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On Pólya-Szegö's inequality

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Abstract
In the paper, we give some new improvements of Pólya-Szegö's integral inequality which in a special case yield some of the recent results related with Pólya-Szegö's inequality.

MSC: 26D15

Keywords: Pólya-Szegö's inequality; Pólya-Szegö's integral inequality; Bellman's inequality

1 Introduction

The well-known Pólya-Szegö's inequality can be stated as follows ([1] or see [2], p.62).

If $0 < m_1 \le u_k \le M_1$ and $0 < m_2 \le v_k \le M_2$, where $k = 1, 2, \dots, n$, then

$$\left(\sum_{k=1}^{n} u_k^2\right)\left(\sum_{k=1}^{n} v_k^2\right) \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{k=1}^{n} u_k v_k\right)^2.$$

An integral analogue of Pólya-Szegö's inequality easy follows.

If $(E, \mathcal{A}, x)$ is a measure space and $f(x)$, $g(x)$ are non-negative measurable functions and $f^2(x)$, $g^2(x)$ are integrable on E, if $0 < m_1 \le f(x) \le M_1$ and $0 < m_2 \le g(x) \le M_2$, then

$$\left(\int_E f^2(x)\,dx\right)\left(\int_E g^2(x)\,dx\right) \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\int_E f(x)g(x)\,dx\right)^2. \tag{1.1}$$

Pólya-Szegö's inequality was studied extensively and numerous variants, generalizations, and extensions appeared in the literatures (see [3–7] and the references cited therein). The aim of this paper is to give some new improvements of Pólya-Szegö's integral inequality which are generalizations of Pólya-Szegö's integral inequality and interrelated result.

Theorem 1.1 *Let $(E, \mathcal{A}, x)$ be a measure space and $f(x)$, $g(x)$, $u(x)$, $v(x)$ be non-negative measurable functions. Let $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f^{1/p}(x)g^{1/q}(x)$, $u^{1/p}(x)v^{1/q}(x)$ be integrable on E and $u(x)$ and $v(x)$ be proportional. If $0 < m_1 \le f(x), u(x) \le M_1$ and $0 < m_2 \le g(x), v(x) \le M_2$, and $f(x) > u(x)$, $g(x) > v(x)$, then*

$$\begin{aligned}&\left(\int_E \big(f(x) - u(x)\big)\,dx\right)^{1/p}\left(\int_E \big(g(x) - v(x)\big)\,dx\right)^{1/q}\\ &\quad\le \Gamma_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_E \big(f^{1/p}(x)g^{1/q}(x) - u^{1/p}(x)v^{1/q}(x)\big)\,dx\end{aligned} \tag{1.2}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional and

$$\left(\int_E f(x)\,dx, \int_E u(x)\,dx\right) = \mu\left(\int_E g(x)\,dx, \int_E v(x)\,dx\right)$$

for some constant μ and where

$$\Gamma_{p,q}(\xi) = \left(\sqrt[p]{p}\cdot\sqrt[q]{q}\right)^{-1}\frac{1-\xi}{(1-\xi^{1/p})^{1/p}(1-\xi^{1/q})^{1/q}}\cdot\xi^{-1/pq}. \tag{1.3}$$

Remark 1.1 Taking for $p = q = 2$ and $u(x) = v(x) \equiv 0$ in (1.2), (1.2) changes to the following result:

$$\left(\int_E f(x)\,dx\right)^{1/2}\left(\int_E g(x)\,dx\right)^{1/2} \le \frac{1}{2}\left(\sqrt[4]{\frac{M_1M_2}{m_1m_2}} + \sqrt[4]{\frac{m_1m_2}{M_1M_2}}\right)\int_E f^{1/2}(x)g^{1/2}(x)\,dx \tag{1.4}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional.

Replace $f^{1/2}(x)$ and $g^{1/2}(x)$ by $f(x)$ and $g(x)$ in (1.4), respectively, and hence $m_i^{1/2}(x)$ and $M_i^{1/2}(x)$ are replaced by m_i and M_i ($i = 1, 2$), respectively. Therefore

$$\left(\int_E f^{1/2}(x)\,dx\right)^{1/2}\left(\int_E g^{1/2}(x)\,dx\right)^{1/2} \le \frac{1}{2}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)\int_E f(x)g(x)\,dx.$$

This is just Pólya-Szegö integral inequality (1.1). In fact, Theorem 1.1 is just a special case of Theorem 2.1 stated in Section 2.

