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Confined Brownian Motion

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Abstract

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List of Abbreviations

fps	Frames per second
MRSE	Mean Relative Squared Error
MSD	Mean Squared Displacement
PDF	Probability Density Function
RICM	Reflection Interference Contrast Microscopy
SDE	Stochastic Differential Equations

1 Introduction

Since the observations of Gordon Moore in the 60's we know that the technological progress is bound to our ability to miniaturize. It's indeed due to the miniaturization that we are able to have more computational power leading to the rise of knew technologies like the Deep Learning [1] that showed the need of large computational capabilities by having the computer program *AlphaGo* beating *Lee Sedol* one of the greatest player of *Go* in 2016. Since this powerful demonstration AIs using the same technologies are showing up in every field, from the language translator to autonomous cars and is know starting to be extensively used in physics with in 2020 the first focus session on machine learning at the *March Meeting* that continued this year with presentations at every sessions. The success of Deep learning is not due to the fact that it's new and fancy algorithm since it known for several decade but only the fact that the miniaturization permitted to do the stunning amount of computation needed to have a smart AI. Our ability to use this technologies is finally bound to our ability to understand the surface physics at the manometer scale.

On another side we have microfluidic since the 80s which is an incredible multidisciplinary field involving chemistry, engineering, soft matter physics and also biotechnology. Microfluidic permitted the development of daily life technologies like the ink-jet printers or more advanced tools such as DNA chips [2] or lab-on-a-chip technology [3]. The ability to compose with a lot of different system to build microfluidic systems is a wonderful playground for physicists which gave a lot of complex systems in confinement to study and understand how different boundaries can change the dynamic properties of a system. At a time of miniaturization and nanotechnologies, the need of tools permitting the systematic study of complex confined system is a key.

In order to address these challenges my work in the past three years focused on using the confined Brownian motion. Brownian Motion is a central paradigm in modern science. It has implications in fundamental physics, biology, and even finance, to name a few. By understanding that the apparent erratic motion of colloids is a direct consequence of the thermal motion of surrounding fluid molecules, pioneers like Einstein and Perrin provided decisive evidence for the existence of atoms [4, 5]. Specifically, free Brownian motion in the bulk us characterized by a typical spatial extent evolving as the square root of time, as well as Gaussian displacements. At a time of miniaturization and interfacial science, and moving beyond the idealized bulk picture, it is relevant to consider the added roles of boundaries to the above context. Indeed, Brownian motion at interfaces and in confinement is a widespread practical situation in microbiology and nanofluidics. In such case, surface effects become dominant and alter drastically the Brownian statistics, with key

implications towards: i) the understanding and smart control of the interfacial dynamics of microscale entities; and ii) high-resolution measurements of surface forces at equilibrium. Interestingly, a confined colloid will exhibit non-Gaussian statistics in displacements, due to the presence of multiplicative noises induced by the hindered mobility near the wall [6–8]. Besides, the particle can be subjected to electrostatic or Van der Waals forces [9] exerted by the interface, and might experience slippage too [10, 11]. Considering the two-body problem, the nearby boundary can also induce some effective interaction [12]. Previous studies have designed novel methods to measure the diffusion coefficient of confined colloids [13–18], or to infer surface forces [19–24].

In the first part of the manuscript I will present the history of the Brownian motion and its basic theory. In a second part I will present particle tracking using Mie holography and our experimental setup. Then the third part will focus on one trajectory analysis in order to infer the surface induced effects on the Brownian motion. In a last chapter I will present more complex inference.



2 Brownian Motion

2.1 The Brownian motion discovery

in 1827 the Scottish botanist Robert Brown published an article [25] on his observation on the pollen of *Clarkia pulchella* with a lot of details on his reflection processes. His experiments were made to understand the flower reproduction, but, as he was looking through the microscope he observed some minute particles ejected from the pollen grains. At first, he thought the goal of this agitation was to test the presence of a male organ. To test this theory, he extended his observations to Mosses and *Equiseta*, which were drying for a hundred years. However, the fact that this peculiar motion was still observable made him invalidate his theory. Interestingly, each time that he encountered a material that he could reduce to a fine enough powder to be suspended in water, he observed the same type of motion, although, he never understood its particle's movement.

The difficulty at this time to observe and capture such a movement made the study of what we call contemporarily Brownian motion difficult and the first theoretical work was done by Louis Bachelier in his PhD thesis “The theory of speculation,” where he described a stochastic analysis of the stock and option market. Nowadays, the mathematical description of random movement is still used in the modern financial industry.

It is finally in 1905 that Albert Einstein theoretically state that “bodies of microscopically visible size suspended in a liquid will perform movements of such a magnitude that they can be easily observed in a microscope” [4]. A remark to make here is that in 1948 Einstein wrote a letter to one of his friends where he stated having deduced the Brownian motion “from mechanics, without knowing that anyone had already observed anything of the kind” [26].

It is in 1908 that Jean Perrin published his experimental work on Brownian motion. That way he could measure the Avogadro number and prove the kinetic theory that Einstein developed. I would also cite Chaudesaigues and Dabrowski, who helped Perrin to track the particles manually, half-minutes by half-minutes, for more than 3000 displacements (25 hours) and several particles. This impressive and daunting work is highly detailed in “*Mouvement brownien et molécules*” [27]. This is partly due to the results this work that Perrin received the Nobel award in 1926.

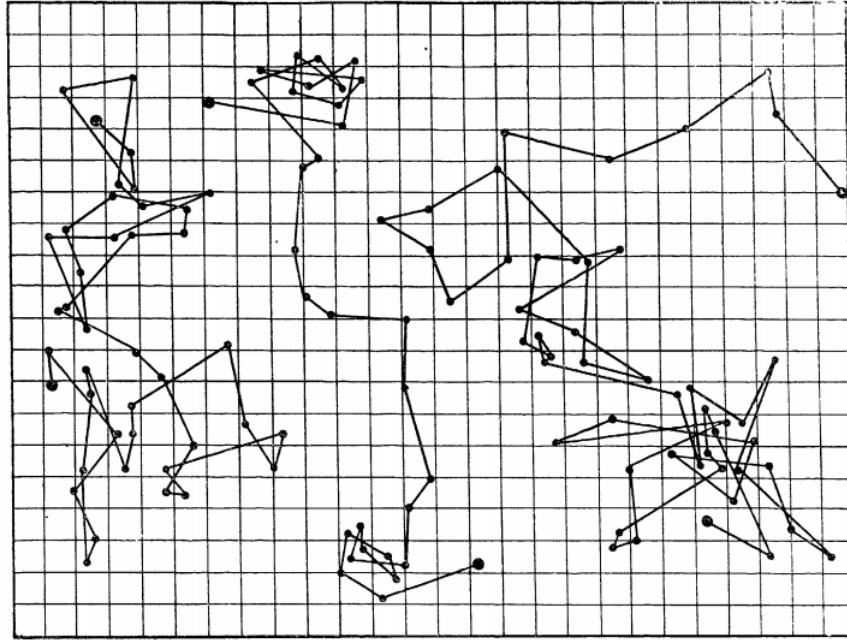


Figure 1: Brownian motion of $1 \mu\text{m}$ particles in water tracked manually by Jean Perrin and his colleagues. The points are spaced in time by 30 seconds, and 16 divisions represent $50 \mu\text{m}$.

2.2 Einstein's Brownian theory

In this section we will derive the main characteristics of bulk Brownian motion in the manner of Einstein in 1905 by summarizing the section 4 of [4]. We will then examine the random motion of particles suspended in a liquid and its relation to diffusion, caused by thermal molecular motion. We assume that each particle motion is independent of other particles; also the motions of one particle at different times are assumed to be independent of one another provided that the time interval is not too small. Furthermore, we now introduce a time interval τ which is small compared to the observation time but large enough so that the displacements in two consecutive time intervals τ may be taken as independent events.

For simplicity, we will here look only at the Brownian motion of n particles in 1D along the x axis. In a time interval τ the position of each particle will increase by a displacement Δ , positive or negative. The number of particles dn experiencing a displacement lying between Δ and $\Delta + d\Delta$ in a time interval τ is written as:

$$dn = n\varphi(\Delta)d\Delta , \quad (2.2.1)$$

where

$$\int_{-\infty}^{\infty} \varphi(\Delta) d\Delta = 1 , \quad (2.2.2)$$

and φ is the probability distribution of displacement. We assume for now, that φ is a Gaussian distribution, with a variance scaling linearly with τ . Additionally, since such a distribution is even, it satisfies: $\varphi(\Delta) = \varphi(-\Delta)$.

Let $f(x, t)$ be the number of particles per unit volume. From the definition of the function $\varphi(\Delta)$ we can obtain the distribution of particles found at time $t + \tau$ from their distribution at a time t , through:

$$f(x, t + \tau) = \int_{-\infty}^{+\infty} f(x - \Delta, t) \varphi(\Delta) d\Delta . \quad (2.2.3)$$

Since we suppose that τ is very small with respect to t , we have at first order in time:

$$f(x, t + \tau) \simeq f(x, t) + \tau \frac{\partial f}{\partial t} . \quad (2.2.4)$$

Besides, we can Taylor expand $f(x + \Delta, t)$ in powers of Δ since only small values of Δ contribute. We obtain:

$$f(x - \Delta, t) = f(x, t) - \Delta \frac{\partial f(x, t)}{\partial x} + \frac{\Delta^2}{2!} \frac{\partial^2 f(x, t)}{\partial x^2} . \quad (2.2.5)$$

Combining Eqs.2.2.4, 2.2.5 and 2.2.3 we obtain:

$$f + \frac{\partial f}{\partial t} \tau = f \int_{-\infty}^{+\infty} \varphi(\Delta) d\Delta + \frac{\partial f}{\partial x} \int_{-\infty}^{+\infty} \Delta \varphi(\Delta) d\Delta + \frac{\partial^2 f}{\partial x^2} \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \varphi(\Delta) d\Delta . \quad (2.2.6)$$

On the right-hand side, since $\varphi(x)$ is an even function, the second term vanishes. Considering Eq.2.2.2 and invoking the definition:

$$\frac{1}{\tau} \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \varphi(\Delta) d\Delta = D , \quad (2.2.7)$$

Eq.2.2.6 finally becomes:

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}. \quad (2.2.8)$$

We can here recognize a partial equation of diffusion with D the diffusion coefficient. We will now initiate the same position $x = 0$ for all the particles at $t = 0$ as in Fig.2. $f(x, t)dx$ denotes the number of particles whose positions have increased between the times 0 and t by a quantity lying between x and $x + dx$ such that we must have:

$$f(x \neq 0, t = 0) = 0 \text{ and } \int_{-\infty}^{+\infty} f(x, t)dx = n. \quad (2.2.9)$$

The solution Eq.2.2.8 is then the Green's function of the heat equation in the bulk:

$$f(x, t) = \frac{1}{\sqrt{4\pi D}} \frac{\exp\left(\frac{-x^2}{4Dt}\right)}{\sqrt{t}}. \quad (2.2.10)$$

From this solution we can see that the mean value of the displacement along the x axis is equal to 0 and the square root of the arithmetic mean of the squares of displacements (that we commonly call the Root Mean Square Displacement (RMSD)) is given by:

$$\lambda_x = \sqrt{\langle \Delta^2 \rangle} = \sqrt{2Dt}. \quad (2.2.11)$$

The mean displacement is thus proportional to the square root of time. This result is generally the first behavior that we check when we study Brownian motion. In 3D, the square root of the MSD is given by $\lambda_x\sqrt{3}$.

Previously in his article [4], Einstein had found by writing the thermodynamic equilibrium of a suspension of particles that the diffusion coefficient of a particle should read:

$$D = \frac{RT}{N_A} \frac{1}{6\pi\eta a} = \frac{k_B T}{6\pi\eta a}, \quad (2.2.12)$$

with R the gas constant, T the temperature, N_A the Avogadro number, η the fluid viscosity and k_B the Boltzmann constant. Thus, an experimental measurement of D could lead to a measurement of the Avogadro number since:

$$N_A = \frac{t}{\lambda_x^2} \frac{RT}{3\pi\eta a} . \quad (2.2.13)$$

Furthermore, measuring N_A also gives us the mass of atoms and molecules since the mass of a mole is known; as an example the mass of an oxygen atom will be given by $\frac{16}{N_A}$ and the mass a water molecule by $\frac{18}{N_A}$. Finally, Einstein ends up in article [4] by writing, “*Let us hope that a researcher will soon succeed in solving the problem posed here, which is of such importance in the theory of heat!*” I would like here to emphasize the importance of solving this problem at the very beginning of the 20th century. At this time two hypotheses about the fundamental matter components existed, one involving energy and a continuum description in terms of field, and the other one, discrete atoms, especially supported by Boltzmann and his kinetic theory of gases, used by Einstein. Due to a lot of conceptual misunderstandings and experimental error scientist such as Svedberg or Henri thought that Einstein’s theory was false [28] by even suggesting that the statistical properties of Brownian motion were changing with the pH of the solution. It is finally in 1908 that Chaudesaigues and Perrin published all the evidence to prove Einstein’s theory mainly by their ability to create particle emulsions of well-controlled radii.

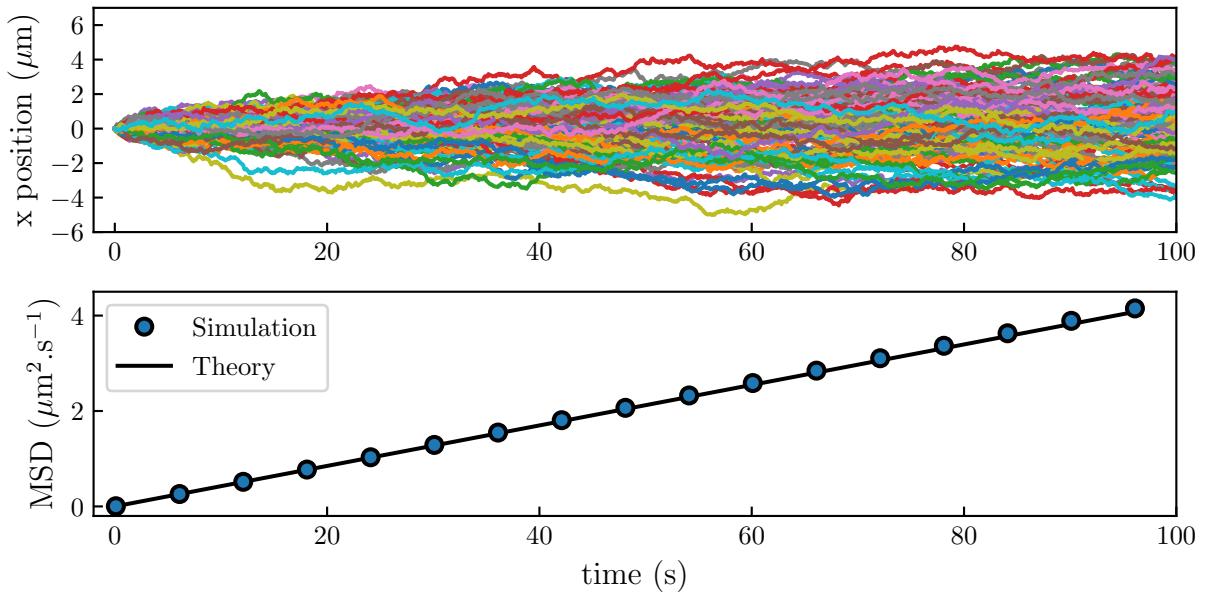


Figure 2: Simulation of over-damped Brownian motion in the bulk (see Eq. (2.4.9)) of $1 \mu\text{m}$ particles in water. On the top each line represents the trajectory of a Brownian particle over 100 seconds. A total of 100 trajectories are shown. On the bottom, bullets represent the Mean Square Displacement (MSD) computed from the simulated trajectories. The plain black line represents Einstein’s theory, which is computed from the square of Eq. 2.2.11. 

2.3 The Langevin Equation

in physics we generally describe Brownian motion through a particular Stochastic Differential Equations (SDE). This model was introduced in 1908 by Langevin [29], this model is now used by the major part of physicists working on random processes. The Langevin equation for a free colloid reads:

$$m dV_t = -\gamma V_t dt + \alpha dB_t , \quad (2.3.1)$$

with m the mass and V_t the velocity of the particle. This SDE is the Newton's second law, relating the particle momentum change on the left-hand side of the equation to forces on the right-hand side. We see that the total force applied on the particle is given by two terms: a friction term, with a Stokes-like fluid friction coefficient γ , a random force with α that we will detail for a spherical particle, dB_t a random noise which has a Gaussian distribution of zero mean thus:

$$\langle dB_t \rangle = 0 , \quad (2.3.2)$$

and variance equal to:

$$\langle dB_t^2 \rangle = dt . \quad (2.3.3)$$

For a spherical particle, the friction term is given by the Stoke's formula: $\gamma = 6\pi\eta a$ with η the fluid viscosity and a the particle radius. Thus, we can derive the mean value of the particle velocity as:

$$\langle \frac{dV_t}{dt} \rangle = -\frac{\gamma}{m} \langle V_t \rangle dt + \frac{\alpha}{m} \langle dB_t \rangle , \quad (2.3.4)$$

with the properties of dB_t given by Eq.2.3.2, it becomes:

$$\langle dV_t \rangle = -\frac{\gamma}{m} \langle V_t \rangle dt . \quad (2.3.5)$$

Moreover, without a loss of generality, the average of a variable x , $\langle x \rangle$, is done over a set

of N observations $\{x_i\}$ such as:

$$\langle x \rangle = \frac{1}{N} \sum_{i=1}^N x_i , \quad (2.3.6)$$

one can then show that:

$$\frac{d}{dt} \langle x \rangle = \frac{d}{dt} \left[\frac{1}{N} \sum_{i=1}^N x_i \right] = \frac{1}{N} \sum_{i=1}^N \frac{d}{dt} x_i = \langle \frac{d}{dt} x \rangle . \quad (2.3.7)$$

The latter thus shows that it is possible to invert average value $\langle \cdot \rangle$ and a derivative. Therefore, Eq.2.3.5 becomes:

$$\frac{d}{dt} \langle V_t \rangle = -\frac{\gamma}{m} \langle V_t \rangle , \quad (2.3.8)$$

which has a familiar solution:

$$\langle V_t(t) \rangle = V_0 e^{-\frac{\gamma}{m} t} , \quad (2.3.9)$$

with V_0 an initial velocity. This result shows that the average of the velocity should decay to zero with a characteristic time $\tau_B = \frac{m}{\gamma}$. For instance, the polystyrene particles used during my experiments which are micrometric we have $\tau_B \approx 10^{-7}$ s. This signifies that if we measure the displacements of a particle with a time interval $\tau \gg \tau_B$ the displacement can be taken as independent events as it was stated by Einstein. In physical terms, this means that we are in the over-damped regime, in this case the Langevin equation reads:

$$-\gamma V_t dt + \alpha dB_t = 0 . \quad (2.3.10)$$

The experiments done during my thesis used a video camera that can reach a maximum of hundreds frames per second (fps) reaching time steps of $\approx 10^{-2}$ s. Therefore, all my work falls into the over-damped regime. Before focusing definitely on Eq.2.3.10, we can use Eq.2.3.4 to characterize further the unknown coefficient α . To do so we compute the mean square value of Eq.2.3.4, starting by taking the second order Taylor expansion:

$$\begin{aligned} d(V_t^2) &\simeq \frac{\partial V_t^2}{\partial V_t} dV_t + \frac{1}{2} \frac{\partial^2 V_t^2}{\partial V_t^2} (dV_t)^2 \\ &= 2V_t dV_t + (dV_t)^2 \end{aligned} \quad (2.3.11)$$

combining Eqs.2.3.1 and 2.3.11, we obtain by only keeping the terms of order dt :

$$d(V_t^2) = 2V_t \left(-\frac{\gamma}{m} V_t dt + \frac{\alpha}{m} dB_t \right) + \frac{\alpha^2}{m^2} dB_t^2 . \quad (2.3.12)$$

Thus, the average value of $d(V_t^2)$ reads:

$$\langle d(V_t^2) \rangle = -2\frac{\gamma}{m} \langle V_t^2 \rangle dt + 2\frac{\alpha}{m} \langle V_t dB_t \rangle + \frac{\alpha^2}{m^2} \langle dB_t^2 \rangle . \quad (2.3.13)$$

Moreover, since dB_t is chosen independently of the velocity V_t , one can write $\langle V_t dB_t \rangle = \langle V_t \rangle \langle dB_t \rangle = 0$. Taking the latter remark into account and the fact that $\langle dB_t^2 \rangle = dt$, Eq.2.3.13 becomes:

$$\langle d(V_t^2) \rangle = \left[-2\frac{\gamma}{m} \langle V_t^2 \rangle + \frac{\alpha^2}{m^2} \right] dt . \quad (2.3.14)$$

Since equilibrium averages in thermodynamics must become time independent, we have $\langle d(V_t^2) \rangle = 0$, thus:

$$\langle V_t^2 \rangle = \frac{\alpha^2}{2\gamma m} . \quad (2.3.15)$$

Besides, from the equipartition of energy we also know that:

$$\langle \frac{1}{2} m V_t^2 \rangle = \frac{1}{2} k_B T . \quad (2.3.16)$$

The latter equation permits a direct determination of the amplitude of the noise α :

$$\alpha = \sqrt{2k_B T \gamma} . \quad (2.3.17)$$

The latter result permits computing the amplitude of the random force in the Langevin equation. Taking the over-damped Langevin equation, it reads:

$$V_t dt = \sqrt{2 \frac{k_B T}{\gamma}} dB_t \quad (2.3.18)$$

Furthermore, one can write the position of the particle X_t at a time t , such as:

$$X_t = \int_0^t V_{t'} dt' , \quad (2.3.19)$$

where we can suppose at the initial time $t = 0$ that $X_0 = 0$. Computing $\langle X_t^2 \rangle$ using Eqs.2.3.18,2.3.19 and 2.3.3 thus gives:

$$\langle X_t^2 \rangle = 2 \frac{k_B T}{\gamma} t = 2 D t \quad (2.3.20)$$

By relating $\langle X_t^2 \rangle$ to the Mean Square Displacement (MSD) to the initial position such as:

$$\text{MSD} = \langle (X_0 - X_t)^2 \rangle = \langle X_t^2 \rangle , \quad (2.3.21)$$

we obtain that the MSD should be linear with the time. This result confirms that using the over-damped Langevin equation, leads to the Einstein's result Eq.2.2.11. Where one can identify the diffusion coefficient of the particle to be $D = k_B T / \gamma$. Additionally, the latter identity is called the Stokes-Einstein relation.

Additionally, the Langevin equation is great to compute correlator such as the velocity correlator $\langle V_t V_{t''} \rangle$ which the simplest to compute and the one that we will detail below. Indeed, if we use the full-Langevin equation, $\langle X_t^2 \rangle$ can't be that easily computed since $m dV_t$ does not vanish. We would thus need to rewrite Eq.2.3.20 using the velocity correletor such as:

$$\langle X_t^2 \rangle = \int_0^t \int_0^t \langle V_{t'} V_{t''} \rangle dt' dt'' . \quad (2.3.22)$$

Let us now study how the two-point correlator function $\langle V_t V_{t''} \rangle$, using the full-Langevin

equation multiplied by V_0 and following the same steps as for Eq.2.3.9, one has:

$$\langle V_t V_0 \rangle = \langle V_0^2 \rangle e^{-t/\tau_B} . \quad (2.3.23)$$

As the equilibrium state is invariant under temporal translation and assuming that V_0 has an equilibrium steady-state distribution with $\langle V_0^2 \rangle = k_B T / m$ we have:

$$\langle V_t V'_t \rangle = \frac{k_B T}{m} e^{-|t-t'|/\tau_B} . \quad (2.3.24)$$

One can solve Eq.2.3.22 by splitting the integral in two parts, where $t' > t''$ and $t' < t''$:

$$\begin{aligned} \langle X_t^2 \rangle &= \frac{k_B T}{m} \int_0^t dt' \int_0^{t'} dt'' e^{-|t'-t''|/\tau_B} = 2 \frac{k_B T}{\gamma} \left(\int_0^t dt' \left[1 - e^{-t'/\tau_B} \right] \right) \\ &= 2 \frac{k_B T}{\gamma} \left(t - \tau_B \left[1 - e^{-t/\tau_B} \right] \right) . \end{aligned} \quad (2.3.25)$$

We can extract two results from that equation. At a short time $t \ll \tau_B$, one has:

$$\begin{aligned} \langle X_t^2 \rangle &\simeq 2 \frac{k_B T}{\gamma} \left(t - \tau_B \left[1 - 1 + \frac{t}{\tau_B} - \frac{t^2}{2\tau_B^2} \right] \right) \\ &= \frac{k_B T}{m} t^2 . \end{aligned} \quad (2.3.26)$$

This is the ballistic regime. If one can experimentally explore times shorter than τ_B one will then measure the real velocity of the particle. At longer times, $t \gg \tau_B$, the MSD is given by:

$$\langle X_t^2 \rangle \simeq 2 \frac{k_B T}{\gamma} t = 2 D t . \quad (2.3.27)$$

This is the diffusive regime where the MSD, as found earlier, Eq.2.3.20 with the over-damped Langevin equation. To study this different result, it can be interesting to simulate the Brownian motion.

2.4 Numerical simulation of bulk Brownian motion

2.4.1 The numerical Langevin Equation

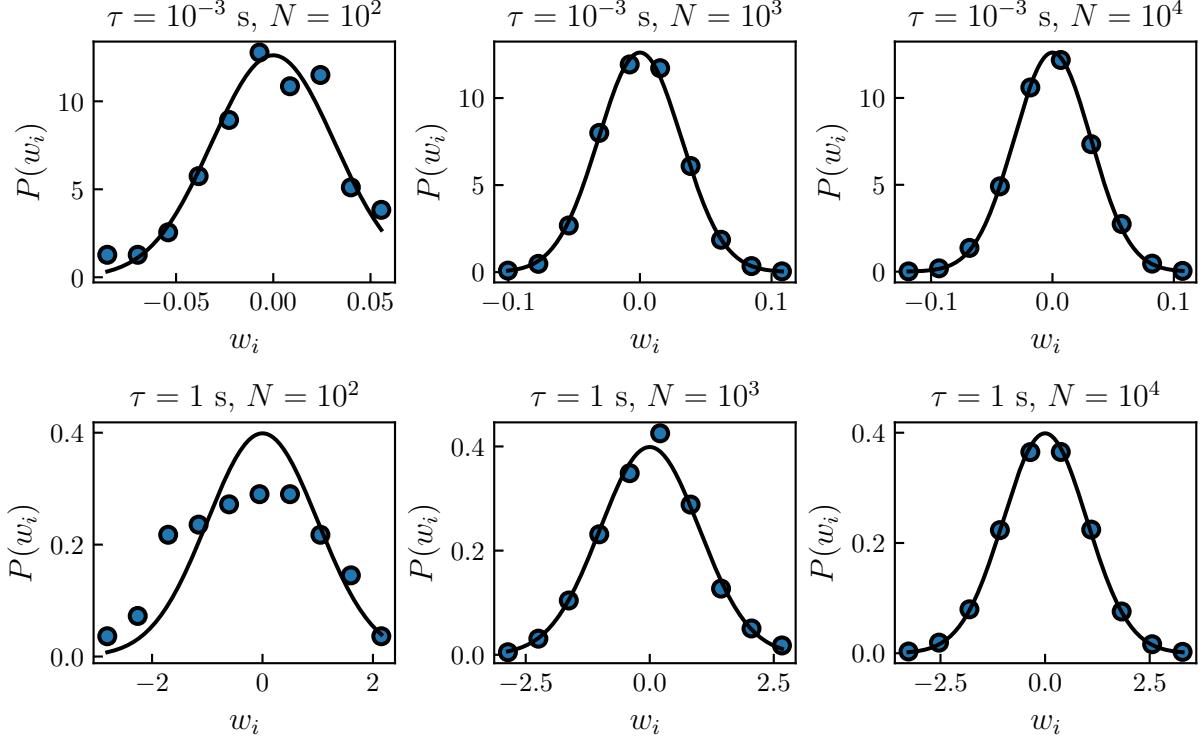


Figure 3: Bullets represents the probability density function of w_i , a Gaussian-distributed number with a mean value $\langle w_i \rangle$ and a variance $\langle w_i^2 \rangle = \tau$. The plain black line is a Gaussian distribution of zero mean and a τ variance (see Eq.2.4.3). On the first line, the simulation is done with $\tau = 10^{-3}$ s and $\tau = 1$ s on the second one. Each column corresponds to a number N of draws. From the left to the right: $N = 10^2$, 10^3 and 10^4 .

The Langevin equation is an ordinary differential equation that can easily be numerically simulated in the bulk case. One approximates the continuous position X_t of a particle at a time t by a discrete-time sequence x_i which is the solution of the equation at a time $t_i = i\tau$, τ being the time step of the simulation. One can then use the Euler method to numerically write V_t as:

$$V_t \simeq \frac{x_i - x_{i-1}}{\tau} , \quad (2.4.1)$$

and dV_t/dt as

$$\begin{aligned}\frac{dV_t}{dt} &\simeq \frac{\frac{x_i - x_{i-1}}{\tau} - \frac{x_{i-1} - x_{i-2}}{\tau}}{\tau} \\ &= \frac{x_i - 2x_{i-1} + x_{i-2}}{\tau^2}.\end{aligned}\quad (2.4.2)$$

The only term remaining to be computed numerically is the random term dB_t . One can thus replace dB_t/dt by w_i/τ ¹ a Gaussian distributed random number generated with a mean $\langle w_i \rangle = 0$ and a variance $\langle w_i^2 \rangle = \tau$. The Probability Density function (PDF) of the Gaussian distribution is thus given by:

$$P(w_i) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{w_i^2}{2\tau}}. \quad (2.4.3)$$

The random number w_i can be generated with the following Python snippet.

```

1 import numpy as np
2
3 tau = 0.5 # time step in seconds
4 wi = np.random.normal(0, np.sqrt(tau))

```

In the latter, `random.normal()` is a built-in Numpy module that permits the generation of Gaussian-distributed random numbers. Finally, by combining Eqs.2.4.1 and 2.4.2, the full-Langevin equation becomes:

$$m \frac{x_i - 2x_{i-1} + x_{i-2}}{\tau^2} = -\gamma \frac{x_i - x_{i-1}}{\tau} + \sqrt{2k_B T \gamma} \frac{w_i}{\tau}. \quad (2.4.4)$$

From the latter, one can write x_i as:

$$x_i = \frac{2 + \tau/\tau_B}{1 + \tau/\tau_B} x_{i-1} - \frac{1}{1 + \tau/\tau_B} x_{i-2} + \frac{\sqrt{2k_B T \gamma}}{m(1 + \tau/\tau_B)} \tau w_i, \quad (2.4.5)$$

where we can observe that two initial conditions are needed, the first two positions of the particle. Numerically, these positions could be randomly generated or set to 0. If enough statics are generated, it will not affect the results.

¹ The notation w was choosen since in mathematical terms, a real-valued continuous-time stochastic process such as dB_t is called a Wiener process in honor of Norbert Wiener [30].

2.4.2 Simulating Brownian Motion Using Python

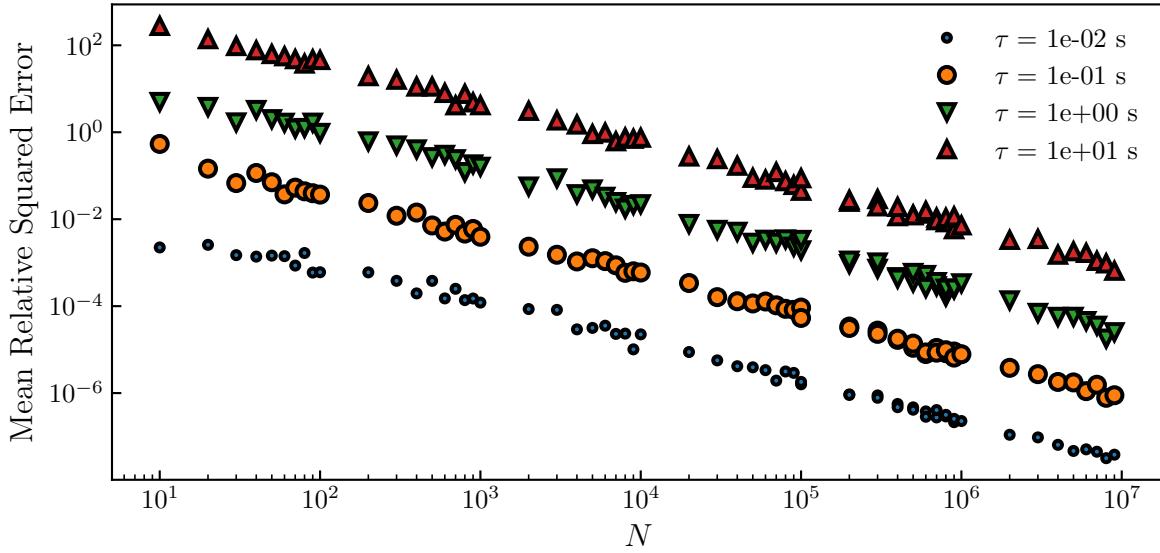


Figure 4: Mean Relative Squared Error (MRSE) of the Probability Density Function PDF measured from a generation of N Gaussian random numbers w_i and the actual Gaussian (see Eq. (2.4.3)) from which the generation is done. The generation is done over a Gaussian which has a mean value $\langle w_i \rangle = 0$ and variance $\langle w_i^2 \rangle = \tau$. We explore parameter ranges from $N = 10$ to 10^7 and $\tau = 10^{-2}$ to 10 s.

Before, diving into the simulation, it could be interesting to ask ourselves how long the simulation should be. Indeed, at equilibrium, the different observables' mean values to remain constant we should wait a sufficient amount of time. It is possible to follow a qualitative approach by generating N numbers w_i , measuring the resulting PDF $P_c(w_i)$ and looking for how much we need to increase N to have $P_c(w_i) \approx P(w_i)$, under some given small-error criterion. As we can see in Fig.3, for simulations made with $\tau = 10^{-3}$ s and $\tau = 1$ s, we observe that as we increase N , the measured PDF, gets closer to the real one given by Eq. (2.4.3).

To have a more quantitative approach, one can compute the Mean Relative Squared Error (MRSE) between the measured PDF $P_c(w_i)$ and the nominal function $P(w_i)$ as a function of the number N of generated numbers, as:

$$\text{MRSE} = \left\langle \frac{(P_c(w_i) - P(w_i))^2}{P(w_i)^2} \right\rangle_N \quad (2.4.6)$$

where the notation $|_N$ denotes the average over N realizations. Additionally, since we measure $P_c(w_i)$ by doing a histogram, the question of how many bins are used should

be answered. It is possible to use the optimal Freedan-Diaconis rule [31] to compute the width of the bins to be used in a histogram. This rule reads:

$$\text{Bin width} = 2 \frac{\text{IQR}(\{w_i\})}{\sqrt[3]{N}}, \quad (2.4.7)$$

where IQR is the interquartile range, and $\{w_i\}$ a sample of N random numbers w_i . Moreover, one should at least take 2 bins as a minimum. The optimal number of bins can be computed using the following Python snippet.

```

1      import numpy as np
2
3
4      def _iqr(wi):
5          """Function to compute interquartile range."""
6          return np.subtract(*np.percentile(x, [75, 25]))
7
8      def optimal_bins(wi):
9          """
10             Function to compute the optimal number of bins using Freedan-Diaconis rule.
11             Input: list of random numbers / Output: optimal bins number
12         """
13
14      n = int(diff(wi) / (2 * _iqr(wi) * np.power(len(wi), -1 / 3)))
15
16      if n <= 2:
17          return 2
18      else:
19          return n

```

As we can see in Fig.4, for τ varying between 10^{-2} and 10 seconds, and N between 10 and 10^6 , the MRSE decreases as N increases. Moreover, it is interesting to observe that the MRSE is greater as τ increases for a fixed N value. As an example, we would need to only generate $N = 10^{-3}$ numbers to obtain an MRSE of 10^{-4} for $\tau = 0.1$ s, while we would need to $N = 10^6$ for $\tau = 1$ s.

Now that the Langevin equation has been numerically implemented, one could use it to simulate Brownian trajectories. A simple way to do the simulation using Python is provided in the appendix.A.1. A set of trajectories simulated for a fictive particle of radius $a = 1 \mu\text{m}$ and mass $m = 10 \mu\text{g}$ in water is shown in Fig.5-a). For such a particle, the diffusive characteristic time is $\tau_B = 0.53$ s. Moreover, as one can see in Fig.5-b), the

MSD is correctly modeled by Eq. (2.3.26) for $\tau \ll \tau_B$, and by Eq. (2.3.27) for $\tau \gg \tau_B$. Note that for non-continuous data such as the simulation data presented here or sampled experimental trajectories, and for a given time increment Δt , the MSD is generally defined by:

$$\langle \Delta x^2 \rangle_t = \langle (x_i(t + \Delta t) - x_i(t))^2 \rangle_t , \quad (2.4.8)$$

where the average $\langle \rangle|_t$ is performed over time t . The following Python function can be used to numerically compute the MSD Eq. (2.4.8).

```

1     def msd(x, Dt):
2         """Function that returns the MSD for a list of time indices Dt for a trajectory x"""
3         _msd = lambda x, t: np.mean((x[:-Dt] - x[Dt:]) ** 2)
4         return [_msd(x, i) for i in t]

```

Additionally, as we have seen earlier, the Langevin Equation can be simplified to it is over-damped version of Eq. (2.3.10). In this case, the time step τ of the simulation should be greater than the characteristic time τ_B . Thus, if one is interested in the long-time statistical properties of Brownian motion one can use the over-damped Langevin equation. In this case, by putting $m = 0$ into Eq. (2.4.5), one can write x_i as:

$$x_i = x_{i-1} + \sqrt{2D}w_i . \quad (2.4.9)$$

The statistical properties at a long time could be retrieved by simulating Brownian motion using the full-Langevin equation. But, since the integration scheme used for Eq. (2.4.5) requires $\tau \ll \tau_B$, long simulation runs are necessary to retrieve the over-damped properties. A simulation of the over-damped Brownian motion trajectories using Eq. (2.4.9) is shown in Fig.2 and is realized using the following Python Snippet.

```

1     import numpy as np
2
3     N = 1000 # trajectory length
4     D = 1   # diffusion coefficient
5     tau = 0.5 # time step
6     trajectory = np.cumsum(np.sqrt(2 * D) * np.random.normal(0, np.sqrt(tau), N))

```

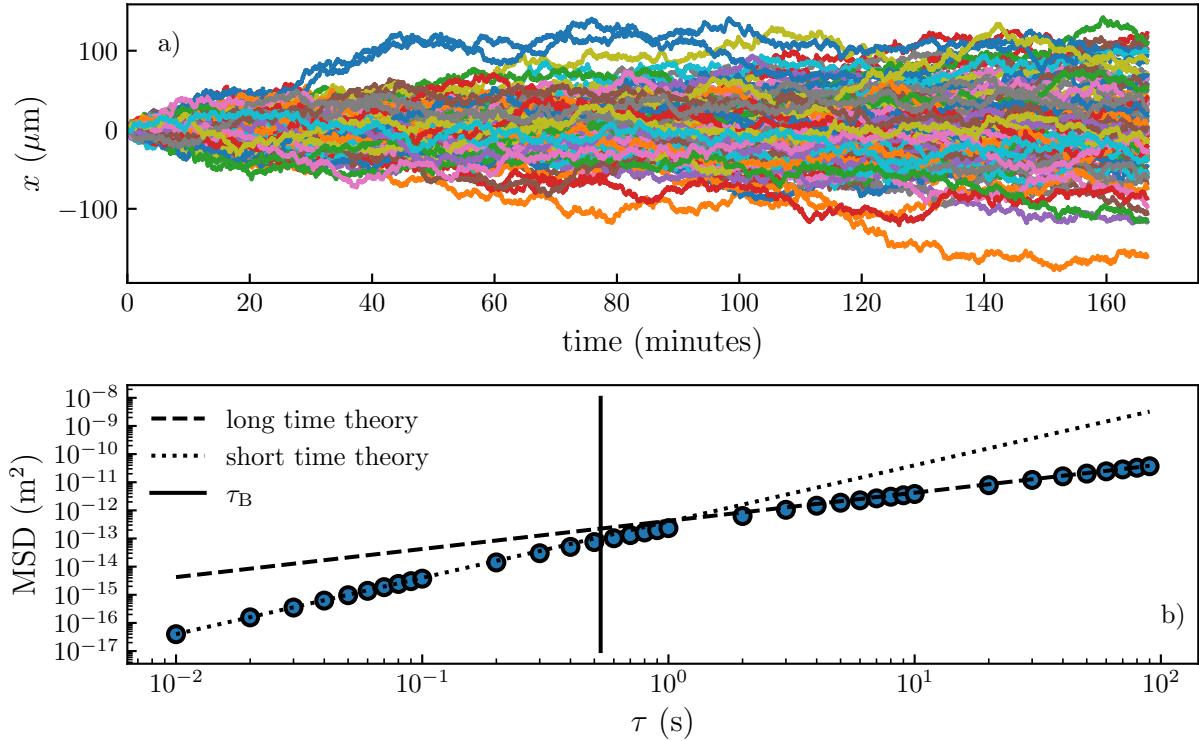


Figure 5: a) Set of 100 trajectories simulated using the full-Langevin equation (see Eq. (2.4.5)) for particles of a radius $a = 1 \mu\text{m}$ and mass $m = 10 \mu\text{g}$ in water, with viscosity $\eta = 0.001 \text{ Pa.s}$. The simulations are done with a time step $\tau = 0.01 \text{ s}$. b) Bullets represent the measured Mean Squared Displacements (MSD) of the simulated trajectories. The plain black line represents the characteristic inertial timescale, $\tau_B = m/\gamma = 0.53 \text{ s}$. The dotted line represents the MSD in the ballistic regime (see Eq. (2.3.26)), when $t \ll \tau_B$. The dashed line represents the MSD in the diffusive regime (see Eq. (??)), when $t \gg \tau_B$, $MSD = 2D\tau$. A detailed explanation of the simulation process can be found in the appendix.A.1.

