This course was developed as a part of the VLIR-UOS Cross-Cutting projects:

- Statistics: 2011-2016, 2017.
- Statistics: 2017.
- Statistics for development : 2018-2022.
- The >eR-BioStat ITP: 2024-2026.

The >eR-Biostat initiative

Making R based education materials in statistics accessible for all

Introduction to Statistical inference and estimation using R:
Foundations of inference (one population)

Developed by

Ziv Shkedy (Hasselt Univesrsity, Belgium) and Tadesse Awoke (Gondar University, Ethiopia).

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twitter @erbiostat

Development team

- Tadele Worku Mengesha (Gondar University).
- Abdisa Gurmessa (Jmma University).
- Ziv Shkedy (Hasselt Univesrsity).
- Tadesse Awoke (Gondar University).
- Thi Huyen Nguyen (Hasselt University).

Recommended reading

Introductory Statistics for the Life and Biomedical Sciences
First Edition

Julie Vu
Preceptor in Statistics
Harvard University

David Harrington
Professor of Biostatistics (Emeritus)
Harvard T.H. Chan School of Public Health
Dana-Farber Cancer Institute

This book can be purchased for $0 on Leanpub by adjusting the price slider.

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The book is available for free online:

https://www.openintro.org/book/biostat/

Chapter 4: Foundations for inference

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- In this part of the course, we cover mainly Chapter 4 and Section 5.1.
- The examples that are used for illustration are not the same as the examples in the book.

Chapter 4 & 5.1: Foundations for inference

Software

- R functions: Hypothesis testing for one/two samples
  - `z.test()`.
  - `t.test()`.

- R program for the examples is available onlne.

Datasets

- Data are given as a part of R programs for the course.

- External datasets (which are not given as a part of the R code) and used for illustration are available online.

Topics

1. Samples and populations.
2. Point estimators.
3. Variability of the sample mean.
4. Confidence intervals.
5. Hypotheses testing in one population.
6. Decision errors.
7. Hypotheses testing using t distribution.

Part 1:

Samples and populations

1.1: Notations and definitions

Populations

- The population is the 'big picture' of which we are certain parameters want to know, but we have all sorts of reasons (practical, physical, financial, ...) can not fully observe.
- examples:

- The birth lengths of all children as last year were recorded in all Flemish maternity hospitals.
- The results of the parliamentary elections in Belgium if today would be voted.
- The blood of the inhabitants of the European Union.

Sample

- A Sample: is a portion of a population taken in such a way that it would represent the population that we would like to investigate.

Populations and samples

1. A population can be described by the random variable X.
2. A sample from that population of X: $X_1, ..., X_n$ have the same distribution as X and independent.

Population and sample: notations

Population : X

$$X_1, X_2, \ldots, X_N$$

$$\mu, \sigma^2$$

sample

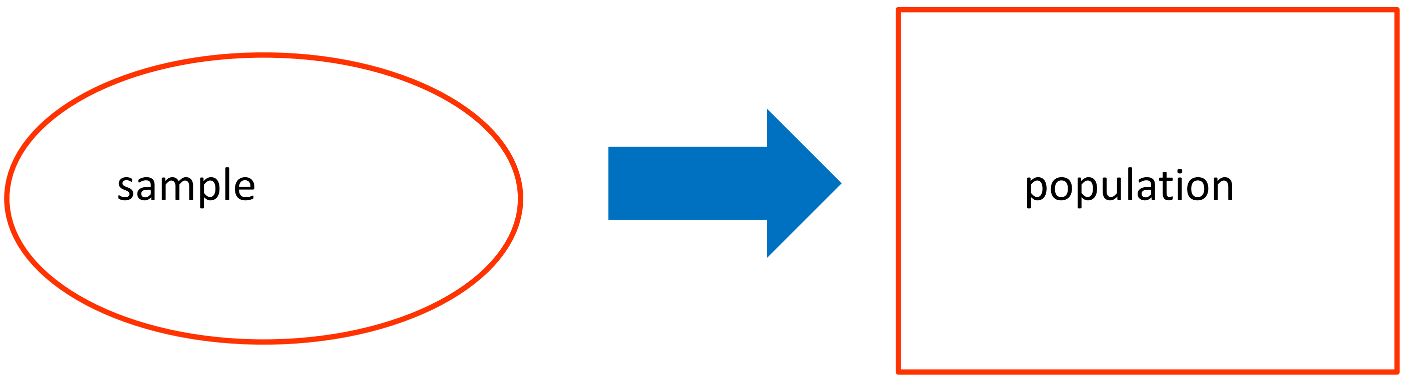
$$X_1, X_2, \ldots, X_n$$

X_i : Random variable from the population.

μ, σ^2 : The unknown parameters.

Population and sample: notaions

- Based on a sample of size n from a population, we try to make a statement about the population.



$\bar{x}, s^2$ μ, σ^2

Example

- Population: ozone level in parts per billion from 1300 to 1500 hours at Roosevelt Island.

$$\boxed{X_1, X_2, \ldots, X_N}$$

uknown parameters

$$E(X) = \mu = ?$$

$$Var(X) = \sigma^2 = ?$$

X_i = ozone levels in parts per billion.

Example: the airquality data in R

Daily air quality measurements in New York, May to September 1973.

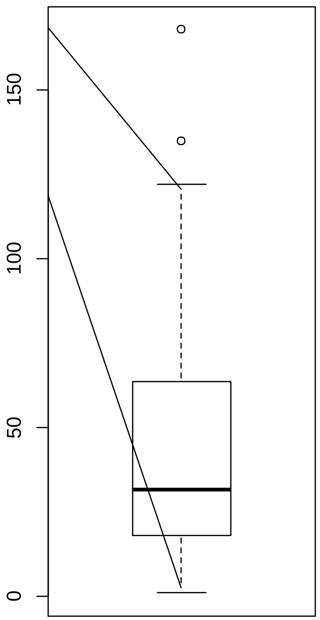
```
> help(airquality)
> airquality$Ozone
  [1]  41  36  12  18  NA  28  23  19   8  NA   7  16  11  14  18  14  34   6
 [19]  30  11   1  11   4  32  NA  NA  NA  23  45 115  37  NA  NA  NA  NA  NA
 [37]  NA  29  NA  71  39  NA  NA  23  NA  NA  21  37  20  12  13  NA  NA  NA
 [55]  NA  NA  NA  NA  NA  NA  NA 135  49  32  NA  64  40  77  97  97  85  NA
 [73]  10  27  NA   7  48  35  61  79  63  16  NA  NA  80 108  20  52  82  50
 [91]  64  59  39   9  16  78  35  66 122  89 110  NA  NA  44  28  65  NA  22
[109]  59  23  31  44  21   9  NA  45 168  73  NA  76 118  84  85  96  78  73
[127]  91  47  32  20  23  21  24  44  21  28   9  13  46  18  13  24  16  13
[145]  23  36   7  14  30  NA  14  18  20
```

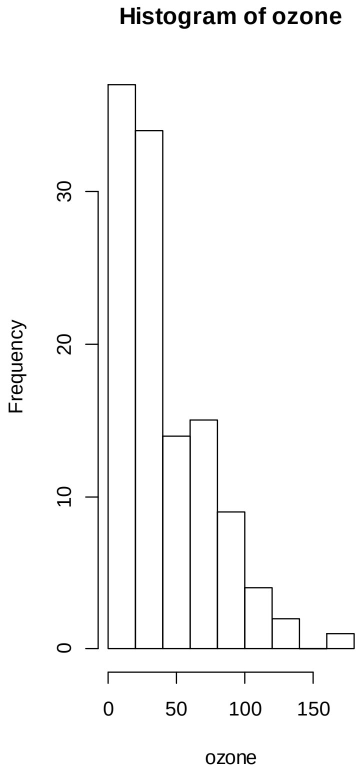
Missing values

Graphical output

```
> ozone<-na.omit(airquality$Ozone)
> length(ozone)
[1] 116
> par(mfrow=c(1,2))
> boxplot(ozone)
> hist(ozone)
```

A skew distribution with "long" right tail.





1.2 Random samples in R

Random sample form a normal distribution

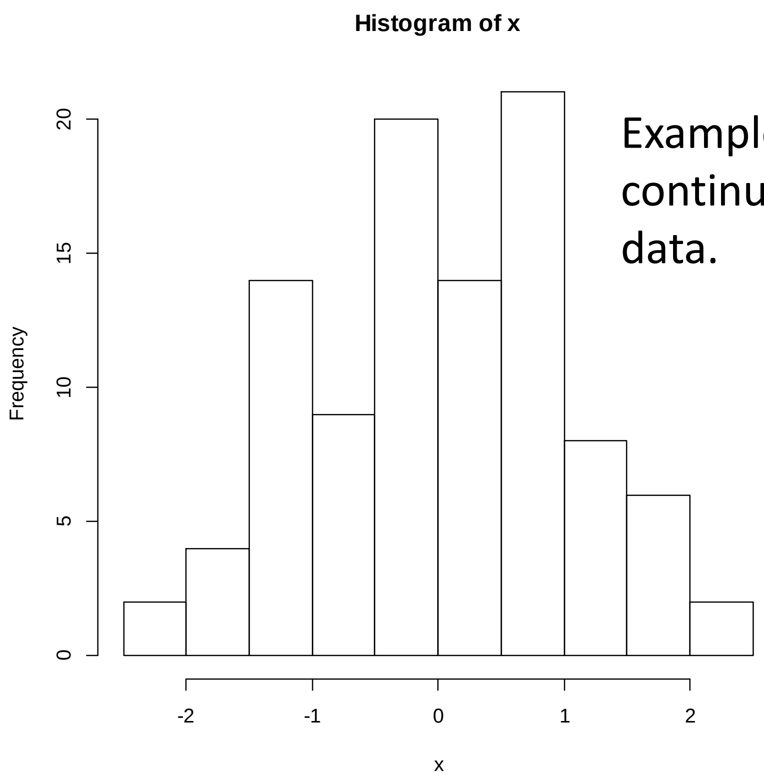
Population:

$$X_i \sim N(\mu = 0, \sigma = 1)$$

Random sample, n=100

$$X_1, X_2, \ldots, X_{100}$$

```
> x<-rnorm(100,0,1)
> hist(x)
```



Example of continuous data.

Histogram of 100 observations from N(0,1).

Random sample form a Binomial distribution

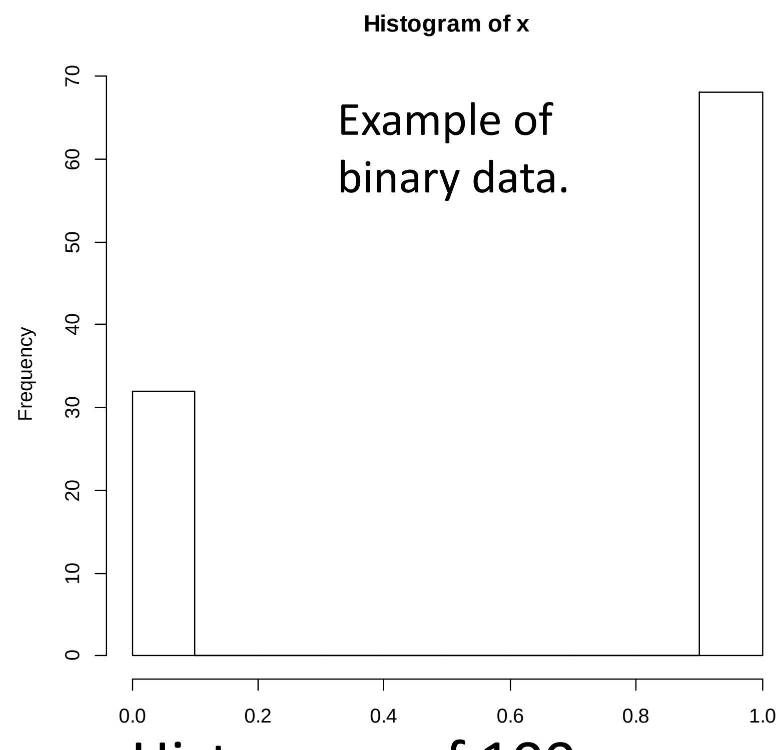
Population:

$$X_i \sim B(100, \pi = 0.7)$$

Random sample, n=100

$$X_1, X_2, \ldots, X_{100}$$

```
>x<-rbinom(100,1,0.7)
>hist(x)
```



Histogram of 100 observations from B(0.7).

Random sample form a Poisson

Population: $X_i \sim P(3)$

Random sample, n=100 $X_1, X_2, \ldots, X_{100}$

```
> x<-rpois(100,3)
> x
  [1] 0 1 0 9 3 3 3 2 5 3 3 1 4 1 3 7 4 3 5 3 1 5 2 0 5 4 0 3 0 1 3 1 3 1 2 2 4
 [38] 4 5 3 5 0 3 6 3 2 3 2 0 2 1 6 4 3 3 1 1 3 5 9 4 5 2 2 3 2 3 3 2 3 3 3 3 0
 [75] 4 4 5 3 4 4 6 2 4 3 3 3 4 3 3 3 4 5 7 2 2 7 5 5 3 2
```

A random sample of 100 values from P(3)

```
> table(x)
x
 0  1  2  3  4  5  6  7  9
 8 10 15 33 14 12  3  3  2
>
```

A frequency table

Example of binary data.

Part 2:

Point estimation for population mean μ and population variance σ^2

Sample Statistics as Estimators of Population Parameters

- A **sample statistic** is a numerical measure of a summary characteristic of a sample.

- A population parameter is a numerical measure of a summary characteristic of a population.

- An estimator of a population parameter is a sample statistic used to estimate the population parameter.

Definition: sample mean and sample variance

If $X_1, \ldots, X_n$ is a random sample from a population X then:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The sample mean

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

The sample variance

The mean and the variance

The mean and the variance of the sample values:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\begin{aligned} s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \\ &= \left(\frac{1}{n-1}\sum_{i=1}^{n} x_i^{\,2}\right) - \left(\frac{n}{n-1}\overline{x}^{\,2}\right) \end{aligned}$$

Example: the airquality data

The estimators for the unknown parameter (μ,σ) in the population

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

Point estimators of the population parameters in R:

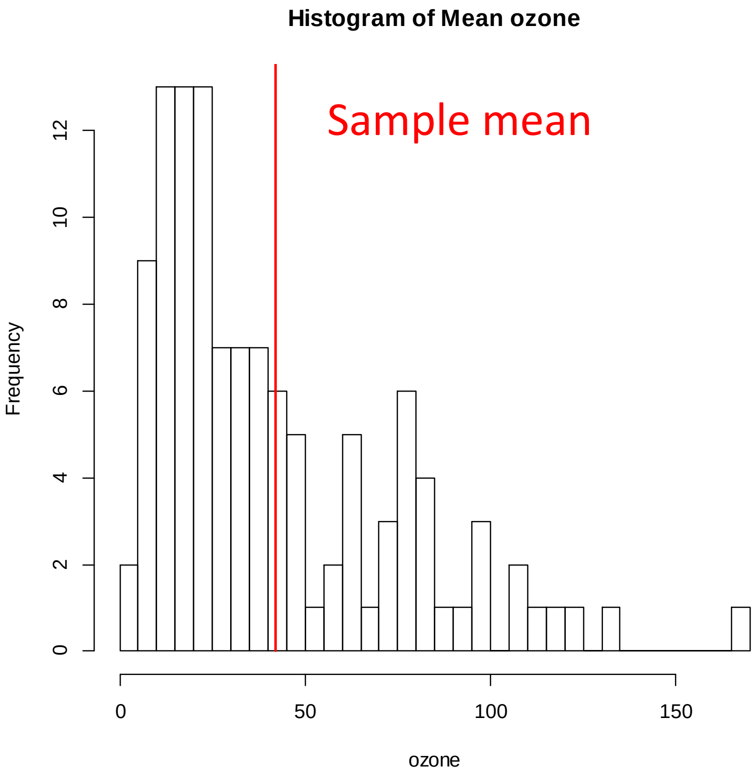
```
> ozone=airquality$Ozone
> meanozone=mean(ozone,na.rm=T)
> meanozone
[1] 42.12931
```

```
> varozone=var(ozone,na.rm=T)
> varozone
[1] 1088.201
```

```
> sqrt(varozone)
[1] 32.98788
```

Graphical output

```
> hist(ozone,breaks=25,main="Histogram of Mean ozone")
> lines(c(meanozone,meanozone),c(0,23),col="red",lwd=2)
> text(meanozone,25,round(meanozone,2))
> text(100,25,"point estimate of mean ozone ",col=3)
```



Population and sample

Population

$$X_1, X_2, \ldots, X_N$$

$$\mu, \sigma^2$$

Unknown parameters of the population.

sample

$$X_1, X_2, \ldots, X_n$$

$$\overline{x}, s^2$$

Parameter estimates for the sample.

Example: Average Heights and Weights for American Women

The data set gives the average heights and weights for 15 American women aged 30–39.

Sample size.

```
> help(women)
> women
   height weight
1      58    115
2      59    117
3      60    120
4      61    123
5      62    126
6      63    129
7      64    132
8      65    135
9      66    139
10     67    142
11     68    146
12     69    150
13     70    154
14     71    159
15     72    164
```

Example: Average Heights and Weights for American Women

Parameter estimates for height:

Estimator for μ

```
> womenheight=women$height
> meanheight=mean(womenheight,na.rm=T)
> meanheight
[1] 65
```

$$\bar{x} = \frac{1}{15}\sum_{i=1}^{15} x_i = 65$$

Estimator for σ^2

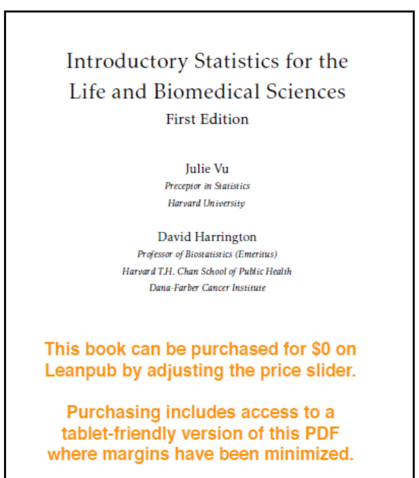
```
> womenheight=women$height
> varheight=var(womenheight,na.rm=T)
> varheight
[1] 20
```

$$s^2 = \frac{1}{14}\sum_{i=1}^{15}(x_i - \bar{x})^2 = 20$$

Part 3: Variability of estimates

4.1.1: The Sampling Distributions of the sample mean
4.1.2: Standard error of the mean

Basic properties for a sample mean μ and a sample variance σ^2


Introductory Statistics for the
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Section 4.1

The sample mean

Let X_1, X_2, ..., X_n is a random sample from a population X with mean μ and variance σ^2 :

$$\underbrace{\begin{gathered} E(X_i) = \mu \\ Var(X_i) = \sigma^2 \end{gathered}}$$

Parameters of the population

$$\underbrace{\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

Random variable in the sample.

$$E(\bar{X}) = ?$$

$$Var(\bar{X}) = ?$$

The variance of the sample mean

If the population X have mean μ and variance σ^2
i.e., E (X) = μ and Var (X) = σ^2 , then the sample mean $\overline{X}$ is a mean and variance given by:

$$\boxed{E(\overline{X}) = \mu}$$

$$\boxed{Var(\overline{X}) = \frac{\sigma^2}{n}}$$

The sample variance

- The sample variance S^2 is an estimator of the population variance σ^2

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

$$\boxed{E(S^2) = \sigma^2}$$

Sampling Distributions of the sample mean

- $\bar{X}$ is a random variable. Its value is determined partly by which people are randomly chosen to be in the sample.
- Many possible samples, many possible $\bar{X}$'s.
- The variance of the sample mean:

$$\boxed{Var(\bar{X}) = \frac{\sigma^2}{n}}$$

- The variance of the sample mean is reduced as the sample size increases.

Example

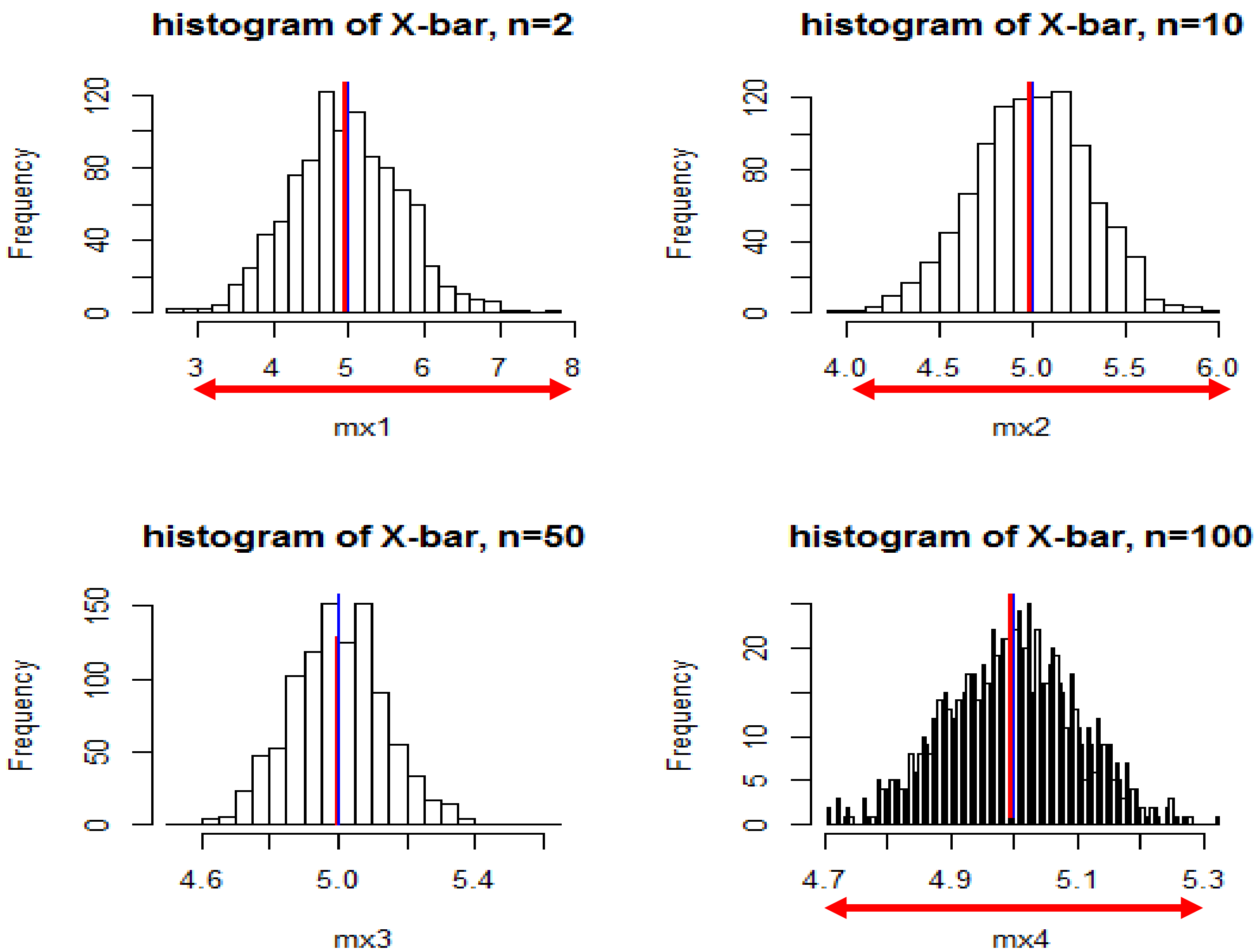
```
> mx1=c(1:1000)
> mx2=c(1:1000)
> mx3=c(1:1000)
> mx4=c(1:1000)
> for(i in 1:1000)
+ {
+ sample1=rnorm(2,5,1)
+ sample2=rnorm(10,5,1)
+ sample3=rnorm(50,5,1)
+ sample4=rnorm(100,5,1)
+ mx1[i]=mean(sample1)
+ mx2[i]=mean(sample2)
+ mx3[i]=mean(sample3)
+ mx4[i]=mean(sample4)
+ }
> mean(mx1)
[1] 4.967702
```

Example:

- 4 samples from normal distribution.
- Sample size=1000.

