CHERNOFF BOUNDS

$X_1, \dots, X_n$ i.i.d r.v.s $\quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad E[X_1] = \mu$

assume $m(t) = E[e^{tX_1}] < \infty$ for $t \in (-\varepsilon, \varepsilon)$. Then

$P(\bar{X}_n \geq \mu + \varepsilon) \leq e^{-n g(\bar{t})}$ UPPER TAIL ESTIMATE

$\exists\, \bar{t}$ s.t. $g(\bar{t}) = \bar{t}(\mu+\varepsilon) - \log(m(\bar{t})) > 0$

$P(\bar{X}_n \leq \mu - \varepsilon) \leq e^{-n h(\bar{t})}$ LOWER TAIL ESTIMATE

$\exists\, \bar{t}$ s.t. $h(\bar{t}) = -\bar{t}(\mu - \varepsilon) - \log(m(-\bar{t})) > 0$

Example 1 NORMAL R.V.s

$X_1, \dots, X_n \sim N(0,1)$ independent

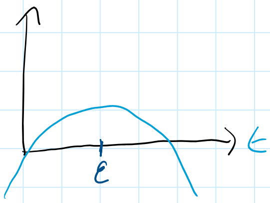
$\mu = E[X_1] = 0 \qquad m(t) = e^{\frac{t^2}{2}} \qquad t \in \mathbb{R}$

Upper tail estimate:

$P(\bar{X}_n \geq \mu + \varepsilon) \leq e^{-n g(\bar{t})}$

$$g(t) = t(\mu + \varepsilon) - \log(m(t))$$
$$= t\varepsilon - \frac{t^2}{2}$$

$g(t)$ has a maximum over $\mathbb{R}$ (parabola)



$g'(t) = \varepsilon - t = 0$

$\Rightarrow \bar{t} = \varepsilon$

$g(\bar{t}) = \varepsilon^2 - \frac{\varepsilon^2}{2} = \frac{\varepsilon^2}{2}$

Thus

$$P(\bar{X}_n \geq \varepsilon) \leq e^{-n\frac{\varepsilon^2}{2}}$$

Similarly, $P(\bar{X}_n \leq -\varepsilon) \leq e^{-n\frac{\varepsilon^2}{2}}$

Example 2

$Z_1, \dots, Z_n$ indep. $N(0,1)$

$X_i = Z_i^2$ CHI-SQUARE DISTRIBUTION

$Z_1, \dots, Z_n$ indep. $N(0,1)$

$X_i = Z_i^2$ CHI-SQUARE DISTRIBUTION (with 1 degree of freedom)

$X_i \sim \Gamma\left(\frac{1}{2}, 1\right)$ GAMMA

$$E[X_i] = E[Z_i^2] = 1$$

$$m(t) = m_{X_1}(t) = E\left[e^{tX_1}\right] = E\left[e^{tZ_1^2}\right]$$

$$= \int_{-\infty}^{+\infty} e^{tz^2} f_Z(z)\,dz = \int_{-\infty}^{+\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \underbrace{e^{-\frac{z^2}{2}(1-2t)}}_{=e^{-\frac{s^2}{2}}}\,dz$$

change of variable $s^2 = z^2(1-2t)$

$s = z\sqrt{1-2t}$ $\qquad ds = dz\,\sqrt{1-2t}$

$\int_{-\infty}^{+\infty} \longrightarrow \int_{-\infty}^{+\infty}$ as $\sqrt{1-2t} \geq 0$, well-defined

if $1-2t > 0 \iff t < \frac{1}{2}$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \frac{1}{\sqrt{1-2t}}\,ds = \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}}_{\text{density of } N(0,1)}\,ds$$

$$= \frac{1}{\sqrt{1-2t}} \cdot 1$$

$$m(t) = \begin{cases} \frac{1}{\sqrt{1-2t}} & \text{if } t < \frac{1}{2} \\ +\infty & \text{if } t \geq \frac{1}{2} \end{cases}$$

