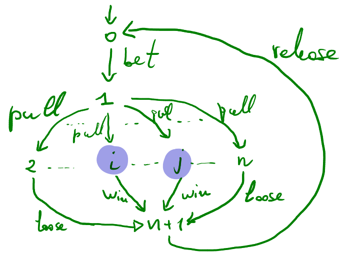
Example. A (simplified) slot machine



$S = \{0, \dots, n+1\}$ & $I = \{0\}$

$Act = \{bet, win, loose, pull, release\}$

Fix an interval $[i,j]$ with $1 \leq i \leq j \leq n$

we can get rid of $n+1$ and those transitions

$$\rightarrow = \{(0, bet, 1)\} \cup \bigcup_{1 \leq h \leq n} \left(\{(1, pull, h)\} \cup \{(h, \alpha, n+1) \mid \alpha = \begin{cases} win, & i \leq h \leq j \\ loose, & otw \end{cases}\}\right) \cup \{(n+1, release, 0)\}$$

3 wheels

$AP = \{ w_i = f \mid 1 \leq i \leq 3 \ \& \ f \in Fruits \} \cup \{ price = n \mid n \in \omega \}$ where $Fruits = \{apple, pear, banana \dots\}$

let $W : \{i, \dots, j\} \longrightarrow Fruit^3$

$L : h \longmapsto \{ price = h, \ w_1 = f_1, \ w_2 = f_2, \ w_3 = f_3 \mid w(h) = f_1, f_2, f_3 \}$

Exercise: Define L on $h \notin \{i, \dots, j\}$

Non-determinism

- crucial modelling mechanism
- under-specification

Deterministic TS $|I| \leq 1$

- **action-deterministic** $\forall s \in S, \alpha \in Act : |Post(s, \alpha)| \leq 1$

  $Post(s,\alpha) := \{s' \in S : s \xrightarrow{\alpha} s'\}$

  $s \xrightarrow{\alpha} s_1$, $s \xrightarrow{\alpha} s_2$, $s_1 = s_2$

- **AP-deterministic** $\forall s \in S \ \forall A \in 2^{AP} : |\{ s' \in Post(s) \mid L(s') = A \}| \leq 1$

  $Post(s) := \bigcup_{\alpha \in Act} Post(s,\alpha)$

  $s \xrightarrow{\alpha} s_1$, $s \xrightarrow{\beta} s_2$, $L(s_1) = L(s_2) \Rightarrow s_1 = s_2$

Executions / Traces

Execution fragment $\rho \in$ $\underbrace{S(Act\ S)^*}_{finite} \cup \underbrace{S(Act\ S)^\omega}_{infinite}$

s.t. $\rho = s_0 \alpha_1 s_1 \alpha_2 s_2 \dots \alpha_n s_n \dots \Rightarrow s_i \xrightarrow{\alpha_{i+1}} s_{i+1}$ for all i

ρ **maximal** if ρ infinite or
$\rho = s_0 \alpha_1 s_1 \alpha_2 s_2 \dots \alpha_n s_n \wedge Post(s_n) = \emptyset$

ρ **initial** if $s_0 \in I$

Execution initial maximal execution fragment.

Reachable states $Reach(TS) = \{ s \mid \exists \rho \text{ initial execution fragment ending in } s \}$

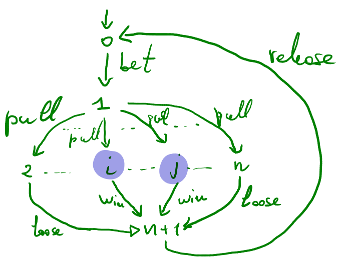
We are now going to see a state-based approach, where algorithms "ignore" actions. Formally:

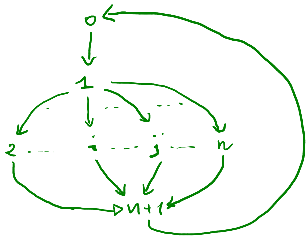
the **state graph** of $TS = (S, Act, \to, I, AP, L)$ is obtained by "removing" the actions from TS

$$G(TS) = \langle S, E \rangle \quad \text{where} \quad E = \bigcup_{s \in S} \{s\} \times post(s)$$

Example

the TS of the slot machine & its state graph





note that state label also "disappear" but that's a sort of illusion :)

> Notation: given a sequence $\sigma = \sigma_0 \sigma_1 \dots \sigma_n \dots$
> - $|\sigma|$ is its lenght (if σ is infinite, $|\sigma| = \infty$)
> - $\sigma[i]$ is the i-th element of σ
> - if σ is finite then $last(\sigma)$ is the last element of σ

From now on we assume TS fixed.

