Innovation Diffusion Processes: Concepts, Models and Predictions

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Abstract

Diffusion phenomena have attracted considerable research attention for their interdisciplinary character, allowing the combination of theories and concepts from many disciplines, such as natural sciences, mathematics, physics, statistics, social sciences, marketing science, economics and technological forecasting.The formal representation of diffusion processes has historically used epidemic models borrowed from biology, namely the logistic or s-shape equation, under the hypothesis that an innovation spreads in a social system through communication between people just like an epidemic does through contagion. Nowadays, we are witnessing many different diffusion processes, highly aided by unprecedented mobility patterns and communication possibilities, that range from the spread of new epidemics to the massive adoption of new technologies and products and from the fast diffusion of news to the wide acceptance of new trends and social norms. This review article describes some basic innovation diffusion models, based on ordinary differential equations (ODE), able to describe different dynamics in diffusion.

Contents

1. Introduction

Modelling and forecasting the diffusion of innovations is a research field devoted to describing and predicting with mathematical models the growth dynamics of an innovation entering into a social system. The term innovation may indicate several concepts, ranging from natural to socio-economic domains, such as a new disease, a new product, a new technology, a new idea or a new way of doing things. This theme has always attracted many scholars due to its very interdisciplinary nature, enabling the combination of theories, concepts and models from the natural sciences, such as physics and biology, and from social sciences, such as sociology, economics and marketing. A famous contribution on the diffusion of innovations is the book by Rogers (2010), *Diffusion of Innovations*, originally published in 1962 and then re-edited several times, which proposed a rich theoretical description of diffusion processes and their key factors, such as characteristics of innovation and types of adopters, drivers and obstacles, with a large variety of examples. Rogers (2010) stressed that this research started with a series of independent studies. The common factor that made it possible to combine apparently distant sciences in understanding phenomena is the observation that a new product, technology or idea spreads into society through *communication* and *imitation*, just like a new virus does among people through *contagion*. Hence, the mathematical laws governing biological processes, such as the spread of a virus or the growth of a biological organism, are similar to those acting in a social system when an innovation starts to spread and be adopted by people. Vespignani (2012) highlighted that phenomena like the growth of a new pathogen, the diffusion of social norms or the spread of black-outs on a nation-wide scale, despite being referred to as very different systems, may be modeled with a similar mathematical description. Vespignani (2012) also noticed that epidemiologists, computer scientists and social scientists share a common interest in studying contagion dynamics, with the purpose of understanding and controlling it. From this perspective, Vespignani stressed the importance of research theories and methods that enable discrimination between what is relevant and what is superflous in the description of socio-technical systems. Marchetti (1980) theorised that society is a learning system and the mathematical model able to capture this learning behaviour is the logistic equation, introduced by Verhulst (1838). The logistic equation has been at the core of many contributions on diffusion and many models extended its structure to account for specific assumptions, starting with the logistic model of Mansfield (1961). The marketing and economic literature has been especially productive in this sense since the publication of the Bass Model, (BM, Bass (1969)), one of the most successful upgrades of the logistic model. In the last thirty years, several reviews on innovation diffusion modelling have been proposed, especially discussing the central role played by the BM, (e.g. Mahajan et al. (1990), Parker (1994), Mahajan et al. (1995), Meade and Islam (1995), Meade and Islam (1998), Meade and Islam (2006), Hauser et al. (2006), Peres et al. (2010)). Mahajan et al. (1990) focused on aggregate models able to relax some of the assumptions underlying the BM. Hauser et al. (2006) proposed an analysis with a special attention to management and marketing issues needing to be addressed by diffusion models. Meade and Islam (2006) reviewed diffusion models, focusing on estimation, inferential and predictive performance. Peres et al. (2010) updated the contributions in the marketing literature, analysing new issues related to diffusion processes, such as the power of communication forces, heterogeneity of consumers, employability of diffusion models to non-durable goods and aspects of competition between products.

While the aggregate modelling of diffusion has basically relied on the logistic model, the BM and subsequent extensions, aimed at understanding the nature and evolutionary behaviour of these processes, a parallel and more recent stream of research developed theories and models based on a network representation and the use of network models to capture their dynamics. Pastor-Satorras et al. (2015) proposed a comprehensive review of the vast body of research produced on epidemic modelling in complex networks, observing that the real-world accuracy of models used in epidemiology was recently improved by the integration of large-scale datasets, making it possible to track human interactions and mobility, which are best represented in the form of networks. Pastor-Satorras et al. (2015) stressed how mathematical computational modelling of diffusion processes in networks started across different disciplines and statistical physics played a central role in defining theories and concepts. In this review, the authors started with the description of classical models in epidemiology based on the SIR and SIS framework, for which an extensive treatment was provided for instance in Anderson and May (1992) and Keeling and Rohani (2011). Pastor-Satorras et al. (2015) observed that this traditional approach to epidemiology, giving rise to a set of aggregate differential equations, is based on some necessary assumptions, which may prove inadequate in real-world situations, and thus network theory may provide a more suitable framework (Newman 2018).

