Discrete Mathematics 1
Set Theory

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1. Sets: a definition

A **set** is an unordered collection of objects, called **elements** or **members** . A set is said to **contain** its elements. If ***x*** is an element of the set **A**, then we write $\boldsymbol{x \in A}$. If ***x*** is not an element of the set ***A***, then we write $\boldsymbol{x \notin A}$.
For example, if ***S*** is the set of states in the United States, then New York is an element of **S** and Ontario is not an element of **S**. If ***E*** is the set of even integers, then $\boldsymbol{2 \in E}$ and $\boldsymbol{3 \notin E}$.

1. Sets: a definition

There are several different ways to describe a set. One way of describing a set is known as the **roster method** . This is where we list all the elements of a set between curly braces. For example:

{-3,-2,-1,0,1,2,3}

The elements of a set may also be described verbally:

{ integers between -3 and 3 inclusive}

The set builder notation may be used to describe sets that are too tedious to list explicitly. To denote any particular set, we use the letter x**:{x|x is an integer and |x|<4} or equivalently {x|x $\in$ Z ,|x|<4}**

1. Sets: a definition

Example 3 - Switching between representations

Consider the following set:

$$\{x \in \mathbb{Z} : -2 \leq x < 4\}.$$

This is the set of all integers x such that -2 is less than or equal x and x is less than 4. Using the roster method, this set can be written as

$$\{-2, -1, 0, 1, 2, 3\}.$$

1. Sets: a definition

Example 3 - Switching between representations

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1. Sets: a definition

You Try

Match each set described using set builder notation in parts (a) through (f) with the same set described using the roster method in parts (A) through (F).

a. $\{x \in \mathbb{Z} : x^2 = 1\}$

b. $\{x \in \mathbb{Z} : x^3 = 1\}$

c. $\{x \in \mathbb{Z} : |x| \leq 2\}$

d. $\{x \in \mathbb{Z} : x^2 < 4\}$

1. Special sets of numbers

- $\mathbb{Z}$, $\{-2, -1, 0, 1, 2, \ldots\}$, the set of integers
- $\mathbb{Z}^+$ or $\mathbb{N}$, $\{1, 2, 3, \ldots\}$, the set of natural numbers or positive integers
- $\mathbb{Q}$, $\left\{\frac{a}{b} \middle| a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$, the set of rational numbers of the form $\frac{2}{3}$
- $\mathbb{R}$, the set of real numbers
- $\mathbb{R}^+$, the set of positive real numbers
- $\mathbb{C}$, $\{a + ib | a \in \mathbb{R}, b \in \mathbb{R}, b \neq 0\}$, the set of complex numbers of the form $2 + 3i$.

1.1. Empty Set

Consider the following set described using set builder notation:

$$\{x \in \mathbb{Z} : x^2 = 2\}.$$

This is the set of all integers whose square is equal to 2. However, no such integers exist. Therefore, using the roster method to describe it, this is the set $\{\}$.

We call the set $\{\}$ the **empty set** and denote this set by $\emptyset$. The empty set has no elements.

1.2. Universal set

The set of all the entities in the current context is called the universal set, or simply the universe. It is denoted by U.

The context may be a homework exercise, for example, where the Universal set is limited to the particular entities under its consideration. Also, it may be any arbitrary problem, where we clearly know where it is applied

1.3. Cardinality

Suppose that a set A contains a finite number of distinct elements. We refer to the number of elements of A as the **cardinality** of A and denote this by $|A|$. If A contains an infinite number of distinct elements, we say that A has infinite cardinality and we write $|A| = \infty$.

Thus, we see that $|\{0, 1, 2\}| = 3$ and $|\mathbb{Z}| = \infty$. Additionally, note that $|\emptyset| = 0$.

1.4. Equality

We say that two sets are **equal** if and only if they contain the same elements. In other words, A and B are equal sets if and only if

$$\forall x(x \in A \iff x \in B).$$

When A and B are equal sets, we write $A = B$. When A and B are not equal sets, we write $A \neq B$.

The sets $\{2, 3, 5\}$ and $\{5, 2, 3\}$ are equal sets, since they contain the same elements. The order in which the elements of a set are listed does not matter. Additionally, it does not matter whether elements are repeated. Thus, the sets $\{a, b, c\}$ and $\{b, b, a, c, b, a, c, c, c\}$ are equal sets as well.