$$P(\bar{X}_n \geq 1 + \varepsilon) \leq e^{-g(t)} \qquad \mu = 1$$

$$g(t) = t(1+\varepsilon) - \log(m(t))$$

$$= t(1+\varepsilon) - \log\left(\frac{1}{\sqrt{1-2t}}\right)$$

$$= t(1+\varepsilon) - \log\left((1-2t)^{-\frac{1}{2}}\right)$$

$$= t(1+\varepsilon) + \frac{1}{2}\log(1-2t)$$

$g'(\bar{t}) = 0 \qquad g'(t) = 1 + \varepsilon + \frac{(-2)}{2(1-2t)} = 0$

$g'(\bar{\epsilon}) = 0 \qquad g'(\epsilon) = 1+\varepsilon + \dfrac{(-2)}{2(1-2\epsilon)} = 0$

$1+\varepsilon = \dfrac{1}{1-2\epsilon} \qquad 1-2\epsilon = \dfrac{1}{1+\varepsilon} \qquad 2\epsilon = 1 - \dfrac{1}{1+\varepsilon}$

$\bar{\epsilon} = \dfrac{\varepsilon}{2(1+\varepsilon)} \qquad \left(\bar{\epsilon} < \frac{1}{2} \text{ if } \varepsilon < 1\right)$

$$g(\bar{\epsilon}) = g\left(\frac{\varepsilon}{2(1+\varepsilon)}\right) = \frac{\varepsilon(1+\varepsilon)}{2(1+\varepsilon)} + \frac{1}{2}\log\left(1 - \frac{\varepsilon}{1+\varepsilon}\right)$$

$$= \frac{\varepsilon}{2} + \frac{1}{2}\log\left(\frac{1}{1+\varepsilon}\right) =$$

$$= \frac{1}{2}\left(\varepsilon - \log(1+\varepsilon)\right)$$

We have the inequality

(☆) $\log(1+x) \leq x - \dfrac{x^2}{2(1+x)} \qquad \forall x > 0$

Using (☆), $g(\bar{\epsilon}) \geq \dfrac{1}{2} \dfrac{\varepsilon^2}{2(1+\varepsilon)} = \dfrac{\varepsilon^2}{4(1+\varepsilon)}$

$\Rightarrow P(\bar{X}_n \geq 1+\varepsilon) \leq e^{-n g(\bar{\epsilon})} \leq \exp\left(-n \dfrac{\varepsilon^2}{4(1+\varepsilon)}\right)$

Similarly, we find the estimate for the lower tail

$P(\bar{X}_n \leq 1-\varepsilon) \leq e^{-n\frac{\varepsilon^2}{4}}$

- To prove (☆),

  denote $f(x) = \log(1+x)$, $\quad g(x) = x - \dfrac{x^2}{2(1+x)}$

  Note $f(0) = g(0) = 0$,

  $f(x) = \int_0^x f'(t)\,dt \qquad g(x) = \int_0^x g'(t)\,dt$

  if we show that $f'(x) \leq g'(x) \quad \forall x$

  then $f(x) \leq g(x) \quad \forall x$

BOUND ON DISTRIBUTION of $N(0,1)$

$P(\bar{X}_n \geq \varepsilon) \leq e^{-n\frac{\varepsilon^2}{2}} \qquad \forall n$

$n = 1 \qquad P(Z \geq \varepsilon) \leq e^{-\frac{n\varepsilon^2}{2}}$

$n=1 \qquad P(Z \geq \varepsilon) \leq e^{-\frac{n\varepsilon^2}{2}}$

$Z \sim N(0,1) \qquad f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ density

$f'(z) = -z f(z)$ ORDINARY DIFFERENTIAL EQUATION

$$P(Z > x) = \int_x^{+\infty} f(z)\,dz = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= \int_x^{+\infty} \frac{1}{z} \cdot z \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{=f(z)}\,dz$$

