Lecture 9
giovedì 23 ottobre 2025 16:25

VARIANCE

$X : \Omega \to \mathbb{R}$ r.v. $\quad X \in L^2(\Omega)$

$$\mathrm{Var}[X] = E[(X - E[X])^2] \qquad \text{VARIANCE of } X$$

$\mathrm{Var}[X] \geq 0$

NOT RANDOM

$$\mathrm{Var}[X] = E[X^2 + (E[X])^2 - 2X\,E[X]]$$
$$= E[X^2] + (E[X])^2 - 2E[X] \cdot E[X]$$
$$= E[X^2] + (E[X])^2 - 2(E[X])^2$$
$$= \underbrace{E[X^2]}_{< +\infty \text{ if } X \in L^2(\Omega)} - (E[X])^2$$

- DISCRETE CASE:

$$E[X^2] = \sum_{x \in \mathbb{R}} x^2 p_X(x)$$

$$\mathrm{Var}(X) = \sum_{x \in \mathbb{R}} x^2 p_X(x) - \left(\sum_x x\, p_X(x)\right)^2$$

COVARIANCE of two random variables

$X, Y \in L^2$

$$\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY - X\,E[Y] - Y\,E[X] + E[X]E[Y]]$$
$$= E[XY] - E[Y]E[X] - \cancel{E[X]E[Y]} + \cancel{E[X]E[Y]}$$
$$= \underbrace{E[XY]}_{< +\infty} - E[X]E[Y]$$

Lemma if $X, Y \in L^2(\Omega)$ then $X \cdot Y \in L^1(\Omega)$

Proof Use inequality $xy \leq \dfrac{x^2 + y^2}{2} \quad \forall\, x, y \in \mathbb{R}$

(follows from $(x-y)^2 \geq 0$)

$\left(\text{follows from } (x-y)^2 \geq 0 \right)$

$$E[|XY|] = E[|X||Y|] \underset{\text{MONOTONICITY}}{\leq} E\left[\frac{X^2}{2} + \frac{Y^2}{2}\right]$$

LINEARITY

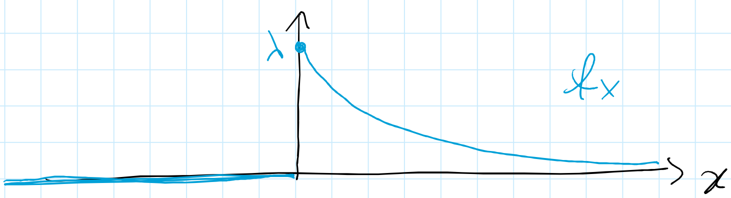
$$\leq \frac{1}{2} E[X^2] + \frac{1}{2} E[Y^2] < +\infty$$

$\Rightarrow XY \in L^1(\Omega)$

Example $\quad X \sim Exp(\lambda) \quad \lambda > 0$

X absolutely continuous r.v with density

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$



$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{0} x \cdot 0 \, dx + \int_0^{+\infty} x \lambda e^{-\lambda x} dx$$

INTEGRATION BY PARTS

$$\int_a^b f' g \, dx = [f g]_a^b - \int_a^b f g' \, dx$$

$$\frac{d}{dx} x = 1 \qquad \int e^{-\lambda x} dx = \frac{1}{-\lambda} e^{-\lambda x} \rightarrow \text{PRIMITIVE}$$

$$= \left[\lambda x \left(-\frac{1}{\lambda} e^{-\lambda x} \right) \right]_{x=0}^{x \to +\infty} - \int_0^{+\infty} \cancel{\lambda}\, 1 \cdot \left(-\frac{1}{\cancel{\lambda}} e^{-\lambda x} \right) dx$$

$$= \underbrace{- \lim_{x \to +\infty} \frac{x}{e^{\lambda x}}}_{=0} + 0 + \int_0^{+\infty} e^{-\lambda x} dx$$

$$= \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_{x=0}^{x \to +\infty} = \underbrace{-\frac{1}{\lambda} e^{-\infty}}_{=0} + \frac{1}{\lambda} e^0 = \frac{1}{\lambda}$$

$E[X] = \frac{1}{\lambda}$

$$\underbrace{\qquad}_{=0}$$

$$E[X] = \frac{1}{\lambda}$$

- $E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = \int_0^{+\infty} \underbrace{x^2}_{\frac{d}{dx}(x^2)=2x} \lambda e^{-\lambda x} dx$ $\quad \int \lambda e^{-\lambda x} = -e^{-\lambda x}$

$= $ INTEGRATION BY PARTS

$$= \left[-x^2 e^{-\lambda x}\right]_{x=0}^{x\to+\infty} - \int_0^{+\infty} 2x\left(-e^{-\lambda x}\right) dx$$

$$= -\underbrace{\lim_{x\to+\infty} \frac{x^2}{e^{\lambda x}}}_{=0} + 0 + 2\int_0^{+\infty} \underbrace{x}_{\frac{d}{dx}=1} \underbrace{e^{-\lambda x}}_{\int = -\frac{1}{\lambda}e^{-\lambda x}} dx$$

