PROBLEMS - SET 3

Problem 1. A telecommunication channel sends sequences of binary digits (0 or 1). Due to transmission noise the receiver may get some digit wrong.

(a) Suppose first each digit is changed in the transmission with probability 0.0002, independently of the other digits. Let X denote the number of changed digits in a transmission of 10000 digits. Compute (possibly with approximations) $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and $\mathrm{P}(X \geq 3)$.

(b) Suppose instead that the noise in the channel increases with time, so that the i-th digit is received wrong with probability $0.0003 \cdot \left(1 - \exp\left[-\frac{i}{1000}\right]\right)$. Let X denote the number of changed digits in a transmission of 10000 digits. Compute (possibly with approximations) $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and $\mathrm{P}(X \geq 3)$.
Hint: use the identity
$$\sum_{i=1}^{N} a^i = a\frac{a^N - 1}{a-1}.$$

Problem 2. Let X, Z e W be independent random variables with $X \sim \mathrm{Be}(p)$ and $Z, W \sim \mathrm{Pois}(\lambda)$. Define $Y := XZ + W$.

(i) Determine the discrete densities of (X,Y) and Y.

(ii) Using p_Y obtained above, compute $\mathrm{E}(Y)$ e $\mathrm{Var}(Y)$.

(iii) Compute $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ *without* using p_Y.

Problem 3. For given $p \in (0,1)$ and $n \geq 2$, let $Z_1, \ldots, Z_n$ be independent random variables with values in $\{-1, 1\}$, with $\mathrm{P}(Z_i = 1) = p$ for all $i = 1, \ldots, n$. Define

$$X := \prod_{i=1}^{n} Z_i = Z_1 \cdot Z_2 \cdots Z_n \, .$$

(i) Determine the distribution of X.

(ii) Show that X is independent of the random vector $(Z_2, \ldots, Z_n)$ if and only if $p = \frac{1}{2}$.

Problem 4. Let X be a point uniformly chosen in the interval $[0,2]$. What is the probability that the area of the equilateral triangle of side X is greater than 1?

Problem 5. Let $X \sim \mathrm{U}(0,1)$ and $Y := 4X(1-X)$. Compute the distribution function F_Y of Y, show that Y is absolutely continuous and compute its density

Problem 6. Let X be a point uniformly chosen in the interval $[0,4]$. Moreover let Q be the square centered in the origin whose side has length X. Compute the probability that Q is contained in the unit circle, i.e. the circle centered in the origin and with radius 1.

Problem 7. Consider the random variables $X \sim \mathrm{Be}(p)$, $Y \sim \mathrm{Exp}(\lambda)$, and assume they are independent. Set $Z := X + Y$. Compute the distribution function F_Z of Z. Is Z an absolutely continuous random variable?