Discrete Mathematics 2
Elementary Mathematical Logic

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1. Propositional Logic

A **proposition** is a sentence that declares a fact that is either True or False.

Examples 1 - Propositions

- Atlanta is the capital of Iowa.
- 1 + 1 is 2.
- 1 + 1 is 3.

Not propositions

- How much is this cookie?
- This sentence is false.

1. Propositional Logic

Propositional logic consists of a set of formal rules for combining propositions in order to derive new propositions.

We can use boolean variables (typically p and q) to represent propositions and define functions for each propositional rule. Each rule can be implemented using the boolean operators (and, or, not) discussed in the section on operators and expressions .

A **truth table** is a method of showing truth values of compound propositions using the truth values of its components. It is typically created with rows representing possible truth values and columns representing the propositions.

1.1. Negation

The **negation** is a statement that has the opposite truth value. The negation of a proposition p, denoted by $\neg p$, is the proposition "It is *not the case* , that p".

For example, the negation of the proposition "Today is Friday." would be "It is not the case that, today is Friday." or more succinctly "Today is not Friday."

p	$\neg p$
True	False
False	True

1.2. Conjunction

"I am a rock **and** I am an island."

Let p and q be propositions. The conjunction of p and q, denoted in mathematics by $p \wedge q$, is True when both p and q are True, False otherwise.

p	q	$p \wedge q$
True	True	True
True	False	False
False	True	False
False	False	False

1.3. Disjunction

"She studied hard **or** she is extremely bright."

Let p and q be propositions. The disjunction of p and q, denoted in mathematics by $p \vee q$, is True when at least one of p and q are True, False otherwise.

p	q	$p \vee q$
True	True	True
True	False	True
False	True	True
False	False	False

1.4. Exclusive Disjunction

"Take **either** 2 Advil **or** 2 Tylenol."

Let p and q be propositions. The exclusive disjunction of p and q (also known as *xor*), denoted in mathematics by $p \oplus q$, is True when exactly one of p and q are True, False otherwise.

p	q	$p \oplus q$
True	True	False
True	False	True
False	True	True
False	False	False

1.5. Implication

" **If** you get a 100 on the final exam, **then** you earn an A in the class."

Let p and q be propositions. The implication of p and q, denoted in mathematics by $p \implies q$, is short hand for the statement "if p then q". As such, implication requires q to be True whenever p is True. If p is not True, then q can be any value. In other words, implication fails (is False) when p is True and q is False. Note, this is different from "p if and only if q".

p	q	$p \implies q$
True	True	True
True	False	False
False	True	True
False	False	True

1.5. Implication

Implication can be considered a "contract" which fails *only when* the conditions are met and the results are not fulfilled.

1.6. Converse, Contrapositive and Inverse of an Implication

We can form new compound propositions from the implication, $p \implies q$. They are

- The **converse** : $q \implies p$
- The **contrapositive** : $\neg q \implies \neg p$
- The **inverse** $\neg p \implies \neg q$

The truth tables for these new propositions are shown in the table.

1.6. Converse, Contrapositive and Inverse of an Implication

The truth tables for these new propositions are shown in the table.

p	q	$p \implies q$ (conditional)	$q \implies p$ (converse)	$\neg q \implies \neg p$ (contrapositive)	$\neg p \implies \neg q$ (inverse)
True	True	True	True	True	True
True	False	False	True	False	True
False	True	True	False	True	False
False	False	True	True	True	True

In the section proposition equivalences we will explain why the truth table shows that the conditional $p \implies q$ and contrapositive $\neg q \implies \neg p$ are logically equivalent, and why the converse $q \implies p$ and inverse $\neg p \implies \neg q$ are logically equivalent.

1.6. Converse, Contrapositive and Inverse of an Implication

Example 6 - Conditional, Converse, Contrapositive and Inverse.

a. Translate the statement, "If an integer n, is divisible by 4, then it is divisible by 2", using a conditional.

b. Form its converse, contrapositive, and inverse and translate.

Solution

a. Let

p: An integer n, is divisible by 4

q: An integer n, is divisible by 2

The sentence, "If an integer n, is divisible by 4, then it is divisible by 2", is translated $p \implies q$.

b. Its converse is $q \implies p$, which may be translated, "If an integer n, is divisible by 2, then it is divisible by 4."

The contrapositive is $\neg q \implies \neg p$, which may be translated, "If an integer n, is not divisible by 2, then it is not divisible by 4."

The inverse is $\neg p \implies \neg q$, which may be translated, "If an integer n, is not divisible by 4, then it is not divisible by 2."

1.7. Bi-Implication

"It is raining outside **if and only if** it is a cloudy day."

Let p and q be propositions. The **bi-implication** of p and q, denoted in mathematics by $p \iff q$, is short hand for the statement "p if and only if q". As such, bi-implication requires q to be True only when p is True. In other words, bi-implication fails (is False) when p is True and q is False or when p is False and q is True.

p	q	$p \iff q$
True	True	True
True	False	False
False	True	False
False	False	True

Bi-implication is True if the propositions have the same truth value and False otherwise.

1.7. Bi-Implication

It is important to contrast implication with bi-implication. Consider the implication example "If you get a 100 on the final exam, then you earn an A in the class." This means that when you get a 100 on the final you also get an A in the class.

As a bi-implication it would say "You get a 100 on the final exam if and only if you earn an A in the class." This becomes a two-way contract where you can earn an A in the class by getting a 100 on the final, but if you do not get a 100 on the final you will not earn an A.