Sampling from normal distributions.

Graphical output: distribution of the sample mean



- n: sample size.
- Distribution of the sample mean for different values of n.

Example for graphical output

```
> par(mfrow=c(2,2))
> hist(mx1,breaks=20, main="histogram of X-bar, n=2")
> lines(c(mean(mx1),mean(mx1)),c(1,129),col="red",lwd=2)
> lines(c(5,5),c(1,1000),col="blue")
> hist(mx2,breaks=20, main="histogram of X-bar, n=10")
> lines(c(mean(mx2),mean(mx2)),c(1,129),col="red",lwd=2)
> lines(c(5,5),c(1,1000),col="blue")
> hist(mx3,breaks=20, main="histogram of X-bar, n=50")
> lines(c(mean(mx3),mean(mx3)),c(1,129),col="red",lwd=2)
> lines(c(5,5),c(1,1000),col="blue")
> hist(mx4,breaks=200, main="histogram of X-bar, n=100")
> lines(c(mean(mx4),mean(mx4)),c(1,129),col="red",lwd=2)
> lines(c(5,5),c(1,1000),col="blue")
```

Distribution of the sample mean

We distinguish between three cases :

1. X has a normal distribution with unknown μ and σ^2 known.
2. X has an unknown distribution, but we have a large sample.
3. X has a normal distribution with unknown μ and σ^2 and small sample.

Distribution of the sample mean: case 1

When the sample comes from a normal population with known σ ², i.e:

$$X_i \sim N(\mu, \sigma^2) \qquad i = 1,..,n$$

Then for the sample mean : $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

And so: $$Z_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$$

Distribution of the sample mean: case 2

If the sample comes from a population whose distribution is unknown, but the sample size is large (in practice n> 30), then approximate:

$$\boxed{\bar{X} \sim N(\mu, \frac{\sigma^2}{n})}$$

$$\boxed{Z_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)}$$

If the variance in the population σ^2 is unknown, use the parameter estimate, the sample variance s^2:

$$Z_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{s^2}{n}}} \sim \mathrm{N}(0,1)$$

σ^2 and S^2

The population variance σ^2 is usually not known.
We can use the estimator for σ^2 : sample variance S^2.

$$\boxed{E(S^2) = \sigma^2}$$

The sample variance S^2 is an unbiased estimator of the population variance

$$Var(\bar{X}) = \frac{\sigma^2}{n} \Rightarrow SE(\bar{X}) = \sqrt{\frac{S^2}{n}}$$

The standard error of the sample mean.

Distribution of the sample mean: case 3

If the sample comes from a population whose distribution is normal and the variance is unknown, and the sample is small , then :

$$\boxed{T_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim t_{(n-1)}}$$

A Student t-distribution with (n-1) degrees of freedom is denoted by t (n-1).

The standard error of the sample mean

An estimate for the standard error of the sample mean.

$$SE = \frac{s}{\sqrt{n}}$$

s → The standard deviation.

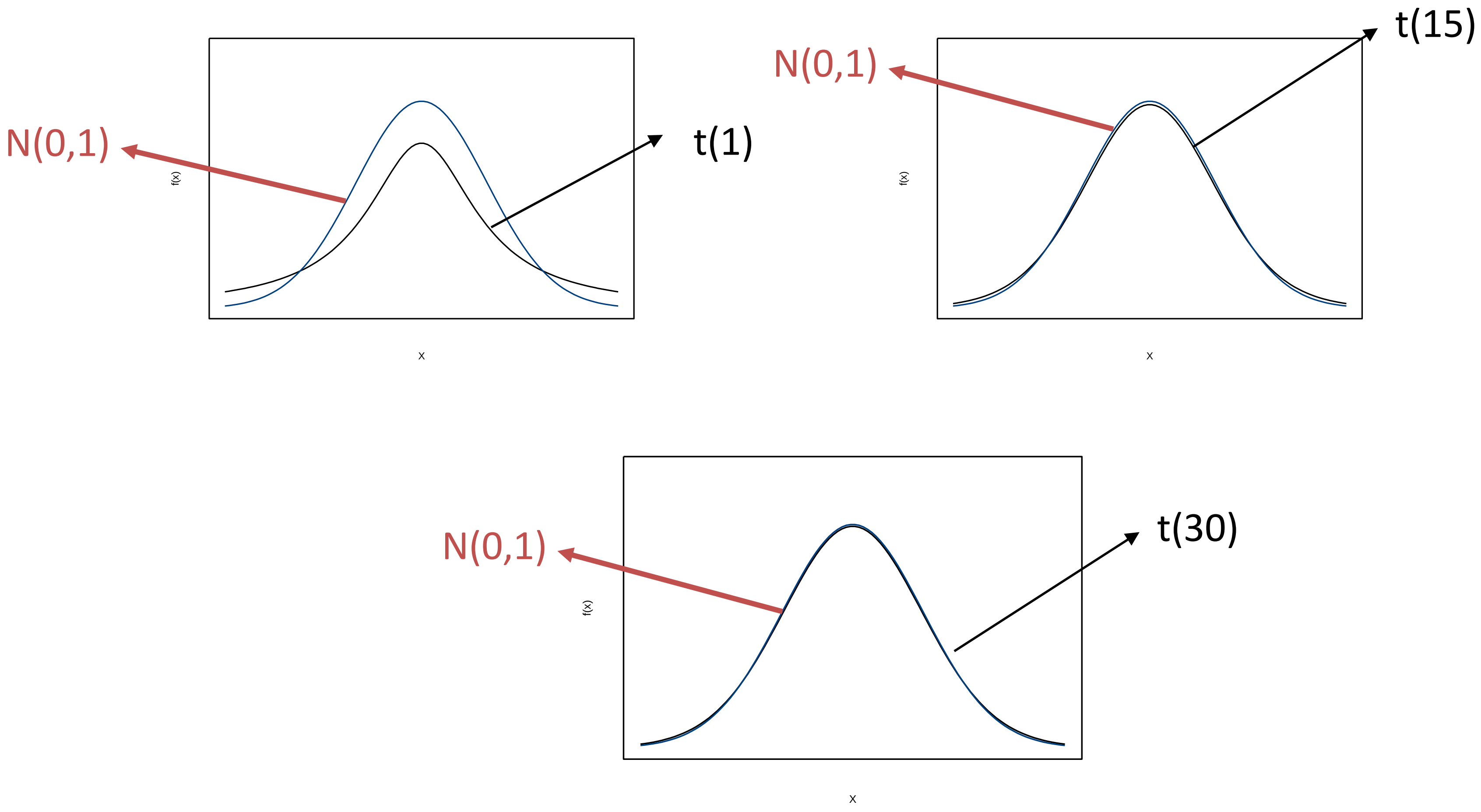
n → The sample size.

If the sample comes from a population whose distribution is normal and the variance is unknown, and the sample size is small , then :

$$\boxed{T_{\overline{X}} = \frac{\overline{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim t_{(n-1)}}$$

A Student t-distribution with (n-1) degrees of freedom is denoted by t (n-1).

Student's t-distributions, and N (0,1)



Example

$$X \sim t_{(13)}$$

What is the value of a, so that

$$P(X > a) = 0.025 \text{ ?}$$

Student's t-distribution

p df	0.25	0.1	0.05	0.025	0.01	0.005	0.001
1	1	3.078	6.314	12.706	31.821	63.657	318.309
2	0.817	1.886	2.92	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.215
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.44	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.86	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.25	4.297
10	0.7	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.696	1.356	1.782	2.179	2.681	3.055	3.93
13	0.694	1.35	1.771	2.16	2.65	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.69	1.337	1.746	2.12	2.583	2.921	3.686
17	0.689	1.333	1.74	2.11	2.567	2.898	3.646
18	0.688	1.33	1.734	2.101	2.552	2.878	3.61
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.08	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.5	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.06	2.485	2.787	3.45
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.31	1.697	2.042	2.457	2.75	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
50	0.679	1.299	1.676	2	2.403	2.678	3.261
60	0.679	1.31	1.671	2.009	2.39	2.66	3.232
120	0.677	1.289	1.658	1.98	2.358	2.61	3.16
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.000

Look in row labeled 13 and column labeled .025 to find P(x>2.16) = 0.025

$$P(X > 2.16) = 0.025$$

Student's t-distribution

p df	0.25	0.1	0.05	0.025	0.01	0.005	0.001
1	1	3.078	6.314	12.706	31.821	63.657	318.309
2	0.817	1.886	2.92	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.215
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
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21	0.686	1.323	1.721	2.08	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.5	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.06	2.485	2.787	3.45
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
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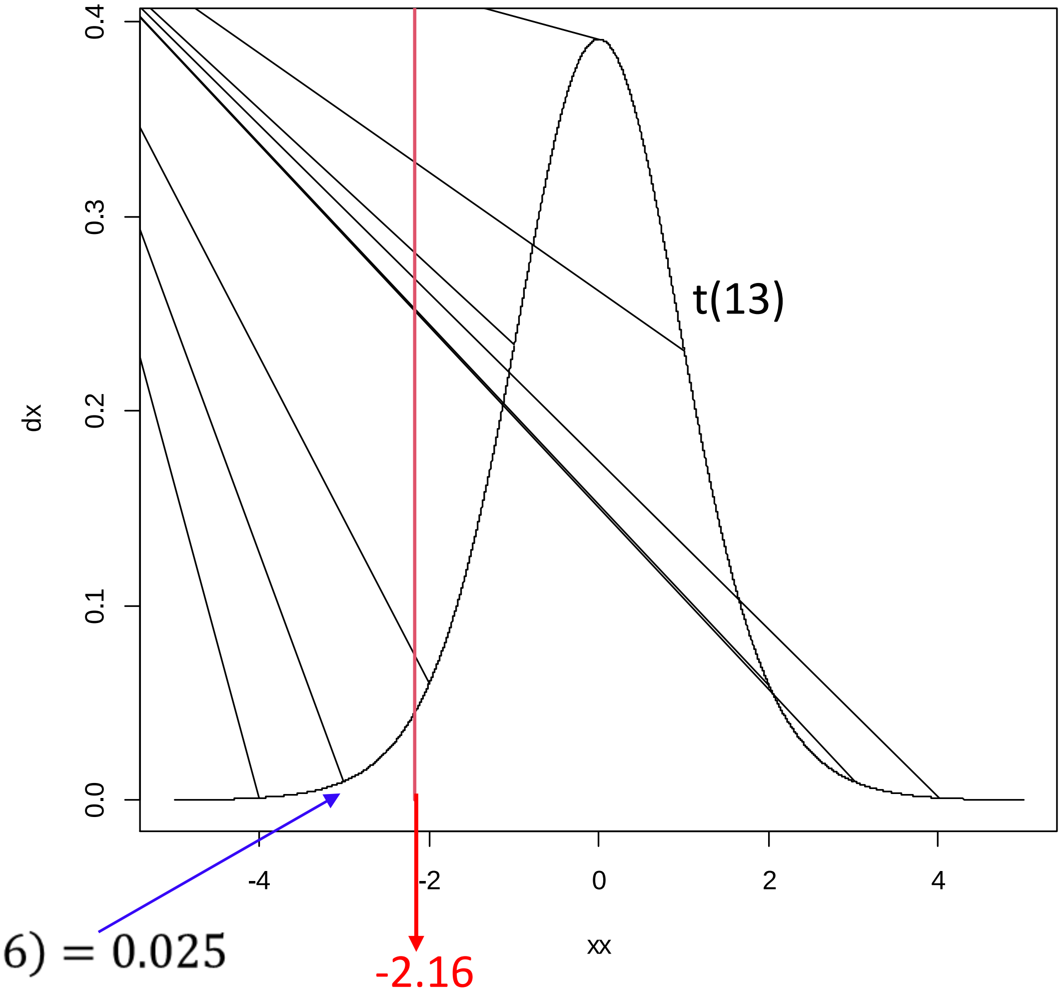
30	0.683	1.31	1.697	2.042	2.457	2.75	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
50	0.679	1.299	1.676	2	2.403	2.678	3.261
60	0.679	1.31	1.671	2.009	2.39	2.66	3.232
120	0.677	1.289	1.658	1.98	2.358	2.61	3.16
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.000

$$P(T > 2.16) = 0.025$$

$$P(T < -2.16) = 0.025$$

Student's t-distribution in R

```
> qt(0.975,13)
[1] 2.160369
> qt(0.025,13)
[1] -2.160369
```

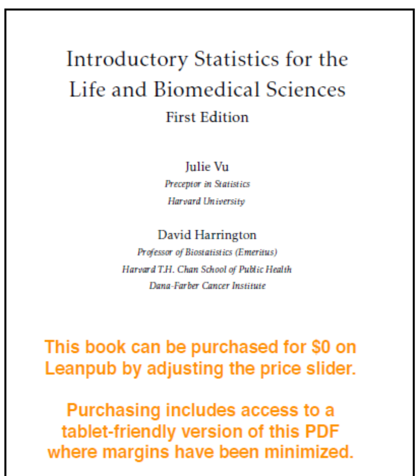


$P(T \leq -2.16) = 0.025$

	Case 1	Case 2	Case 3
Distribution of the population	$X_i \sim N(\mu, \sigma^2)$	$X_i \sim unkown$	$X_i \sim N(\mu, \sigma^2)$
σ^2	Known	Known / Not known	Not known
Sample size	No condition	≥ 30	< 30
Distribution of the sample mean	$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$	$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ $\bar{X} \sim N(\mu, \frac{S^2}{n})$	
Standardized distribution	$\boxed{\frac{\bar{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)}$	$\boxed{\frac{\bar{X} - \mu}{\sqrt{\frac{\sigma^2 of S^2}{n}}} \sim N(0,1)}$	$\boxed{\frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim t_{(n-1)}}$

Part 4:
Confidence intervals

Interval Estimation for a population mean μ


Introductory Statistics for the
Life and Biomedical Sciences
First Edition

Julie Vu
Preceptor in Statistics
Harvard University

David Harrington
Professor of Biostatistics (Emeritus)
Harvard T.H. Chan School of Public Health
Dana-Farber Cancer Institute

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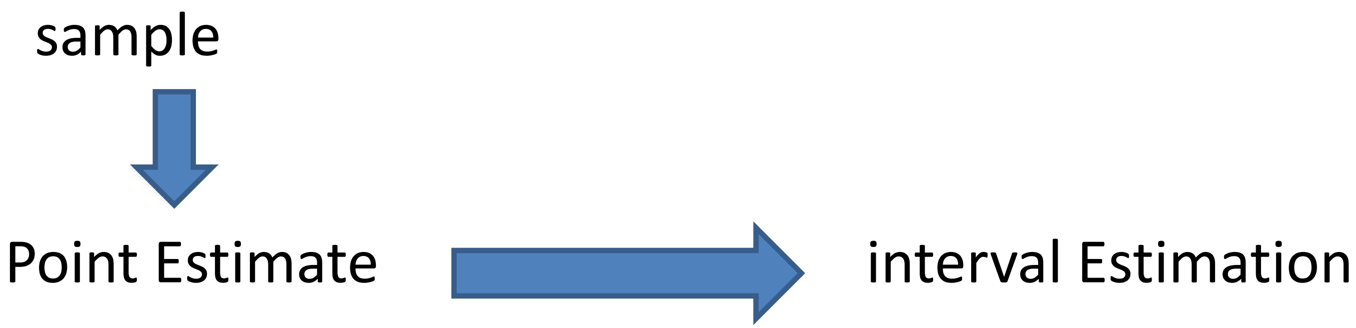
Section 4.2

Interval Estimation



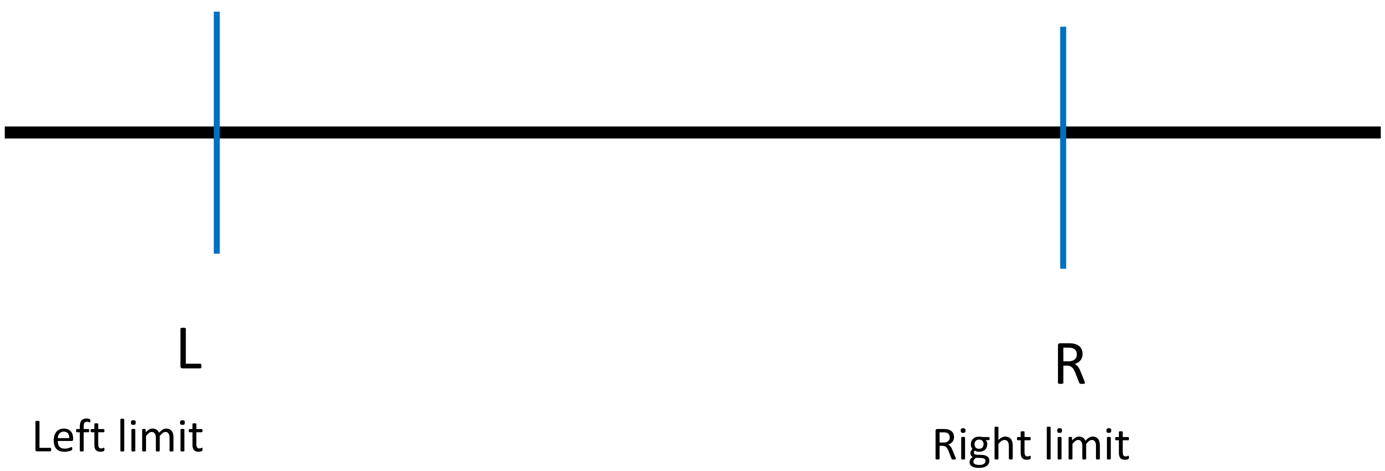
- On the basis of the sample, we find an interval in parameter space which contains this parameter μ “almost always”

- Such an interval is called a confidence interval

Confidence interval

On the basis of the sample, we find an interval [L, R] so :

$$\boxed{P(\mu \in [L, R]) = \text{"large"}}$$



We find an interval [L, R] that contains the value of the population mean (μ) with "high probability"

Confidence interval

Large → 1 - α

$$\boxed{P\left(L \le \mu \le R\right) = 1 - \alpha}$$

Example: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$ Confidence level

Confidence interval

We are looking for an interval that

$$\boxed{\frac{\alpha}{2}} \qquad \boxed{P\left(L \le \mu \le R\right) = 1-\alpha} \qquad \boxed{\frac{\alpha}{2}}$$

L=left limit

R= right limit

$$\alpha = 5\% \Rightarrow 1-\alpha = 95\%$$

$$\frac{\alpha}{2} = 2.5\% \qquad \boxed{P\left(L \le \mu \le R\right) = 95\%} \qquad \frac{\alpha}{2} = 2.5\%$$

A confidence interval for a population mean

- How can we calculate L and R?
- CI calculate in 5 steps:
  - Step 1: choose a confidence level $1-\alpha$

    (e.g. 90%, 95%, 99%)
  - Step 2: decision on the basis of the data in which case you are in (1, 2 or 3)
  - Step 3: Find the critical points in the correct table

    (Standard Normal or t-table)
  - Step 4: calculate the point estimators for μ ($\sigma 2$ and if not known)
  - Step 5: calculate the confidence interval

Case 1

If $X \sim N(\mu, \sigma^2)$

then: $\bar{X} \sim N(\mu, \dfrac{\sigma^2}{n})$

And $Z_{\bar{X}} = \dfrac{\bar{X} - \mu}{\sqrt{\dfrac{\sigma^2}{n}}} \sim N(0,1)$

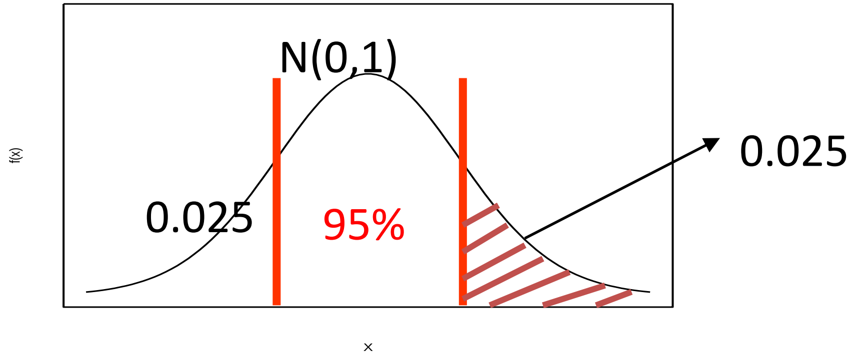
1. X has <u>a normal distribution</u> with unknown μ and <u>known</u> σ^2.