2.4.3 Speedup using Cython

I would like to point out that the optimization of a simple simulation of a Brownian trajectory can be interesting. Indeed, using a pure Python code as presented in the first part of appendix.A.1, the simulation of one trajectory of 10^6 steps, needs 6 s to be computed. Thus, more than 10 minutes are required to compute the 100 trajectories shown in Fig.5. This is long and due to how Python systematically verifies what we do is allowed. Indeed, it verifies at each step of the `for` loop the object type of each variable and if the mathematical operation are possible. This is generally the main drawback [32] of interpreted language, and the only solution is to use a compiled language (*e.g.* C or Fortran).

The difference between an interpreted (*e.g.* Python) and compiled language (*e.g.* C) lies in the result of the process of interpreting or compiling. A compiler (*e.g.* gcc for

the C language  translate the source code into the computer native language, and create an executable file. The execution of compiled language does not require any more translations, and hence run significantly faster. Contrariwise, an interpreted language is not translated in advance, but is done at the execution, line by line, and each time the program is executed. This process is done by the interpreter, such as Python, Matlab or Perl for their eponymous language. At each execution, the time taken by the interpreter to read and execute each line slow the process, causing execution to take more time.

To overcome this problem with Python, the `cython` package has been developed to translate in C and compile the part of the code that is long to execute, especially the `for` loops. Thus, one can transform is Python source code into a hybrid Python-C code. As presented in appendix.A.1, compiling the `for` loop using `cython` in the full-Langevin simulations reduces the time to generate a 10^6 -step trajectory from 6 s to 30 ms thus achieving a speedup factor of 200. Moreover, in the hybrid version, the execution time is limited by the random number generation. Indeed, it takes $\approx 24.0\text{ms}$ using `numpy` to generate $10^6 w_i$ numbers and $\approx 6\text{ ms}$ for the trajectory computation. Additionally, as shown at the end of appendix.A.1, even a pure C implementation of the random generation can be slower than the `numpy` one, thanks to `numpy`'s memory optimization. Thus, by using the above mixed-language strategy, the simulation is optimal with the current tools and language at hand.

2.5 Conclusion

In this chapter, we have covered the history of Brownian motion, from the first observation by Robert Brown in the middle of the 19th century to its mathematical and experimental proofs in the early 20th century. We have then described mathematically the bulk Brownian motion and its important statistical properties. Finally, we have used the latter description to simulate Brownian motion using both the full-Langevin equation, and its over-damped version.

3 Particle Characterization and Tracking Using Optical Interferences

3.1 Introduction

Properties of coherent light to produce interference has been widely deployed in metrology for a long time as illustrated by, for example, the famous Fabry-Pérot [33, 34] and Michelson interferometers [35]. The latter was initially used to evaluate the rotation of the Earth and is still employed today for the recent measurement of gravitational waves [36]. Since the beginning of the century, interest on tracking and characterizing colloidal particles risen thanks to the democratization of micro fluidics and lab-on-a-chip technologies. In the following, I will provide some insights on the three most used tracking methods:

- Reflection Interference Contrast Microscopy (RICM)
- Lorenz-Mie theory
- Rayleigh-Sommerfeld back-propagation

The first one, RICM, uses the principle of optical-path difference in a Michelson interferometer. The other two, use the interferences between the light scattered by the colloid and the incident light. Generally, both sources are collinear, so that we speak of in-line holography.

3.2 Reflection Interference Contrast Microscopy

RICM was first introduced in cell biology by Curtis to study embryonic chick heart fibroblasts [37] in 1964. RICM gained in popularity 40 years after both in biology and physics [38–43]. It was also used recently in soft matter physics to study the elastohydrodynamic lift at a soft wall [44].

When we illuminate a colloid with a plane wave from the bottom, a part of the light is reflected at the surface of the glass substrate and another part, at the colloid surface. The difference of optical paths between two reflections creates an interference pattern. Let us focus on the mathematical description of this phenomenon. In the far field, we can describe two one-dimensional electric field vectors with same angular frequency ω [45] as propagative waves, through:

$$\vec{E}_1(\vec{r}, t) = \vec{E}_{01} \cos(\vec{k}_1 \cdot \vec{r} - \omega t + \epsilon_1), \quad (3.2.1)$$

and:

$$\vec{E}_2(\vec{r}, t) = \vec{E}_{02} \cos(\vec{k}_2 \cdot \vec{r} - \omega t + \epsilon_2), \quad (3.2.2)$$

where the k_i are the wave vector satisfying $|\vec{k}_i| = k = 2\pi n_m / \lambda$, the angular wavenumber, λ being the illumination wavelength, n_m the optical index of the medium, $\epsilon_{1,2}$ the initial phases of each wave and \vec{r} the position. Here, the origin ($\vec{r} = \vec{0}$) is taken at the position of the first reflection (*i.e.* the glass slide). Thus, on the particle, \vec{r} is given by the particle's height such that $|r| = z$.

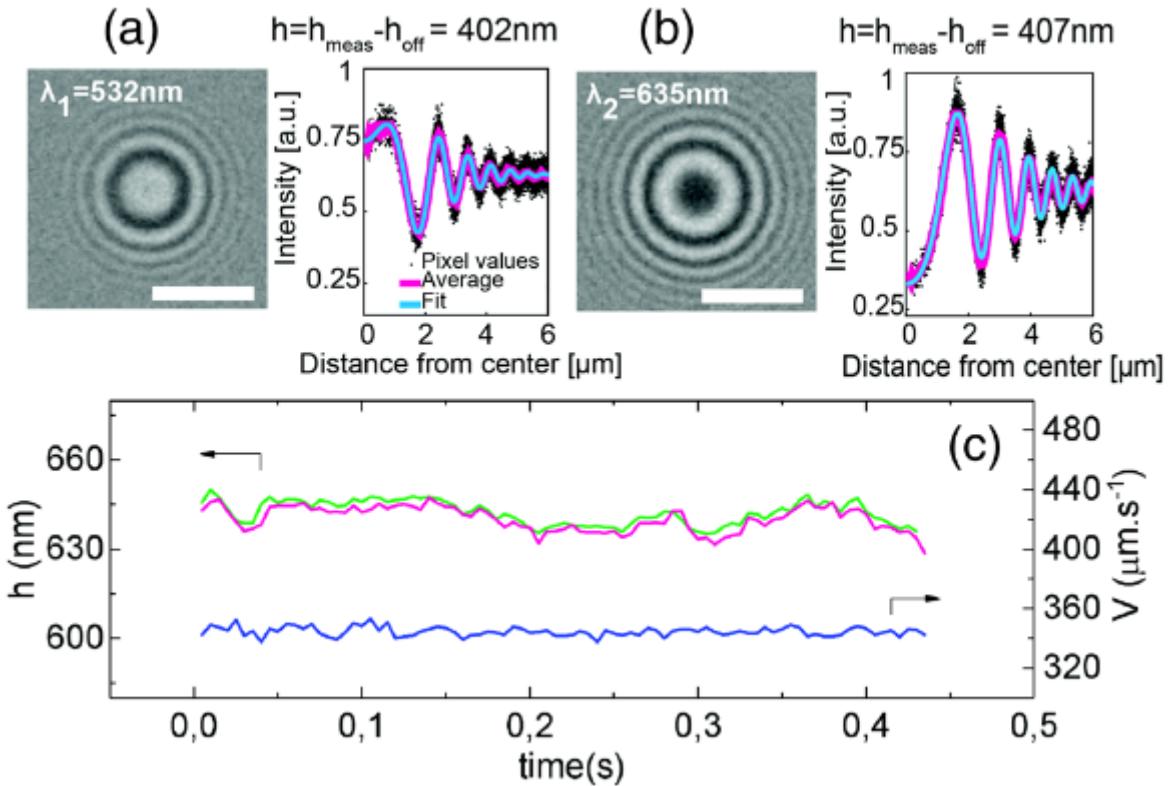


Figure 6: Figure from [44] representing RICM with two wavelengths. (a) Left: interference patterns created with a wavelength $\lambda_1 = 532 \text{ nm}$ (scale bar $5 \mu\text{m}$). Right: radial intensity profile (black dots) extracted from the image, azimuthally averaged (magenta line) and fitted with Eq.3.2.8 to measure the height of the particle (here noted h). (b) Same as (a) with a wavelength $\lambda_2 = 635 \text{ nm}$. (c) Time series of the height h of a particle (green: λ_1 , purple: λ_2) and the particle velocity measured along the flow (in blue).

Experimentally, one measures the intensity of the interference patterns. They can be computed from the time-averaged squared total electric field $\vec{E} = \vec{E}_1 + \vec{E}_2$. The measured intensity is thus given by:

$$I = \langle \vec{E}^2 \rangle = \langle \vec{E}_1^2 + \vec{E}_2^2 + 2\vec{E}_1 \cdot \vec{E}_2 \rangle = \langle \vec{E}_1^2 \rangle + \langle \vec{E}_2^2 \rangle + 2\langle \vec{E}_1 \cdot \vec{E}_2 \rangle , \quad (3.2.3)$$

where $\langle \vec{E}_1^2 \rangle$ and $\langle \vec{E}_2^2 \rangle$ are respectively given by I_1 and I_2 , the incident intensities. Using trigonometry, we have:

$$\left\langle \vec{E}_1 \cdot \vec{E}_2 \right\rangle = \left\langle \frac{1}{2} \vec{E}_{01} \vec{E}_{02} \left[\cos(\vec{k}_1 \cdot \vec{r} - \vec{k}_1 \cdot \vec{r} + \phi) + \cos(2\omega t + \phi') \right] \right\rangle_t . \quad (3.2.4)$$

As we average over time, the second cosine vanishes. Thus one has:

$$\langle \vec{E}_1 \cdot \vec{E}_2 \rangle = \frac{1}{2} \langle \vec{E}_{01} \vec{E}_{02} \rangle \cos(\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} + \phi) , \quad (3.2.5)$$

with ϕ the phase difference between the two fields, which is usually equal to π due to the reflection properties on a higher optical index. Indeed, a colloid has generally a greater optical index than the dilution medium. Finally, the total intensity can be read as:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} + \phi) . \quad (3.2.6)$$

By taking $k_1 = -k_2$ due to the reflection properties, we have:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(\frac{4\pi n_m}{\lambda} z + \phi\right) . \quad (3.2.7)$$

So far we supposed that the reflection occurs at a unique point; however, we would likely be using spherical colloids. Therefore, illuminating from the bottom, the reflection happens on half of the sphere surface. Moreover, thanks to the spherical geometry the holograms exhibit a radial symmetry, we thus write one can write the radial interference intensity $I(x)$, with x the distance from the pattern center, through [43]:

$$I(x) = A_0 + A_1 e^{-b_1 x^2} + A_2^{-b_2 x^2} \cos\left[\frac{4\pi n_m}{\lambda} (g(x) + z) + \phi\right] , \quad (3.2.8)$$

Where A_1 and b_1 are parameters [43] that fit the slightly bent background that arises from diffuse reflection on the upper part of the sphere, A_2 and b_2 the decaying contrast

of the higher order maxima, A_0 background intensity, and

$$g(x) = a - \sqrt{a^2 - x^2} , \quad (3.2.9)$$

is the sphere profile, to consider the increase sphere-wall as x increases. Finally, this method benefits from equations that are computationally light and enable a quick tracking of particles. However, as we can see in Eq. (3.2.8), because of the periodicity of the cosine, the interference pattern is the same for all heights z separated by a distance $\lambda/(2n_m) \approx 200$ nm for $\lambda = 532$ nm and $n_m = 1.33$.

It is possible to extend this separation to $\simeq 1.2 \mu\text{m}$ as used in [44] length by using two different wavelengths. Despite the spatial resolution of this method which can attain 10 nm, the measurement ambiguity is not compatible with the study of Brownian motion due to the periodicity above. Hence RICM it is not usable in our context. As a matter of fact, we experimentally reach height spans of a few microns.

3.3 Lorenz-Mie theory

When a colloid is illuminated with a plane wave, a part of the light is scattered. In consequence, the incident field \vec{E}_0 and scattered field \vec{E}_s interferes. The interference patterns thus obtained are called holograms. If the particle is not smaller than the illumination wavelength, it is not possible to use Rayleigh's approximations [46] to describe the scattering. Instead, one needs to use the Lorenz-Mie theory for dielectric spheres. This theory was developed by Lorenz and independently by Mie in 1880 and 1908, respectively [47, 48].

It is in the early 2000s that the Lorenz-Mie theory was first used in order to track and characterize particles [49, 50]. Since then, a lot of studies have been realized with this method [51]. In the following, I will describe the Lorenz-Mie method. In this part, the height z of the particle is the distance between the particle's center and the focal plane of the objective lens.

Let the incident field be a plane wave uniformly polarized along an axis \hat{e} , with an amplitude E_0 and propagating along the \hat{z} direction :

$$\vec{E}_0(\vec{r}, z) = E_0(\vec{r}) e^{ikz} \hat{e} \quad (3.3.1)$$

Let us consider a particle of radius a at a position \vec{r}_p . In such case, the scattered field can be written using the Lorenz-Mie theory [45], as:

$$\vec{E}_s(\vec{r}, z) = \vec{f}_s(k(\vec{r} - \vec{r}_p)) E_0(\vec{r}) \exp(-ikz) , \quad (3.3.2)$$

with \vec{f}_s , the Lorenz-Mie scattering function. The intensity I that we measure at \vec{r} is given by the intensity of superimposition of the incident and scattered amplitudes. Since the measurements are done at the focal plane, i.e. $z = 0$, I is given by:

$$\begin{aligned} I(\vec{r}) &= |\vec{E}_s(\vec{r}, 0) + \vec{E}_0(\vec{r}, 0)|^2 \\ &= E_0^2(\vec{r}) + 2E_0^2 \Re \left(\vec{f}_s(k(\vec{r} - \vec{r}_p)) \hat{e} \right) + |\vec{f}_s(k(\vec{r} - \vec{r}_p))|^2 . \end{aligned} \quad (3.3.3)$$

Most of the experimental defects on the images are due to spacial illumination variations caused by dust particles. They can be corrected by normalizing the image by the background. In another word, we normalize $I(\vec{r})$ by the intensity of the incident field $I_0 = E_0(\vec{r})^2$ which corresponds the experimental background.

Experimentally, the background can be measured by different methods. One is to have an empty field of view and the other one, which is more convenient, is to compute the median of a stack of images. Additionally, for the latter to work, the movie should be long enough for the particle to diffuse sufficiently. If this condition is not satisfied, a ghost of the particle will appear on the background. Moreover, this process permits getting rid of any immobile particles that could generate any additional noise. Examples of hologram before and after the normalization are shown in Figs.7 a-c).

Finally, we write the normalized intensity as:

$$\frac{I(\vec{r})}{I_0(\vec{r})} = 1 + 2\Re \left(\vec{f}_s(k(\vec{r} - \vec{r}_p)) \hat{e} \right) + |\vec{f}_s(k(\vec{r} - \vec{r}_p))|^2 \quad (3.3.4)$$

Now that we have the analytical form of the holograms' intensity, it is possible to fit an experimental one to Eq.3.3.4 as shown in Figs.7 d-e). For the sake of completeness, I will detail the Lorenz-Mie scattering function $\vec{f}_s(k\vec{r})$, which is given by the series:

$$\vec{f}_s(k\vec{r}) = \sum_{n=1}^{n_c} \frac{i^n (2n+1)}{n(n+1)} \left(ia_n \vec{N}_{\text{eln}}^{(3)}(k\vec{r}) - b_n \vec{M}_{\text{ohn}}^{(3)}(k\vec{r}) \right) \quad (3.3.5)$$

where $\vec{N}_{eln}^{(3)}(k\vec{r})$ and $\vec{M}_{oln}^{(3)}(k\vec{r})$ are the vectorial spherical harmonics, and a_n and b_n are coefficients depending on the particle and illumination properties. For a spherical and isotropic particle of radius a and refractive index n_p , which is illuminated by a linearly polarized plane wave, the a_n and b_n coefficients are expressed in terms of spherical Bessel functions j_n and Hankel functions h_n , as [45]:

$$a_n = \frac{\zeta^2 j_n(\zeta ka) k a j'_n(ka) - j_n(ka) [\zeta k a j_n(\zeta ka)]'}{\zeta^2 j_n(\zeta ka) k a h_n^{(1)'}(ka) - h_n^{(1)}(ka) \zeta k a j'_n(\zeta ka)}, \quad (3.3.6)$$

and:

$$b_n = \frac{j_n(\zeta ka) k a j'_n(ka) - j_n(ka) \zeta k a j'_n(mka)}{j_n(\zeta ka) k a h_n^{(1)'}(ka) - h_n^{(1)}(ka) \zeta k a j'_n(mka)}, \quad (3.3.7)$$

where $\zeta = n_p/n_m$, and where the prime notation denotes differentiation with respect to the argument.

Finally, an hologram is mainly given by the Lorenz-Mie scattering function of Eq.3.3.5. Moreover, as we can observe in Eqs.3.3.6 and 3.3.7, an hologram depends on a lot of parameters and variables (λ , n_m , n_p , a and \vec{r}_p). The parameters can be fitted by comparison to experimental data. In general, the illumination wavelength λ and medium index n_m are known and do not need to be fitted. From only one hologram, one can measure precisely the position \vec{r}_p of the particle and simultaneously characterize the radius and optical index of the colloid. As a side note, it is even possible to characterize a particle without a priori knowledge of its characteristics using a Bayesian approach [52, 53].

Computing Eq.3.3.5 numerically brings another interesting question, as it is analytically written as a sum over n . One could ask after which number n_c of terms the series converges. It has actually been found that the series converges after a number of terms given by [54]:

$$n_c = ka + 4.05(ka)^{1/3} + 2. \quad (3.3.8)$$

Consequently, the holograms of larger particles require more terms to converge and, hence, are longer to fit. As an example, the largest particles used during my thesis have a radius $a = 2.5 \mu\text{m}$ leading to $n_c = 55$ in water and for an illumination wavelength $\lambda = 532 \text{ nm}$. For the smallest ones, where $a = 0.5 \mu\text{m}$ we find $n_c = 18$ which makes a huge difference in practice. Indeed, if each of the terms of the sum takes the same time to be computed; a $2.5 \mu\text{m}$ particle's hologram is $55/18 \simeq 3$ times longer to be fitted compared to the hologram

of a $0.5 \mu\text{m}$ particle.

If a reader wants to evaluate an hologram given by the Lorenz-Mie theory for a peculiar particle and position, it can be done in a few lines with the `holopy` module using the following Python snippet which was used to make Fig.13 and 14:

```
1 import holopy as hp
2 from holopy.scattering import calc_holo, Sphere
3
4 sphere = Sphere(n=1.59, r=1.5, center=(4/0.1, 4/0.1, 10))
5 # n is the optical index of the particle, r its radius in microns
6 # center is its center position in microns.
7
8 medium_index = 1.33
9 illum_wavelen = 0.532
10 illum_polarization = (1, 0)
11 detector = hp.detector_grid(shape=100, spacing=0.1)
12 # shape is the size in pixels of the camera and the spacing is the pixel's size in microns.
13
14 holo = calc_holo(
15     detector, sphere, medium_index, illum_wavelen, illum_polarization, theory="auto"
16 )
17 #the hologram can directly be plotted using:
18 hp.show(holo)
```

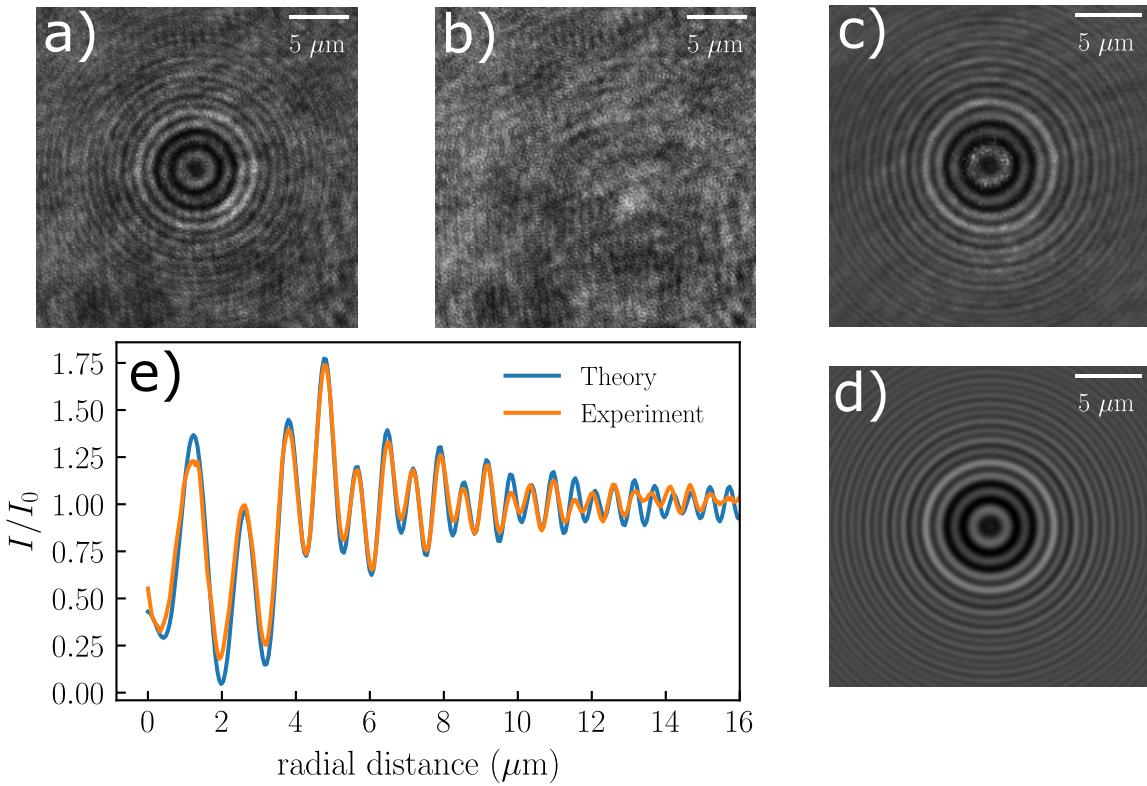


Figure 7: a) Raw hologram of a $2.5 \mu\text{m}$ polystyrene particle measured experimentally with the setup detailed in section 3.5. b) Background obtained by taking the median value of an image time series. c) Normalized hologram given by dividing a) by b). d) Result of the fit of c) using Eq.3.3.4, from which the particle is found to be at a height $z = 14.77 \mu\text{m}$. e) Comparison of the normalized radial intensities, obtained experimentally from c) and theoretically from d).

3.3.1 Hologram dependence on the particule's characteristics

As we can see with the Eq.3.3.5, the in-line holograms vary with the position, radius and optical index of the particle. For in-line holograms, as both both incident and scattered field are collinear, the x and y positions of the particle are given by the center of the hologram. Thus, it is possible to track the motion of a colloid only in 2 dimensions by using algorithm such as the Hough transforms to find the center. As a side note, in that case, it would be optimal to place the particle just above the focal plane to have an Airy disk like hologram, as shown in Fig.14 for $a = 2.5 \mu\text{m}$ and $z = 5 \mu\text{m}$.

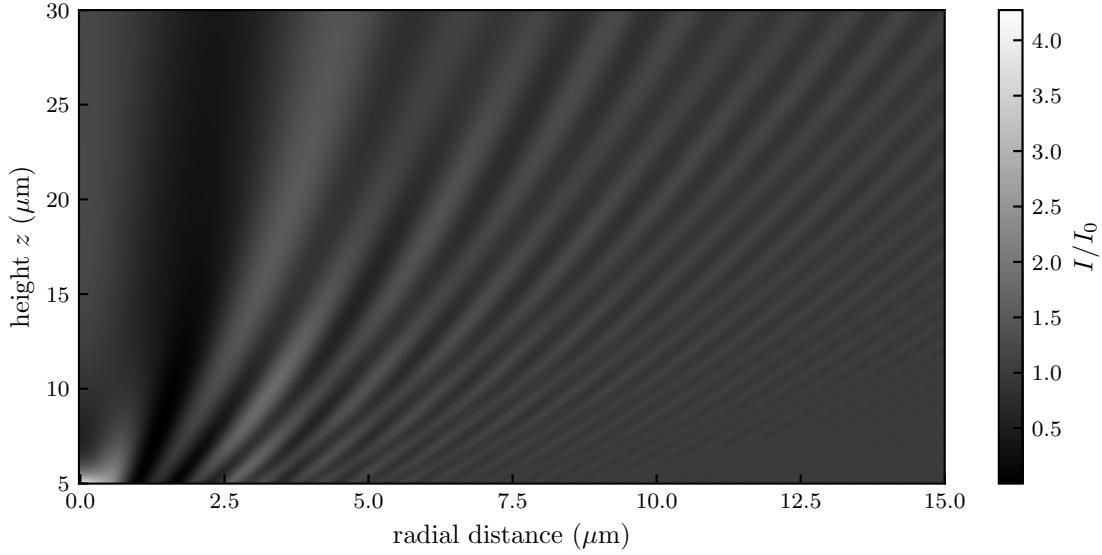


Figure 8: Radial intensity profile stack as a function of the distance between the particle center and the focal place of the objective lens, generated for a particle of radius $a = 1.5 \mu\text{m}$ and optical index $n = 1.59$.

In order to gain some insights on how the holograms vary with the different parameters, one can compute holograms for particles of different size and height. We will start by looking at a particle of a radius $a = 1.5 \mu\text{m}$ and $n_p = 1.59$ as shown in the Fig.8. In this case, one can observe that as the distance between the particle and the focal plane z increases, the hologram's rings get larger. Unlike a Michelson interferometer or RICM we do not observe the rings scrolling.

Additionally, this thickening of the rings can also be observed on the Fig.12, where hologram's intensity profile are plotted as a function of the height z both theoretically and experimentally for a polystyrene colloidal particle of radius $a = 1.5 \mu\text{m}$, and, for different couples of parameters on the Fig.14.

Also, we can note that if z is not large enough compared to the radius of the particle, the center of an hologram can be so bright that if the camera does not have a large enough dynamic range, the rings could not be seen. Thus, for having an optimal condition for the fits, one should take care to defocus the enough the objective lens to have $z \gg a$.

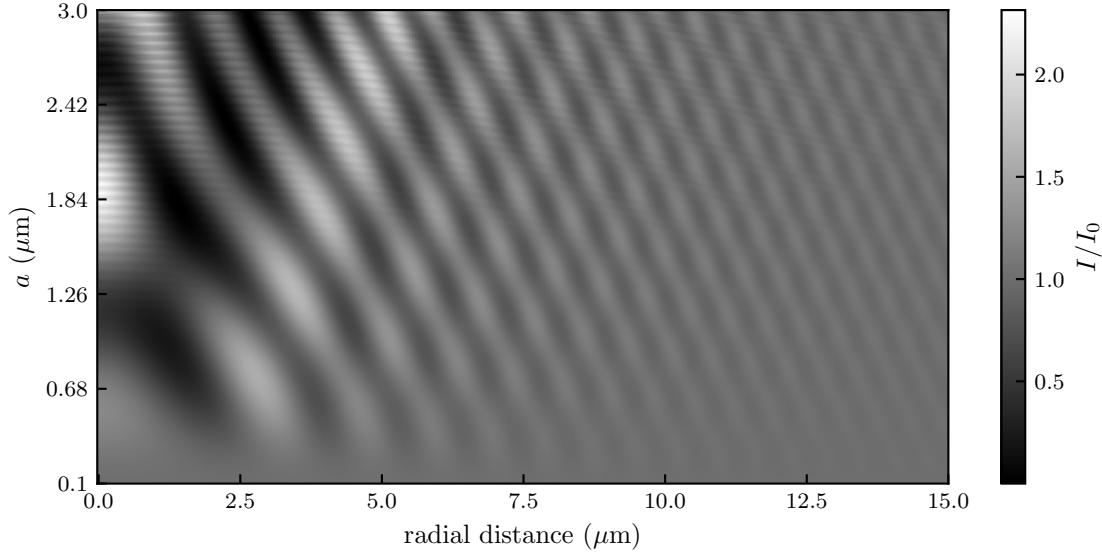


Figure 9: Radial intensity profile stack as a function of the particle radius, generated for a particle of optical index $n = 1.59$ with a distance between the particle center and the focal place of the objective lens $z = 15 \mu\text{m}$.

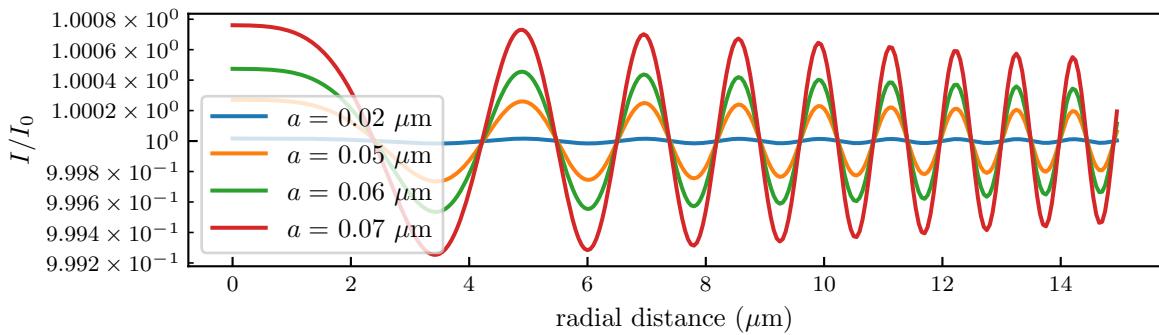


Figure 10: Radial intensity profile for the particle radius $a \ll \lambda$, generated for a particles of optical index $n = 1.59$ with a distance between the particle center and the focal place of the objective lens $z = 15 \mu\text{m}$ and $\lambda = 532 \text{ nm}$.

We can now take a look at the variation with respect to the radius of the particle as shown on the Fig.9 for a particle of optical index $n = 1.59$ and at a distance $z = 15 \mu\text{m}$. One can observe that for small particles compared to the wavelength $a \ll \lambda$ we do not observe the rings this is due to the fact that for the small particles, the scattering can be approximated using the Rayleigh theory which tells us that the scattering is isotropic. Thus, the variation of intensity around I_0 will be smaller for small particle.

Also, in this small particle regime, the particle size will not affect the general shape of the hologram but just the intensity of the hologram as it can be seen on the Fig.10, for particle of radius $a = 0.02 \mu\text{m}$ to $a = 0.07 \mu\text{m}$ for a wavelength $\lambda = 0.532 \mu\text{m}$.

Additionally, since the noise to signal ratio will be lower than for bigger particles, it will be less precise to characterize small colloids compared to the wavelength.

As the particle gets bigger, the scattering become anisotropic and it mostly towards the incident plane wave direction. This effects leads to an increase of the amplitude of the rings I/I_0 , as one can see on the Fig.9. Thus, the noise to signal ratio is high enough to easily discern the hologram on top of the noise as one can see on the experimental picture fig.7-a). One who wants to use this method should thus take care to use large enough particle for the holograms' intensity to be greater than the camera noise level.

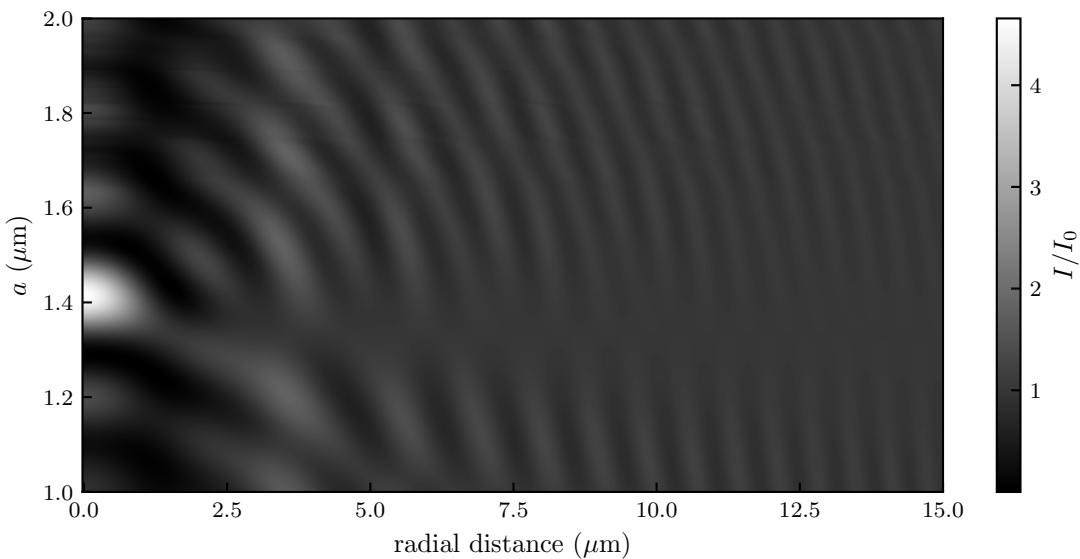


Figure 11: Radial intensity profile stack as a function of the particle optical index, generated for a particle of radius $a = 1.5 \mu\text{m}$ with a distance between the particle center and the focal place of the objective lens $z = 15 \mu\text{m}$.

Finally, one can also check how the holograms are varying with the optical index a particle. In this case it is not the particle's optical index n_p which will matter the most but the ratio $\zeta = n_p/n_m$ which can be found in the a_n and b_n formulas, Eq.3.3.6 and 3.3.7. Indeed, for the scatter to happen, the optical index n_p of the colloid needs to be different from the optical index of the surrounding medium n_m . Additionally, the numerical solution of the Lorenz-Mie scattering is not stable for $n_p \simeq n_m$. In the Fig.11, we can observe holograms of a particle of radius $a = 1.5 \mu\text{m}$ at fix distance between the particle and the focal plane of the objective lens $z = 15 \mu\text{m}$ with a varying colloid's optical index, in water $n_m = 1.33$. On the Fig.11, one can thus observe than for $n_p \simeq n_m$ we do not obervve any holograms. Additionally, one can observe that the noise to signal ratio gradually increases as n_p became different from n_m . One who wants to use this method should thus take care to the particle material or the solvent to have n_m different enough to n_p for the

holograms' intensity to be greater than the camera noise level.

3.3.2 Lorenz-Mie conclusion

The combination of the height, optical index and radius of colloid thus gives unique holograms. This uniqueness of the holograms permits extracting precisely the position, optical index and radius of a colloid. In order to see how holograms are for different couples of parameters on the Figs.13 and 14, one can see possible holograms for different size and height. Additionally, one can use the Jupyter Notebook on my github repository in order to plot any hologram [Q](#).

Finally, Lorenz-Mie is the most versatile in-line holographic method, indeed, it permits tracking and characterize unique particles even without a priori knowledge. Besides, it is possible to write the Lorenz-Mie function \vec{f}_s for particular cases such as anisotropic [55], non-spherical particles [56] or particle clusters [55, 57] to name a few; such possibilities open the door to a lot of experimental studies. Additionally, it can reach really high precision as the tenth of nanometer on the position and radius as well as 10^{-3} on the optical index [50].

Unfortunately, the Lorenz-Mie fitting suffer from a major drawback which is the time needed to fit one image. For example, a 200 by 200 pixels image, of a $2.5 \mu\text{m}$ particle's hologram, can take up to two minutes to be fitted using a pure and straightforward python algorithm. A lot of work as been done to have faster tracking, such as random-subset fitting [58], GPU (graphical processing unit) acceleration, machine-learning [59, 60] and deep neural networks [61].

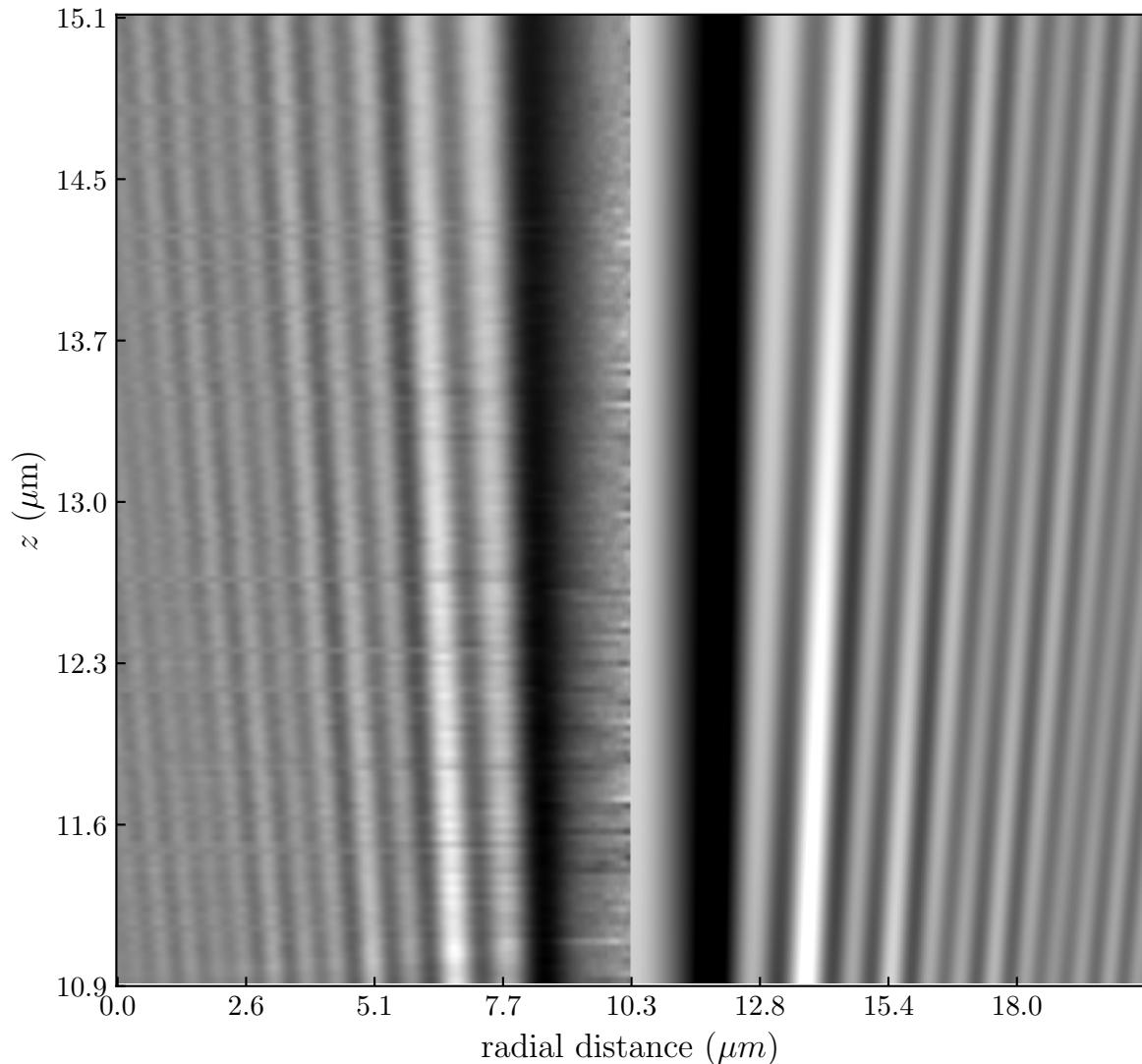


Figure 12: On the left, experimentally measured holograms' radial intensity profile stack, generated from a polystyrene bead of nominal radius $a = 1.5 \pm 0.035 \mu m$ using the experimental setup explained in chapter 3.5. The calibration of this particle radius and optical index is shown in Fig.20. On the right, the corresponding theoretical stack using the result of each individual hologram's fit.

