Lecture 2

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Example 1 TOSS A COIN

1) $\Omega = \{H, T\}$
2) $\mathcal{A} = 2^{\Omega} = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$
3) $P : \mathcal{A} \to [0,1]$

PROPERTY i) • $P(\{H,T\}) = 1$

Consequence of i) : • $P(\emptyset) = 0$

$(\text{as } P(A^c) = 1 - P(A))$

• Let $P(\{H\}) = p \in [0,1]$

as a consequence, $P(\{T\}) = 1 - P(\{T\}^c)$

$= 1 - P(\{H\}) = 1 - p$

• $P(\{T\}) = 1 - p$

Example 2

Consider Ω FINITE with $|\Omega| = n$

1) $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$
2) $\mathcal{A} = 2^{\Omega}$ (always for finite sets)
3) How to define the probability?

• Let $P(\{\omega_1\}) = p_1 \in [0,1]$

$P(\{\omega_2\}) = p_2$

$\dots$

$P(\{\omega_n\}) = p_n$

We are given numbers $p_1, \dots, p_n \in [0,1]$

Property i) implies

$1 = P(\Omega) = P(\{\omega_1, \omega_2, \dots, \omega_n\})$

$= P(\{\omega_1\} \cup \{\omega_2\} \cup \dots \cup \{\omega_n\})$

DISJOINT UNION
$\{\omega_i\} \cap \{\omega_j\} = \emptyset$ if $i \neq j$

PROPERTY (ii)

$$= P(\{\omega_1\}) + P(\{\omega_2\}) + \dots + P(\{\omega_n\})$$
$$= \sum_{i=1}^{n} p_i$$

$\Rightarrow$ To define a probability on a finite set with n elements we have to assign to each element a value $P(\{\omega_i\}) = p_i \in [0,1]$ such that $\sum_{i=1}^{n} p_i = 1$

- $n-1$ parameters $p_1, \dots, p_{n-1}$
  as $p_n = 1 - \sum_{i=1}^{n-1} p_i$
- for a set $A \subseteq \Omega$ $\quad (A \in \mathcal{A} = 2^{\Omega})$
  $A = \{\omega_{A_1}, \dots, \omega_{A_K}\}$
  $$P(A) = \sum_{i=1}^{K} P(\{\omega_{A_i}\}) = \sum_{i=1}^{K} p_{A_i} = \sum_{\omega \in A} P(\{\omega\})$$

Example 3

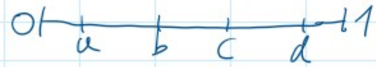
$\Omega = [0,1]$ NOT finite set (uncountable)

for example Ω could be the set of outcomes of a measurement, or the price of a stock in the market

- How to define a probability?
  What σ-algebra?
  We want to define P such that
  $$[a,b] \subseteq [0,1]$$
  $$P([a,b]) = b - a \qquad (\text{LENGTH of interval})$$
  Is it a probability? It has to satisfy the 2 properties

Is it a probability? It has to satisfy the 2 properties

i) $P([0,1]) = 1 - 0 = 1$ ✓

ii) We want, for  $a < b < c < d$

- $P([a,b] \cup [c,d]) = P([a,b]) + P([c,d])$

Is it possible to define a probability satisfying this property? In what σ-algebra?

Fact: A function $P : 2^{[0,1]} \to [0,1]$ with σ-additivity property can not exist!

$\Rightarrow$ P has to be defined on a σ-algebra $\mathcal{A} \subsetneq 2^{\Omega}$, thus P is not given for any $A \subseteq [0,1]$.

P is defined in $\mathcal{B}([0,1])$: the BOREL σ-algebra, "generated" by the intervals

$$P : \mathcal{B}([0,1]) \to [0,1]$$

There exists P with σ additivity property, which is the lenght when restricted to intervals.

