DATA SCIENCE    **Stochastic Methods**

January 22, 2025

**Problem 1.** [12] Let $X_1, X_2, \ldots, X_n$ be n independent random variables, each uniformly distributed on $(0,1)$. Define $Z = \min\{X_1, X_2, \ldots, X_n\}$ and $W = \max\{X_1, X_2, \ldots, X_n\}$.

  (i) Compute the distribution functions of the random variables $Z$ and $W$;

  (ii) Compute $E[Z]$;

  (iii) Compute $E[W]$;

  (iv) Compute $P[\max\{X_1, X_2\} < \min\{X_3, X_4\}]$.


**Problem 2.** [12] Let $\{A_n, n \geq 1\}$ be a countable partition of events in a probability space $\Omega$, i.e. their union is equal to $\Omega$ and they are mutually disjoint. Define for each $n$ the random variables $X_n = \mathbb{I}_{A_n}$, $Y_n = \mathbb{I}_{A_n^c}$, $Z_n = \sum_{i=1}^n X_i$ and $W_n = \sum_{i=1}^n Y_i$.

  (i) Prove that $Y_n = 1 - X_n$ and $W_n = n - Z_n$;

  (ii) Compute the discrete densities of $X_n$, $Y_n$, $Z_n$ and $W_n$ and their expectations;

  (iii) Are $X_2$ and $Y_1$ independent? Compute $E[X_2|Y_1]$;

  (iv) Compute the almost sure convergence of $X_n$, $Y_n$, $Z_n$ and $W_n$ as $n \to \infty$.


**Problem 3.** [12] Five balls are distributed between two boxes, labeled A and B. In each period, one of the boxes is selected at random. If the selected box is empty, all the balls are moved into it. If the selected box contains balls, one ball is drawn from it. With probability $1/3$ the ball is placed back into the same box, and with probability $2/3$ it is placed into the other box. Let $X_n$ denote the number of balls in box A after $n$ periods.

  (i) Determine the state space $S$ and the transition probability matrix;

  (ii) Classify the states (communication, periodicity and recurrence);

  (iii) Does the Markov Chain admit a unique invariant distribution? is this distribution reversible?

  (iv) In the long run, what fraction of the time is box A empty?

  (v) Compute the invariant distribution.