Clearly, the use of network models relies on the availability of suitable data, able to provide information at *individual level.* In this sense, *aggregate models*, based on ordinary differential equations, ODE, appear to be a more parsimonious approach, needing only historical data of the diffusion process to describe and forecast it, as observed by Harvey and Kattuman (2020). Despite their often simplified structure, aggregate diffusion models may serve as powerful tools to shed light on the evolution of a process both in time and scale. Nowadays, many different diffusion processes may be observed, ranging from the spread of new epidemics to the massive adoption of new products, technologies and life-styles. As some of these processes are showing, diffusion processes require a strong effort in prediction and management.

This paper is aimed at presenting a class of aggregate models, based on ordinary differential equations, for innovation diffusion, starting from the BM, with the discussion of applications to various contexts, such as technological forecasting, marketing and epidemiology. The BM serves as a basis for some modelling extensions, obtained by relaxing some of its typical assumptions. To this end, some motivating examples are illustrated in Section 2. The key objectives of the article are to review the main mathematical features of some diffusion models, both for univariate and multivariate processes, (Section 3 and 4); to present the statistical techniques involved in model estimation and selection, underlying some crucial aspects for identification (Section 5); to show and discuss the proposed models with real-data applications, ranging from commercial markets to epidemiological contexts (Section 6) and to propose some perspectives for future achievements in research and practice (Section 7).

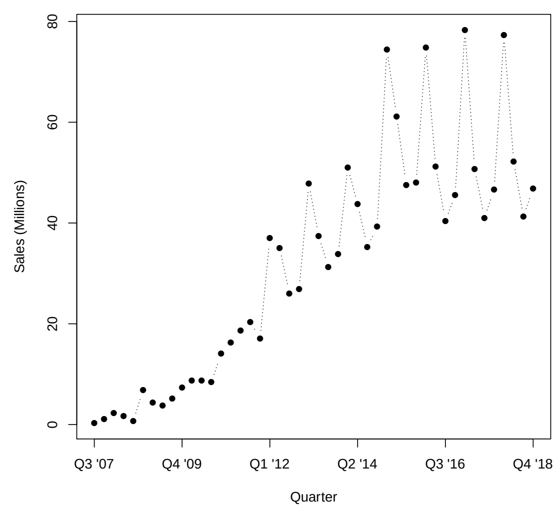

Figure 1

Global sales of Apple iPhone in millions of units (source: Apple Inc.)

2. Diffusion processes: Motivating examples

In this section three examples of diffusion processes are presented to illustrate common traits and specificities of applications related to different contexts, including commercial products, energy technologies and epidemiology. Figure 1 displays global sales of the Apple iPhone from Q3/07 to Q4/18, expressed in millions of units. Apart from evident seasonal peaks, the series exhibits a clear nonlinear trend, typical of products with a finite life-cycle, characterized by phases of launch, growth, maturity and decline. The iPhone series appears to have just reached the maturity phase.

Figure 2 shows the yearly consumption of energy produced from renewable sources, specifically photovoltaic and wind power, in Australia, from 1990 to 2019. In this case, the data pattern has an exponential behaviour and the technology appears in the growth phase of its diffusion process.

Figure 3 displays daily new cases of COVID-19 infections from February 19 to June 1, 2020, in Italy. The series represents what has been called the ‘first wave’ of the diffusion

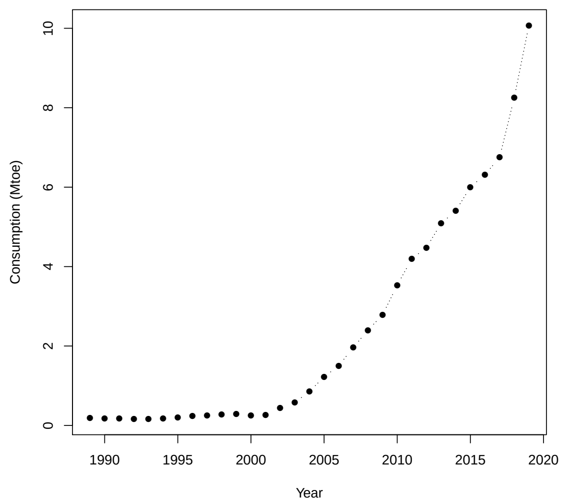

Figure 2

Consumption of renewable energy in Mtoe in Australia (source: BP Statistical Review of World Energy, 2020)

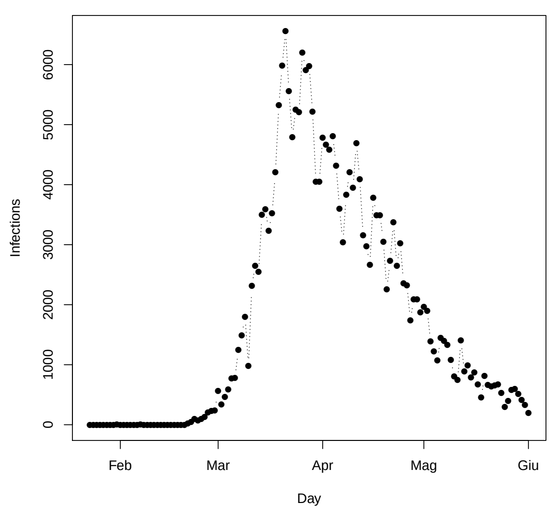

Figure 3

Daily infections from COVID-19 in Italy from 22/01/20 to 01/06/20 (source: Worldometer, 2020)

and similar patters may be found in most affected countries. The first wave shows a finite cycle and an asymmetric shape. Indeed, while the first part of the process has been very fast until the maximum peak, the decline phase has been much slower.

