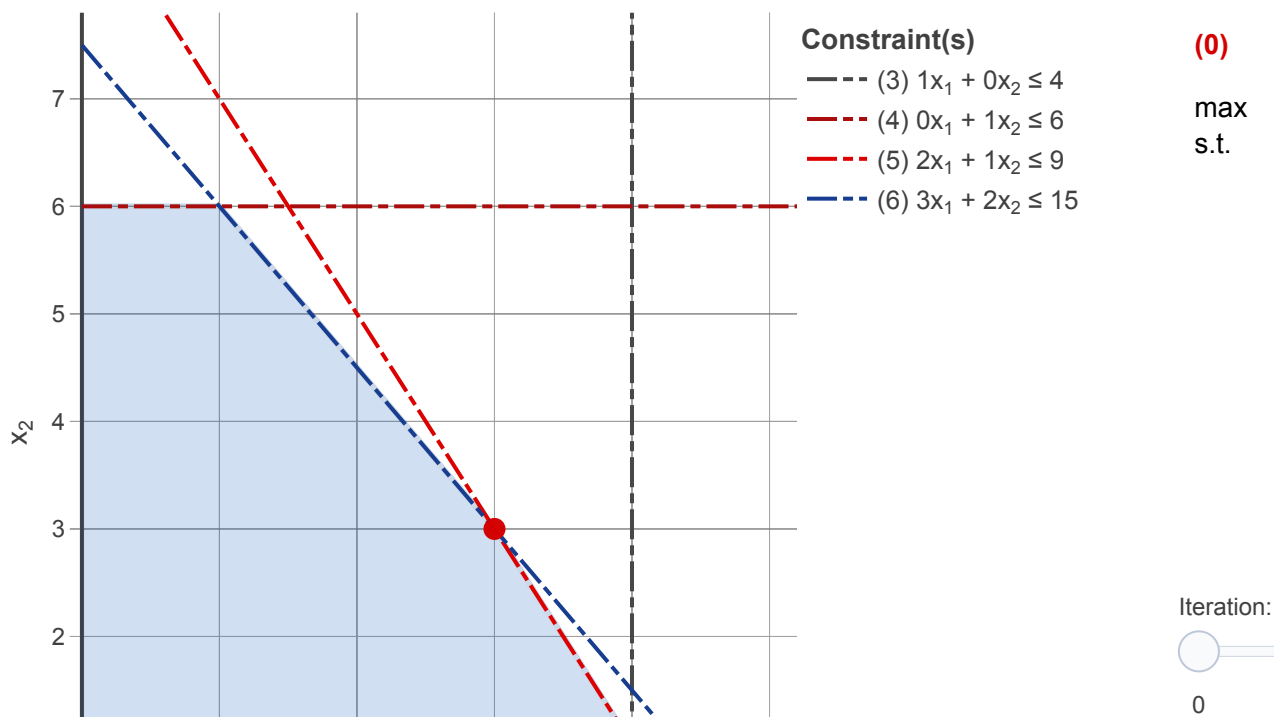
Q18: We have now completed an iteration of simplex! What is the corresponding feasible solution of the new LP?

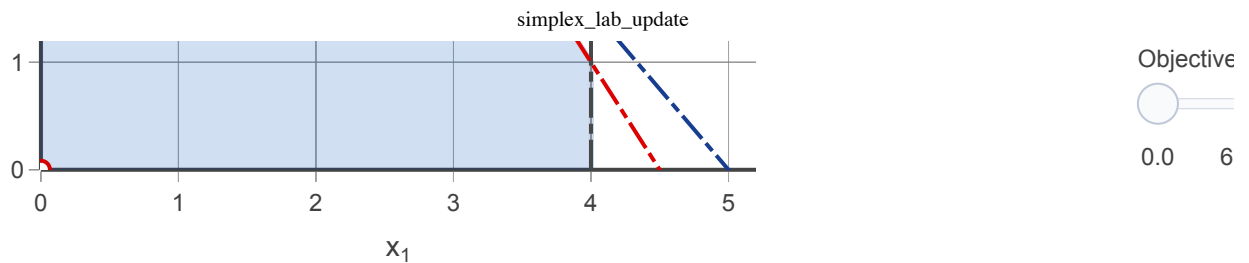
A: (4,0,0,6,1,3)