Linear Time Beh & Properties

A **path fragment of TS** is a path in its state graph:

$$\pi \in S^* \cup S^\omega \quad \text{s.t.} \quad \forall\, 0 \leq i < |\pi| : \pi[i+1] \in Post(\pi[i])$$

π maximal if $\pi \in S^*$ & $Post(last(\pi)) = \emptyset$ or $\pi \in S^\omega$

π initial if $\pi[0] \in I$

π path if initial & maximal

$$\bigcup_{\{\pi \text{ path}(TS) : \pi[0] = s\}} trace(\pi)$$

Trace of π $\{L(\pi[i])\}_{0 \leq i < |\pi|}$

$$\boxed{Traces(TS) := \bigcup_{s \in I} traces(s)}$$

An **LT property** (on AP) is an element P of $(2^{2^{AP}})^\omega$ i.e. $P \subseteq (2^{AP})^\omega$

Examples. Let $AP = \{red, green, yellow\}$ and P_{light} = "the traffic light is infinitely often red"

$P_{light} \ni \{red\}\{red, yellow\}\{green, yellow\}\{red\}\{red, yellow\}\{green, yellow\}$

$\not\ni \{red\}\{green\}\emptyset\,\emptyset\,\emptyset \ldots$

$\ni \{red\}^\omega$

$\ni X^\omega$ if $red \in X \subseteq AP$

$\ni \{X_i\}_{i \in \omega}$ if $red \in X_i \iff i$ prime

Let $AP = \{c_1, \ldots, c_n\}$

thread h is in the critical section

$P_{mutex} = \{\{A_i\}_{i \in \omega} \in (2^{AP})^\omega \mid \forall i \geq 0,\ 1 \leq h \leq k \leq n : \{c_h, c_k\} \subseteq A_i \Rightarrow h = k\}$

$= \bigwedge_{1 \leq h < k \leq n} \{c_h, c_k\} \not\subseteq A_1 \wedge \ldots \wedge \bigwedge_{1 \leq h < k \leq n} \{c_h, c_k\} \not\subseteq A_n$

Exercise: What does $P' = \{\{A_i\}_{i \geq 0} \in (2^{AP})^\omega \mid \forall i \geq 0\ \exists\, 1 \leq h \leq n.\ c_h \in A_i\}$ state? Give two different traces satisfying $P'(n)$

Exercise: Let P_{slot}: "always $(price = 0 \to$ eventually $\bigvee_{i=1}^{10} price = i)$". Give an example of an element of P_{slot} and one of $(2^{AP})^\omega \setminus P_{slot}$

$$s_i \xrightarrow{\alpha_i} s_{i+1}$$

$$\begin{array}{ccccccccccc}
TS & & I & & & & & & & & \\
\uplus & & \uplus & & & & & & & & \\
\pi & = & s_0 & \alpha_1 & s_2 & \cdots & \alpha_n & s_n & \alpha_{n+1} & \cdots & \\
 & & \downarrow & & \downarrow & & & \downarrow & & & \\
 & & L(s_0) & & L(s_2) & \cdots & & L(s_n) & & \cdots & \in P \quad \text{LT property}
\end{array}$$

$$TS \models P$$

The importance of Traces

WLOG: no terminal states in TS (hence all maximal paths are infinite)

the trace of a maximal path of TS is $trace(\pi) = \{L(\pi[i])\}_{i \geq 0}$

Notice that $trace(\pi) \in (2^{AP})^{\omega}$

$$\boxed{TS \models P \iff Traces(TS) \subseteq P}$$

$s \in S$, $s \models P \iff traces(s) \subseteq P$

Read $\underbrace{Traces(TS)}_{\text{refinement}} \subseteq \underbrace{Traces(TS')}_{\text{abstract model}}$ as "TS correctly implements TS'"

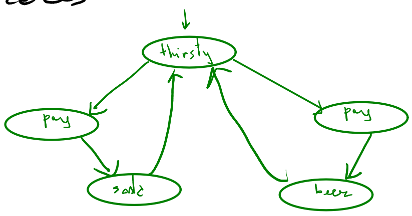
Thm TS & TS' t.s. on the same atomic propositions then
$Traces(TS) \subseteq Traces(TS') \iff \forall$ LT prop. P $TS' \models P \implies TS \models P$

Proof ($\Rightarrow$) $TS' \models P \overset{def}{\iff} Traces(TS') \subseteq P$
$\overset{hyp}{\implies} Traces(TS) \subseteq P \overset{def}{\iff} TS \models P$

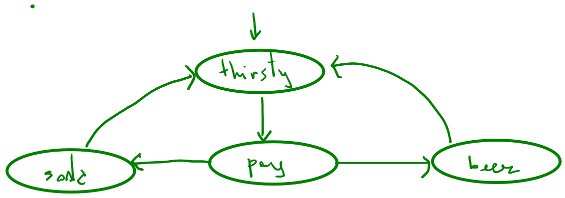
($\Leftarrow$) $TS' \models Traces(TS')$ since $Traces(TS') \subseteq Traces(TS')$
$\overset{hyp}{\implies} TS \models Traces(TS') \overset{def}{\iff} Traces(TS) \subseteq Traces(TS')$ □

Cor $Traces(TS) = Traces(TS') \iff \forall P$ LT f.b : $TS \models P \iff TS' \models P$

Exercise: Is



equivalent to



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