These three examples display different situations in terms of the development of a diffusion process over time. However, one common trait is the evident nonlinear growth, typical of these processes. The underlying hypothesis is that these dynamics are fueled by individuals' behavior, by means of communication (word of mouth) or contagion. On the other hand, these different cases suggest the need for external management aimed at controlling the speed and scale of diffusion. This point is especially evident in the case of renewables in Australia and COVID-19 in Italy, where public policy actions have been implemented to control the process, although to achieve opposite results. In the case of renewables, incentive measures such as feed-in tariffs have been implemented to stimulate the growth of this technology, whose widespread adoption was hindered by important barriers related to the dominance of non-renewables and the inherent characteristics of renewables, which make them difficult to integrate in power systems. On the contrary, a strong policy measure, a two-month lockdown, has put in place to contain the diffusion process of COVID-19, both its speed and size. In this case, reaching the maximum peak (in the first wave), after which the process started to decline, has been clearly a desirable outcome.
These three examples suggest some first considerations that will be at the core of the modelling approach proposed in this paper and based on aggregate diffusion models: individuals' behaviour is the driver of any diffusion process and defines its boundaries in time and space. At the same time, management of these processes appears to be a necessary condition.

3. Univariate diffusion models

3.1. Innovation and imitation: The BM

The Bass model, BM, describes the life-cycle of an innovation, capturing its typical phases of launch, growth and maturity and decline. Originally conceived in the marketing domain, its purpose is to model the development of a new product growth over time, as result of the purchase choices of a given set of potential customers. These decisions are assumed to be influenced by two sources of information, an external one, such as mass media and advertising and an internal, including social interactions and word of mouth. These are 'competing' sources of information, whose effect creates two distinct groups of adopters. One group is influenced only by the external source, the *innovators*; the other only by the internal one and these are the *imitators*. One of the great advantages associated with the BM is the concrete possibility to explain the initialising phase of diffusion due to the presence of innovators. Indeed, there exists a huge body of literature on the role of innovators, also called 'early adopters' by Rogers (2010) or 'opinion leaders' by Katz and Lazarsfeld (1966), but the first model taking into account their role in a formal way is the BM. In particular, it is assumed that there exists a constant level of innovators buying the product at the beginning of the diffusion. In this sense, the BM accounts for the role of all the communication efforts realized by firms, whereas a pure logistic approach like that of the Mansfield model does not.
The formal representation of the BM is a first-order differential equation

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t)). \qquad 1.$$

In equation 1, the variation over time of adoptions, $z'(t)$, is proportional to the residual market, $(m - z)$ where m is the market potential or carrying capacity, and $z(t)$ is the cumulative number of adoptions at time t. The market potential m represents the maximum number of adoptions within the life-cycle and its value is assumed to be constant along the whole diffusion process. The residual market is multiplied by parameters p and q. Parameter p represents the effect of the external influence, due to the mass media communication, while parameter q is the coefficient of imitation, whose influence is modulated by the ratio $z(t)/m$, giving rise to the so-called word-of-mouth effect. By posing $z(t)/m = y(t)$ the model may be re-written as

$$y'(t) = (p + qy(t))(1 - y(t)). \tag{2.}$$

The BM may also be interpreted as a hazard function, that is the probability that an event will occur at time t given that it has not occurred

$$\frac{y'}{(1-y)} = (p + qy) \qquad p, q > 0. \tag{3.}$$

Equation 3 describes the conditional probability of an adoption at time t, resulting from the sum of the probabilities of two incompatible events, p and $q(z/m)$: so the model excludes adoptions due to both innovation and imitation. This separation of effects generates the two classes of adopters previously mentioned, although these are latent categories, as aggregate data on adoptions obviously do not provide evidence on this difference.

The closed-form solution of the BM is

$$y(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} \qquad t > 0. \tag{4.}$$

or, equivalently, posing $z = ym$,

$$z(t) = m\,\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} \qquad t > 0. \tag{5.}$$

Cumulative sales $z(t)$ are a function of p and q. The market potential m acts as scale parameter of the process.

The BM assumes a finite life-cycle for a product. However, in many cases we observe successive generations of the same product: this has been studied for instance by Norton and Bass (1987) and Jiang and Jain (2012), proposing some modeling options for successive generations of the same product.

