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COVER SHEET FOR TECHNICAL MEMORANDA

RESEARCH DEPARTMENT

SUBJECT: A Theorem on Color Coding - Case 20878

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ABSTRACT

A proof is given to the following topological theorem: A necessary and sufficient number of colors to color the lines of an arbitrary linear graph with not more than m lines starting at any one junction point in such a way that no two lines to the same junction point have the same color is the greatest integer $\leq \frac{3}{2}$ m. The theorem is of practical interest in connection with the wiring of units with a pre-formed cable, for it gives a limit to the number of color codings of wire necessary.

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MEMORANDUM FOR FILE

In this memorandum a solution is given to a certain topological coloring problem arising in connection with the color coding of wires in units such as relay panels. In this work there are a number of relays, switches, and other devices A, B, ..., K to be interconnected. The connecting wires are first formed in a cable with the leads associated with A coming out at one point, those with B at another, etc., and it is necessary, in order to distinguish the different wires, that all those coming out of the cable at the same point be differently colored. There may be any number of leads joining the same two points. We might have, for example, four wires from A to B, two from B to C, three from C to D and one from A to D. The four from A to B must all be of different colors, and all different from those from B to C and A to D, but the three from C to D can be the same as three of those from A to B. Also the one from A to D can be the same as one from B to C. If we assume that not more than m leads start at any one point, the question arises as to the least number of different colors that is sufficient to color any network. We shall prove the following proposition:

Theorem: A sufficient number of colors for any network with not more than m lines from any point is the greatest integer $\leq \frac{3}{2}m$. This number of colors is necessary for some networks.

It is easy to construct graphs for which this number of different colors is necessary, for consider the networks of Figs. 1 and 2.

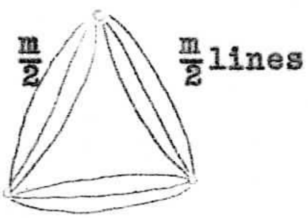

Fig. 1

(m even)

Fig. 2

(m odd = 2n + 1)

In each of these networks all the lines must clearly have different colors. In Fig. 1, m is even and there are exactly $\frac{3}{2}m$ lines. In Fig. 2, $m = 2n + 1$ is odd and the number of lines is $3n + 1$, the greatest integer in $\frac{3}{2}m$.

The proof of sufficiency is considerably more difficult. Let us first suppose m even. Now if N is our given network it is well known that we may add lines and junction points to get a regular network N' of degree m.* If we can color N' we can surely color N. A theorem due to Petersen states that any regular network of even degree $m = 2n$ can be factored into n regular second degree graphs. In our case let the factors of N' be $N_1, N_2, \ldots, N_n$. Each of these is merely a collection of polygons which do not touch each other, and each N_i, therefore, can be colored with three colors. This gives a total of $3n = \frac{3}{2}m$ colors.

Unfortunately it is not known under what conditions a regular graph of odd degree can be factored and we must use a different attack. The theorem will be proved by mathematical induction, making the coloring of N depend on coloring a network with one less junction point. Let us eliminate from N one junction point and the $m = 2n + 1$ lines coming to it and assume the remaining network to be satisfactorily colored with $3n + 1$ colors. Let the points that were connected to P in the original network be numbered $1, 2, \ldots, k$, and suppose there were p_1 parallel lines in the first group G1 connecting P to point 1, etc. Now after coloring the reduced network we have left available at junction 1 at least $[(3n + 1) - (2n + 1 - p_1)] = n + p_1$ colors, at junction 2, $(n + p_2)$, etc. By choosing properly from these available colors and by suitable interchanges of certain colors in the part of the network already colored we will show that the lines from P can be satisfactorily colored.

Let us arrange the data in a table as follows.

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*A regular network of degree m is one in which exactly m lines end at each junction point.

Colors

	1	2	3	.	.	.	.	.	.	(3n + 1)
1	1	1	0	1	.	.	.	.	.	.
2	1	0	1	0	.	.	.	.	.	.
lines ⋮	.			.			.			.
(2n+1)	.			.			.			.
	.			.			.			.

