# Nonlinear models for new product growth

# New product life cycle: phases

1. Introduction
2. Growth
3. Maturity
4. Decline

What are the variables influencing a product's life cycle?
Marketing strategies play an essential role . . .
but the success of a new product ultimately depends on consumers accepting them.

# Diffusion of innovations

Diffusion is the process by which an innovation is communicated through certain channels over time among the members of a social system (Rogers, 2003).

Four key elements for describing an innovation diffusion process:

- innovation
- communication channels
- time
- social system

# Innovation

An innovation is:

- New product, new service, new technology, new production process, new way of doing things (Schumpeter, 1947).
- Typical distinction: radical vs incremental innovations.
- Radical innovations could be hindered from barriers and social inertia.

# New product growth models

General aim: depict the successive increases in the number of adopters and predict the continued development of a diffusion process already in progress (Mahajan and Muller, 1979).

- Fourt and Woodlock model (1960)
- Mansfield model (1961)
- Bass model (1969)
- Generalized Bass model (1994)

# Bass Model

The Bass Model is defined by a first order differential equation

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))$$

# Bass Model

innovation

$$z'(t) = \left( p + q\frac{z(t)}{m} \right) (m - z(t))$$

# Bass Model

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))$$

imitation

# Bass Model

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))$$

word-of-mouth

# Bass Model

If we pose $\frac{z(t)}{m} = y(t)$ the model becomes

$$y'(t) = (p + qy(t))(1 - y(t))$$

# Bass Model: solution

The Bass Model has a closed-form solution

$$y(t) = F(t; p, q) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} \qquad t > 0.$$

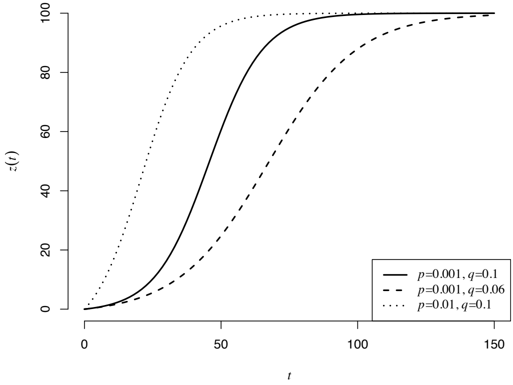
or, by posing $z = ym$

$$z(t) = m\, F(t; p, q) = m\, \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} \qquad t > 0.$$

Cumulative sales $z(t)$ 'depend' on parameters $p$ and $q$.
The market potential $m$ is a scale parameter and is assumed constant.

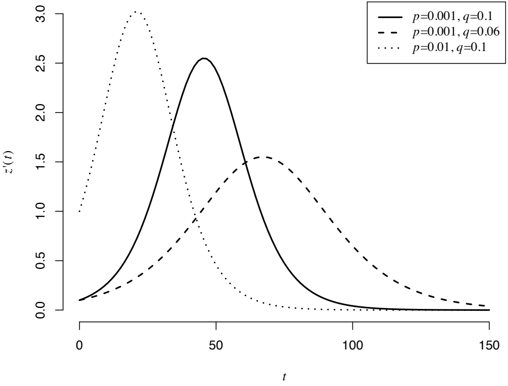
# Bass Model

## Cumulative process



# Bass Model

## Instantaneous process



# Bass Model: estimation

The Bass Model is a nonlinear model

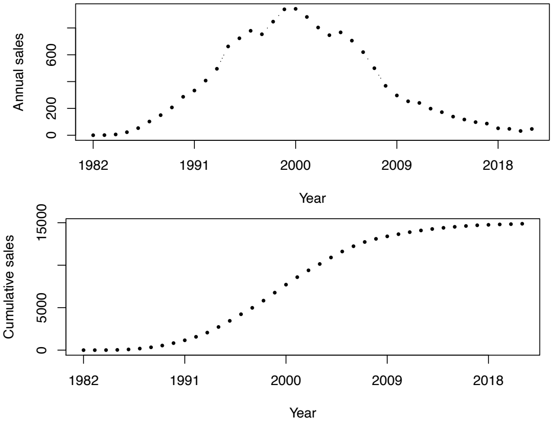
$$Z(t) = f(\beta, t) + \varepsilon(t)$$

where $Z(t)$ is the dependent variable, $f(\beta, t)$ is the deterministic term, function of $\beta \in R^k$ and of time $t$.
The second term, $\varepsilon(t)$, is the error term, for which usual assumptions hold, namely $M(\varepsilon(t)) = 0$,$Var(\varepsilon(t)) = \sigma^2$, $Cov(\varepsilon(t), \varepsilon(t')) = 0, t \neq t'$.