Now that we have seen an iteration of simplex algebraically, let's use GILP to visualize it! The LP example we have been using is called `LIMITING_CONSTRAINT_2D_LP`. To visualize simplex, we must import a function called `simplex_visual()`.

```
In [8]: lp = gilp.examples.LIMITING_CONSTRAINT_2D_LP # get the LP example
gilp.simplex_visual(lp, initial_solution=np.array([[0],[0]])).show() # show the
```

Geometric Interpretation of LPs





This visualization is much the same as the previous one but we now have an additional slider which allows you to toggle through iterations of simplex. Furthermore, the corresponding dictionary at every iteration of simplex is shown in the top right. If you toggle between two iterations, you can see the dictionary form for both the previous and next LP at the same time.

Q19: Starting from point $(0,0)$, by how much can you increase x_1 before the point is no longer feasible? Which constraint do you *hit* first? Does this match what you found algebraically?

A: You can increase x_1 by 4 before the point is not longer feasible. you hit $x_1=4$ constraint first. This matches what what we found algebraically.

Q20: Which variable will be the next increasing variable and why? (Hint: Look at the dictionary form LP at iteration 1)

A: The next increasing variable is x_2 because because it has a positive coefficient in the max function.

Q21: Visually, which constraint do you think is the most limiting constraint? How much can x_2 increase? Give the corresponding feasible solution and its objective value of the next dictionary form LP. (Hint: hover over the feasible points to see information about them.)

A: I think the most limiting constraint is x_5 . x_2 can increase by at most a value of 1 given that $x_1=4$. the corresponding feasible solution would be $(4,1,0,5,0,1)$ and the objective value would be 23.

Q22: Move the slider to see the next iteration of simplex. Was your guess from **Q21** correct? If not, describe how your guess was wrong.

A: My guess from question 21 was right.

Q23: Look at the dictionary form LP after the second iteration of simplex. What is the increasing variable? Identify the most limiting constraint graphically and algebraically. Show your work and verify they are the same constraint. In addition, give the next feasible solution and its objective value.

A: The increasing variable is x_3 . The most limiting constraint is x_6 . Algebraically x_3 cannot be bigger than 1, which is the smallest value out of all the other constraints. Graphically the most limiting constraint is $3x_1+2x_2 \leq 15$. This is because going along the border line, this is the next constraint that is hit. The next feasible solution would be $(3,3,1,3,0,0)$ and its objective value would be 24).

Q24: Is the new feasible solution you found in **Q23** optimal? (Hint: Look at the dictionary form LP)

A: Yes because there are no futher variables that can be increased.

Q25: In **Q21** and **Q23**, how did you determine the most limiting constraint graphically?

A: By travelling along the border of the feasible region and determining the next constraint that was hit on the path.

(BONUS): In 2D, we can increase a variable until we hit a 2D line representing the most limiting constraint. What would be the analogous situation in 3D?