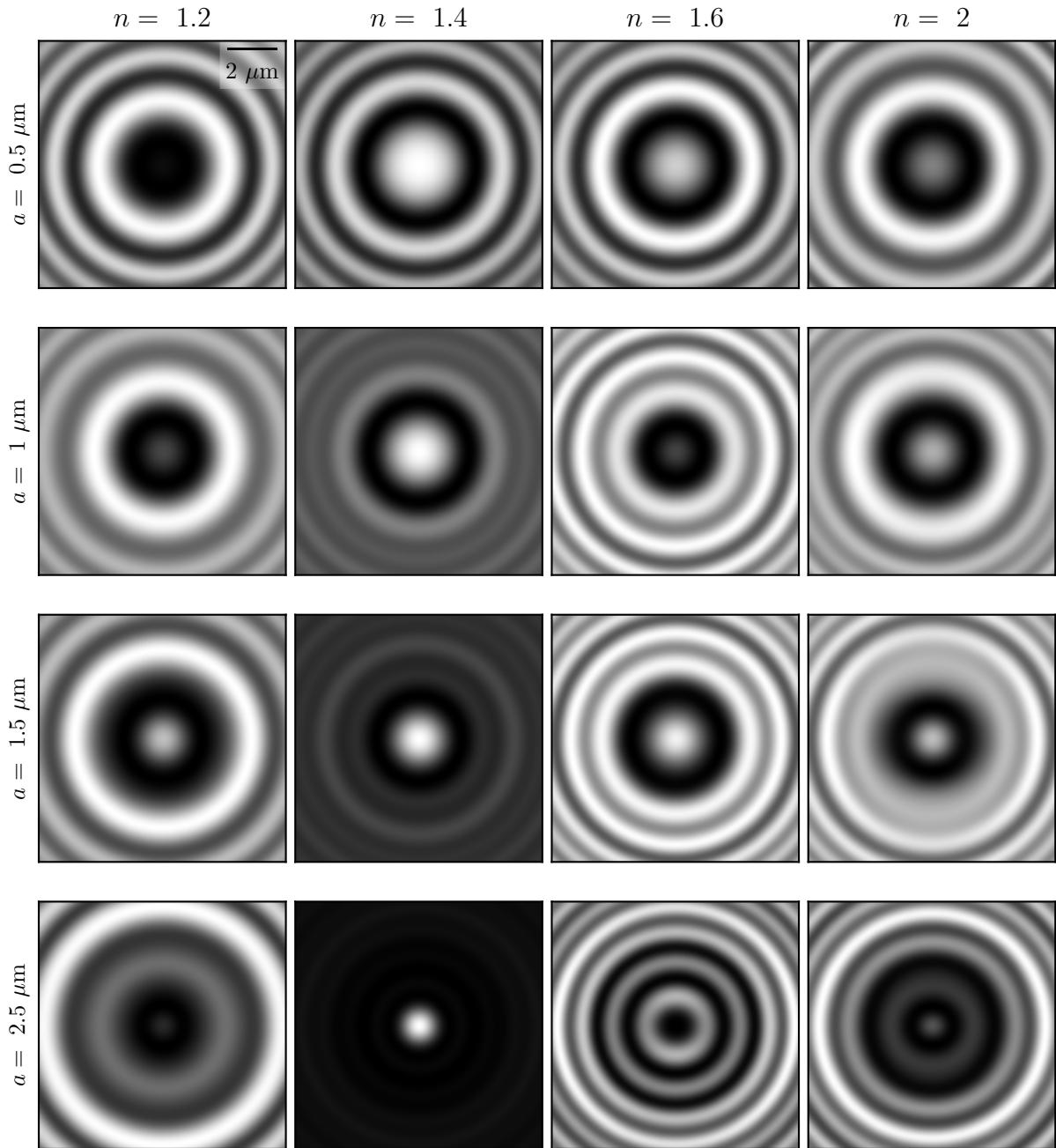


Figure 13: On the left, experimentally measured holograms' radial intensity profile stack, generated from a polystyrene bead of nominal radius $a = 1.5 \pm 0.035 \mu\text{m}$ using the experimental setup explained in chapter 3.5. The calibration of this particle radius and optical index is shown in Fig.20. On the right, the corresponding theoretical stack using the result of each individual hologram's fit.

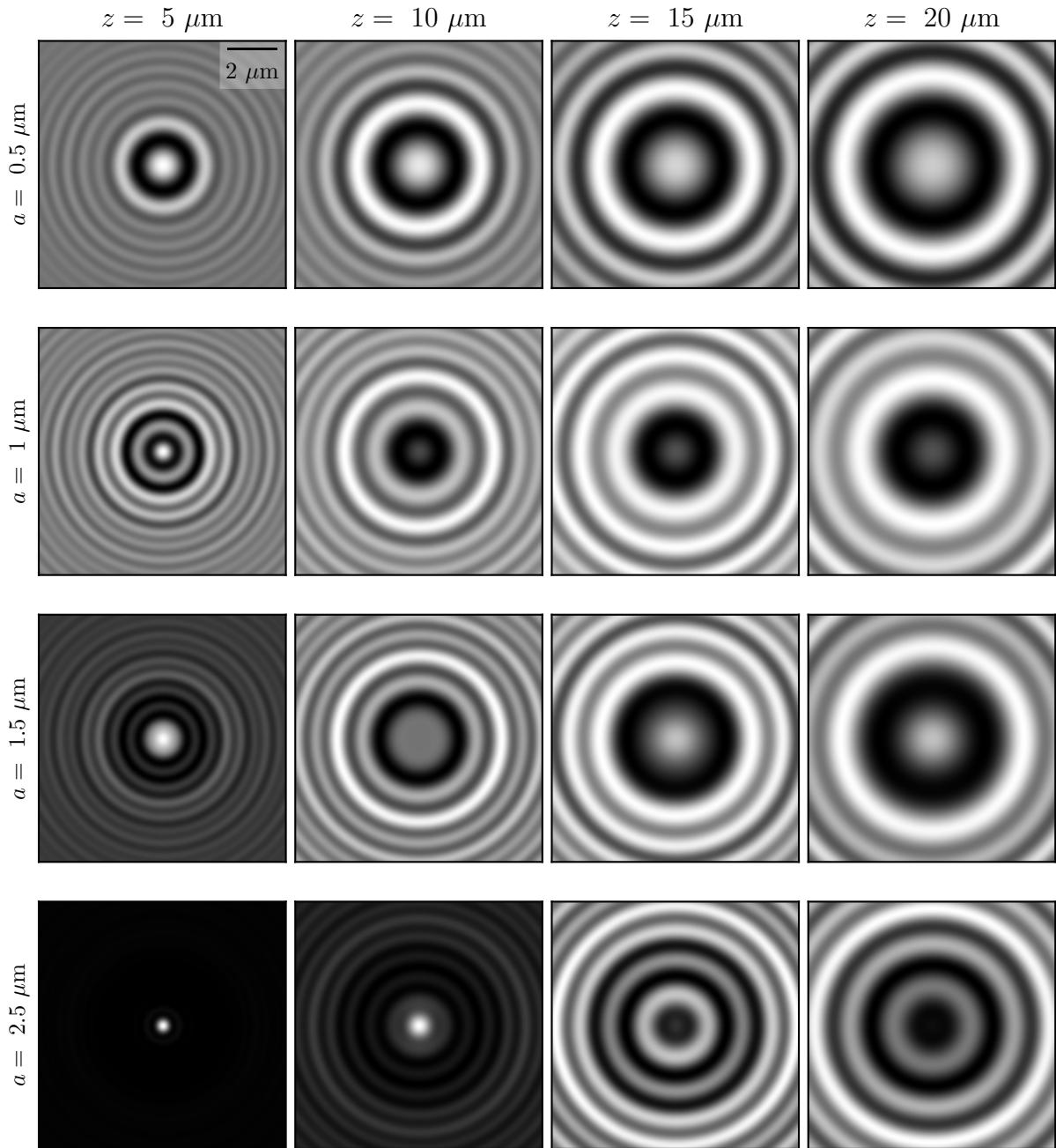


Figure 14: On the left, experimentally measured holograms' radial intensity profile stack, generated from a polystyrene bead of nominal radius $a = 1.5 \pm 0.035 \mu\text{m}$ using the experimental setup explained in chapter 3.5. The calibration of this particle radius and optical index is shown in Fig.20. On the right, the corresponding theoretical stack using the result of each individual hologram's fit.

3.4 Rayleigh-Sommerfeld back-propagation

Rayleigh-Sommerfeld back-propagation [62] works on the same principle as the Lorenz-Mie fitting but assumes that we have small scatterers, or, a low difference of optical index., such as :

$$|\zeta - 1| \ll 1 \text{ and } ka|\zeta - 1| \ll 1 . \quad (3.4.1)$$

In this case, at the focal plane, the intensity of the scattered field is smaller than the incident field, hence, the term $|\vec{E}_s|^2$ can be ignored. Thus, the normalized intensity, Eq.3.3.4 can be rewritten as:

$$\frac{I(\vec{r})}{I_0(\vec{r})} = 1 + 2\Re \left(\frac{E_s(\vec{r}, 0)}{E_0(\vec{r})} \right) . \quad (3.4.2)$$

If one can retrieve completely the scattered field from an image, it is possible to reconstruct it above the focal plane by convolution using the Rayleigh-Sommerfeld propagator [63]:

$$h_{-z}(\vec{r}) = \frac{1}{2\pi} \frac{\partial}{\partial z} \frac{e^{ikR}}{R} , \quad (3.4.3)$$

where $R^2 = r^2 + z^2$ and the sign convention on the propagator indicates if the particle is below or above the focal plane. Using this propagator we have:

$$E_s(\vec{r}, z) = E_z(\vec{r}, 0) \otimes h_{-z}(\vec{r}) \quad (3.4.4)$$

By using the convolution theorem [63–66] and supposing a uniform illumination, one can write the reconstructed scattered field at a height z as:

$$E_s(\vec{r}, z) \approx \frac{e^{ikz}}{4\pi^2} \int_{-\infty}^{\infty} B(\vec{q}) H(\vec{q}, -z) e^{i\vec{q} \cdot \vec{r}} d^2 q , \quad (3.4.5)$$

where $B(\vec{q})$ is the Fourier transform of I/I_0 and $H(\vec{q}, -z)$ is given by

$$H(\vec{q}, -z) = e^{iz\sqrt{k^2 - q^2}} . \quad (3.4.6)$$

Finally, using Eq.3.4.5 one can reconstruct the scattered field and intensity since $I(\vec{r}) = |E_s(\vec{r})|^2$ as shown in Fig.15. Moreover, by finding the position where we have an inversion of the center from bright to dark Fig.15, we measure the position of the particle. Those equations are way less computational intensive than the Lorenz-Mie function Eq.3.3.5. Thus tracking can be way faster, moreover, Fourier transforms can be largely accelerated using GPU.

Additionally, as the propagator Eq.3.4.3 take only into account the intensity of the image, this method does not require any information on the particle and number of particles. As a matter of fact, to write Eq.3.4.5, one just need to assume that we have spherical colloids. Thus, this method is great to reconstruct the 3D position of a lot of particles or clusters formations.

However, the major drawback is that it is the less precise of the presented measurements and that we can't use it to characterize the particles generating the holograms.

3.4.1 Numerical Rayleigh-Sommerfeld back-propagation

The `holopy` Python module also provide a set of method that permits to user the Rayleigh-Sommerfeld back-propagation. Given the `hologram` variable containing all the needed metadata about the hologram such as the pixel size, medium index n_n illumination wavelength λ and the actual image. Then, one can use the `propagate` method to back-propagate an hologram over a set of height `zstack` using the following Python snippet.

```

1 import holopy as hp
2 import numpy as np
3
4 zstack = np.linspace(0, 20, 11)
5 rec_vol = hp.propagate(holo, zstack)
```

Please note that using the `propagate` each propagation will be done by performing a convolution of the reference hologram over the distance to be propagated. However, better reconstruction can be obtained iteratively propagating holograms over short distance. The latter method is called Cascaded Free Space Propagation, and is particularly useful when the reconstructions have fine features or when propagating over large distances [67]. It can be done by specifying the argument `cfsp` to the `propagate` method. For example, to propagate three steps over each distance, we can use `hp.propagate(holo, zstack, cfsp=3)`.

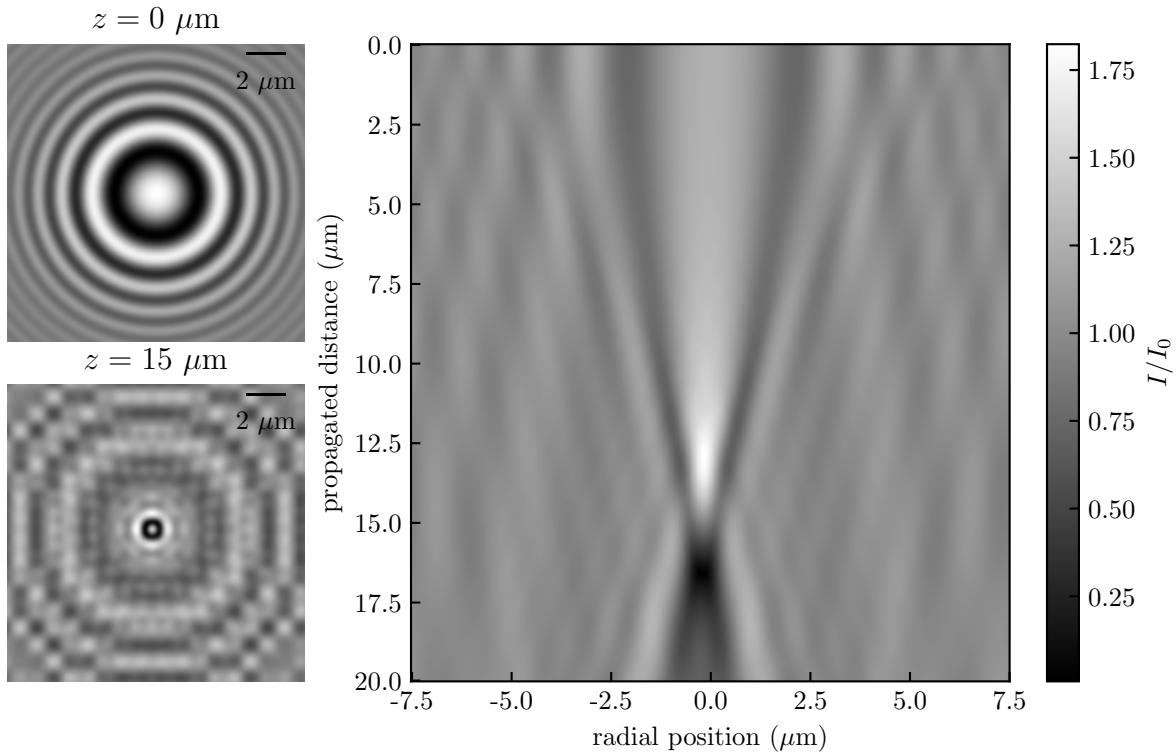


Figure 15: On the left: the original hologram on the top and propagated along $15 \mu\text{m}$ on the bottom. On the right: Reconstruction using Eq.3.4.5 of the scattered intensity of single colloidal sphere of a radius $a = 0.1 \mu\text{m}$ polystyrene spheres, $n_p = 1.59$ in water at a original $n_m = 1.59$, height of $15 \mu\text{m}$.

3.5 Experimental setup

The experimental setup I used during my PhD could be employed in order to use both Lorenz-Mie and Rayleigh-Sommerfeld Back propagation methods. In order to observe the holograms we use an homemade inverted microscope as shown on the Fig.16 and schematized in Fig.17. This microscope is built using Thor Labs cage system, as it permits doing vertical construction easily. Using this microscope, we observe the holograms coming form the interactions between a laser sourced and the beads present in a sample.

A sample consists of a parallelepipedic chamber ($1.5 \text{ cm} \times 1.5 \text{ cm} \times 150 \mu\text{m}$), made from two glass covers, a parafilm spacer, and sealed with vacuum grease, containing a dilute suspension of spherical polystyrene beads. Sealing the sample with vacuum grease permits to drastically decrease the evaporation, this permits to reduce the possible evaporation driven flow in the sample.

We used 3 different colloidal sizes, of nominal radii $0.56 \mu\text{m}$, $1.5 \mu\text{m}$ and $2.5 \mu\text{m}$, at room temperature T , in distilled water (type 1, MilliQ device) of viscosity $\eta = 1 \text{ mPa.s}$. The particles are made of polystyrene of density $\rho = 1050 \text{ kg.m}^{-3}$ and optical index $n_p = 1.598$

for a wavelength of 532 nm.

The sample is illuminated by a collimated laser beam with a 532 nm wavelength. The used laser delivers a power of 4 W and has a centimetric waist. Since the laser is collimated, it has a near zero exentricity such as it can be seen as a plane wave. With this characteristics, we can assume that there is no optical force exerted on the particles [NEED TO CHECK HOW TO EVALUATE THAT]. As depicted in the chapter 3.3, the light scattered, by one colloidal particle at a given time t , interferes with the incident beam.

An oil-immersion objective lens (x60 magnification, 1.30 numerical aperture) collects the resulting instantaneous interference pattern, and relays it to a camera (Basler acA1920-155um) with a 51.6 nm/pixel resolution (see Fig.7a)).

The exposure time of the camera is set to $\tau_{\text{expo}} = 3$ ms to avoid motion-induced blurring of the image. As a general rule, the particle should not diffuse more than the pixel size during that time, such that here $2D\tau_{\text{expo}} < 51.6$ nm.

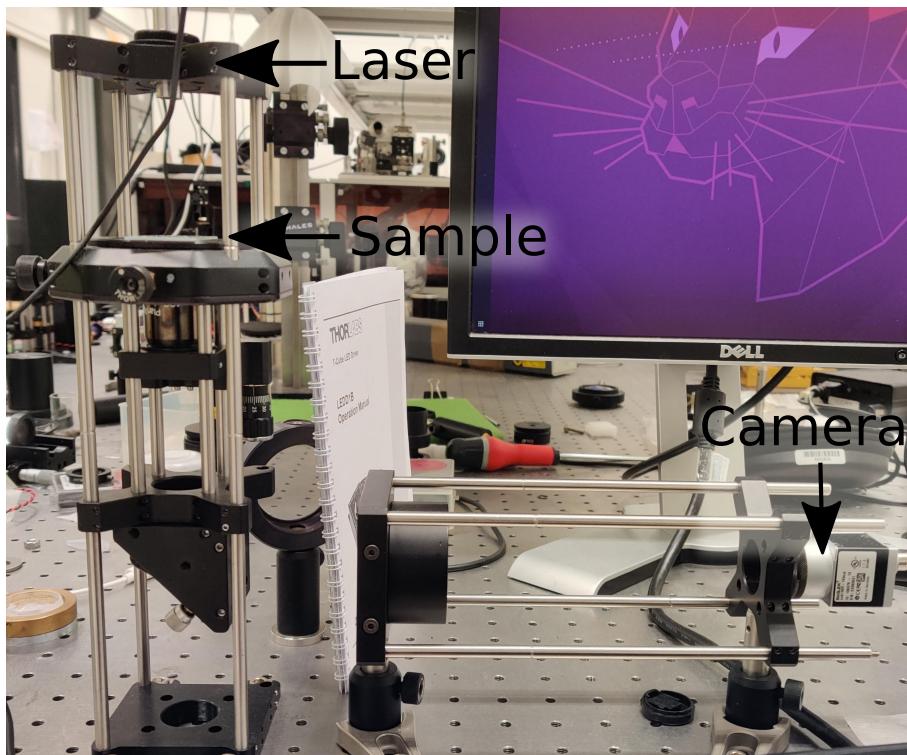


Figure 16: Photo of the custom build microscope used along my thesis. It is mainly composed of Thorlabs cage system. The camera used is a Basler acA1920-155um, we use a x60 magnification and 1.30 numerical aperture oil-immersion objective lens. The light source is a collimated 521 μm wavelength laser.

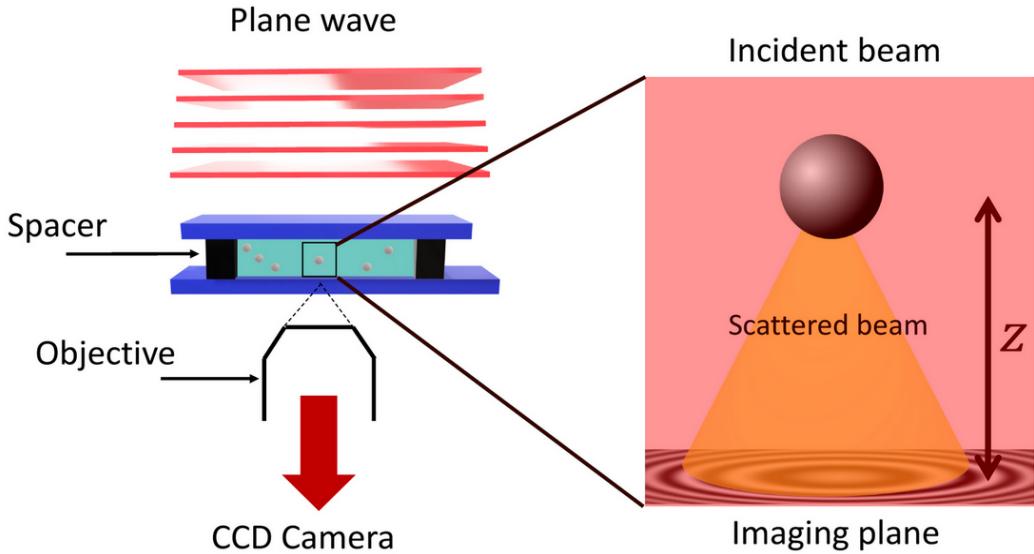


Figure 17: Schematic of the experimental setup. A laser plane wave of intensity I_0 illuminates the chamber containing a dilute suspension of micro-spheres in water. The light scattered by a particle interferes with the incident beam onto the focal plane of an objective lens, that magnifies the interference pattern and relays it to a camera.

3.6 Optical forces

As we illuminate the particle with a laser, it is important to know if the optical forces that arises from the interactions between the light and the particle needs to be taken into account. When a plane wave is incident on a sphere, it scatters and absorbs light. This process depends both on the light wavelength λ and on the sphere properties, its radius a and refractive index $n_p = n_r - jn_i$. As for polystyrene the imaginary part $n_i \ll 1$ as shown in Fig.18 we neglect it, however, since we want here compute the optical forces, and hence, the quantity of the light which is absorbed we will consider it in this section. Additionally, in the Mie theory, the particle is characterized by the size parameter $\tilde{x} = 2\pi a/\lambda$. The optical force F_{opt} is given by [45]:

$$F_{\text{opt}} = \frac{I_r n_m \pi a^2}{c} (Q_{\text{ext}} - g Q_{\text{sca}}) , \quad (3.6.1)$$

where I_r is the irradiance in W.m^{-2} on the sphere, c is the speed of light in vacuum, $g = \pi r^2$ the sphere cross section and Q_{ext} and Q_{sca} being respectively the extinction and scattering efficiency given by:

$$Q_{\text{ext}} = \frac{2}{k^2 a^2} = \sum_{n=1}^{\infty} (2n+1)(|a_n|^2 + |b_n|^2) , \quad (3.6.2)$$

and,

$$Q_{\text{ext}} = \frac{2}{k^2 a^2} = \sum_{n=1}^{\infty} (2n+1)\Re(a_n + b_n) , \quad (3.6.3)$$

where the a_n and b_n coefficient are given by the Eqs.3.3.6 and 3.3.7 respectively.

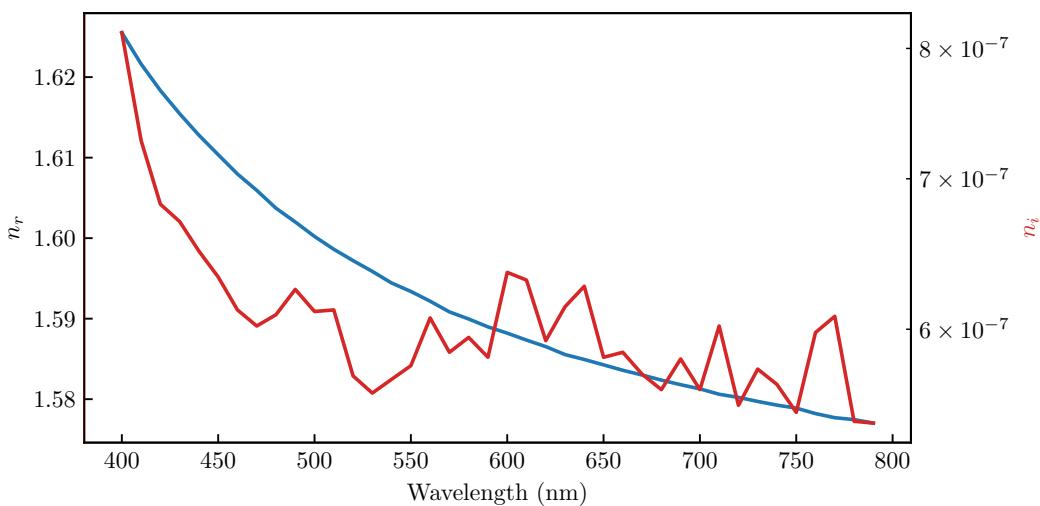


Figure 18: Real (left axis) and imaginary (right axis) part of the refractive index of polystyrene as a function of the incident wavelength. Data obtained from [68]

To compute the optical force I personally use the `miepython` python's module and retrieving the optical index data using the refractiveindex.info website, using the following Python snippet.

```

1 import miepython as mp
2 import numpy as np
3
4 # Download the data on the refractiveindex.info website
5 poly = np.genfromtxt(
6     r"https://refractiveindex.info/tmp/data/organic/(C8H8)n%20-%20polystyren/Zhang.txt",
7     delimiter="\t",
8 )
9 N = len(poly) // 2

```

```

10 poly_lam = poly[1:N, 0] # wavelength
11 poly_nre = poly[1:N, 1] # real part
12 poly_nim = poly[N + 1 :, 1] # imaginary part
13
14 x = 2 * np.pi * a / poly_lam
15 n = poly_nre - 1.0j * poly_nim
16 qext, qsca, qback, g = mp mie(n, x) # compute the efficiencies
17 E0 = 4.5e-3 / (np.pi * 1.75e-3 ** 2) # compute the irradiience
18 c = 299792458 / 1.33 # light velocity in the medium
19
20 F = E0 * np.pi * r0 ** 2 * (qext - g * qsca) / c # compute the optical force

```

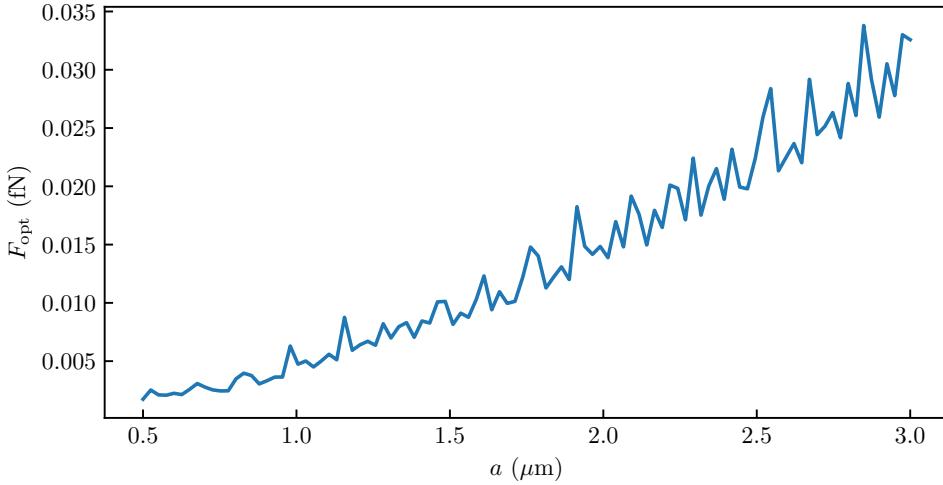


Figure 19: Optical force F_{opt} (see Eq.3.6.1) exerted on a spherical particle of radius a by a plane wave of wavelength $\lambda = 532 \text{ nm}$, and of irradiance $I_r = 467.7 \text{ W.m}^{-2}$. The force is computed using the `miepython` python's module and the refractive index of polystyrene [68].

Using Python, one can thus compute the optical force F_{opt} as a function of the particle radius. As shown on Fig.19 the optical exerted on the colloidal particle from the microscope's illumination source, is of the order of magnitude $F_{\text{opt}} \simeq 10^{-2} \text{ fN}$. By comparing this result with the buoyant of a colloidal particle F_g :

$$F_g = \frac{4}{3}\pi(\rho_m - \rho_p)g , \quad (3.6.4)$$

we found that for a particle of radius $a = 0.5 \mu\text{m}$, $F_g = -0.3 \text{ fN}$ and for $a = 1.5 \mu\text{m}$, $F_g = -7 \text{ fN}$. Thus, we can conclude that $F_{\text{opt}} \ll F_g$, hence, in the following of the manuscript we neglect the optical force as it is lower than the other external forces acting on the

colloids. However, experiments on Brownian motion with has been done by increasing the irradiance to 10^6 W.cm^{-2} and they were able to measure the optical force from the motion statistics [19].

3.7 Experimental pipeline

The method we choose is the Lorenz-Mie fitting method, since this it permits the characterization of single particles. Indeed, since we are interested to fine effects near the surface, we need to know perfectly the radius of the particle we have recorded. This feature also make our all process calibration free, as we don't need to assume any physical properties. An example of the pipeline that permits to track a single particle trajectory is shown in the appendix A.2. In the following, the different steps of the pipe line are going to be described.

3.7.1 Recording the holograms

Since we use a Basler camera, we use the provided Pylon software in order to record the holograms. The latter permits to adjust the parameters of the camera, such as the region of interest (ROI), frames per second (fps) or the opturation time to name a few. Also, movies can be recorded as a time series of images, AVI or MP4 files. AVI files or times series are a great way to save the movies since they are lossless. However, in general we use an ROI of 1000×1000 pixels to record the particle for a long time enough.

Additionally, since the recording is done using 8 bits per pixel (or 256 gray levels), an image of 1000×1000 pixels needs a disk space of 1 MB². One can thus see that image sequences and AVI files are not suitable for our case because of several points: i) at 100 fps one would need 108 GB to store a 30 minutes film, it could lead to TB of data per experiment which was not manageable. ii) it would require a sequential writing speed of 60 MB/s, which is just below the limit of the better HDD. iii) AVI files are bound to 2 GB maximum thus dramatically reducing the length of the experiments.

To conclude, for all of these reasons, we choosed to use the MP4 files format (MPEG-4 encoding) for the video recording. Using the lowest compression, we did not observe any

² An uppercase B denotes Byte which is equivalent to 8 bits, such as $1 \text{ B} = 8 \text{ b}$. For storage indications, Bytes are generally used, since historically a set of 8 bits encodes a single text character, and, are for this reason the smallest addressable memory unit in most of the computer architectures. As an example, in binary “LOMA” would be encoded by “01001100 01001111 01001101 01000001”

impact on the fitting process due to quality loss. Finally, a video of 30 minutes will size $\simeq 3$ GB.

3.7.2 Fitting the holograms

Once the holograms are recorded, we fit all of the images to retrieve the trajectory of the particle. To do so, we choosed to use the Pylorenzmie module developed by the Grier's lab at the New York University. Although this module present a lot of capabilities, it is quite cumbersome to use, thus, I developed a wrapper around the module that I called wraphorenzmie which can be found on my github repository [\(9\)](#).

This wrapper permits to directly load the MP4 files, compute the background and choose if what parameters should be fitted. While fitting a series of images, it uses the results of the previous image as the initial fit parameters.

However, as presented in the section 3.3 about the Lorenz-Mie fitting, the main drawback is the time to fit an image. Indeed, using a Python algorithm, one needs 30 seconds to fit images of 100×100 pixels and a few minutes for a 500×500 pixels hologram. We can directly see a bottleneck, indeed, if one want to track one trajectory made of 100 000 images one would need to wait a minimum of $\simeq 70$ days; for a series of images that need only a few minutes to be shot experimentally.

When I started my PhD, two groups, the Grier's lab and the Manoharan's lab, had already introduced python packages, respectively, Pylorenzmie and Holopy in order to inverse holograms. They had introduced ways to only fit a set of randomly chosen pixels, and, demonstrated that taking only 1 % of the image pixels, could lead to similar precision and improve considerably the fit's execution time [58].

Unfortunately, even if fitting a random subset of pixels is faster, it leads to a few images per second, and, is still too long for the amount of data we wanted to have. Ironically, this part of my project is certainly the one where I spent the most my time, and I certainly learned a lot about code optimization and computer cluster usage.

It is around the half of my thesis, that Pylorenzmie got a new commit on their github repository which was telling that they succeeded on using GPU acceleration using CUDA³.

³ CUDA is the acronym of Compute Unified Device Architecture. It is a parallel computing platform and programming model made to permits an easier use the GPU for general purpose. CUDA is developed by NVIDIA since 2012, thus all recent NVIDIA's GPUs are CUDA enable. It is possible to use it with every language as long as a library as been developed, such as cupy for Python [\(9\)](#).

This was not an easy task since they needed to reconstruct the Bessel functions in an understandable way for the GPU, fortunately it is possible to do so by using continued fractions [54]. This humongous update permits fitting whole images at a whooping speed improvement of 20 fps. At this speed, we fit the tridimension position of the particle, the radius and optical index.

As a remark, the fits are done by solving a least square problem using the Levenberg-Marquardt algorithm [69]. This algorithm, is largely used in any curve-fitting applications due to its capabilities to find a minimum even by starting far from it. As mentioned, it is also possible to use various models of Machine Learning or Deep Learning to do the fits [61]. However since we can write analytically the holograms, the deep learnings models can't be more precise than a standard least-square fitting process. Deep Learning however, could be a great option if one wants to prioritize the computation time over the fit's precision.

Finally, to have a more reliable and fast tracking, we begin by fitting the first 10 000 with \vec{r}_p , a and n_p as free parameters. Using the results of this fit we can characterize the physical properties of the observed colloid with high precision. Then, using the later result we can fit all the images with only the particle position \vec{r}_p as a free parameter.

3.7.3 Radius and optical index characterization

Once the data of the radius and optical index retrieved, the quantity we can look at is the distribution of measurements. Using 10 000 measurements one can do a 2D histogram of the a and n_p as presented in the fig.20 here smoothed using a Gaussian KDE (Kernel Density Estimator). Using the latter using the top 10% measurements couple possibility, we measure that the radius of the observed particle is $a = 1.514 \pm 0.003 \mu\text{m}$ and its optical index $n_p = 1.585 \pm 0.002$.

Finally, using this measurement of radius and optical index, we fit the whole movie by removing them from the free parameters. Doing so, we have measured the trajectory of the particle as shown in Fig.21 in tridimension for the particle previously characterized.

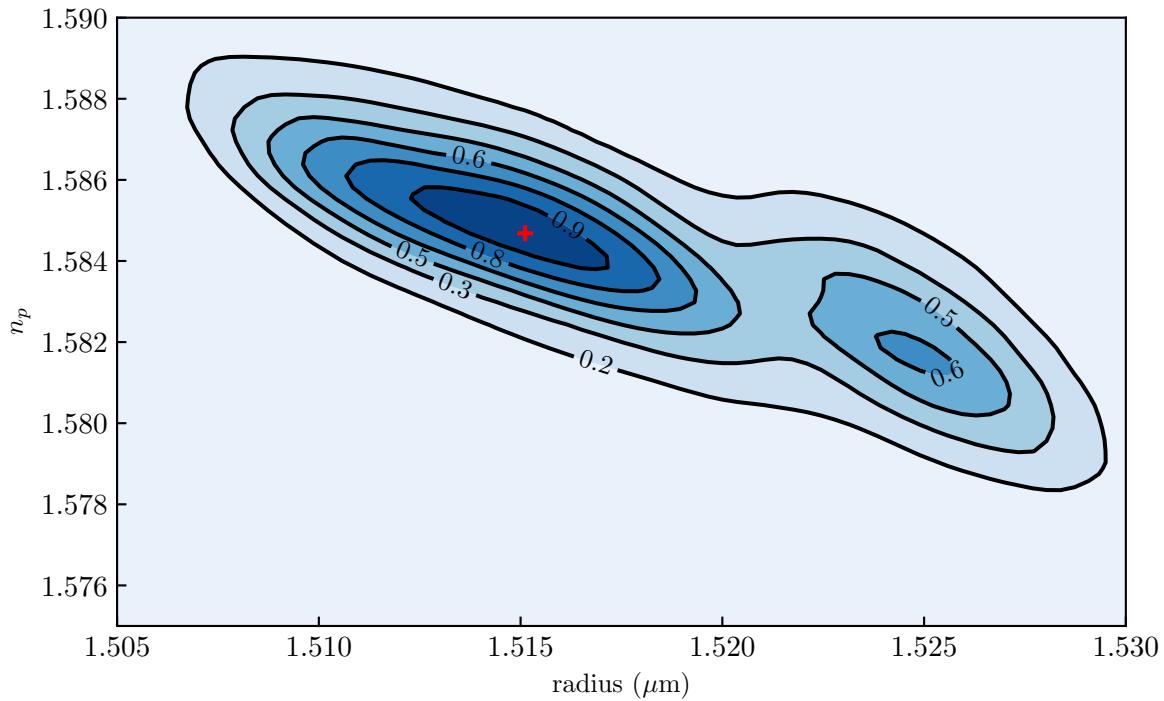


Figure 20: 2D Probability density function of the measurements of the optical index n_p and radius a . Black lines indicate iso-probability. Taking the 10% top probability, we measure $n_p = 1.585 \pm 0.002$ and $a = 1.514 \pm 0.003 \mu\text{m}$.