3.2. Exogenous actions: The GBM

Reviews on diffusion models, such as that provided in Mahajan et al. (1990), pointed out that a significant limitation of the BM was the failure to incorporate marketing mix variables in the model under managerial control, such as price strategies and advertising. Peres et al. (2010) observed that this omission raised a conceptual conflict because the model provides a high level of fit and reliable forecasts just making some hypotheses about consumers' behaviour and without marketing mix variables, but at the same time it is clear that marketing mix decisions exert a notable impact on new product growth. Additionally, the shortening of life-cycles due to successive generations, analyzed in Norton and Bass (1987), increased the need for a model incorporating control variables. Bass et al. (2000) provided a review on several attempts to introduce control variables into diffusion models and listed some desirable properties of a diffusion model with decision variables: it should have empirical support and be managerially useful, allowing a direct interpretation of parameters and comparisons with other situations and it should have a closed-form solution and be easy to implement. The model presenting all these properties, formalized by Bass et al. (1994), is the Generalized Bass Model, GBM. Conceived for taking into account both price and advertising strategies, the Generalized BM enlarges the basic structure of the BM by multiplying its basic structure by a very general intervention function $x(t)$, assumed to be nonnegative and integrable. The GBM has a very simple structure

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))x(t). \qquad 6.$$

The original form of function $x(t)$ designed by Bass et al. (1994) jointly considers the percentage variation of prices and advertising efforts has the form $x(t) = 1 + Pr(t) + A(t)$, where $Pr(t)$ and $A(t)$ are price and advertising at time t. One interesting feature of the GBM is that it reduces to the BM, when $x(t) = 1$, i.e. when there are no changes in price and advertising. Besides, if the percentage changes in price and advertising remain the same from one period to the next, then function $x(t)$ reduces to a constant, yielding again the BM. This would explain why the BM provides good parameter estimates, even without marketing mix variables. This generalization allows testing the effect of marketing mix strategies on diffusion and making scenario simulations based on the modulation of function $x(t)$. Interestingly, Bass et al. (1994) noticed that the model's internal parameters m, p and q are not modified by these external actions. Function $x(t)$ acts on the natural shape of diffusion, modifying its temporal structure and not the value of its internal parameters. Consequently, the important effect of $x(t)$ is to anticipate or delay adoptions, but not to increase or decrease them. Hence, function $x(t)$ may represent all those strategies applied to control the timing of a diffusion process but not its size. A more general perspective on the usability of function $x(t)$ was proposed by Guseo et al. (2007). For example, a strong and fast shock may take an exponential form

$$x(t) = 1 + c_1 e^{b_1(t-a_1)} I_{t \geq a_1}, \qquad 7.$$

where parameter c_1 is the intensity and sign of the shock, b_1 is the 'memory' of the effect and is typically negative, and a_1 is the starting time of the shock. The use of exponential shock is suitable for identifying the positive effect of marketing strategies or incentive measures, to speed up the diffusion process.

Interestingly, the possibility to define a flexible function $x(t)$ has highlighted a much larger perspective on the usability of the GBM, which may be applied as an efficient diagnostic tool for detecting all kinds of external actions affecting a diffusion process. In particular it has proven to be crucial in cases in which innovation dynamics are significantly influenced by institutional aspects, policies, cultural and economic factors (Guseo et al. (2007), Guidolin&Mortarino (2010), Bunea et al. (2020)).

3.3. Dynamic market potential

One of the characteristic assumptions of the BM relates to the size of the market potential m, whose value is assumed to be determined at the time of introducing the new product and remains constant along the whole diffusion process. This is a simplifying assumption that may prove reasonable in some circumstances, while in other cases a time-dependent market potential may appear to be a more suitable choice. Mahajan et al. (1990) observed that theoretically there is no rationale for a static adopter population and on the contrary, a dynamic adopters population seems reasonable. The possibility of a dynamic potential has been addressed in literature since the 1970s. Some works introduced a variable structure by modifying the residual market $(m(t) - z(t))$ (Mahajan and Peterson (1978), Horsky (1990), Kamakura and Balasubramanian (1988), Mesak and Darrat (2002)). Other researchers also considered a modification of the word of mouth ratio, $z(t)/m(t)$, including Sharif and Ramanathan (1981), Jain and Rao (1990), Centrone et al. (2007), Goldenberg et al. (2010). Following Guseo and Guidolin (2009), a generalization of the BM, considering a dynamic market potential, $m(t)$, may be simply described as

$$z'(t) = m(t)\left\{\left(p + q\frac{z(t)}{m}\right)\left(1 - \frac{z(t)}{m(t)}\right)\right\} + z(t)\frac{m'(t)}{m(t)} \qquad 8.$$

$$\frac{z'(t)m(t) - z(t)m'(t)}{m^2(t)} = \left(\frac{z(t)}{m(t)}\right)' = \left(p + q\frac{z(t)}{m(t)}\right)\left(1 - \frac{z(t)}{m(t)}\right)$$

and setting $y(t) = z(t)/m(t)$ yields

$$y'(t) = p + qy(t)(1 - y(t)) \qquad 9.$$

which is the standard BM. This generalization of the BM where $m(t)$ is time-dependent has closed-form solution

$$z(t) = m(t)\,\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} \qquad 10.$$

where $m(t)$ is a free function that may take several forms depending on suitable modelling choices. In Guseo and Guidolin (2009) $m(t)$ is made depend on a communication process about the new product, which typically precedes the adoption phase and serves the purpose of building the market, so that the final model, called GGM, takes the form

$$z(t) = K\,\sqrt{\frac{1 - e^{-(p_c+q_c)t}}{1 + \frac{q_c}{p_c}e^{-(p_c+q_c)t}}}\,\frac{1 - e^{-(p_s+q_s)t}}{1 + \frac{q_s}{p_s}e^{-(p_s+q_s)t}}. \qquad 11.$$