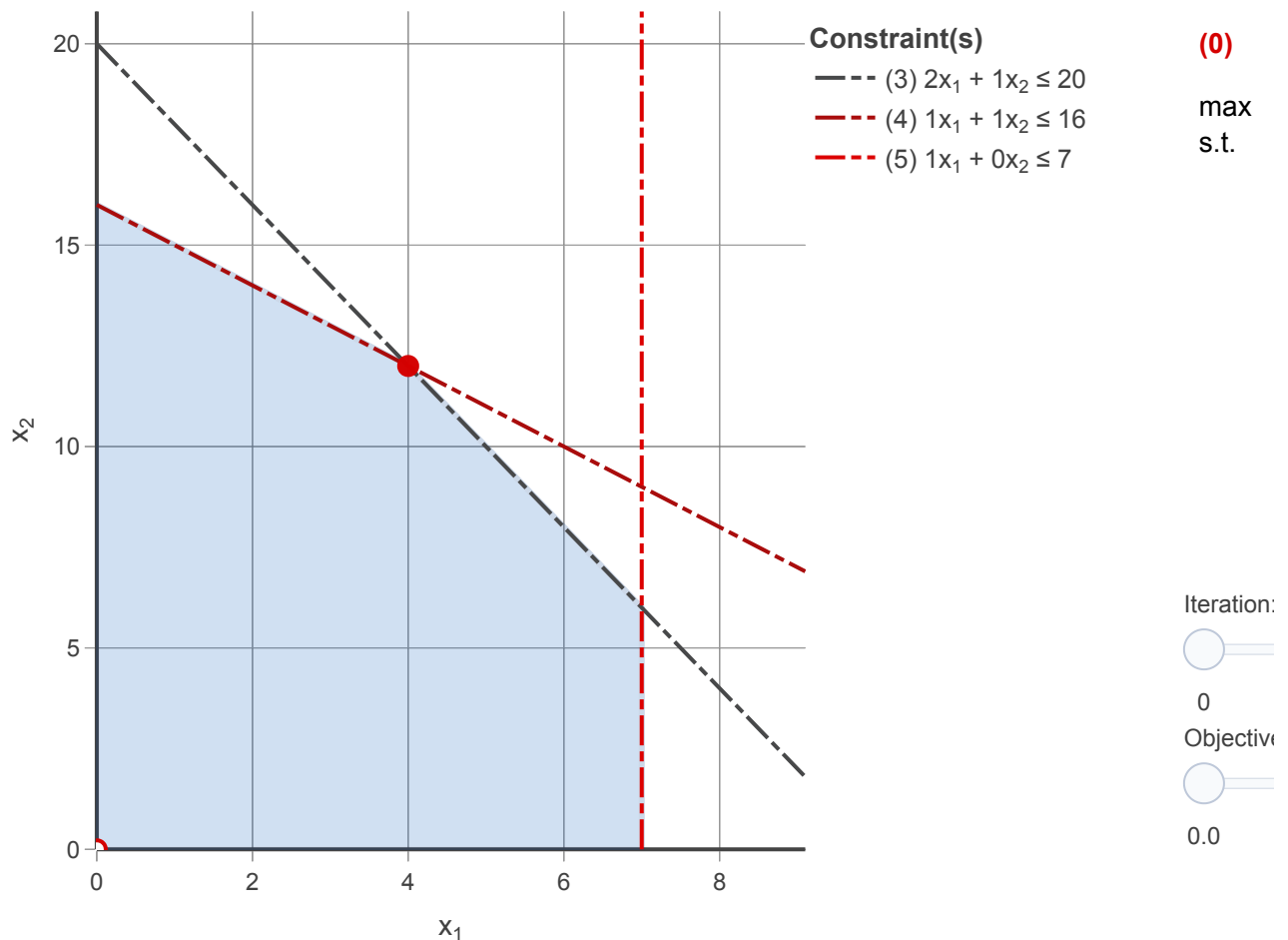
A:

Part III: Geometrical Interpretation of the Dictionary

We have seen how the simplex algorithm transforms an LP from one dictionary form to another. Each dictionary form has a corresponding dictionary defined by the variables on the LHS of the constraints. Furthermore, each dictionary form has a corresponding feasible solution obtained by setting all non-dictionary variables to 0 and the dictionary variables to the constants on the RHS. In this section, we will explore the geometric interpretation of a dictionary.

```
In [9]: lp = gilp.examples.ALL_INTEGER_2D_LP # get LP example
gilp.simplex_visual(lp, initial_solution=np.array([[0],[0]])).show() # visualize
```

Geometric Interpretation of LPs



Recall, we can hover over the corner points of the feasible region. **BFS** indicates the feasible

solution corresponding to that point. For example, $(7,0,6,9,0)$ means $x_1 = 7, x_2 = 0, x_3 = 6, x_4 = 9$, and $x_5 = 0$. **B** gives the indices of the variables “being defined” in that dictionary – that is, the variables that are on the LHS of the constraints. For simplicity, we will just say these variables are *in the dictionary*. For example, if **B** = $(1, 3, 4)$, then x_1, x_3 , and x_4 are in the dictionary. Lastly, the objective value at that point is given.

Q26: Hover over the point $(7,6)$ where $x_1 = 7$ and $x_2 = 6$. What is the feasible solution at that point ?

A: The feasible solution at that point is $(7,6,0,3,0)$

We have a notion of *slack* for an inequality constraint. Consider the constraint $x_1 \geq 0$. A feasible solution where $x_1 = 7$ has a slack of 7 in this constraint. Consider the constraint $2x_1 + 1x_2 \leq 20$. The feasible solution with $x_1 = 7$ and $x_2 = 6$ has a slack of 0 in this constraint.