BY PARTS

$$= \underbrace{\left[-2x e^{-\lambda x}\right]_0^{+\infty}}_{=0} - 2\int_0^{+\infty} 1\cdot\left(-\frac{1}{\lambda} e^{-\lambda x}\right) dx$$

$$= \frac{2}{\lambda}\int_0^{+\infty} e^{-\lambda x} dx = \frac{2}{\lambda}\left[-\frac{1}{\lambda} e^{-\lambda x}\right]_{x=0}^{x\to+\infty}$$

$$= \frac{2}{\lambda^2}\left(-\underbrace{e^{-\infty}}_{=0} + \underbrace{e^0}_{=1}\right) = \frac{2}{\lambda^2}$$

$$\Rightarrow E[X^2] = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$Var(X) = \frac{1}{\lambda^2}$$

<u>Exercise</u> $\quad X \sim Exp(\lambda)$

$Y = \min(X, 1) \qquad (Y(\omega) = \min(X(\omega), 1))$

note $X \geq 0 \quad , 1 \geq 0 \quad \Rightarrow Y \geq 0$

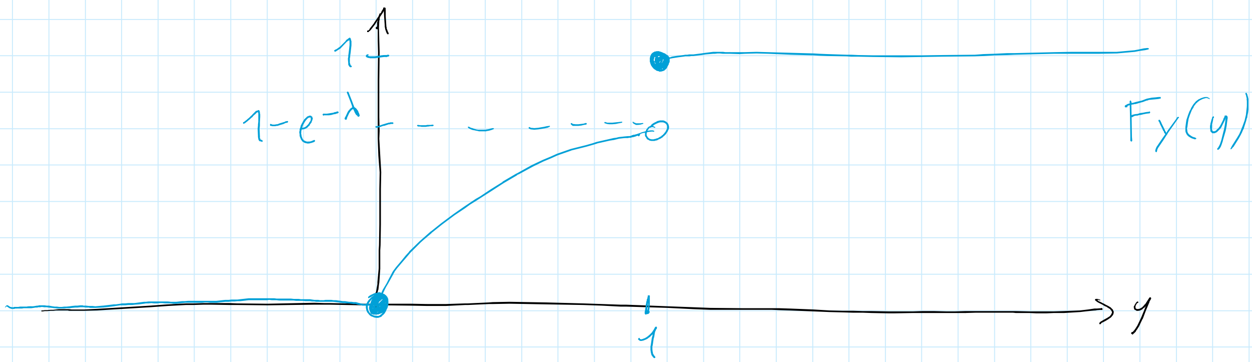
Compute the distribution function of Y

$F_Y(y) = P(Y \leq y) \qquad \forall y \in \mathbb{R}$

Recall $F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}$

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ P(\min(X,1) \leq y) = P(X \leq y) = 1 - e^{-\lambda y} \end{cases}$$

$$F_Y(y) = \begin{cases} P(\min(X,1) \leq y) = P(X \leq y) = 1 - e^{-\lambda y} & \text{if } 0 \leq y < 1 \\ \{\omega : \min(X(\omega),1) \leq y\} = \{\omega : X(\omega) \leq y\} & \\ P(\min(X,1) \leq y) = 1 & \text{if } y \geq 1 \\ \{\omega : \min(X(\omega),1) \leq y\} = \{\omega : 1 \leq y\} = \Omega & \end{cases}$$



F_Y has a jump in 1 $\Rightarrow$ F_Y not continuous

$\Rightarrow$ Y not absolutely continuous

$\Rightarrow$ Need another method to compute expectation

<u>Proposition</u> $X : \Omega \to \mathbb{R}$ r.v (general)

i) if $X \geq 0$ then

$$E[X] = \int_0^{+\infty} (1 - F_X(x))\,dx = \int_0^{+\infty} P(X > x)\,dx$$

$[\ F_X(x) = P(X \leq x) \qquad (X \leq x)^c = (X > x)\ ]$

ii) if $X \in L^1(\Omega)$ then

$$E[X] = -\int_{-\infty}^{0} F_X(x)\,dx + \int_0^{+\infty} (1 - F_X(x))\,dx$$

<u>Proof</u>

of i) in case X is absolutely continuous

$X \geq 0 \qquad E[X] = \int_0^{+\infty} x\, f_X(x)\,dx$

recall $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$

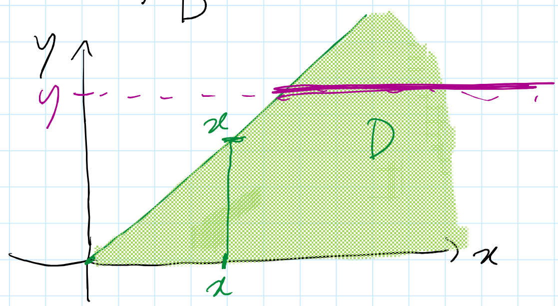
write $x = \int_0^x 1\,dy$

$E[X] = \int_0^{+\infty}\left(\int_0^x 1\,dy\right) f_X(x)\,d$

write $x = \int_0^x 1 \, dy$

$$E[X] = \int_0^{+\infty} \left(\int_0^x 1 \cdot dy \right) f_X(x) \, dx$$ DOUBLE INTEGRAL