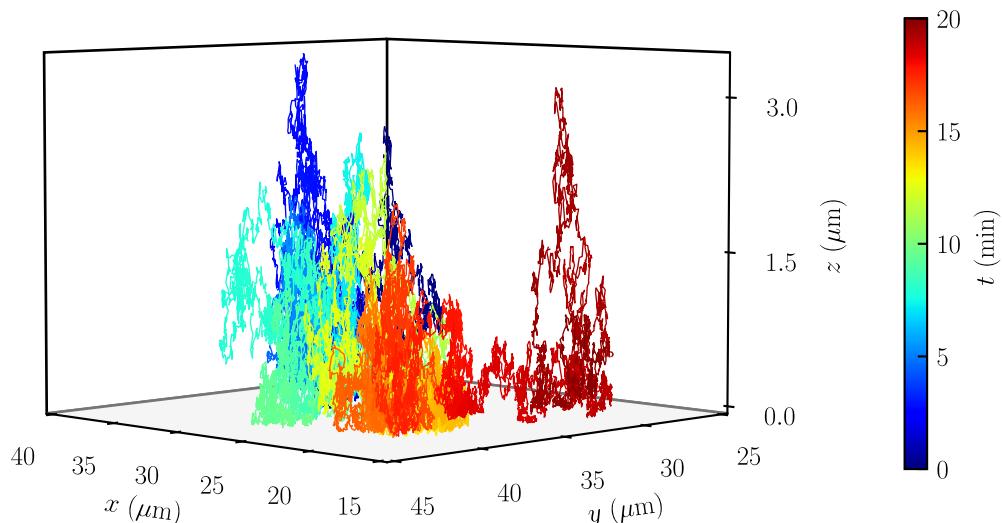


Figure 21: 3D plot of an experimental trajectory measured in water for a particle of optical index $n_p = 1.585$ and radius $a = 1.514 \mu\text{m}$.

3.7.4 Conclusion

In this chapter, we have covered different method that enable the tracking of individual particles. All the different method each come with pros and cons. We decided to use Lorenz-Mie since it is requires no calibration. Then we have shown how we use it in practice, from the experimental setup to the numerical treatment. Also, an exemple of the Jupyter notebooks used for the traking can be found in the appendix A.2. We have discussed on how to have fast and accurate fits to retrieve the particle trajectory. To do so, we first characterize fully the particle, namely, its radius and optical index; before tracking a whole movie.

Now that we have an understanding on the tracking of single colloids, we can use the later measured trajectory in order to understand how the Brownian motion is affected in various configurations. In the following, use confined Brownian motion in order to do a stochastic inference of surface-induced effects.

4 Stochastic inference of surface-induced effects using Brownian motion

4.1 Confined Brownian motion theory

By observing the trajectory along the z axis of a particle of $1.5 \mu\text{m}$ as shown on the fig.22, one can see that the particle height does not get higher than $\simeq 4 \mu\text{m}$. Indeed due to gravity, the particle is confined near the surface. Brownian motion in confinement and at interfaces is a canonical situation, encountered from fundamental biophysics to nanoscale engineering. This confinement induces near-wall effects, such as hindered mobility and electrostatic interactions.

In the first part of this chapter, I will detail the theory background of the confined Brownian motion and how to numerically simulate it. In a second part, I will present how to analyse experimental data. In particular, I will detail a multi-fitting procedure that allows a thermal-noise-limited inference of diffusion coefficients spatially resolved at the nanoscale, equilibrium potentials, and forces at the femtowatt resolution.

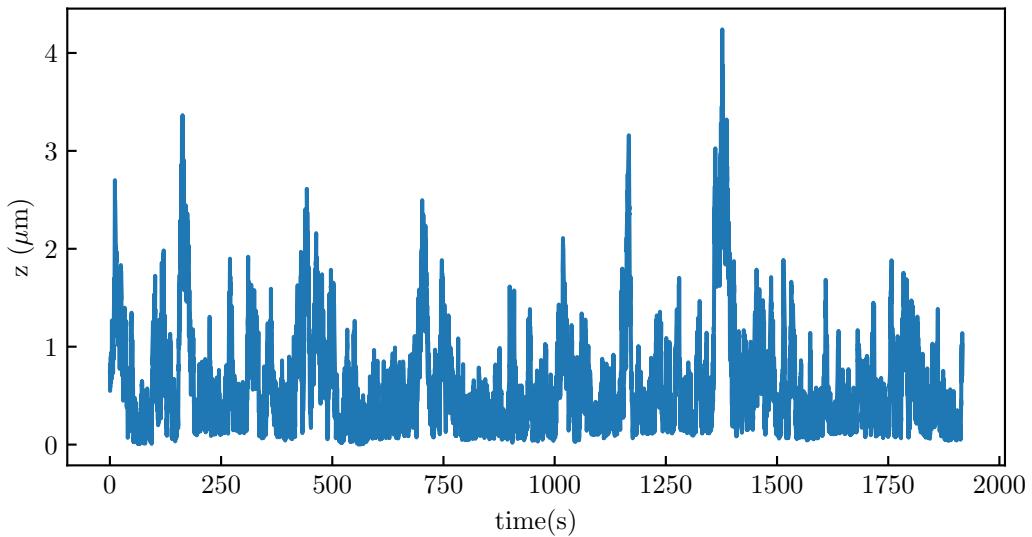


Figure 22: Experimental trajectory of a particle of polystyrene of radius $a = 1.5 \mu\text{m}$ near a wall ($z = 0$) along the z axis — perpendicular to the wall.

4.1.1 Gravitational interactions

In our experiment, we observe confined Brownian motion since the colloids are subject to gravity. Indeed, the density of the observed colloid ρ_p is different of the medium ρ_m — in our experiment water, $\rho_m = 1000 \text{ kg.m}^{-3}$. Thus, the particle lies into a gravitational potential given by:

$$U_g(z) = \Delta m g z = \frac{4}{3} \pi a^3 g \Delta \rho z , \quad (4.1.1)$$

where Δm is the mass difference of the particle and a fluid sphere of the same size, $\Delta \rho$ the corresponding density difference such as $\Delta \rho = \rho_m - \rho_p$ and g the gravitational acceleration. By invoking the definition of a distance that we call the Boltzmann length,

$$\ell_B = \frac{k_B T}{4/3 \pi a^3 \Delta \rho g} , \quad (4.1.2)$$

one can rewrite the gravitational potential Eq.4.1.1 as:

$$U_g = \frac{k_B T}{\ell_B} . \quad (4.1.3)$$

The Boltzmann length ℓ_B is the typcal gravitational decay length and represents the balance between the gravitational potential and thermal energy. This distance was first measured by Perrin [5], by enumerating the number of particles as a function of height to reconstruct the concentration of the colloidal suspension that exponentially decays as e^{-z/ℓ_B} . As an exemple, in water, for a particle polystyrene, $\rho_p = 1050 \text{ kg.m}^{-3}$ and of radius $a = 1.5 \mu\text{m}$ we have $\ell_B = 0.58 \mu\text{m}$.

For particle with $\ell_B \gg a$, one can consider that the particle does not feel the gravity. This is particulary the case when the density of the colloids and fluid matches, in this particular case $\ell_B = 0$. Thus density matching can be a way to do gravitation free experiments. In the case of our experiment, we want to measure confinement induced effects, therefore, we need this gravitational interaction to have the particles near the surface. Indeed, as a particle gets larger, or, denser ℓ_B decreases and the particle will be, in average, closer to the surface.

4.1.2 Sphere-wall interactions

As we have seen, external forces acts on the particle such as the gravity, however it is not the only one. As the Brownian particle is close to a wall we can expect some interactions between the surfaces. In our case, we suppose that the Brownian particles do not interact between each other, which is the case for the dilute solution used. Indeed, the particles studied are at a minimum $50 \mu\text{m}$ apart which correspond to 10 times their size for the larger beads.

To describes the interaction between, the Brownian particle and the wall, we use the DLVO⁴ theory. This theory was first developed to describe the interactions between colloids to explain the stability of colloidal suspension. It describes the interactions using two forces components; the Van Der Waals force which arise form the interactions between surface's molecules and electrostatic interactions due the a double-layer formed with the ions present in the solution.

Double-Layer interactions

When a surface is immersed in water are usually charged [70] due to a high dielectric constant $\epsilon \simeq 80$ that permit the build up of charges for low energetic price. Commonly, surface charging is done through ionization of dissociation of surface groups⁵, from the binding of ions from the solution — for example, adsorption of $-\text{OH}^-$ onto the water-air interface that charge it negatively. In the bulk, a fluid should be electrically neutral, thus the fluid contains as many ions of opposite charge. However, when a surface is charged negatively, negative ions are repelled from the surface, while positive ions are attracted towards the surface. Therefore, a double-layer charge distribution is formed near the surface, as shown Fig.30-a). Experimentally, we use glass slides and polystyrene beads, that are both negatively charged in water, thus leading to repulsive double-layer. This repulsive force prevent the colloids to stick together or to the surface of the substrate.

If the solution contains an electrolyte (ions of positive and negative charges), for example a salty solution, containing Na^+ and Cl^- ions. In the DLVO theory, the electrostatic field $\Psi(\vec{r})$ generated by the double layer satisfies the mean field Poisson's equation [70]:

⁴ The DLVO theory is named after Derjaguin, Landau, Verwey, and Overbeek [70].

⁵ For example, the dissociation of protons from surface carboxylic groups [70] ($-\text{COOH} \rightarrow -\text{COO}^- + \text{H}^+$) which charge negatively the surface.

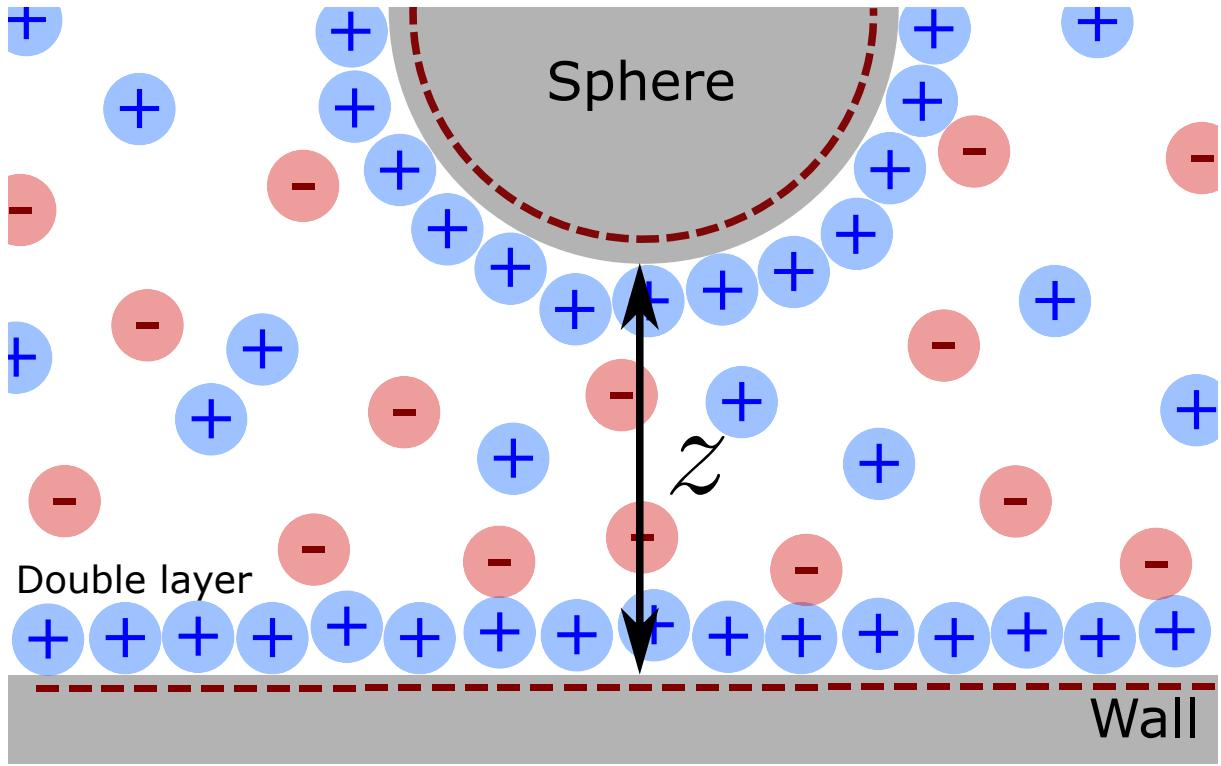


Figure 23: A Brownian colloid diffusing near a wall. Both wall and colloid's surface charge negatively, in consequence, a layer of positively charge ions are towards the surfaces, forming a double-layer charge distribution.

$$\nabla^2 \Psi(\vec{r}) = -\frac{1}{\epsilon_r \epsilon_0} \rho_e(\vec{r}) , \quad (4.1.4)$$

where ϵ_0 the vacuum permittivity, ϵ_r the relative permitivity of the fluid, $\rho_e(\vec{r})$ the local charge density. The latter can be written as:

$$\rho_e(\vec{r}) = e \sum_i z_i c_i(\vec{r}) , \quad (4.1.5)$$

where e is the elementary charge, i denotes an ionic species of valence z_i and local ionic concentration $c_i(\vec{r})$ (number density). If the solution is at the thermodynamic equilibrium, the Boltzmann equation is used to calculate the local ion density such that:

$$c_i(\vec{r}) = c_i^0 \exp \left(\frac{z_i e \Psi(\vec{r})}{k_B T} \right) , \quad (4.1.6)$$

where c_i^0 is the bulk concentration (number density) of the ionic species i . By combining Eqs.4.1.4, 4.1.5 and 4.1.6, one can obtain the Poisson-Boltzmann equation:

$$\nabla^2 \Psi(\vec{r}) = \sum_i \frac{z_i e c_i^0}{\epsilon_0 \epsilon_r} \exp\left(-\frac{z_i e \Psi(\vec{r})}{k_B T}\right) . \quad (4.1.7)$$

Since the Poisson-Boltzmann is non-linear, it is most likely to be solve numerically. However, for simple geometry such as uniformly charged plane or sphere it can be solve analytically. Let consider, to simplify, that we have a monovalent electrolyte, meaning that the electrolyte is composed of two ions of valence equal to one — Na^+ Cl^- for example — and c_i^0 is equal to the electrolyte solution concentration c_s^0 . In that case Eq.4.1.7 simplifies to:

$$\begin{aligned} \nabla^2 \Psi(\vec{r}) &= \frac{e c_s^0}{\epsilon_0 \epsilon_r} \left[\exp\left(\frac{-e \Psi(\vec{r})}{k_B T}\right) - \exp\left(\frac{+e \Psi(\vec{r})}{k_B T}\right) \right] \\ &= 2 \frac{e c_s^0}{\epsilon_0 \epsilon_r} \sinh\left(\frac{e \Psi(\vec{r})}{k_B T}\right) . \end{aligned} \quad (4.1.8)$$

In the case, where the Ψ is small enough everywhere to have the electrostatic potential energy $e\Psi \ll k_B T$, which generally the case when using salty solution. In that case, it is possible, using the a Taylor approximation at the second order to write:

$$\exp\left(-\frac{z_i e \Psi(\vec{r})}{k_B T}\right) \simeq 1 + \frac{z_i e \Psi(\vec{r})}{k_B T} . \quad (4.1.9)$$

Thus, the Poisson-Boltzmann equation (Eq.4.1.7) becomes:

$$\nabla^2 \Psi(\vec{r}) = \sum_i \frac{z_i e c_i^0}{\epsilon_0 \epsilon_r} \left(1 + \frac{z_i e \Psi(\vec{r})}{k_B T} \right) . \quad (4.1.10)$$

Since the fluid in the bulk, is electrically neutral, the first term vanishes as $\sum_i z_i c_i^0 = 0$. One thus have a linearized version of Eq.4.1.7, which is known as the Debye-Hückel equation:

$$\nabla^2 \Psi(\vec{r}) = \left[\sum_i \frac{z_i^2 e^2 c_i^0}{\epsilon_0 \epsilon_r k_B T} \right] \Psi(\vec{r}) . \quad (4.1.11)$$

From this approximation, one can identify that the term between brackets is the inverse of a distance squared. We can thus define a distance ℓ_D , the Debye length such as:

$$\ell_D = \sqrt{\sum_i \frac{\epsilon_0 \epsilon_r k_B T}{z_i^2 e^2 c_i^0}} . \quad (4.1.12)$$

The Debye length is the characteristic length of the double-layer, and, the electrostatic interactions. For a monovalent electrolyte, at 25 °C (298 K), the Debye length of aqueous solution is:

$$\begin{aligned} \ell_D &= \sqrt{\frac{\epsilon_0 \epsilon_r k_B T}{2c_s^0 e^2}} = \sqrt{\frac{8.854 \times 10^{-12} \times 78.4 \times 1.381 \times 10^{-23} \times 298}{2 \times (1.602 \times 10^{-19})^2 \times 6.022 \times 10^{26} M}} \\ &= 0.304 \times 10^{-9} / \sqrt{M} \text{ m ,} \end{aligned} \quad (4.1.13)$$

with M the molar concentration ($1 \text{ M} = 1 \text{ mol.L}^{-1}$ corresponding to a number density of $c_s^0 = 6.022 \times 10^{26} \text{ m}^{-3}$). Thus, for a salty concentration we have $\ell_D = 0.304 / \sqrt{[\text{NaCl}]} \text{ nm}$. For example, for NaCl solution, one can have $\ell_D = 100 \text{ nm}$ for a concentration $[\text{NaCl}] = 9.2 \mu\text{M}$ and $\ell_D = 10 \text{ nm}$ for a concentration $[\text{NaCl}] = 9.2 \text{ mM}$.

Finally, the Debye-Hückel approximation finally writes:

$$\nabla^2 \Psi(\vec{r}) = \kappa^2 \Psi(\vec{r}) , \quad (4.1.14)$$

with $\kappa = 1/\ell_D$. Using the latter approximation one can compute the electrostatic potential around a sphere. Let us consider a perfect sphere of radius a and charge Qe of charge density $\sigma = Qe/4\pi a^2$. With Q being the number of charge on the surface. Since the system has a spherical symmetry, one has $\Psi(\vec{r}) = \Psi(r)$ with $r = |\vec{r}|$. Using the Laplacian operator ∇^2 in the spherical coordinates, Eq.4.1.14 writes:

$$\frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi(r)}{\partial r} \right) \right] = \kappa^2 \Psi(r) , \quad (4.1.15)$$

which has a general solution:

$$\Psi(r) = C_1 \frac{\exp(\kappa r)}{r} + C_2 \frac{-\exp(\kappa r)}{r} \quad (4.1.16)$$

For one sphere, the electrostatic field vanishes at infinity such as $C_1 = 0$, such it has the form of a Yukawa potential:

$$\Psi(r) = C_2 \frac{-\exp(\kappa r)}{r} . \quad (4.1.17)$$

Additionally, at the surface of a charged sphere, the electrostatic potential satisfies:

$$\left. \frac{\partial \Psi(r)}{\partial r} \right|_{r=a} = \frac{Qe}{4\pi\epsilon_0\epsilon_r a^2} = \frac{\sigma}{\epsilon_0\epsilon_r} . \quad (4.1.18)$$

By applying the latter boundary condition to Eq.4.1.17 we find:

$$\Psi(r) = \frac{\sigma a^2}{\epsilon_0\epsilon_r} \frac{\exp(\kappa a)}{1 + \kappa a} \frac{\exp(-\kappa r)}{r} \quad (4.1.19)$$

This solution can be used to determine the electrostatic field between two spheres. Indeed, if we suppose that the presence of a second sphere, do not interfere with the distribution of ions in the double layer of the other. We can thus use the superposition approximation to write the potential $\Psi_2(z)$ between two spheres surfaces. Where z is the distance between the 2 colloids. For two spheres of charge σ_1 and σ_2 and radius a_1 and a_2 , $\Psi_2(z)$ writes [71]:

$$\Psi(z) = \frac{4\pi}{\epsilon_0\epsilon_r} \left(\frac{\sigma_1 a_1^2}{1 + \kappa a_1} \right) \left(\frac{\sigma_2 a_2^2}{1 + \kappa a_2} \right) \frac{\exp(-\kappa z)}{a_1 + a_2 + z} . \quad (4.1.20)$$

From the latter equation, it is possible to write the electrostatic field between a wall and a spherical colloid, by setting one of the radius to infinity. Doing so and multiplying by e , one can have the electrostatic potential $E_{elec}(z)$ between a Brownian particle and the wall:

$$E_{elec}(z) = B e^{-\frac{z}{\ell_D}} . \quad (4.1.21)$$

Where B is the constant that represent the surface charges, for sphere of radius a and charge σ_1 and a wall of charge σ_2 , one has:

$$B = \frac{4\pi}{\epsilon_0\epsilon_r} \left(\frac{\sigma_1 a^2}{1 + \kappa a} \right) \frac{\sigma_2}{\kappa} . \quad (4.1.22)$$

Additionally, B is often written as a function of the surfaces potential as [CITE PRIEVES]:

$$B = 16\epsilon_r\epsilon_0 \left(\frac{k_B T}{e} \right) \tanh \left(\frac{e\phi_1}{4k_B T} \right) \tanh \left(\frac{e\phi_2}{4k_B T} \right), \quad (4.1.23)$$

where ϕ_1 and ϕ_2 are the Stern potentials of the sphere and the wall surfaces. Typical values for B ranges from $1 k_B T$ to $50 k_B T$. In our study we will keep B to describe the electrostatic energy potential as it complicated to decouple the σ_1 and σ_2 when the colloid and wall's surface materials are different [72]. However, this is not impossible and is the idea of future work.

Van der Waals interactions

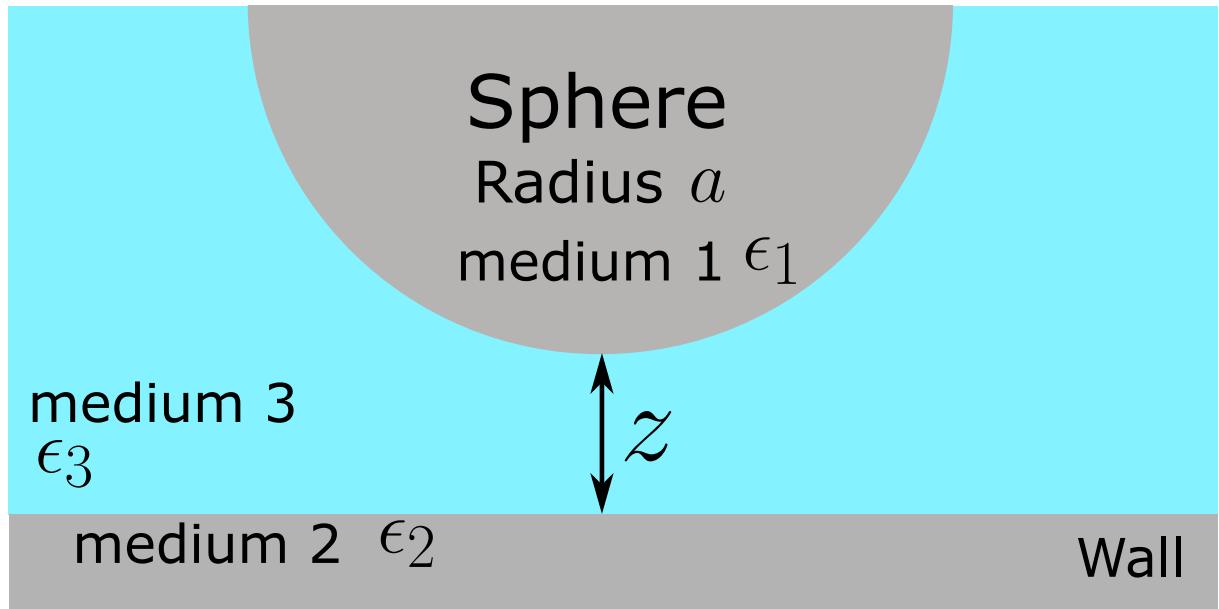


Figure 24: A colloid of radius a separated from a wall from a distance z . The colloid material has a static dielectric constant ϵ_1 , the wall ϵ_2 and the solution ϵ_3 .

In the DLVO theory, the Van der Waals forces describes the interaction between colloids at very short range. This forces are mainly attractive and in our case having Van der Waals interactions would lead to the particles stick to the wall. The Van der Waals potential energy E_{VdW} for a spherical colloid of radius a , at a height z at a few nanometers (< 5 nm) to the surface [70]:

$$E_{VdW} = -\frac{Aa}{6z} \quad (4.1.24)$$

where A is the retarded Hamaker constant. For our system, where the particle, medium and wall are different medium as schematize in Fig.24. In that case the Hamaker constant

writes [70]:

$$A = \frac{3}{4}k_B T \left(\frac{\epsilon_1 - \epsilon_3}{\epsilon_1 + \epsilon_3} \right) \left(\frac{\epsilon_2 - \epsilon_3}{\epsilon_2 + \epsilon_3} \right) + \frac{3h}{4\pi} \int_{\nu_1}^{\infty} \left(\frac{\epsilon_1(i\nu) - \epsilon_3(i\nu)}{\epsilon_1(i\nu) + \epsilon_3(i\nu)} \right) \left(\frac{\epsilon_2(i\nu) - \epsilon_3(i\nu)}{\epsilon_2(i\nu) + \epsilon_3(i\nu)} \right) d\nu , \quad (4.1.25)$$

where ϵ_1 , ϵ_2 and ϵ_3 are the static dielectric constants of the three media, $\epsilon(i\nu)$ are the imaginary dielectric constant at a frequency ν and $\nu_n = (2\pi k_B T n / h) = 4 \times 10^{13} \text{ ns}^{-1}$ at 300 K; and; h is the Plank's constant. The first term fives the zero-frequency energy of the van der Waals interaction and the second term the dispersion energy. However, since the static dielectric of water is large compared to the other terms [70], one can approximate the Hamaker constant of our system to only the zero-frequency term:

$$A = \frac{3}{4} 1.381 \times 10^{-23} \times 298 \left(\frac{2.5 - 80}{2.5 + 80} \right) \left(\frac{5 - 80}{5 + 80} \right) \simeq 2.48 \times 10^{-21} \text{ J} = 0.62 k_B T . \quad (4.1.26)$$

Since A is positive we correctly have an attractive Van der Waals force, this case will always be the case due to the high dielectric constant of water. More over, the zero-frequency Van der Waals force is mainly due to electrostatic interaction and is thus screened by electrolytes, we thus need to add a factor $\approx e^{-z/\ell_D}$ to the Van der Waals forces. Also, it has been proposed [73] that to compute the force at higher distances, one should add a second factor, to account for the retarded Van der Waals force of $\simeq (1 + 11z/100\text{nm})^{-1}$.

All the effects added together we can see that the Van der Waals forces will play a role only a few nanometer (< 10 nm) as it is commonly observed [M,P]. In our experiments, the Debye length ℓ_D (> 20 nm) is large enough for the particle to avoid this region, in the following of this work the Van der Waals interactions will be neglected. To study the Van der Waals interactions with Brownian motion, it is possible, if one add enough salt to have $\ell_D \simeq 1$ nm. However, with such a short Debye length, all the colloids would stick to the surface and between each other. However, we have experimentally observed some dynamics on the sucked particles, further work on this movement could lead to interesting determination of the near-wall potential.

Total potential and equilibrium distribution

If we combine the gravitational and electrostatic interactions the particules lies into a total energy potential $U(z)$:

$$U(z) = U_g + U_{\text{elec}} \quad (4.1.27)$$

By combining Eqs.4.1.3, 4.1.21 and 4.1.27, and also adding the condition that the particle can't go inside the wall $U(z)$ finally writes:

$$\frac{U(z)}{k_B T} = \begin{cases} B e^{-\frac{z}{\ell_D}} + \frac{z}{\ell_B}, & \text{for } z > 0 \\ +\infty, & \text{for } z \leq 0 \end{cases}, \quad (4.1.28)$$

From this total potential energy, one can then construct the Gibbs-Boltzmann distribution to write the equilibrium PDF of position $P_{\text{eq}}(z)$:

$$P_{\text{eq}}(z) = A \exp\left(-\frac{U(z)}{k_B T}\right), \quad (4.1.29)$$

where A is a normalization constant such that $\int_0^\infty P_{\text{eq}}(z) dz = 1$. Given an array of heights z_i one can use compute P_{eq} using the following Python snippet, where the A is computed using the `np.trapz` function. An example of a theoretical energy potential and PDF of position can be seen Fig.25 for $\ell_B = 500$ nm, $B = 4 k_B T$ and $\ell_D = 50$ nm.

```

1 import numpy as np
2
3 def _Peq(z):
4     if z <= 0:
5         return 0
6     else:
7         return np.exp(-(B * np.exp(-z / 1d) + z / 1b))
8
9
10 def Peq(z):
11     P = np.array([_Peq(zi) for zi in z])
12     return P / np.trapz(P,z)

```

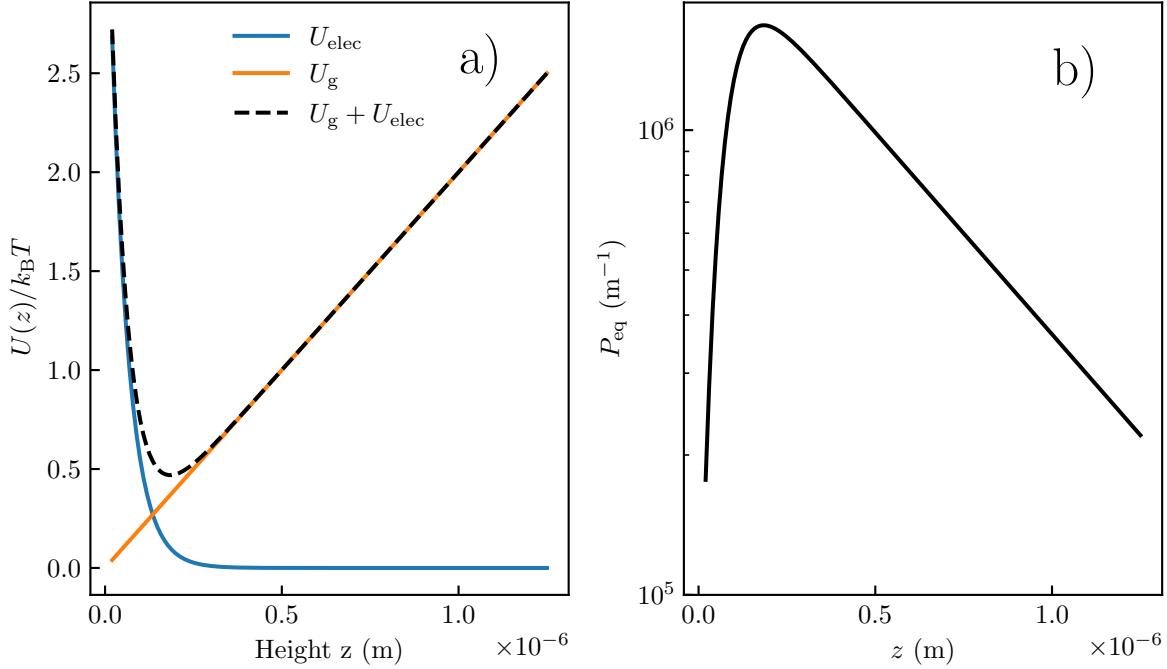


Figure 25: a) Potential energy of a colloid of Boltzmann length $\ell_B = 500$ nm. The electrostatic potential U_{elec} is here characterized by a surface charge constant $B = 4 k_B T$ and a Debye length $\ell_D = 50$ nm. b) Corresponding Gibbs-Boltzmann equilibrium distribution of position.

4.1.3 Local diffusion coefficient

As we have seen in the Chapter 2, for a freely diffusing colloid in bulk the diffusion coefficient is given by Eq.2.2.12 and is a constant. However, when a particle is confined, the diffusion is hindered, this means that the diffusion coefficient vary with the height of the particle and becomes anisotropic. One of the first measurement that has been done by Faucheux and Libchaber [13]. As we can see on the Fig.26, using a microscope, they tracked Brownian colloids in two dimensions, and, measured the average coefficient diffusion for different confinement constant $\gamma = (\langle z \rangle_t - a)/a$. Finally, on the Fig.26, one can observe that the diffusion coefficient parallel to the surface decreases as the particle get closer to the wall, and, seems to saturate around $0.3D_0$ for high confinement.

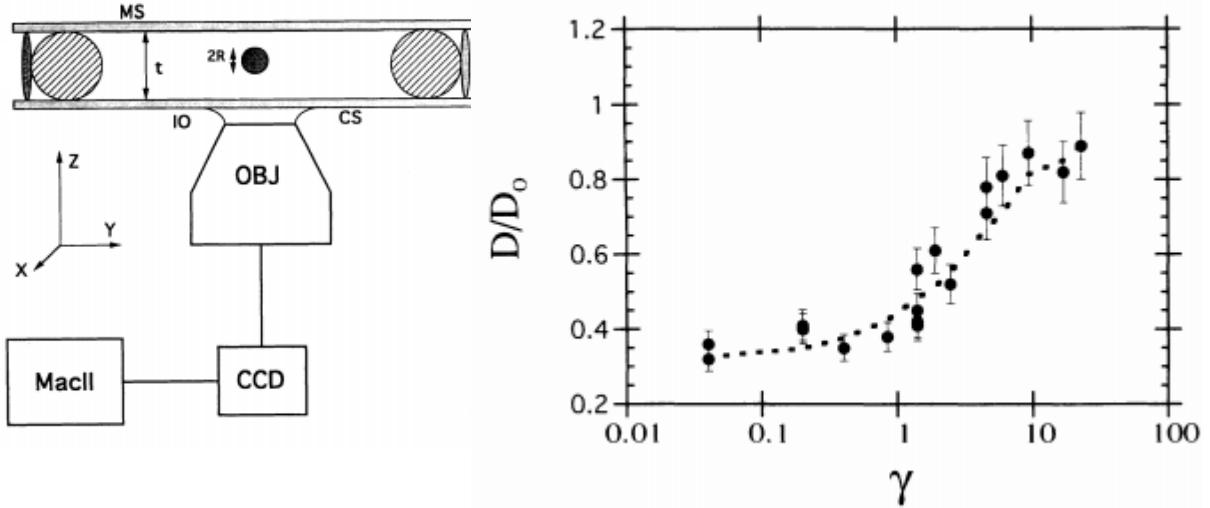


Figure 26: Figure extracted from [13], on the left is the experimental setup used. It is an inverted microscope used in order to track particle of size $2R$ inside a cell of thickness t . On the right is their final result, where they measure the diffusion parallel coefficient D_{\perp} given by Eq.4.1.41, here normalized by D_0 the bulk diffusion coefficient as a function of γ a confinement constant $\gamma = (\langle z \rangle - a)/a$.

To understand the reason of this hindered diffusion coefficient, let us start by writing the diffusion coefficient D using the fluctuation dissipation theorem:

$$D = \mu k_B T , \quad (4.1.30)$$

with:

$$\mu = \frac{v_{\text{sphere}}}{F_{\text{drag}}} , \quad (4.1.31)$$

where v_{sphere} is the terminal velocity to an applied force F_{drag} . For a spherical colloid of radius R moving at a velocity v_{sphere} the drag force in bulk $F_{\text{drag}}^{\text{B}}$ is given by the Stockes' law:

$$F_{\text{drag}}^{\text{B}} = c\pi\eta av_{\text{sphere}} , \quad (4.1.32)$$

where c is a constant that depends on the boundary conditions imposed at the surface of

the colloid, typically $c = 6$ for no-slip and $c = 4$ (such as air bubbles for example) for slip boundary conditions. Combining Eqs.4.1.30, 4.1.31 and 4.1.32 for a freely diffusing hard sphere in bulk we retrieve Eq.2.2.12:

$$D_0 = \frac{k_B T}{6\pi\eta a} . \quad (4.1.33)$$

The Stocke's drag force can be computed by solving the Navier-Stokes equation:

$$\rho \left[\frac{\partial \vec{v}}{\partial} + (\vec{v} \cdot \nabla \vec{v}) \right] + \nabla p = \eta \nabla^2 \vec{v} , \quad (4.1.34)$$

and the continuity equation for incrompressible fluids:

$$\nabla \cdot \vec{v} = 0 \quad (4.1.35)$$

where \vec{v} and p is the velocity and pressure fields ρ is the density of the fluid. When the Reynolds number $Re = \rho a v_{\text{sphere}} / \eta \ll 1$, the first two terms are inertial and are negligibly small compared to the viscous term $\eta \nabla^2 \vec{v}$. In that case, the Navier-Stokes Eq.4.1.34, is simplified to the Stockes flow formula:

$$\nabla p = \eta \nabla^2 \vec{v} \quad (4.1.36)$$

The diffusion coefficient for a freely diffusing spherical colloid in bulk can thus be found by solving Eqs.4.1.36 and 4.1.35 by having a no-slip boundary condition on the particle surface and the velocity vanishing at infinity. However, in the case of a confined particle, there is an additional no-slip condition at the wall surface. Indeed, as shown on the Fig.27, the fluid velocity drop to zero at the wall surface, with a shear rate:

$$\dot{\gamma} = \frac{\partial v_{\text{sphere}}}{\partial z} \quad (4.1.37)$$

This shear rate, due to the presence of the wall introduce a shear force $F(z) = \simeq \eta \dot{\gamma}(z)$ acting on the particle. Hence, the drag force acting on the particle will increase inversely with the distance to the wall, due to additional hydrodynamic pressure arising from velocity gradients. At the macro scale, this effect can be seen with a frisbee, indeed, as it

gets closer to the grounds, hydrodynamic pressure increases due to air velocity gradient and one can observe that the frisbee seems to stop falling and continue to glide really near the ground's surface.

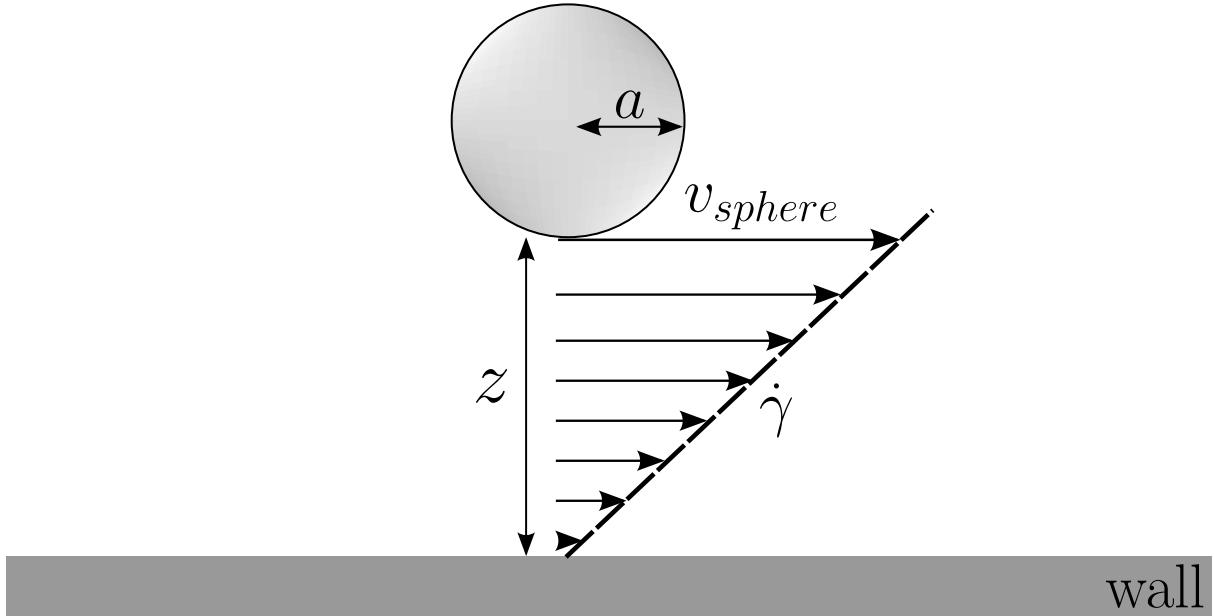


Figure 27: Schematic representation of a spherical colloid moving near a wall and the induced shear rate $\dot{\gamma}$.