Q27: What is the slack in constraint $1x_1 + 1x_2 \leq 16$ when $x_1 = 7$ and $x_2 = 6$?

A: The slack would be 3

Q28: Look at the constraint $2x_1 + 1x_2 \leq 20$. After rewriting in dictionary form, the constraint is $x_3 = 20 - 2x_1 - 1x_2$. What does x_3 represent?

A: The slack of the constraint.

Q29: What do you notice about the feasible solution at point $(7,6)$ and the slack in each constraint?

A: The slack value of the constraint is the same value for that x value in the feasible solution.

It turns out that each decision variable is really a measure of slack in some corresponding constraint!

Q30: If the slack between a constraint and a feasible solution is 0, what does that tell you about the relationship between the feasible solution and constraint geometrically?

A: That the feasible solution is at a corner point on that constraint.

Q31: For $(7,6)$, which variables are **not** in the dictionary? For which constraints do they represent the slack? (Hint: The **B** in the hover box gives the indices of the variables in the dictionary)

A: 3 and 5 are not in the dictionary.

Q32: For $(7,6)$, what are the values of the non-dictionary variables? Using what you learned from **Q30**, what does their value tell you about the feasible solution at $(7,6)$?

A: The value at those non-dictionary variables is 0. This tells us that the feasible solution is at the intersection of those two constraint.

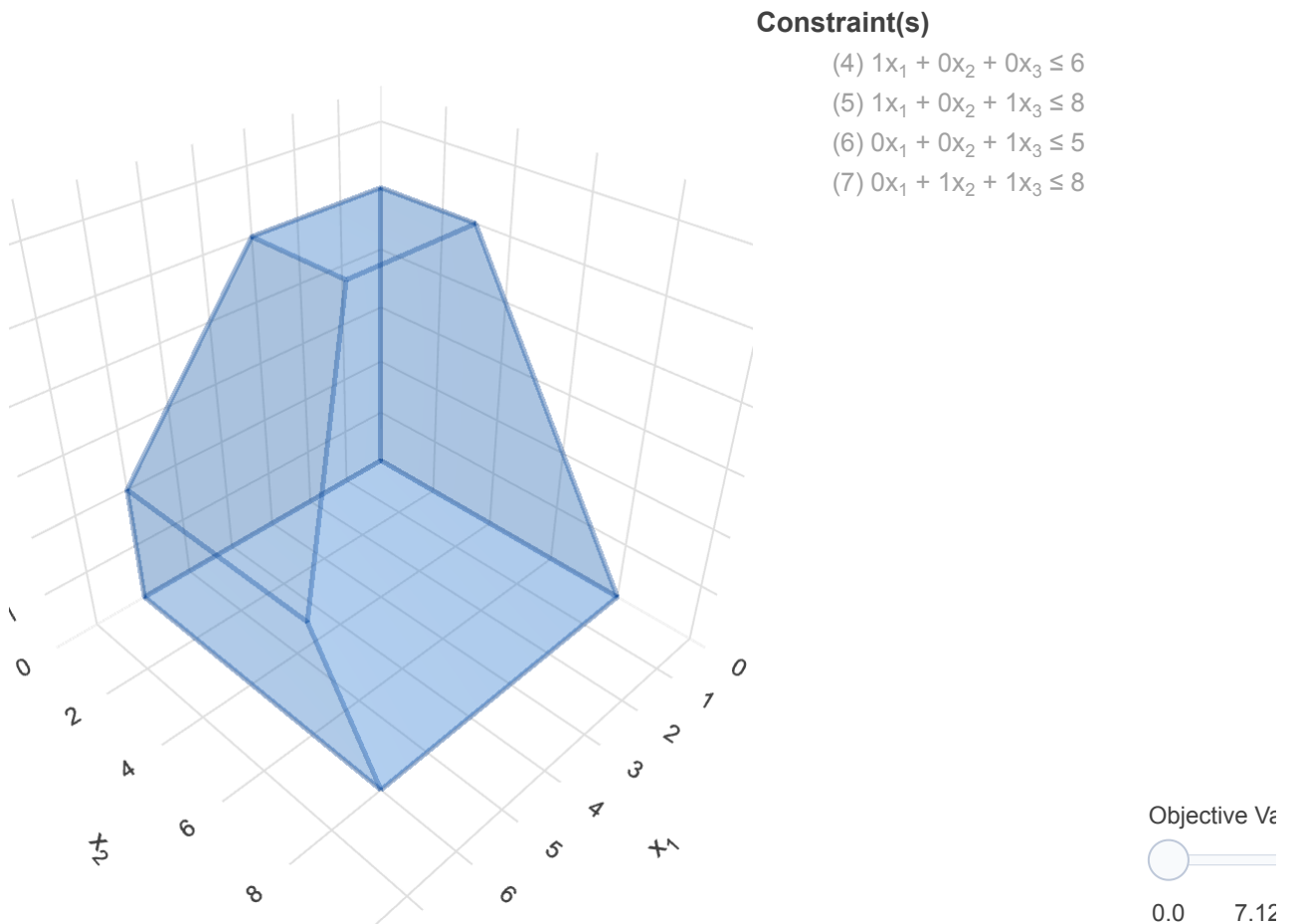
Q33: Look at some other corner points with this in mind. What do you find?