In the GGM cumulative adoptions $z(t)$ depend on parameters K, p_c, q_c, p_s, q_s. Parameters p_c and q_c describe the dynamics of spread of information, contributing to the creation of the market potential, while parameters p_s and q_s refer to the adoption phase.
The BM is a special case of GGM where the spread of information is so fast that there is a set of potential adopters ready to purchase as soon as the product enters the market, $m(t) = K$.

4. Multivariate diffusion models

One of the key points to consider in diffusion processes is the possible presence of competitors. The case of a monopolistic market is very rare and in general a competitive environment characterizes a diffusion process: new products or technologies entering a new market soon gain some concurrents, competing for the same market niche. Accounting for this fact may be very important since the action of concurrents may limit a product's sales growth. On the other hand, competition may also be beneficial, by enlarging the market for all players. To date, competition modelling in diffusion has essentially been limited to duopolistic situations, where no more than two diffusion processes are modeled simultaneously. This is partly due to the complexity of systems of differential equations, accounting for the possible interactions among market players: as the number of equations grows, so does the number of parameters involved, making the mathematical and statistical management of these systems practically unfeasible. A traditional approach to diffusion modelling under competition relied on the famous Lotka–Volterra models, a class of models based on the independent contributions of Lotka (1920) and Volterra (1926). These equations, originally employed in the natural sciences for describing interactions between species and especially the so-called 'predator-prey' relationship, have been also largely applied to the technological domain. From the first contributions by Abramson and Zanette (1998), Baláž and Williams (2012) and Morris and Pratt (2003), the literature on Lotka–Volterra models has expanded to analyze the competitive dynamics in innovative markets (Chakrabarti (2016), Guidolin and Guseo (2015), Guidolin et al. (2019), Gupta and Jain (2016), Hung et al. (2017), Kreng and Wang (2009), Tseng et al. (2014)). The typical structure of the Lotka-Volterra model is

4.1. Within and cross imitation: The UCRCD model

By generalizing to the multivariate case the basic structure of the BM, diffusion models under duopolistic competition have been developed, for instance by Guseo and Mortarino (2010), Guseo and Mortarino (2012), Guseo and Mortarino (2014), Guseo and Mortarino (2015), Krishnan et al. (2000), Laciana et al. (2014) and Savin and Terwiesch (2005). A common feature of these models is accounting for the interplay between products by splitting the imitation effect into two parts: the *within-product* imitation, due to a product's specific sales, and the *cross-product* imitation, due to sales of the concurrent.

In addition, competitors may enter the market at the same time so that their life-cycles are essentially simultaneous, or, more generally, a product starts as a monopolist and gains concurrent brands along the way. The situation of sequential market entry, also called *diachronic competition*, is more common in reality, although it is less treated in literature. The UCRCD model by Guseo and Mortarino (2014) postulates a diffusion process characterized by two phases: monopoly and competition. Borrowing some terms from game theory, the first market player, may be termed the *incumbent*, while the second, entering the market at a second stage, may be referred as the *entrant*. Given these different phases, the market potential may have different levels: m_a, the market potential of the incumbent in the monopolistic phase, and m_c, the market potential under competition. The residual market $m - z(t)$ is assumed to be common, where $z(t) = z_1(t) + z_2(t)$ are common cumulative adoptions. The second market player enters the market at time $t = c_2$ with $c_2 > 0$.

The model is a system of differential equations where $z_1'(t)$ and $z_2'(t)$ indicate instantaneous adoptions of the first and of the second market player, respectively, and I_A is an indicator function of event A,

$$\begin{aligned} z_1'(t) \;=\; m\Bigg\{ & \left[p_{1a} + q_{1a}\frac{z(t)}{m}\right](1 - I_{t>c_2}) \\ & + \left[p_{1c} + (q_{1c}+\delta)\frac{z_1(t)}{m} + q_{1c}\frac{z_2(t)}{m}\right] I_{t>c_2} \Bigg\} \left[1 - \frac{z(t)}{m}\right], \qquad 12. \end{aligned}$$

$$
\begin{aligned}
z_2'(t) &= m\left[p_2 + (q_2-\gamma)\frac{z_1(t)}{m} + q_2\frac{z_2(t)}{m}\right]\left[1-\frac{z(t)}{m}\right] I_{t>c_2},\\
m &= m_a(1-I_{t>c_2}) + m_c I_{t>c_2}\\
z(t) &= z_1(t) + z_2(t) I_{t>c_2}.
\end{aligned}
$$