A colloid diffusing near a wall thus experience a local drag force that depends on both it's distance to the wall z and the movement direction. Thanks to the linearity of the Stokes equation, one can separate this local drag force for motion parrallel and perpendicular to the wall. As the presence of the wall correct the drag force from a multiplicative coefficient, the confinement effect often expressed as an effective viscosity:

$$\eta_{\perp}(z) = \eta\lambda_{\perp}(z) , \text{ and, } \eta_{\parallel}(z) = \eta\lambda_{\parallel}(z) , \quad (4.1.38)$$

where λ_{\perp} and λ_{\parallel} are respectively perpendicular and parallel correction factor to the drag force due to the presence of the wall. Due to this correction, the diffusion coefficients for parallel and perpendicular motion relative to the wall writes:

$$D_{\perp}(z) = \frac{D_0}{\lambda_{\perp}(z)} , \text{ and, } D_{\parallel}(z) = \frac{D_0}{\lambda_{\parallel}(z)} . \quad (4.1.39)$$

For a no-slip boundary conditions imposed at the wall and the surface of the colloid, Brenner [74] obtained for perpendicular motion:

$$\lambda_{\perp}(z) = \frac{4}{3} \sinh \beta \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left[\frac{2 \sinh(2n+1)\beta + (2n+1)\sinh 2\beta}{4 \sinh^2(n+1/2)\beta - (2n+1)^2 \sinh^2 \beta} - 1 \right], \quad (4.1.40)$$

where $\beta = \cosh^{-1}((z+a)/a)$. The solution for the diffusion parallel to the wall, Faxén found [75]:

$$\lambda_{\parallel}(z) = \left[1 - \frac{9}{16}\xi + \frac{1}{8}\xi^3 - \frac{45}{256}\xi^4 - \frac{1}{16}\xi^5 \right]^{-1}, \quad (4.1.41)$$

where $\xi = a/(z+a)$. Eqs.4.1.40 and 4.1.41 are precise for all z . However, the solution for the perpendicular motion can be quite complex to compute as it is an infinite series, to be computed numerically it requires a software that enable arbitrary-precision floating-point arithmetic⁶ — such as Mathematica or the `mpmath` Python's module for example. D_{\perp} can be evaluated using the following Python snippet, where `nsum` function is used to compute the infinite sum:

```

1 from mpmath import nsum
2
3 def Dz(eta, z, a):
4     a = (z + a) / a
5     beta = float(acosh(a))
6     summ = nsum(
7         lambda n: (n * (n + 1) / ((2 * n - 1) * (2 * n + 3)))
8         *
9         (
10            (2 * sinh((2 * n + 1) * xi) + (2 * n + 1) * sinh(2 * beta))
11            /
12            (4 * (sinh((n + 1 / 2) * beta) ** 2)
13             - ((2 * n + 1) ** 2) * (sinh(beta) ** 2)
14            )
15        )
16        -
17    ),
18    [0, inf],
19    )
20     summ = float(summ)
21 return kT / (6 * pi * eta * 4 / 3 * float(sinh(beta)) * summ * a)

```

⁶ Arbitrary-precision floating-point arithmetic enables to evaluate mathematical expression with any precision, in other words, any number of digits.

To simplify the computation of λ_{\perp} , Goldman *et al.* [76] showed that Eq.4.1.40 can be Padé approximated, giving:

$$\lambda_{\perp} = \frac{6z^2 + 9az + 2a^2}{6z^2 + 2az} . \quad (4.1.42)$$

In the near-wall regime, such that $z \ll a$, it is also possible to further approximate λ_{\perp} to:

$$\lambda_{\perp} = \frac{a}{z} . \quad (4.1.43)$$

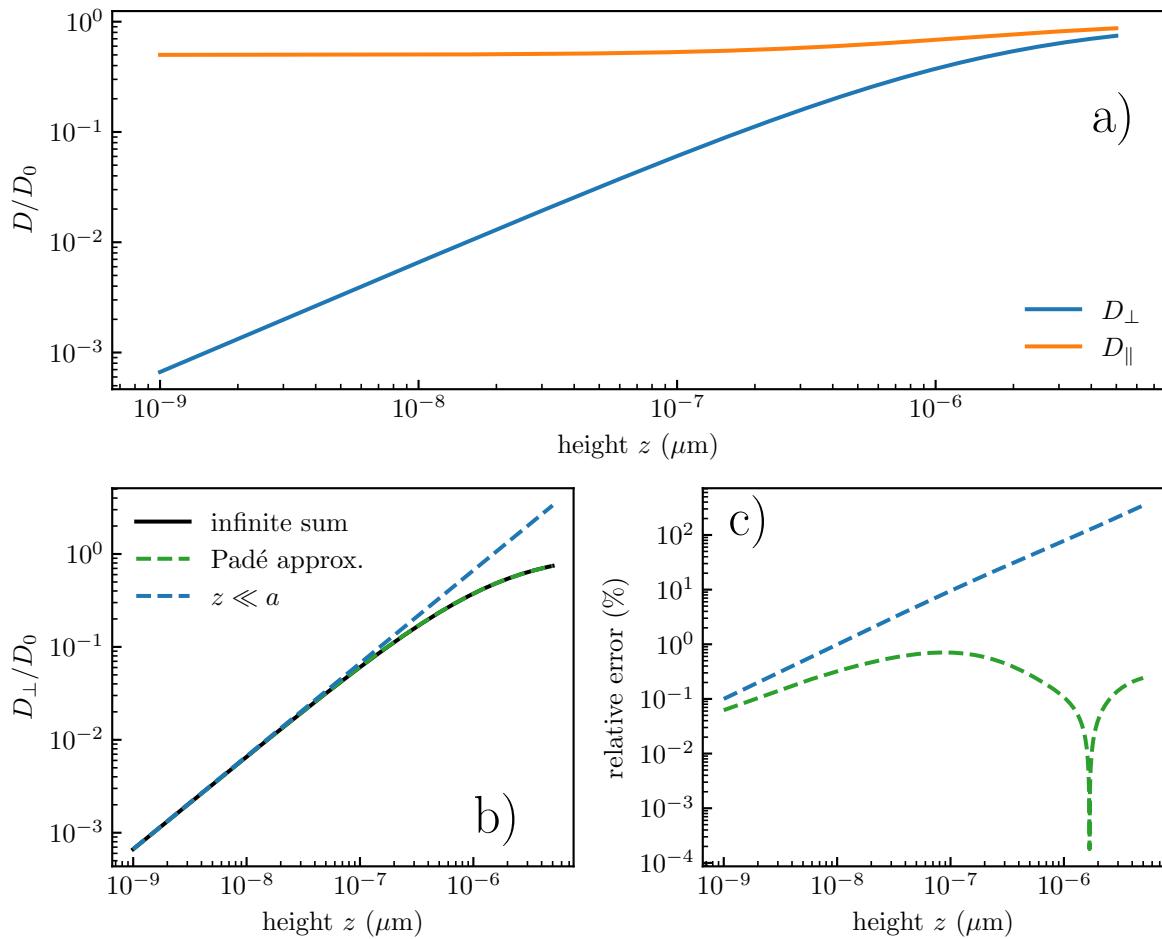


Figure 28: a) Parallel and perpendicular hindered relative diffusion coefficient for a colloidal particle of radius $a = 1.5 \mu\text{m}$. b) Perpendicular hindered relative diffusion coefficient for a colloid particle of radius $a = 1.5 \mu\text{m}$. In black the exact solution given by the infinite sum Eq.4.1.40. In green the Padé approximation, Eq.4.1.42 and in blue the near wall regime Eq.4.1.43. c) Relative error of the perpendicular hindered coefficient.

The Padé approximation and the near-wall approximation for the hindered diffusion are

plotted on Fig.28-b). The Padé approximation fits really well with the exact solution given by the infinite sum Eq.4.1.40, the near-wall approximation fits well when $z < a/10$. To check how precise the approximation are, it is possible to plot the relative error as in the Fig.28-c). The Padé approximation shows a precision up to 1%, thus, in the following of this work, when we refer to any evaluation of the hindered perpendicular diffusion, the Padé approximation will be used.

4.1.4 Langevin equation for the confined Brownian motion

Now that the external forces acting on the particle and hindered diffusion coefficient is known, we rewrite the overdamped Langevin Eq.2.3.10 equation as:

$$V_t^i dt = -\frac{1}{\gamma(z)} \frac{\partial U(z)}{\partial x_i} dt + \sqrt{2D_i(z)} dB_t . \quad (4.1.44)$$

where $\gamma(z) = 6\pi\eta_i(z)a$ and i denotes three spatial directions, x , y and z ⁷, and, dB_t still is a Gaussian distribution satisfying $\langle dB_t \rangle = 0$ and $\langle dB_t^2 \rangle = dt$. As we have discussed previously, the potential U only varies along the z axis, thus, the external forces only acts on the particle on the z axis while the particle diffuses freely along the x and y axis.

4.1.5 Spurious drift

It is interesting to observe that due to the hindered diffusion the magnitude of the Langevin force, $\sqrt{2D_i(z)}$, is not anymore a constant, but varies with the height of the particle. This effect is called multiplicative noise and will have some interesting effects on the dynamics properties of the Brownian motion. To show the effects of the multiplicative noise, one can integrate over a time τ the Eq.4.1.44, in the absence of the external force one has:

$$\Delta x_i = \int_{t_0}^{t_0+\tau} \sqrt{2D_i(z)} dB_t \quad (4.1.45)$$

where Δx_i is a space increment. However, the noise term is not well-defined and the time at which the magnitude of the force $\sqrt{2D_i(z)}$ in the integration Eq.4.1.45 needs to

⁷ Where the previously determined $\eta_{||}$ and $D_{||}$ correspond to the x and y axis and η_{\perp} and D_{\perp} correspond to the z axis.

be specified. It is thus necessary to determine where the Diffusion coefficient $D(z)$ in Eq.4.1.45 is evaluated in the time interval $[t_0, t_0 + \tau]$. In general, $D(z)$ is represented by $D(z + \alpha\Delta z)$, or in the same way $D(z(t_0 + \alpha\tau))$, with $\alpha \in [0, 1]$. The value of α determines at which time in the interval $[t_0, t_0 + \tau]$ the local diffusion is evaluated, hence, the magnitude of the random force. The physics will thus change on how the noise is calculated, however, the requirement is that the long-time thermal equilibrium must be consistent with the Boltzmann distribution and, hence, constrain the choice of α . Taking into account α , the langevin equation without external 4.1.45 forces along the z axis becomes:

$$\Delta z = \sqrt{2D_{\parallel}(z + \alpha\Delta z)}dB_t . \quad (4.1.46)$$

By Taylor expendig D_{\parallel} to the first order one has:

$$D_{\parallel}(z + \alpha\Delta z) \simeq D_{\parallel}(z) + \alpha \frac{dD_{\parallel}(z)}{dz} \Delta z . \quad (4.1.47)$$

Substituting the later in Eq.4.1.46 leads to:

$$\begin{aligned} \Delta z &\simeq \sqrt{2D_{\parallel}(z) + \alpha \frac{dD_{\parallel}(z)}{dz} \Delta z dB_t} \\ &= \sqrt{2D_{\parallel}(z)} \left[1 + \alpha \frac{dD_{\parallel}(z)}{dz} \frac{\Delta z}{D_{\parallel}(z)} \right]^{-1/2} dB_t . \end{aligned} \quad (4.1.48)$$

Additionally, since the last term satisfies:

$$\alpha \frac{dD_{\parallel}(z)}{dz} \frac{\Delta z}{D_{\parallel}(z)} \ll 1 , \quad (4.1.49)$$

one can thus Taylor expend at the first order the last term of Eq.4.1.48 as:

$$\left[1 + \alpha \frac{dD_{\parallel}(z)}{dz} \frac{\Delta z}{D_{\parallel}(z)} \right]^{-1/2} \simeq 1 + \frac{\alpha}{2} \frac{dD_{\parallel}(z)}{dz} \frac{\Delta z}{D_{\parallel}(z)} . \quad (4.1.50)$$

By finally substituting the first-order expansion of $\Delta z \simeq \sqrt{2D(z)}dB_t$, we obtain:

$$\begin{aligned}
\Delta z &\simeq \sqrt{2D_{\parallel}(z)} \left[1 + \frac{\alpha}{2} \frac{dD_{\parallel}(z)}{dz} \frac{\Delta z}{D_{\parallel}(z)} \right] dB_t \\
&\simeq \sqrt{2D_{\parallel}(z)} \left[1 + \frac{\alpha}{2} \frac{dD_{\parallel}(z)}{dz} \frac{\sqrt{2D(z)}dB_t}{D_{\parallel}(z)} \right] dB_t \\
&= \sqrt{2D_{\parallel}(z)}dB_t + \alpha \frac{dD_{\parallel}(z)}{dz} dB_t^2
\end{aligned} \tag{4.1.51}$$

Since for short time $dB_t \rightarrow 0$ it is possible to replace dB_t^2 by its average value $\langle dB_t^2 \rangle_t = \tau$, the latter thus become [77]:

$$\begin{aligned}
\Delta z &= \alpha \frac{dD_{\parallel}(z)}{dz} \tau - \frac{1}{\gamma(z)} \frac{\partial U(z)}{\partial z} \tau + \sqrt{2D_{\parallel}(z)}dB_t \\
&= \left(-\frac{1}{\gamma(z)} \frac{\partial U(z)}{\partial z} + \alpha \frac{dD_{\parallel}(z)}{dz} \right) \tau + \sqrt{2D_{\parallel}(z)}dB_t \\
&= \bar{v}_d \tau + \sqrt{2D_{\parallel}(z)}dB_t ,
\end{aligned} \tag{4.1.52}$$

where:

$$\bar{v}_d = -\frac{1}{\gamma(z)} \frac{\partial U(z)}{\partial z} + \alpha \frac{dD_{\parallel}(z)}{dz} = v_d + v_{\text{noise}} , \tag{4.1.53}$$

are the total drifts acting on the particle. The first term v_d is the drift due to “real” deterministic forces due to the electrostatic double-layer and gravity interactions, while the second term v_{noise} represents a noise induced drift. This spurious drift does disappear when the coefficient diffusion is homogeneous as for the x and y axis as the coefficient diffusion only depends on the colloid’s height. Theoretically, α can take any value between 0 and 1. However, α generally takes 3 different values [78]: $\alpha = 0$, the Itô integral, corresponding to the use of the initial value of $D(z)$; $\alpha = 1/2$, the Stratonovich integral, corresponding to the mid-point value; and $\alpha = 1$ the anti-Itô or isothermal integral, corresponding to the use of the final value.

In particular, the Itô integral $\alpha = 0$ is commonly used in economics and biology as integrals are approximated by a sum, the first point of the integral is chosen. The Stratonovich integral $\alpha = 1/2$ arises in Physical systems where noise correlation time $\tau_c > 0$, and, where the velocities are often computed at mid-points using three measurements. Finally, the isothermal integral $\alpha = 1$ is used in physical systems at equilibrium with a heat bath [22] such as our case with Brownian motion. Using the Padé approximation Eq.4.1.42, the spurious drift v_{noise} writes:

$$v_{\text{noise}}(z) = 2\alpha D_0 a \frac{2a^2 + 12az + 21z^2}{(2a^2 + 9az + z^2)^2}, \quad (4.1.54)$$

and the deterministic part of the drift writes:

$$v_d = -\frac{k_B T}{\gamma(z)} \left[-\frac{1}{\ell_D} B \exp\left(-\frac{z}{\ell_D}\right) + \frac{1}{\ell_B} \right]. \quad (4.1.55)$$

Finally, the langevin equation for a confined colloid near a wall writes:

$$V_t^i dt = \bar{v}_d^i dt + \sqrt{2D_i(z)} dB_t, \quad (4.1.56)$$

where the drifts \bar{v}_d^i are non-zero only for the z axis. A typical example of drift velocity for a colloidal particle of radius $a = 1.5 \mu\text{m}$ in water, confined in interaction potential with a Debye length $\ell_D = 50 \text{ nm}$, $B = 4 k_B T$ and a Boltzmann length $\ell_B = 500 \text{ nm}$, is plotted in Fig.29. As one can observe, the spurious drift is not negligible, and, even the presence of this additional drift doubles the total drift \bar{v}_d near the surface.

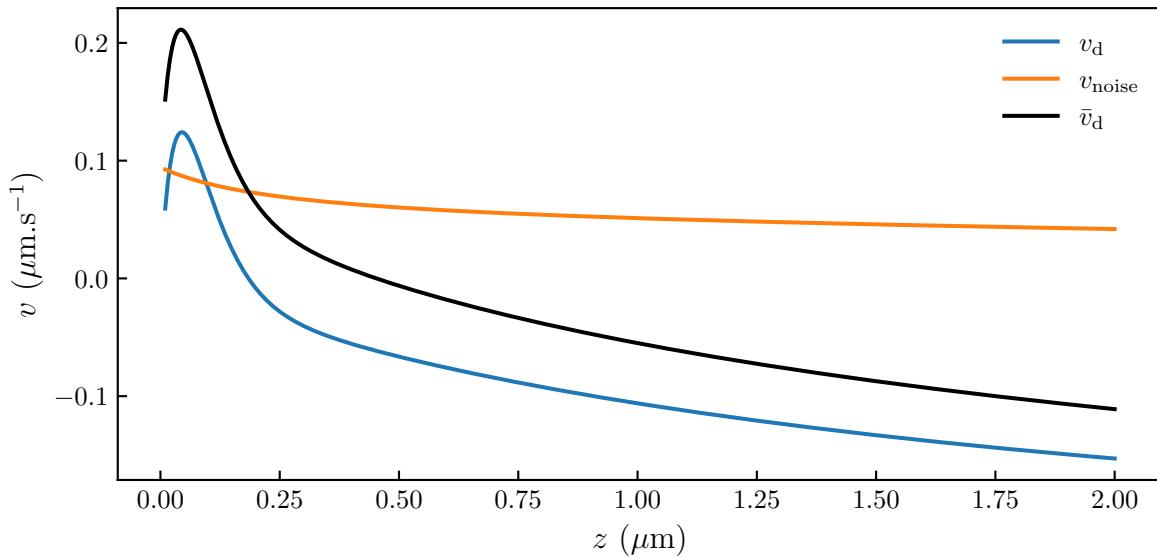


Figure 29: Typical drift velocity for a confined colloidal particle of radius $a = 1.5 \mu\text{m}$ in water. The physical properties of the interaction are $\ell_D = 50 \text{ nm}$, $B = 4 k_B T$ and $\ell_B = 500 \text{ nm}$.

4.1.6 Fokker-Plank equation

The Fokker-Plank equation is an alternative way to describe Brownian motion. Instead of explicitly calculating a Brownian trajectory by solving the Langevin equation, Fokker-Plank equation describes the particle distribution function $P(x, x_0; t)$ where x denotes the particle position and x_0 its initial position. To derive the Fokker-Plank plank equation, let us start by taking generic Langevin equation:

$$dX_t = u(X_t)dt + a(X_t)dB_t , \quad (4.1.57)$$

where X_t is the particle position, u is the drift due to external forces and a the magnitude of the random force. Let consider the average value of an arbitrary function $f(X_t)$ for a stochastic process obeying Eq.4.1.57, that started at position x_0 at time $t = 0$, by definition [79]:

$$\langle f(X_t) \rangle = \int dx p(x, x_0; t) f(x) , \quad (4.1.58)$$

with the initial conditions that can be written as:

$$p(x, x_0; 0) = \delta(x - x_0) . \quad (4.1.59)$$

We now take the time derivative of Eq.4.1.58, we start by expending f at the dt as:

$$\begin{aligned} \left\langle \frac{df(X_t)}{dt} \right\rangle &= \frac{d}{dt} \left\langle \frac{\partial f(X_t)}{\partial x} u(X_t) dt + \frac{1}{2} \frac{\partial^2 f(X_t)}{\partial x^2} a^2(X_t) dB_t^2 \right\rangle \\ &= \frac{d}{dt} \left(\frac{\partial f(X_t)}{\partial x} u(X_t) dt + \frac{1}{2} \frac{\partial^2 f(X_t)}{\partial x^2} a^2(X_t) dt \right) \\ &= \frac{\partial f(X_t)}{\partial x} u(X_t) + \frac{1}{2} \frac{\partial^2 f(X_t)}{\partial x^2} a^2(X_t) . \end{aligned} \quad (4.1.60)$$

By combining Eqs.4.1.58 and 4.1.60, we get:

$$\begin{aligned}
\left\langle \frac{df(X_t)}{dt} \right\rangle &= \int dx \frac{\partial p(x, x_0; t)}{\partial t} f(x) \\
&= \frac{\partial f(X_t)}{\partial x} u(X_t) + \frac{1}{2} \frac{\partial^2 f(X_t)}{\partial x^2} a^2(X_t) \\
&= \int dx p(x, x_0; t) Gf(x),
\end{aligned} \tag{4.1.61}$$

where G is an operator called the generator and is defined by its action on a function f by:

$$Gf = \frac{1}{2} a^2(x) \frac{\partial^2 f(x)}{\partial x^2} + u(x) \frac{\partial f(x)}{\partial x}. \tag{4.1.62}$$

Using the definition of the adjoint of G , denoted by G^\dagger , one has:

$$\begin{aligned}
\int dx \frac{\partial p(x, x_0; t)}{\partial t} f(x) &= \int dx p(x, x_0; t) Gf(x) \\
&= \int dx G^\dagger p(x, x_0; t) f(x).
\end{aligned} \tag{4.1.63}$$

From the latter, we thus have:

$$\frac{\partial p(x, x_0; t)}{\partial t} = G^\dagger p(x, x_0; t). \tag{4.1.64}$$

Which leads to the Forward-Fokker-Planck equation:

$$\frac{\partial p(x, x_0; t)}{\partial t} = \frac{\partial^2}{\partial x^2} \left[\frac{a^2(x)}{2} p(x, x_0; t) \right] - \frac{\partial}{\partial x} [u(x)p(x, x_0; t)] \tag{4.1.65}$$

The latter is called Forward because the partial differential equation is in terms of the variable x , the position of the particle at which the stochastic process ends up, at time t . For a confined Brownian motion near a wall, using the previously determined drifts \bar{v}_d Eq.4.1.53, the Fokker-Plank equation for the movement along the z axis, perpendicular to the wall writes:

$$\frac{\partial p(z, z_0; t)}{\partial t} = \frac{\partial^2}{\partial z^2} [D(z)p(z, z_0; t)] - \frac{\partial}{\partial z} [\bar{v}_d p(z, z_0; t)]. \tag{4.1.66}$$

As an example, the stationary solution of the latter, $\partial p/\partial t = 0$, is given by the Gibbs-Boltzmann distribution $P_{\text{eq}}(z)$ (Eq.4.1.29). The solution for the Eq.4.1.64 with the particle starting at position z_0 , at time $t_0 = 0$ writes:

$$\begin{aligned} P(z, z_0; t) &= \exp(G^\dagger t)p(z, z_0, 0) \\ &= \exp\left(\frac{\partial^2}{\partial z^2}D(z)t - \frac{\partial}{\partial z}\bar{v}_d t\right)\delta(z - z_0) \end{aligned} \quad (4.1.67)$$

4.1.7 Numerical simulation of confined Brownian motion

We had previously determined with that the simulation of a bulk Brownian motion, without external forces can by the simple formula (Eq.2.4.9):

$$x_i = x_{i-1} + \sqrt{2D}w_i . \quad (4.1.68)$$

However, with the confined Brownian motion one needs to take into account the hindered diffusion, external forces due to gravity and double-layer interactions and the noise-induced drift. Putting all that together, leads to a new equation for x_i which writes for the movement parallel to wall:

$$x_i = x_{i-1} + \sqrt{2D_{\parallel}}w_i , \quad (4.1.69)$$

where i represents x and y axis. For the particle movement perpendicularly to the wall (z axis), one needs to add the total drift \bar{v}_d Eq.4.1.53, such that:

$$z_i = z_{i-1} + \bar{v}_d(z_{i-1})\tau + \sqrt{2D_{\perp}}w_i , \quad (4.1.70)$$

where τ is the simulation time step and w_i being a Gaussian distribution of mean value $\langle w_i \rangle = 0$ and variance $\langle w_i^2 \rangle = \tau$. Unlike the bulk Brownian motion where the time step τ can be chosen for the desired precision as shown previously on Fig.4; for an accurate simulation, the time-step should here be short enough for the drifts \bar{v}_d and local diffusion coefficient to be relatively constant in the time periods $t_{i+1} - t_i = \tau$ and in the displacement range $\Delta z = z_{i+1} - z_i$, such that:

$$\bar{v}_d(z \in [z_i, z_{i+1}]) \simeq \bar{v}_d(z_i) , \quad (4.1.71)$$

and,

$$D_{\perp,\parallel}(z \in [z_i, z_{i+1}]) \simeq D_{\perp,\parallel}(z_i) . \quad (4.1.72)$$

Since that the diffusion coefficient vary less for the movement parallel one can only consider the perpendicular motion to determine the simulation time step. Also, as it can be seen on the Fig.?? where the drift v_d and diffusivity gradient are plotted, the diffusion gradient are negligible compared to the drift gradient, we thus focus on the drifts condition Eq.4.1.71. Moreover, the drifts vary more when the colloid is near the surface where one can approximate the diffusion coefficient D_{\perp} by the approximation Eq.4.1.43 such that:

$$D_{\perp}^{z \ll a}(z) = D_0 \frac{z}{a} . \quad (4.1.73)$$

As near the surface, the gravitational interaction can be neglected as it is smaller than the electrostatic interactions has it has been shown on the potential plot Fig.25. In that case, by taking $\alpha = 1$, the total drifts Eq.4.1.53 near the surface simplifies to:

$$\begin{aligned} \bar{v}_d &= \frac{k_B T}{\gamma_0} \frac{z}{a} \left[\frac{B}{\ell_D} \exp\left(-\frac{z}{\ell_D}\right) \right] + \frac{\partial}{\partial z} D_0 \frac{z}{a} \\ &= D_0 \frac{z}{a} \left[\frac{B}{\ell_D} \exp\left(-\frac{z}{\ell_D}\right) \right] + \frac{D_0}{a} \\ &= \frac{D_0}{a} \left[1 + \frac{Bz}{\ell_D} \exp\left(-\frac{z}{\ell_D}\right) \right] \end{aligned} \quad (4.1.74)$$

By expending the exponential term to the order z/ℓ_D order, we get:

$$\begin{aligned} \bar{v}_d &= \frac{D_0}{a} \left[1 + \frac{Bz}{\ell_D} \exp\left(-\frac{z}{\ell_D}\right) \right] = \frac{D_0}{a} \left[1 + \frac{Bz}{\ell_D} \left(1 - \frac{z}{\ell_D}\right) \right] \\ &= \frac{D_0}{a} \left(1 + \frac{Bz}{\ell_D} \right) \end{aligned} \quad (4.1.75)$$

To satisfy Eq.4.1.71, we need to have small relative change of drifts in an interval $[z, z+\Delta z]$ which can be written as:

$$\frac{\bar{v}_d(z + \Delta z) - \bar{v}_d(z)}{\bar{v}_d(z)} \ll 1 \quad (4.1.76)$$

Combining Eqs.4.1.75 and 4.1.76, we get:

$$\Delta z \ll \frac{\ell_D}{B} + z . \quad (4.1.77)$$

Using the MSD, it is possible to link the latter to the simulation time step τ by using the average value of Δz^2 , such that:

$$\langle \Delta z^2 \rangle(z) = 2D_{\perp}(z)\tau = 2D_0 \frac{z}{a}\tau \quad (4.1.78)$$

Combining Eqs.4.1.77 and 4.1.78 thus leads to:

$$\tau = \frac{a\langle \Delta z^2 \rangle}{2D_0 z} < \frac{a}{2D_0} \frac{\left(\frac{\ell_D}{B} + z\right)^2}{z} = \tau_{\max}(z) , \quad (4.1.79)$$

where τ_{\max} is thus the maximal time step that satisfies Eq.4.1.77. At this point, there is two different way to do the simulation: the first one is to do an adaptive time step where τ_{\max} for each step of the simulation ; the second one is to find the smallest $\tau_{\max}(z)$ and use for all the simulation a time step τ satisfying $\tau < \min(\tau_{\max})$. The latter can be evaluated by finding the height z_{\min} , at which the derivative of τ_{\max} nullifies, such that:

$$\left. \frac{\partial \tau_{\max}}{\partial z} \right|_{z_{\min}} = 0 . \quad (4.1.80)$$

Solving the latter gives:

$$z_{\min} = \frac{\ell_D}{B} . \quad (4.1.81)$$

which finally gives a maximal simulation time step, $\min(\tau_{\max})$:

$$\min(\tau_{\max}) = \frac{2a}{D_0} \frac{\ell_D}{B} , \quad (4.1.82)$$

this time should be the maximal time step τ used for a confined Brownian simulation near a wall to ensure an accurate simulation of the near-wall effects. In the Fig.30-b) some typical τ_{\max} are plotted for a colloid of radius $a = 1.5 \mu\text{m}$, $B = 4$ and ℓ_D varying between 20 and 100 nm. We can observe that for this range of values that well represents the experiments that I have done during my thesis, taking a simulation time step $\tau < 0.01$ s can be used for all set of parameters.

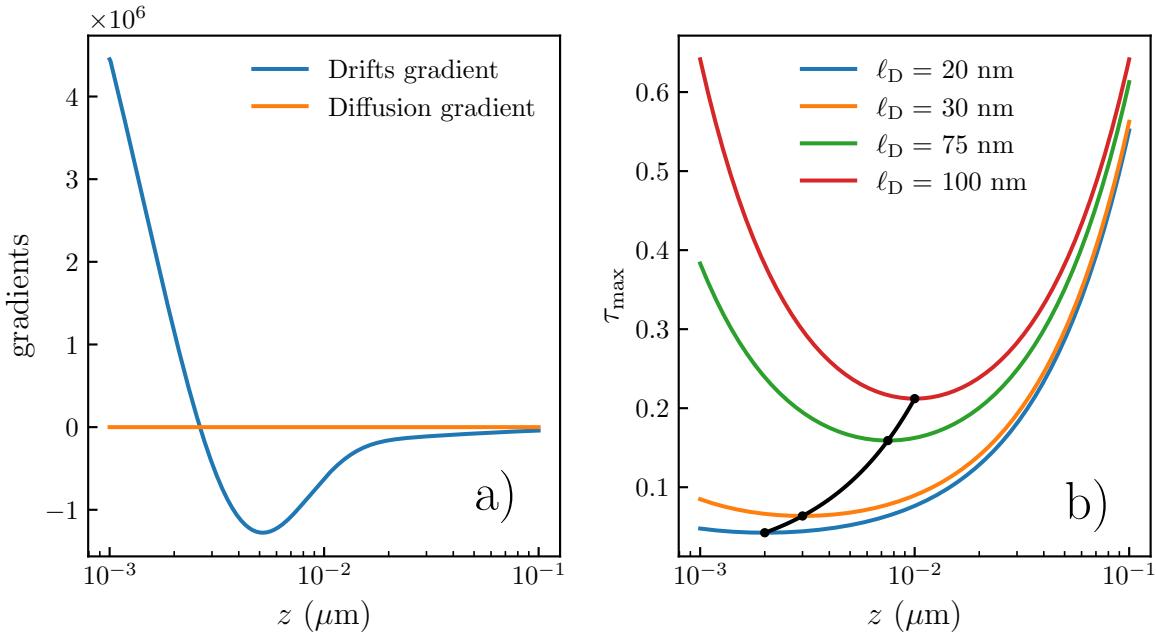


Figure 30: a) Drift and diffusive gradients for a confined colloidal particle of radius $a = 1.5 \mu\text{m}$ in water. The physical properties of the interaction are $\ell_D = 50 \text{ nm}$, $B = 4 k_B T$ and $\ell_B = 500 \text{ nm}$. b) τ_{\max} for a particle of radius $a = 1.5 \mu\text{m}$ and $B = 4$ for different Debye length. The black line represent the minimum value τ_{\max} for ℓ_D varying between 20 and 100 nm, this minimal time represent the maximal simulation time step that should be use for an accurate simulation.

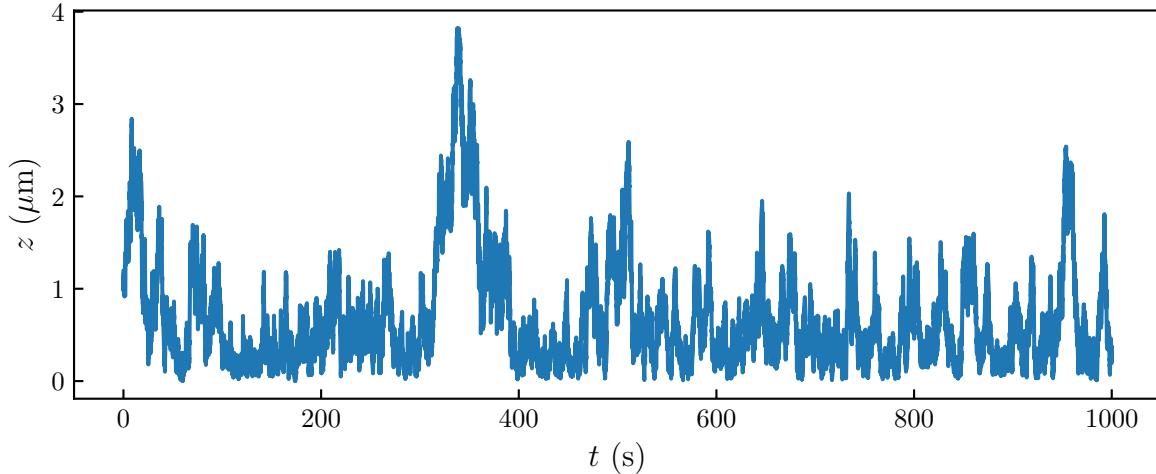


Figure 31: Simulated confined Brownian motion height trajectory of a colloidal particle of radius $a = 1.5 \mu\text{m}$ of density $\rho_p = 1050 \text{ kg.m}^{-3}$, $\alpha = 1$ and the potential is characterized by $\ell_D = 50 \text{ nm}$ and $B = 4$.

We have developed the simulation of the Confined Brownian motion using Python, as part of the Master’s internship of Élodie Millan, the interested reader will find more information on confined Brownian motion in more complex systems in her forthcoming thesis. A trajectory of a colloidal particle of radius $a = 1.5 \mu\text{m}$ of density $\rho_p = 1050 \text{ kg.m}^{-3}$ and the potential characterized by $\ell_D = 50 \text{ nm}$ and $B = 4$, is plotted in the Fig.31 which do look like the experimental trajectory that was shown in Fig.22 as an introduction to the chapter. In this trajectory, the noise induced lift is taken in account by using the isothermal approach $\alpha = 1$. However, to check if it is the correct way to take into account the spurious drift, the constraint we have is that the long time statistics should satisfy the Gibbs-Boltzmann equation Eq.4.1.29. To compute an experimental probability density function from a set of points, one can use the following Python snippet.

```

1 def pdf(data, bins=10, density=True):
2
3     pdf, bins_edge = np.histogram(data, bins=bins, density=density)
4     bins_center = (bins_edge[0:-1] + bins_edge[1:]) / 2
5
6     return pdf, bins_center

```

The Gibbs-Boltzmann distribution for $\alpha = 0, 0.5$ and 1 is shown in Fig.32. We see that the Isothermal integral of the noise gives the correct distribution, in the other two cases, we observe that the particle is more likely to be found nearer the surface, this is corrected by the additional spurious drift Eq.4.1.54.

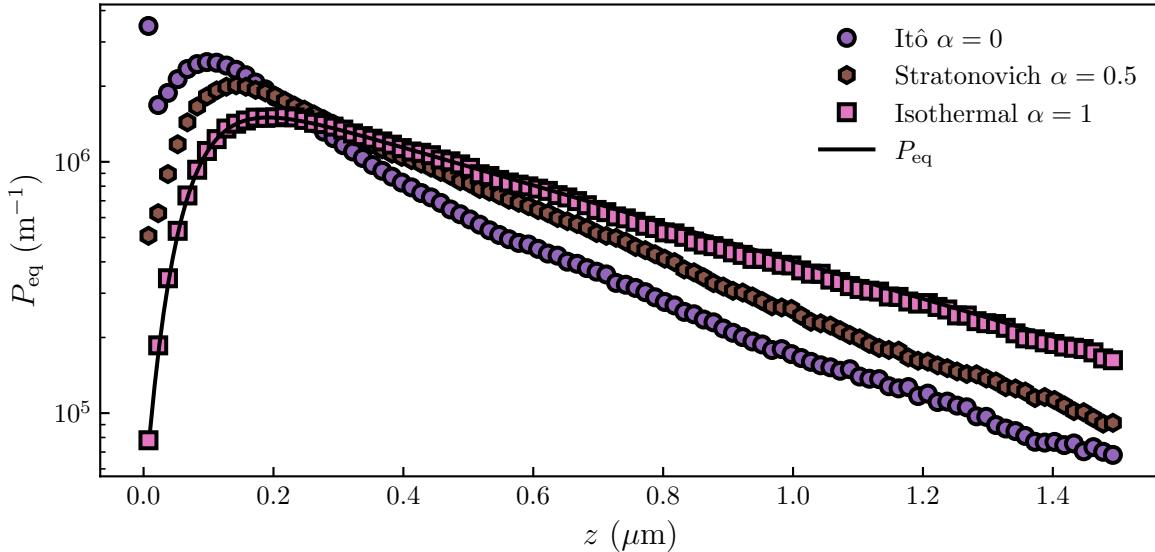


Figure 32: Probability Density Function of the height of the particle for different computation of the spurious drift, Itô $\alpha = 0$, Stratonovich $\alpha = 0.5$ and Isothermal $\alpha = 1$. The plain black line represents the expected Gibbs-Boltzmann distribution. The simulation parameters : $a = 1.5 \mu\text{m}$, $\rho_p = 1050 \text{ kg.m}^{-3}$, $\ell_D = 50 \text{ nm}$, $B = 4$ and $\ell_B = 577 \text{ nm}$.

4.2 Experimental study

Let us now analyze the experimental data obtain through the Mie tracking. In the theory, we wrote the height of the particle z being the shortest distance between the wall and the colloidal particle surface. However, it is not the height measured by the Mie tracking, since it measure the distance between the objective lens focal plane and the particle center. To have the correct measured height, we suppose that the particle do approach the very closely to the wall. From that assumption, we then a moving minimum along the trajectory to set the minimal value of the trajectory to zero. The moving minimum can be calculated using the following Python function:

```

1 def movmin(z, window):
2     result = np.empty_like(z)
3     start_pt = 0
4     end_pt = int(np.ceil(window / 2))
5
6     for i in range(len(z)):
7         if i < int(np.ceil(window / 2)):
8             start_pt = 0
9         if i > len(z) - int(np.ceil(window / 2)):
10            end_pt = len(z)
11         result[i] = np.min(z[start_pt:end_pt])
12         start_pt += 1

```

```

13     end_pt += 1
14
15     return result

```

In the latter, `window` represent the number of points is used to compute the minimum. As an example, if one chose `window = 10000`, the first value of `result` is the minimum of the first 10000 points of `z`. If there is enough data around the point where the minimum is calculated, the ensemble is centered, with a window of 100, the minimal value at the 100th elements is between the 50th and 150th (*e.g.* `result[100] = np.min(z[50:150])`). The raw and rescaled trajecotry are shown Fig.33. Of course, that technique is not perfect and we are working on a method that could directly measure the wanted wall-particle distance, also using Mie (Next chapter ?). However, subtract the moving minimum has a benefit, indeed it can remove some experimental drift that can be due of the physical movement of the optical pieces of the microscope. Moreover, as the exact location of the $z = 0$ origin is thus *a priori* undetermined we add to the physical parameters B , ℓ_D and ℓ_B a height offset z_{off} that account for the correction of the wall position.