CI for case 1

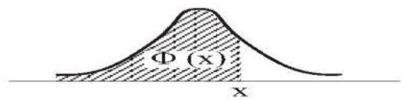
Stap 1: example, choose 1-α = 0.95

Stap 2: Case 1, thus: $$Z_{\overline{X}} = \frac{\overline{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$$

Stap 3: critical point:



Tabel 3 : Standaard normale verdeling



$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

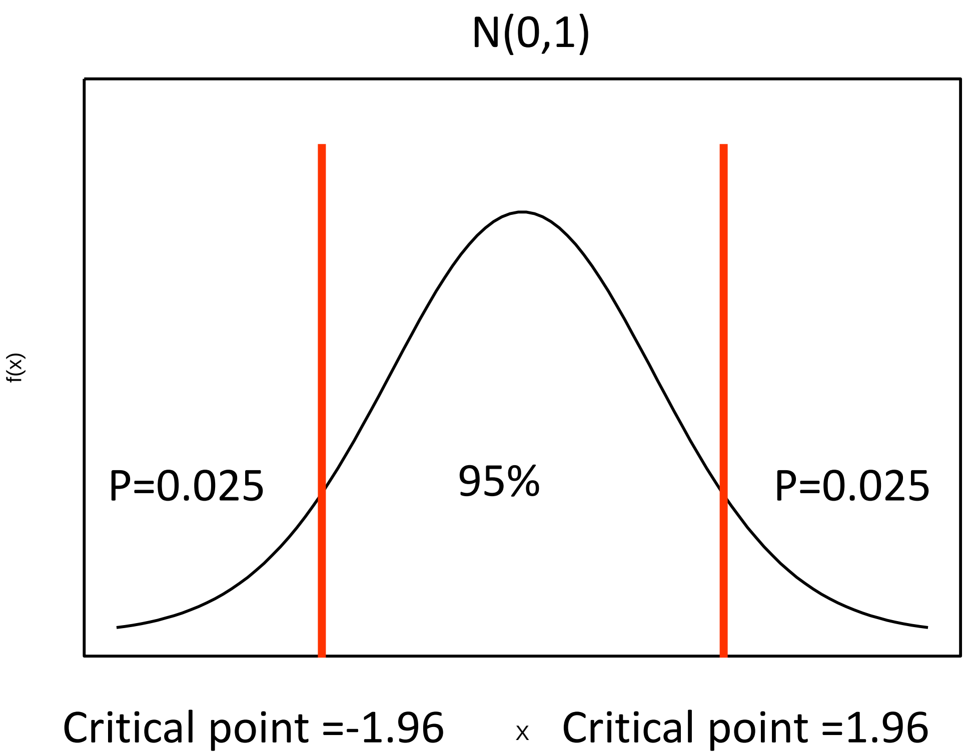
$$P(Z < 1.96) = 0.975$$

$$P(Z > 1.96) = 0.025$$

Critical value in R

```
> qnorm(0.025,0,1)
[1] -1.959964
> qnorm(0.975,0,1)
[1] 1.959964
>
```

CI for case 1



From the table of the standard normal distribution, we find that :

$$\boxed{P\left(-1.96 \leq \frac{\bar{X}-\mu}{\sqrt{\frac{\sigma^2}{n}}} \leq 1.96\right) = 0.95}$$

Thus, critical points: -1.96 and 1.96

CI for case 1

Step 4: calculate the point estimator : $\bar{X}$

Step 5: calculate the CI

For this, we know:

$$P\left(-1.96 \leq \frac{\bar{X}-\mu}{\sqrt{\frac{\sigma^2}{n}}} \leq 1.96\right) = 0.95$$

or, after the conversion of the formula:

$$P\left(\bar{X} - 1.96 \times \sqrt{\frac{\sigma^2}{n}} \leq \mu \leq \bar{X} + 1.96 \times \sqrt{\frac{\sigma^2}{n}}\right) = 0.95$$

CI for case 1

$$\boxed{P\left(\underbrace{\bar{X}-1.96\times\sqrt{\frac{\sigma^2}{n}}}_{L}\le\mu\le\underbrace{\bar{X}+1.96\times\sqrt{\frac{\sigma^2}{n}}}_{R}\right)=0.95}$$

$$\boxed{P(L\le\mu\le R)=1-\alpha}$$

So, a (1-α) CI for μ is :

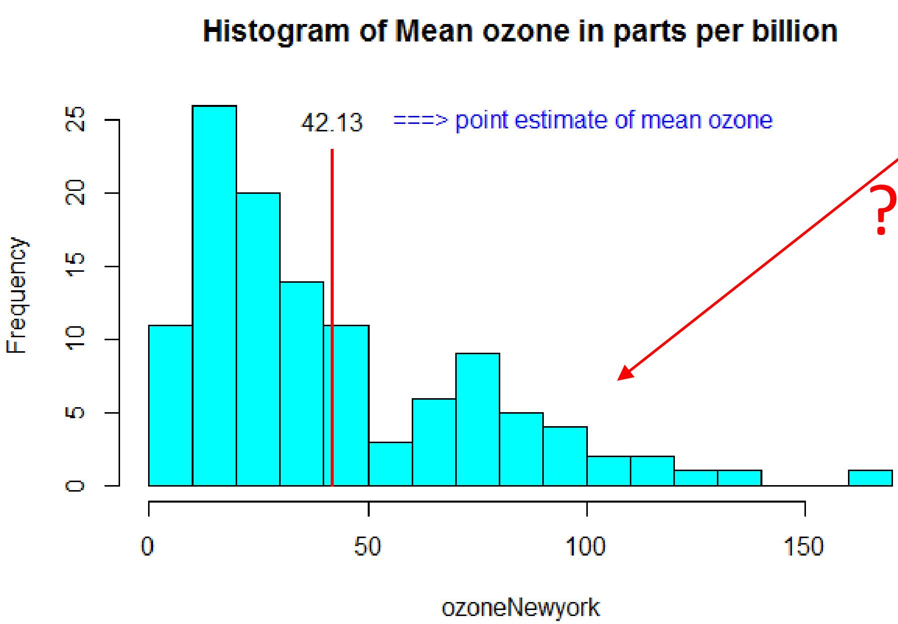
$$\left[\bar{x}-z\sqrt{\frac{\sigma^2}{n}},\bar{x}+z\sqrt{\frac{\sigma^2}{n}}\right]$$

Example for case 1

The airquality data

- Daily reading of the mean ozone in parts per billion value at Roosevelt Island from May 1, 1973 to Sep 30, 1973 were use.
- X = Daily reading of ozone value

$$X \sim N(\mu, \sigma^2)$$



???

and $\sigma^2 = 1024$ known

Sample (n = 116) from population

The sample mean
$\bar{x} = 42.13$

The mean

$$X_i \sim N(\mu, \sigma^2 = 1024)$$

- The mean is unknown parameter.

1. Determine a point estimator for ozone value
2. Determine a 95% CI for the mean of the ozone values

Example for case 1

The 95% CI for μ : the mean ozone value in the population

Step 1: *Choose a confidence level 1-α = 0.95*

Stap 2: *: Decide on the basis of the data in which case you are in:*

population is normally distributed

σ^2 known

→ Case 1, so normal distribution

Stap 3 Find the critical points in the appropriate table N (0,1): -1.96 and 1.96

Stap 4: *Calculate the point estimator for μ:*

Stap 5: *calculate the confidence interval for* $\bar{x} = 42.13$

Example for case 1

$$\boxed{P\left(\underbrace{\bar{X} - 1.96 \times \sqrt{\frac{\sigma^2}{n}}}_{L} \le \mu \le \underbrace{\bar{X} + 1.96 \times \sqrt{\frac{\sigma^2}{n}}}_{R}\right) = 0.95}$$

$$42.13 - 1.96 \times \sqrt{\frac{1024}{116}}$$

$$= 36.306$$

$$42.13 + 1.96 \times \sqrt{\frac{1024}{116}}$$

$$= 47.973$$

A 95% CI for the population mean μ of the ozone value [36.306, 47.953]

Interpretations:

Based on our sample, we are 95% confident that the true mean of ozone value lie in between 36.306 and 47.973

Confidence interval using R

For Case 1

```
> ozone=na.omit(airquality$Ozone)
> sigma=32
> sem=sigma/sqrt(n)
> E=qnorm(0.975)*sem
> xbar=mean(ozone)
> xbar+c(-E,E)
```

[1] 36.30601 47.95261

Analysis using the R package TeachingDemos

```
> library(TeachingDemos)
> ozone=na.omit(airquality$Ozone)
> sigma=32
> z.test(ozone,sd=sigma)
```

One sample test with normal distribution using the function z.test()

```
        One Sample z-test

data:  ozone
z = 14.1796, n = 116.000, Std. Dev. = 32.000, Std. Dev. of the
sample mean = 2.971, p-value <2.2e-16

alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 36.30601  47.95261
sample estimates:
mean of ozone
     42.12931
```

Case study 1a:

The `airquality` data: analysis of the average wind speed

Confidence interval for the population mean

Page 187

Case 2

If $X \sim F$

Then: $\bar{X} \sim N(\mu, \frac{S^2}{n})$

and $T_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)$

3. X has an unknown distribution, but we have a large sample (n> 30)

$$E(X) = \mu$$

$$Var(X) = \sigma^2$$

The same as case 1 but we replace σ^2 by S^2.

CI for case 2

Step 1: example, choose 1-α = 0.95

Step 2: case 2, so : $\boxed{\dfrac{\overline{X}-\mu}{\sqrt{\dfrac{\sigma^2}{n}}} \sim N(0,1)}$ or $\boxed{\dfrac{\overline{X}-\mu}{\sqrt{\dfrac{S^2}{n}}} \sim N(0,1)}$

Step 3: critical points: -1.96 and 1.96
(the same as in Case 1, since we are still using the standard normal distribution function)

Step 4: Calculate the point estimator (s) $\overline{x}$ (and possibly s²)

CI for case 2

Step 5: In the same manner as in Case 1:

The (1-α) CI for μ is :

$$\left[\bar{x} - z\sqrt{\frac{\sigma^2}{n}}, \bar{x} + z\sqrt{\frac{\sigma^2}{n}}\right] \quad \text{or} \quad \left[\bar{\boldsymbol{x}} - \boldsymbol{z}\sqrt{\frac{\boldsymbol{s}^2}{\boldsymbol{n}}}, \bar{\boldsymbol{x}} + \boldsymbol{z}\sqrt{\frac{\boldsymbol{s}^2}{\boldsymbol{n}}}\right]$$

Example for case 2

The airquality Data

- Suppose X = Daily reading of ozone value.
- X has an unknown distribution with unknown variance.
- But large sample (n = 116 >> 30).

The 95% CI for μ : the mean ozone value in the population

Step 1: choose confidence level 1-α = 0.95

Step 2: case 2, so :

$$\boxed{\frac{\overline{X}-\mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)}$$

Step 3: critical points: -1.96 and 1.96

(the same as in Case 1, since we are still using the standard normal distribution function)

Example for case 2

Step 4 : Calculate the point estimators:

$\bar{x}$ *= 42.13 and* s^2 *= 1088.201*

Step 5 : In the same manner as in Case 1:

The (1-α) CI for μ is :

=> $$\left[\bar{x} - z\sqrt{\frac{s^2}{n}}, \bar{x} + z\sqrt{\frac{s^2}{n}}\right]$$

=> $$\left[42.13 - 1.96\sqrt{\frac{1088.2}{116}},\quad 42.13 + 1.96\sqrt{\frac{1088.2}{116}}\right]$$

=> $$\left[36.126,\quad 48.132\right]$$

Example for case 2

A 95% CI for the population mean μ of the ozone value [36.126, 48.132]

Interpretations:

Based on our sample, we are 95% confident that the true mean of ozone value lie between 36.126 and 48.132.

Confidence Interval using R

For Case 2

```
> ozone=na.omit(airquality$Ozone)
> sigma=sd(ozone)
> sem=sigma/sqrt(n)
> E=qnorm(0.975)*sem
> xbar=mean(ozone)
> xbar+c(-E,E)
```

[1] 36.12624 48.13238

Analysis using the R package TeachingDemos

```
> library(TeachingDemos)
> ozone=na.omit(airquality$Ozone)
> z.test(ozone, sd=sd(ozone),conf.level = 0.95)


        One Sample z-test

data:  ozone
z = 13.7549, n = 116.000, Std. Dev. = 32.988, Std.
Dev. of the sample mean = 3.063, p-value <
2.2e-16
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 36.12624 48.13238
sample estimates:
mean of ozone
     42.12931
```

CI for case 3

If $$X \sim N(\mu, \sigma^2)$$

then: $$\cancel{\bar{X} \sim N(\mu, \frac{S^2}{n})}$$

And $$T_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim t(n-1)$$

3. X has a normal distribution with <u>unknown</u> μ and σ2.
n is small.

$$E(S^2) = \sigma^2$$

We use t(n-1) instead of N(0,1).

CI for case 3

Stap 1: example, choose 1-α = 0.95

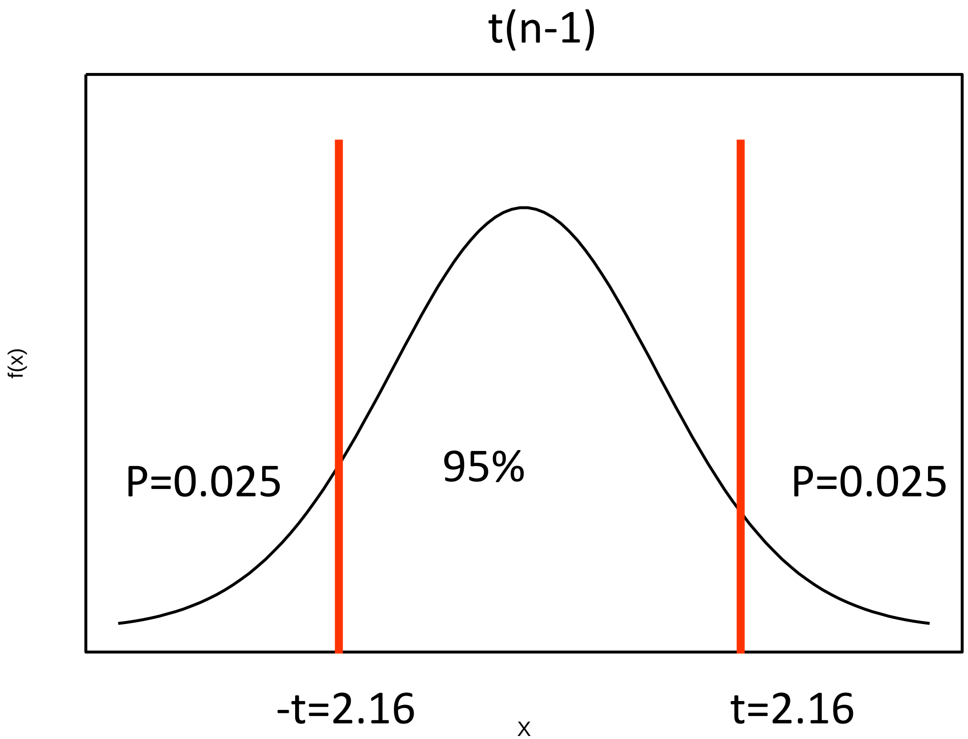
Stap 2: case 3, so:

$$\boxed{\frac{\bar{X}-\mu}{\sqrt{\frac{S^2}{n}}} \sim t_{(n-1)}}$$

Stap 3: critical points: gives you the t-table
value will depend on the number of degrees of freedom
(and hence on the size of the sample)

Take for example, n = 14

CI for case 3



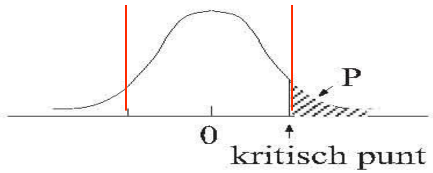
Example: n=14 and α=0.05

From t table with 13 df, we find that

$$P\left(-2.16 \leq \frac{\bar{X}-\mu}{\sqrt{\frac{\sigma^2}{n}}} \leq 2.16 \right) = 0.95$$

Student t-distribution

Tabel 4 : Kritische punten student t verdeling



P / v.g.	.25	.10	.05	.025	.010	.005	.001
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	.816	1.886	2.920	4.303	6.965	9.925	22.326
3	.765	1.638	2.353	3.182	4.541	5.841	10.213
4	.741	1.533	2.132	2.776	3.747	4.604	7.173
5	.727	1.476	2.015	2.571	3.365	4.032	5.893
6	.718	1.440	1.943	2.447	3.143	3.707	5.208
7	.711	1.415	1.895	2.365	2.998	3.499	4.785
8	.706	1.397	1.860	2.306	2.896	3.355	4.501
9	.703	1.383	1.833	2.262	2.821	3.250	4.297
10	.700	1.372	1.812	2.228	2.764	3.169	4.144
11	.697	1.363	1.796	2.201	2.718	3.106	4.025
12	.695	1.356	1.782	2.179	2.681	3.055	3.930
13	.694	1.350	1.771	2.160	2.650	3.012	3.852
14	.692	1.345	1.761	2.145	2.624	2.977	3.787
15	.691	1.341	1.753	2.131	2.602	2.947	3.733
16	.690	1.337	1.746	2.120	2.583	2.921	3.686
17	.689	1.333	1.740	2.110	2.567	2.898	3.646
18	.688	1.330	1.734	2.101	2.552	2.878	3.610
19	.688	1.328	1.729	2.093	2.539	2.861	3.579
20	.687	1.325	1.725	2.086	2.528	2.845	3.552
21	.686	1.323	1.721	2.080	2.518	2.831	3.527
22	.686	1.321	1.717	2.074	2.508	2.819	3.505
23	.685	1.319	1.714	2.069	2.500	2.807	3.485
24	.685	1.318	1.711	2.064	2.492	2.797	3.467
25	.684	1.316	1.708	2.060	2.485	2.787	3.450
26	.684	1.315	1.706	2.056	2.479	2.779	3.435
27	.684	1.314	1.703	2.052	2.473	2.771	3.421
28	.683	1.313	1.701	2.048	2.467	2.763	3.408
29	.683	1.311	1.699	2.045	2.462	2.756	3.396
30	.683	1.310	1.697	2.042	2.457	2.750	3.385
40	.681	1.303	1.684	2.021	2.423	2.704	3.307
60	.679	1.296	1.671	2.000	2.390	2.660	3.232
120	.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	.674	1.282	1.645	1.960	2.326	2.576	3.090

$$P(T > 2.16) = 0.025$$

$$P(T < -2.16) = 0.025$$

$$C.I: 95\% \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$$

CI for case 3

$$\boxed{P\left(\underbrace{\bar{X}-t\times\sqrt{\frac{S^2}{n}}}_{L}\le\mu\le\underbrace{\bar{X}+t\times\sqrt{\frac{S^2}{n}}}_{R}\right)=1-\alpha}$$

$$\boxed{P\left(L\le\mu\le U\right)=1-\alpha}$$

So: a (1-α) CI for μ is :

$$\boxed{\left[\bar{X}-t\times\sqrt{\frac{S^2}{n}},\bar{X}+t\times\sqrt{\frac{S^2}{n}}\right]}$$

Example for case 3

The women Data:

Suppose X = the height of woman aged 30 – 39.
We assume that:

1. $X \sim N(\mu, \sigma^2)$
2. σ^2 unknown

Sample Size: $n = 15$

Mean: $\bar{x} = 65$

Variance: $s^2 = 20$

Example for case 3

The 95% CI for μ, the average height of women aged 30 - 39 :

Step 1: *Choose a confidence level 1-α = 0.95*

Step 2: *Decide on the basis of the data in which case you are in:*

normal distribution, small sample (n = 15 <30) and unkown σ^2

→ Case 3 , so t distribution: t(15-1)

Step 3: *Find the critical points in the correct table :* -2.145 and 2.145

Step 4: *Calculate the point estimators for μ and σ^2 :*

$$\bar{x} = 65 \text{ } and \text{ } s^2 = 20$$

Example for case 3

Step 5: *calculate the confidence interval :*

$$\left[\bar{x} - t\sqrt{\frac{s^2}{n}}, \bar{x} + t\sqrt{\frac{s^2}{n}}\right] = \left[65 - 2.145\sqrt{\frac{20}{15}},\ 65 + 2.145\sqrt{\frac{20}{15}}\right]$$

$$= [62.52;\ 67.48]$$

A 95% CI for the population mean μ of the women height is [62.52, 67.48]

Interpretation:

Based on our sample, we are 95% confident that the true mean of the women height aged 30 – 39 will lies in between 62.52 and 67.48

Confidence Interval using R

For Case 3

```
> Height=women$height
> n=length(Height)
> SE=s/sqrt(n)
> E=qt(0.975,df=n-1)*SE
> xbar=mean(Height)
> xbar+c(-E,E)

[1] 62.52341 67.47659

OR

> library(TeachingDemos)
> t.test(women$height))
```

Analysis using the R function `t.test()`

```
> t.test(women$height))

One Sample t-test

data:  women$height
t = 56.2917, df = 14, p-value < 2.2e-16
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 62.52341 67.47659
sample estimates:
mean of x
       65
```

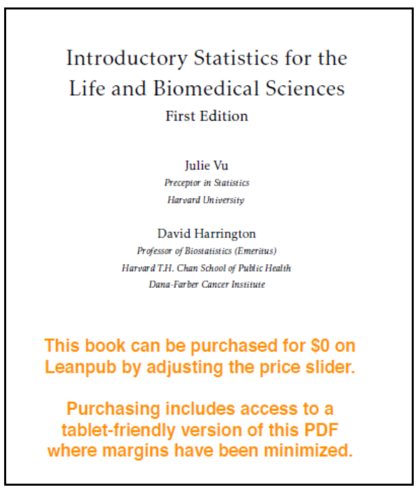
Case study 2:
The The NHANES dataset: BMI

Confidence interval for the population mean

Page 198

Part 5:

Hypotheses testing



Introductory Statistics for the
Life and Biomedical Sciences
First Edition

Julie Vu
Preceptor in Statistics
Harvard University

David Harrington
Professor of Biostatistics (Emeritus)
Harvard T.H. Chan School of Public Health
Dana-Farber Cancer Institute

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Section 4.3

5.3.1: A Formal Approach to Hypothesis Testing

Testing a hypothesis about a population parameter

Hypothesis tests

- In the testing theory we try, on the basis of a sample, a hypothesis about the population keys.
- In practice, we test a hypothesis about a parameter of a population distribution.
- Example:
  - The parameter of interest: the population mean.
  - The parameter μ in a population with distribution N (μ, σ^2).

Example: population and sample

- Two zoologists study the population of Horse shoe crabs.
- The variable of interest is the number of satellites, E,g the number of satellites in the horse shoe crab population.
- In the sample:

$$s^2 = 9.9119$$
$$\bar{x} = 2.91$$
$$n = 173$$

The zoologists collected data about 173 horse shoe crabs and in the sample the mean is equal to 2.91 and the variance to 9.911.

Notation

X_i **:** the number of satellites.

$X_1, X_2, \ldots, X_N$

$$E(X_i) = \mu$$

$$Var(X_i) = \sigma^2$$

The two hypotheses of zoologists

- The hypothesis of the first zoologist (about population mean) is that

$$E(X_i) = \mu = 2.5$$

- The hypothesis of the second zoologist (about population mean) is that

$$E(X_i) = \mu = 2.9$$

both zoologists know the variance in the population mean is not known.

What we want to do in this section ?

- We look for a procedure we can use to make a decision about the parameter in the population.

Population

$$X_1, X_2, \ldots, X_N$$

$$E(X_i) = \mu$$

What we want to do in this section ?

We look for a procedure we can use to make a decision about the parameter in the population:

$\mu = 2.5$ or $\mu = 2.9$

The decision rule

Population

$$X_1, X_2, \dots, X_N$$

$$E(X_i) = \mu$$

sample

$$X_1, X_2, \dots, X_n$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Based on the mean in the sample, we would like to make a decision about the parameter (the mean) in the population

The estimate from the sample of the unknown parameter

$$\bar{X}$$

$$\mu = 2.5$$

$$or$$

$$\mu = 2.9$$

?

The decision rule

sample

$$X_1, X_2, \ldots, X_n$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

previous sections

Point estimator

$$E(\bar{X}) = \mu$$

confidence interval

$$\bar{X} \pm a \times \sqrt{Var(\bar{X})}$$

this section

On the basis of the sample and the sample mean we want to test hypotheses about the population parameter

The sample

- The two zoologists decide to a large sample taking the population and the number of satellites per crab to count.