# Bass Model: estimation

- Typical starting values for $p$ and $q$ are 0.01 and 0.1.
- Estimating $m$ is the most difficult task.
- Parameter estimates are very sensitive to the number of available data.
- Reliable estimates are obtained after the maximum peak, but ... *"By the time sufficient observations have been developed for realiable estimation, it is too late to use the estimates for forecasting purposes"* (Mahajan, Muller, Bass, 1990).

# Compact Discs in USA



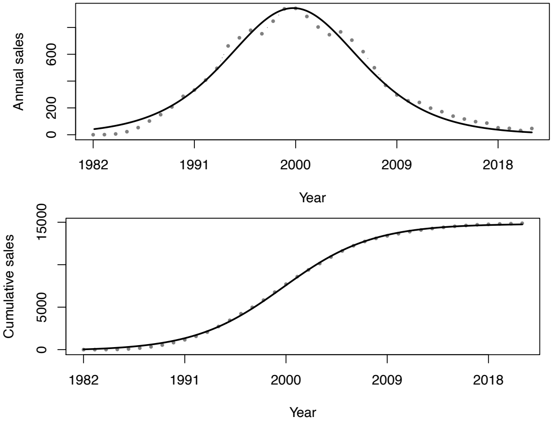
# Compact Discs in USA

Bass Model for CD: estimates and 95% CIs

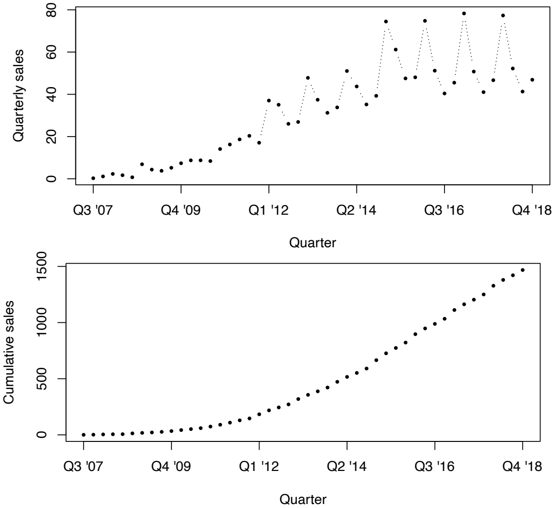
| | Estimate | Std.Error | Lower | Upper | $p$-value |
|---|---|---|---|---|---|
| $m$ | 14814 | 49 | 14716 | 14911 | <0.0001 |
| $p$ | 0.0022 | 0.0001 | 0.0020 | 0.0024 | <0.0001 |
| $q$ | 0.25 | 0.0035 | 0.24 | 0.26 | <0.0001 |

$R^2 = 0.9998$

# Compact Discs in USA



# Apple iPhone



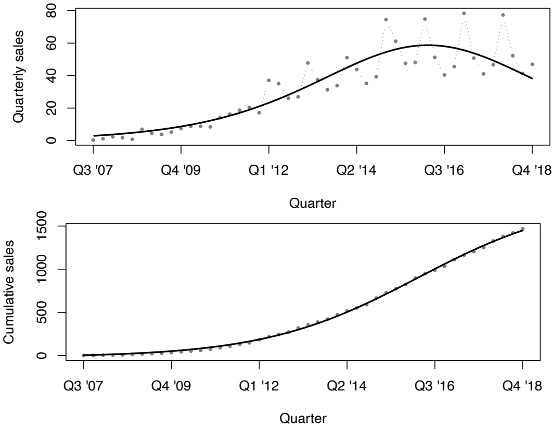
# Apple iPhone

Bass Model for iPhone: estimates and 95% CIs

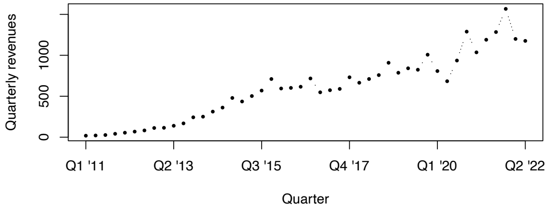
| | Estimate | Std.Error | Lower | Upper | $p$-value |
|---|---|---|---|---|---|
| $m$ | 1823.7 | 34.12 | 1756.8 | 1890.6 | <0.0001 |
| $p$ | 0.0014 | 0.0001 | 0.0013 | 0.0015 | <0.0001 |
| $q$ | 0.1259 | 0.0027 | 0.1206 | 0.1311 | <0.0001 |