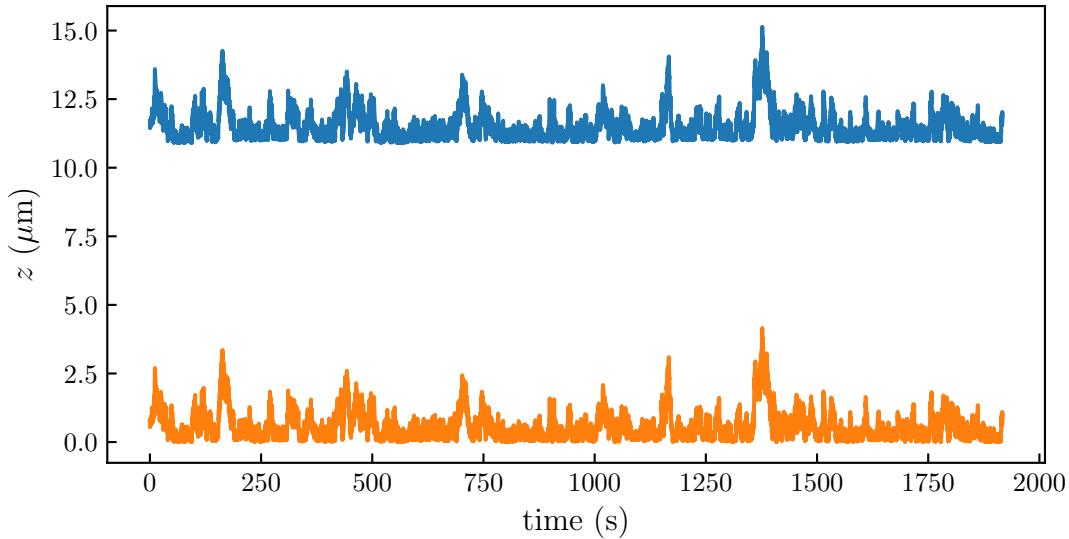


Figure 33: Raw trajectory measured using the Mie tracking technique, and it's rescaled value using moving minimum algorithm with a window of 10000 points.

4.2.1 Equilibrium distribution

As we have done for the simulated trajectory, one can construct the equilibrium probability density function $P_{\text{eq}}(z)$ of the position of the particle. As seen in Fig.34, and explain in the previous section, an exponential tail is observed at large distance, which is identified

to the sedimentation contribution in Perrin's experiment [5], but here with the probability density function of a single particle instead of the concentration field. In contrast, near the wall, we observe an abrupt depletion, indicating a repulsive electrostatic contribution. Additionally, we see that the Gibbs-Boltzmann distribution Eq.?? fits the data very well.

Moreover, as shown in Fig.35, we verified that we recover the Debye relation $\ell_D = 0.304/\sqrt{[\text{NaCl}]}$, with ℓ_D in nm, and where $[\text{NaCl}]$ is the concentration of salt in mol/L, with a prefactor corresponding to a single monovalent salt in water at room temperature [70]. Besides, we have verified, as shown on the Fig.35, that the dimensionless parameter B related to surface charges is constant in the studied salt-concentration range, thus excluding any nonlinear effect [80, 81] in our case.

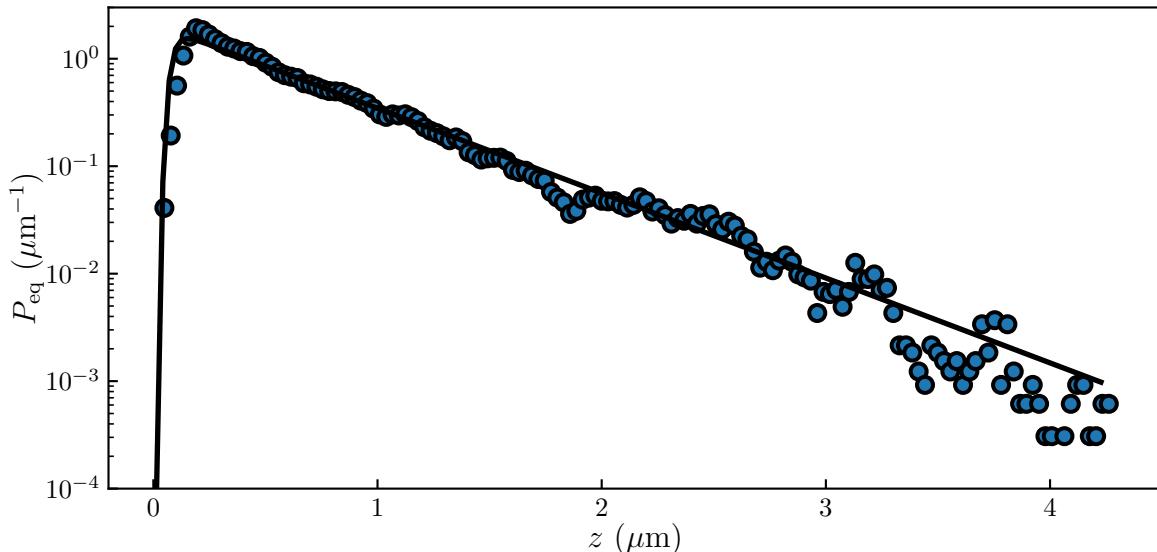


Figure 34: Measured equilibrium probability density function P_{eq} of the distance z between the particle and the wall. The solid line represents the best fit to the normalized Gibbs-Boltzmann distribution in position, using the total potential energy $U(z)$ of Eq.4.1.28, with $B = 4.8$, $\ell_D = 21 \text{ nm}$, and $\ell_B = 530 \text{ nm}$.

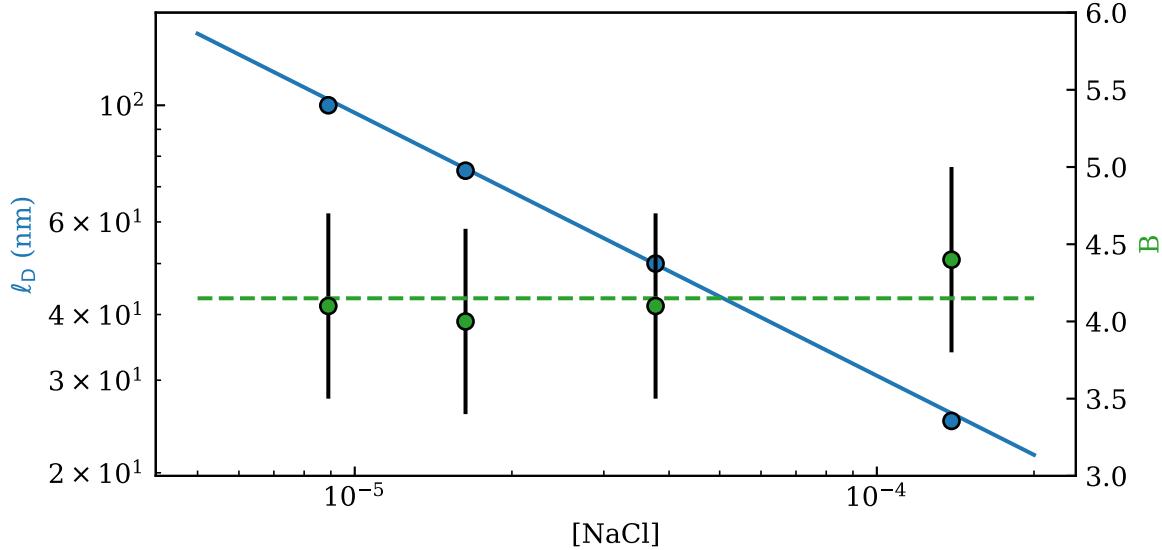


Figure 35: In blue, left axis, measured Debye length ℓ_D as a function of salt concentration [NaCl]. The solid line is the expected Debye relation $\ell_D = 0.304/\sqrt{[NaCl]}$, for a single monovalent salt in water at room temperature. In green, right axis, measured B as a function of salt concentration [NaCl]. The dashed line represents the mean value of the measured B values.

4.2.2 Mean Square Displacement

We now turn to dynamical aspects, by considering the mean-squared displacement (MSD). For the three spatial directions, indexed by $i = x, y$, and z , corresponding to the coordinates $r_x = x$, $r_y = y$, and $r_z = z$, of the position \vec{r} , and for a given time increment Δt , the MSD is defined as:

$$\langle \Delta r_i(t)^2 \rangle_t = \langle [r_i(t + \Delta t) - r_i(t)]^2 \rangle_t , \quad (4.2.1)$$

where the average $\langle \rangle_t$ is performed over time t . For a free Brownian motion in the bulk, and in the absence of other forces than the dissipative and random ones, the MSD is linear in time, *i.e.* $\langle \Delta r_i(t)^2 \rangle_t = 2D_0\Delta t$, where $D_0 = k_B T / (6\pi\eta a)$ is the bulk diffusion coefficient given by the Stokes-Einstein relation [4], and η is the liquid viscosity. Further including sedimentation restricts the validity of the previous result along z to short times only, *i.e.* for $\Delta t \ll \ell_B^2/D_0$ such that the vertical diffusion is not yet affected by the gravitational drift.

The presence of a rigid wall at $z = 0$ adds a repulsive electrostatic force along z . It also decreases the mobilities nearby through hydrodynamic interactions, leading to effective viscosities $\eta_{\parallel}(z) = \eta_x(z) = \eta_y(z)$, and $\eta_{\perp}(z) = \eta_z(z)$. Interestingly, despite the previous

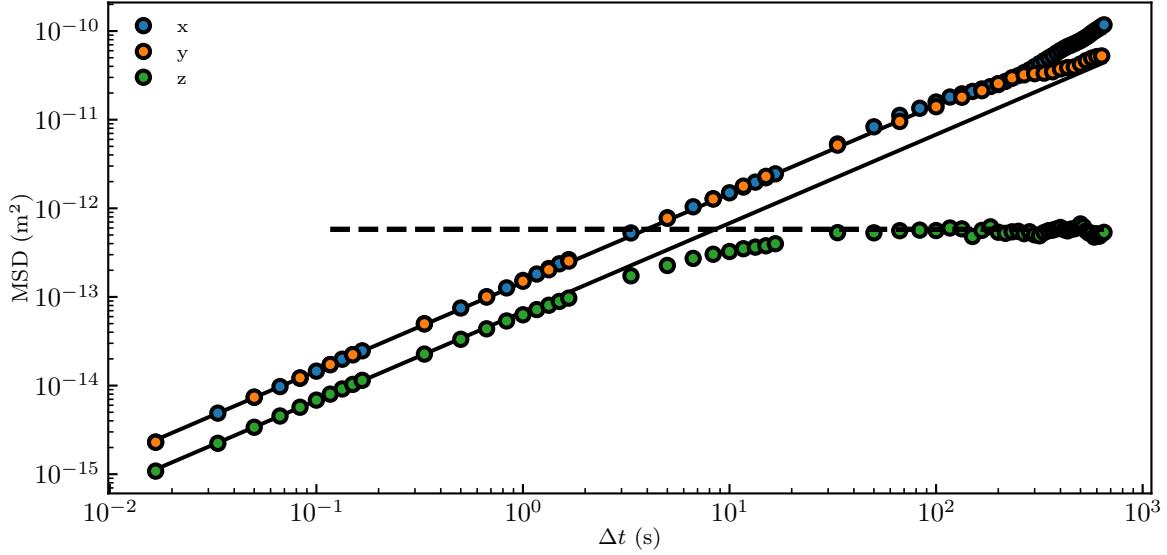


Figure 36: Measured mean-squared displacements (MSD, see Eq.4.2.1) as functions of the time increment Δt , for the three spatial directions, x , y , and z . The solid lines are best fits to Eq. (4.2.2), using Eqs.4.1.28, 4.1.40, and 4.1.41, with $B = 4.8$, $\ell_D = 21$ nm, and $\ell_B = 530$ nm, providing the average diffusion coefficients $\langle D_{\parallel} \rangle = \langle D_x \rangle = \langle D_y \rangle = 0.52 D_0$ and $\langle D_z \rangle = 0.24 D_0$. The dashed line is the best fit to Eq. (4.2.10), using Eq. 4.1.28, with $B = 4.8$, $\ell_D = 21$ nm, and $\ell_B = 530$ nm.

modifications, the temporal linearity of the MSD is not altered by the presence of the wall [7, 19] for x and y , as well as at short times for z . In such cases, the MSD reads:

$$\langle \Delta r_i(t)^2 \rangle_t = 2\langle D_i \rangle \Delta t , \quad (4.2.2)$$

where for each spatial direction we introduced the local diffusion coefficient $D_i(z) = D_0 \eta / \eta_i(z)$, and its average

$$\langle D_i(z) \rangle = \int_0^\infty dz D_i(z) P_{\text{eq}}(z) , \quad (4.2.3)$$

against the Gibbs-Boltzmann distribution in position. As shown in Fig. 36, the MSD measured along x or y is indeed linear in time. By fitting to Eq.4.2.2, using Eqs.4.1.28 and 4.1.41, we extract an average transverse diffusion coefficient $\langle D_{\parallel} \rangle = \langle D_x \rangle = \langle D_y \rangle = 0.52 D_0$. In contrast, along z , we identify two different regimes: one at short times, where the MSD is still linear in time, with a similarly-obtained best-fit value of $\langle D_z \rangle = 0.24 D_0$; and one at long times, where the MSD saturates to a plateau. This latter behaviour indicates that the equilibrium regime has been reached, with the particle having essentially explored all the relevant positions given by the Gibbs-Boltzmann distribution.

4.2.3 Non-gaussian dynamics - Displacement distribution

Having focused on the MSD, *i.e.* on the second moment only, we now turn to the full probability density function P_i of the displacement Δr_i . Since, the diffusion coefficient $D_i(z)$ varies as a result of the variation of z along the particle trajectory, P_i exhibits a non-Gaussian behavior, as seen in Figs. 37-a,b,c,d). We even resolve the onset of a non-Gaussian behavior in P_x , by zooming on the large- $|\Delta x|$ wings. At short times, the diffusion coefficient D_i and the drift velocity \bar{v}_d , can be considered constant. By writing the initial condition $\delta(x_i - x_i^0)$, the solution of the Forward Fokker-Plank Eq.4.1.67, becomes:

$$\begin{aligned} P(x_i, z_0, \Delta t) &= \exp \left[\frac{\partial^2}{\partial z^2} D_i(z_0) \Delta t - \frac{\partial}{\partial z} \bar{v}_d^i(z_0) \Delta t \right] \frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp(ju(z - z_0)) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} du \left[D_i(z_0) \Delta t \frac{\partial^2}{\partial z^2} \exp(ju(z - z_0)) - \bar{v}_d^i(z_0) \Delta t \frac{\partial}{\partial z} \exp(ju(z - z_0)) \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp \left[-u^2 D(z_0) \Delta t + ju(z - z_0) - ju \bar{v}_d^i(z_0) \Delta t \right] , \end{aligned} \quad (4.2.4)$$

where $\bar{v}_d^i(z_0)$ is non zero only for the z axis. The latter can be reduced to [82, 83]:

$$P_i(\Delta r_i, z_0, \Delta t) = \frac{1}{\sqrt{4\pi D_i(z_0) \Delta t}} \exp \left[-\frac{(\Delta r_i - \bar{v}_d^i \Delta t)^2}{4D_i(z) \Delta t} \right] . \quad (4.2.5)$$

Which is a Gaussian distribution with a 0 mean value for the x and y axis and

$$\langle P_z(\Delta z, z_0, \Delta t) \rangle = \bar{v}_d \Delta t , \quad (4.2.6)$$

for the z axis. Additionally, it has a variance $\sigma_i(z_0) = \sqrt{2D_i(z_0) \Delta t}$. From Eq.4.2.5, we can observe than the total drift \bar{v}_d induces an asymmetry on the displacement along the z axis. However, in our experiment, as we have access to long enough trajectory, we are interested about the distribution which is not conditioned by the initial position but by the Gibbs-Boltzmann distribution Eq.4.1.28. At short times, P_i can thus be modeled by the averaged diffusion Green's function [18, 84]:

$$\begin{aligned} P_i(\Delta r_i, \Delta t) &= \int_0^{\infty} dz_0 P_{\text{eq}} P(x_i, z_0, \Delta t) \\ &= \int_0^{\infty} dz P_{\text{eq}}(z) \frac{1}{\sqrt{4\pi D_i(z) \Delta t}} e^{-\frac{\Delta r_i^2}{4D_i(z) \Delta t}} , \end{aligned} \quad (4.2.7)$$

against the Gibbs-Boltzmann distribution. Which can alternatively be written as an integral over the diffusion such that:

$$P(\Delta r_i, \Delta t) = \int_0^\infty dD_i P(D_i) \frac{1}{\sqrt{4\pi D_i \Delta t}} e^{-\frac{\Delta r_i^2}{4D_i \Delta t}} \quad (4.2.8)$$

The latter can be evaluated using the following Python snippet.

```

1 def P_D(B, ld, lb):
2     # Computing the D PDF.
3     z = np.linspace(1e-9, 15e-6, 1000)
4     P_D = Dz(z) * P_b_off(z, offset, B, ld, lb)
5     P_D = P_D / np.trapz(P_D, z)  # extra step to ensure PDF normalization
6     return Dz, P_D
7
8
9 def _P_Dz_short_time(Dz, Dt, B, ld, lb):
10    # Using the D PDF to compute the P()
11    D_z, P_D = P_D(B, ld, lb)
12    P = P_D / np.sqrt(4 * np.pi * D_z * Dt) * np.exp(-(Dz ** 2) / (4 * D_z * Dt))
13    P = np.trapz(P, D_z)
14    return P
15
16
17 # Wrapping the previous function of easier use for Dz arrays.
18 def P_Dz_short_time(Dz, Dt, B, ld, lb):
19     P = np.array([_P_Dz_short_time(i, Dt, B, ld, lb) for i in Dz])
20     P = P / np.trapz(P, Dz) # extra step to ensure PDF normalization
21     return P

```

In the latter snippet, the evaluation is done for Δz , however to compute $P_x(\Delta x)$ one should just change the $Dz(z)$ function to compute the parallel diffusion coefficient $D_{||}$. As a PDF P should be normalize such that $\int P = 1$, I added extra step to ensure PDF noralization along the evaluation. Since, we have reach equilibrium, the particle the overall drift of the particle should be equal to 0 thus leading to a mean value of the distribution Eq.4.2.7, $\langle P_i(\Delta r_i, \Delta t) \rangle = 0$. As shown in Figs.37-a,c,b,d) Eq.4.2.7 captures the early data very well. At long times, Eq.4.2.7 remains valid only for P_x and P_y . Nevertheless, the equilibrium regime being reached, P_z only depends on the Gibbs-Boltzmann distribution and can eventually be written as:

$$\lim_{\Delta t \rightarrow \infty} P_z(\Delta z) = \int_0^\infty dz P_{eq}(z + \Delta z) P_{eq}(z) , \quad (4.2.9)$$

which contains in particular the second moment:

$$\lim_{\Delta t \rightarrow \infty} \langle \Delta z^2 \rangle = \int_{-\infty}^{+\infty} d\Delta z \Delta z^2 \int_0^{\infty} dz P_{\text{eq}}(z + \Delta z) P_{\text{eq}}(z) . \quad (4.2.10)$$

As shown in Fig. 37-e), Eq.4.2.9 captures the long-term data along z very well. Additionally, Eq.4.2.10 permits to fit the plateau of the MSD as shown in the Fig.36. Eq.4.2.9 can be evaluated using the following Python function:

```

1 def _Pdeltaz_long(DZ, B, ld, lb):
2     z = np.linspace(0, 20e-6, 1000)
3     dP = P_eq(z, B, ld, lb) * P_eq(z + DZ, B, ld, lb)
4     P = trapz(dP,z)
5     return P
6
7 def Pdeltaz_long(DZ, B, ld, lb):
8     pdf = np.array([_Pdeltaz_long(i,B, ld, lb) for i in DZ])
9     pdf = pdf / trapz(pdf,DZ)
10    return pdf

```

Where the P_{eq} function has been described in the section 4.1.2.

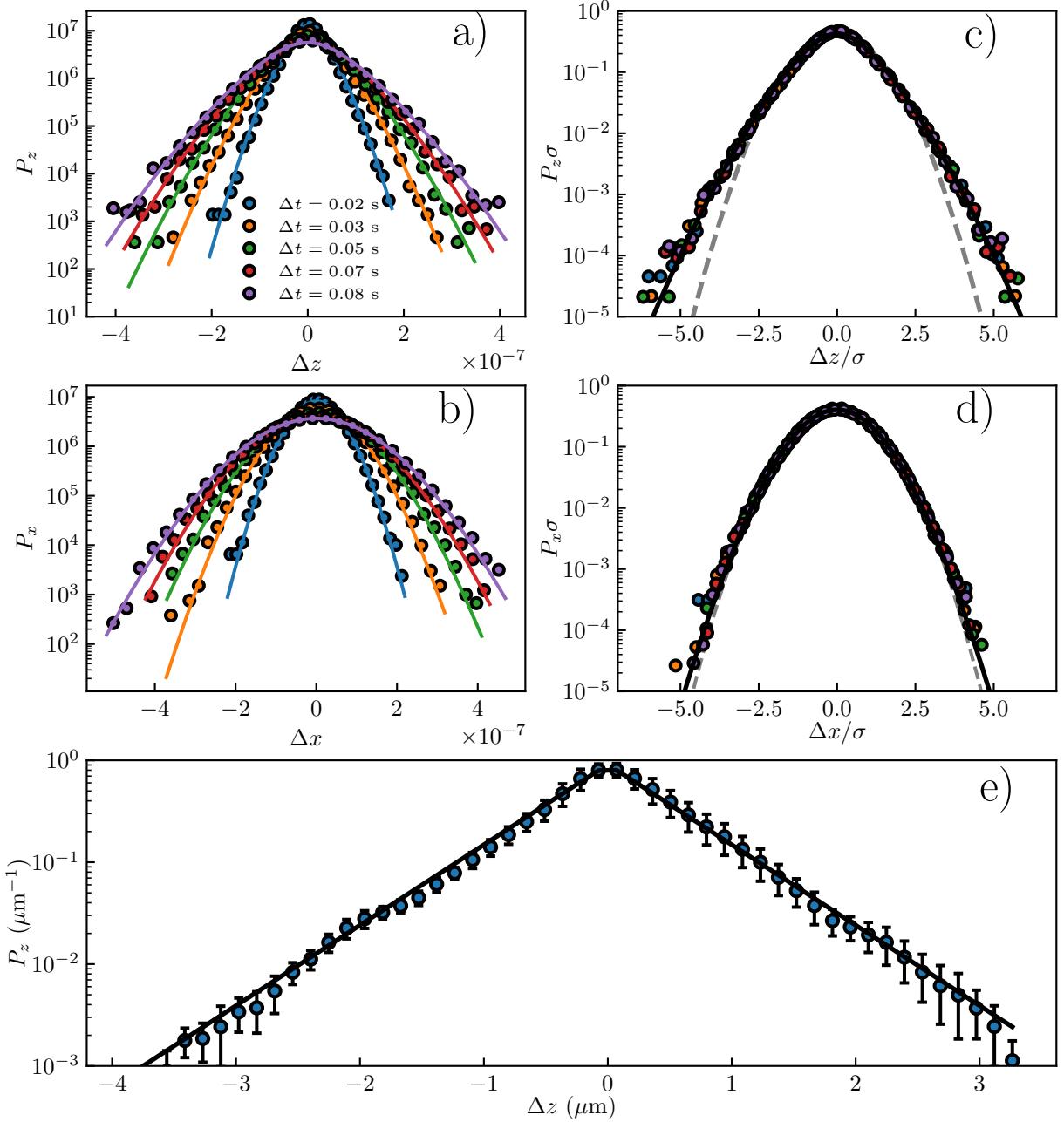


Figure 37: a, b) Probability density functions P_i of the displacements Δx and Δz , at short times. The solid lines are the best fits to Eq.4.2.7, using Eqs.4.1.28, 4.1.40, and 4.1.41, with $B = 4.8$, $\ell_D = 21 \text{ nm}$, and $\ell_B = 530 \text{ nm}$. c,d) Normalized probability density functions $P_i\sigma$ of the normalized displacements $\Delta x/\sigma$ and $\Delta z/\sigma$, at short times, with σ^2 the corresponding MSD (see panel Fig.36), for different time increments Δt ranging from 0.0167 s to 0.083 s, as indicated with different colors. The solid lines are the best fits to Eq.4.2.7, using Eqs.4.1.28, 4.1.40, and 4.1.41, with $B = 4.8$, $\ell_D = 21 \text{ nm}$, and $\ell_B = 530 \text{ nm}$. For comparison, the grey dashed lines are normalized Gaussian distributions, with zero means and unit variances. d) Probability density function P_z of the displacement Δz , at long times, averaged over several values of Δt ranging between 25 and 30 s. The solid line is the best fit to Eq. (4.2.9), using Eq. (4.1.28), with $B = 4.8$, $\ell_D = 21 \text{ nm}$, and $\ell_B = 530 \text{ nm}$.

4.2.4 Local diffusion coefficient inference

We now wish to go beyond the previous average $\langle D_i \rangle$ of Eq. (4.2.2), and resolve the local diffusion coefficient $D_i(z)$. To measure local viscosity from experimental trajectories, a binning method is generally employed [85]. This method consists on constructing the displacement PDF conditioned on the particle height, and measure the distribution's variance $\sigma_i(z_0) = \sqrt{2D_i(z_0)\Delta t}$ as in Eq. 4.2.5.

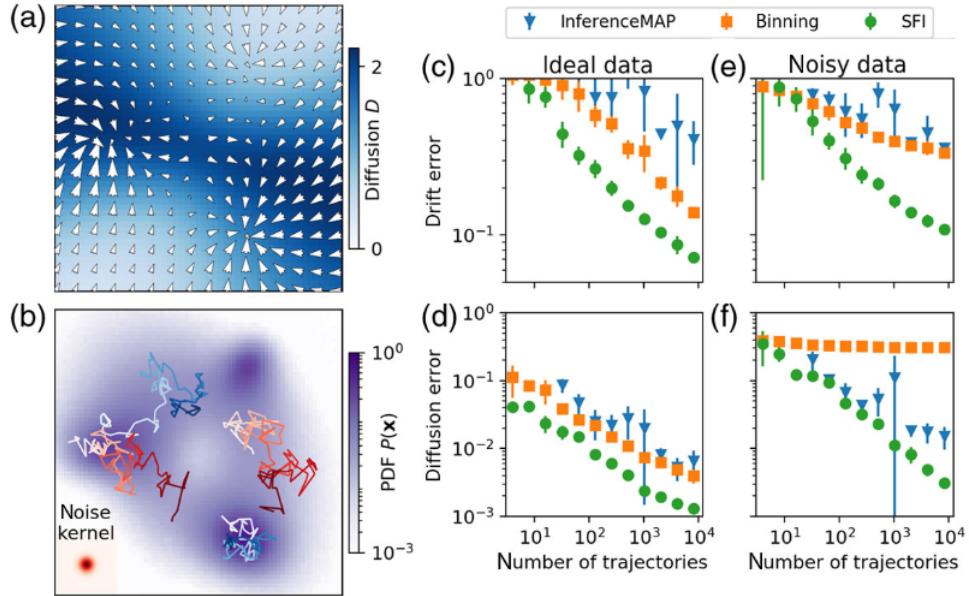


Figure 38: Figure from [86]. Quantitative comparison of Surface Force Inference (SFI) with other methods on a simulated system mimicking 2D single-molecule trajectories in a complex environment with space-dependent isotropic diffusion. a) The diffusion field (blue gradient) and drift field (white arrows). b) The steady-state distribution function (PDF) of the process. The traces are representative trajectories of 100 time steps. c-f) Comparison of the performance of SFI and two widely used inference methods: InferenceMAP, a method for single-molecule inference (blue triangles) [87], and grid-based binning with maximum-likelihood estimation [85, 88] (orange squares). They evaluated the performance of these methods on the approximation of the drift field (c,e)) and diffusion field (d,f)) as a function of the number N of single-molecule trajectories (similar to the ones in panel b)) used. With ideal data (c,d)) and in the presence of measurement noise(e,f)). The performance is evaluated as the average mean-squared error on the reconstructed field along trajectories. SFI outperforms both other methods in all cases. More information about the parameters of their simulation and analysis can be found in their work [86].

Although the binning method is well suited for drift measurements, it suffers from a lack of convergence and precision when second moments or local diffusion coefficients have to be extracted. In particular, the binning method did not allow us to measure specifically the local diffusion coefficient in the key interfacial region corresponding to

$z < 100$ nm. Indeed, as we can observe in the Fig.38-f) the diffusion error on noisy (such as experimental) data does saturate of the binning method is outperformed by a robust developed method recently developed by Frishman and Ronceray [86]. This method uses Stochastic Force Inference (SFI), in order to evaluate spatially-varying force fields and diffusion coefficients, from the information contained within the trajectories.

In practice, SFI can reconstruct force and diffusion fields and measure entropy production from Brownian trajectories. SFI is based on the communication-theory notion of capacity which is an information-theoretic bound, when the system is viewed as a communication channel, that limit at which rate information about the fields can be extracted from a Brownian trajectory. To explain the method, let us consider a Brownian particle that obey to the equation similar to Eq.4.1.56:

$$\dot{x}_\mu = F_\mu(x) + \partial_n u D(x)_{\mu\nu} + \sqrt{2D(x)}_{\mu\nu} \partial dB_t , \quad (4.2.11)$$

where the force field $F_\mu(x)$ and the diffusion tensor field $D_{\mu\nu}$ are assumed to be time-independent. This method thus is build to measure the entire diffusion matrix, which for a particle diffusing above a wall has non-zero the diagonal term (*i.e.* D_{xx} , D_{yy} and D_{zz}). However, for more complex environment such as cellular environment as shown on Fig.38-a), non-diagonal terms exists leading to correlations between the displacement axis. As an example, it has been observed in elastohydrodynamics [89] that the a parallel movement near a soft wall can induce perpendicular forces, hence leading in that peculiar case, to correlation between the x and z movement. The SFI idea is to approximate the diffusion field as a linear combination of ginite basis of n_b kown functions $b = b_\alpha(x)_{\alpha=1\dots n_b}$. For the diffusion of spherical spheres near a wall where the diffusion coefficient is given by the Eqs.4.1.40 and 4.1.41 we got robust results using their built-in polynomial basis:

$$b = \{1, x_\mu, x_\mu x_\nu, \dots\} \text{ (up to the order } n_b \text{).} \quad (4.2.12)$$

They perfrom this approximation by projecting the diffusion field onto the space spanned by $b_\alpha(x)$ using the position PDF P_{eq} as a measure. This corresponds to a least-squares fit of the diffusion field by linear combinations of the b_α 's. To do this fit, they define a projector:

$$c_\alpha(x) = B_{\alpha\beta}^{1/2} b_\beta(x) , \quad (4.2.13)$$

where $B_{\alpha\beta}^{1/2}$ is an orthonormalization matrix such that $\int c_\alpha c_\beta P_{\text{eq}}(x)dx = \delta_{\alpha\beta}$. They then approximate $D_{\mu\nu}$ by its projection as a linear combination of known functions:

$$D_{\mu\nu}(x) \simeq D_{\mu\nu\alpha} c_\alpha(x) , \quad (4.2.14)$$

with,

$$D_{\mu\nu\alpha} = \int D_{\mu\nu}(x) c_\alpha(x) P_{\text{eq}} dx . \quad (4.2.15)$$

As the equation here are over a whole trajectory which is long enough to satisfy P_{eq} , the empirical projector \hat{c}_α can be approximated using the trajectory averages. To measure the local diffusion they construct a local estimator $\hat{d}_{\mu\nu} = \Delta x_\mu(t_i)\Delta x_\nu(t_i)/2\Delta t$, so that $D_{\mu\nu\alpha}$ now reads:

$$\hat{D}_{\mu\nu\alpha} = \frac{1}{\tau} \sum_i \hat{d}_{\mu\nu}(t_i) \hat{c}_\alpha(x(t_i)) \Delta t . \quad (4.2.16)$$

The latter equation corresponds to a linear regression of $\hat{d}_{\mu\nu}(t_i)$, and permits to project the diffusion tensor onto a finite set of basis functions. We implemented this method, using fourth-order polynomials in our case. to simplify the use of their method with out data, I developed a simple Python function around their method **Q** which can infer the local diffusion coefficient by only one function call:

¹ `Dx, Dy, Dz, z_D = Compute_diffusion(pos)`

Where in the latter, `pos` is the 3D trajectory of a Brownian colloid. It allowed us to infer the local diffusion coefficients $D_i(z)$, down to $z = 10$ nm, as shown in Fig. 39. The results are in excellent agreement with the theoretical predictions, $D_{\parallel}(z)$ and $D_z(z)$, using Eqs.4.1.40 and 4.1.40, thus validating the method.

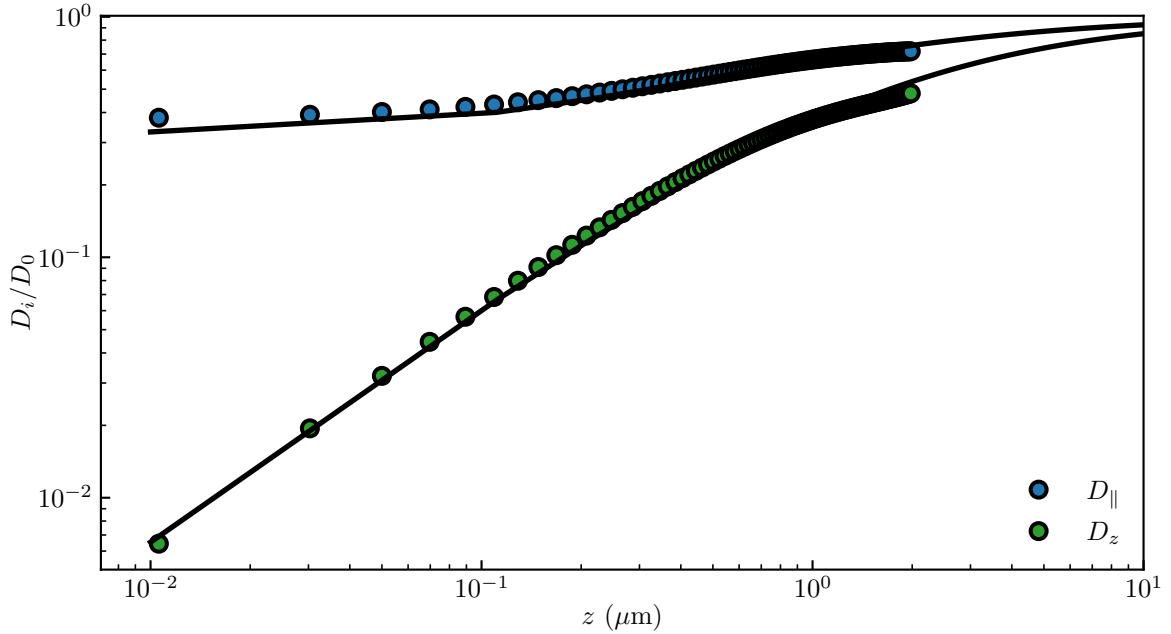


Figure 39: Measured local short-term diffusion coefficients D_i of the microparticle, normalized by the bulk value D_0 , as functions of the distance z to the wall, along both a transverse direction x or y ($D_i = D_{\parallel} = D_x = D_y$, blue) and the normal direction z ($D_i = D_z$, green) to the wall. The solid lines are the theoretical predictions, $D_{\parallel}(z) = D_0 \eta_{\perp}(z)$ and $D_z(z) = D_0 \eta_z(z)$, using the local effective viscosities $\eta_{\perp}(z)$ and $\eta_{\parallel}(z)$ of Eqs.4.1.40 and 4.1.40, respectively.

4.2.5 Precise potential inference using multi-fitting technique

So far, through Figs.34 to 39, we have successively presented the various measured statistical quantities of interest, as well as their fits to corresponding theoretical models. Therein, we have essentially three free physical parameters, B , ℓ_B , ℓ_D , describing the particle and its environment, as well as the *a priori* undetermined location of the $z = 0$ origin. These four parameters are actually redundant among the various theoretical models. Therefore, in order to measure them accurately, we in fact perform all the fits simultaneously, using a Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm that is well suited for unconstrained nonlinear optimization [90]. To do so, we construct a global minimizer:

$$\chi^2 = \sum_{n=1}^N \chi_n^2 , \quad (4.2.17)$$

where we introduce the minimizer χ_n^2 of each set n among the N sets of data, defined as:

$$\chi_n^2 = \sum_{i=1}^{M_n} \frac{[y_{ni} - f_n(x_{ni}, \mathbf{b})]^2}{f_n(x_{ni}, \mathbf{b})^2} , \quad (4.2.18)$$

with $\{x_{ni}, y_{ni}\}$ the experimental data of set n , M_n the number of experimental data points for set n , f_n the model for set n , and $\mathbf{b} = (b_1, b_2, \dots, b_p)$ the p free parameters. In our case, $p = 4$, and $\{x_{ni}, y_{ni}\}$ represent all the experimental data shown in Figs.34 to 39.

Due to strong dependence of the normal diffusion coefficient D_z with z , it is possible to find the wall position with a 10-nm resolution, thus overcoming a drawback of the Lorenz-Mie technique which only provides the axial distance relative to the focus of the objective lens. Besides, the three physical parameters globally extracted from the multifitting procedure are: $B = 4.8 \pm 0.6$, $\ell_D = 21 \pm 1$ nm, and $\ell_B = 530 \pm 2$ nm. Using the particle radius $a = 1.518 \pm 0.006$ μm calibrated from the preliminary fits of the interference patterns to the Lorenz-Mie scattering function (see section 3.7.3), and the 1050 kg.m^{-3} tabulated bulk density of polystyrene, we would have expected $\ell_B = 559$ nm instead, which corresponds to less than 2% error, and might be attributed to nanometric offsets, such as *e.g.* the particle and/or wall rugosities.

4.2.6 Measuring external forces using the local drifts

Finally, we investigate the total conservative force $F_z(z)$ acting on the particle along z . The first method way to measure it is to calculate the gradient of the potential U which is experimentally measure from the position PDF giving:

$$F_z^{\text{eq}} = -\nabla U = k_B T \frac{\partial}{\partial z} \ln(P_{\text{eq}}) , \quad (4.2.19)$$

where one can use the experimentally measured P_{eq} (See Fig.34) the results of this method is shown in fig.40. However, it can be interesting to measure the forces using the local drifts as for more complex systems, some non-conservative forces could arise. As the Eq.4.2.19 only take into account to the potential U , only convervative forces can be extracted from the measurement of P_{eq} . For more complex system, non-conservative forces could be measured by the difference between forces obtained through P_{eq} and the local drifts. By averaging the overdamped Langevin equation over a fine-enough z -binning grid and short enough time interval Δt , one gets in the Itô convention (corresponding to our definition of Δz):

$$F_z(z) = 6\pi\eta_z(z)a \frac{\langle \Delta z \rangle}{\Delta t} - k_B T \frac{D'_z(z)}{D_z(z)} , \quad (4.2.20)$$

where the last term corresponds to the additional contribution due to the non-trivial integration of the multiplicative noise [22, 91–93], with the prime denoting the derivative with respect to z . From the averaged measured vertical drifts $\langle \Delta z \rangle$, and invoking Eq. (??),

one can reconstruct $F_z(z)$ from Eq. (4.2.20), as shown in Fig. ??b). We stress that the statistical error on the force measurement is comparable to the thermal-noise limit [94]:

$$\Delta F = \sqrt{24\pi k_B T \eta_z(z) a / \tau_{\text{box}}(z)} , \quad (4.2.21)$$

where $\tau_{\text{box}}(z)$ is the total time spent by the particle in the corresponding box of the z -binning grid. To corroborate these measurements, we invoke Eq. (4.1.28) and express the total conservative force $F_z(z) = -U'(z)$ acting on the particle along z :

$$F_z(z) = k_B T \left(\frac{B}{\ell_D} e^{-\frac{z}{\ell_D}} - \frac{1}{\ell_B} \right) . \quad (4.2.22)$$

Using the physical parameters extracted from the above multifitting procedure, we plot Eq. 4.2.22 in Fig. 40. The agreement with the data is excellent, thus showing the robustness of the force measurement. In particular, we can measure forces down to a distance of 40 nm from the surface. Besides, far from the wall, we are able to resolve the actual buoyant weight $F_g = -7 \pm 4$ fN of the particle. This demonstrates that we reach the femtoNewton resolution, and that this resolution is solely limited by thermal noise.

