PROBLEMS - SET 4

Problem 1. Let $X \sim Bin(n, p)$. Compute

$$\mathrm{P}[X \text{ is odd}]$$

Problem 2. A box contains k balls numbered from 1 to k. We extract n balls from the box and let X denote the maximum number that we obtain. In both the cases with or without replacement, compute the distribution F_X.

Problem 3. Compute the characteristic function and the moment generating function of the absolutely continuous random variables with densities

i.
$$f(x) = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{2}|x|\right) & \text{if } |x| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

ii.
$$f(x) = \frac{1}{b-a} 1_{[a,b]}(x)$$

iii.
$$f(x) = \frac{1}{2} \exp[-|x-a|].$$

Problem 4. We know that for a real, discrete random X variable with density p, the characteristic function is given by

$$\varphi_X(u) = \sum_{n \in \mathbb{Z}} e^{iun} p(n).$$

Determine the densities of the discrete random variables having the following characteristic functions:

i.
$$\varphi(u) = \frac{1}{4}\left(1 + e^{iu}\right)^2$$

ii.
$$\varphi(u) = \frac{1}{2 - e^{iu}}$$

iii.
$$\varphi(u) = cos(u)$$

iv.
$$\varphi(u) = cos^2(u)$$

v.
$$\varphi(u) = \sum_{k=0}^{+\infty} a_k cos(kt)$$

where $a_k > 0$ and $\sum_k a_k = 1$.

Problem 5. Let $X \sim \mathrm{N}(1,4)$ and $Y \sim \mathrm{N}(2,1)$ be independent random variables.

(a) Set $U := X + Y$ and $V := X - 2Y$. Find μ and Σ such that $(U,V) \sim \mathrm{N}(\mu, \Sigma)$.

(b) Find a 2×2 matrix A such that defining

$$\begin{pmatrix} Z \\ W \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix},$$

Z and W are independent and have both variance 1.

Problem 6. Set $V := \{1, 2, \dots, N\}$. To each *unordered* pair $\{i, j\}$ with $i, j \in V$ and $i \neq j$ we assign a random variable $X_{ij} \sim N(0,1)$ and we assume that they are all independent. You can interpret it as follows: if the elements of V are geographical locations, $e^{-X_{ij}}$ is the time needed to travel from i to j of viceversa. We say that i and j are *far* if $X_{ij} < 0$. For $i \in V$, set

$$X_i := \sum_{j : j \neq i} X_{ij};$$

$$N_i := \text{ number of locations far from } i \ = |\{j : X_{ij} < 0\}|.$$

(a) Find the distribution of N_i.

(b) What is the joint distribution of X_1 and X_2? Are they independent?

(c) We say that i is *isolated* if it is far from all other elements of V. Let p_N be the probability that there exists at least one isolated point. Show that

$$\lim_{N \to +\infty} p_N = 0.$$