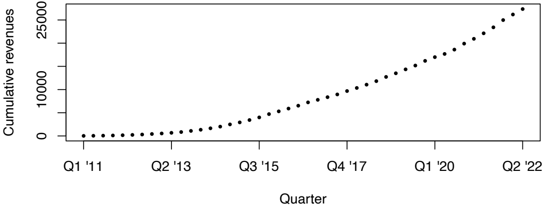
$R^2 = 0.9995$

# Apple iPhone



# Twitter revenues



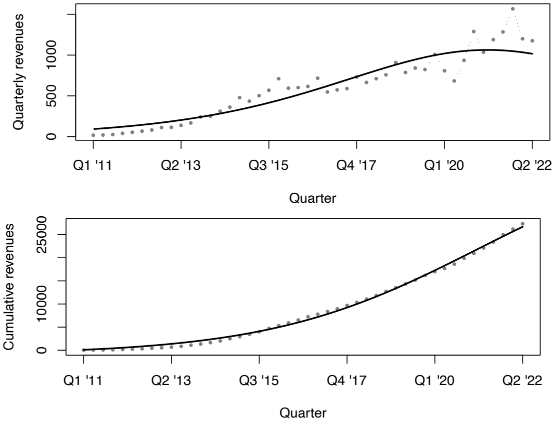
# Twitter revenues

Bass Model for Twitter: estimates and 95% CIs

| | Estimate | Std.Error | Lower | Upper | $p$-value |
|---|---|---|---|---|---|
| $m$ | 44633.7 | 3557.9 | 37660.3 | 51607.0 | <0.0001 |
| $p$ | 0.0019 | 0.0001 | 0.0018 | 0.0021 | <0.0001 |
| $q$ | 0.09 | 0.004 | 0.08 | 0.10 | <0.0001 |

$R^2 = 0.9995$

# Twitter revenues



# Bass Model: interesting properties

- Parsimonious model with just three parameters $m$, $p$, $q$.
- Only needs aggregate sales data.
- Easy to interpret.

# Bass Model: limitations

- The market potential $m$ is constant along the whole life cycle.
- The Bass Model does not account for marketing mix strategies.
- It is a model for products with a limited life cycle: needs a hypothesis.

# Generalized Bass Model

The Bass Model does not account for the effect of exogenous variables, such as marketing mix, public incentives, environmental shocks. Besides, in some cases the diffusion process does not have a bell shape curve, but a more complex structure.

# Generalized Bass Model

The Generalized Bass Model (Bass et al., 1994) adds an intervention function $x(t)$

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))x(t).$$

where $x(t)$ is an integrable, non negative function.

- The Bass Model is a special case where $x(t) = 1$.
- if $0 < x(t) < 1$ the process slows down,
- if $x(t) > 1$ the process accelerates.

# Generalized Bass Model: closed-form solution

The closed-form solution of the model is

$$z(t) = m\,\frac{1 - e^{-(p+q)\int_0^t x(\tau)d\tau}}{1 + \frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}}, \qquad t > 0.$$

Interesting: function $x(t)$ does not modify the market potential $m$!
Function $x(t)$ modifies the speed of the process.

# Modelling $x(t)$: exponential shock

Function $x(t)$ may take several forms in order to describe various types of shock.
A strong and fast shock may take an exponential form

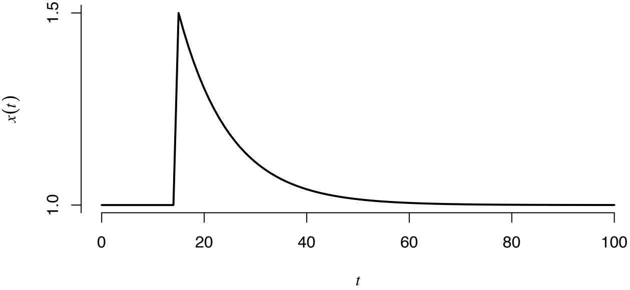
$$x(t) = 1 + c_1 e^{b_1(t-a_1)} I_{t \geq a_1},$$

where parameter $c_1$ is intensity and sign of the shock, $b_1$ is the 'memory' of the effect and is typically negative, and $a_1$ is the starting time of the shock.

# Modelling $x(t)$: exponential shock

The use of exponential shock is suitable for identifying the positive effect of marketing strategies or incentive measures, in order to speed up the diffusion process.
Also, a negative shock may represent a fast slowdown in sales due to the entrance of a competitor.