Fig. 3

In this array the $2n+1$ lines from P are listed vertically, the $3n+1$ colors horizontally. If a certain color is available for a certain line a 1 is placed at the intersection of the corresponding row and column, otherwise a 0. In a row corresponding to a line in G_i there will be $(n+p_i)$ 1's. By the use of three operations on this array we will arrange to obtain a series of 1's along the main diagonal and this will represent a coloring scheme for the network. These operations are:

1. Interchange of columns. This corresponds to merely renumbering the colors.

2. Interchange of rows. This corresponds to renumbering the lines from P.

3. Interchange of colors in a chain of two colors. Two points will be said to be chained together for two colors if we can go from one point to the other along lines which alternate these colors. If we have a satisfactorily colored network and interchange the two colors in a chain of that network over its entire length (Note that in a correctly colored network a chain cannot branch out) then it is clear that the network will still be satisfactorily colored. We will also use the fact that if only one of the two colors in question appears at each of three distinct points then only one pair of the points (at most) can be chained together for these two colors, since a chain can only have two ends. Now consider the array of Fig. 4. Suppose that in the first row there is a 0 in one column and a 1 in another. Interchanging these two colors in the chain starting from the first terminal will be seen to be equivalent to interchanging these two columns for this row and all other rows that are chained to it.

Let us suppose that we have arranged to get 1's on the main diagonal D down to a certain point. We will show that we can get another 1 on the diagonal.

3n+1 colors

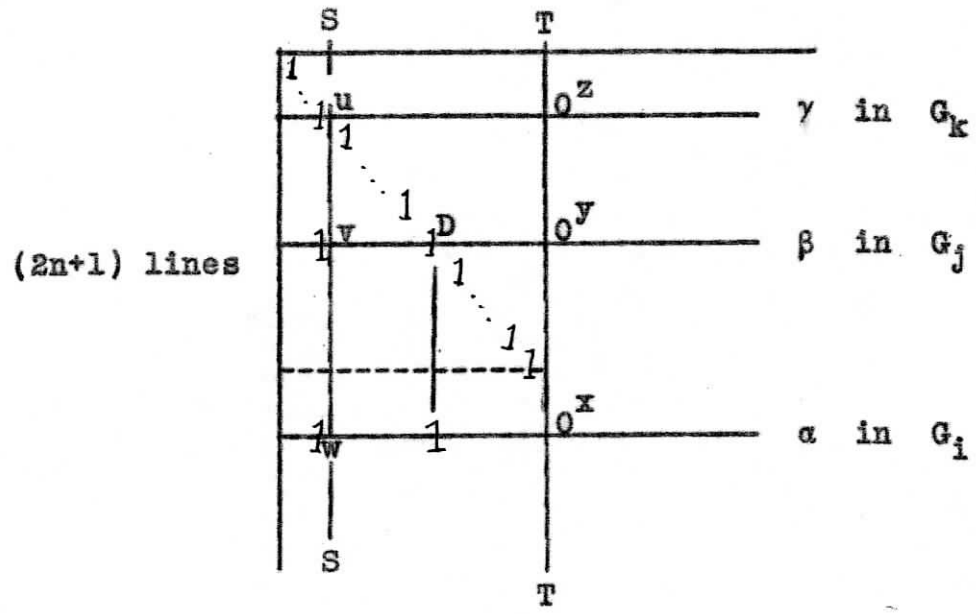

Fig. 4

Referring to Fig. 4, if there are any 1's on the next line α to the right of T-T, one of these may be brought, by an interchange of columns, to position x. Assuming this is not true there will be $n+p_i$ 1's to the left of T-T in α (assuming α is in G_i). Hence there are $n+p_i$ rows above α having a 1 in D in the same column as a 1 in α. At least n+1 of these rows are not in G_i, since G_i has p_i members and we have accounted for one already, namely α. Let β be one of these, belonging, say, to G_j. If β has a 1 to the right of T-T by an interchange first of columns, then of α and β this one may be moved to x without affecting the 1's along D. Assuming this is not true, there are $n+p_j$ 1's on β to the left of T-T and hence $n+p_j$ rows above α have a 1 in D in the same column as a 1 in β, and of these at least n do not belong to G_j (as it only has p_j members). Now there are not more than 2n rows above α and therefore the n rows we have associated with β and the $n+1$ we have associated with α must have at least one in common, i.e., there exists a row not belonging to G_i or G_j and having a 1 in D in the same column that α and β also have a 1. Call this row γ, and suppose it belongs to G_k. If γ has a 1 to the right of T-T it may be moved to x by first interchanging columns and then rows α and γ as before. Assuming this is not true, then there are 0's at x,y,z and 1's at u,v,w as shown. Hence at least one of α, β, γ is not chained to either of the others in the two colors of the columns T-T and S-S. If it is α, interchange the colors in the chain starting at i and the 1 at w moves to x without affecting u or v. If it is β, interchange the colors starting at j and then the rows α and β. If it is γ, interchange the colors starting at k and then interchange α and γ so that the 1 at w takes the place of the 1 at u, and the 1 at u is moved to x. This completes the proof.

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