A: That the feasible solutions at those points also have 0 for the x values that are not part of the dictionary.

Now, let's look at a 3 dimensional LP!

```
In [10]: lp = gilp.examples.ALL_INTEGER_3D_LP # get LP example
gilp.lp_visual(lp).show() # visualize it
```

Geometric Interpretation of LPs



Q34: Hover over the point $(6,6,2)$ where $x_1 = 6$, $x_2 = 6$, and $x_3 = 2$. Note which variables are not in the dictionary. Toggle the corresponding constraints on. What do you notice?

A: The variables not in the dictionary are 4,5, and 7. The point $(6,6,2)$ is at the intersection of the constraints 4,5, and 7.

Q35: Look at some other corner points and do as you did in Q34. Do you see a similar pattern? Combining what you learned in Q33, what can you say about the relationship between the variables not in the dictionary at some corner point, and the corresponding constraints?

A: The constraints not in the dictionary all intersect at the corner point.

Q36: What geometric feature do feasible solutions for a dictionary correspond to?

A: Feasible solutions for a dictionary correspond to a corner point.

Part IV: Choosing an Increasing Variable

The first step in an iteration of simplex is to choose an increasing variable. Sometimes, there are multiple options since multiple variables have a positive coefficient in the objective function.

Here, we will explore what this decision translates to geometrically.

In this section, we will use a special LP commonly referred to as the Klee-Minty Cube.