- The sample size is 173.

$$\left.\begin{aligned} &X_i \sim unknown \\ &n > 30 \\ &\sigma^2 : unkown \end{aligned}\right\} \quad \boxed{\frac{\bar{X}-\mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)}$$

The two hypotheses

- We call the hypothesis of the first zoologist: the null hypothesis

- We call the hypothesis of the second the zoologist: the
- alternative hypothesis

$$\boxed{\begin{aligned} H_0 &: \mu = 2.5 \\ H_1 &: \mu = 2.9 \end{aligned}}$$

$$\bar{X} \sim N(\underset{\uparrow}{2.5}, \frac{S^2}{n})$$

μ under H_0

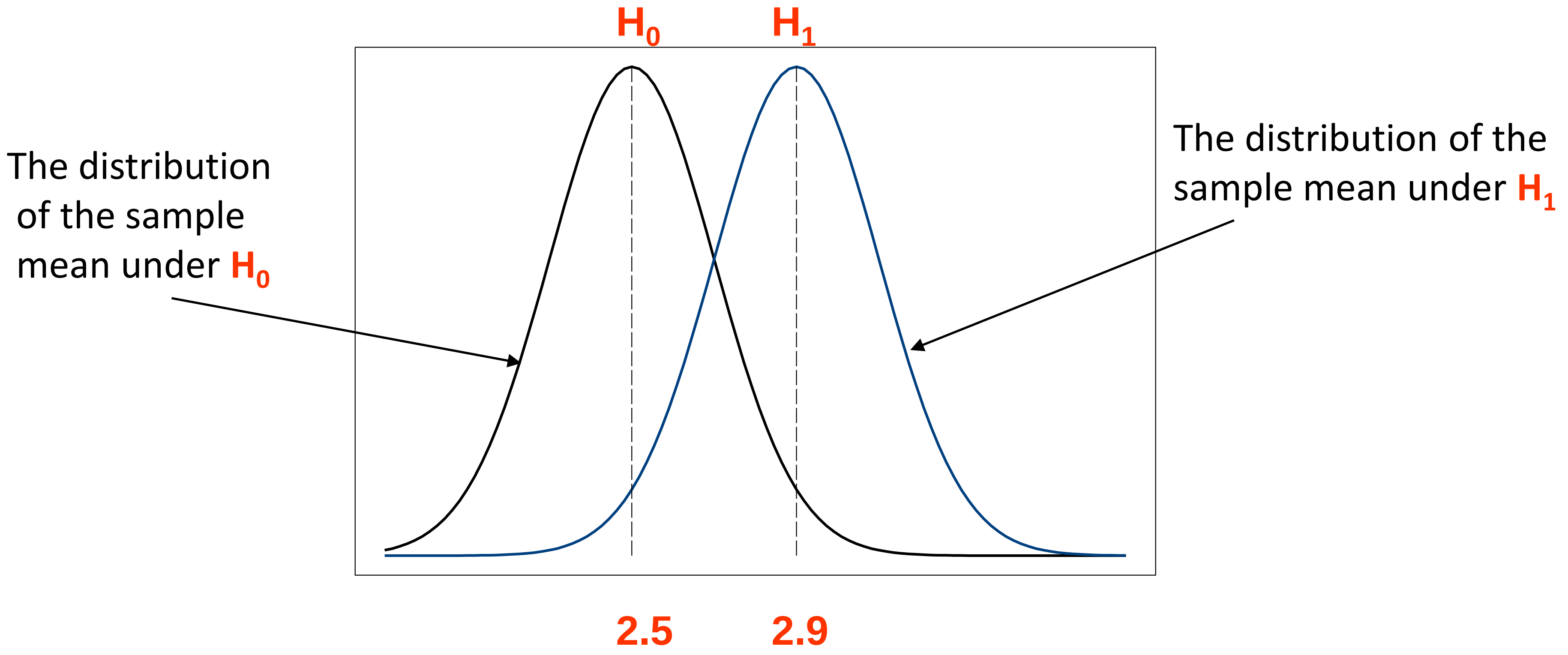
$$\bar{X} \sim N(\underset{\uparrow}{2.9}, \frac{S^2}{n})$$

μ under H_1

The distribution of the sample mean

$$\bar{X} \sim N\left(2.5, \frac{S^2}{n}\right) \qquad under \quad H_0$$

$$\bar{X} \sim N\left(2.9, \frac{S^2}{n}\right) \qquad under \quad H_1$$



The rejection region

$$H_0 : \mu = 2.5$$
$$H_1 : \mu = 2.9$$

$$\mu_0 = 2.5 < \mu_1 = 2.9$$

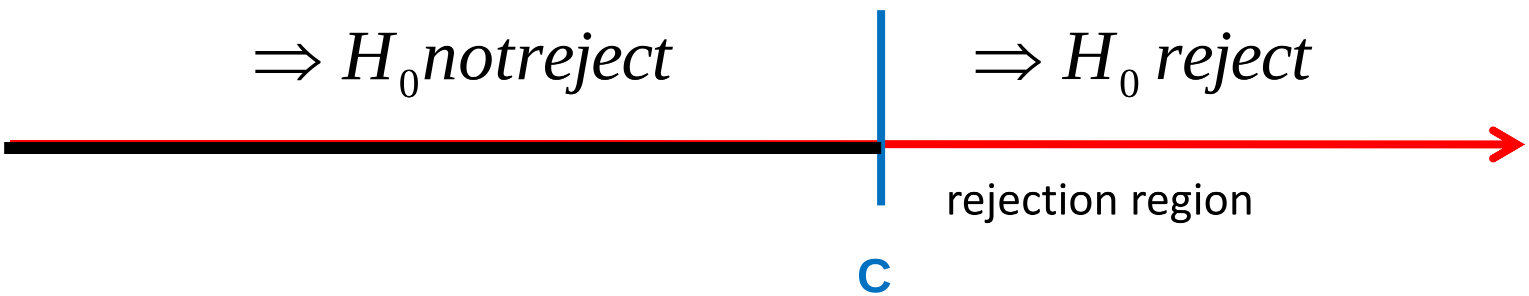
- From the nature of the alternative hypothesis it follows that we will reject the null hypothesis if we find in our sample a value of $\overline{X}$ which is `` too large ''.
- We determine c so :

To a `` large'' value of $\overline{X}$ (which we reject the null hypothesis) $\overline{X}$ means a value that is greater than c.

The rejection region: the decision rule

when we find a $\overline{X}$ value that is greater than c, then we reject the null hypothesis.

The decision rule $\quad \overline{X} > c \Rightarrow H_0 \quad reject$



If the $\overline{X}$ rejection region is, we reject the null hypothesis.

"The Question"

The decision rule is:

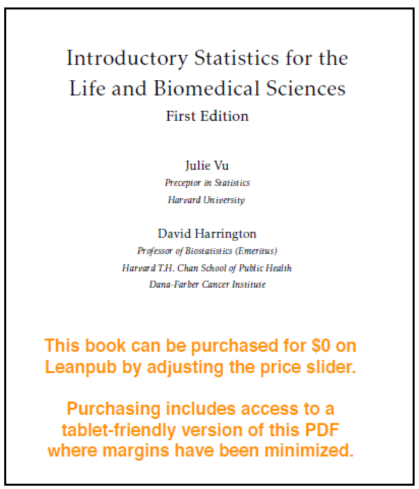
$$\overline{X} > c \Rightarrow H_0 \qquad reject$$

How can we choose the value of c????

The two types of errors

Part 6:
Decision errors

The two types of errors


Introductory Statistics for the
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Professor of Biostatistics (Emeritus)
Harvard T.H. Chan School of Public Health
Dana-Farber Cancer Institute

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Section 4.3.4: **Decision errors**

6.1 Two types of errors

Decision errors

Definition: the two types of errors

- The two types of errors that can be made by taking a decision:
    1. if we reject the null hypothesis when it is correct then we say that we committed a type I error.
    2. if we accept the null hypothesis when it is wrong, then we say that we committed a type II error.

Type I error

- Example: If the first zoologist is correct about the value of μ (2.5) and we reject the null hypothesis ($\mu = 2.5$) then we make a Type I error.

- General: If the null hypothesis is true and we reject the null hypothesis we make a Type I error.

Type II error

- Example: If the second zoologist is correct about the value of μ ($\mu = 2.9$), and we do not reject the null hypothesis ($\mu = 2.5$) we make a type II error.

- General: If the alternative hypothesis is correct and we do not reject the null hypothesis then we make a Type II error.

The two errors

Population

Test	H_0 is true	H_0 not true
reject H_0	incorrect statement Type I error	correct statement
Not reject H_0	correct statement	incorrect statement Type II error

The two errors

- We implement a "decision rule" for a given value of the probability of a Type I error.

- Decision rule for a given value of “small” Type I error (usually 0.01 or 0.05).

- If the researcher uses the probability of a type I error at 0.05 then there is 5% that the null hypothesis is rejected, while it is true.

The significance level

- The Probability of a Type I error is denoted by α
- We call α the significance level.
- We say that the null hypothesis H_0 is rejected at a significance level of α (or cannot be rejected at significance level α).

Choosing the value of critical point for the test

6.2 The decision rule

The distribution of the sample mean

The null and alternative hypotheses

$$H_0 : \mu = 2.5$$
$$H_1 : \mu = 2.9$$

The distribution of the sample mean under the null and the alternative hypotheses

$$\overline{X} \sim N(2.5, \frac{S^2}{n})$$

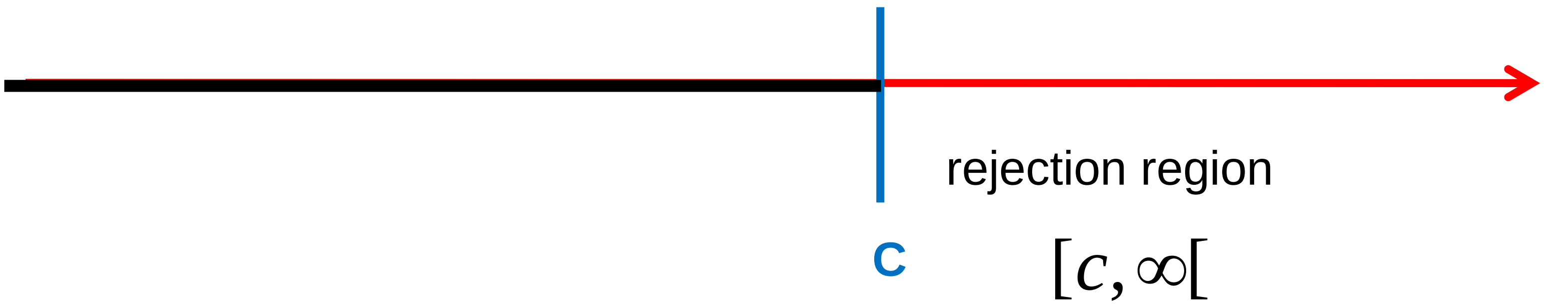
under H_0

$$\overline{X} \sim N(2.9, \frac{S^2}{n})$$

under H_1

The rejection region

When the value of $\overline{X}$ is greater than C, we reject null hypothesis



$$\overline{X} > c \Rightarrow H_0 \qquad reject$$

How can we choose c ?

Type I error & c

- If the null hypothesis is true and we reject the null hypothesis we make a type I error.

$$\alpha = 0.05$$

$$\bar{X} > c \Rightarrow H_0 \qquad reject$$

$$P(\bar{X} > c) = 0.05$$

if the null hypothesis is true!!

The critical point and the rejection region

Determine c under the null hypothesis that

$$P(\bar{X} > c) = 0.05 \quad \text{if} \quad \bar{X} \sim N\left(2.5, \frac{S^2}{n}\right)$$

$$\boxed{P(\bar{X} > c) = P\left(\frac{\bar{X} - 2.5}{\sqrt{\frac{S^2}{n}}} > \frac{c - 2.5}{\sqrt{\frac{S^2}{n}}}\right) = 0.05}$$

The critical point and the rejection region

$$P\left(\frac{\bar{X}-2.5}{\sqrt{\frac{S^2}{n}}} > \frac{c-2.5}{\sqrt{\frac{S^2}{n}}}\right) = 0.05$$

$$P\left(Z > \frac{c-2.5}{\sqrt{\frac{S^2}{n}}}\right) = 0.05$$

The test statistic

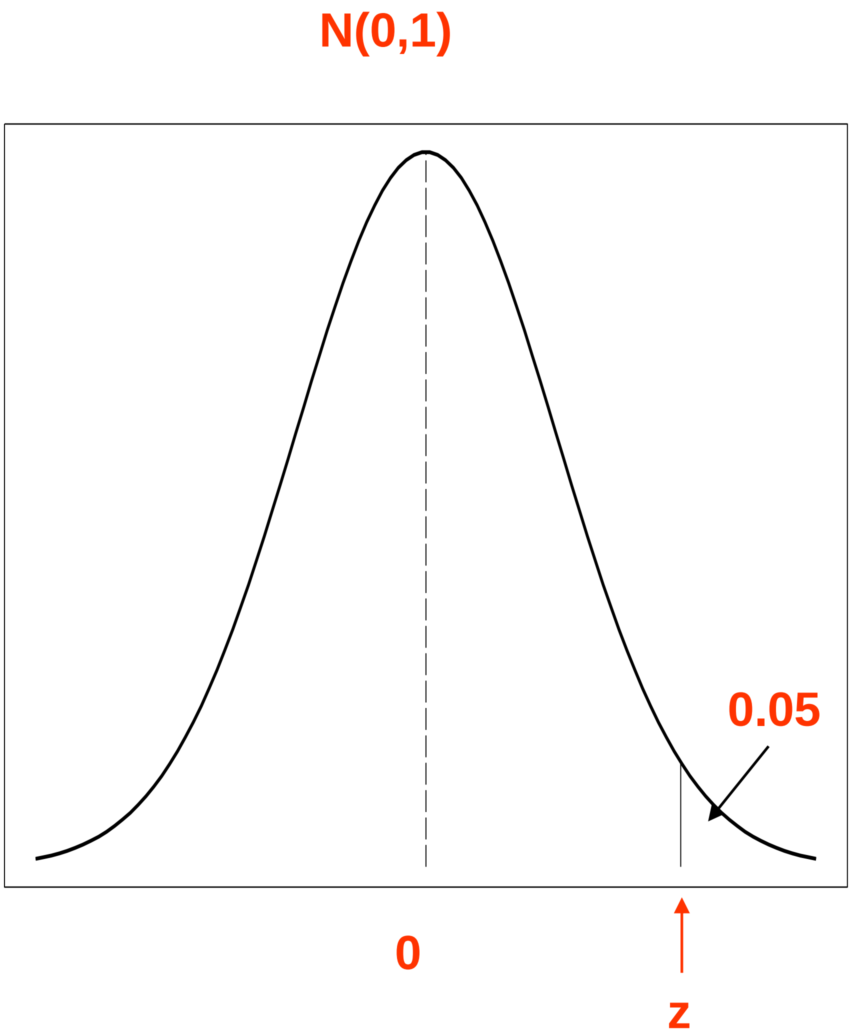
The sample mean

$$\frac{\bar{x}-\mu_0}{\sqrt{\frac{s^2}{n}}}$$

Population mean under H_0

Standard error of the sample mean.

The critical point and the rejection region

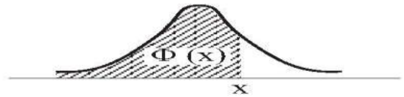


We choose the significance level α = 0.05.

This means that we reject the null hypothesis with probability 0.05 if it is correct, or with a probability of 95% the null hypothesis will be acceptable (not reject) if they are indeed correct.
This leads to the determination of a critical point z

$$P(Z < z) = 0.95$$

Tabel 3 : Standaard normale verdeling



	$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	[illegible]	[illegible]	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	[illegible]	[illegible]	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$$P(Z < 1.645) = 0.95$$

$$P(Z > 1.645) = 0.05$$

1

In R:

```
> qnorm(0.95,0,1)
[1] 1.644854
```

The critical point and the rejection region

$$P\left(Z > \frac{c - 2.5}{\sqrt{\frac{S^2}{n}}} \right) = 0.05$$

$$P(Z > 1.645) = 0.05$$

$$\frac{c - 2.5}{\sqrt{\frac{S^2}{n}}} = 1.645$$

The critical point and the rejection region

$$\frac{c-2.5}{\sqrt{\frac{S^2}{n}}}=1.645 \Rightarrow c=1.645\times\sqrt{\frac{S^2}{n}}+2.5$$

From the Sample (see page 91):

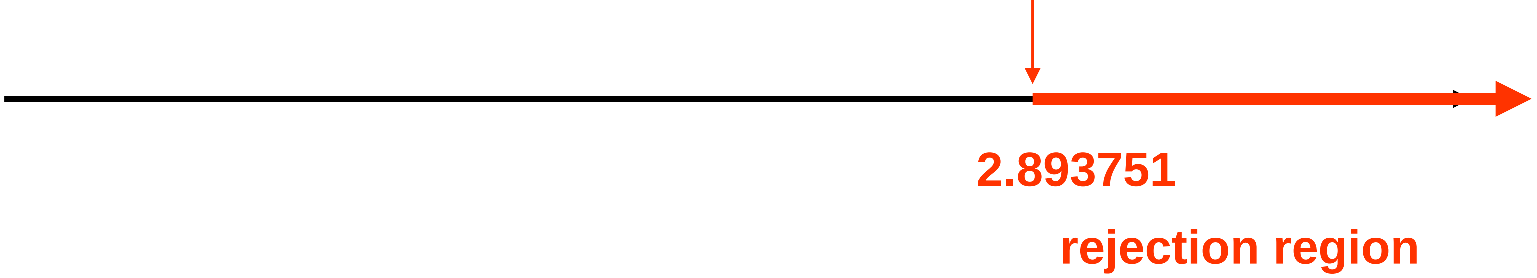
$$s^2=9.9119$$

$$\bar{x}=2.91$$

$$n=173$$

$$\Rightarrow c=1.645\times\sqrt{\frac{9.9119}{173}}+2.5=2.893751$$

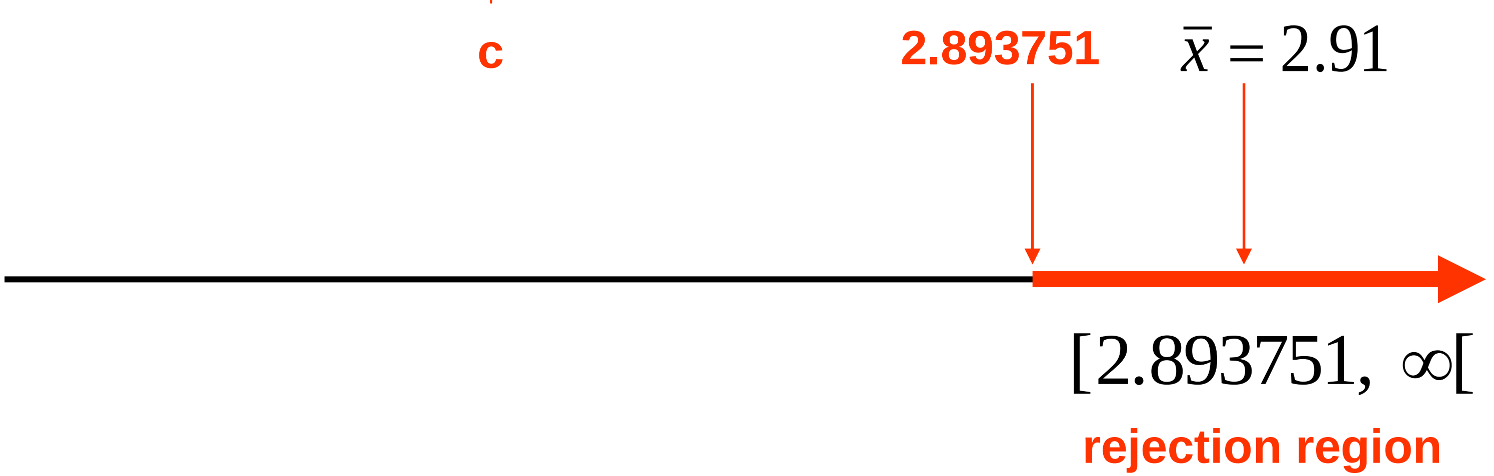
$$[2.893751,\ \infty[$$



The critical point and the rejection region

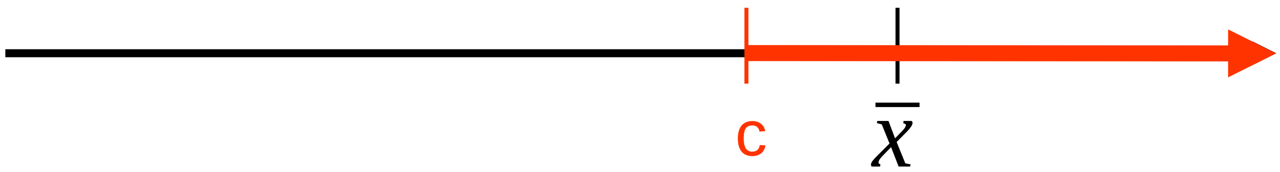
$$\frac{\bar{x}-2.5}{\sqrt{\frac{S^2}{n}}} > \frac{c-2.5}{\sqrt{\frac{S^2}{n}}} \Rightarrow \ldots \qquad c = z \times \sqrt{\frac{S^2}{n}} + \mu_0$$

$$\bar{x} = 2.91 > \underbrace{2.893}_{\text{c}}\ldots \Rightarrow H_0 \quad reject$$



The critical point and the test statistic

We reject H_0 because $\bar{x} > c$



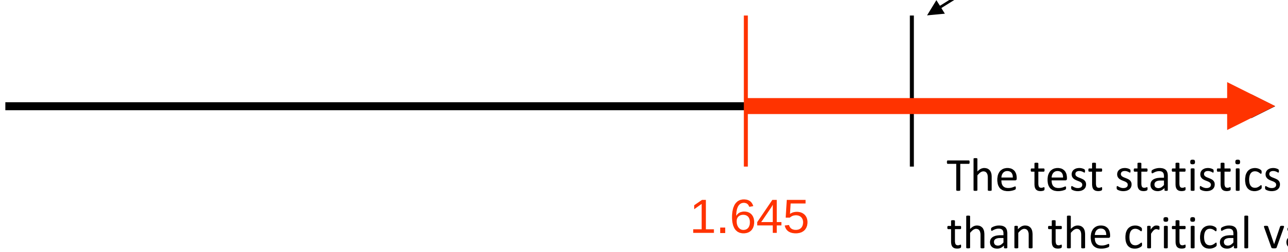
We can use the value of the test statistic comparing the critical point

$$\frac{\bar{x}-2.5}{\sqrt{\frac{S^2}{n}}} > 1.645$$

$$\frac{\bar{x}-2.5}{\sqrt{\frac{9.9119}{173}}} = 1.712886$$

$$\frac{\bar{x}-\mu_0}{\sqrt{\frac{s^2}{n}}}$$

The critical point



The test statistics is greater than the critical value !!!

the checklist

Stap	information	example
1	The hypotheses (the qualifying problem)	$H_0 : \mu = 2.5$ $H_1 : \mu = 2.9$
2	The distribution in the population & σ^2	$X \sim unknown$ σ^2 not known
3	sample size	$n = 173 > 30$
4	The distribution of the sample mean under the null hypothesis	$\bar{X} \sim N\left(2.5, \frac{S^2}{n}\right)$
5	The level of significance	$\alpha = 0.05$
6	The test statistic	$\boxed{\frac{\bar{X} - 2.5}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)}$
7	The distribution of the review greatness under the null hypothesis	
8	The critical point	1.645 N(0,1)

Examples by R

```
> xbar=2.91;s=sqrt(9.9119)
> n=173
> H0=2.5
> cc=qnorm(0.95)*(s/sqrt(n))+H0
> cc
[1] 2.893716
```

$$c = z \times \sqrt{\frac{S^2}{n}} + \mu_0$$

$$c = 1.645 \times \sqrt{\frac{9.9119}{173}} + 2.5 = 2.893751$$

We reject H_0

$$\bar{x} > c$$

$$\bar{x} = 2.91 > 2.893\ldots \Rightarrow reject H_0$$

By using critical point

```
> xbar=2.91;s=sqrt(9.9119);n=173;H0=2.5
> test.stat=(xbar-H0)/(s/sqrt(n))
> test.stat
[1] 1.712886
> crit.point=qnorm(0.95)
> crit.point
[1] 1.644854
```

$$\frac{\bar{x}-2.5}{\sqrt{\frac{9.9119}{173}}}=1.712886$$

We reject H_0

$$t=1.71>1.644..=t_{\alpha}$$

Test statistic

Critical value

Example: the air quality data

The estimators for the unknown parameter (μ,σ) in the population

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

Point estimators of the population parameters in R:

```
> ozone=airquality$Ozone
> meanozone=mean(ozone,na.rm=T)
> meanozone
[1] 42.12931

> varozone=var(ozone,na.rm=T)
> varozone
[1] 1088.201


> sqrt(varozone)
[1] 32.98788
```

Example: the air quality data

- Mean ozone in parts per billion from 1300 to 1500 hours at Roosevelt Island.

```
> z.test(ozone,sd=s,mu=40)
```