Properties

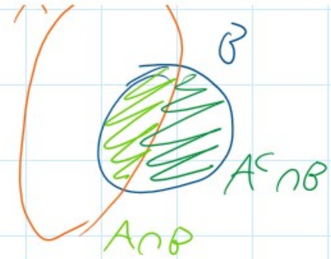
3) $P(A \cup B) = 1 - P(A^c \cap B^c)$

- Recall $(A \cup B)^c = A^c \cap B^c$
  $(A \cap B)^c = A^c \cup B^c$

4) $P(B) = P(A \cap B) + P(A^c \cap B)$ $\quad \forall A, B \in \mathcal{A}$

Indeed, $B = (A \cap B) \cup (A^c \cap B)$ 

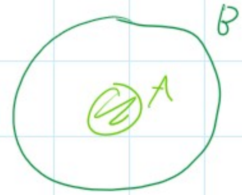
Indeed, $B = (A \cap B) \cup (A^c \cap B)$ — disjoint



5) if $A \subseteq B$ then $P(A) \leq P(B)$ MONOTONICITY of P

from property 4) we have

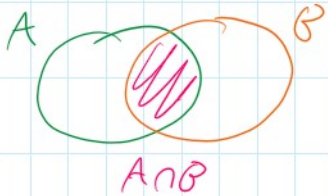
$$P(B) = \underbrace{P(A \cap B)}_{= P(A)} + \underbrace{P(A^c \cap B)}_{\geq 0}$$



$$\Rightarrow P(B) \geq P(A)$$

6) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) \leq P(A) + P(B)$



$A \cup B = A \cup (A^c \cap B)$ — DISJOINT

$$P(A \cup B) = P(A) + P(A^c \cap B)$$

$B = (A \cap B) \cup (A^c \cap B)$ — disjoint $\Rightarrow P(B) = P(A \cap B) + P(A^c \cap B)$

$$\Rightarrow P(A^c \cap B) = P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(A^c \cap B) = P(A) + P(B) - P(A \cap B)$$

7) $A, B, C \in \mathcal{A}$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B \cap C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$$

Proof $P(A \cup B \cup C) = P((A \cup B) \cup C)$

$$\overset{(6)}{=} P(A \cup B) + P(C) - \underbrace{P((A \cup B) \cap C)}$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C)$$

$= P(A) + P(B) - P(A\cap B) + P(C) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$

$(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$

$P((A\cap C)\cup(B\cap C)) = P(A\cap C) + P(B\cap C) - P(\underbrace{(A\cap C)\cap(B\cap C)}_{= A\cap B\cap C})$

8) $A_1, \dots, A_n$

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) - \sum_{k\neq j} P(A_k\cap A_j) + \sum_{k\neq j\neq i} P(A_k\cap A_j\cap A_i) - \dots + (-1)^{n+1} P(A_1\cap A_2 \dots \cap A_n)$$

$$\Rightarrow P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$$

UNIFORM PROBABILITY

finite Ω, $|\Omega| = n$

In general $P(A) = \sum_{\omega\in A} P(\{\omega\})$

The case when all elements in Ω have the same probability is called UNIFORM

$P(\{\omega\}) = \frac{1}{n} = \frac{1}{|\Omega|} \quad \forall\omega\in\Omega$

In this case $\boxed{P(A) = \frac{|A|}{|\Omega|}}$

$P(A) = \sum_{\omega\in A} P(\{\omega\}) = \sum_{\omega\in A} \frac{1}{n} = \frac{|A|}{n} = \frac{|A|}{|\Omega|}$

Problem We throw n balls randomly into 3 boxes, initially empty. Compute P(at least one box remains empty)

Solución

$\Omega = \{ (\omega_1, \dots, \omega_n) : \omega_i \in \{1,2,3\} \}$

ω_i is ball i-th , $\omega_i = m$ means that ball i falls in box m, with $m = 1$ or 2 or 3

UNIFORM PROBABILITY

$\Omega = \{ (1,1,\dots,1), (2,1,\dots,1), (3,1,\dots,1), (1,2,1), (1,3,\dots) \dots \}$

