Term Algebras

$(\{f_1,\dots,f_n\}, ar)$ $\quad ar: f_i \mapsto \omega$

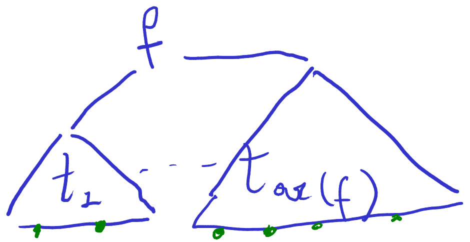
assume $\quad V \cap \{f_1,\dots,f_n\} = \emptyset$

- Fix

Term Algebra The term algebra on a signature Σ and a countable set V of variables is the smallest set $Term_{\Sigma,V}$ s.t.

- $V \in T$
- $\forall f \in \Sigma \ \forall t_1,\dots,t_{ar(f)} \in Term_{\Sigma,V} : f(t_1,\dots,t_{ar(f)}) \in Term_{\Sigma,V}$

$T_\Sigma \subseteq Term_{\Sigma,V}$ is the set of closed terms



- are either variables or "constants" (i.e $c \in \Sigma$ s.t $ar(c) = 0$)

Q: why is $Term_{\Sigma,V}$ required to be the "smallest" set?

Transition System Specification

A TSS is a set of (inference) rules H/α where

H is a finite set of transitions of the form

LITERALS

$\underbrace{t \xrightarrow{a} t' \quad \text{or} \quad t \in X}_{\text{positive premises}}$

or

$\underbrace{t \notin X \quad \text{or} \quad t \stackrel{a}{\not\to}}_{\text{negative premises}}$

α is a finite set of transitions of the form

$t \xrightarrow{a} t' \quad \text{or} \quad t \in X$

where $t \in Term_{\Sigma,V}$ & $X \subseteq Term_{\Sigma,V}$

"The first systematic study of TSSs may be found in [208], while the first study of TSSs with negative premises appeared in [57]." (Aceto et al.)

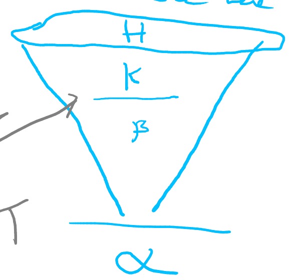
[208] R. d. Simone, Calculabilité et Expressivité dans l'Algèbre de Processus Parallèles Meije, thèse de 3 e cycle, Univ. Paris 7, 1984.
[57] B. Bloom, S. Istrail, and A. Meyer, Bisimulation can't be traced: preliminary report, in Conference Record 15th ACM Symposium on Principles of Programming Languages, San Diego, California, 1988, pp. 229–239. Preliminary version of Bisimulation can't be traced, J. Assoc. Comput. Mach., 42 (1995), pp. 232–268.

TSS

A proof of a closed transition rule H/α from T is an upwardly branching tree without infinite branches, whose
- nodes are labelled by literals
- the root is labelled by α, and
- if K is the set of labels of the nodes directly above a node with label β, then
  1. either $K = \emptyset$ and $\beta \in H$,
  2. or K/β is a closed substitution instance of a transition rule in T .

If a proof of H/α from T exists, then H/α is provable from T , notation T |-- H/α.

are the leaves



closed subst instance of a rule in T

Regular Expressions

BNF-like syntax $\quad A$, finite alphabet

$$E ::= 0 \mid 1 \mid a \mid E+E \mid E\cdot E \mid E^*$$

Exercise 6 Give the term algebra for reg. exp

Denotational semantics : $\mathcal{L} : E \longrightarrow 2^{A^*}$ $\quad$ Term-Algebra homomorphism

$\mathcal{L}(0) = \emptyset \qquad \mathcal{L}(1) = \{\varepsilon\} \qquad \mathcal{L}(a) = \{a\}$

$\mathcal{L}(E_1 + E_2) = \mathcal{L}(E_1) \cup \mathcal{L}(E_2)$

$\mathcal{L}(E_1 \cdot E_2) = \mathcal{L}(E_1) \cdot \mathcal{L}(E_2) \triangleq \{vw \in A^* \mid v \in \mathcal{L}(E_1),\ w \in \mathcal{L}(E_2)\}$