Testing the null hypothesis that mu =40

$$\frac{\bar{x}-\mu_0}{\sqrt{\frac{s^2}{n}}}$$

```
One Sample z-test

data:  ozone
z = 0.6952, n = 116.000, Std. Dev. = 32.988, Std. Dev. of the sample
mean = 3.063, p-value = 0.4869
alternative hypothesis: true mean is not equal to 40
95 percent confidence interval:
 36.12624 48.13238
sample estimates:
mean of ozone
     42.12931
```

```
>
sqrt(1088.201)
[1] 32.98789
```

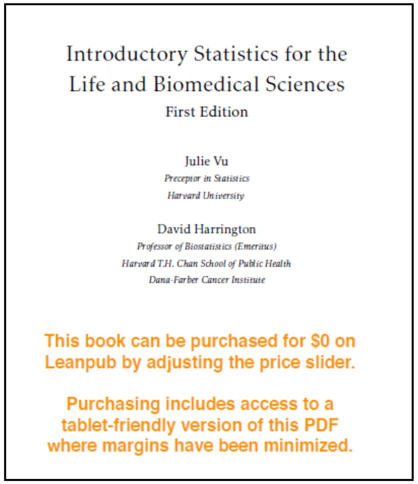
Case study 1b:

The `airquality` data: analysis of the average wind speed

Test of hypothesis about the population mean (two sided test)

Page 193

Part 7: Inference for one-sample means with the t distribution


Introductory Statistics for the Life and Biomedical Sciences
First Edition

Julie Vu
Preceptor in Statistics
Harvard University

David Harrington
Professor of Biostatistics (Emeritus)
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Section 5.1: **Single-sample inference with the *t*-distribution**

7.1: hypothesis testing using a t distribution

t-test for a population

- We assume that X~N(μ,σ²) & n is small
- For this test, we used the Student t distribution.

as $$X \sim N(\mu, \sigma^2)$$

than: $$\cancel{\bar{X} \sim N(\mu, \frac{S^2}{n})}$$

and $$T_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim t(n-1)$$

X has a normal distribution with unknown μ and σ². n is small

$$E(S^2) = \sigma^2$$

Example

- A researcher would like the following hypotheses :

$$H_0 : \mu = 21$$
$$H_1 : \mu = 22$$

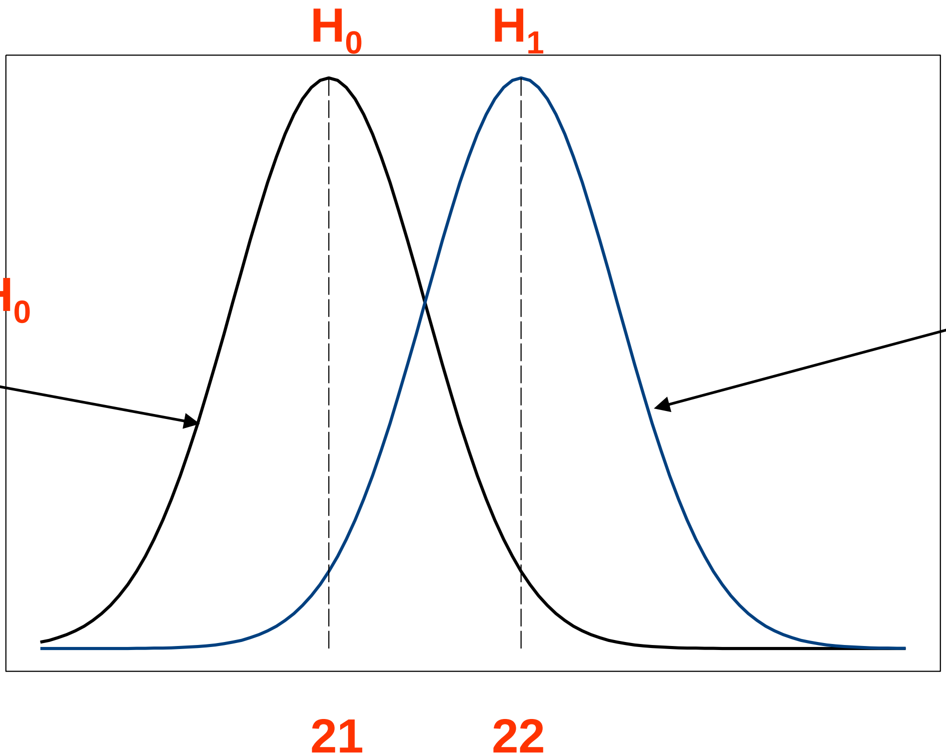
- We assume that

$$X \sim N(\mu, \sigma^2)$$

The distribution of the population

$$X \sim N(21, \sigma^2) \qquad under \quad H_0$$
$$X \sim N(22, \sigma^2) \qquad under \quad H_1$$

The distribution of population among H_0



The distribution of population among H_1

The sample

- To test the hypotheses, we draw a sample of size 9 (n = 9) from the population.
- X has a normal distribution with unknown μ and σ^2.

  n is small

$$\left.\begin{array}{l} X_i \sim N(\mu, \sigma^2) \\ n = 9 \qquad (small) \\ \sigma^2 : unknown \end{array}\right\} \quad \boxed{\frac{\bar{X} - 21}{\sqrt{\frac{S^2}{n}}} \sim t(n-1)}$$

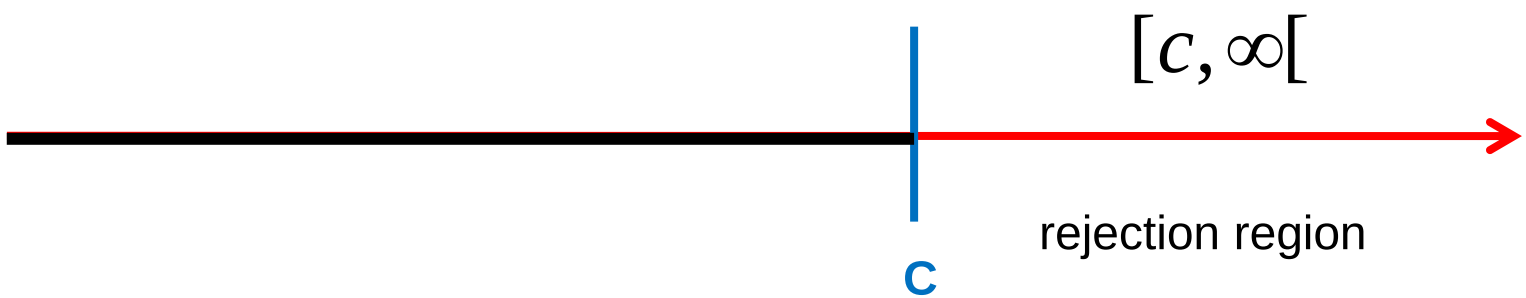
The distribution of the test statistic population under **H_0**

The rejection region

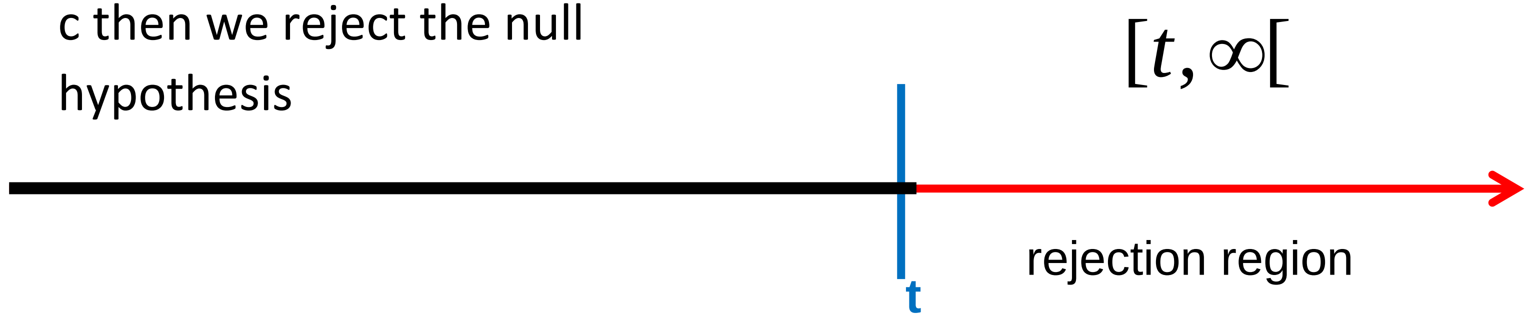
$$H_0 : \mu = 21$$
$$H_1 : \mu = 22$$

$$\mu_0 < \mu_1$$

when the value of $\overline{X}$ is greater than c then we reject the null hypothesis



when the value of T is larger than c then we reject the null hypothesis



The choice of c

- We choose c so that Type I error 0.05.

$$\alpha = 0.05$$

$$\bar{X} > c \Rightarrow H_0 \quad reject \quad \Longleftrightarrow \quad T > t \Rightarrow H_0 \quad reject$$

$$P(\bar{X} > c) = P(T > t) = 0.05$$

if the null hypothesis is correct

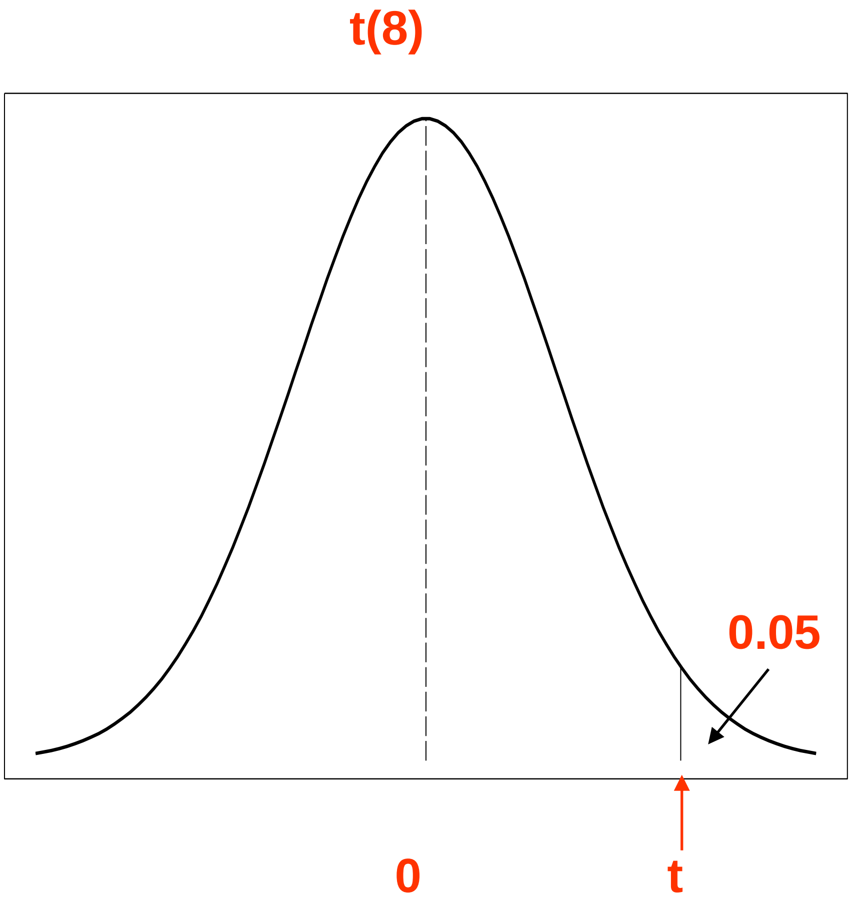
The choice of c

Determine c so that Type I error =5%

$$P(\bar{X} > c) = 0.05$$

$$\boxed{P(\bar{X} > c) = P\left(\frac{\bar{X} - 21}{\sqrt{\frac{S^2}{n}}} > \frac{c - 21}{\sqrt{\frac{S^2}{n}}}\right) = 0.05} \Rightarrow \boxed{P\left(T > \frac{c - 21}{\sqrt{\frac{S^2}{n}}}\right) = 0.05}$$

The critical point



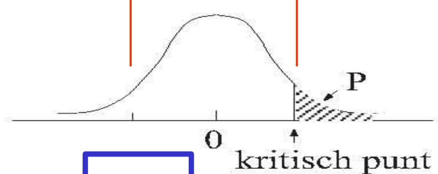
$$P(T > t) = \alpha$$

The distribution of the test statistic under H_0

$$\boxed{\frac{\bar{X} - 21}{\sqrt{\frac{S^2}{n}}} \sim t(n-1)}$$

Student's t-distribution and critical point

Tabel 4 : Kritische punten student t verdeling



P / v.g.	.25	.10	.05	.025	.010	.005	.001
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	.816	1.886	2.920	4.303	6.965	9.925	22.326
3	.765	1.638	2.353	3.182	4.541	5.841	10.213
4	.741	1.533	2.132	2.776	3.747	4.604	7.173
5	.727	1.476	2.015	2.571	3.365	4.032	5.893
6	.718	1.440	1.943	2.447	3.143	3.707	5.208
7	.711	1.415	1.895	2.365	2.998	3.499	4.785
8	.706	1.397	1.860	2.306	2.896	3.355	4.501
9	.703	1.383	1.833	2.262	2.821	3.250	4.297
10	.700	1.372	1.812	2.228	2.764	3.169	4.144
11	.697	1.363	1.796	2.201	2.718	3.106	4.025
12	.695	1.356	1.782	2.179	2.681	3.055	3.930
13	.694	1.350	1.771	2.160	2.650	3.012	3.852
14	.692	1.345	1.761	2.145	2.624	2.977	3.787
15	.691	1.341	1.753	2.131	2.602	2.947	3.733
16	.690	1.337	1.746	2.120	2.583	2.921	3.686
17	.689	1.333	1.740	2.110	2.567	2.898	3.646
18	.688	1.330	1.734	2.101	2.552	2.878	3.610
19	.688	1.328	1.729	2.093	2.539	2.861	3.579
20	.687	1.325	1.725	2.086	2.528	2.845	3.552
21	.686	1.323	1.721	2.080	2.518	2.831	3.527
22	.686	1.321	1.717	2.074	2.508	2.819	3.505
23	.685	1.319	1.714	2.069	2.500	2.807	3.485
24	.685	1.318	1.711	2.064	2.492	2.797	3.467
25	.684	1.316	1.708	2.060	2.485	2.787	3.450
26	.684	1.315	1.706	2.056	2.479	2.779	3.435
27	.684	1.314	1.703	2.052	2.473	2.771	3.421
28	.683	1.313	1.701	2.048	2.467	2.763	3.408
29	.683	1.311	1.699	2.045	2.462	2.756	3.396
30	.683	1.310	1.697	2.042	2.457	2.750	3.385
40	.681	1.303	1.684	2.021	2.423	2.704	3.307
60	.679	1.296	1.671	2.000	2.390	2.660	3.232
120	.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	.674	1.282	1.645	1.960	2.326	2.576	3.090

$$n = 9\,(small)$$

$$df. = 8$$

$$\alpha = 0.05$$

$$P(T > t) = 0.05$$

$$P(T > 1.86) = 0.05$$

The sample

subject	X_i
1	22
2	19
3	17
4	26
5	21
6	20
7	29
8	27
9	22

n=9

$$\left.\begin{aligned}\bar{x} &= \frac{1}{9}\sum_{i=1}^{9} x_i = 22.556 \\ s^2 &= \frac{1}{9-1}\sum_{i=1}^{9}\left(x_i - \bar{x}\right) = 3.972^2\end{aligned}\right\}$$

The estimators for the unknown parameters (μ and σ ²) in the population

The rejection region & statistic

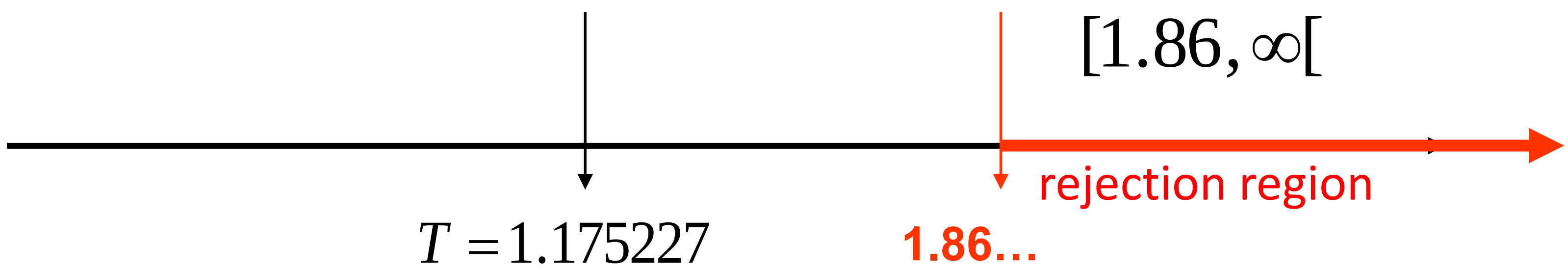
$$s^2 = 3.972$$

$$\bar{x} = 22.556$$

$$n = 9$$

$$\frac{\bar{x} - 21}{\sqrt{\frac{3.972^2}{9}}} = 1.175227$$

$T < t \Rightarrow$ We do not reject H_0



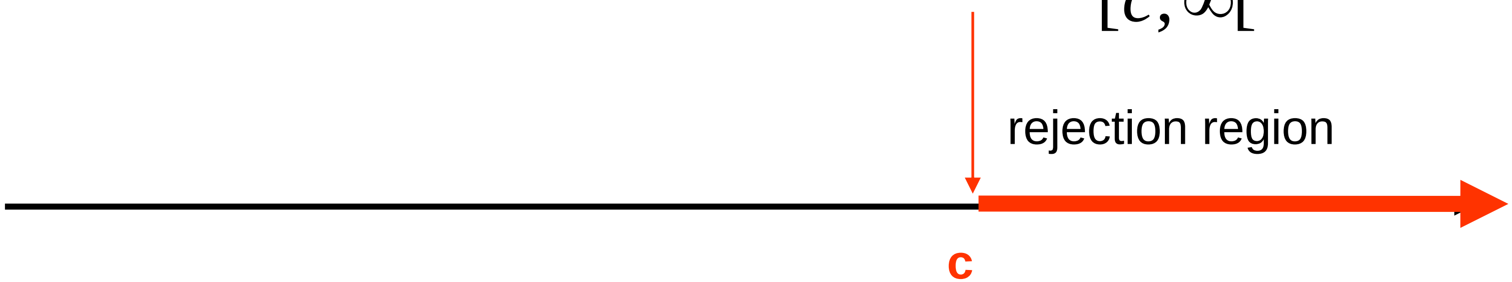
The rejection region

$$P\left(T > \frac{c-21}{\sqrt{\frac{S^2}{n}}}\right) = 0.05$$

$$P(T > 1.86) = 0.05$$

$$\frac{c-21}{\sqrt{\frac{S^2}{n}}} = 1.86 \Rightarrow c = 1.86 \times \sqrt{\frac{S^2}{n}} + 21$$

$$[c, \infty[$$



$$c = t \times \sqrt{\frac{S^2}{n}} + \mu_0$$

The rejection region

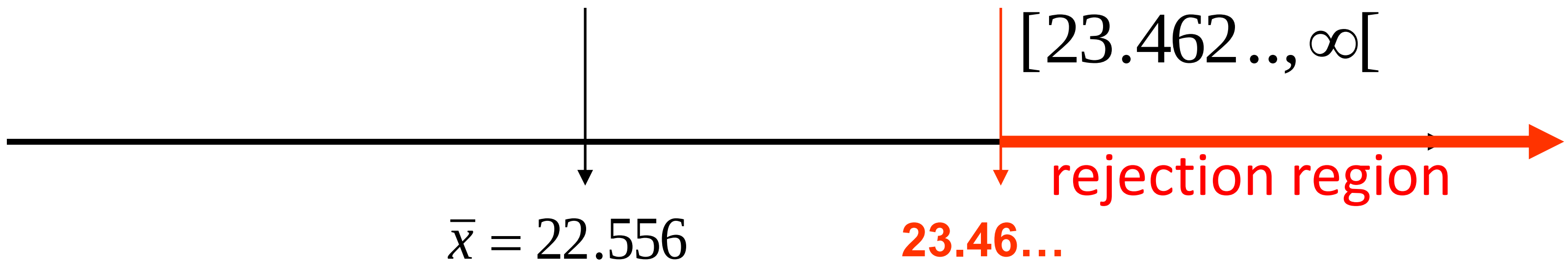
$s^2 = 3.972$

$\bar{x} = 22.556$

$n = 9$

$$c = 1.86 \times \sqrt{\frac{3.972^2}{9}} + 21 = 23.46264$$

$\bar{x} < c \Rightarrow$ We do not reject H_0



the checklist

Step	information	example
1	The hypotheses (the qualifying problem)	$H_0 : \mu = 21$ $H_1 : \mu = 22$
2	The distribution in the population and σ2	$X \sim N(\mu, \sigma^2)$ σ^2 not known
3	sample size	$n = 9 < 30$
4	The distribution of the sample mean	Unknown
5	The level of significance	$\alpha = 0.05$
6	The test statistic	$\boxed{\dfrac{\overline{X} - 21}{\sqrt{\dfrac{S^2}{n}}} \sim t(8)}$
7	The distribution of the test statistic	
8	The critical point (or points)	1.86 t(8)

R code

```
> x=c(22,19,17,26,21,20,29,27,22)
> xbar=mean(x)
> mu = 21
>  s = sd(x)
>  n = length(x)
>  t = (xbar-mu)/(s/sqrt(n))
>  t                      # test statistic
[1] 1.174854
> crit.val = qt(1-alpha, n-1, lower.tail = TRUE)
> crit.val     # critical value
[1] 1.859548
```

- The test statistic 1.174854 is greater than the critical value of 1.859548.
- Hence, at 0.05 significance level, we can reject the null hypothesis.

Example:

The women data:

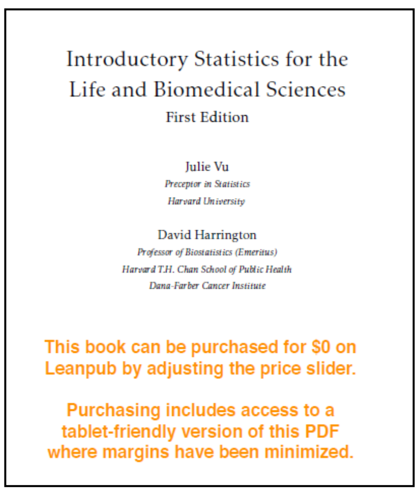
heights and weights for American women aged 30–39.

```
> womenheight=women$height
> t.test(womenheight,mu=60,conf.level=0.90)

        One Sample t-test

data:  womenheight
t = 4.3301, df = 14, p-value = 0.000692
alternative hypothesis: true mean is not equal to 60
90 percent confidence interval:
 62.96621 67.03379
sample estimates:
mean of x
       65
```

$$\boxed{\begin{array}{l} H_0: \mu = 60 \\ H_1: \mu \neq 60 \end{array}}$$

Testing a hypothesis about a Population parameter

7.2: One sided and two-sided testing problems



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Section 4.3.1 The Formal Approach to Hypothesis Testing

The hypothesis and the alternative hypothesis

- In the previous example, we tested the hypothesis that the mean of a normal distribution with unknown variance equal to a certain value (21).
- As an alternative hypothesis we mean that it was equal to another specified
- value (22).

$$H_0 : \mu = 21$$
$$H_1 : \mu = 22$$

- In practice, the researcher usually do not know the exact details of the alternative hypothesis.