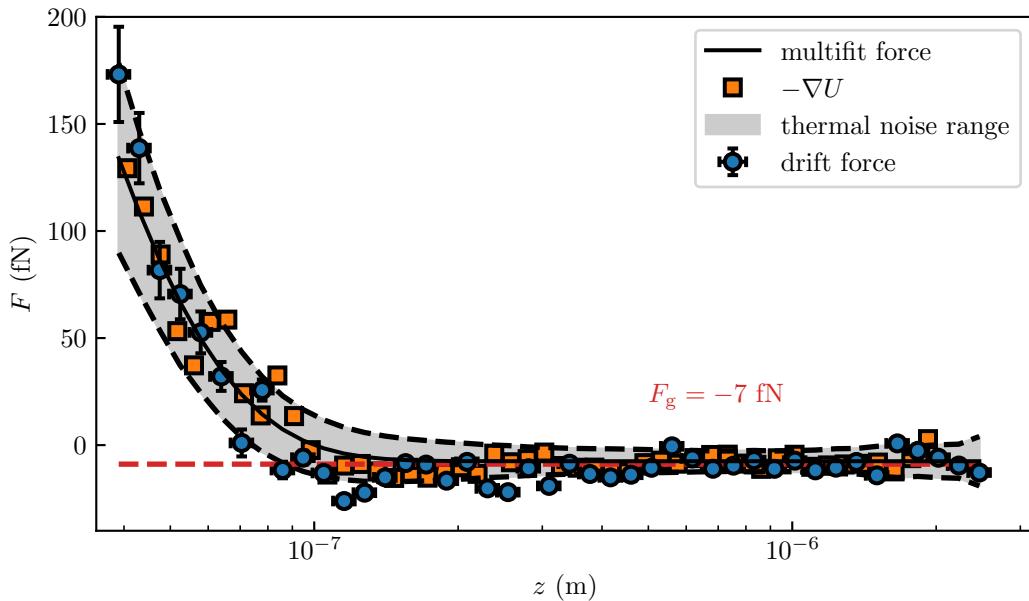


Figure 40: Total normal conservative force F_z exerted on the particle as a function of the distance z to the wall, reconstructed from Eq. 4.2.20, using Eq. 4.1.42 in circles and using Eq. 4.2.19 in squares. The solid line corresponds to Eq. (4.2.22), with $B = 4.8$, $\ell_D = 21$ nm and $\ell_B = 530$ nm. The black dashed lines and grey area indicate the amplitude of the thermal noise computed from Eq. 4.2.21. The horizontal red dashed line indicates the buoyant weight $F_g = -7$ fN of the particle.

4.3 conclusion

To conclude, we have successfully built a multi-scale statistical analysis for the problem of freely diffusing individual colloids near a rigid wall. Combining the equilibrium distribution in position, time-dependent non-Gaussian statistics for the spatial displacements, a novel method to infer local diffusion coefficients, and a multifitting procedure, allowed us to reduce drastically the measurement uncertainties and reach the nanoscale and thermal-noise-limited femtoNewton spatial and force resolutions, respectively. The ability to measure tiny surface forces, locally, and at equilibrium, as well the possible extension of the method to non-conservative forces and out-of-equilibrium settings [95, 96], opens fascinating perspectives for nanophysics and biophysics.

5 Work in progress and observations

In this section, I will present some of the work that has been started but not finished yet, or, observations made during the experiments.

5.1 Elastohydrodynamic lift at soft wall

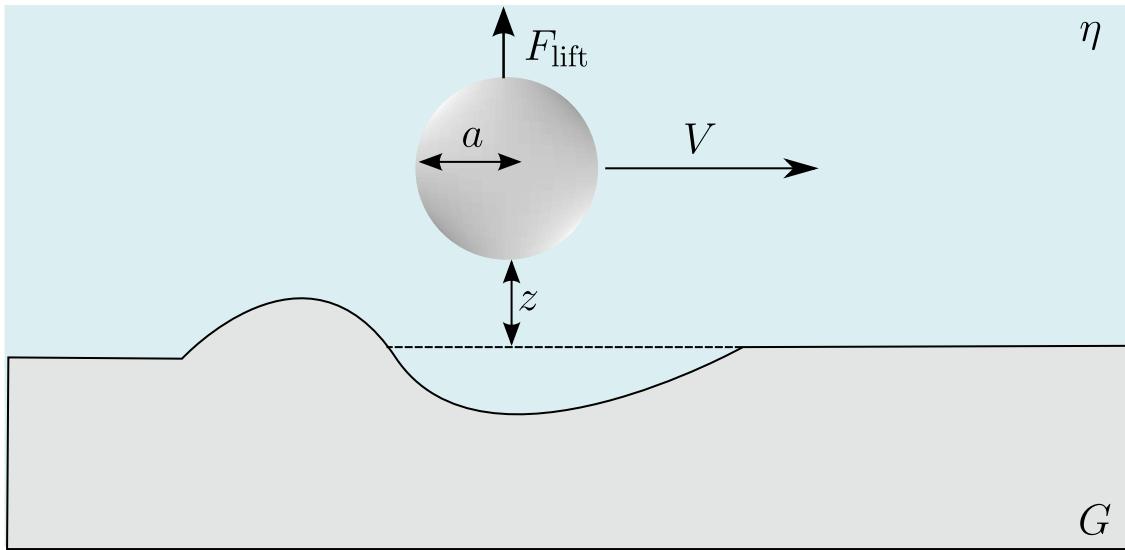


Figure 41: Schema of a spherical colloid of radius a immersed in a fluid of viscosity η sliding at a velocity V above an incompressible and linear-elastic substrate of shear elastic modulus G . From the elastohydrodynamic interactions between the particle and the soft wall arises a net lift force F_{lift} (See Eq.5.1.1).

Elastohydrodynamic (EHD) is the branch in physics that permit to model the interaction between elasticity and hydrodynamics. We will here focus particularly on EHD in lubrication, which permits to model the physics of sliding motion between objects with a fluid-lubricated contact. Lubrication EHD is present in many length scales and timescales, including for example, landslides, roller bearings or blood cells motion in microfluidic devices. It is recently, in 2015, that the problem of a free particle that can sediment, slide or roll near a soft surface has been treated. As the particle slides near the surface, the moved fluid deforms the soft wall surface's. One of the main results of [Salez] is that sliding induced a symmetry breaking of the deformation, hence, a net force is applied to the sliding object. They further show that this force is oriented towards the particle, and, thus act as a self-sustaining EHD lift force. The first experimental proofs of this force have been done at the macro scale using negatively buoyant centimetric cylinders immersed in a viscous fluid sliding down a tilted wall that is coated with an elastic layer. They show

that the self-sustained EHD lift reduces the friction by nearly an order of magnitude and suggests that this EHD force could partially explain phenomena such as reduced wear in animal joints and long-runout landslides. The EHD lift force has also recently been measured at the micro-scale, using micron-sized colloidal spheres in micro channels in a shear flow. However, all of this experiments uses system which are out of equilibrium, and, we would like to test if Brownian motion could trigger such an effect. In the soft lubrication theory, and, taking a sphere of radius a moving at constant velocity V , in a solvent of viscosity η , at a distance z for a thick (by respect to the particle radius), incompressible, linear-elastic substrate of shear elastic modulus G , The lift force F_{lift} reads:

$$F_{\text{lift}} \simeq \frac{\eta^2 V^2}{G} \frac{a^{5/2}}{z^{5/2}} . \quad (5.1.1)$$

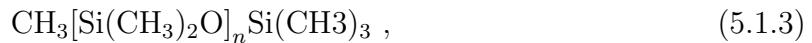
To incorporate fluctuations into this deterministic picture, an idea is to replace the velocity V in Eq.5.1.1 by the typical Brownian velocity obtain through the Maxwell-Boltzmann distribution $\sqrt{k_B T/m}$, leading the the following estimate of the Brownian EHD lift force:

$$F_{\text{lift,Brown}} \simeq \frac{\eta^2 k_B T}{G \rho_p a^{1/2} z^{5/2}} . \quad (5.1.2)$$

From this equation, we can observe a counterintuitive effect, as the particle radius decreases (the surface of contact), the larger the EHD force $F_{\text{lift,Brown}}$ is. Taking typical biophysics value such as $G \simeq 10$ kPa, $\rho_p = 1350$ kg.m⁻³ and $a = 100$ nm, the EHD force reaches the picoNewton order of magnitude. The latter means that microscopic entities in biology and nanoscience may spontaneously trigger large Brownian EHD couplings, drastically affecting their dynamics. However, it is importante to note that it is a simple estimate and, contains a high risk of conceptual failure associated, for example, to the lake of compensating drift at equilibrium as we have with a hard wall (See Eq.4.1.54). To do the soft coating experimentally, we use Polydimethylsiloxane (PDMS), as a soft surface.

5.1.1 PDMS

PDMS is a silicone-based organic polymer which is widely use due to its versatility and ease of use [97]. Its chemical formula is:



where n is the number of $\text{Si}(\text{CH}_3)_2\text{O}$ dimethyl groups. By mixing a solution of PDMS chains with curring agent containing, for example hydrosilane groups (SiH), bonds, or crosslink between different PDMS chains are appears as shown in Fig.42.

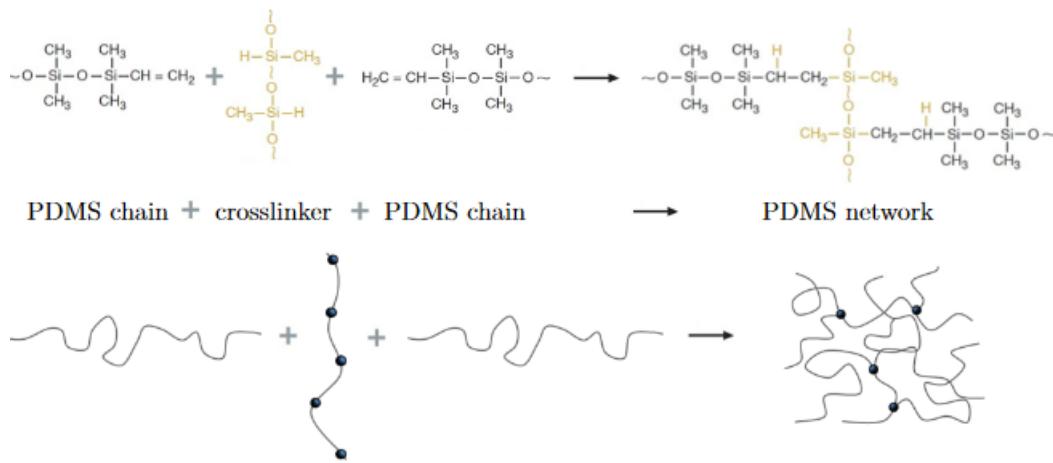


Figure 42: Figure taken from [98]. Exemple of crosslinking reaction between the PDMS chains and a curing agent containing hydrosilane groups.

Due to this crosslink, the PDMS turns into an incomprehensible and linear-elastic solid. Some of its characteristics is to be hydrophobic and to exhibit strong gas permeability [99]. The elasticity modulus G of the crosslinked PDMS can be tune by changing the mixing ratio of base polymer solution and curing agent. For example, for one of the most used PDMS Sygard 184, a mixing ratio of 10 : 1 leads to a elastic modulus $G = 1.5 \text{ MPa}$ and $G \simeq 100 \text{ kPa}$ for a 35 : 1 ratio [100]. To prepare experimental samples, it possible to spin coat the microscope slides with base:agent mix before it is cured in order to have a thick soft surface. However, for simplicity and check if we can observe any forces with our experiment we first decided to use already prepared samples sold by Ibidi, these came as soft coated dishes with a coverslip on the bottom that we can directly fit onto our microscope.

5.1.2 Measuring non-conservative forces

To measure the non-conservative forces that a felt by a Brownian particle diffusing on top of a soft surface, we do the exact same experiment and data analysis as developed in the section 4.2. As the EHD force do not derive from a potential, we need to extract the

non-conservative forces F_{NC} at a distance of 100 nm from the wall, to do so, we combine Eq.4.2.19 and Eq.4.2.22, F_{NC} thus reads:

$$F_{NC} = F_z(z) - F_z^{\text{eq}}(z) \quad (5.1.4)$$

As seen on the fig.43 the measured F_{NC} for two different elastic modulus $G = 15$ and 28 kPa, only gets out of the noise for $F_{NC} > 10$ fN, as the plot is in logarithmic scale negative values do not show. These first experiment seems to show the that the particle do trigger some non-conservative forces. However, this first experiment are not sufficient to tell if it's the EHD lift force but give some support to continue experiment.

In the forthcoming experiment, we will vary three parameters, η , G and a to check if we can a universal plot showing that $F_{\text{lift},\text{Brown}}$ correctly varies as a function of $\frac{\eta^2}{Ga^{1/2}}$ as shown in Fig.44 with the first experiments.

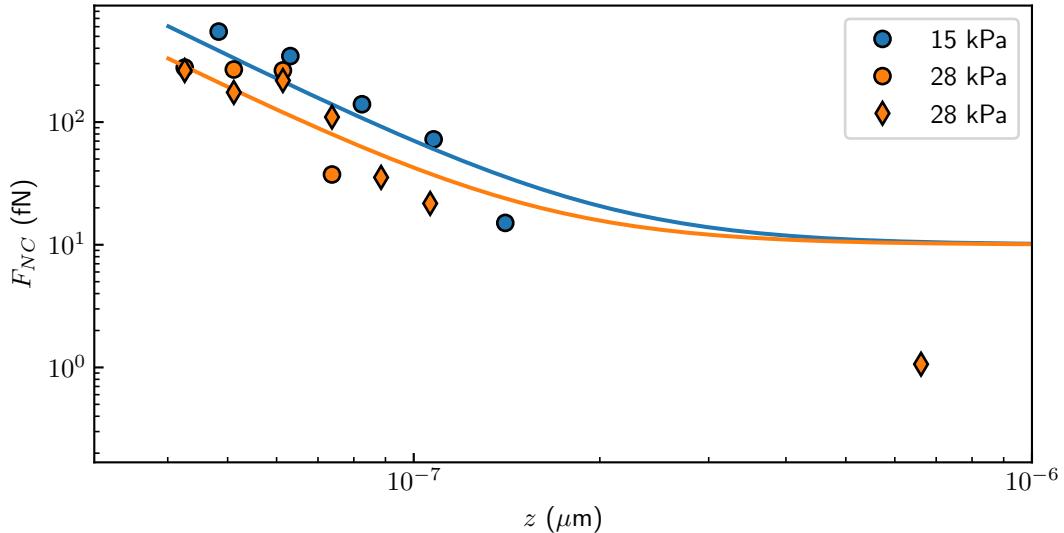


Figure 43: Non-conservative forces measured experimentally for colloidal particle of radius $a = 1.5 \mu\text{m}$ diffusing above an incompressible and linear-elastic substrate of shear elastic modulus $G = 15$ and 28 kPa. Plain line corresponds to the Brownian model of the EHD lift force $F_{\text{lift},\text{Brown}}$ (See Eq.5.1.2) with an added noise-level of 10 fN.

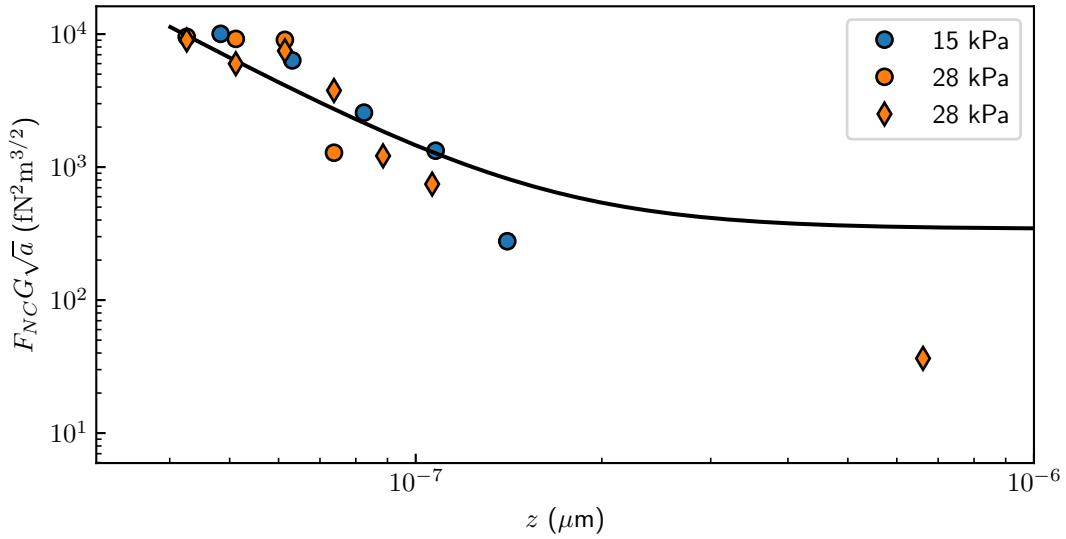


Figure 44: Non-conservative forces normalized $G\sqrt{a}$ by measured experimentally for colloidal particle of radius $a = 1.5 \mu\text{m}$ diffusing above an incompressible and linear-elastic substrate of shear elastic modulus $G = 15$ and 28 kPa . Plain line corresponds to the Brownian model of the EHD lift force $F_{\text{lift},\text{Brown}}$ (See Eq.5.1.2) with an added noise-level of 10 fN .

5.2 Close wall stuck motion

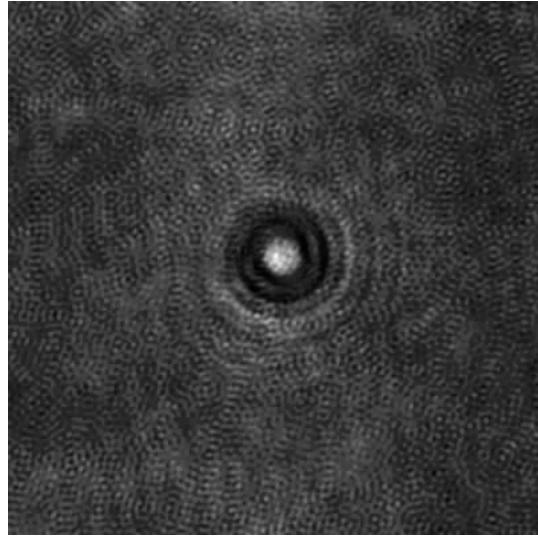


Figure 45: Median image of a stuck particle. The median is calculated over 120 images taken every 30 s.

As we have done the experiments to measure the Debye length ℓ_D as a function of the concentration of NaCl, we observed that some particle were stuck at the surface. As we

first expected, we were not observing any movement from the stucked particle. However, surprisingly we observe that some particle were slightly diffusing. This slight diffusion can be observed directly from the raw data, by looking at the median of the movie, as shown Fig.45, one can observe the “ghost” of the particle due to its movement in time. Moreover, as we can’t properly have the background in this experiment since the particle do not diffuse enough, the statistical error of the tracking is increased. The measured trajectory is shown Fig.46, we observe that mechanical drift happened during the experiment, this could be due to a drift of the sample or the objective for example. This drift is of the order of magnitude of $2 \mu\text{m.h}^{-1}$ along the x and y axis and $6 \mu\text{m.h}^{-1}$, in the following we look at the short time dynamics ($t < 1 \text{ s}$), since the drift at this time scale is of the order of the nm it is not necessary to remove the drift from the trajectory.

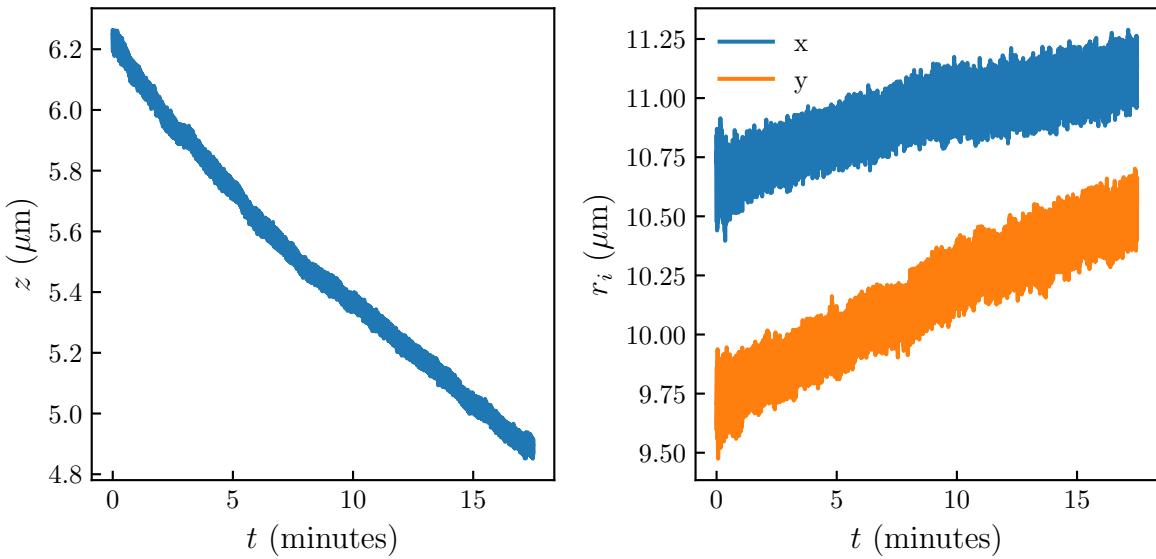


Figure 46: Raw trajectory measured using the Mie tracking technique for the x , y and z axis of a particle of radius $a = 1.5 \mu\text{m}$. The time between each frame was the to $1/200 \text{ s}$.

Let’s focus directly on the dynamics of the system, the MSD along the x , y and z axis is shown in Fig.47. We observe that the MSD along the z axis is a constant, it can be the case because it moves less than the statistical error tracking, the experiment could thus be bound by the experimental precision, or, a diffusion regime could be at shorter time. As we here can’t observe any diffusion regime on the z axis, we can’t determine if we are bound the tracking, hence, I will not physically comment the results obtained along the z axis. However, on the x axis, interestingly we can observe two regimes as had for the z before meaning that the particle is diffusing in a potential which isotropic along the x and y axis, as if the particle was rolling on the surface due to the rugosities for example (NEED TO ASK YACINE FOR REF). The diffusive regime, at short time, is linear with Δt showing a normal diffusion with average diffusion coefficients (see Eq.4.2.2)

$\langle D_{\parallel} \rangle = \langle D_x \rangle = \langle D_y \rangle = 0.14 D_0$. This diffusion coefficient is lower than what is obtainable using the previous defined equation of D_{\parallel} , Eq.4.1.41, which we do not explain.

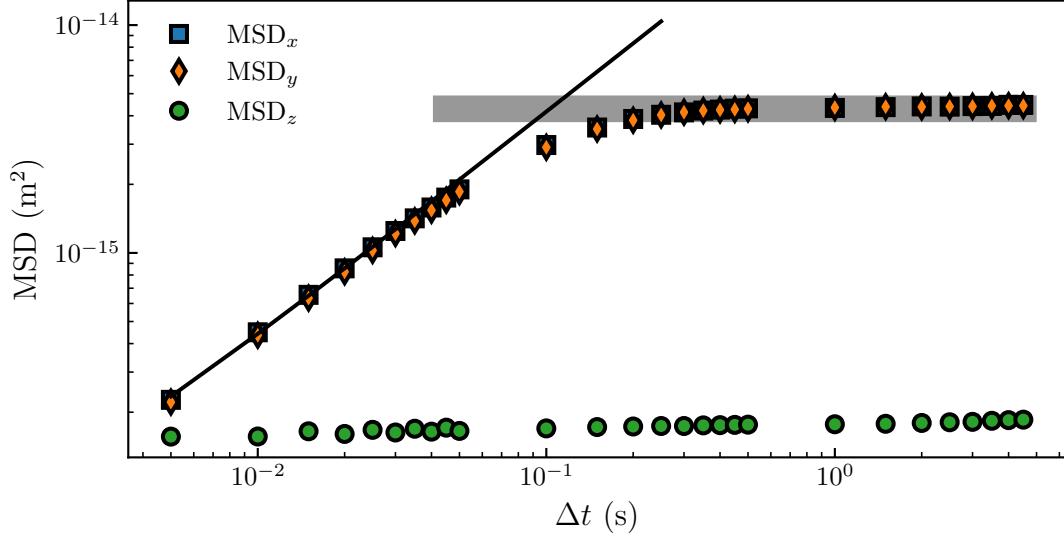


Figure 47: Measured mean-squared displacements of a stuck particle (MSD, see Eq.4.2.1) as functions of the time increment Δt , for the three spatial directions, x , y , and z . The solid line is best fit to Eq. (4.2.2), having $\langle D_i \rangle$ as a free parameter, providing the average diffusion coefficients $\langle D_{\parallel} \rangle = \langle D_x \rangle = \langle D_y \rangle = 0.14 D_0$. The plain gray line is the average value of the plateau of the MSD along the x and y axis axis, providing $\lim_{\Delta t \rightarrow \infty} \langle \Delta x^2 \rangle = 4.3 \times 10^{-15} \text{ m}^2$.

The plateau of the MSD along the x and y gives a value of $\lim_{\Delta t \rightarrow \infty} \langle \Delta x^2 \rangle = 4.3 \times 10^{-15} \text{ m}^2$. By supposing that the particle is in an harmonic oscillator potential, we can give an estimate of the spring constant k_H , using the relation:

$$k_H = \frac{k_B T}{\lim_{\Delta t \rightarrow \infty} \langle \Delta x^2 \rangle} \simeq \frac{4 \times 10^{-21}}{4.3 \times 10^{-15}} \simeq 1 \text{ } \mu\text{N.m}^{-1}. \quad (5.2.1)$$

An idea we have, is to try to observe the same phenomenon using a soft surface and see if we can observe a change of k_H as a function of the elastic modulus G . If the latter assumption is possible this experiment could lead to local determination of elastic modulus using Brownian probes. Additionally, we can look at displacement PDF P_i as shown on the Fig.48. Contrary to the result we had for a free diffusing particle (See Fig.37) we do not observe non-Gaussianity. Moreover, distribution P_z seems to corroborate with the fact that we are bound to the fit error on the fit along the z axis. Indeed, if the particle is at equilibrium, which should be the case with a MSD being constant, P_z should be given by exponential as shown in Fig.37-e). Here P_z looks like a Gaussian distribution of the

fit statistical error.

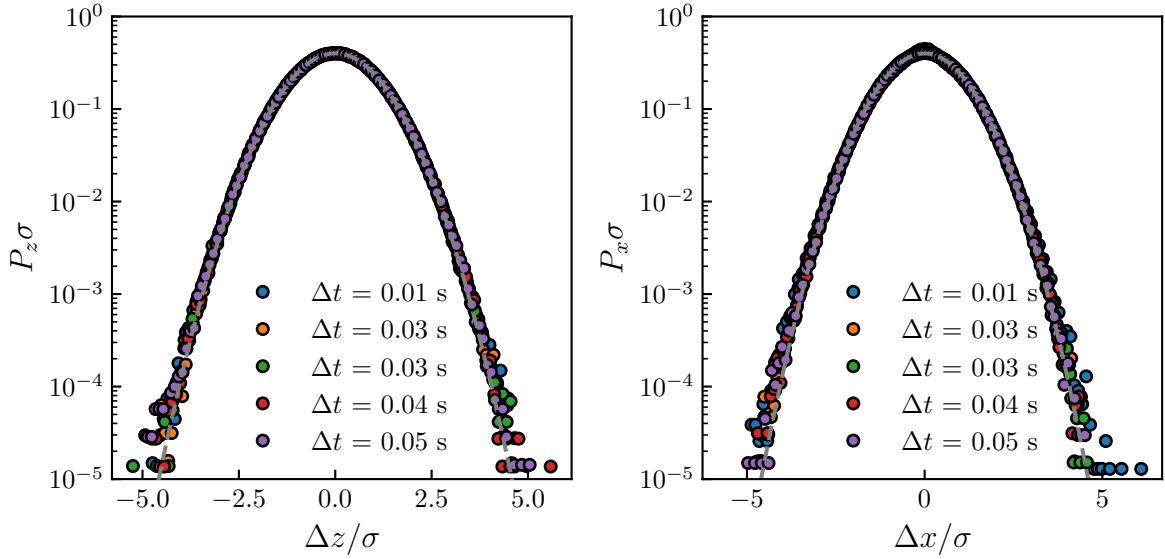


Figure 48: Normalized probability density functions $P_i \sigma$ of the normalized displacements $\Delta x / \sigma$ and $\Delta z / \sigma$, at short times, with σ^2 the corresponding MSD (see Fig.47), for different time increments Δt ranging from 0.01 s to 0.05 s, as indicated with different colors. The grey dashed lines are normalized Gaussian distributions, with zero means and unit variances.

5.3 Direct relative distance measurement using Mie

One of the effect that is not taken into account in the Mie theory presented in the section 3.3 is the presence of the glass slide. Indeed the focal plane, where the holograms are recorded is in the glass slides, thus, the holograms should be refracted at the glass interface. However, it is not a problem, for the realized fit, indeed, to have faster fit, we use only the few maxima leading to radial distance of $\simeq 5 \mu\text{m}$ for a height $z \simeq 15 \mu\text{m}$, leading to a maximal angle of incidence $\theta = \tan^{-1}(5/15) \simeq 18.5^\circ$. As we are working with small angles, we can suppose that for the first maxima, the is not refracted.

However, the small angle approximation does not hold for higher order maxima, thus as the refraction angle depends on the wall-particle distance, adding it to the Mie theory could lead to a direct measurement of the particle height. Moreover, the Mie fitting would even more robust since a mechanical drift of the sample of the objective would not change the refractive angle of the particle. However, the main difficulty is to be able to have a more stable imaging device to detect with a high accuracy the highest order maxima which are less intense.

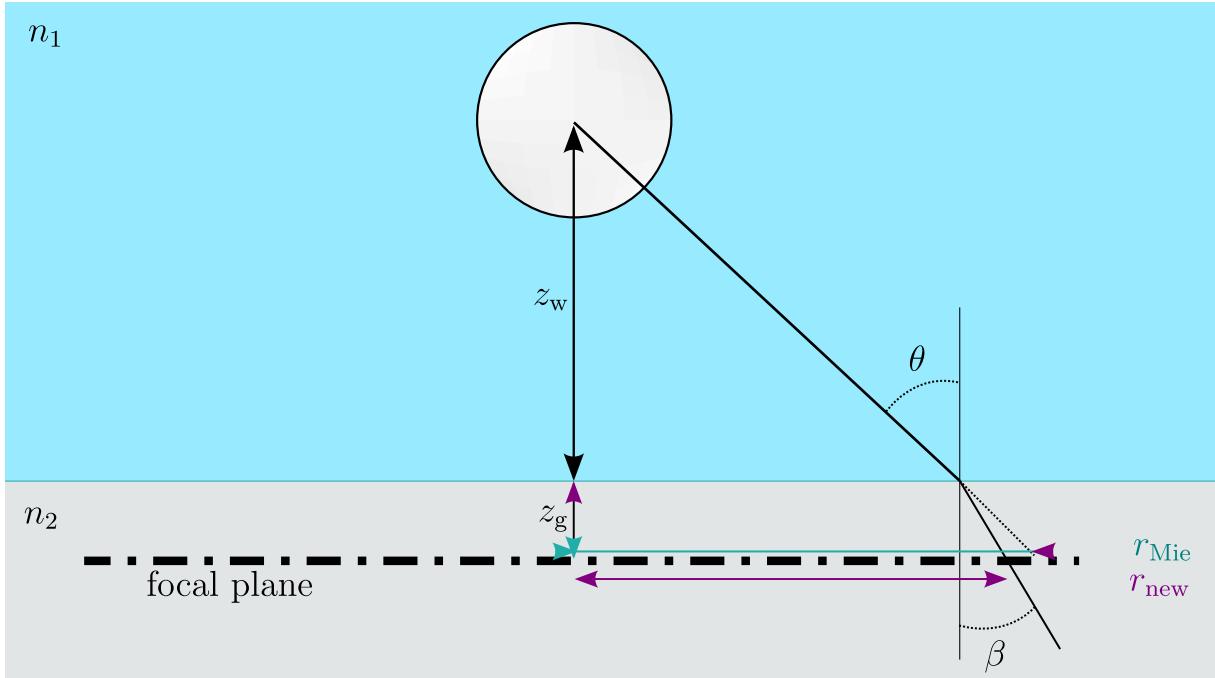


Figure 49: Scheme of a particle immersed in a fluid of index n_1 above a glass substrate of optical index n_2 . The particle is at a distance z_w above the glass substrate, and the objective focal plane z_g below the interface. Due to the Snell-Descartes law, light with an incident angle θ_1 is refracted with an angle β .

Let us build a simple model using the Snell-Descartes law to take into account the refraction at the glass interface, the system is schematized in Fig.49. We decompose the height of the particle z in two parts such that:

$$z = z_w + z_g , \quad (5.3.1)$$

where z_g is the interface-focal plane interface and z_w the interface-particle distance, the latter being the distance we want to measure. We want to write the radial intensity of the hologram I , taking into account the refraction, let us write I_{Mie} the Mie solution without the refraction and I_{new} the modified theory. An incident ray coming from the particle with an angle θ is refracted with an angle β , as shown in Fig.49. Using trigonometry and the Snell-Decartes's law ($n_1 \sin(\theta) = n_2 \sin(\beta)$), those angles writes:

$$\theta = \tan^{-1} \left(\frac{r_{\text{Mie}}}{z_w + z_g} \right) , \quad (5.3.2)$$

and,

$$\beta = \sin^{-1} \left(\frac{n_1 \sin(\theta)}{n_2} \right) . \quad (5.3.3)$$

Without refraction, the light arrives at the focal plane at a distance r_{Mie} as shown in Fig.49, due to the refraction this distance is modified to r_{new} :

$$r_{\text{new}} = r_{\text{Mie}} + z_g (\tan(\beta) - \tan(\theta)) . \quad (5.3.4)$$

Due to the change of optical path, the Mie scattering field (see Eq.3.3.2) undergoes a phase difference $\Delta\varphi_w$ in the fluid and $\Delta\varphi_g$ in the glass substrate:

$$\Delta\varphi_w = \frac{2\pi}{\lambda} n_1 z \tan(\theta') , \quad (5.3.5)$$

and,

$$\Delta\varphi_g = \frac{2\pi}{\lambda} n_2 z \tan(\beta') , \quad (5.3.6)$$

where θ' and β' are given by:

$$\theta' = \tan^{-1} \left(\frac{r_{\text{new}}}{z_w + z_g} \right) , \quad (5.3.7)$$

and,

$$\beta' = \sin^{-1} \left(\frac{n_1 \sin(\theta')}{n_2} \right) . \quad (5.3.8)$$

Moreover, a part of the light is reflected onto the surface, thus attenuating the scattering field by a factor $T = (n_1 + n_2)/2n_2$. Finally, taking the phase difference, the corrected scattered field E_s^{new} writes:

$$E_s^{\text{new}} = T E_s \exp(-j(\Delta\varphi_w - \Delta\varphi_g)) , \quad (5.3.9)$$

with the hologram normalized radial intensity given by:

$$\frac{I(r)}{I_0(r)} = \frac{|E_s^{\text{new}}(r) + E_0(r)|}{E_0(r)} \quad (5.3.10)$$

We worked on this project with Mathias Perrin, a specialist of the numerical simulation of Mie systems and he numerically shows that the simple correction made to correctly describes the change made by the presence of the glass slide as shown in the Fig.XX. Thus, proving that it is theoretically possible to directly fit the wall-particle distance using Mie holography tracking. However, we still not succeed on applying to the experimental data, probably due to the lack of accuracy on the higher order hologram's maxima.