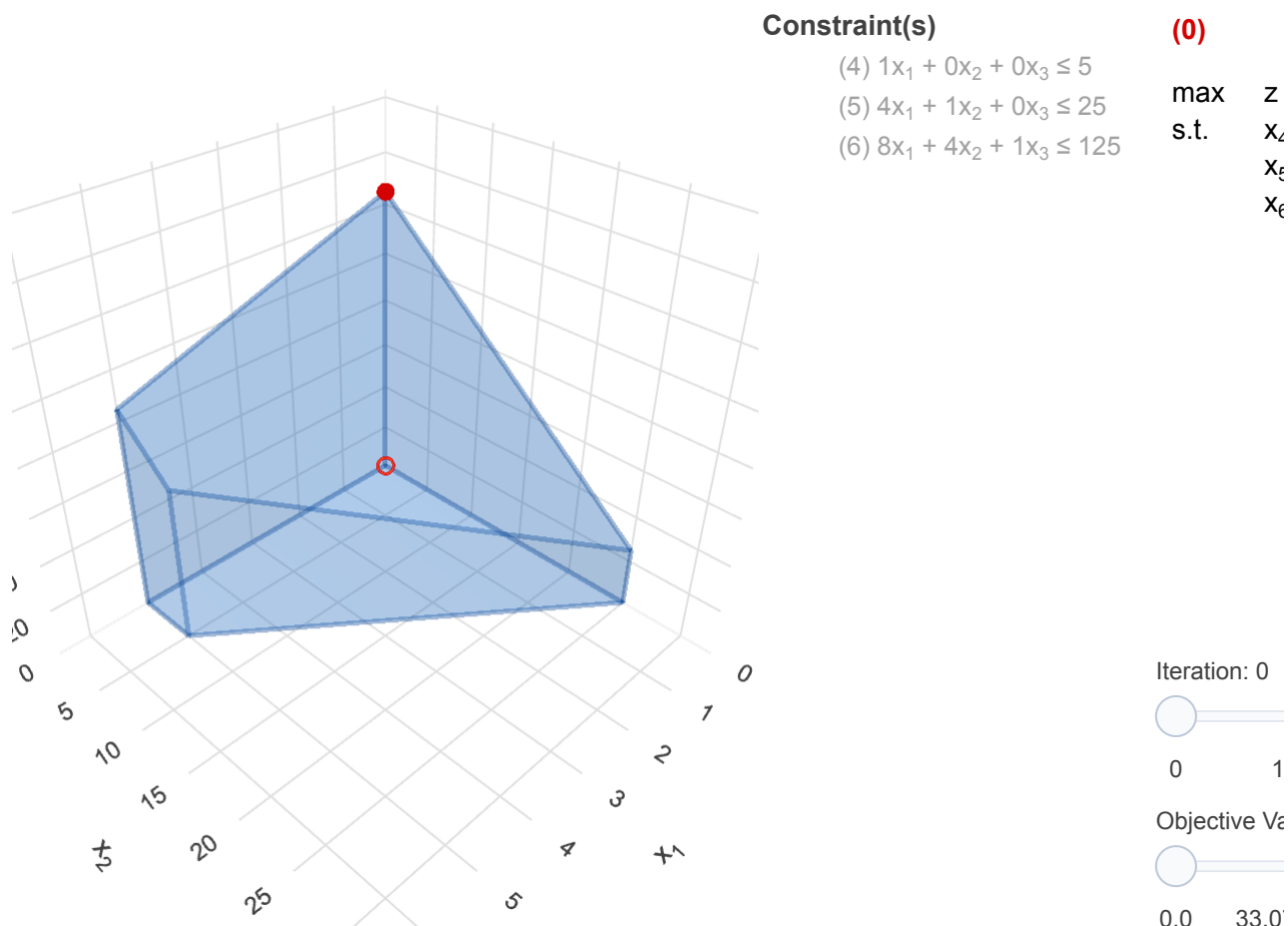
$$\begin{aligned} \max \quad & 4x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 \leq 5 \\ & 4x_1 + x_2 \leq 25 \\ & 8x_1 + 4x_2 + x_3 \leq 125 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Furthermore, we will use an optional parameter called `rule` for the `simplex_visual()` function. This rule tells simplex which variable to choose as an increasing variable when there are multiple options.

```
In [11]: klee_minty = gilp.examples.KLEE_MINTY_3D_LP
```

```
In [12]: gilp.simplex_visual(klee_minty, rule='dantzig', initial_solution=np.array([[0],[0],[0]]))
```

Geometric Interpretation of LPs



Q37: Use the iteration slider to examine the path of simplex on this LP. What do you notice?

A: x_1 is maximized first.

Above, we used a rule proposed by Dantzig. In this rule, the variable with the *largest* positive coefficient in the objective function enters the dictionary. Go through the iterations again to

verify this.

Let us consider another rule proposed by Bland, a professor here at Cornell. In his rule, of the variables with positive coefficients in the objective function, the one with the smallest index enters. Let us examine the path of simplex using this rule! Again, look at the dictionary form LP at every iteration.