# Modelling $x(t)$: exponential shock



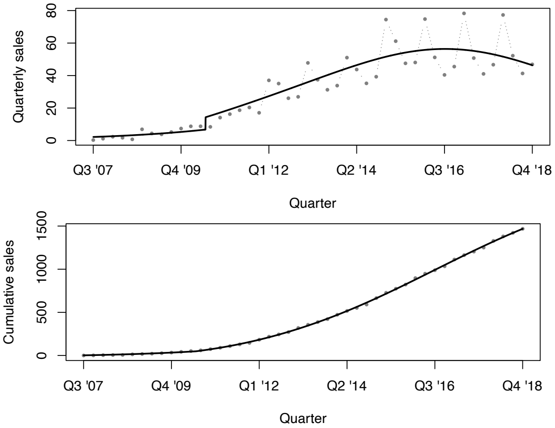
# Apple iPhone

GBM for iPhone: estimates and 95% CIs

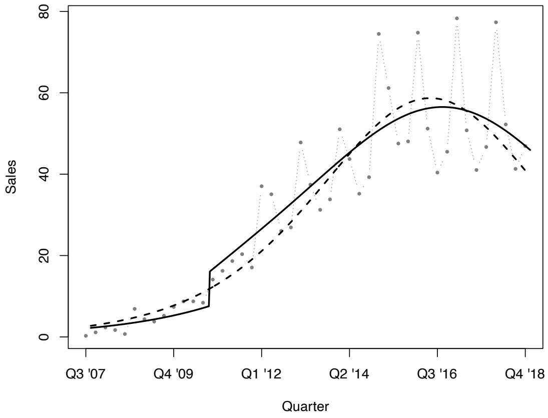
| | Estimate | Std.Error | Lower | Upper | $p$-value |
|---|---|---|---|---|---|
| $m$ | 2108.9 | 124.9 | 1864.1 | 2353.8 | $< 0.001$ |
| $p$ | 0.0009 | 0.0001 | 0.0008 | 0.0011 | $< 0.001$ |
| $q$ | 0.10 | 0.001 | 0.08 | 0.12 | $< 0.001$ |
| $a_1$ | 12.5 | 0.99 | 10.56 | 14.44 | $< 0.001$ |
| $b_1$ | $-0.14$ | 0.06 | $-0.25$ | $-0.03$ | 0.02 |
| $c_1$ | 1.13 | 0.17 | 0.78 | 1.47 | $< 0.001$ |

$R^2 = 0.9998$

# Apple iPhone



# Apple iPhone



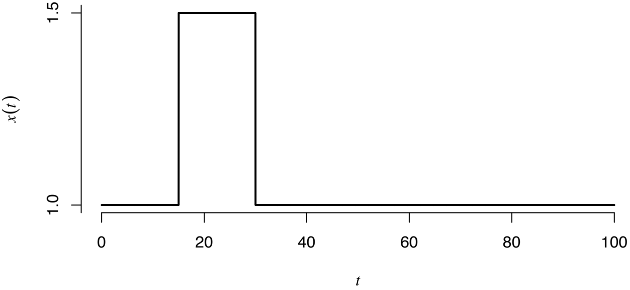
# Modelling $x(t)$: rectangular shock

A more stable shock, acting on a longer period of time, may be modeled through a rectangular shock

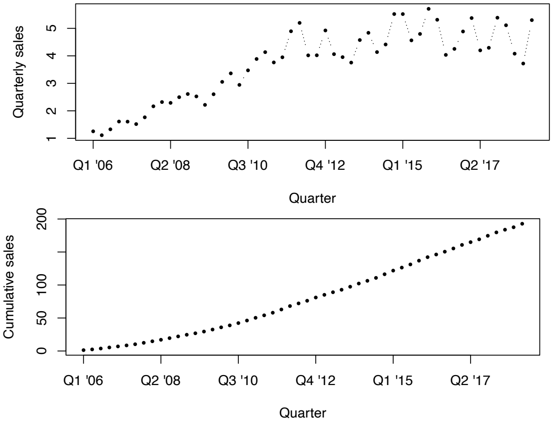
$$x(t) = 1 + c_1 I_{t \geq a_1} I_{t \leq b_1},$$

where parameter $c_1$ describes intensity of the shock, either positive or negative, parameters $a_1$ and $b_1$ define beginning and end of the shock (con $a_1 < b_1$).
The rectangular shock is useful to identify the effect of policies and measures within a limited time interval.