Theorem 1.2 *Let $(E, \mathcal{A}, x)$ be a measure space and $f(x)$, $g(x)$, $u(x)$, $v(x)$ be non-negative measurable functions, and let $f^{1/p}(x)$, $g^{1/p}(x)$, $u^{1/p}(x)$, $v^{1/p}(x)$ be integrable on E, and $u(x)$ and $v(x)$ be proportional. If $p > 1$, $0 < m_1 \le \frac{f(x)}{(f(x)+g(x))^{p-1}}, \frac{u(x)}{(u(x)+v(x))^{p-1}} \le M_1$ and $0 < m_2 \le \frac{g(x)}{(f(x)+g(x))^{p-1}}, \frac{v(x)}{(u(x)+v(x))^{p-1}} \le M_2$, and $f(x) > u(x)$, $g(x) > v(x)$, then*

$$\begin{aligned}&\left(\int_E\left[f^p(x) - u^p(x)\right]dx\right)^{1/p} + \left(\int_E\left[g^p(x) - v^p(x)\right]dx\right)^{1/p}\\ &\quad\le \Gamma_{p,\frac{p}{p-1}}\left(\frac{m_1m_2}{M_1M_2}\right)\left(\int_E\left(\left[f(x)+g(x)\right]^p - \left[u(x)+v(x)\right]^p\right)dx\right)^{1/p}\end{aligned} \tag{1.5}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional and

$$\left(\int_E f^p(x)\,dx, \int_E u^p(x)\,dx\right) = \mu\left(\int_E g^p(x)\,dx, \int_E v^p(x)\,dx\right)$$

for some constant μ and $\Gamma_{p,\frac{p}{p-1}}(\xi)$ is as in (1.3).

Remark 1.2 Taking for $u(x) = v(x) \equiv 0$ in (1.5), (1.5) changes to the following inequality:

$$\left(\int_E f^p(x)\,dx\right)^{1/p} + \left(\int_E g^p(x)\,dx\right)^{1/p} \le \Gamma_{p,\frac{p}{p-1}}\left(\frac{m_1m_2}{M_1M_2}\right)\left(\int_E\left(f(x)+g(x)\right)^p\,dx\right)^{1/p}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional.

This is just the inequality in Lemma 2.2 (see Section 2). In fact, Theorem 1.2 is just a special case of Theorem 2.2 stated in Section 2.

2 Main results

We need the following lemmas to prove our main results.

Lemma 2.1 [8] *Let $(E,\mathcal{A},x)$ be a measure space and $f(x)$, $g(x)$ be non-negative measurable functions. Let $p,q>0$, $\frac{1}{p}+\frac{1}{q}=1$ and $f^{1/p}(x)g^{1/q}(x)$ be integrable on E. If $0<m_1\le f(x)\le M_1$ and $0<m_2\le g(x)\le M_2$, then*

$$\left(\int_E f(x)\,dx\right)^{1/p}\left(\int_E g(x)\,dx\right)^{1/q}\le \Gamma_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_E f^{1/p}(x)g^{1/q}(x)\,dx \tag{2.1}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional.

Lemma 2.2 [9] *Let $(E,\mathcal{A},x)$ be a measure space and $f(x)$, $g(x)$ be non-negative measurable functions, and $f^{1/p}(x)$, $g^{1/p}(x)$ be integrable on E. If $p>1$, $0<m_1\le\frac{f(x)}{(f(x)+g(x))^{p-1}}\le M_1$ and $0<m_2\le\frac{g(x)}{(f(x)+g(x))^{p-1}}\le M_2$, then*

$$\left(\int_E f^p(x)\,dx\right)^{1/p}+\left(\int_E g^p(x)\,dx\right)^{1/p}\le \Gamma_{p,\frac{p}{p-1}}\left(\frac{m_1m_2}{M_1M_2}\right)\left(\int_E\bigl(f(x)+g(x)\bigr)^p\,dx\right)^{1/p} \tag{2.2}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional.

Lemma 2.3 (Bellman's inequality [10]) *If*

$$\phi(x)=\bigl(x_1^p-x_2^p-\cdots-x_n^p\bigr)^{1/p},\quad p>1$$

for x_i in the region $\mathbb{R}$ defined by

(a) $x_i\ge 0$,

(b) $x_1\ge(x_2^p+x_3^p+\cdots+x_n^p)^{1/p}$.

Then, for $x,y\in\mathbb{R}$, we have

$$\phi(x+y)\ge\phi(x)+\phi(y), \tag{2.3}$$

with equality if and only if $x=\mu y$, where μ is a constant.