In the monopolistic phase, $t \le c_2$, the trajectory of the incumbent, $z_1'(t)$, is described according to a standard Bass model with parameters p_{1a}, q_{1a}, and m_a. When $t > c_2$, both concurrents exist in the market and influence each other. The incumbent is characterized by new parameters: the innovation coefficient under competition, p_{1c}, and the imitation coefficient, which is divided into two parts, the *within* imitation coefficient $q_{1c}+\delta$, measuring internal growth through the ratio z_1/m, and the *cross* imitation one, q_{1c} which is powered by z_2/m and measures the effect of the diffusion of the entrant on the incumbent. The entrant has three corresponding parameters: the innovation coefficient p_2, the *within* imitation coefficient q_2, modulating internal growth through the ratio z_2/m and the *cross* imitation coefficient $q_2-\gamma$, which measures the effect, of the incumbent. Typically parameters δ and γ are assumed to be different, and the model is called *unrestricted* UCRCD. If the restriction $\delta = \gamma$ applies, the model takes a reduced form, called *standard* UCRCD, see Guseo and Mortarino (2014), and a symmetric behavior between the two competitors is assumed.
A possible generalization of the UCRCD model has been proposed in Guidolin and Guseo (2015) and Guidolin and Guseo (2020), with a Lotka-Volterra model with churn effects.

5. Statistical inference for diffusion models

The statistical implementation of diffusion models is quite sensitive to the amount of data available and reliable estimates are obtained if non-cumulative data include the peak, as observed by Srinivasan and Mason (1986). However, this clearly reduces model usefulness for forecasting. Mahajan et al. (1990) effectively synthesized the problem, stating that 'parameter estimation for diffusion models is primarily of historical interest; by the time sufficient observations have been developed for reliable estimation, it is too late to use the estimates for forecasting purposes'. Van den Bulte and Lilien (1997) considered some bias in parameter estimation, including the tendency to underestimate the market potential, whose value is generally close to the latest observed data. Estimation aspects were also discussed in Venkatesan and Kumar (2002), Venkatesan et al. (2004) and Jiang et al. (2006). Empirical experience demonstrated that ordinary least squares technique (OLS) is non-optimal for estimating diffusion models, because of some shortcomings including the tendency to yield negative sign parameters. Srinivasan and Mason (1986) proposed using the nonlinear least squares approach (NLS), which is generally accepted to be the more reliable method of estimation. Specifically, the structure of a nonlinear regression model, following Seber and Wild (1989), may be considered

$$
w(t) = \eta(\beta, t) + \varepsilon(t), \qquad 13.
$$

where $w(t)$ is the observed response, $\eta(\beta, t)$ is the deterministic component describing instantaneous or cumulative processes, depending on parameter set β and time t, and $\varepsilon(t)$ is a residual term, not necessarily independent and identically distributed (i.i.d.).

Model global goodness-of-fit is evaluated through the R^2, the value of which is typically greater than 0.95, because it is calculated on cumulative data. From this perspective, comparison and selection between concurrent models becomes essential. The performance of an extended model, m_2, compared with a nested one, m_1, may be evaluated through a squared multiple partial correlation coefficient $\tilde{R}^2$ in the interval $[0;1]$, namely,

$$\tilde{R}^2 = (R^2_{m_2} - R^2_{m_1})/(1 - R^2_{m_1}), \tag{14.}$$

where $R^2_{m_i},\ i = 1,2$ is the standard determination index of model m_i.
The $\tilde{R}^2$ coefficient has a monotone correspondence with the F-ratio

$$F = [\tilde{R}^2(n - v)]/[(1 - \tilde{R}^2)u], \tag{15.}$$

where n is the number of observations, v the number of parameters of the extended model m_2, and u the incremental number of parameters from m_1 to m_2. Under strong conditions on the distributional shape of the error term $\varepsilon(t)$, particularly i.i.d. and normality, the statistic F-ratio, for the null hypothesis of equivalence of the two models, is a central Snedecor's F with u degrees of freedom for the numerator and $n - v$ degrees of freedom for the denominator, $F \sim F_{u,n-v}$.

6. Applications

This section discusses the application of the models discussed previously to the datasets described as motivating examples.

6.1. BM

The first case refers to the estimation of a standard BM to Apple iPhone sales data. The results are displayed in Table 1 and Figure 4. As may be observed in Figure 4, which illustrates instantaneous predictions realized with the BM, the mean trajectory of the data is well captured. The seasonal component is obviously not described through this model and may be treated in different ways, according to modelling needs, both by performing a prior smoothing of data with moving-averages or non-parametric smoothing methods (i.e. loess), or by describing it at a second stage of modelling with a SARMAX refinement. Parameter estimates in Table 1 describe a typical case for a standard BM with imitation coefficient q much larger than innovation coefficient p. This reflects the typical behaviour of most products, where the imitative component dominates the life-cycle. According to its structure, the BM predicts a maximum peak, after which the series starts to decline. It may be observed that the last data points are not well fitted with the BM, which tends to underestimate the last observations available. Despite this, the model appears to be a suitable choice for this product life-cycle.
A less satisfactory situation concerns the series of consumption of renewable energy, displayed in Figure 5 and Table 2. Despite an apparent good fit, evidenced by parameter estimates in Table 2, it may be easily observed that the model fails to capture the last part of the trajectory, which is still increasing. This problem may be related to a double regime within the data: while the first part is characterized by flat behavior, also confirmed by the weak influence of innovators ($p = 0.00025$), the second has clear exponential growth. This pattern is typical of the products and technologies that experience difficulties in taking off and require exogenous actions, stimulating the adoption process. In the case of energy

consumption, these exogenous actions generally take the form of incentive measures, whose purpose is to facilitate the final choice of consumers.

