ABSOLUTELY CONTINUOUS RANDOM VARIABLES

X discrete $\longrightarrow$ F_X piecewise constant

We say that X is a CONTINUOUS R.V.
if $F_X : \mathbb{R} \to [0,1]$ is continuous $(X : \Omega \to \mathbb{R})$

__Def__ $X : \Omega \to \mathbb{R}$ r.v. We say that X is ABSOLUTELY CONTINUOUS if there exists a function $f_X : \mathbb{R} \to [0, +\infty)$ s.t.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt \qquad \forall x \in \mathbb{R}$$

f_X is called the DENSITY of the r.v.

__Note__ X absolutely continuous $\Rightarrow X$ continuous
- In general f_X not continuous, just integrable.

Properties of density

1) $$\int_{-\infty}^{\infty} f_X(t)\,dt = 1$$

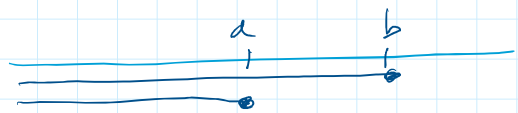
$$\overset{\shortparallel}{P}(X \leq +\infty) = P(X \in \mathbb{R}) = 1 = \mu_X(\mathbb{R})$$

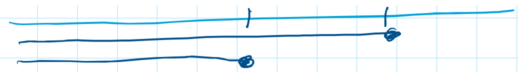
2) $$P(a < X \leq b) = F_X(b) - F_X(a) \qquad \forall a < b, \in \mathbb{R}$$

$\hookrightarrow$ for any r.v.

$F_X(b) = P(X \leq b) \qquad F_X(a) = P(X \leq a)$

$\{a < X \leq b\} = \{X \leq b\} \setminus \{X \leq a\}$





and $\{X \le a\} \subset \{X \le b\}$

$$\Rightarrow P(a < X \le b) = P(X \le b) - P(X \le a)$$

if X is absolutely continuous then

$$P(a \le X \le b) = \int_a^b f_X(t)\,dt$$

$$\hookrightarrow = \int_{-\infty}^b f_X(t)\,dt - \int_{-\infty}^a f_X(t)\,dt = \int_a^b f_X(t)\,dt$$

3) for a general set $B \in \mathcal{B}(\mathbb{R})$

$$P(X \in B) = \int_B f_X(t)\,dt$$

4) $P(X = a) = 0 \quad \forall a \in \mathbb{R}$

$$\hookrightarrow = P(\{\omega : X(\omega) = a\}) = \int_a^a f_X(t)\,dt = 0$$

<u>Note</u>

$$1 = P(X \in \mathbb{R}) = P\Big(\bigcup_{x \in \mathbb{R}} \{X = x\}\Big)$$

$\bigcup_{x \in \mathbb{R}} \{X = x\}$ disjoint union

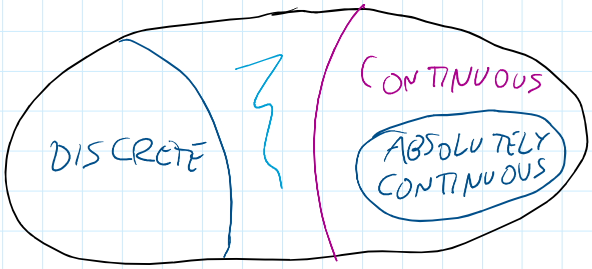
NOT COUNTABLE

$$\neq \sum_{x \in \mathbb{R}} P(X = x) = \sum_{x \in \mathbb{R}} 0 = 0$$

We see that $P\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \sum_{n \in \mathbb{N}} P(A_n)$

ONLY for COUNTABLE UNIONS of sets

NOT for uncountable unions



<u>Fact</u>

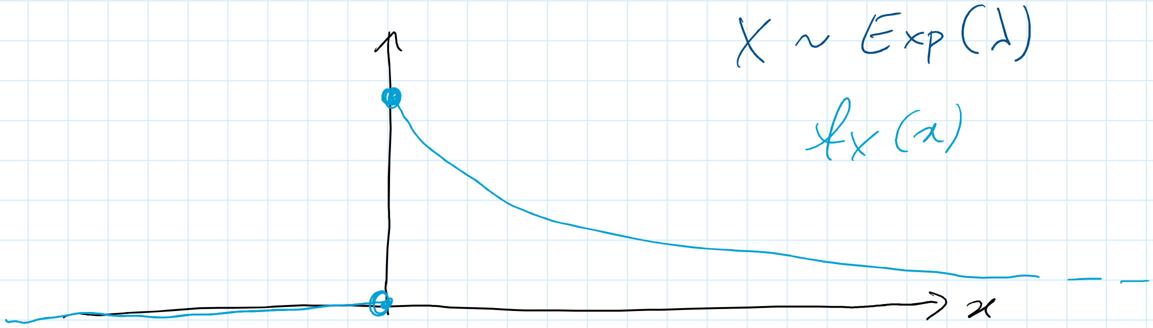
- there exist continuous r.v. which are not absolutely continuous
- there exist r.v. neither discrete nor continuous

