Lecture 23

giovedì 27 novembre 2025 16:34

MARTINGALES & CONDITIONAL EXPECTATION

→ MARKOV PROPERTY : $(X_n)_{n\geq 0}$

$$P(X_{n+1} = x \mid \boxed{X_n = y}, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0)$$
$$= P(X_{n+1} = x \mid \boxed{X_n = y})$$

$$P(A\mid B) = \frac{P(A\cap B)}{P(B)} \quad \text{if} \quad P(B) > 0$$

Markov property → Condition on exact position at time n

$$P(\dots \mid X_n = y)$$

$$P(X_{n+1} = x \mid X_n \geq 0, X_{n-1} = i_{n-1}, \dots)$$
$$\neq P(X_{n+1} = x \mid X_n \geq 0)$$

In general not true for a Markov chain.

$\{X_n \geq 0\}$ event carries less information than $\{X_n = y\}$

MARTINGALE ≃ a stochastic process $(X_n)_{n\geq 0}$ (DISCRETE TIME)

such that expectation of X_{n+1}, knowing "the information" of X_n is equal to X_n

CONDITIONAL EXPECTATION

Let $X : \Omega \to \mathbb{R}$ r.v. on $(\Omega, \mathcal{A}, P)$

- $E[X] \simeq$ best approximation of r.v. X with a constant.

Given a r.v. $Y : \Omega \to \mathbb{R}$, we want to define

$E[X \mid Y]$ CONDITIONAL EXPECTATION of X given Y

$E[X\mid Y] \simeq$ approximation of X as a random variable which looks like Y

$\simeq$ best approximation as a function of Y

- if Y takes 2 values then $E[X\mid Y]$ is the best

- if Y takes 2 values then $E[X|Y]$ is the best approximation of X as a r.v. taking 2 values.

Example RANDOM IMAGE. 

$X : \Omega \to \{PIXELS\}$ $\quad 10^8 = \#$ PIXELS

$X(\omega) = $ color

approximation with random image with less pixels ~ 100

is $E[X|Y] : \Omega \to \{100 \text{ PIXELS}\}$

↳ conditional expectation

- DISCRETE RANDOM VARIABLES

X and Y discrete $X, Y : \Omega \to \mathbb{R}$

finite or countable state space

What is $E[X|Y] = h(Y)$? Function of Y

$E[X | Y = y] = ?$

Consider the r.v. $X | Y = y$

What is its density ?

joint density

$$p_{X|Y=y}(x) = P(X = x \mid Y = y)$$
$$= \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

denote $p_{X|Y}(x|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$ CONDITIONAL DENSITY

Compute expectation of the r.v. $X | Y = y$

Note $\sum_x p_{X|Y}(x|y) = \dfrac{\sum_x p_{X,Y}(x,y)}{p_Y(y)} = \dfrac{p_Y(y)}{p_Y(y)} = 1$

$$E[X | Y = y] = \sum_x x \, p_{X|Y}(x|y) = h(y)$$

↳ function of y

Then $E[X|Y] = h(Y)$

- $X \mid Y = y$ is a r.v. with density

$$p_{X|Y=y}(x) = p_{X|Y}(x, y)$$

- $E[X \mid Y = y]$ is a number, is the expectation of the r.v. $X \mid Y = y$

$$E[X \mid Y = y] = h(y) = \sum_x x \, p_{X|Y}(x)$$

$h : \mathbb{R} \to \mathbb{R}$ function (on the discrete sets)

- $E[X \mid Y]$ is a random variable

$$E[X|Y] : \Omega \to \mathbb{R}$$
$$\omega \mapsto E[X|Y](\omega) = h(Y(\omega))$$

Property $E[\,E[X|Y]\,] = E[X]$ TOWER PROPERTY

Proof $E[\,E[X|Y]\,] = E[h(Y)] = \sum_y h(y) \, p_Y(y)$

$$= \sum_y \left(\sum_x x \, p_{X|Y}(x|y) \right) p_Y(y)$$

$$= \sum_y \sum_x x \, \frac{p_{X,Y}(x,y)}{\cancel{p_Y(y)}} \cdot \cancel{p_Y(y)}$$

$$= \sum_x x \underbrace{\sum_y p_{X,Y}(x,y)}_{= p_X(x)} = \sum_x x \, p_X(x) = E[X]$$

- ABSOLUTELY CONTINUOUS R.V.

(X, Y) is absolutely continuous random vector

- $X \mid Y = y$ is a absolutely continuous r.v. with density

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{if } f_Y(y) = 0 \end{cases}$$

CONDITIONAL DENSITY

- $E[X \mid Y = y] = \int^{+\infty} x \, f_{X|Y}(x|y) \, dx = h(y)$

- $E[X \mid Y=y] = \int_{-\infty}^{+\infty} x \, f_{X|Y}(x|y)\, dx = h(y)$

  $h: \mathbb{R} \to \mathbb{R}$

- $E[X|Y] : \Omega \to \mathbb{R}$ (random variable)