The third application of the BM refers to the diffusion of COVID-19 in Italy in the period from 22/01/20 to 01/06/20. In this case, the model is interpreted within an epidemiological context, so innovators, reflected by parameter p, are those infected first and imitators are those being infected due to contagion dynamics, and represented by parameter q. The market potential m may be interpreted as the maximum number of people affected by COVID-19 within the period considered. As shown in Figure 6 the BM does not efficiently describe the asymmetric shape of the series.

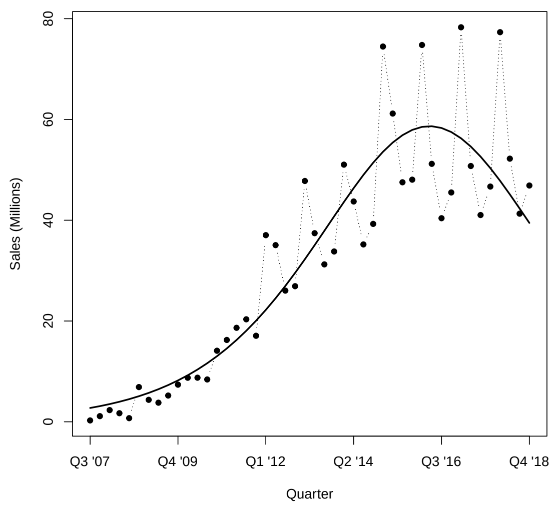

Figure 4

BM for global sales of Apple iPhone

Table 1 BM for global sales of Apple iPhone

	Estimate	Std.Error	Lower	Upper
m	1823	34	1756	1890
p	0.0014	0.00005	0.0013	0.0015
q	0.13	0.002	0.12	0.13
$R^2 = 0.9995$				

Table 2 BM for consumption of renewable energy in Australia

	Estimate	Std.Error	Lower	Upper
m	153	10	133	173
p	0.00025	0.00001	0.00023	0.00028
q	0.21	0.005	0.20	0.22
$R^2 = 0.9994$				

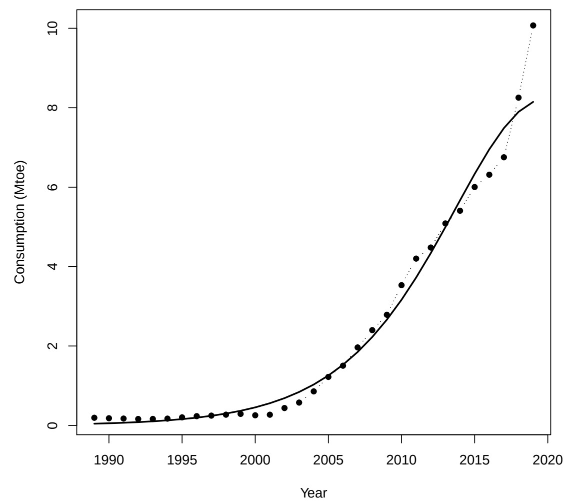

Figure 5

BM for consumption of renewable energy in Australia

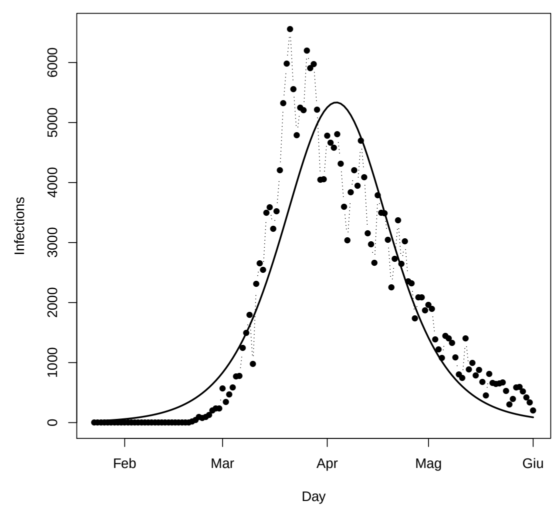

Figure 6

BM for daily infections from COVID-19 in Italy from 22/01/20 to 01/06/20

6.2. GBM

In order to provide a better description of the series of renewable energy consumption, a GBM with one exponential shock has been estimated. The results of this procedure are presented in Table 3 and Figure 7; the GBM efficiently describes the data, as confirmed by parameter estimates. Specifically, the coefficients concerning the exponential shock, a_1, b_1 and c_1, are all significant, highlighting the positive effect of an external perturbation on consumption dynamics. This incidentally emphasizes the role of the GBM as an evaluation tool for testing the effect of external shocks. Parameter $a_1 = 19.12$ suggests that the positive shock to consumption, reasonably connected with energy policy support to renewables, started around the year 2007 and rapidly led to a dramatic increase of renewables' diffusion in that country. Interestingly, the estimate of parameter m has notably increased with respect to the one of the BM, implying a longer trajectory: so capturing the behaviour of data in a more efficient way implies a better model performance in predictive terms.

