DATA SCIENCE **Stochastic Methods**

January 22, 2025

Problem 1. [12] Let $X_1, X_2, \ldots, X_n$ be n independent random variables, each uniformly distributed on $(0,1)$. Define $Z = \min\{X_1, X_2, \ldots, X_n\}$ and $W = \max\{X_1, X_2, \ldots, X_n\}$.

(i) Compute the distribution functions of the random variables Z and W;

(ii) Compute $\mathrm{E}[Z]$;

(iii) Compute $\mathrm{E}[W]$;

(iv) Compute $\mathrm{P}[\max\{X_1, X_2\} < \min\{X_3, X_4\}]$.

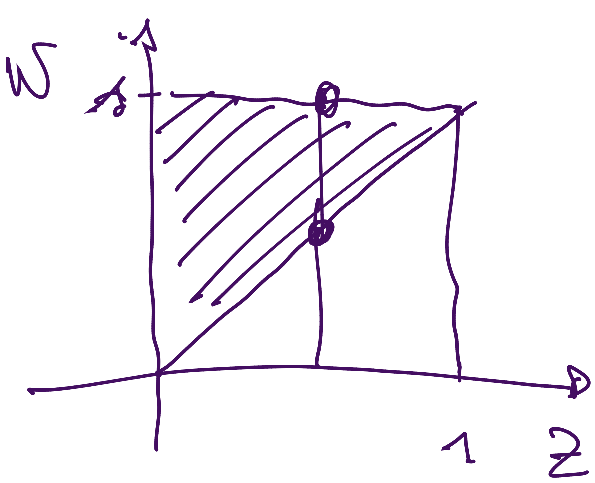
(i) $$F_W(w) = \begin{cases} 0 & w < 0 \\ w^n & 0 \le w < 1 \\ 1 & w \geq 1 \end{cases} \qquad F_Z(z) = \begin{cases} 0 & z < 0 \\ 1-(1-z)^n & 0 \le z < 1 \\ 1 & z \geq 1 \end{cases}$$

(ii) $$\mathbb{E}[Z] = \int_0^1 (1-z)^n \, dz = \left[-\frac{(1-z)^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$$

(iii) $$\mathbb{E}[W] = \int_0^1 (1-w^n) \, dw = \left[w - \frac{w^{n+1}}{n+1} \right]_0^1 = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

(iv) Let $Z = \max(X_1, X_2)$, $W = \min(X_1, X_2)$

$$\mathbb{P}[Z < W] = \int_0^1 2z \left(\int_z^1 2(1-w) \, dw \right) dz =$$



$$= \int_0^1 (2z - 4z^2 + 2z^3) \, dz = 1/6$$

Problem 2. [12] Let $\{A_n, n \geq 1\}$ be a countable partition of events in a probability space Ω, i.e. their union is equal to Ω and they are mutually disjoint. Define for each n the random variables $X_n = \mathbb{I}_{A_n}$, $Y_n = \mathbb{I}_{A_n^c}$, $Z_n = \sum_{i=1}^n X_i$ and $W_n = \sum_{i=1}^n Y_i$.

(i) Prove that $Y_n = 1 - X_n$ and $W_n = n - Z_n$;

(ii) Compute the discrete distributions of X_n, Y_n, Z_n and W_n and their expectations;

(iii) Are X_2 and Y_1 independent? Compute $\mathrm{E}[X_2|Y_1]$;

(iv) Compute the almost sure convergence of X_n, Y_n, Z_n and W_n as $n \to \infty$.

(i) $X_n + Y_n = \mathbb{I}_{A_n} + \mathbb{I}_{A_n^c} = \mathbb{I}_{A_n \cup A_n^c} = 1$, $W_n = \sum_{i=1}^n (1 - X_i) = n - Z_n$

(ii) $X_n \sim Bin(1, \mathbb{P}[A_n])$, $Y_n \sim Bin(1, \mathbb{P}[A_n^c])$, $Z_n \sim Bin(1, \mathbb{P}[A_1 \cup \dots \cup A_n])$

$$W_n = \begin{cases} n & \omega \notin A_1 \cup \dots \cup A_n \\ n-1 & \omega \in A_1 \cup \dots \cup A_n \end{cases}$$

(iii) $\mathbb{P}[X_2 = 1 | Y_1 = 0] = \dfrac{\mathbb{P}[X_2 = 1, Y_1 = 0]}{\mathbb{P}[Y_1 = 0]} = \dfrac{\mathbb{P}[A_2 \cap A_1]}{\mathbb{P}[A_1]} = 0 \neq \mathbb{P}[X_2 = 1]$

(iv) Fix ω : $\exists \bar{n}$: $\omega \in A_{\bar{n}} \Rightarrow X_1 = \dots = X_{\bar{n}-1} = 0$, $X_{\bar{n}} = 1$, $X_{\bar{n}+1} = \dots = 0$

$\Rightarrow \lim_{n \to +\infty} X_n(\omega) = 0 \quad \forall \omega \in \Omega \Rightarrow X_n \xrightarrow{a.s} 0$

$Y_n = 1 - X_n \longrightarrow 1 - 0 = 1$

$Z_n = \sum_{i=1}^n X_i = \mathbb{I}_{A_1 \cup \dots \cup A_n} \Rightarrow \forall \omega \in \Omega$, $\exists \bar{n}$: $\omega \in A_1 \cup \dots \cup A_n$

$\forall n \geq \bar{n} \Rightarrow Z_n \xrightarrow{a.s} 1$

$W_n = \underbrace{n}_{\to +\infty} - \underbrace{Z_n}_{\to 1} \xrightarrow{a.s} +\infty$

Problem 3. [12] Five balls are distributed between two boxes, labeled A and B. In each period, one of the boxes is selected at random. If the selected box is empty, all the balls are moved into it. If the selected box contains balls, one ball is drawn from it. With probability $1/3$ the ball is placed back into the same box, and with probability $2/3$ it is placed into the other box. Let X_n denote the number of balls in box A after n periods.

(i) Determine the state space S and the transition probability matrix;

(ii) Classify the states (communication, periodicity and recurrence);

(iii) Does the Markov Chain admit a unique invariant distribution? is this distribution reversible?

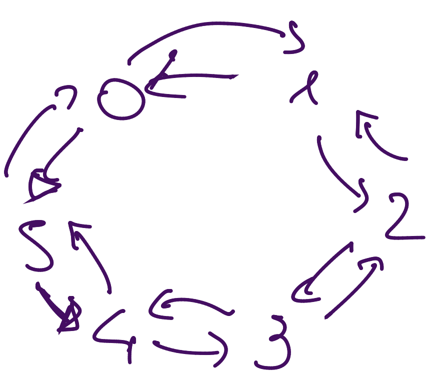
(iv) In the long run, what fraction of the time is box A empty?

(v) Compute the invariant distribution.

(i) $S = \{0,1,2,3,4,5\}$

$$P = \begin{bmatrix} 1/6 & 1/3 & 0 & 0 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 0 & 1/3 & 1/6 \end{bmatrix}$$

(ii)



X_n is irreducible

Aperiodic $P_{00}^{6} > 0$ and $P_{00}^{7} > 0$

Since $\#S = 6 < +\infty$, all states are recurrent

(iii) YES (in this case we have a unique invariant distrib.) π

YES $\pi_i P_{ij} = \pi_j P_{ji}$ $\forall i \neq j$

(iv) The fraction of time is equal to π_0

(v) $\pi P = \pi \iff \pi_0 = \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 1/6$

(P is doubly stochastic matrix).