# Modelling $x(t)$: rectangular shock



# Apple iMac


Quarterly sales
1
2
3
4
5
Q1 '06
Q2 '08
Q3 '10
Q4 '12
Q1 '15
Q2 '17
Quarter
Cumulative sales
0
50
100
200
Q1 '06
Q2 '08
Q3 '10
Q4 '12
Q1 '15
Q2 '17
Quarter

# Apple iMac

## GBM for iMac: estimates and 95% CIs

| | Estimate | Std.Error | Lower | Upper | $p$-value |
|---|---|---|---|---|---|
| $m$ | 281.66 | 3.58 | 274.65 | 288.68 | $< 0.0001$ |
| $p$ | 0.0047 | 0.0042 | 0.0047 | 0.0048 | $< 0.0001$ |
| $q$ | 0.061 | 0.001 | 0.059 | 0.063 | $< 0.0001$ |

$R^2 = 0.9999088$

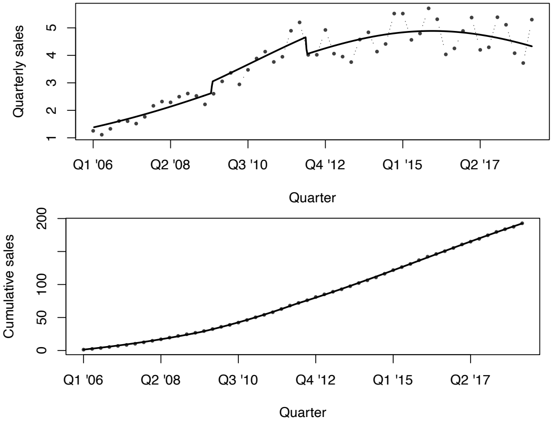
# Apple iMac

GBM for iMac: estimates and 95% CIs

| | Estimate | Std.Error | Lower | Upper | P-value |
|---|---|---|---|---|---|
| $m$ | 304.1 | 3.67 | 296.9 | 311.3 | $< 0.0001$ |
| $p$ | 0.0043 | 0.00001 | 0.0042 | 0.0044 | $< 0.0001$ |
| $q$ | 0.055 | 0.00 | 0.053 | 0.056 | $< 0.0001$ |
| $a_1$ | 14.67 | 0.96 | 12.79 | 16.54 | $< 0.0001$ |
| $b_1$ | 25.95 | 0.71 | 24.55 | 27.35 | $< 0.0001$ |
| $c_1$ | 0.16 | 0.02 | 0.13 | 0.20 | $< 0.0001$ |

$R^2 = 0.9999$

# Apple iMac



Quarterly sales
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Q2 '08
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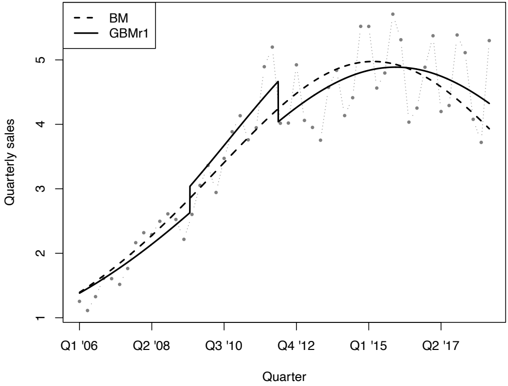
# Apple iMac




Model comparison . . .

# Modelling $x(t)$: mixed shock

It may be useful to have more than one shock of different nature. A simple case is made of a couple of shocks, rectangular and exponential,

$$x(t) = 1 + c_1 I_{t \geq a_1} I_{t \leq b_1} + c_2 e^{b_2(t-a_2)} I_{t \geq a_2}$$

Other combinations are possible.

# Model performance and selection

The usual performance indicator is the $R^2$

$$R^2 = \frac{\mathsf{SST} - \mathsf{SSE}}{\mathsf{SST}} = \frac{\sum(y_i - \bar{y})^2 - \sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2}$$

where $y_i$, $i = 1, 2, ..., n$ are calculated with the selected model.
Further evaluations are performed through analysis of residuals (e.g. residual plots, Durbin-Watson statistic).

# Model selection: $\tilde{R}^2$

In order to select between two 'nested' models, a suitable tool is the $\tilde{R}^2$

$$\tilde{R}^2 = \frac{\mathsf{SSE}_{m_1} - \mathsf{SSE}_{m_2}}{\mathsf{SSE}_{m_1}} = (R^2_{m_2} - R^2_{m_1})/(1 - R^2_{m_1}),$$

where $R^2_{m_i},\ i = 1, 2$ is the $R^2$ of model $m_i$.
If $\tilde{R}^2 > 0.3$ then the more complex model is significant.