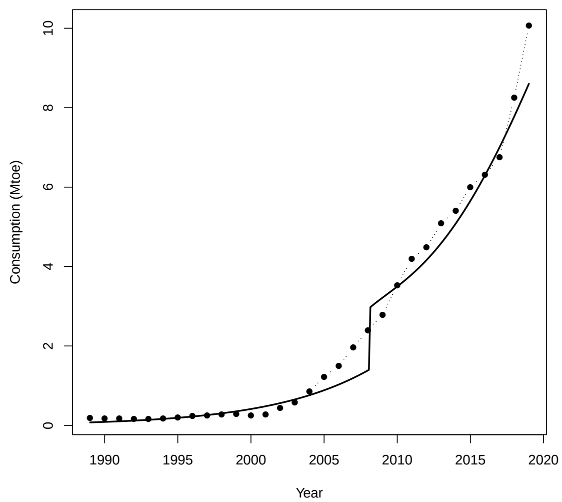

Figure 7

GBM for consumption of renewable energy in Australia

Table 3 GBM for consumption of renewable energy in Australia

	Estimate	Std.Error	Lower	Upper
m	382	140	107	656
p	0.00019	0.00004	0.0001	0.0002
q	0.16	0.01	0.13	0.18
a_1	19.12	0.46	18.21	20.02
b_1	-0.40	0.13	-0.65	-0.14
c_1	1.11	0.24	0.63	1.58
$R^2 = 0.9997$				
$\tilde{R}^2 = 0.5$				

6.3. GGM

A dynamic market potential model, GGM, is applied to the series of iPhone sales and the results are presented in Figure 8 and Table 4. It may be observed that all parameters are statistically significant and provide evidence that the diffusion process has been characterized by two phases, one for the spread of information regarding the product, creating the market potential, and one for the proper adoption process. Both phases are governed by imitative behavior, i.e. parameters q_c and q_s. The greater flexibility of the GGM with respect to the BM can be seen in Figure 8, where the predictions according to the GGM capture the behaviour of data much better, especially in the last part. The improvement obtained over the simpler BM is also confirmed by the index $\tilde{R}^2 = 0.6$.

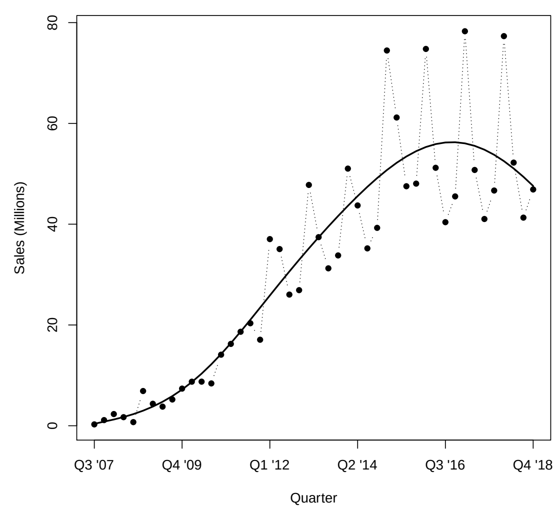

Figure 8

GGM for global sales of Apple iPhone

Table 4 GGM for global sales of Apple iPhone

	Estimate	Std.Error	Lower	Upper
K	2116	97	1925	2307
p_c	0.0059	0.0015	0.0028	0.01
q_c	0.21	0.04	0.13	0.28
p_s	0.0021	0.0002	0.0015	0.0026
q_s	0.10	0.01	0.09	0.11
$R^2 = 0.9998$				
$\tilde{R}^2 = 0.6$				

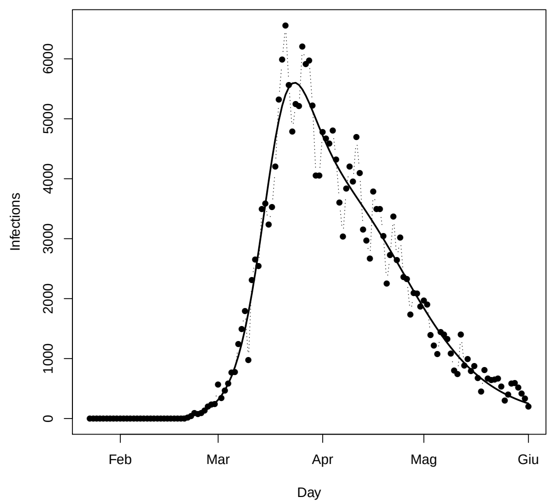

Figure 9

GGM for daily infections from COVID-19 in Italy from 22/01/20 to 01/06/20

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