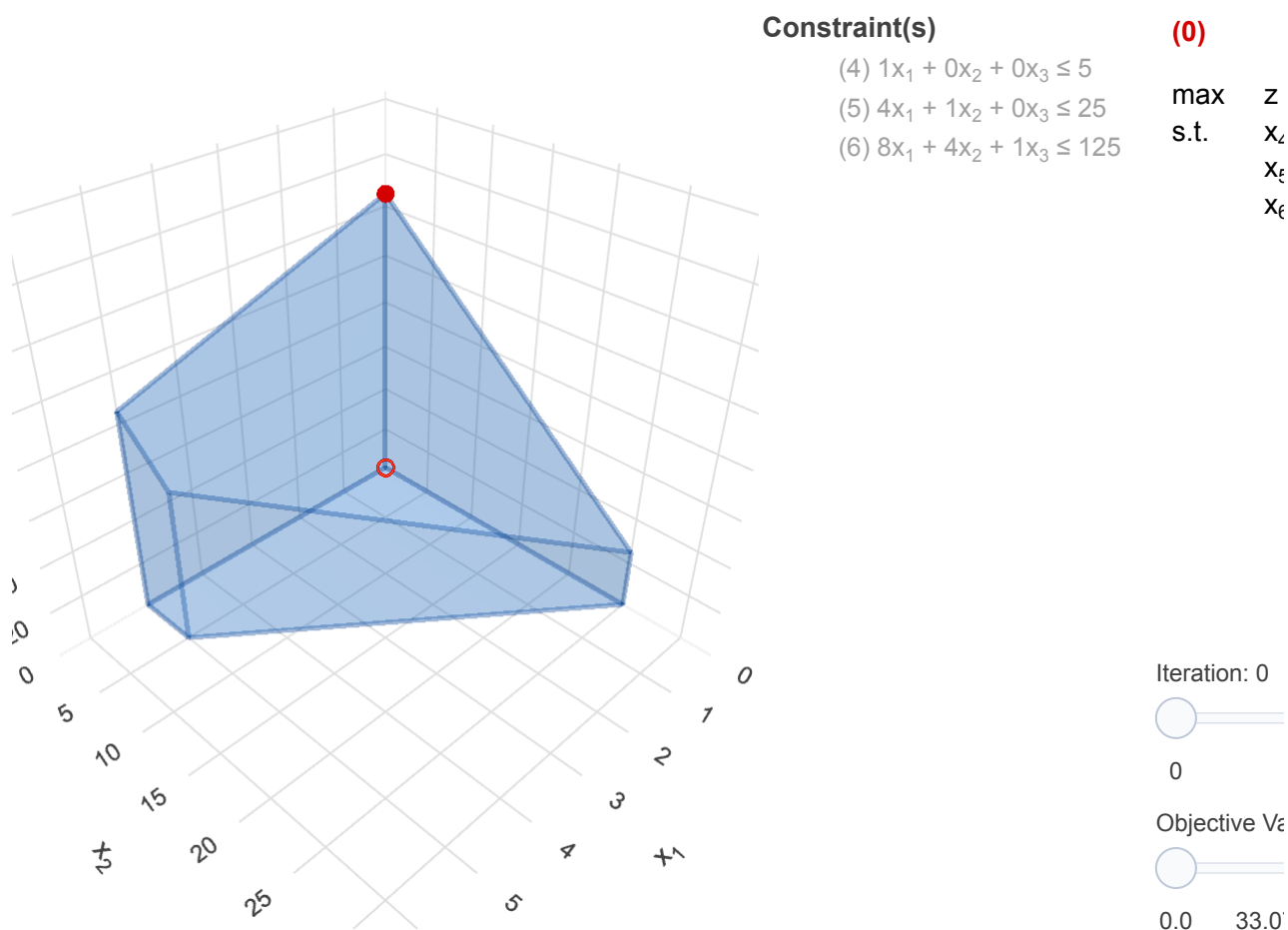
```
In [15]: gilp.simplex_visual(klee_minty, rule='manual', initial_solution=np.array([[0],[0]]))
```

INSTRUCTIONS

At each iteration of simplex, choose one of the variables with a positive coefficient in the objective function. The list of indices for possible variables (also called entering variables) is given.

Pick one of [1, 2, 3]

Geometric Interpretation of LPs



Q38: What is the difference between the path of simplex using Dantzig's rule and Bland's rule?

A: Dantzig's rule maximizes x_4 first at iteration 3 where Bland's rule maximizes x_3 first at iteration 3.

Can you do any better? By setting `rule='manual'`, you can choose the entering variable explicitly at each simplex iteration.

Q39: Can you do better than 5 iterations? How many paths can you find? (By my count, there are 7)

A: Yes, by increasing x_3 first, we only do 1 iteration. I found 7 paths.

```
In [17]: gilp.simplex_visual(klee_minty, rule='manual', initial_solution=np.array([[0],[0]]
```

INSTRUCTIONS

At each iteration of simplex, choose one of the variables with a positive coefficient in the objective function. The list of indices for possible variables (also called entering variables) is given.