Case (a)

The average under H_1 is smaller than the average under H_0

$$\boxed{\begin{aligned} H_0 &: \mu = \mu_{H_0} \\ H_1 &: \mu < \mu_{H_0} \end{aligned}}$$

null hypothesis

alternative hypothesis

One sided test problem

Case (b)

The average under H_1 is greater than the average under H_0 :

$$\boxed{\begin{aligned} H_0 &: \mu = \mu_{H_0} \\ H_1 &: \mu > \mu_{H_0} \end{aligned}}$$

null hypothesis

alternative hypothesis

One sided test problem

Case (c)

The average under H_0 is not equal to the mean under H_1 :

$$\boxed{\begin{aligned} H_0 &: \mu = \mu_{H_0} \\ H_1 &: \mu \neq \mu_{H_0} \end{aligned}}$$

null hypothesis

alternative hypothesis

two sided test problem

Example (case a)

$$\boxed{\begin{aligned} H_0 &: \mu = \mu_{H_0} \\ H_1 &: \mu < \mu_{H_0} \end{aligned}}$$

One sided test

Example: one-tailed test

- A gynecologist says that girls at birth, averaging less than 51 cm.
- His colleague Judge reproach him that his claim is based on a prejudice, and that the average length is 51 cm indeed.
- They draw a sample of 100 girls.
- In the sample:

$$\left.\begin{array}{l} \bar{x} = 50.8 \quad \& \quad s^2 = 1.6 \\ n=100 \end{array}\right\}$$

The variance σ^2 is unknown but large n.

$$\boxed{\frac{\bar{X}-\mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)}$$

The testing problem

The choice of H_1 reflects here the assertion of the first gynecologist

$$H_0 : \mu = 51$$ null hypothesis

$$H_1 : \mu < 51$$ alternative hypothesis

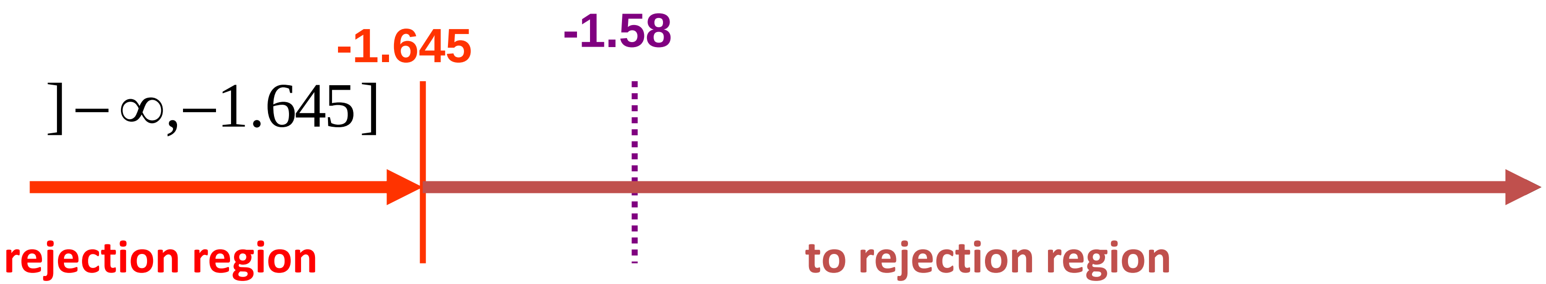
One-sided test

The test statistic

- We now supplement the sample values and find:

$$\frac{\bar{x}-51}{\sqrt{\frac{s^2}{n}}}=\frac{50.8-51}{\sqrt{\frac{1.6}{100}}}=-1.58$$

- Conclusion: at significance level of 5%, the length of girls at birth 51 cm.



The checklist

Stap	information	example
1	The hypotheses (the qualifying problem)	One-sided test
2	The distribution in the population & σ^2	
3	sample size	
4	The distribution of the sample mean under H_0	
5	The level of significance	
6	The test statistic	$\boxed{\dfrac{\bar{X}-51}{\sqrt{\dfrac{S^2}{n}}} \sim ?}$
7	The distribution of the review greatness	
8	The critical point (or points)	

R code

```
> xbar=50.8;s=sqrt(1.6);n=100;H0=51
> test.statgy=(xbar-H0)/(s/sqrt(n))
> test.statgy
[1] -1.581139
> crit.point1=qnorm(0.95,lower.tail=TRUE)#p=0.05 one tailed
> -crit.point1
[1] -1.644854
```

Example (case c)

$$\boxed{\begin{aligned} H_0 &: \mu = \mu_{H_0} \\ H_1 &: \mu \neq \mu_{H_0} \end{aligned}}$$

two-tailed test

Example: two-tailed test

- At a certain university one takes many years an intelligence-off normally distributed with mean score results yields 115 (115 = μ under H_0).
- An administrator wants now for the new class to test the hypothesis that the mean is the same as in previous years.
- He takes a sample of size 50, and: mean 118 and variance 98.

n $\bar{X}$ S^2

The testing problem

The choice of H_1 is to be determined by the consideration that the administrator has no idea whether the new crop is better or worse than the previous

$$H_0 : \mu = 115$$ Null hypothesis

$$H_1 : \mu \neq 115$$ Alternative hypothesis

two-tailed test

The rejection region (sided test)

$$\frac{\bar{x}-115}{\sqrt{\frac{s^2}{n}}} = \frac{118-115}{\sqrt{\frac{98}{50}}} = 2.14$$

The test statistic

2.14 > 1.96 ⇨ the administrator rejects H_0 at significance level 0.05.

The checklist

Stap	information	Example
1	The hypotheses (the testing problem)	One-sided test
2	The distribution in the population& σ^2	
3	sample size	
4	The distribution of the sample mean under H_0	
5	The level of significance	
6	The test statistic	$\boxed{\dfrac{\bar{X}-115}{\sqrt{\dfrac{S^2}{n}}} \sim ?}$
7	The distribution of the test statistic	
8	The critical point (or points)	

R code

```
> xbar=118;s=sqrt(98);n=50;H0=115
> test.statcrop=(xbar-H0)/(s/sqrt(n))
> test.statcrop
[1] 2.142857
> alpha = 0.05
> crit.pointcrop = qnorm(1-alpha/2)
> crit.pointcrop
[1] 1.959964
> -crit.pointcrop
[1] -1.959964
> c(-crit.pointcrop,crit.pointcrop)
[1] -1.959964  1.959964
```

Population, n & σ^2

- from above examples show that we are always three things to keep in mind:

1. which assumption we make about the distribution of the population?
2. is the variance σ^2 is known or should they be estimated using S^2?
3. how big is the sample (which is the value of n)?

1: n large

For n large

$$\boxed{\frac{\bar{X}-\mu_{H_0}}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)}$$

If σ^2 is known

$$\boxed{\frac{\bar{X}-\mu_{H_0}}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)}$$

If σ^2 is unknown

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

2: n small & normal distribution

2: n Small & normal distribution

$$\boxed{\frac{\bar{X}-\mu_{H_0}}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)}$$

If σ^2 is known

$$\boxed{\frac{\bar{X}-\mu_{H_0}}{\sqrt{\frac{S^2}{n}}} \sim t_{(n-1)}}$$

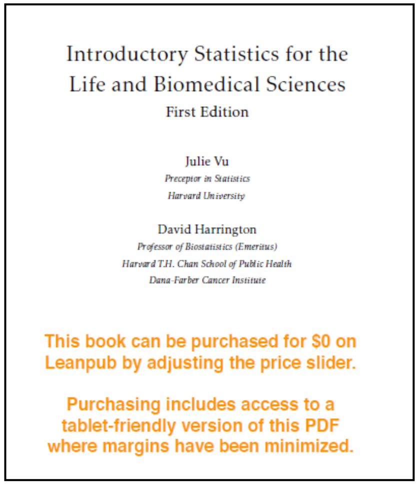
If σ^2 is unknown

The choice of the setting

n	σ^2	statistics	distribution of the population	Distribution for statistical
large	known	$\boxed{\dfrac{\overline{X}-\mu_{H_0}}{\sqrt{\dfrac{\sigma^2}{n}}}}$	normal distribution or not known	Z(0,1)
large	not known	$\boxed{\dfrac{\overline{X}-\mu_{H_0}}{\sqrt{\dfrac{S^2}{n}}}}$	normal distribution or not known	Z(0,1)
small	known	$\boxed{\dfrac{\overline{X}-\mu_{H_0}}{\sqrt{\dfrac{\sigma^2}{n}}}}$	normal distribution	Z(0,1)
small	not known	$\boxed{\dfrac{\overline{X}-\mu_{H_0}}{\sqrt{\dfrac{S^2}{n}}}}$	normal distribution	t(n-1)
small	not known		normal distribution	Not for this course

$$\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i \qquad S^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$$

7.3 The P-value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Section 4.3.1: The Formal Approach to Hypothesis Testing

The significance level and the critical point

- In the examples on hypothesis testing, we have until now always pre specified significance level α (usually α = 0.05).
- We determine the rejection region so:

$$P_{H_0}\left(\bar{x} \in [c, \infty[\right) = \alpha$$

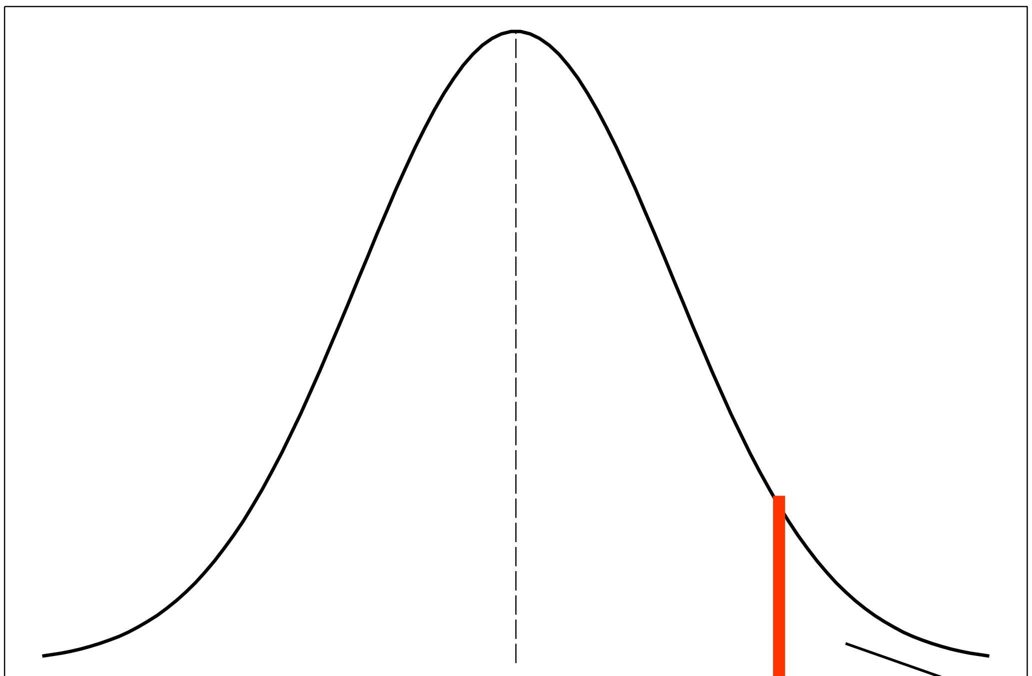
The level of significance and the p-value

- The relationship between the p-value and the level of significance is clear:

H_0 is rejected on
significance level α if and only if
the p-value $<\alpha$

Right-sided test $\mu_0 < \mu_1$

$$[c, \infty[$$



The distribution of the test statistic under H_0

observed value of $\dfrac{\bar{x} - \mu_{H_0}}{\sqrt{\dfrac{\sigma^2}{n}}}$ or $\dfrac{\bar{x} - \mu_{H_0}}{\sqrt{\dfrac{S^2}{n}}}$

p-value

Example 1 (p-value): right-tailed test

population

$$X_i \sim N(\mu, \sigma^2)$$
$$n = 9 \qquad (small)$$
$$\sigma^2 : unknown$$

sample

$$\bar{x} = 22.556$$
$$s^2 = 3.972^2$$

$$H_0 : \mu = 21$$
$$H_1 : \mu > 21$$

$$\frac{\bar{x} - 21}{\sqrt{\frac{3.972^2}{9}}} = 1.175227$$

We look at student t distribution with 8 df

$$p - value = P(T > 1.175227) = 0.1481026$$

P-value = 0.1481026 > 0.05, we can not reject H_0.

R code

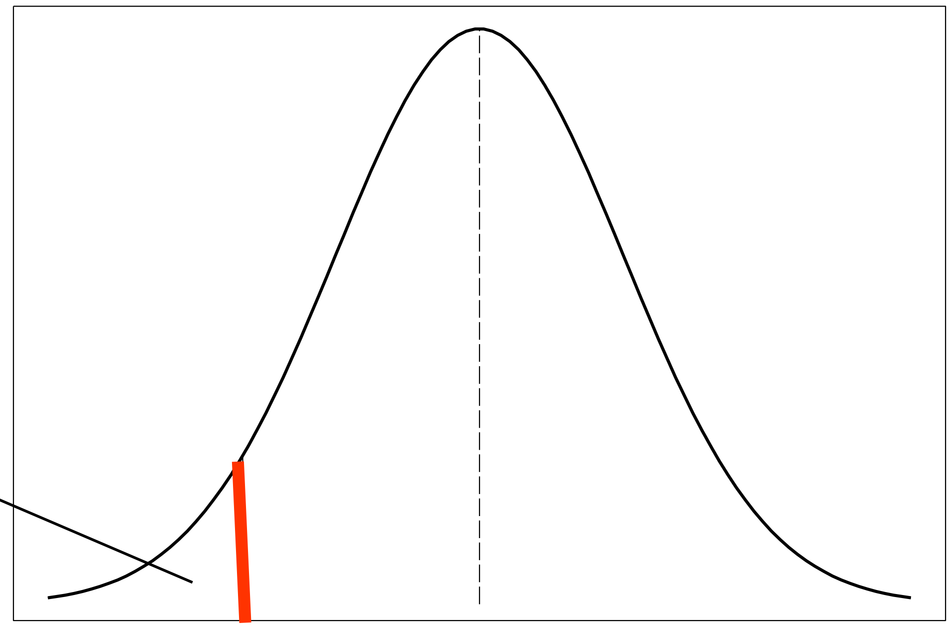
```
> x=c(22,19,17,26,21,20,29,27,22)
> xbar=mean(x)
> mu = 21
>  s = sd(x)
>  n = length(x)
>  t = (xbar-mu)/(s/sqrt(n))
> alpha = .05
>  pval = pt(t,df=n-1, lower.tail=FALSE)
>  pval
[1] 0.1369174
```

P value

Left-sided test $\mu_0 > \mu_1$

$$]-\infty,-c]$$

The p-value



observed value of $\dfrac{\vec{x}-\mu_{H_0}}{\sqrt{\dfrac{\sigma^2}{n}}}$ of $\dfrac{\vec{x}-\mu_{H_0}}{\sqrt{\dfrac{S^2}{n}}}$

Example 2 (p-value): left-tailed test

$$H_0 : \mu = 51$$
$$H_1 : \mu < 51$$

$$\boxed{\frac{50.8 - 51}{\sqrt{\frac{1.6}{100}}} = -1.58}$$

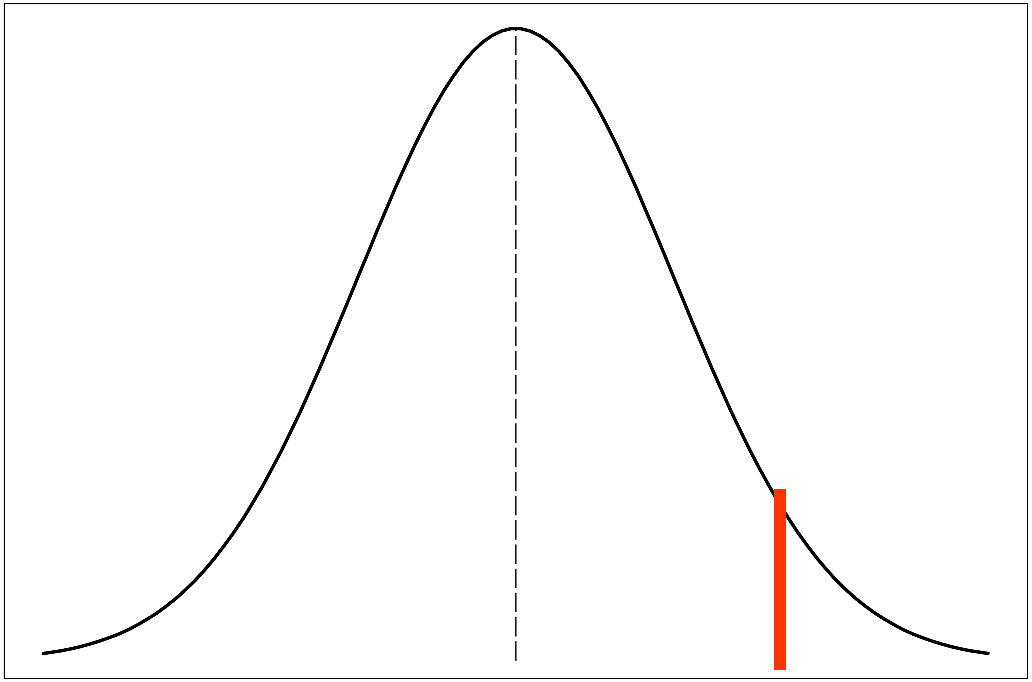
$$p - value = P(Z < -1.58) = 0.0571$$

for each significance level > 0.0571 H_0 will be rejected

R code

```
> xbar=50.8;s=sqrt(1.6);n=100;H0=51
> test.statgy=(xbar-H0)/(s/sqrt(n))
> test.statgy
[1] -1.581139
> pval1 = pt(test.statgy,df=n-1,
+ lower.tail=TRUE)
> pval1
[1] 0.05851802
```

two-tailed test



$$]-\infty,-c]\cup[c,\infty[$$

observed value of $\dfrac{\bar{x}-\mu_{H_0}}{\sqrt{\dfrac{\sigma^2}{n}}}$ of $\dfrac{\bar{x}-\mu_{H_0}}{\sqrt{\dfrac{S^2}{n}}}$

$$p-value = 2\times P\left(Z > \frac{\bar{x}-\mu_{H_0}}{\sqrt{\frac{s^2}{n}}} \right)$$

Example 3 (p-value)

$$H_0 : \mu = 115$$
$$H_1 : \mu \neq 115$$

$$\boxed{\frac{98-115}{\sqrt{\frac{98}{50}}} = 2.14}$$

$$p - value = 2 \times P\left(Z > 2.14\right) = 2 \times [1 - \Phi(2.14)] = 0.0324$$

for each significance level> 0.0324 H_0 will be rejected

R code

```
> bar=118;s=sqrt(98);n=50;H0=115
> test.statcrop=(xbar-H0)/(s/sqrt(n))
> test.statcrop
[1] 2.142857
>  2*(1-pnorm(test.statcrop))
[1] 0.03212457
```

The level of significance and the p-value

- Statistical computer packages give as output a hypothesis test the p-value.
- A generally accepted criterion (e. g in scientific publications) is as follows

1. If the P-value <0.05, then H0 is rejected, and then the results are significant.
2. if the p-value> 0.05, then H0 is not rejected, and then the results are not significant.

Case study 3:

The The NHANES dataset: number of sleep hours per night

Test of hypothesis about the population mean (one sided test)

Page 203

Case studies

Examples from the online book

Case study 1a:

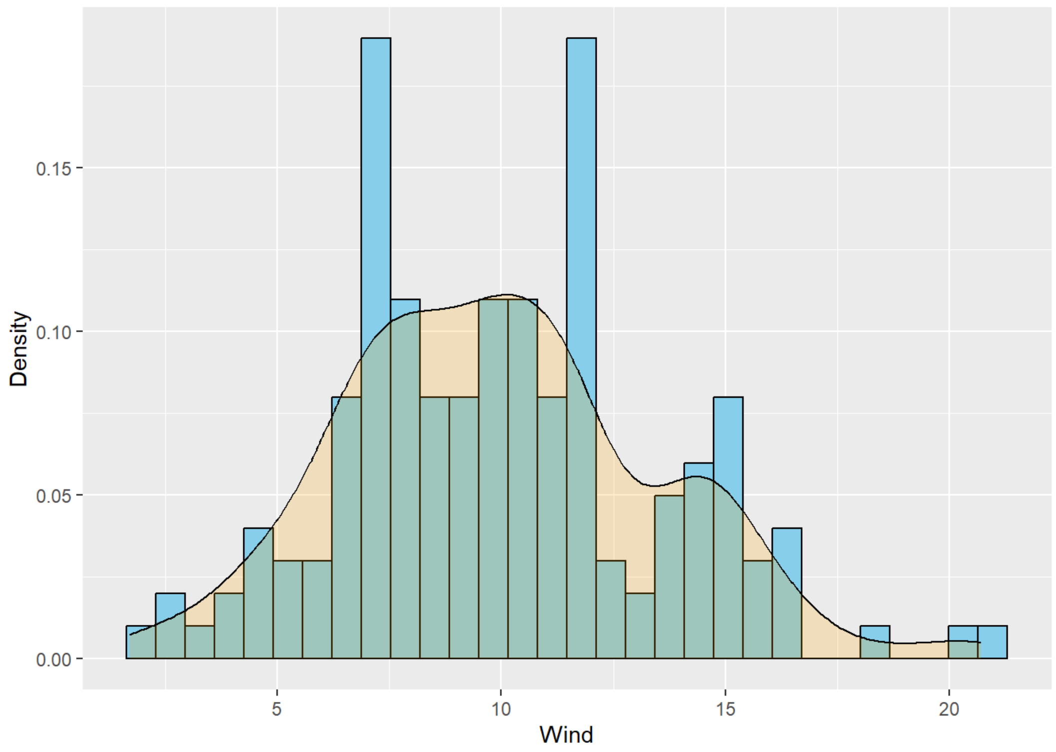
The `airquality` data: analysis of the average wind speed

Confidence interval for the population mean

The average wind speed per day

- The `airquality` dataset gives information about 153 daily air quality measurements in New York, May to September 1973.
- The variable Wind is the average wind speed in miles per hour at 0700 and 1000 hours at LaGuardia Airport.

The average wind speed per day



$$n = 153$$

$$\bar{x} = 9.55$$
$$s = 3.52$$

Distribution of the test statistic

Case 2

If $X \sim F$

Then: $\bar{X} \sim N(\mu, \frac{S^2}{n})$

and $T_X = \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)$

3. X has an unknown distribution, but we have a large sample (n> 30)

$E(X) = \mu$

$Var(X) = \sigma^2$

The same as case 1 but we replace σ^2 by S^2.