$$= \iint_D f_X(x) \, dx \, dy$$ $\nearrow$ iteration formula



projection of D on x is $[0,+\infty)$
and for x fixed
$0 \le y \le x$

→ projection of D on y is $[0, +\infty)$
→ for y fixed, x belongs to $[y, +\infty)$
from iteration formula of double integral

$$\iint_D f_X(x) \, dx \, dy = \int_0^{+\infty} \left(\int_y^{+\infty} f_X(x) \, dx \right) dy$$

$$= \int_0^{+\infty} P(X \in [y, +\infty)) \, dy$$

$$= \int_0^{+\infty} P(X \geq y) \, dy \qquad \boxtimes$$

Back to the exercise, compute $(Y \geq 0)$

$$E[Y] = \int_0^{+\infty} (1 - F_Y(y)) \, dy$$

$$= \int_0^1 \left(1 - (1 - e^{-\lambda y})\right) dy + \int_1^{+\infty} (1 - 1) \, dy$$

$$= \int_0^1 e^{-\lambda y} \, dy = \left[-\frac{1}{\lambda} e^{-\lambda y} \right]_{y=0}^{y=1}$$

$$= -\frac{1}{\lambda} \left(e^{-\lambda} - e^0 \right) = \frac{1}{\lambda} \left(1 - e^{-\lambda} \right)$$

RANDOM VECTORS

$X : \Omega \to E$ random variable, $(E, \mathcal{E})$ measurable space

$E = \mathbb{R}^d \rightarrow$ RANDOM VECTOR

Def A RANDOM VECTOR is a random variable taking values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

- $\mathcal{B}(\mathbb{R}^d)$ is the BOREL σ-algebra of $\mathbb{R}^d$
  $\rightarrow$ GENERATED by products of interval
  $\mathcal{C} = \{ [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_d, b_d] : a_i, b_i \in [-\infty, +\infty] \}$

$\rightarrow$ random vector is such that $X : \Omega \rightarrow \mathbb{R}^d$
$(\Omega, \mathcal{A}, P)$ probability space
$X^{\leftarrow}([a_1, b_1] \times \dots \times [a_d, b_d]) \in \mathcal{A}$

$$X : \Omega \rightarrow \mathbb{R}^d$$
$$\omega \mapsto (X_1(\omega), X_2(\omega), \dots, X_d(\omega)) = X(\omega)$$

denote $X = (X_1, \dots, X_d)$
$X_i : \Omega \rightarrow \mathbb{R}$ are real random variables

LAW of a random vector
$$\mu_X(B) = P(X^{-1}(B)) = P(X \in B)$$
for any $B \in \mathcal{B}(\mathbb{R}^d)$

- μ_X is a Probability on $\mathcal{B}(\mathbb{R}^d)$ and is called the JOINT DISTRIBUTION of $X_1, \dots, X_d$
  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu_X)$ is a probability space

Remark It can be proved that the values of μ_X on multiple intervals $[a_1, b_1] \times \dots \times [a_d, b_d]$ uniquely determines the value of μ_X on any $B \in \mathcal{B}(\mathbb{R}^d)$.

- the joint distribution $\mu_X = \mu_{X_1, \dots, X_d}$ gives the MARGINAL DISTRIBUTIONS μ_{X_i} $\forall i = 1, \dots, d$

$A \in \mathcal{B}(\mathbb{R}) \qquad \mu_{X_1}(A) = P(X_1 \in A)$

$$\mu_{X_1,\dots,X_d}(A_1 \times A_2 \times \dots \times A_d) = P(X_1$$
$$= P((X_1,\dots,X_d) \in A_1 \times \dots \times A_d)$$
$$= P(\{X_1 \in A_1\} \cap \{X_2 \in A_2\} \cap \dots \cap \{X_d \in A_d\})$$
$$= P(X_1 \in A_1, X_2 \in A_2, \dots, X_d \in A_d)$$

$$\mu_{X_1}(A) = P(X_1 \in A) = P(X_1 \in A_1, X_2 \in \mathbb{R}, \dots, X_d \in \mathbb{R})$$
$$= \mu_{X_1,\dots,X_d}(A_1 \times \mathbb{R} \times \dots \times \mathbb{R})$$

- Having the joint distribution $\mu_{X_1,\dots,X_d}$
  We get the marginal distributions by

$$\mu_{X_i}(A) = \mu_{X_1,\dots,X_d}(\underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{i-1} \times \underset{\text{index } i}{A} \times \overbrace{\mathbb{R} \times \dots \times \mathbb{R}}^{d-i})$$