1.8. Compound Propositions

To find truth values of compound propositions, it is useful to break them up into smaller parts.

When creating your own truth table it is crucial to be systematic about ensuring you have all possible truth values for each of the simple propositions. Each simple proposition has two possible truth values, so the number of rows in the table should be $\mathbf{2^n}$ where **n** is the number of propositions. You should also consider breaking complex propositions into smaller pieces.

1.8. Compound Propositions

Example 9

Create a truth table for the compound proposition:

$(p \wedge q) \implies (p \wedge r)$ for all values of p, q, r.

Solution

It should have 8 rows - since there are three simple propositions and each one has two possible truth values.

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \implies (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

2. Proposition Equivalences

Two propositions are considered **logically equivalent** (or simply **equivalent**) if they have the same truth values in every instance. It is often easiest to see this by constructing a truth table for the two propositions and comparing.

Example 10

Consider the propositions $\neg p \vee q$ versus $p \implies q$.

p	q	$\neg p \vee q$	$p \implies q$
True	True	True	True
True	False	False	False
False	True	True	True
False	False	True	True

Since the truth table in all rows is the same for the two compound propositions, they are equivalent.

2. Proposition Equivalences

Example 11

Consider three compound propositions:

1. $(p \wedge q) \implies r$
2. $(p \implies q) \wedge (p \implies r)$
3. $p \implies (q \wedge r)$

2.1. De Morgan's Laws

Two important logical equivalences are De Morgan's Law. These describe how we "distribute" the negation across the and and or operators.

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

We use the symbol $\equiv$ to denote two statements which are logically equivalent.

2.2. Tautologies, Contradictions and Contingencies

A proposition is a **tautology** if its truth value is *always* True.

A proposition is a **contradiction** if its truth value is *always* False.

A proposition that is neither a tautology nor a contradiction is said to be a **contingency** .

Example 13 - Tautology and Contradiction

$p \vee \neg p$ is an example of a tautology.

$p \wedge \neg p$ is an example of a contradiction.

This can be seen in the truth table.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
True	False	True	False
False	True	True	False

2.3. Predicates and Quantifiers

2.3.1 Predicate

A **predicate** is a statement involving a variable.

Example 14 - Predicates

- $x \leq 3$
- computer c is infected
- country x is on continent y

Predicates are denoted as $P(x)$ or $Q(x, y)$ where P and Q represent the statements and x and y represent the possible values. After a value is assigned to each variable, the predicate becomes a proposition which has a truth value.

2.3.1 Predicate

Example 16

Let $Q(x, y)$ be the statement $x - y = 4$.

Edit the Python tutor below to find the truth values of $Q(6, 2)$, $Q(1, 5)$, and $Q(-2, 2)$.

2.3.2 Quantifier

Consider the statements

- For all integers x, $x^2 \geq 0$.
- Some student in the class has a birthday in July.

Each of these statements considers a proposition over an entire population or set, called the domain, and quantifies how many elements (or people) in the set satisfy the proposition. To represent this idea, we use two main quantifiers, the **universal quantifier** and the **existential quantifier** .

The **Universal Quantifier** , $\forall$, represents the statement "for all", "for every", "for each". When it comes before a statement, it means that statement is true *for all values* in the domain.

2.3.2 Quantifier

Example 18

Universal Quantifier $\forall x, x + x > x$

Let $P(x)$ be the statement $x + x > x$. Is this true for all integers x?

2.3.2 Quantifier

The **Existential Quantifier** , $\exists$, represents the statement "there exists", "for some", "at least one". When it comes before a statement, it means the statement is true for *at least one value* in the domain.

Example 19

Existential Quantifier $\exists x, x^2 = 4$

Let $P(x)$ be the statement $x^2 = 4$. Is this true for at least one integer x?

2.3.2 Negation of Quantifiers

It is important to consider the negation of a quantified expression.

- "Every student in this class has taken Programming Fundamentals."

This is a universally quantified statement and can be expressed as $\forall x P(x)$ where $P(x)$ is the statement "x has taken Programming Fundamentals" and the domain consists of all the students in this class. The negation of the statement would be "It is not true that every student in this has taken Programming Fundamentals." Equivalently,

- "There is a student in this class who has NOT taken Programming Fundamentals."

This is an existentially quantified statement expressed as $\exists x \neg P(x)$.

2.3.2 Negation of Quantifiers

De Morgan's Laws with Quantifiers

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

Example 21

- Someone in the class can speak Latin.

Using quantifiers, we write this statement as $\exists x L(x)$ where $L(x)$ is the proposition "x speaks Latin." and the domain is the students in the class. Its negation would be $\forall x \neg L(x)$.

- All the students in the class can not speak Latin.

2.3.2 Negation of Quantifiers

The predicate of a quantified statement could be a compound statement. For instance,

- Some dogs are big and fluffy.

This is written as $\exists x(B(x) \wedge F(x))$ where $B(x)$ is the proposition "x is big." and $F(x)$ is the proposition "x is fluffy." and the domain is dogs. Negating this statement would give

$$\neg\exists x(B(x) \wedge F(x)) \equiv \forall x \neg(B(x) \wedge F(x)) \equiv \forall x(\neg B(x) \vee \neg F(x))$$

In words,

- All dogs are not big or not fluffy.

2.8. Exercises

Reference:

https://ggc-discrete-math.github.io/logic.html#_logic

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