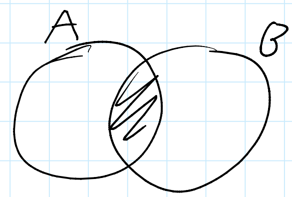
$(A_n)_{n \in \mathbb{N}} \quad A_n \in \mathcal{A}$

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right)$$

Example

A and $B \in \mathcal{A}$



Sequence $A_1 = A$, $A_2 = B$,
$A_3 = A$, $A_4 = B$, $A_5 = A, \dots$

$$A_n = \begin{cases} A & \text{if } n \text{ ODD} \\ B & \text{if } n \text{ EVEN} \end{cases}$$

What are limsup and liminf ?

- $\bigcup_{k=n}^{\infty} A_k = A \cup B$ for any n
  $\Rightarrow \limsup_{n\to\infty} A_n = A \cup B$

- $\bigcap_{k=n}^{\infty} A_k = A \cap B$ for any n
  $\Rightarrow \liminf_{n\to\infty} A_n = A \cap B$

Note $\boxed{\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n}$

Example Consider a sequence of TRIALS (independent) in which the probability of success at any n is $p_n \in [0,1]$.

Set $A_n = \{$ the n-th trial is a success $\}$

$P(A_n) = p_n$

The events A_n are independent.

the events A_n are independent.

$\limsup_n A_n = \{ \omega \in A_n \text{ infinitely often} \}$

$= \{ \text{infinitely many successes occur} \}$

- if $\sum_{n=1}^{\infty} p_n < \infty \Rightarrow P(\limsup_n A_n) = 0$
- if $\sum_{n=1}^{\infty} p_n = +\infty \Rightarrow P(\limsup_n A_n) = 1$

$\Rightarrow$ if $P(A_n) = p$ independent of n

then $\sum_{n=1}^{\infty} p = +\infty$ and thus

$P(\text{infininely many success}) = 1$

<u>Example</u> play "10" in roulette in casino

$p_n = p = \frac{1}{37}$ constant

$P(\text{infinitely many wins}) = 1$

RANDOM VARIABLES

Let E be a set.

We want to define a RANDOM VARIABLE

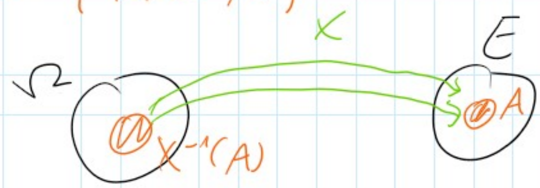
on a probability space $(\Omega, \mathcal{A}, P)$

$$X : \Omega \to E$$
$$\omega \mapsto X(\omega)$$

Given $A \subseteq E$ we can define the preimage

$$X^{-1}(A) = \{ \omega \in \Omega : X(\omega) \in A \}$$
$$\underset{\Omega}{\cap|} \qquad = \{ X \in A \}$$

$\overset{\shortparallel}{\Omega}$ $\qquad = \{X \in A\}$



Consider $E = \mathbb{R}$ $\qquad X : \Omega \to \mathbb{R}$

We want to compute probability of the event

$P($ X takes values larger than 1 $)$

$= P(\{\omega : X(\omega) \geq 1\})$

$= P(X \geq 1)$

$= P(\{X \in [1, +\infty)\})$

We have to require that such probability can be computed $\Rightarrow$ this means that the event $\{X \geq 1\}$ must belong to the σ-algebra $\mathcal{A}$.

__Def__ Given

$(\Omega, \mathcal{A}, P)$ probability space

$(E, \mathcal{E})$ measurable space ($\mathcal{E}$ is a σ-algebra)

$X : \Omega \to E$ is a RANDOM VARIABLE

if $X^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{E}$.

- X is a measurable function
- The BOREL σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ is the σ-algebra generated by either
  - the set of intervals
  - the sets $[a, +\infty)$ for $a \in \mathbb{R}$
  - the open sets

- the open sets

- Property if $\mathcal{E} = \sigma(\mathcal{C})$, i.e. $\mathcal{B}$ is the σ-algebra generated by $\mathcal{C}$, $(E, \mathcal{E})$ measurable space
  $X : \Omega \to E$ is a random variable if and only if
  $X^{-1}(A) \in \mathcal{A} \quad \forall A \in \mathcal{C}$ [meaning that it is enough to consider sets in $\mathcal{C}$, than $\mathcal{E}$]

- In particular, $X : \Omega \to \mathbb{R}$ is a random variable iff $X^{-1}([a, +\infty)) = \{X \geq a\} \in \mathcal{A}$

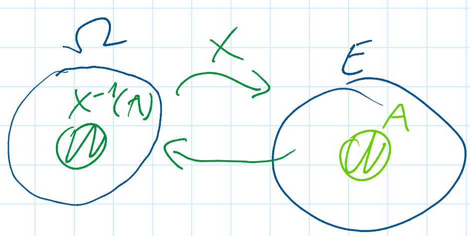
– – –

$$(\Omega, \mathcal{A}, P) \xrightarrow{X} (E, \mathcal{E})$$

We define μ_X a probability on $\mathcal{E}$

$$\mu_X(A) = P(X \in A) = P(X^{-1}(A))$$

LAW or DISTRIBUTION of the random variable X



$\mu_X : \mathcal{E} \to [0,1]$ is a probability on $(E, \mathcal{E})$.