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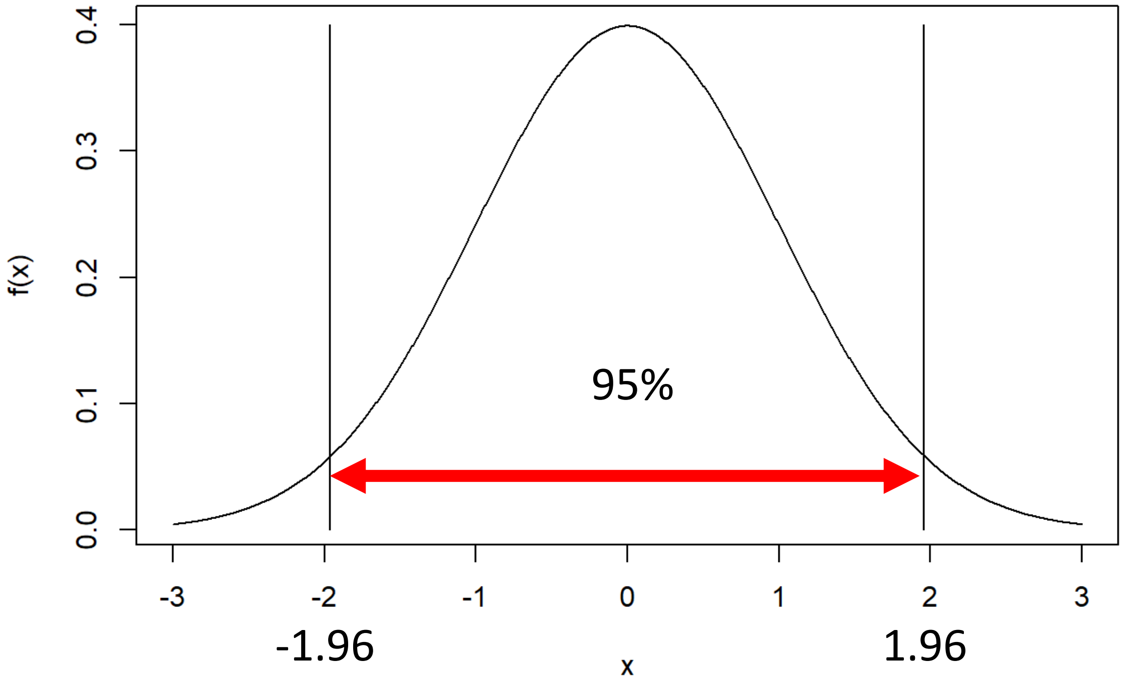
Our example:

n=153

Use N(0,1) to choose the critical values for the Upper and Lower limits.

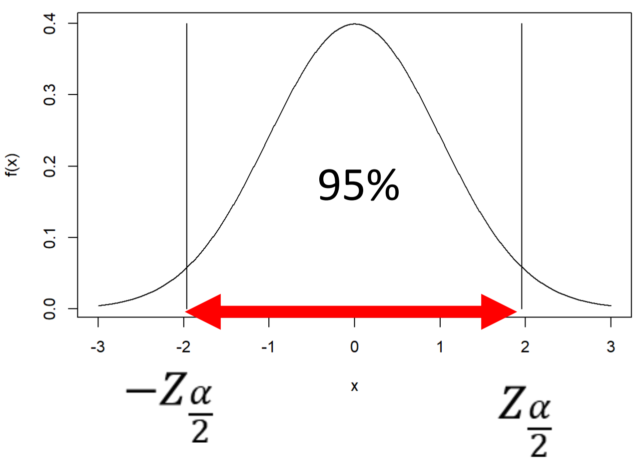
Critical values from N(0,1)

- For α=0.025, we are looking for two values for which the probability to be between these value is 0.95=(1-2α).

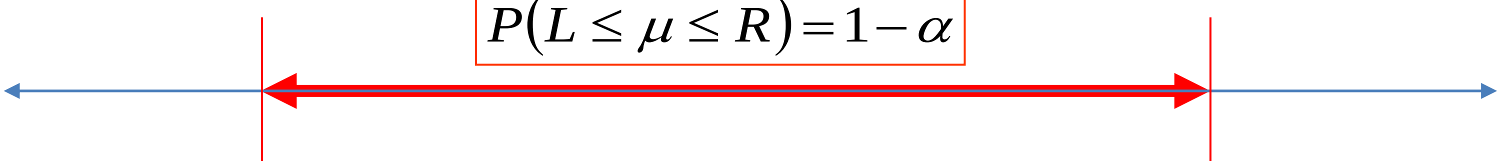


Lower (L) and upper (R) limits

- For a given value of α, we are looking for two values for which the probability to be between these value is 0.95=(1-α).



$$\boxed{P(L \le \mu \le R) = 1 - \alpha}$$



$\bar{x} - Z_{\frac{\alpha}{2}} \times SE$=L

$\bar{x} + Z_{\frac{\alpha}{2}} \times SE$=R

The average wind speed per day

- A 95% confidence interval:

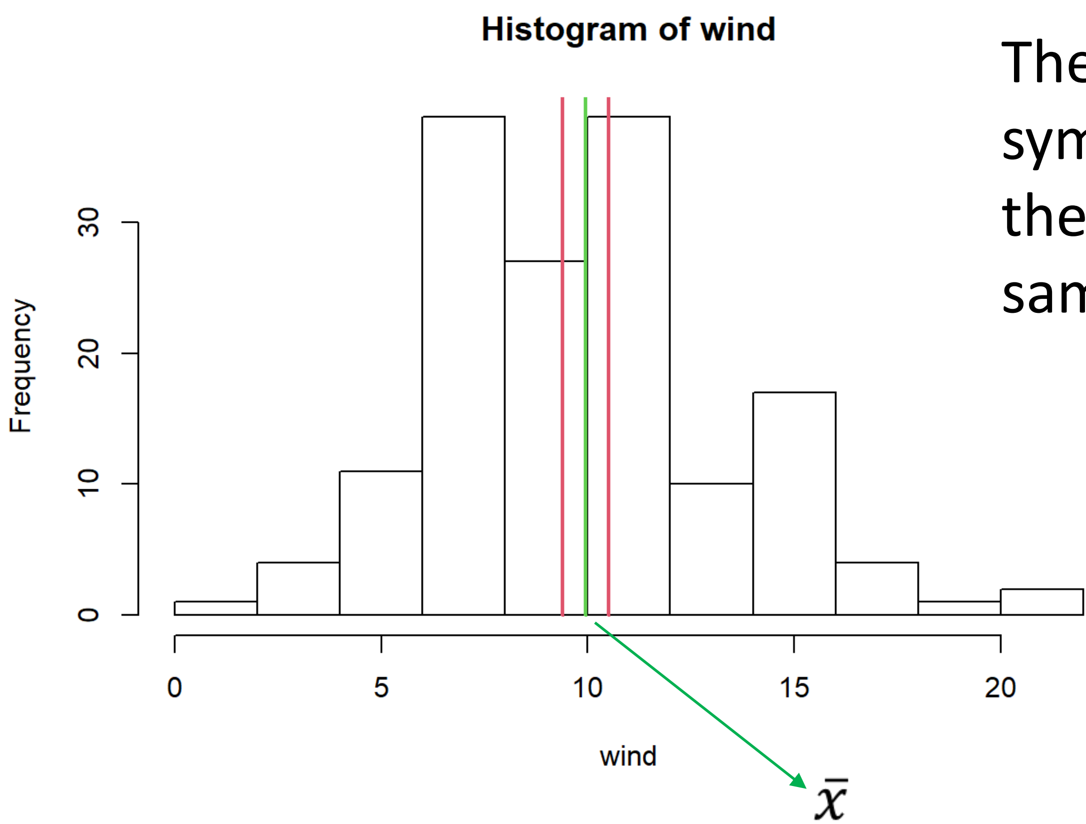
$$(\bar{x} - m, \bar{x} + m)$$

$$m = Z_\alpha \times SE = Z_\alpha \times \frac{s}{\sqrt{n}} = 1.96 \times \frac{3.52}{\sqrt{153}} = 0.5582$$

SE of the sample mean.

$$(\bar{x} - m, \bar{x} + m) = (9.391, 10.508)$$

95% confidence interval



The interval is symmetric around the value of the sample mean.

$9.391 = \bar{x} - Z_{\frac{\alpha}{2}} \times SE = L$

$$\boxed{P(L \le \mu \le R) = 1 - \alpha}$$

$\bar{x} + Z_{\frac{\alpha}{2}} \times SE = R = 10.508$

Case study 1b:

The `airquality` data: analysis of the average wind speed

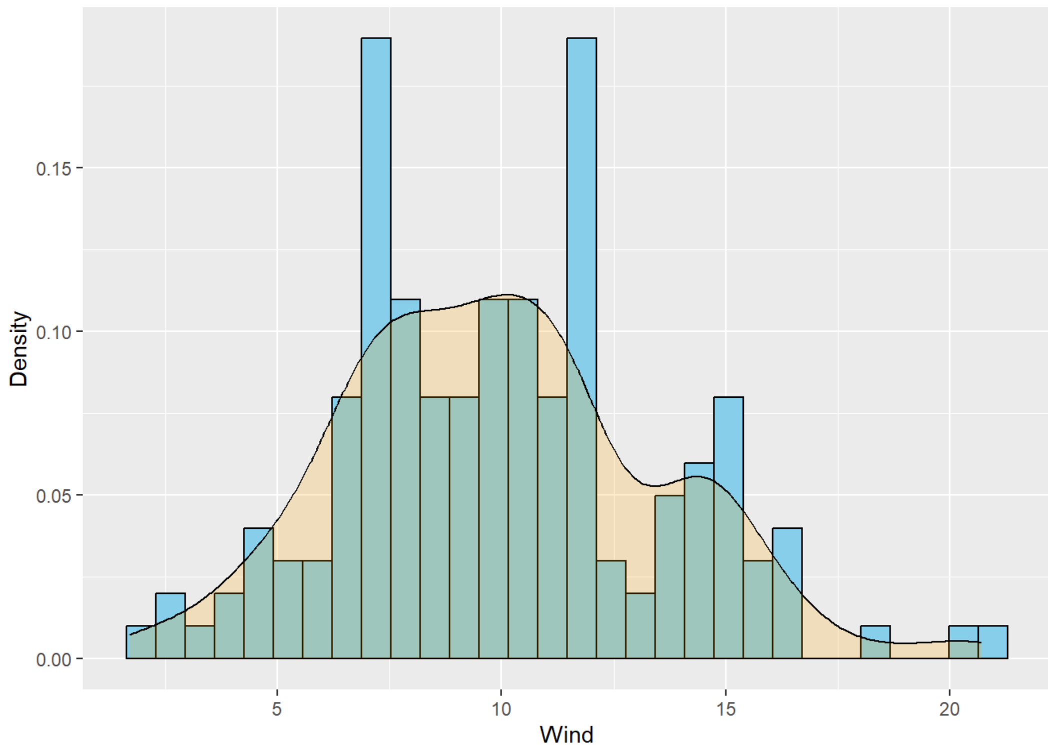
Test of hypothesis about the population mean (two sided test)

The average wind speed per day

$$H_0: \mu = 9$$
$$H_A: \mu \neq 9$$

- We test the null hypothesis that the population mean is equal to 9.
- The alternative hypothesis: the mean is not equal to 9 (we do not specify a value).
- For the analysis, we assume that the variance in the population is known.

The average wind speed per day



- Sample size:

  $n = 153$
- Point estimators in the sample:

  $\bar{x} = 9.55$

  $s = 3.52$
- We assume that in the population:

  $\sigma = 3.52$

Distribution of the test statistic

Case 2

If $X \sim F$

Then: $\bar{X} \sim N(\mu, \frac{S^2}{n})$

and $T_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)$

3. X has an unknown distribution, but we have a large sample (n> 30)

$E(X) = \mu$

$Var(X) = \sigma^2$

The same as case 1 but we replace σ^2 by S^2.

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Our example:

$n = 153$

⇨ Use N(0,1) to choose the critical value.

Test statistic

n=153, under H_0 the distribution of the test statistic:

$$t=\frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}\sim N(0,1)$$

Test statistic:

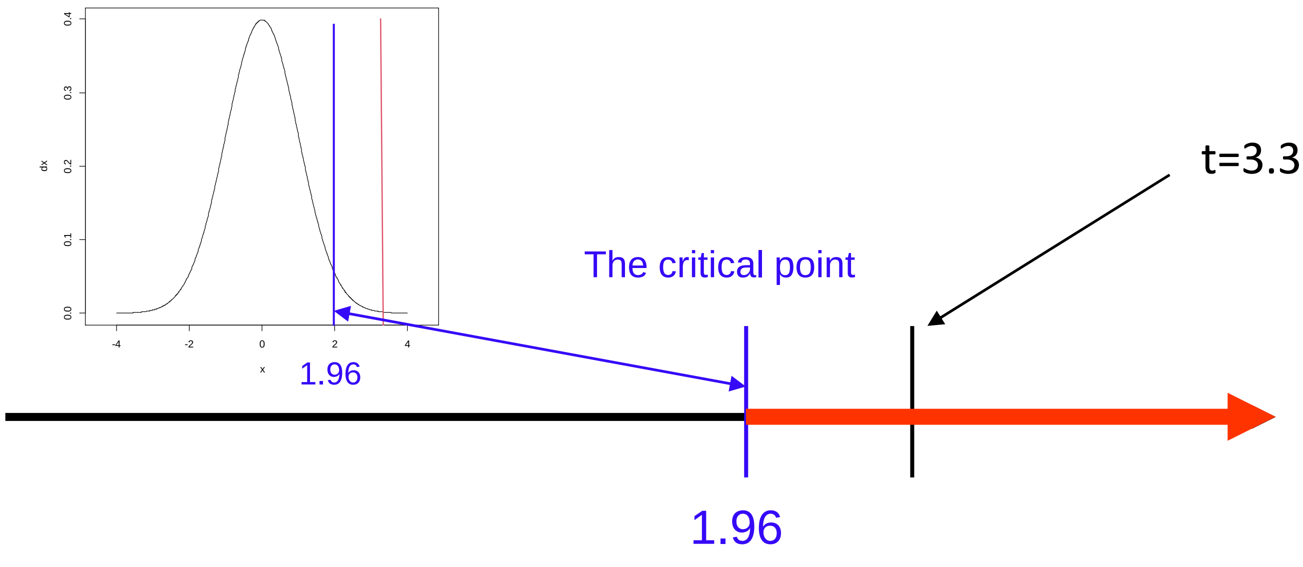
$$t=\frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}=\frac{9.95-9}{\frac{3.523}{\sqrt{153}}}=3.3619$$

The critical point and the test statistic

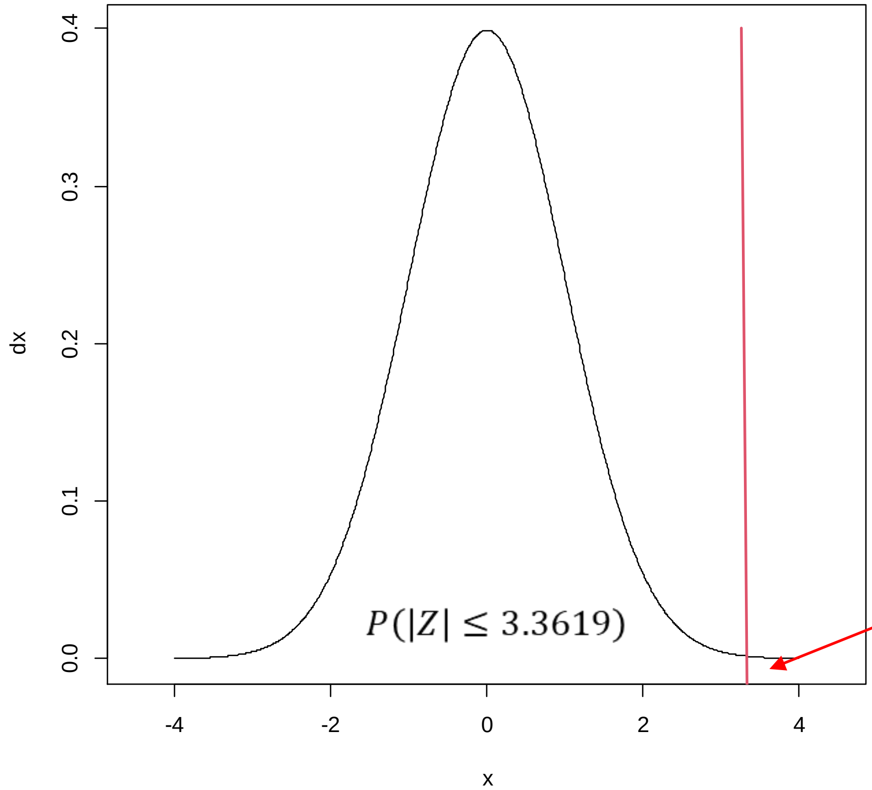
We can use the value of the test statistic comparing the critical point.

For two sided test and α=0.05, $Z_{0.975}$=1.96.

We reject H_0 : 3.3619 > 1.96.



p-value



$$2\times\left(1 - P(|Z| \leq 3.3619)\right)$$

$$2 \times 0.00038 = 0.000775 < 0.05 = \alpha$$

- For α=0.05, p-value < 0.05.
- We reject the null hypothesis and conclude that the mean in the population is not 9.

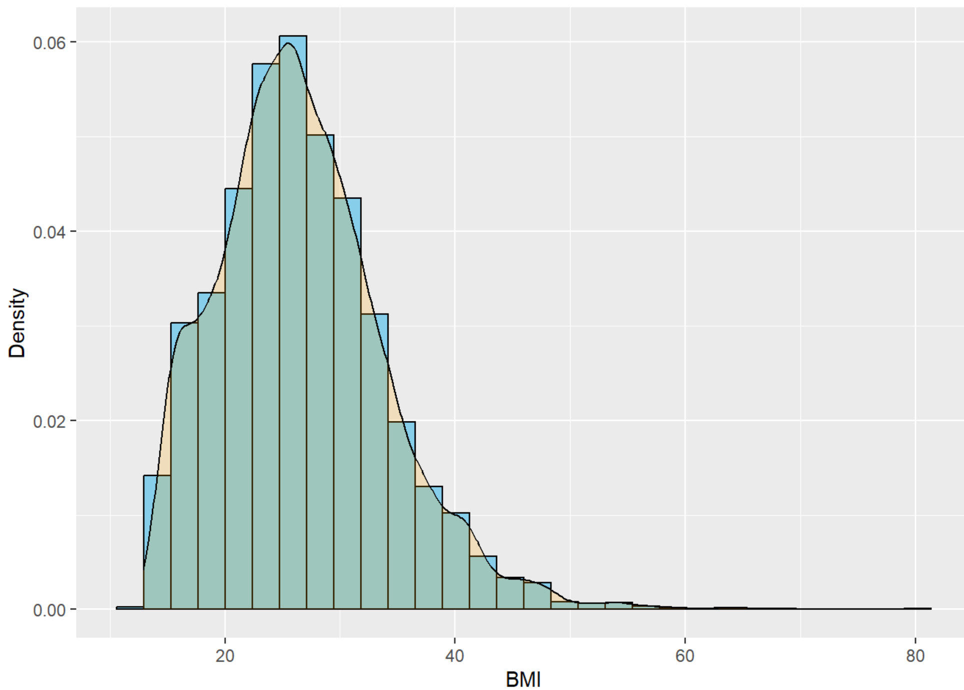
$P(|Z| > 3.3619)$

Case study 2:
The The NHANES dataset: BMI

The The NHANES dataset: BMI

- The NHANES dataset consists of data from the US National Health and Nutrition Examination Study.
- Information about 76 variables is available for 10000 individuals included in the study.
- The 10000 individuals are considered as the population.
- Some individuals excluded due to missing values.
- In this part we focus on the BMI (the R object BMI).

The The NHANES dataset: BMI



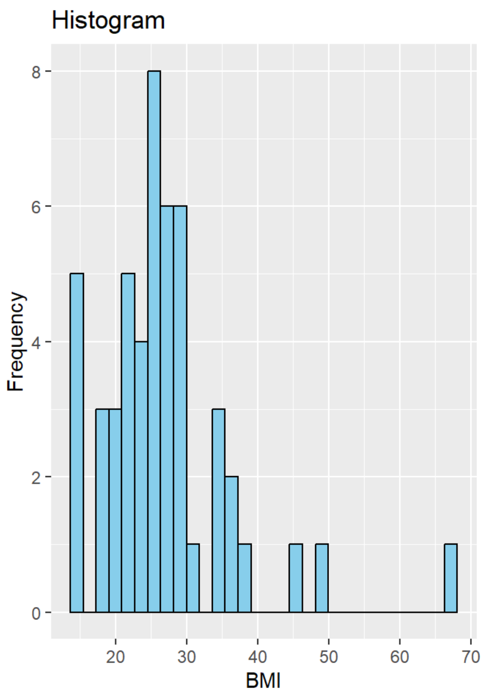
- The population:
  - N=9634.
  - Population mean:

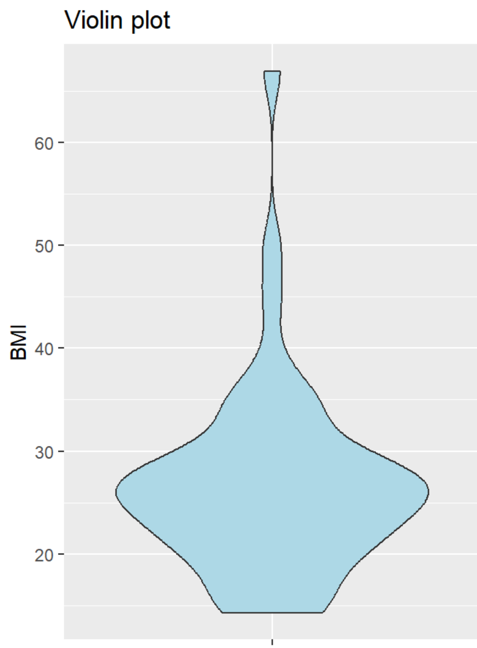
$$\mu = 26.76$$

  - Population variance:

$$\sigma^2 = 54.41$$

The The NHANES dataset: BMI





- A sample of 50 individuals from the population.
- Sample mean:

$$\bar{x} = 26.76$$

- Sample variance:

$$S^2 = 87.67$$

Distribution of the test statistic

Case 2

If $X \sim F$

Then: $\bar{X} \sim N(\mu, \frac{S^2}{n})$

and $T_{\bar{X}} = \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)$

3. X has an unknown distribution, but we have a large sample (n> 30)

$E(X) = \mu$

$Var(X) = \sigma^2$

The same as case 1 but we replace σ^2 by S^2.

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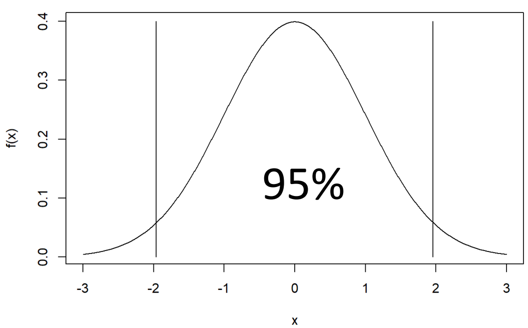
Our example:

$n = 50$

Use N(0,1) to choose the critical value for the upper and lower limits.