$z > x \longrightarrow \frac{1}{z} < \frac{1}{x} \qquad z \cdot f(z) = -f'(z)$

$$\leq \int_x^{+\infty} \frac{1}{x} \left(-f'(z)\right) dz$$

$$= \frac{1}{x} \left[-f(z)\right]_x^{+\infty} = \frac{1}{x}\left[-\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right]_{z=x}^{z\to+\infty}$$

$$= \frac{1}{x}\left(0 + \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right)$$

Then $P(Z > x) \leq \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Remark

$\log m(t) = \log\left(E\left[e^{tX}\right]\right) \neq E[tX]$

BUT $\log E\left[e^{tX}\right] \geq E[tX]$

This is a consequence of JENSEN's inequality

$E[\varphi(X)] \geq \varphi(E[X]) \qquad \forall \varphi : \mathbb{R} \to \mathbb{R}$ CONVEX

- for $\varphi = \exp$:

$\log E\left[e^{tX}\right] \geq \log e^{tE[X]} = tE[X]$

Example 3 BERNOULLI

$X_1, \dots, X_n$ i.i.d $\qquad X_1 \sim Ber(p)$

$E[X_1] = p \qquad m(t) = 1 - p + p e^t$

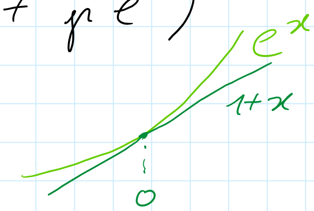
$P(\overline{X}_n \geq p + \varepsilon) \leq e^{-n g(\varepsilon)}$

x

$$P(\bar{X}_n \geq \mu + \varepsilon) \leq e^{-n g(\epsilon)}$$

$$g(\epsilon) = \epsilon(\mu+\varepsilon) - \log\big(\overbrace{1-\mu+\mu e^{\epsilon}}^{x}\big)$$

Use inequality $e^x \geq 1+x \quad \forall x$



$$\log(1+\mu e^{\epsilon}-\mu) \leq \log(\exp(\mu e^{\epsilon}-\mu)) = \mu e^{\epsilon}-\mu$$

$$g(\epsilon) \geq \epsilon(\mu+\varepsilon) - \mu e^{\epsilon} + \mu = \tilde{g}(\epsilon)$$

$$\tilde{g}'(\epsilon) = \mu+\varepsilon - \mu e^{\epsilon} = 0$$

$$\mu e^{\epsilon} = \mu + \varepsilon \qquad e^{\epsilon} = \frac{\mu+\varepsilon}{\mu}$$

Call $\varepsilon = \delta\mu$ $\qquad \bar{\epsilon} = \log(1+\delta)$

$$\tilde{g}(\bar{\epsilon}) = \log(1+\delta)\,\mu(1+\delta) - \mu(1+\delta) + \mu$$
$$= \mu(1+\delta)\log(1+\delta) - \mu\delta$$

Use inequality $\log(1+x) \geq \dfrac{2x}{2+x} \quad \forall x$

Multiply by $(1+\delta)$ $\qquad x = \delta$

$$\tilde{g}(\bar{\epsilon}) \geq \mu \frac{2\delta}{2+\delta}(1+\delta) - \mu\delta$$

$$= \frac{\cancel{2\delta\mu} + 2\delta^2\mu - \cancel{2\mu\delta} - \delta^2\mu}{2+\delta}$$

$$= \frac{\mu\delta^2}{2+\delta}$$

Thus $P(\bar{X}_n \geq \mu(1+\delta)) \leq \exp\left(-\mu n \dfrac{\delta^2}{2+\delta}\right)$

For the lower tail estimate, we get the bound

$$P(\bar{X}_n \leq \mu(1-\delta)) \leq \exp\left(-\mu n \frac{\delta^2}{2}\right)$$