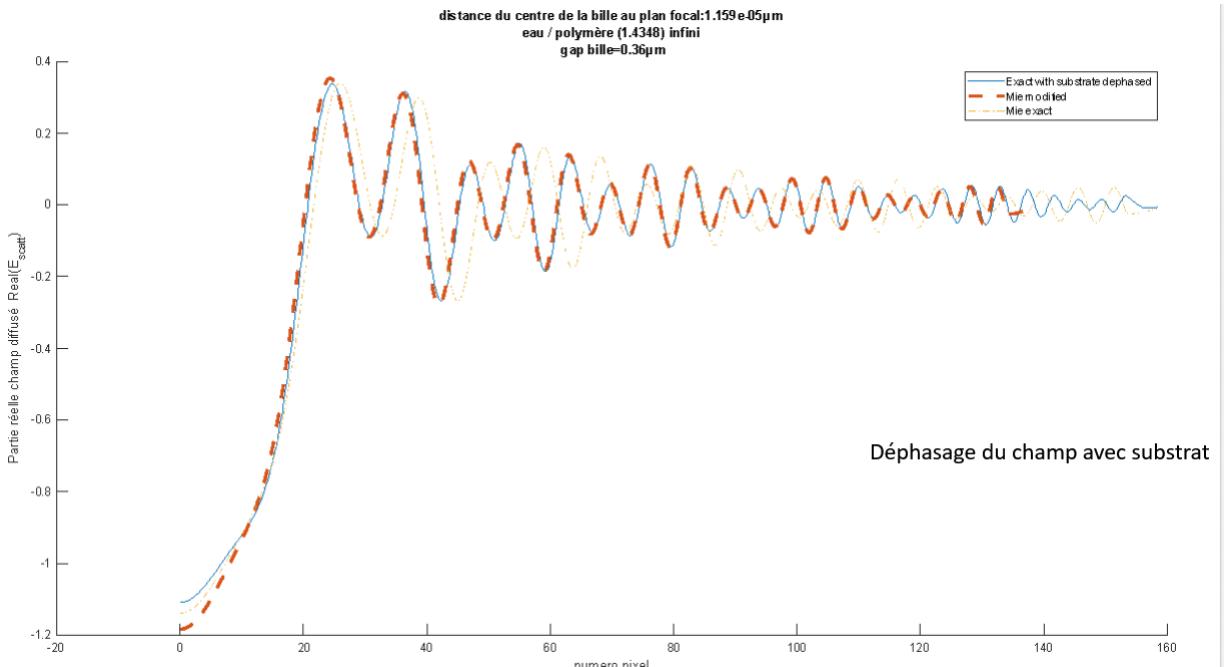


Figure 50: Numerical proof of the hologram correction due to the presence of the glass slide. The yellow dashed line represent the Mie theory without taking into account the substrate. the plain line represent the exact numerical simulation of the phase difference. The red dashed line represents the simple correction made to the Mie theory Eq.5.3.10

5.4 Long time 4th moment - Dean project

5.5 PDMS ageing

A Appendix

1 Intertial Brownian motion simulation

One can write the Langevin equation as:

$$m\ddot{x} = -\gamma\dot{x} + \sqrt{2k_B T \gamma} dB_t \quad (1)$$

By replacing with the Euler method \dot{x} by:

$$\dot{x} \simeq \frac{x_i - x_{i-1}}{\tau}, \quad (2)$$

\ddot{x} by:

$$\begin{aligned} \ddot{x} &\simeq \frac{\frac{x_i - x_{i-1}}{\tau} - \frac{x_{i-1} - x_{i-2}}{\tau}}{\tau} \\ &= \frac{x_i - 2x_{i-1} + x_{i-2}}{\tau^2}. \end{aligned} \quad (3)$$

and finally, dB_t by a Gaussian random number w_i with a zero mean value and a τ variance, one can write x_i as:

$$x_i = \frac{2 + \tau/\tau_B}{1 + \tau/\tau_B} x_{i-1} - \frac{1}{1 + \tau/\tau_B} x_{i-2} + \frac{\sqrt{2k_B T \gamma}}{m(1 + \tau/\tau_B)} \tau w_i, \quad (4)$$

We will in the following use Python to simulate such a movement and check the properties of the mean squared displacement. In the end I will propose a Cython implementation that permits a 1000x speed improvement on the simulation.

```
[1]: # Import important libraries
import numpy as np
import matplotlib.pyplot as plt
```

```
[2]: # Just some matplotlib tweaks
import matplotlib as mpl

mpl.rcParams["xtick.direction"] = "in"
mpl.rcParams["ytick.direction"] = "in"
mpl.rcParams["lines.markeredgecolor"] = "k"
mpl.rcParams["lines.markeredgewidth"] = 1.5
mpl.rcParams["figure.dpi"] = 200
from matplotlib import rc

rc("font", family="serif")
rc("text", usetex=True)
rc("xtick", labelsize="medium")
rc("ytick", labelsize="medium")
rc("axes", labelsize="large")
```

```
def cm2inch(value):
    return value / 2.54
```

[3]:

```
N = 1000000 # length of the simulation
tau = 0.01 # simulation time step
m = 1e-8 # particle mass
a = 1e-6 # radius of the particle
eta = 0.001 # viscosity (here water)
gamma = 6 * np.pi * eta * a
kbT = 4e-21
tauB = m / gamma
```

[4]:

```
print(
    "With such properties we have a characteristic diffusion time of {:.2f} s".
    format(
        tauB
    )
)
```

With such properties we have a characteristic diffusion time of 0.53 s

[5]:

```
def xi(xi1, xi2):
    """
    Function that compute the position of a particle using the full Langevin
    Equation
    """
    t = tau / tauB
    wi = np.random.normal(0, np.sqrt(tau))
    return (
        (2 + t) / (1 + t) * xi1
        - 1 / (1 + t) * xi2
        + np.sqrt(2 * kbT * gamma) / (m * (1 + t)) * np.power(tau, 1) * wi
    )
```

[6]:

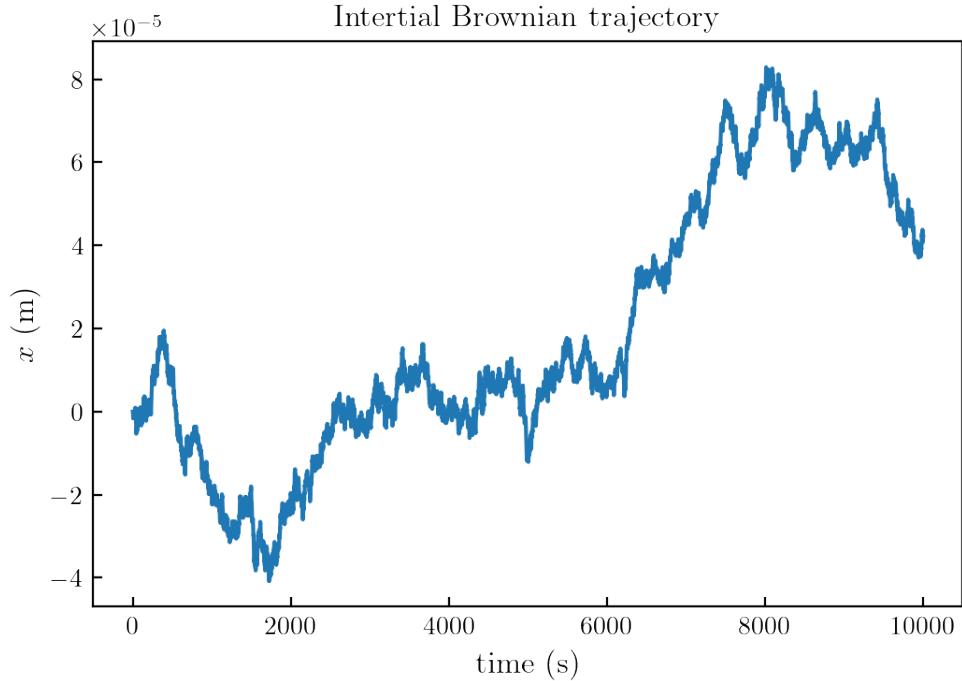
```
def trajectory(N):
    x = np.zeros(N)
    for i in range(2, len(x)):
        x[i] = xi(x[i - 1], x[i - 2])
    return x
```

Now that the functions are setup one can simply generate a trajectory of length N by simply calling the function `trajectory()`

[7]:

```
# Generate a trajectory of 10e6 points.
x = trajectory(1000000)
```

```
[8]: plt.plot(np.arange(len(x))*tau, x)
plt.title("Intertial Brownian trajectory")
plt.ylabel("$x$ (m)")
plt.xlabel("time (s)")
plt.show()
```



1.1 Cross checking

As we are dealing with inertial Brownian motion, the later is characterize by a characteristic time $\tau_B = m/\gamma$. We will check that the simulated trajectory gives us the correct MSD properties to ensure the simulation si done properly. The MSD given by:

$$\text{MSD}(\tau) = \langle (x(t) - x(t + \tau))^2 \rangle \Big|_t , \quad (5)$$

with Δt a lag time. The MSD, can be computed using the function defined in the cell below. For times $\tau \ll \tau_B$ we should have:

$$\text{MSD}(\tau) = \frac{k_B T}{m} \tau^2 , \quad (6)$$

and for $\tau \gg \tau_B$:

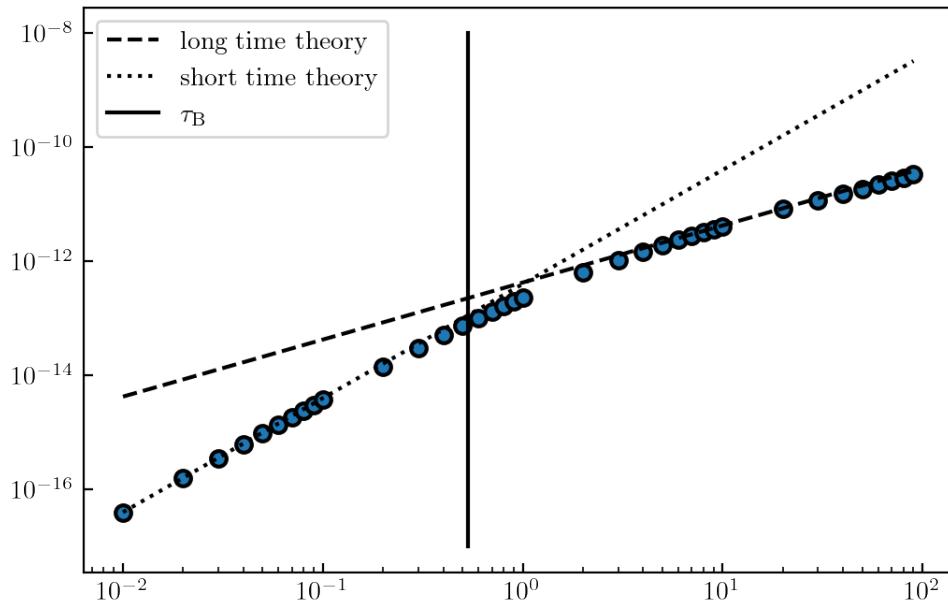
$$\text{MSD}(\tau) = 2D\tau , \quad (7)$$

with $D = k_B T / (6\pi\eta a)$.

```
[9]: t = np.array([*np.arange(1,10,1), *np.arange(10,100,10), *np.
    ↪arange(100,1000,100), *np.arange(1000,10000,1000)])
def msd(x,t):
    _msd = lambda x, t : np.mean((x[:-t] - x[t:])**2)
    return [_msd(x,i) for i in t]
MSD = msd(x,t)
```

```
[10]: D = kbT/(6*np.pi*eta*a)
t_tau = t*tau
plt.loglog(t*tau,MSD, "o")
plt.plot(t*tau, (2*D*t_tau), "--", color = "k", label="long time theory")
plt.plot(t*tau, kbT/m * t_tau**2, ":" , color = "k", label="short time theory")

horiz_data = [1e-8, 1e-17]
t_horiz = [tauB, tauB]
plt.plot(t_horiz, horiz_data, "k", label="$\\tau_B$")
plt.legend()
plt.show()
```



Our simulation is giving the expected results but how much time do we need to generate this trajectory of 1000000 points

```
[11]: %timeit trajectory(1000000)
```

6.32 s ± 101 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)

So we need about 6 seconds to generate this trajectory, which is in the cases of someone who want to look at fine effects and need to generate millions of trajectories is too much, in order to fasten the process i will in the following use Cython to generate the trajectory using C.

1.2 Cython acceleration

```
[12]: %load_ext Cython
```

```
[13]: %%cython
import cython
cimport numpy as np
import numpy as np
from libc.math cimport sqrt
ctypedef np.float64_t dtype_t

cdef int N = 1000000 # length of the simulation

cdef dtype_t tau = 0.01 # simulation time step
cdef dtype_t m = 1e-8 # particle mass
cdef dtype_t a = 1e-6 # radius of the particle
cdef dtype_t eta = 0.001 # viscosity (here water)
cdef dtype_t gamma = 6 * 3.14 * eta * a
cdef dtype_t kbT = 4e-21
cdef dtype_t tauB = m/gamma
cdef dtype_t[:] x = np.zeros(N)

@cython.boundscheck(False)
@cython.wraparound(False)
@cython.nonecheck(False)
@cython.cdivision(True)
cdef dtype_t xi_cython( dtype_t xi1, dtype_t xi2, dtype_t wi):
    cdef dtype_t t = tau / tauB
    return (
        (2 + t) / (1 + t) * xi1
        - 1 / (1 + t) * xi2
        + sqrt(2 * kbT * gamma) / (m * (1 + t)) * tau * wi
    )

@cython.boundscheck(False)
@cython.wraparound(False)
@cython.nonecheck(False)
cdef dtype_t[:] _traj(dtype_t[:] x, dtype_t[:] wi):
    cdef int i
    for i in range(2, N):

        x[i] = xi_cython(x[i-1], x[i-2], wi[i])
    return x
```

```
def trajectory_cython():

    cdef dtype_t[:, ] wi = np.random.normal(0, np.sqrt(tau), N).astype('float64')

    return _traj(x, wi)
```

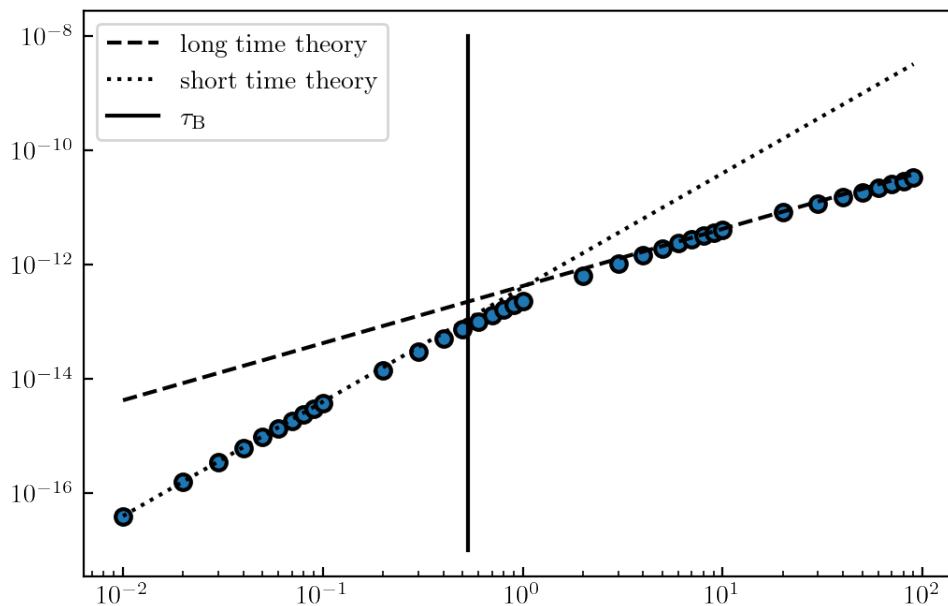
[14]: %timeit trajectory_cython()

28.9 ms ± 416 µs per loop (mean ± std. dev. of 7 runs, 10 loops each)

Rapid check if the Cython code properly works.

```
x=np.asarray(trajectory_cython())
D = kbT/(6*np.pi*eta*a)
t_plot = t*tau
plt.loglog(t*tau,MSD, "o")
plt.plot(t*tau, (2*D*t_plot), "--", color = "k", label="long time theory")
plt.plot(t*tau, kbT/m * t_plot**2, ":" , color = "k", label="short time theory")

horiz_data = [1e-8, 1e-17]
t_horiz = [tauB, tauB]
plt.plot(t_horiz, horiz_data, "k", label="$\tau_B$")
plt.legend()
plt.show()
```



1.2.1 Conclusion

We finally only need $\simeq 6$ ms to generate the trajectory instead of $\simeq 6$ s which is a $\simeq 1000\times$ improvement speed. The simulation si here bound to the time needed to generate the array of random numbers which is still done using numpy function. After further checking, Numpy random generation si as optimize as one could do so there is no benefit on cythonizing the random generation. For the sake of completeness one could fine a Cython version to generate random numbers. Found thanks to Senderle: <https://stackoverflow.com/questions/42767816/what-is-the-most-efficient-and-portable-way-to-generate-gaussian-random-numbers>

```
[16]: %%cython
from libc.stdlib cimport rand, RAND_MAX
from libc.math cimport log, sqrt
import numpy as np
import cython

cdef double random_uniform():
    cdef double r = rand()
    return r / RAND_MAX

cdef double random_gaussian():
    cdef double x1, x2, w

    w = 2.0
    while (w >= 1.0):
        x1 = 2.0 * random_uniform() - 1.0
        x2 = 2.0 * random_uniform() - 1.0
        w = x1 * x1 + x2 * x2

    w = sqrt((-2.0 * log(w)) / w) ** 0.5
    return x1 * w

@cython.boundscheck(False)
cdef void assign_random_gaussian_pair(double[:] out, int assign_ix):
    cdef double x1, x2, w

    w = 2.0
    while (w >= 1.0):
        x1 = 2.0 * random_uniform() - 1.0
        x2 = 2.0 * random_uniform() - 1.0
        w = x1 * x1 + x2 * x2

    w = sqrt((-2.0 * log(w)) / w)
    out[assign_ix] = x1 * w
    out[assign_ix + 1] = x2 * w
```

```
@cython.boundscheck(False)
def my_uniform(int n):
    cdef int i
    cdef double[:] result = np.zeros(n, dtype='f8', order='C')
    for i in range(n):
        result[i] = random_uniform()
    return result

@cython.boundscheck(False)
def my_gaussian(int n):
    cdef int i
    cdef double[:] result = np.zeros(n, dtype='f8', order='C')
    for i in range(n):
        result[i] = random_gaussian()
    return result

@cython.boundscheck(False)
def my_gaussian_fast(int n):
    cdef int i
    cdef double[:] result = np.zeros(n, dtype='f8', order='C')
    for i in range(n // 2): # Int division ensures trailing index if n is odd.
        assign_random_gaussian_pair(result, i * 2)
    if n % 2 == 1:
        result[n - 1] = random_gaussian()

    return result
```

```
[17]: %timeit my_gaussian_fast(1000000)
```

```
28.7 ms ± 963 µs per loop (mean ± std. dev. of 7 runs, 10 loops each)
```

```
[18]: %timeit np.random.normal(0,1,1000000)
```

```
24 ms ± 768 µs per loop (mean ± std. dev. of 7 runs, 10 loops each)
```

One can thus see, that even a pure C implementation can be slower than the Numpy one, thanks to impressive optimization.

```
[ ]:
```

1 Fitting pipeline using pylorenzmie

In order to fit an hologram, I used the pylorenzmie model which provides a set of python classes in order to analyse holographic microscopy data.

Pylorenzmie can be download on the David Grier's github repository: <https://github.com/davidgrier/pylorenzmie>.

What I actually get from the experiments are mp4 movies, in order to analyze them easily, I constructed a wrapper around the pylorenzmie module which can be found on my repository: <https://github.com/eXpensia/wraplorenzmie>.

This wrapper permits to do the following pipeline:

- Directly load the movies
- Compute the back ground.
- Use the first image in order to get the pre guesses
- Fit the 10 000 first images to determine precisely the radius and index of a particle.
- Use the later information in order to fit the whole movie (and save the data in the same time)

Once that done, the trajectory be analyzed separately.

```
[1]: # We first start by import the important modules

import wraplorenzmie.utilities.utilities as utilities
import wraplorenzmie.fits.fit as fit
import imageio
# For Plotting.
import matplotlib.pyplot as plt
import seaborn as sns
import numpy as np
#sns.set(style='white', font_scale=2)
%matplotlib inline
import matplotlib as mpl

mpl.rcParams["figure.dpi"] = 200
from matplotlib import rc
rc('font', family='serif')
rc('text', usetex=True)
rc('xtick', labelsize='x-small')
rc('ytick', labelsize='x-small')

def cm2inch(value):
    return value/2.54
```

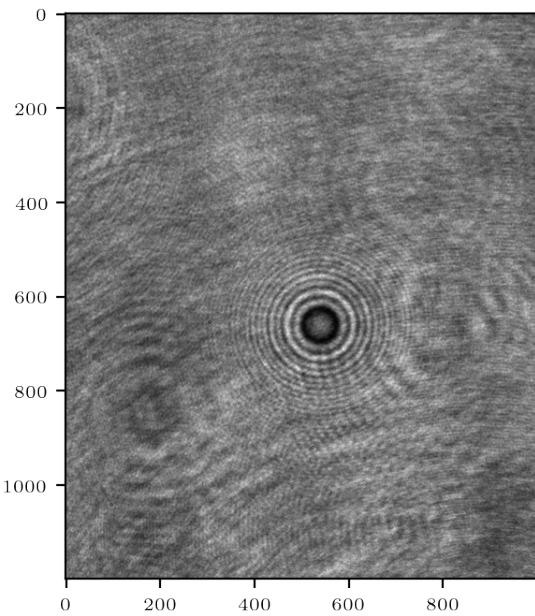
No module named 'pylorenzmie.fitting.cython.cminimizers'

```
[2]: #We load the movie  
vid = utilities.  
→video_reader("Basler_acA1920-155um__22392621__20200527_162231224.mp4")
```

```
[3]: # A function that permits to compute de radial profile of an image this will later be used in order to see if the fits are done correctly  
def radial_profile(data, center=None):  
    if center==None:  
        center = np.array(np.shape(data)) / 2  
  
    y, x = np.indices((data.shape))  
    r = np.sqrt((x - center[0])**2 + (y - center[1])**2)  
    r = r.astype(int)  
  
    tbin = np.bincount(r.ravel(), data.ravel())  
    nr = np.bincount(r.ravel())  
    radialprofile = tbin / nr  
  
    T = data.ravel()  
    V = r.ravel()  
  
    err = [np.std(T[V == u]) for u in np.unique(V)]  
  
    return radialprofile, err
```

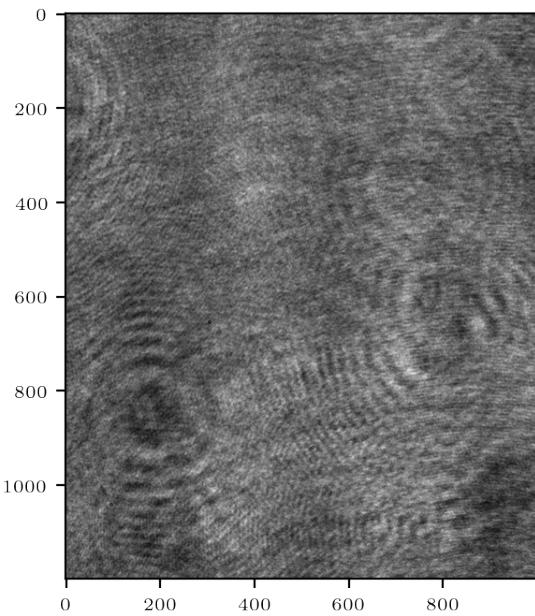
```
[4]: # We take a look at the first image of the movie  
image = vid.get_image(1)  
plt.imshow(image,cmap="gray")
```

```
[4]: <matplotlib.image.AxesImage at 0x1a70b337be0>
```



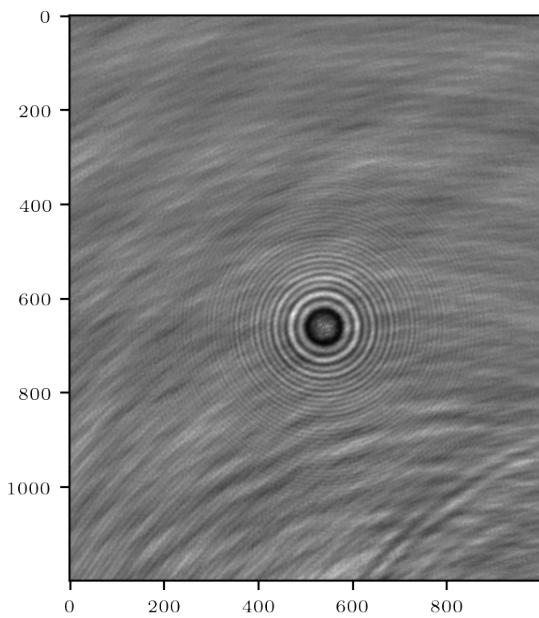
```
[5]: # set the background image (it can also be computed using vid.  
    ↪get_background method)  
vid.number = 125000  
vid.background = np.array(imageio.imread("background.tiff"))  
#vid.background = vid.get_background(n=50) # n is the number of image to  
    ↪use to compute the background  
plt.imshow(vid.background,cmap="gray")
```

```
[5]: <matplotlib.image.AxesImage at 0x1a70bc56490>
```

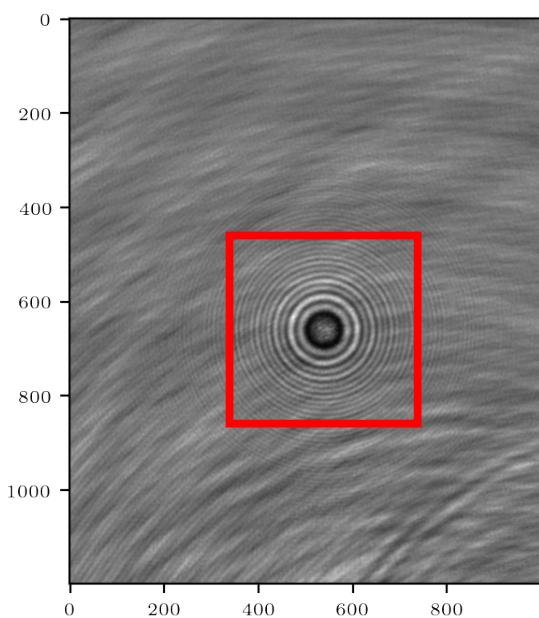


```
[6]: imageio.imwrite("background.tiff",vid.background) # We save the background  
      ↪for possible later use.
```

```
[7]: # the normalized image, we can see that there is some movement in the  
      ↪background.  
      # This could be avoided by computing the background as a function of the  
      ↪time, if the particle diffuses enough.  
normed_image = utilities.normalize(image,vid.background)  
plt.imshow(normed_image,cmap="gray")  
normed_image = normed_image
```



```
[8]: # We found the position of the particle  
feature = utilities.center_find(image)[0]  
utilities.plot_bounding(normed_image, feature)
```

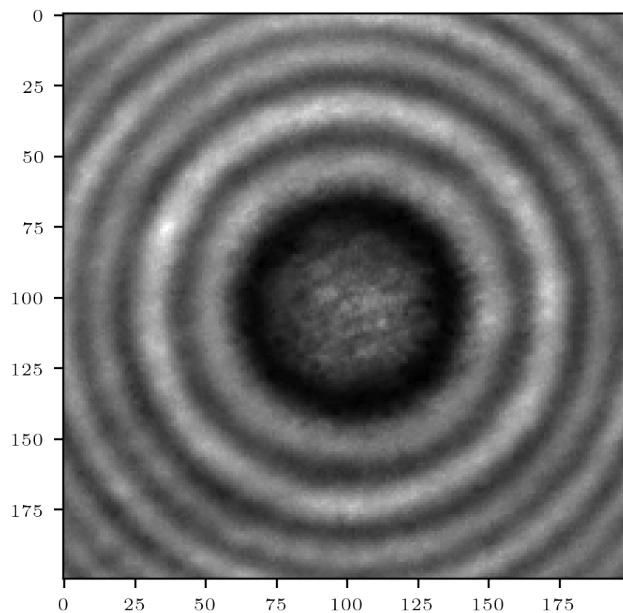


1.1 Fitting the first image

We fit the first image in order to get the preguess. We first start by croping the hologram.

```
[9]: xc, yc, w, h = feature[0]
x_center = xc
y_center = yc
h=200
im_c = fit.crop(image, int(xc), int(yc), int(h))
bk_c = fit.crop(vid.background, int(xc), int(yc), int(h))
cropped = utilities.normalize(im_c,bk_c, dark_count = np.min(im_c))
cropped = cropped / np.mean(cropped)
plt.imshow(cropped,cmap = "gray")
```

[9]: <matplotlib.image.AxesImage at 0x1a71d7e6d00>



```
[10]: # We setup the fitting method.
fitter = fit.fitting(cropped,0.532,0.0513)
fitter.make_guess(1.50,1.59,12,alpha = 1,fit_r=True, fit_n=True,fit_alpha=True)
```

```
[11]: # We do the actual fit.
result = fitter.fit_single(cropped, method = "lm")
```

```
[12]: zo = result.result["x"][2]*0.0513
print(result.result["x"][2]*0.0513)
print(result.redchi)
print(result.result["x"])
```

```
11.427616273713154
7.2459765196825305
[101.23514587 103.00299474 222.76055114 1.5310255 1.58239091
 1.00198476]
```

We can plot the result to see if the fit worked properly, and, for a more quantitative comparison we can compute the radial intensity profile of both hologram and compare them.

```
[13]: center = np.array(np.shape(fitter.image))
```

```
[14]: radial_exp, err = radial_profile(fitter.image)
theo_exp, err = radial_profile(fitter.fitter.model.hologram().
    reshape(fitter.shape))
# computing first the hologram using the fit result
```

```
[15]: fit_data = {}
radius_radial = np.arange(len(radial_exp)) * 0.0513
plt.figure(figsize = (15,15))
fig = plt.figure(figsize=(cm2inch(8.6),1.65*cm2inch(8.6)))
fig.subplots_adjust(left=0.14, bottom=.12, right=.99, top=.98)

plt.subplot(2,2,1)
plt.imshow(fitter.image, cmap = "gray")
#plt.title('subplot(2,2,1)')

fit_data["exp_image"] = fitter.image

plt.subplot(2,2,2)
plt.imshow(fitter.fitter.model.hologram().reshape(fitter.shape), cmap = "gray")
frame1 = plt.gca()
frame1.axes.yaxis.set_ticklabels([])

fit_data["th_image"] = fitter.fitter.model.hologram().reshape(fitter.
    shape)

#plt.title('subplot(2,2,2)')

plt.subplot(2,2,(3,4))
```

```

plt.plot(radius_radial, radial_exp, label="Experimental")
plt.fill_between(radius_radial, radial_exp - err, radial_exp + err, alpha=0.3)
plt.plot(radius_radial, theo_exp, label="Theory")
plt.legend()
plt.xlabel("radius [pixel]")
plt.ylabel("Intensity [a.u.]")

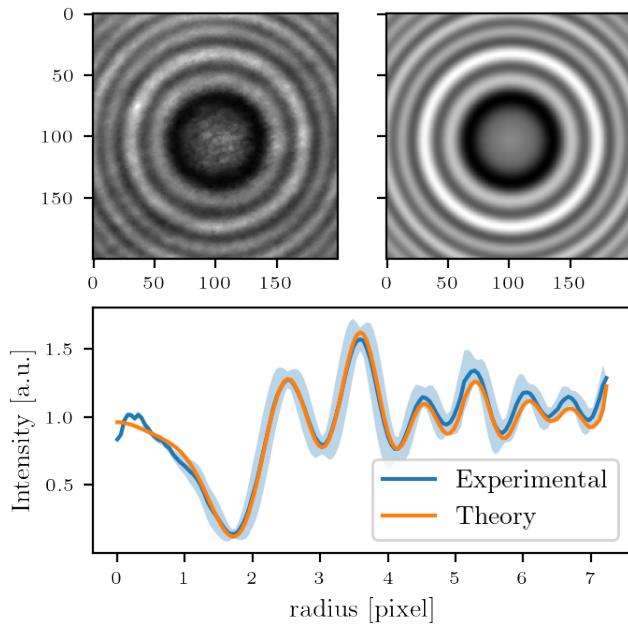
fit_data["I_r_exp"] = radial_exp
fit_data["I_errr_exp"] = err

fit_data["theo_exp"] = theo_exp
fit_data["I_radius"] = radius_radial

fig.set_size_inches(cm2inch(8.6), cm2inch(1.6 * 8.6/1.618))
plt.savefig("fit_fig.pdf")

```

<Figure size 3000x3000 with 0 Axes>



```
[16]: fitter.fit_video(vid =
    vid, savefile="find_nrfit_result_dur_27052020_n_r_fix_0p0513_wav532.
    dat", xc = x ,yc= y, h = 200, n_end=10000,method = "lm")
```

100% 9999/9999 [12:39<00:00, 13.17it/s]

```
[17]: # Since the measurement or not saved into the ram we need to load it
n_r = np.fromfile('find_nr_exame.dat', dtype=np.float64)
n_r = n_r.reshape(len(n_r)//10,10)
r = n_r[:,3]
n = n_r[:,4]
```

1.2 Fitting the n, r distributiton using a KDE estimator

To find the most probable couple of r/n we use a kde estimator using seaborn

```
[18]: import numpy as np
import scipy.stats as st
import matplotlib.ticker as ticker

data = np.random.multivariate_normal((0, 0), [[0.8, 0.05], [0.05, 0.7]], ↴
                                     100)
x = r[(r>1.5) & (r<1.555)]
y = n[(r>1.5) & (r<1.555)]
xmin, xmax = np.min(x), np.max(x)
ymin, ymax = np.min(y), np.max(y)

# Perform the kernel density estimate
xx, yy = np.mgrid[xmin:xmax:100j, ymin:ymax:100j]
positions = np.vstack([xx.ravel(), yy.ravel()])
values = np.vstack([x, y])
kernel = st.gaussian_kde(values)
f = np.reshape(kernel(positions).T, xx.shape)
f = f/np.max(f)
```

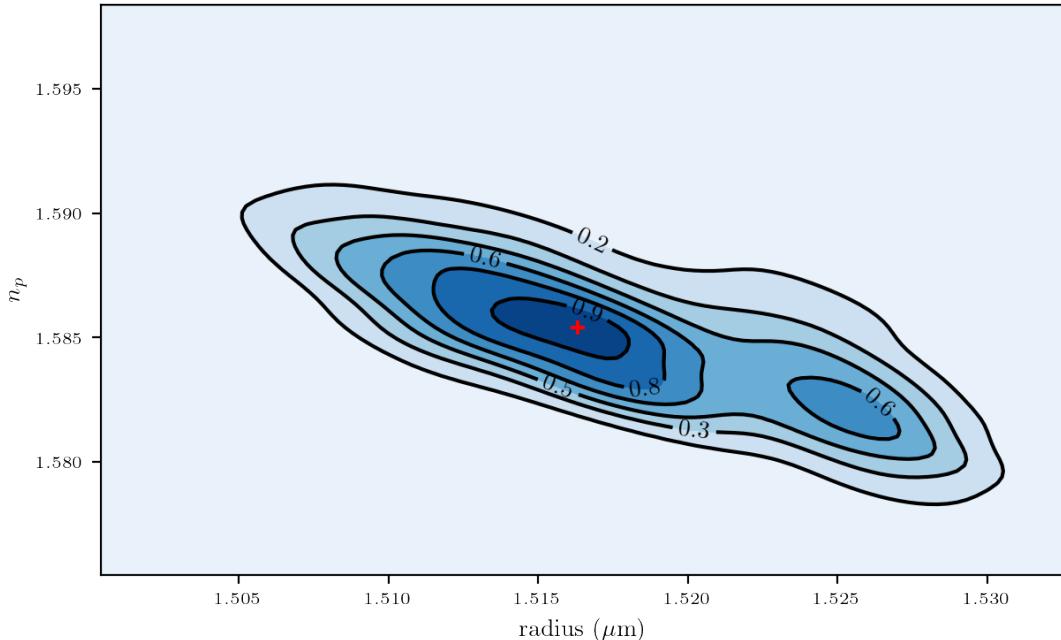
```
[19]: np.round(np.max(f))
```

```
[19]: 1.0
```

```
[20]: fig = plt.figure()
fig.subplots_adjust(left=0.16, bottom=.20, right=.99, top=.99)
ax = fig.gca()
#ax.set_xlim(1.505, 1.53)
#ax.set_ylim(1.575, 1.6)
# Contourf plot
cfset = ax.contourf(xx, yy, f, cmap='Blues')
## Or kernel density estimate plot instead of the contourf plot
#ax.imshow(np.rot90(f), cmap='Blues', extent=[xmin, xmax, ymin, ymax])
# Contour plot
cset = ax.contour(xx, yy, f, colors='k', levels=6)
```

```
# Label plot
ax.xaxis.set_major_formatter(ticker.FormatStrFormatter('%.1.3f'))
ax.yaxis.set_major_formatter(ticker.FormatStrFormatter('%.1.3f'))
ax.clabel(cset, inline=1, fontsize=10, fmt="%1.1f")
plt.scatter(xx[np.where(f == 1)], yy[np.where(f == 1)], color = "red", marker="+")
ax.set_xlabel("radius ($\mathbf{\mu m}$)")
ax.set_ylabel("$n_p$")
# plt.title("KDE r n")
fig.set_size_inches(cm2inch(16), cm2inch(9.9))

plt.tight_layout()
fig.savefig('KDErn.pdf')
# plt.show()
```



```
[21]: print(" n determined with : mu={0}, sigma={1}".format(np.mean(yy[np.where(f > 0.1)]), np.std(yy[np.where(f > 0.1)])))
print(" r determined with : mu={0}, sigma={1}".format(np.mean(xx[np.where(f > 0.1)]), np.std(xx[np.where(f > 0.1)])))

n determined with : mu=1.5851200393768743, sigma=0.003267685282504072
r determined with : mu=1.5181266656310368, sigma=0.00682411690457934
```

```
[22]: (mu_n, mu_r) = np.mean(yy[np.where(f > 0.1)]), np.mean(xx[np.where(f > 0.1)])
```

1.3 Fitting the whole movie

Now that the measurement of n and r is one we can move on the measurement of the whole trajectory by simply using `fitter.fit_video`. For demonstration purposes, I only fit here at $\simeq 22$ image per seconds, if can goes up to at least 60 with recent GPU.

```
[23]: del fitter
fitter = fit.fitting(cropped, 0.532, 0.0513)
fitter.make_guess(mu_r, mu_n, zo, alpha = 1, fit_r=False, fit_n=False, fit_alpha=False)
#result = fitter.fit_single(cropped, method = "lm")
fitter.fit_video(vid = vid, savefile="fit_result_1p5kPa_18122019_n_r_fix_0p0883_wav_532_ex1.dat", xc = xc, yc= yc, h = 200, n_end=10000, method = "lm")
```

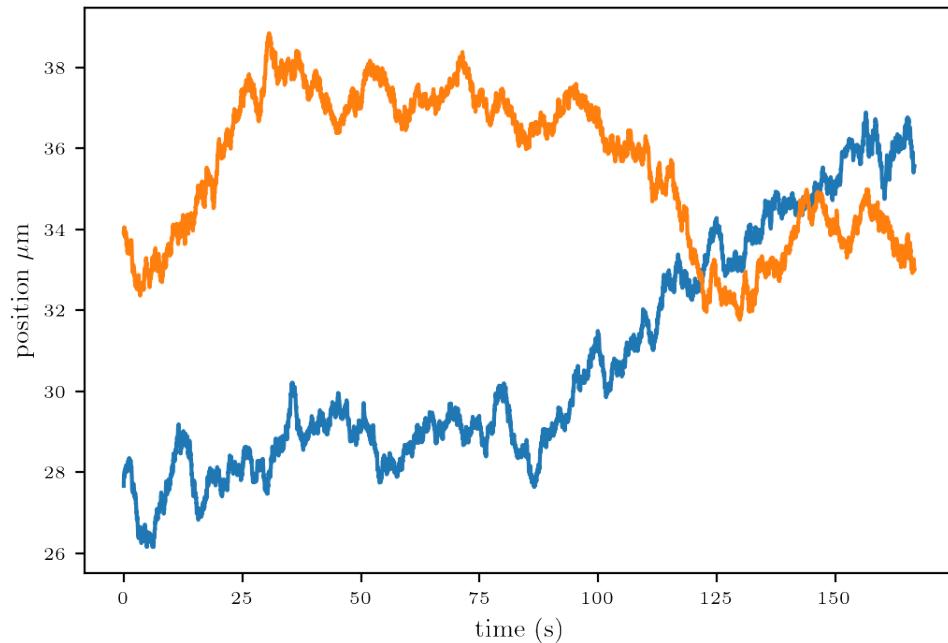
100% 9999/9999 [07:24<00:00, 22.47it/s]

```
[24]: import numpy as np
data = np.fromfile('fit_result_1p5kPa_18122019_n_r_fix_0p0883_wav_532_ex1.dat', dtype=np.float64)
data = data.reshape(len(data)//10, 10)
x = data[:,0]*0.0513
y = data[:,1]*0.0513
z = data[:,2]*0.0513
```

1.4 Plot the trajectory

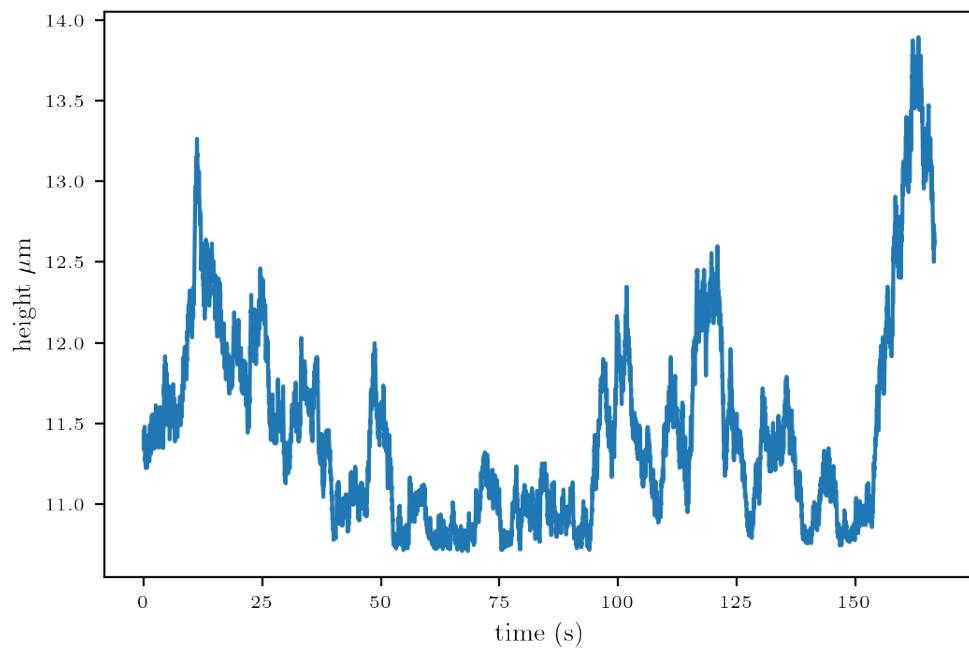
```
[25]: plt.plot(np.arange(len(z))/60, x)
plt.plot(np.arange(len(z))/60, y)
plt.ylabel("position $\mathbf{\mu m}$")
plt.xlabel("time (s)")
```

```
[25]: Text(0.5, 0, 'time (s)')
```



```
[26]: plt.plot(np.arange(len(z))/60, z)
plt.ylabel("height $\mathbf{\mu m}$")
plt.xlabel("time (s)")
```

[26]: `Text(0.5, 0, 'time (s)')`



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