  $\omega \longmapsto E[X|Y](\omega) = h(Y(\omega))$

Exercise

$X \perp\!\!\!\perp Y \qquad X, Y \sim Bin(n,p)$

$E[X \mid X+Y] = ?$

$X \mid X+Y=m$ is r.v. with density

$$p_{X|X+Y}(k|m) = P(X=k \mid X+Y=m)$$

$$= \frac{P(X=k,\ X+Y=m)}{P(X+Y=m)}$$

$X, Y \in \{0,1,\dots,n\} \qquad m \in \{0,\dots,2n\} \qquad 0 \le k \le m$

$0 \le k \le \min\{n,m\} \qquad 0 \le k \le n$

Note that $X+Y \sim Bin(2n,p)$

$$p_{X|X+Y}(k|m) = \frac{P(X=k,\ Y=m-k)}{P(X+Y=m)}$$

$$\overset{\text{INDEPENDENT}}{=} \frac{P(X=k)\,P(Y=m-k)}{P(X+Y=m)}$$

$$= \frac{P(Bin(n,p)=k)\ P(Bin(n,p)=m-k)}{P(Bin(2n,p)=m)}$$

$$= \frac{\binom{n}{k} \cancel{p^k} \cancel{(1-p)^{n-k}} \binom{n}{m-k} \cancel{p^{m-k}} \cancel{(1-p)^{n-(m-k)}}}{\binom{2n}{m} \cancel{p^m} \cancel{(1-p)^{2n-m}}}$$

$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}} \quad \to \text{density of HYPERGEOMETRIC r.v.}$$

$$\binom{2n}{m}$$

$$Z \sim Hp(\underset{\#\text{ draws}}{m}, \underset{\#\text{ balls}}{2n}, \underset{\#\text{ RED balls}}{n}) \rightsquigarrow P(Z=k) =$$

Z = # RED balls in m draws without replacement in an urn with $2n$ balls and n RED balls

$$E[Z] = m \cdot \frac{n}{2n} = \frac{m}{2}$$

$$\to X \mid X+Y=m \sim Hp(m, 2n, n)$$

$$E[X \mid X+Y=m] = \frac{m}{2} = h(m)$$

$$\Rightarrow E[X \mid X+Y] = h(X+Y) = \frac{X+Y}{2}$$

Exercise

(X,Y) ABS. CONT. $\quad f_{X,Y}(x,y) = \dfrac{e^{-\frac{x}{y}} e^{-y}}{y} \mathbb{1}_{\{x>0, y>0\}}$

$E[X \mid Y] = ?$

- $X \mid Y=y$ is r.v. with density $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$y>0$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_0^{+\infty} \frac{e^{-\frac{x}{y}} e^{-y}}{y}\,dx$$

$$= \frac{e^{-y}}{y} \int_0^{+\infty} e^{-\frac{1}{y}x}\,dx = \frac{e^{-y}}{y}\left[\frac{1}{-\frac{1}{y}} e^{-\frac{1}{y}x}\right]_{x=0}^{x\to+\infty}$$

$$= \frac{e^{-y}}{y}\left[-y\underbrace{e^{-\infty}}_{=0} + y\right] = e^{-y}$$

$$f_Y(y) = e^{-y}\mathbb{1}_{\{y>0\}} \Rightarrow Y \sim Exp(1)$$

$$f_{X|Y}(x|y) = \frac{\frac{e^{-\frac{x}{y}} e^{-y}}{y}}{e^{-y}} = \frac{1}{y} e^{-\frac{x}{y}} \mathbb{1}_{\{x>0\}}$$

(if $x,y>0$)

$$\Rightarrow X \mid Y=y \sim Exp\left(\frac{1}{y}\right)$$

$$\left[Z \sim Exp(\lambda) \Rightarrow f_Z(z) = \lambda e^{-\lambda z} \mathbb{1}_{\{z>0\}}\right]$$

$$\left[Z \sim Exp(\lambda) \Rightarrow f_Z(z) = \lambda e^{-\lambda z} \mathbb{1}_{\{z>0\}} \right]$$

$$E[X \mid Y = y] = \frac{1}{\frac{1}{y}} = y = h(y)$$

$$\Rightarrow E[X \mid Y] = h(Y) = Y$$

- CONDITIONAL EXPECTATION $E[X|Y]$

$X, Y : \Omega \to \mathbb{R}$ r.v. $\to$ general setting

$(\Omega, \mathcal{A}, P)$ probability space $X, Y \in L^1(\Omega)$

$\hookrightarrow E|X| < \infty$

$E[X|Y] = h(Y)$

$\hookrightarrow$ best approximation of X as a function of Y

How to define $E[X/Y]$ in general?

Consider $\sigma(Y)$ σ-ALGEBRA GENERATED by Y

$Y : \Omega \to \mathbb{R}$ s.t. the event $Y^{-1}(B) = \{Y \in B\} \in \mathcal{A}$

for any $B \in \mathcal{B}(\mathbb{R})$ (enough to consider B intervals)

$\sigma(Y) =$ σ-algebra generated by (minimal σ-algebra which contains)

the set $\{ A : \exists B \text{ s.t. } A = \{X \in B\} \}$

$\sigma(Y)$ minimal σ-algebra for which Y is measurable

(i.e. a random variable)

$\sigma(Y)$ contains all the information carried by Y

$E[cX] = c\,E[X] \quad \forall c \in \mathbb{R}$

Let $A \in \sigma(Y)$

$$E[X \mathbb{1}_A] \underset{\text{TOWER PROPERTY}}{=} E[E[X \mathbb{1}_A | Y]] = E[\mathbb{1}_A E[X|Y]]$$

$A \in \sigma(Y)$

expect to move outside of conditional expectation

Def $X, Y \in L^1(\Omega)$. The CONDITIONAL EXPECTATION of

X given Y, denoted $E[X|Y]$, is a function of Y
$E[X|Y] = h(Y)$ having the property that

$$E[X \mathbb{1}_A] = E[h(Y) \mathbb{1}_A] \quad \forall A \in \sigma(Y)$$

<u>Property</u> $E[X|Y]$ is $\sigma(Y)$ measurable (= function of Y)
$E[X|Y]$ exists and is unique