1.5. Subsets

We say that a set A is a **subset** of a set B if and only if every element of A is an element of B. In other words, A is a subset of B if and only if

$$\forall x(x \in A \implies x \in B).$$

When A is a subset of B, we write $A \subseteq B$. When A is not a subset of B, we write $A \nsubseteq B$.

In order to show that A is a subset of B, we must show that, whenever $x \in A$, it is also the case that $x \in B$. In order to show that A is not a subset of B, we must find a single x such that $x \in A$ but $x \notin B$.

1.4. Subsets

Example 7

If we let $S = \{1, 5\}$ and $T = \{1, 3, 5\}$, then $S \subseteq T$, since each element of S is an element of T, but $T \nsubseteq S$, since 3 is an element of T that is not an element of S.

If we let A be the set of professional athletes and let F be the set of professional football players, we have $F \subseteq A$, since every professional football player is a professional athlete, but $A \nsubseteq F$, since not every professional athlete is a professional football player.

1.4. Subsets

Note that, for any set A, it is always the case that $\emptyset \subseteq A$ and $A \subseteq A$. For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

If $A \subseteq B$ and B contains at least one element that is not in A, then we say A is a **proper subset** of B, denoted $A \subset B$.

1.6. Power Set

Given a set A, we refer to the **power set** of A as the set of all subsets of A. The power set of A is denoted by $\mathcal{P}(A)$.

$\mathcal{P}(A)$ is a set whose elements are all sets.

If we let $A = \{a, b, c\}$, we see that

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

The empty set only has the empty set as a subset. Thus, we see that

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

2. Set Operations

We can obtain new sets by performing operations on other sets. When performing set operations, it is often helpful to consider all of our sets as subsets of a **universal set** U. We can think of the universal set as the set of all of the objects under consideration.

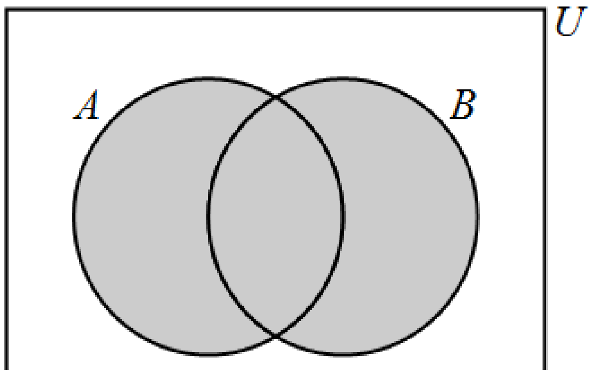
We can represent set operations visually using **Venn diagrams** , named after the English mathematician John Venn. A Venn diagram will consist of a rectangle, which represents the universal set, and one or more circles, which represent the sets under consideration. We will then shade in the regions of the diagram that correspond to one or more set operations.

2.1. Union

The **union** of the sets A and B is the set containing those elements that are in A or B or both, and is denoted by $A \cup B$. More formally,

$$A \cup B = \{x \in U : x \in A \vee x \in B\}.$$

We have the following Venn Diagram for $A \cup B$:



Note that, for any sets A and B,

$$A \cup B = B \cup A.$$

2.1. Union

Example 9

If we let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$, then

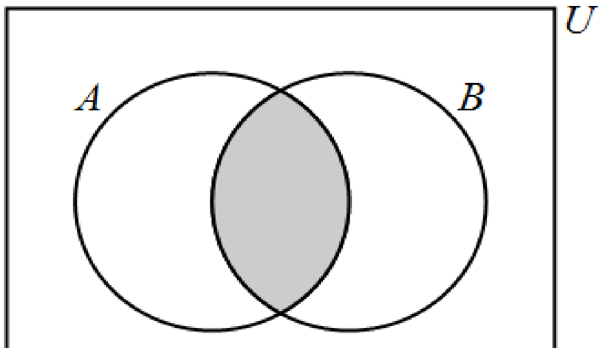
$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}.$$

2.2. Intersection

The **intersection** of the sets A and B is the set containing those elements that are in A and B and is denoted by $A \cap B$. More formally,

$$A \cap B = \{x \in U : x \in A \wedge x \in B\}.$$

We have the following Venn Diagram for $A \cap B$:



Note that, for any sets A and B,

$$A \cap B = B \cap A.$$

If it is the case that $A \cap B = \emptyset$, then we say that A and B are **disjoint** . In other words, two sets are disjoint if and only if they contain no elements in common.