$|\Omega| = \underbrace{3 \cdot 3 \cdots 3}_{n \text{ times}} = 3^n$

$A = \{$ at least one box remains empty $\}$

$A_i = \{$ BOX i remains empty $\}$, for $i = 1,2,3$

e.g. $A_1 = \{$ all balls fall in boxes 2 or 3 $\}$

$A = A_1 \cup A_2 \cup A_3$

$A_1 = \{ (\omega_1, \dots, \omega_n) : \omega_i \in \{2,3\} \}$

$|A_1| = 2^n$

$$P(A) \overset{(7)}{=} P(A_1) + P(A_2) + P(A_3) + P(A_1 \cap A_2 \cap A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3)$$

$$P(A_1) = P(A_2) = P(A_2) = \frac{|A_1|}{|\Omega|} = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

$A_1 \cap A_2 = \{$ Box 1 AND Box 2 remain empty $\}$
$= \{ (3,3,\dots,3) \}$

$|A_1 \cap A_2| = 1 \Rightarrow P(A_1 \cap A_2) = \frac{1}{|\Omega|} = \frac{1}{3^n}$

$P(A_2 \cap A_3) = P(A_1 \cap A_3) = \frac{1}{3^n}$

$A_1 \cap A_2 \cap A_3 = \emptyset \Rightarrow P(A_1 \cap A_2 \cap A_3) = 0$

Then $P(A) = 3 \cdot \frac{2^n}{3^n} - 3 \cdot \frac{1}{3^n} + 0$

$$= \frac{3}{3^n}\left(2^n - 1\right) = \frac{2^n - 1}{3^{n-1}}$$

To calculate the numerical solution, let's proceed by breaking it down step by step.

Step 1: Total number of possible outcomes

The total number of ways to distribute n balls into 3 boxes is:

$$\text{Total outcomes} = 3^n$$

Step 2: Number of favorable outcomes (i.e., at least one empty box)

We will calculate the complement: the number of ways in which no box is empty. This can be done by using inclusion-exclusion.

Inclusion-Exclusion Formula:

Let $N(n)$ represent the number of ways to place n balls into 3 boxes such that all boxes are non-empty.

- There are 3^n ways to place all n balls into 3 boxes (the total number of outcomes).
- Let A_1, A_2, and A_3 represent the events that a particular box is empty.

By inclusion-exclusion, the number of favorable outcomes where no box is empty is:

$$N(n) = 3^n - 3 \cdot 2^n + 3 \cdot 1^n$$

Where:

- 3^n counts all possible outcomes.
- $3 \cdot 2^n$ subtracts the cases where one box is empty (leaving 2 boxes for the n balls).
- $3 \cdot 1^n$ adds back the cases where two boxes are empty (since we subtracted them twice in the previous step).

Step 3: Probability calculation

The number of ways to have at least one box empty is:

$$\text{Favorable outcomes} = 3^n - N(n) = 3^n - (3^n - 3 \cdot 2^n + 3 \cdot 1^n) = 3 \cdot 2^n - 3$$

Thus, the probability of having at least one empty box is:

$$P(\text{at least one empty box}) = \boxed{\frac{3 \cdot 2^n - 3}{3^n}}$$

Example Calculation:

Let's compute the probability for a few values of n.

- **For $n = 1$:**

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^1 - 3}{3^1} = \frac{3 \cdot 2 - 3}{3} = \frac{6 - 3}{3} = 1$$

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- **For** $n = 2$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^2 - 3}{3^2} = \frac{3 \cdot 4 - 3}{9} = \frac{12 - 3}{9} = \frac{9}{9} = 1$$

- **For** $n = 3$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^3 - 3}{3^3} = \frac{3 \cdot 8 - 3}{27} = \frac{24 - 3}{27} = \frac{21}{27} = \frac{7}{9} \approx 0.7778$$

- **For** $n = 4$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^4 - 3}{3^4} = \frac{3 \cdot 16 - 3}{81} = \frac{48 - 3}{81} = \frac{45}{81} = \frac{5}{9} \approx 0.5556$$

So, you can compute the probability for any number n using this method!