<u>Example of abs. cont. r.v.</u>

$\lambda > 0$, X is EXPONENTIAL r.v. with parameter λ

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

$X \sim Exp(\lambda)$



Compute $\int_{-\infty}^{\infty} f_X(t)\,dt$ has to be 1

$$\int_{-\infty}^{+\infty} f_X(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{+\infty} \lambda e^{-\lambda x}\,dx$$

$$= 0 + \cancel{\lambda}\left[-\frac{1}{\cancel{\lambda}} e^{-\lambda x}\right]_{x=0}^{x \to +\infty} = -\underbrace{e^{-\lambda \infty}}_{=0} + e^{0} = 1$$

Compute distribution function

$F_X(x) = \int^{x} f_X(t)\,dt$

Compute distribution function

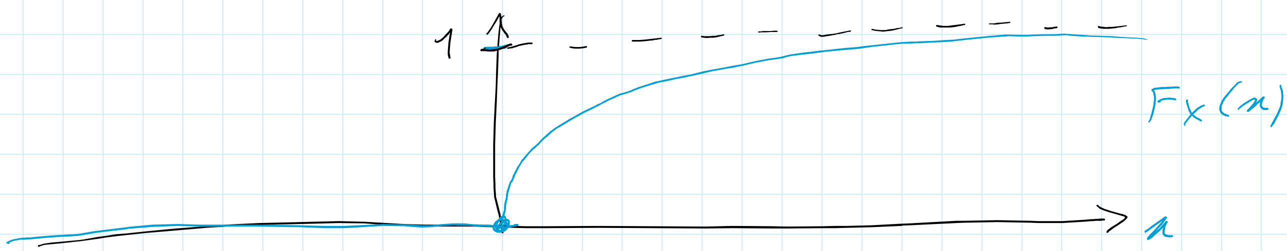
$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

- $= \int_{-\infty}^{x} 0 \, dt = 0 \quad$ if $\quad x < 0$

- if $x > 0$ $\quad F_X(x) = \int_{-\infty}^{0} 0\, dt + \int_0^x \lambda e^{-\lambda t}\, dt$

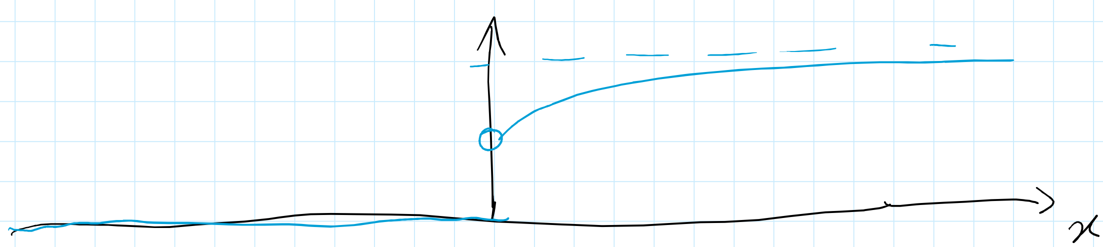
$$= 0 + \lambda \left[-\frac{1}{\lambda} e^{-\lambda t} \right]_{t=0}^{t=x} = -e^{-\lambda x} + 1$$

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$



F_X is continuous

Example of a distribution function of a r.v. neither discrete nor continuous



has a jump → NOT continuous

not piecewise constante for $x > 0$ → NOT discrete.