2.2. Intersection

Example 11

If we let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$, then

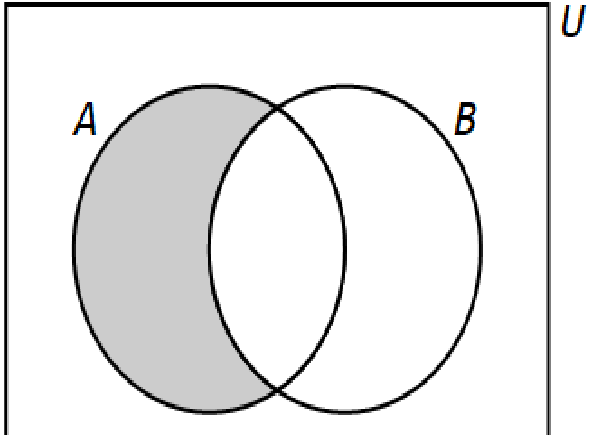
$$A \cap B = \{1, 3, 5\}.$$

2.3. Difference

The **difference** of the sets A and B is the set containing those elements that are in A but not in B and is denoted by $A \setminus B$. Set difference is also denoted by $A - B$. More formally,

$$A \setminus B = \{x \in U : x \in A \wedge x \notin B\}.$$

We have the following Venn Diagram for $A \setminus B$:



2.3. Difference

Note that, for any sets A and B, if $A = B$, then $A \setminus B = \emptyset$ and $B \setminus A = \emptyset$. Thus, when $A = B$,

$$A \setminus B = B \setminus A.$$

However, if $A \neq B$, then

$$A \setminus B \neq B \setminus A.$$

Example 13

If we let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$, then

$$A \setminus B = \{2, 4, 6\}$$

and

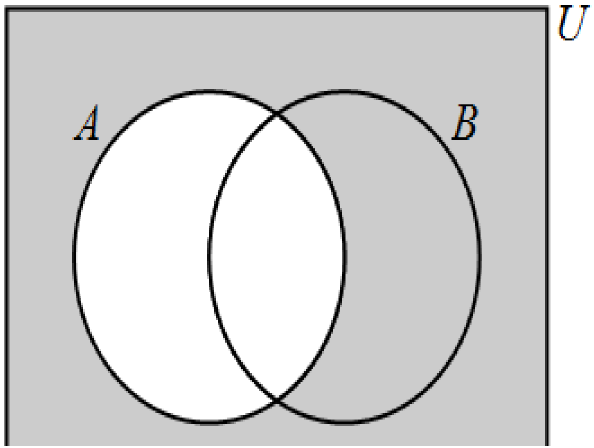
$$B \setminus A = \{7, 9\}.$$

2.4. Complement

The **complement** of a set A is the set of all elements in the universal set U which are not elements of A and is denoted by $\overline{A}$. More formally,

$$\overline{A} = \{x \in U : x \notin A\}.$$

We have the following Venn Diagram for $\overline{A}$:



2.4. Complement

For any set A,

$$\overline{A} = U \setminus A.$$

Example 15

Suppose that our universal set is $U = \{0,1,2,3,4,5,6,7,8,9\}$, the set of all decimal digits. If we let $A = \{1,2,3,4,5,6\}$ and $B = \{1,3,5,7,9\}$, then

$$\overline{A} = \{0,7,8,9\}$$

and

$$\overline{B} = \{0,2,4,6,8\}.$$

Example 16

Suppose that our universal set is $\mathbb{Z}$. If we let E be the set of all even integers, then $\overline{E}$ is the set of all odd integers.

2.5. Multiple Set Operations

We can also perform more than one set operation on a collection of sets. For example, let A, B, and C be sets and consider the following set:

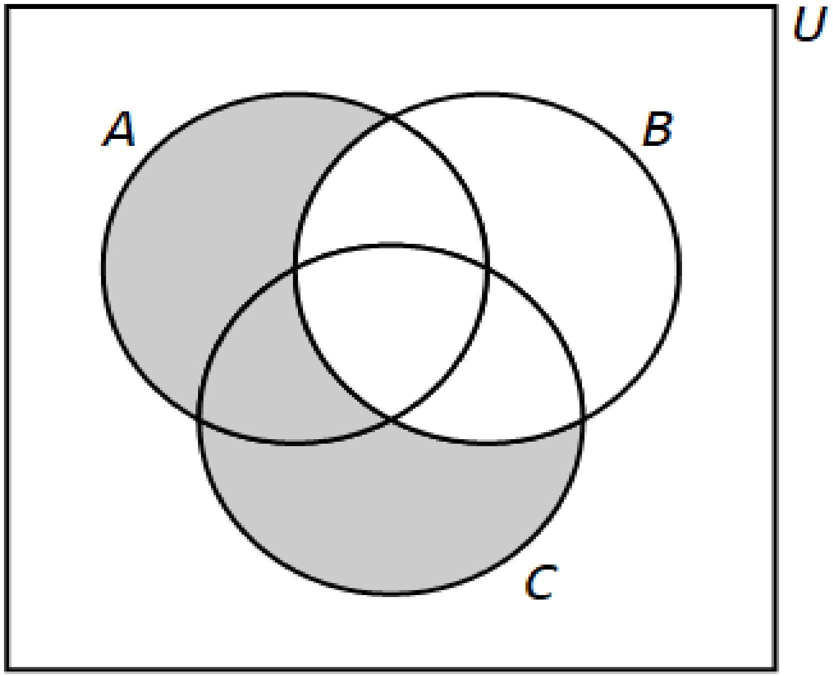
$$(A \setminus B) \cup (C \setminus B).$$

This is the set that is obtained by taking the union of the sets $A \setminus B$ and $C \setminus B$. We have

$$(A \setminus B) \cup (B \setminus A) = \{x \in U : (x \in A \wedge x \notin B) \vee (x \in C \wedge x \notin B)\}.$$

2.5. Multiple Set Operations

We have the following Venn Diagram for $(A \setminus B) \cup (C \setminus B)$:



2.5. Multiple Set Operations

You Try

Draw Venn Diagrams for each of these combinations of the sets A, B, and C.

1. $A \cap (B \cup C)$
2. $(A \cap B) \cup C$
3. $(\overline{A} \cap \overline{C}) \cup B$
4. $(B \cup C) \setminus A$

2.6. Properties of set operations

1. Commutative [edit | edit source]

- $A \cup B = B \cup A$
  $A \cap B = B \cap A$

2. Associative [edit | edit source]

- $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$

3. Distributive [edit | edit source]

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. Special properties of complements

- $(A')' = A$
  $U' = \phi$
  $\phi' = U$
  $A \cap B' = A - B$

5. De Morgan's Law [edit | edit source]

- $(A \cap B)' = A' \cup B'$
  $(A \cup B)' = A' \cap B'$.

2.7. The Cartesian Product

The **Cartesian product** of two sets A and B is the set of *ordered pairs* defined by,

$$A \times B = \{(a, b) | a \in A \wedge b \in B)\},$$

Example 17

Consider the sets, $B = \{0, 1\}$, $T = \{0, 1, 2\}$, and, $C = \{a,\ b,\ c, d\}$. Determine the Cartesian products, and their cardinalities.

a. $B\ \times\ C$

b. $C \times B$

c. $B\ \times\ T$

d. $B\ \times\ B$

e. $B\ \times\ B \times B$

2.7. The Cartesian Product

Solution

For the set, $B \times C$, notice that this will be all ordered pairs of the form, (a, b), with $a \in B$, and $b \in C$, giving,

$B \times C = \{(0, a), (0, b), (0, c), (0, d), (1, a), (1, b), (1, c), (1, d))\}$, which has $2 \times 4 = 8$, elements.

For $C \times B$, switch the ordering, for $B \times C$, to obtain the set with 8, elements,

$C \times B = \{(a, 0), (b, 0), (c, 0), (d, 0), (a, 1), (b, 1), (c, 1), (d, 1)\}$,

The set $B \times T$, will be all ordered pairs of the form, (a, b), with $a \in B$, and $b \in T$, giving, the set with $2 \times 3 = 6$, elements,

$B \times T = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$,

2.7. The Cartesian Product

$$B \times T = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\},$$

The set $B \times B$, will be all order pairs of the form, (a,b), with $a,b \in B$, giving the set with $2 \times 2 = 4$, elements,

$$B \times T = \{(0,0),(0,1),(1,0),(1,1)\},$$

Finally the set $B \times B \times B$, will be the set of all ordered triples of the form, (a,b,c), with $a,b,c \in B$, giving the set with $2 \times 2 \times 2 = 8$, elements,

$$B \times B \times B = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\},$$

2.7. The Cartesian Product

Because Cartesian products are created using ordered pairs, $B \times C$, is, in general, different from $C \times B$.