CONVERGENCE in DISTRIBUTION

$(X_n)_n$ sequence of r.v. $X_n, X : \Omega \to \mathbb{R}$

$X_n \xrightarrow{D} X$ means that

$\lim_{n\to\infty} E[f(X_n)] = E[f(X)] \quad \forall f : \mathbb{R} \to \mathbb{R}$ BOUNDED and CONTINUOUS

Theorem

The following are equivalent:

1) $X_n \xrightarrow{D} X$

2) $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ for any x where F_X is continuous

(F_{X_n}, F_X distribution functions)

3) $\lim_{n\to\infty} \varphi_{X_n}(u) = \varphi_X(u) \quad \forall u \in \mathbb{R}$

(φ_{X_n}, φ_X characteristic functions)

$\varphi_{X_n} = E[e^{iuX_n}]$ $\quad F_{X_n}(x) = P(X_n \le x)$

Other properties

i) if $X = c$ constant then

$X_n \xrightarrow{D} c \iff X_n \xrightarrow{P} c$

Proof $\varepsilon > 0$ We suppose $X_n \xrightarrow{D} c$ and show $X_n \xrightarrow{P} c$

$P(|X_n - c| \ge \varepsilon) \xrightarrow{?} 0$

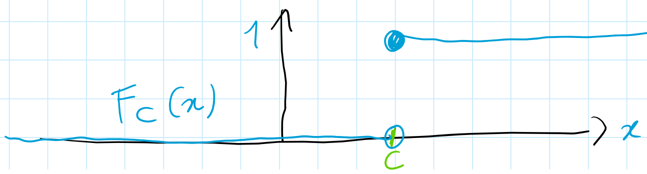
$= P(X_n - c \ge \varepsilon) + P(X_n - c \le -\varepsilon)$

$= P(X_n \ge c + \varepsilon) + P(X_n \le c - \varepsilon)$

$= 1 - F_{X_n}(c+\varepsilon) + F_{X_n}(c - \varepsilon)$

What is the distribution function of $X = c$

$F_c(x) = P(c \le x) = \begin{cases} 0 & \text{if } c < x \\ 1 & \text{if } c \ge x \end{cases}$



$F_c(x)$ is continuous at any $x \neq c$
then by convergence in distribution
$$\lim_{n\to\infty} F_{X_n}(x) = F_c(x) \quad \forall x \neq c$$
Thus $$\lim_{n\to\infty} F_n(c+\varepsilon) = F_c(c+\varepsilon) = 1$$
$$\lim_{n\to\infty} F_n(c-\varepsilon) = F_c(c-\varepsilon) = 0$$
then $$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 1 - 1 + 0 = 0$$
The opposite implication: $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$
is always true. ☒

Remark 2

Convergence $X_n \to X$ in distribution
is defined as WEAK CONVERGENCE of the
distributions of the random variables
$$\mu_{X_n} \xrightarrow{w} \mu_X \qquad \mu_X(A) = P(X \in A)$$
means that $\int f \, d\mu_{X_n} \xrightarrow{n} \int f \, d\mu_X \quad \forall f$ bounded and continuous
$$\underset{\displaystyle E[f(X_n)]}{\parallel} \to \underset{\displaystyle E[f(X)]}{\parallel}$$

CENTRAL LIMIT THEOREM

$(X_n)_n$ i.i.d. r.v.s $\qquad \mu = E[X_1]$
$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \overline{X}_n \to \mu \quad \text{LAW LARGE NUMBERS}$$
Consider $\overline{X}_n - \mu$ is a random variable
What is its distribution? Does it converge to something? as $n \to \infty$

Consider $Z_n = \sqrt{n}\,\underline{\overline{X}_n - \mu}$

as $n \to \infty$

Consider $Z_n = \sqrt{n}\,\frac{\overline{X}_n - \mu}{\sigma}$

$(X_n)_n$ are square integrable with variance $\mathrm{Var}(X_1) = \sigma^2$

CLT : $Z_n \xrightarrow[n \to \infty]{D} Z$, $Z \sim N(0,1)$

Convergence in distribution

Proof uses characteristic functions