95% C. I. for the mean BMI

N(0,1)



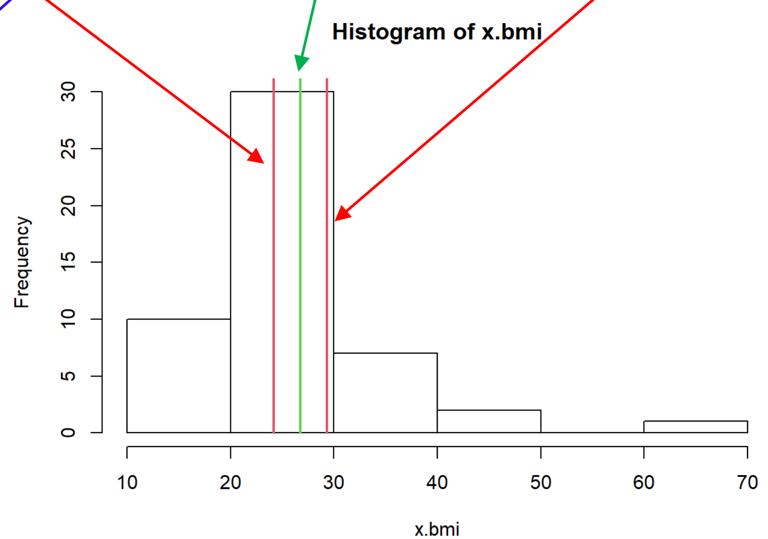
1.96

$$(\bar{x} - m, \bar{x} + m)$$

$$\left(\underbrace{\bar{x} - 1.96 \times \sqrt{\frac{87.67}{50}}}, \underbrace{\bar{x} + 1.96 \times \sqrt{\frac{87.67}{50}}}\right)$$

$$\alpha = 0.025$$
$$Z_\alpha = 1.96$$
$$(1 - 2\alpha) = 0.95$$

The standard error of the sample mean



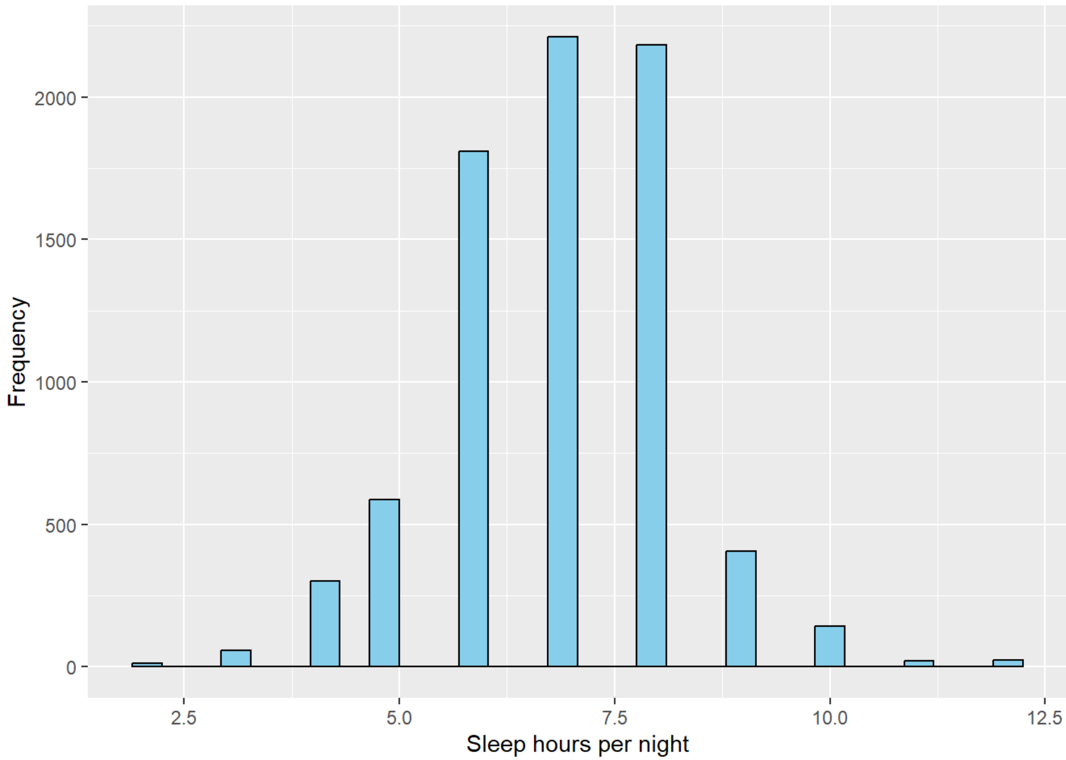
Case study 3:
The The NHANES dataset: number of sleep hours per night

Test of hypothesis about the population mean (one sided test)

The NHANES data set: analysis of the number of sleep hours per night

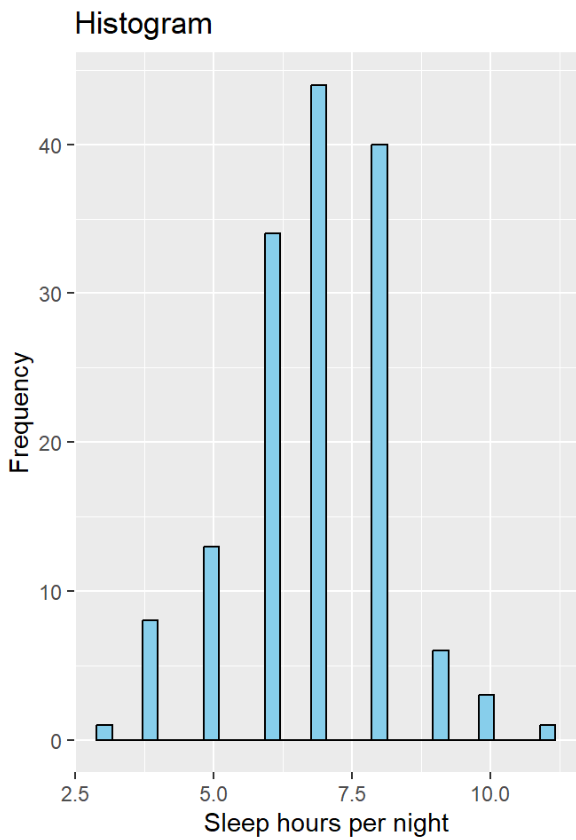
- In this section, the variable of interest is the number of sleeping hours per night (the variable SleepHrsNight).
- Information about the number of sleeping hours per night is available for 7755 individuals (i.e., the population).

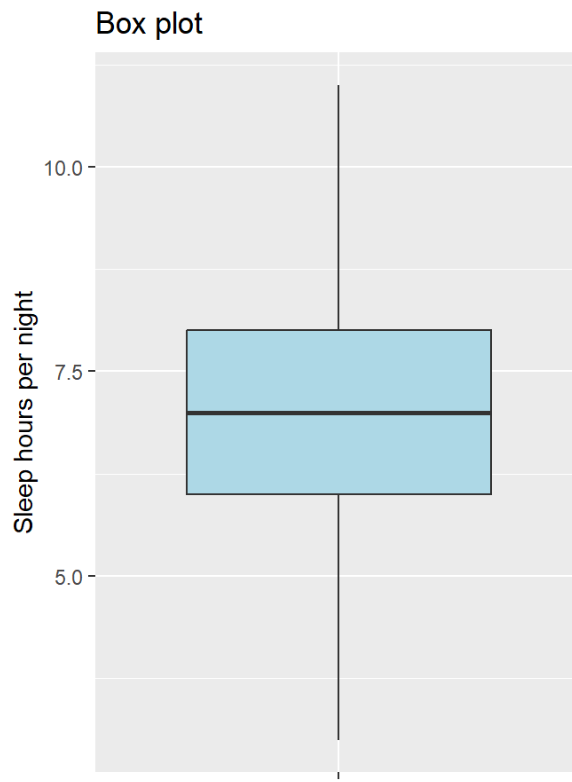
The number of sleep hours per night



$$n = 7755$$
$$\mu = 6.927$$
$$\sigma = 1.813$$

A random sample from the population





- A random sample from the population:

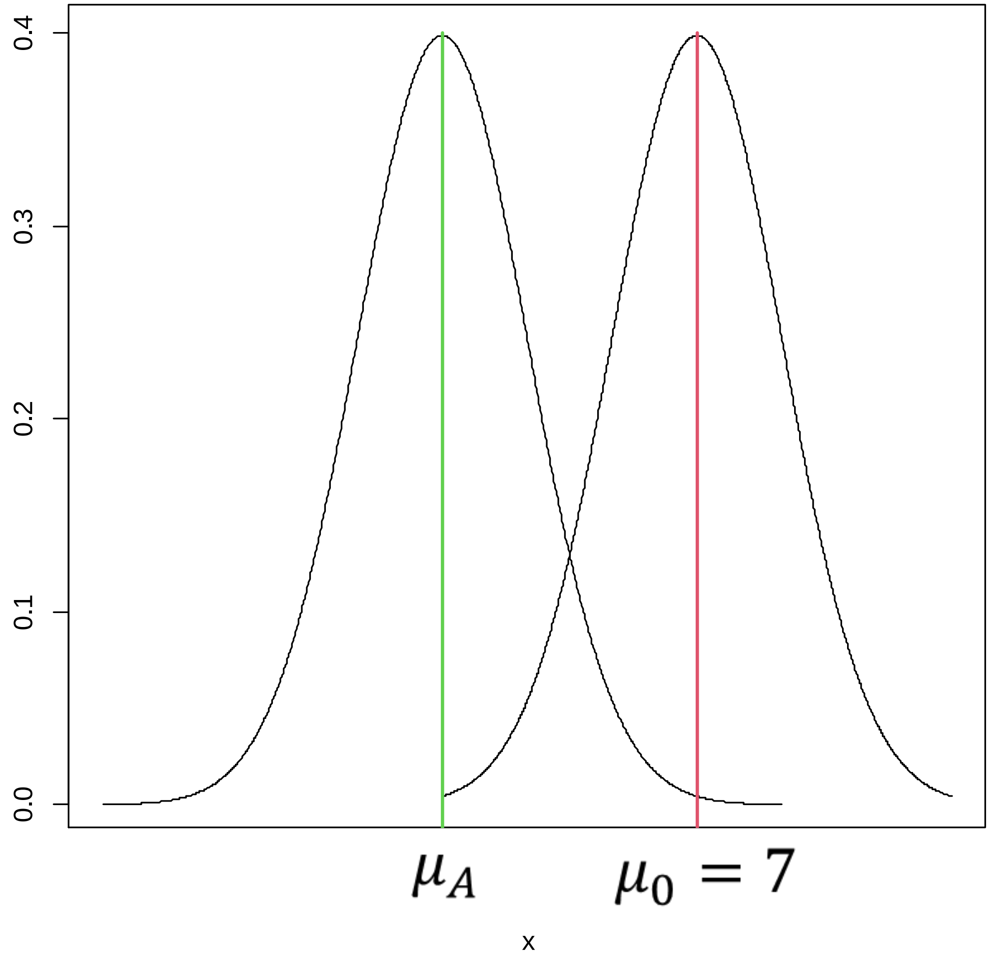
$$n = 150$$
$$\bar{x} = 6.846$$
$$s^2 = 1.862$$

Test of hypothesis: a one sided test

$$H_0: \mu = 7$$
$$H_A: \mu < 7$$

- We test the null hypothesis versus a one sided alternative.

- In our case, under the alternative the mean is smaller than 7 (but not specified).



The null hypothesis

Test statistic

$$t = \frac{\bar{x} - \mu_0}{\sqrt{\frac{1.3646^2}{150}}} = -1.3761$$

The population variance σ^2 is unknown but…n=150.

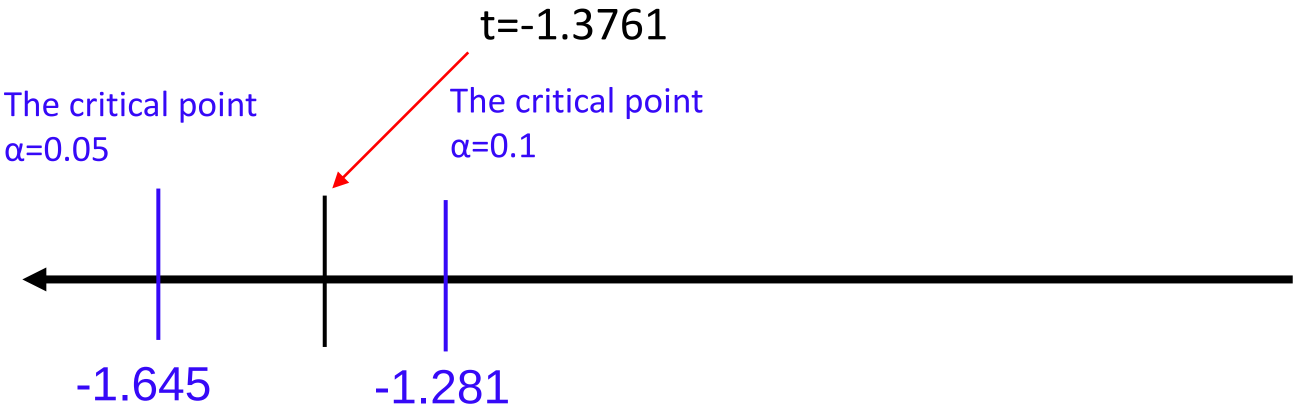
$$\boxed{\frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)}$$

The critical points and the test statistic

For one sided test and α=0.05, Z=-1.645.

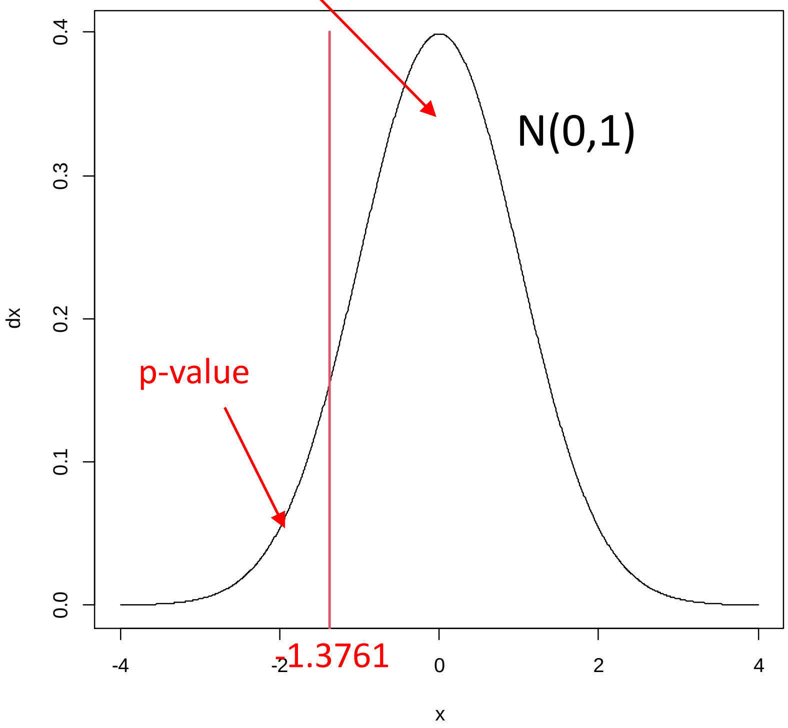
For one sided test and α=0.1, Z=-1.281.

For α=0.1 We reject H_0 : -1.3761< -1.281.



p-value

$$\frac{\bar{X}-\mu}{\sqrt{\frac{S^2}{n}}} \sim N(0,1)$$



$$H_0\colon\ \mu = 7$$
$$H_A\colon \mu < 7$$

$$P(Z < -1.3761) = 0.08439$$

- For α=0.05, we DO NOT reject the null hypothesis.
- For α=0.1, we reject the null hypothesis.

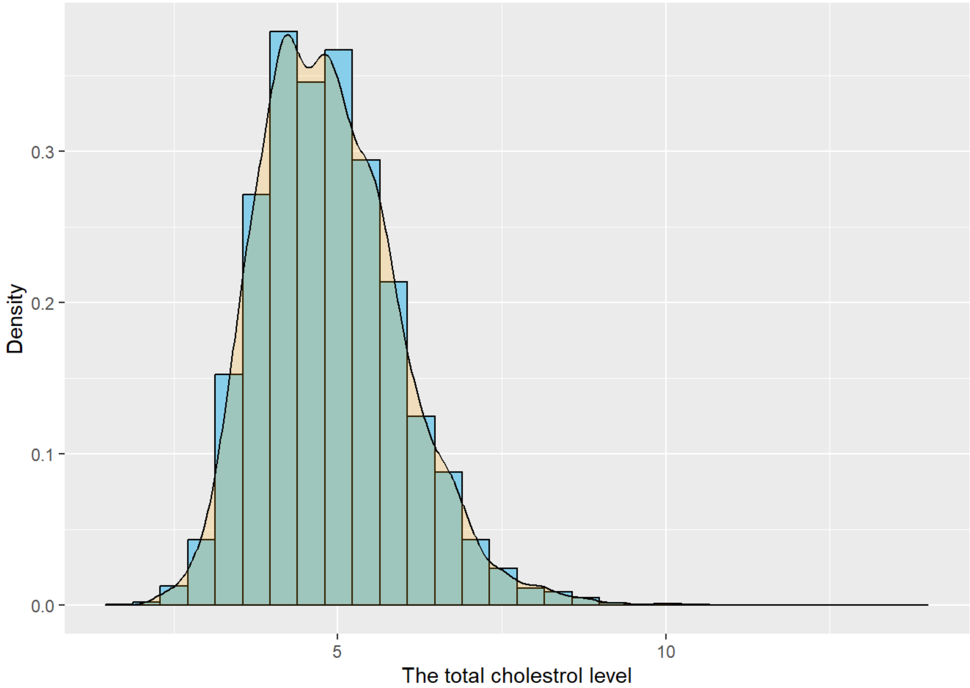
Case study 4:
The The NHANES dataset: cholesterol level

Test of hypothesis about the population mean (two sided test and confidence interval)

The NHANES data set: analysis of the total cholesterol level

- In this section we focus on the total cholesterol level (the variable `TotChol`).
- After omitting all individuals with missing values, 8474 individuals are included in the analysis.

Total cholesterol level



- The population and parameters:

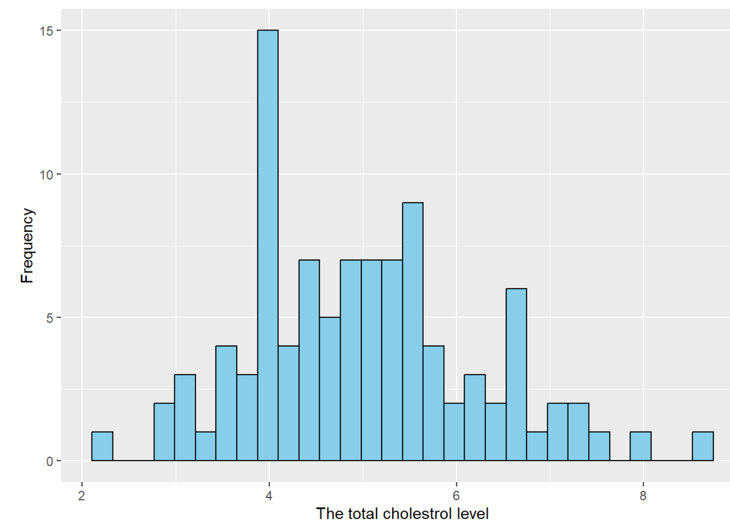
$$n = 8474$$
$$\mu = 4.879$$
$$\sigma^2 = 1.0755^2$$

A random sample of 100 individuals

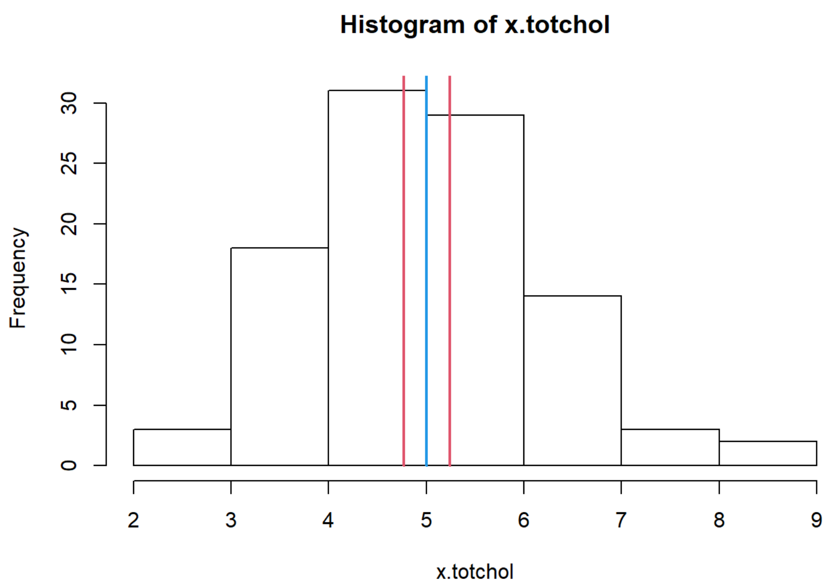
For a sample of 100 individuals:

$$\bar{x} = 5.004$$
$$s = 1.207$$

$$\frac{s}{\sqrt{n}} = \frac{1.2072}{10} = 0.12073$$



A 95% confidence interval for the population mean



$$\alpha = 0.025$$
$$Z_{\alpha} = 1.96$$
$$m = 1.96 \times 0.12073$$

$$\bar{x} \pm m = 5.0041 \pm 1.96 \times 0.12073$$

$$(4.7674, 5.2407)$$

Test of hypothesis for the population mean (two-sided test)

$$H_0: \mu = 5.1$$
$$H_A: \mu \neq 5.1$$

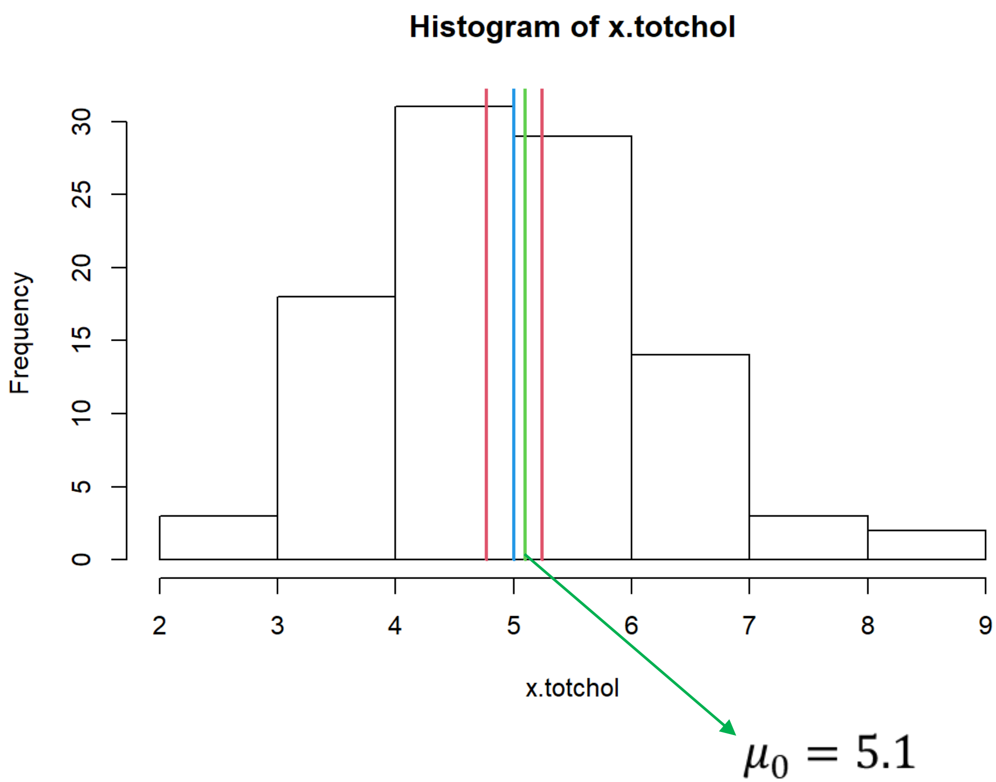
Test statistics and p-value

$\frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{5.0041-5.1}{\frac{1.2073}{10}}$=-0.7943

$2\times\left(1-P(|Z|>0.7943)\right)$=0.427

$$p-value = 0.427 > 0.05 = \alpha$$

A two sided test and a confidence interval



- A 95% confidence interval for the population mean:

$$(4.7674, 5.2407)$$

- The value of μ_0=5.1 is covered by the 95% confidence interval.