Lemma 2.4 [11] *Let $a,b,c,d>0$, $0<\alpha<1$, $0<\beta<1$ and $\alpha+\beta=1$. If $a>b$ and $c>d$, then*

$$a^\alpha c^\beta-b^\alpha d^\beta\ge(a-b)^\alpha(c-d)^\beta \tag{2.4}$$

with equality if and only if $a/b=c/d$.

Our main results are given in the following theorems.

Theorem 2.1 *Let $(E,\mathcal{A},x)$ be a measure space and $f(x)$, $g(x)$, $u(x)$, $v(x)$ be non-negative measurable functions. Let $p,q>0$, $\frac{1}{p}+\frac{1}{q}=1$, and $f^{1/p}(x)g^{1/q}(x)$, $u^{1/p}(x)v^{1/q}(x)$ be integrable* *on E, and $u(x)$ and $v(x)$ be proportional. If $0 < m_1 \le f(x) \le M_1$, $0 < m_2 \le g(x) \le M_2$, $0 < n_1 \le u(x) \le N_1$ and $0 < n_2 \le v(x) \le N_2$, and $f(x) > u(x)$, $g(x) > v(x)$, then*

$$\left(\int_E \bigl(f(x) - u(x)\bigr)\,dx\right)^{1/p} \left(\int_E \bigl(g(x) - v(x)\bigr)\,dx\right)^{1/q} \le \Gamma_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)\int_E f^{1/p}(x) g^{1/q}(x)\,dx - \Gamma_{p,q}\left(\frac{n_1 n_2}{N_1 N_2}\right)\int_E u^{1/p}(x) v^{1/q}(x)\,dx \tag{2.5}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional and

$$\left(\int_E f(x)\,dx, \int_E u(x)\,dx\right) = \mu\left(\int_E g(x)\,dx, \int_E v(x)\,dx\right)$$

for some constant μ.

Proof From the hypotheses and Lemma 2.1, we obtain

$$\Gamma_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)\int_E f^{1/p}(x) g^{1/q}(x)\,dx \ge \left(\int_E f(x)\,dx\right)^{1/p}\left(\int_E g(x)\,dx\right)^{1/q} \tag{2.6}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional, and

$$\Gamma_{p,q}\left(\frac{n_1 n_2}{N_1 N_2}\right)\int_E u^{1/p}(x) v^{1/q}(x)\,dx = \left(\int_E u(x)\,dx\right)^{1/p}\left(\int_E v(x)\,dx\right)^{1/q}. \tag{2.7}$$

From (2.6), (2.7) and in view of $1/p + 1/q = 1$, by using Lemma 2.4, we have

$$\begin{aligned}
&\Gamma_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)\int_E f^{1/p}(x) g^{1/q}(x)\,dx - \Gamma_{p,q}\left(\frac{n_1 n_2}{N_1 N_2}\right)\int_E u^{1/p}(x) v^{1/q}(x)\,dx \\
&\quad \ge \left(\int_E f(x)\,dx\right)^{1/p}\left(\int_E g(x)\,dx\right)^{1/q} - \left(\int_E u(x)\,dx\right)^{1/p}\left(\int_E v(x)\,dx\right)^{1/q} \\
&\quad \ge \left(\int_E \bigl(f(x) - u(x)\bigr)\,dx\right)^{1/p}\left(\int_E \bigl(g(x) - v(x)\bigr)\,dx\right)^{1/q}.
\end{aligned} \tag{2.8}$$

In view of the equality conditions of (2.4) and (2.6), it follows that the sign of equality in (2.5) holds if and only if $f(x)$ and $g(x)$ are proportional and

$$\left(\int_E f(x)\,dx, \int_E u(x)\,dx\right) = \mu\left(\int_E g(x)\,dx, \int_E v(x)\,dx\right)$$

for some constant μ. □

Remark 2.1 If $0 < n_2 \le u(x) \le N_2$ and $0 < n_2 \le v(x) \le N_2$ change to $0 < m_1 \le u(x) \le M_1$ and $0 < m_2 \le v(x) \le M_2$, respectively, then (2.5) reduces to (1.2) stated in the Introduction.