Pick one of [1, 2, 3]1

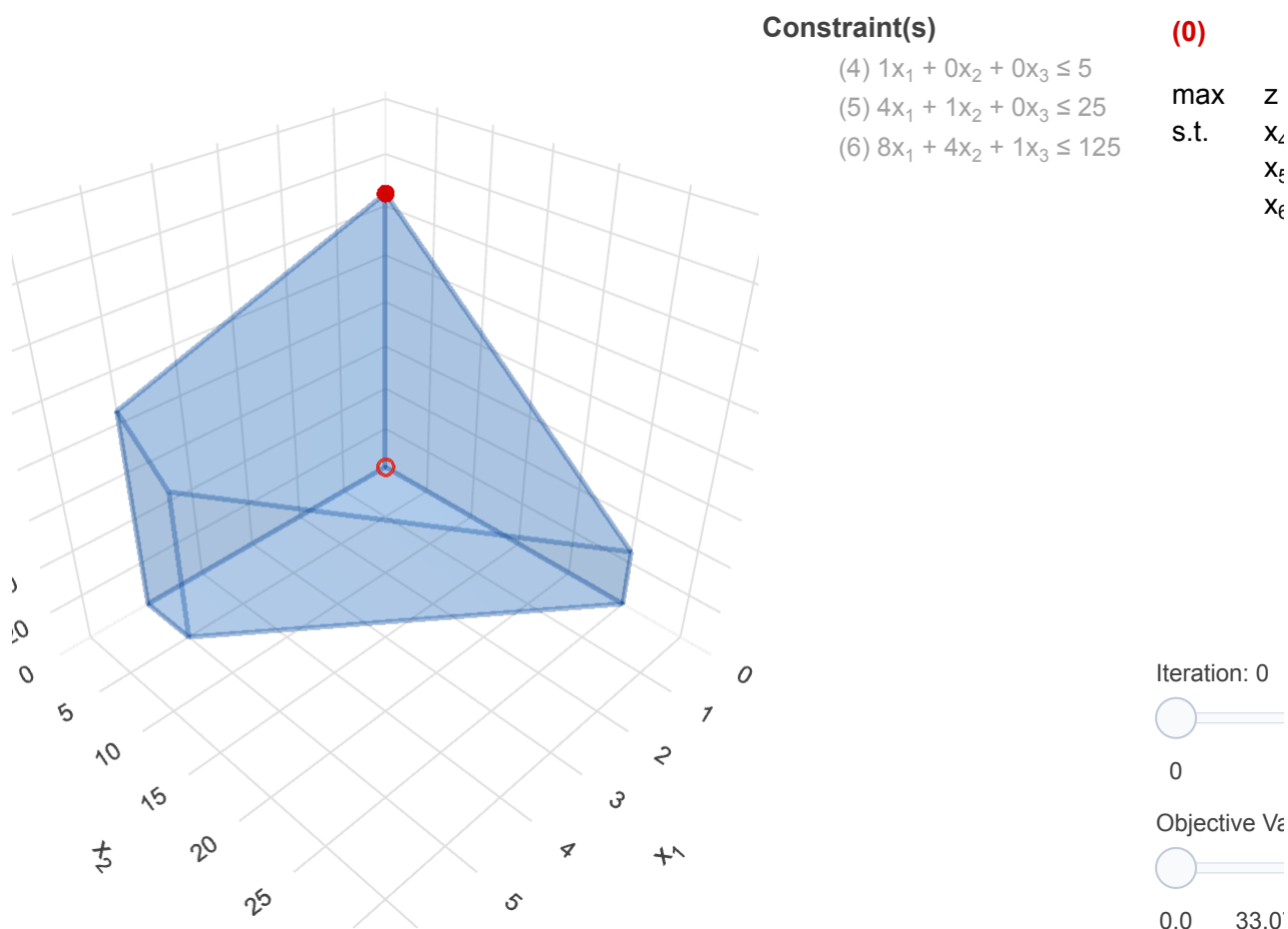
Pick one of [2, 3]2

Pick one of [3, 4]3

Pick one of [5]5

Pick one of [4]4

Geometric Interpretation of LPs



Q40: What does the choice of increasing variable correspond to geometrically?

A: The choice of increasing variable corresponds to the geometrical path that is taken.

Q41: Are there any paths you could visualize taking to the optimal solution that `rule='manual_select'` prevented you from taking? If yes, give an example and explain why it is not a valid path for simplex to take. (Hint: Look at the objective value after each simplex iteration.)

A: Yes, this is because out of the options, simplex picks the path that gives that largest objective value. So if there are two options for a possible path, simplex will only give the option of one with the highest objective value out of the two.