Remark

When X is absolutely continuous r.v., it means that

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$F_X : \mathbb{R} \to [0,1]$ is absolutely continuous function

μ_X is a measure absolutely continuous

(with respect to Lebesgue measure on $\mathbb{R}$)

EXPECTATION (MEAN) of RANDOM VARIABLES

$X : \Omega \to \mathbb{R}$ "expected value"

$X \sim Ber(p)$ $X(\omega) \in \{0,1\}$

$P(X=0) = 1-p$ $P(X=1) = p$

"the value of X is 1 p-times"

0 $(1-p)$-times

expected value $= 0 \cdot (1-p) + 1 \cdot p$

$$E[X] = 1\, P(X=1) + 0\, P(X=0)$$
$$= p$$

- Consider X s.t. $P(X=a) = p$, $P(X=b) = 1-p$

$a, b \in \mathbb{R}$, $a \neq b$

$$E[X] = a\, P(X=a) + b\, P(X=b)$$
$$= a\, p + b(1-p)$$

$p = \frac{1}{2} \longrightarrow E[X] = \frac{a+b}{2}$

Let X be a FINITE r.v with density p_X

$\exists\, N \subseteq \mathbb{R}$, N finite $P(X \in N) = 1$

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$\forall\, k \in N \qquad P(X = k) = p_X(k)$

$$E[X] = \sum_{k \in N} k\, P(X=k) = \sum_{k \in N} k\, p_X(k)$$

Example $\quad X \sim Bin(n, p)$

$k \in \{0, 1, \dots, n\} \qquad p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$E[X] = \sum_{k=0}^{n} k\, p_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$k \binom{n}{k} = k \frac{n!}{k!\,(n-k)!} = \frac{n \cdot (n-1)!}{(k-1)!\,((n-1)-(k-1))!}$$

$$k! = k \cdot (k-1)(k-2)\cdots 1 = k\,(k-1)!$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} p \cdot p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$= n p \underbrace{\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}}_{=1}$$

because $\quad (p + (1-p))^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} p^\ell (1-p)^{(n-1)-\ell}$

$$\underset{(\ell = k-1)}{=} \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$\Rightarrow E[X] = np \quad$ if $X \sim Bin(n, p)$

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if X is discrete we could define

$E[X] = \sum_{x \in N} x\, P(X = x) \qquad N$ countable

sum of series → can be finite or infinite
can be absolutely convergent or not
BETTER to consider case when
$$\sum_{x \in N} |x| \, P(X = x) \text{ is finite}$$
and thus to define expectation in such case

X discrete, $N \subseteq \mathbb{R}$ countable $P(X \in N) = 1$
$$S_+ = \sum_{\substack{x \in N \\ x \geq 0}} x \, p_X(x)$$
$$S_- = \sum_{\substack{x \in N \\ x < 0}} (-x) \, p_X(x)$$

S_+ and S_- are always defined, might be infinite
$S_+, S_- \in [0, +\infty]$

Def If at least one of S_+ and S_- are finite
then we can define
$E[X] = S_+ - S_-$

Note
$$E[X] = \begin{cases} < +\infty & \text{if } S_+ < +\infty, S_- < +\infty \\ +\infty & \text{if } S_+ = +\infty, S_- < +\infty \\ -\infty & \text{if } S_+ < +\infty, S_- = +\infty \end{cases}$$

If both S_+ and S_- are FINITE, we say that
the r.v. X is INTEGRABLE and we have
$E[X] = S_+ - S_-$ is finite

Remark With this definition

X is integrable $\Longleftrightarrow$ $|X|$ is integrable

and $E[|X|] = S_+ + S_-$

Proof if $S_+, S_- < \infty$ then $S_+ + S_-$ is finite

$$E|X| = \sum_{x \in \mathbb{R}} |x| \, p_X(x)$$

$$= \sum_{x \geq 0} x \, p_X(x) + \sum_{x < 0} (-x) \, p_X(x)$$

$$= S_+ + S_- \quad \blacksquare$$

Instead $E[X] = \sum_{x \in \mathbb{R}} x \, p_X(x)$ (if series is absolutely convergent)

$$= \sum_{x \geq 0} x \, p_X(x) + \sum_{x \leq 0} x \, p_X(x)$$

$$= \sum_{x \geq 0} x \, p_X(x) - \sum_{x < 0} (-x) \, p_X(x)$$

$$= S_+ - S_-$$

$L^1(\Omega)$ = space of integrable random variables on Ω

$X \in L^1(\Omega) \Longleftrightarrow \sum_{x \in \mathbb{R}} |x| \, p_X(x) < +\infty$

Note $Y \geq 0$ discrete

$E[Y]$ is always defined, $E[Y] \in [0, +\infty]$

Example $X \sim Geo(p)$

GEOMETRIC r.v. of parameter $p \in [0,1]$

$P(X=k) = (1-p)^k p \qquad k \in \mathbb{N}, k = 0, 1, \ldots$

$X \geq 0 \rightarrow E[X]$ exists

$$E[X] = \sum_{k=0}^{\infty} k (1-p)^k p = \frac{1}{p}$$