Theorem 2.2 *Let $(E, \mathcal{A}, x)$ be a measure space and $f(x)$, $g(x)$, $u(x)$, $v(x)$ be non-negative measurable functions, and let $f^{1/p}(x)$, $g^{1/p}(x)$, $u^{1/p}(x)$, $v^{1/p}(x)$ be integrable on E and $u(x)$*

and $v(x)$ be proportional. If $p > 1$, $0 < m_1 \leq \frac{f(x)}{(f(x)+g(x))^{p-1}} \leq M_1$, $0 < m_2 \leq \frac{g(x)}{(f(x)+g(x))^{p-1}} \leq M_2$, $0 < n_1 \leq \frac{u(x)}{(u(x)+v(x))^{p-1}} \leq N_1$ and $0 < n_2 \leq \frac{v(x)}{(u(x)+v(x))^{p-1}} \leq N_2$, and $f(x) > u(x)$, $g(x) > v(x)$, then

$$\begin{aligned}
&\left(\int_E \left[f^p(x) - u^p(x)\right] dx\right)^{1/p} + \left(\int_E \left[g^p(x) - v^p(x)\right] dx\right)^{1/p} \\
&\quad \leq \left[\Gamma^p_{p,\frac{p}{p-1}}\left(\frac{m_1 m_2}{M_1 M_2}\right)\left(\int_E \left(f(x) + g(x)\right)^p dx\right)\right. \\
&\qquad \left. - \Gamma^p_{p,\frac{p}{p-1}}\left(\frac{n_1 n_2}{N_1 N_2}\right)\left(\int_E \left(u(x) + v(x)\right)^p dx\right)\right]^{1/p} \qquad (2.9)
\end{aligned}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional and

$$\left(\int_E f^p(x)\,dx, \int_E u^p(x)\,dx\right) = \mu\left(\int_E g^p(x)\,dx, \int_E v^p(x)\,dx\right)$$

for some constant μ.

Proof From the hypotheses and Lemma 2.2, it is easy to obtain

$$\begin{aligned}
&\Gamma_{p,\frac{p}{p-1}}\left(\frac{m_1 m_2}{M_1 M_2}\right)\left(\int_E \left(f(x) + g(x)\right)^p dx\right)^{1/p} \\
&\quad \geq \left(\int_E f^p(x)\,dx\right)^{1/p} + \left(\int_E g^p(x)\,dx\right)^{1/p} \qquad (2.10)
\end{aligned}$$

with equality if and only if f and g are proportional, and

$$\begin{aligned}
&\Gamma_{p,\frac{p}{p-1}}\left(\frac{n_1 n_2}{N_1 N_2}\right)\left(\int_E \left(u(x) + v(x)\right)^p dx\right)^{1/p} \\
&\quad = \left(\int_E u^p(x)\,dx\right)^{1/p} + \left(\int_E v^p(x)\,dx\right)^{1/p}. \qquad (2.11)
\end{aligned}$$

From (2.10), (2.11) and by using Lemma 2.3, we have

$$\begin{aligned}
&\left[\Gamma^p_{p,\frac{p}{p-1}}\left(\frac{m_1 m_2}{M_1 M_2}\right)\left(\int_E \left(f(x) + g(x)\right)^p dx\right) - \Gamma^p_{p,\frac{p}{p-1}}\left(\frac{n_1 n_2}{N_1 N_2}\right)\left(\int_E \left(u(x) + v(x)\right)^p dx\right)\right]^{1/p} \\
&\quad \geq \left\{\left[\left(\int_E f^p(x)\,dx\right)^{1/p} + \left(\int_E g^p(x)\,dx\right)^{1/p}\right]^p\right. \\
&\qquad \left. - \left[\left(\int_E u^p(x)\,dx\right)^{1/p} + \left(\int_E v^p(x)\,dx\right)^{1/p}\right]^p\right\}^{1/p} \\
&\quad \geq \left(\int_E \left[f^p(x) - u^p(x)\right] dx\right)^{1/p} + \left(\int_E \left[g^p(x) - v^p(x)\right] dx\right)^{1/p}. \qquad (2.12)
\end{aligned}$$

In view of the equality conditions of (2.10) and (2.3), it follows that the sign of equality (2.9) holds if and only if f and g are proportional and

$$\left(\int_E f^p(x)\,dx, \int_E u^p(x)\,dx\right) = \mu\left(\int_E g^p(x)\,dx, \int_E v^p(x)\,dx\right)$$

for some constant μ. $\square$

Remark 2.2 If $0 < n_1 \leq \frac{u(x)}{(u(x)+v(x))^{p-1}} \leq N_1$, $0 < n_2 \leq \frac{v(x)}{(u(x)+v(x))^{p-1}} \leq N_2$ change to $0 < m_1 \leq \frac{u(x)}{(u(x)+v(x))^{p-1}} \leq M_1$, $0 < m_2 \leq \frac{v(x)}{(u(x)+v(x))^{p-1}} \leq M_2$, respectively, then (2.9) reduces to (1.5) stated in the Introduction.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CJZ and WSC jointly contributed to the main results Theorems 1.1-1.2 and Theorems 2.1-2.2. All authors read and approved the final manuscript.

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