DISCRETE RANDOM VARIABLE

DISCRETE = finite or countable

Def We say that a random variable $X : \Omega \to E$ is DISCRETE if there exists $N \in \mathcal{E}$, N at most countable such that

$$P(X \in N) = \mu_X(N) = 1$$

N is called the SUPPORT of the law μ_X.

" Means that X takes countably (or finitely) many values, with probability 1."

If X is discrete, we can define the map

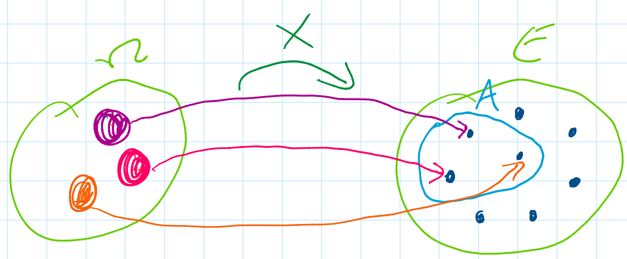
$$p_X : N (\subseteq E) \to [0,1]$$
$$x \mapsto P(X = x) = \mu_X(\{x\})$$

p_X is called the DISCRETE DENSITY of the r.v. X

<u>Thm</u> If X is an E-valued discrete r.v, then

$$\forall A \in \mathcal{E}, \quad P(X \in A) = \mu_X(A) = \sum_{x \in A} \underbrace{p_X(x)}_{= P(X = x)}$$

$$= \sum_{x \in A \cap N}$$



$\Rightarrow$ To describe discrete random variables it is enough to describe densities

<u>Lemma</u> Any function $p : E \to [0,1]$ such that there exists $N \subseteq E$ finite or countable s.t.:

1) $p(x) = 0 \quad \forall x \notin N$
2) $p(x) \geq 0 \quad \forall x \in N$
3) $\sum_{x \in E} p(x) = \sum_{x \in N} p(x) = 1$

is a possible density for a discrete random variable whose support is N.

<u>example</u> $E = \mathbb{R}$ $\qquad X : \underset{\{H,T\}}{\Omega} \to \mathbb{R}$

$N = \{0,1\}$ $\qquad X(H) = 0$

$X(T) = 1$ $\qquad N = \{0,1\}$

$p(x) = P(X = x)$

Consider $X : \Omega \to \{0,1\} \subseteq \mathbb{R}$

$P(X = 1) = p \in [0,1]$

$P(X = 0) = 1 - P(X = 1) = 1 - p$

X is called BERNOULLI random variable

$X \sim Ber(p)$

<u>Example</u>

Consider a countably infinite sequence of independent trials, each trial having a probability of success $p \in [0,1]$.

As sample space consider

$\Omega = \{ (\omega_1, \omega_2, \dots, \omega_n, \dots) : \omega_i \in \{0,1\} \}$

$\omega_i = \begin{cases} 1 & \text{means } i\text{-th trial is success} \\ 0 & \text{" } \qquad \text{failure} \end{cases}$

$P(\omega : \omega_1 = x_1, \omega_2 = x_2, \dots, \omega_n = x_n) = p^k (1-p)^{n-k}$

if there are k successes and $n-k$ failures

"in the vector $(x_1, \dots, x_n)$ there are k ONEs and $n-k$ ZEROs"

<u>Fact</u>: it is possible to define a probability P on

Fact: it is possible to define a probability P on the full set Ω by defining it on the the n-truncation of the sequence

$\boxed{\text{Examples of random variables}}$:

i) for any n, define

$X_n(\omega) = \omega_i$ → outcome of the n-th trial

$X_n : \Omega \to \{0,1\}$ is a discrete random variable

$X_n \sim Ber(p)$ → $P(X_n = 0) = 1 - p$

$P(X_n = 1) = p$

ii) $Y(\omega) = \min\{n \in N : \omega_n = 1\}$

Y is the index of the trial in which there is the first success

$Y : \Omega \to N$ is a discrete random variable

$[\omega = (0,0,1,0,1,1,\ldots) \longrightarrow Y(\omega) = 3]$

iii) $S_n(\omega) = \sum_{i=1}^{n} X_n(\omega) = \omega_1 + \omega_2 + \ldots + \omega_n$

S_n is the number of successes in the first n trials

$S_n \sim Bin(n, p)$ BINOMIAL RANDOM VARIABLE

$S_n : \Omega \to \{0, 1, \ldots, n\}$ discrete r.v.

$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$

iv) $a, b > 0$. Suppose we win a € for each success, and lose b € for each loss

$C_n(\omega) = a S_n(\omega) - b(n - S_n(\omega))$

$C_n : \Omega \to \mathbb{R}$ is discrete r.v.

C_n is the amount won after n trials.

v) We start with a sum of s euros

$$R(\omega) = \min \{ n : C_n \leq -s \}$$

is the time (i.e. the index of the trial) at which we are RUINED.