If the cardinality of set $|A| = a$, and the cardinality of set $|B| = b$, then the cardinality of the Cartesian product is $|A \times B| = ab$

The Cartesian coordinate systems are natural sets that are naturally Cartesian products. The two-dimensional plane, and the three-dimensional space are represented by the following Cartesian product sets,

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}$, and,

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$

2.8. Exercises

1. Consider as universal set, the set of all 26, lowercase letters of the English alphabet, $U = \{a, b, c, \ldots, v, w, x, y, z\}$, and the sets $A = \{a, b, c, d, e, f, g, h\}$, $B = \{f, g, h, i, j, k\}$, and $C = \{x, y, z\}$. For the sets given below:

   a. List the sets below using roster form, and

   b. Draw Venn Diagrams for each of the sets

i. $A \cup B$

ii. $A \cap B$

iii. $A \cup C$

iv. $A \cap C$

v. $A \setminus B$

vi. $B \setminus A$

viii. $C \setminus A$

ix. $A \cup C$

x. $A \cap C$

xi. $\overline{A}$

xii. $\overline{B}$

xiii. $\overline{C}$

xiv. $\overline{B} \cap \overline{C}$

xv. $(\overline{A} \cap \overline{B}) \cup (\overline{B} \cap \overline{C})$

2.8. Exercises

2. Using Venn Diagrams, determine which of the following are equivalent

a. $A \setminus (A \setminus B))$,

$A \cup B$, and

$A \cap B$

b. $A \cup \overline{A}$,

$A \cap \overline{A}$,

U, and

$\emptyset$

c. $\overline{A} \cap \overline{B}$,

$\overline{A \cap B}$,

$\overline{A} \cup \overline{B}$, and

$\overline{A \cup B}$

d. $A \cup (B \cap C)$,

$A \cap (B \cup C)$,

$(A \cap B) \cup (A \cap C)$, and

$(A \cup B) \cap (A \cup C)$,

2.8. Exercises

3. Write each of the following sets using set builder notation

a. $\{\ldots, -9, -7, -5, -3, -2, -1, 1, 3, 5, 7, 9, \ldots\}$

b. $\{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, \ldots\}$

c. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

d. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

e. $\{0, 1, 4, 9, 16, 25, 36, 49, \ldots\}$

f. $\{\ldots, -10, -6, -2, 2, 6, 10, 14, 18, 22, \ldots\}$

g. $\{3, 9, 27, 81, 243, \ldots\}$

2.8. Exercises

4. Write each of the following sets in roster form

a. $\{x \in \mathbb{R} : |2x + 5| = 7\}$

b. $\{10n : n \in \mathbb{N}\}$

c. $\{10n : n \in \mathbb{Z}\}$

d. $\{2^n : n \in \mathbb{N}\}$

e. $\{2^n : n \in \mathbb{Z}\}$

f. $\{x \in \mathbb{R} : x^2 = 4\}$

2.8. Exercises

5. Consider the sets $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 5, 7, 11\}$, and $C = \{1, 2, 3\}$,

a. Determine the cardinalities of following sets,

i. $|A|$

ii. $|A \cup B|$

iii. $|A \cap C|$

iv. $|\mathcal{P}(A)|$

2.8. Exercises

7. Consider the sets, $B = \{0, 1\}$, $S = \{spring, summer, fall, winter\}$, and $C = \{a, b, c, d, e\}$. For each of the following sets:

a. Determine the following Cartesian products.

b. Calculate the cardinality of each Cartesian product.

i. $B \times S$

ii. $S \times B$

iii. $B \times C$

iv. $C \times B$

2.8. Exercises

8. Determine the following power sets,

a. $\mathcal{P}(\{Alabama, Georgia, Florida, Louisiana\})$

b. $\mathcal{P}(\emptyset)$

c. $\mathcal{P}(\{\emptyset\})$

d. $\mathcal{P}(\{Alabama\})$

e. $\mathcal{P}(\{Alabama, Georgia, Florida\})$

2.8. Exercises

Reference:

https://ggc-discrete-math.github.io/logic.html#_set_theory

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