$\mathcal{L}(E^*) = \mathcal{L}(E)^* = \bigcup_{n \geq 0} \mathcal{L}(E)^n$

Exercise 7 Given $X \subseteq A^*$ & $n \in \omega$, define X^n

Exercise 8 Prove or disprove that $(a+b)^* = (a^* + b^*)^*$

Exercise 9 Give an inductive definition of the set of regular expressions

Regular Expressions

Operational semantics

(Act) $$\frac{a \in A}{a \xrightarrow{a} 1}$$

(Cho1) $$\frac{x \xrightarrow{a} x' \quad x' \neq 1}{x + y \xrightarrow{a} x'}$$

(Cho2) $$\frac{x \xrightarrow{a} 1}{x + y \xrightarrow{a} 1}$$

(Cho3) $$\frac{y \xrightarrow{a} y' \quad y' \neq 1}{x + y \xrightarrow{a} y'}$$

(Cho4) $$\frac{y \xrightarrow{a} 1}{x + y \xrightarrow{a} 1}$$

(Seq1) $$\frac{x \xrightarrow{a} x' \quad x' \neq 1}{x \cdot y \xrightarrow{a} x' \cdot y}$$

(Seq1) $$\frac{x \xrightarrow{a} 1}{x \cdot y \xrightarrow{a} y}$$

Basic Process Algebras with $a \in A \cup \{\varepsilon\}$

(Iter1) $$\frac{}{x^* \xrightarrow{\varepsilon} 1}$$

(Iter2) $$\frac{x \xrightarrow{a} x'}{x^* \xrightarrow{a} x' \cdot x^*}$$

Exercise 10 Give the LTS of $a^*(b+c)$

Note that

- x & y range over the set of reg exp
- these rules form a TSS
- there is a set of rules for each operator
- For 0, the set is empty!

... & their operational semantics

We saw that we can define the language of an FSA $M = (Q, \Sigma, q_0, \delta, F)$ as

$$\mathcal{L}_M = \{ a_1 \dots a_n \in \Sigma^* \mid \exists q_1, \dots, q_n \mid q_0 \xrightarrow{a_1} \dots q_{n-1} \xrightarrow{a_n} q_n \xrightarrow{\checkmark} \bullet \}$$

modulo the error you were supposed to fix

where $\rightarrow$ is the relation of the LTS corresponding to M

This can be generalised to ANY LTS e.g.

Since the TSS of reg exp induces an LTS, we can use the very same definition to define the language $\mathcal{L}_E$ of a reg exp E; so

$$\mathcal{L}_E = \{ a_1 \dots a_n \in \Sigma^* \mid \exists E_1, \dots, E_n : E \xrightarrow{a_1} E_1 \dots E_{n-1} \xrightarrow{a_n} E_n = 1 \}$$

where now $\xrightarrow{a_i}$ are transition to be proved by applying the rules of our TSS!

Example

Show that $aab \in \mathcal{L}_E$ where $E = a^*(b + c)$ where $A = \{a, b, c, d\}$

1. find E_1 s.t. $E \xrightarrow{a} E_1$ & there are E_2, E_3 s.t. $E_2 \xrightarrow{a} E_3 \xrightarrow{b} 1$

- a candidate for E_3 is b since (Act) $\dfrac{b \in A}{b \xrightarrow{b} 1}$
- likewise a candidate for E_2 is ab WHY? $\rightsquigarrow$

$$(seq_2)\ \dfrac{(Act)\ \dfrac{a \in A}{a \xrightarrow{a} 1}}{ab \xrightarrow{a} b}$$

$$seq_2\ \dfrac{act\ \dfrac{a \in A}{a \xrightarrow{a} 1}}{aab \xrightarrow{a} ab} \quad seq_2\ \dfrac{act\ \dfrac{a \in A}{a \xrightarrow{a} 1}}{\xrightarrow{a} b} \quad act\ \dfrac{b \in A}{\xrightarrow{b} 1}$$