Lecture 10

venerdì 24 ottobre 2025 08:33

INDEPENDENT RANDOM VARIABLES

Def $X, Y : \Omega \to \mathbb{R}$ random variables are INDEPENDENT if

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

$$\forall\ A, B \in \mathcal{B}(\mathbb{R})$$

Def $X_1, \dots, X_d : \Omega \to \mathbb{R}$ r.v. are INDEPENDENT if for any $A_1, \dots, A_d \in \mathcal{B}(\mathbb{R})$

$$P(X_1 \in A_1, \dots, X_d \in A_d) = \prod_{i=1}^{d} P(X_i \in A_i)$$

$$\mu_{X_1, \dots, X_d}(A_1 \times \dots \times A_d) = \prod_{i=1}^{d} \mu_{X_i}(A_i)$$

"joint distribution is product of marginals"

- DISCRETE RANDOM VECTORS

$X : \Omega \to \mathbb{R}^d$ is DISCRETE if $\exists\ N \in \mathbb{R}^d$ N finite or countable s.t. $P(X \in N) = 1$

$$p_X(x_1, \dots, x_d) = P((X_1, \dots, X_d) = (x_1, \dots, x_d))$$

$$p_{X_1, \dots, X_d}(x_1, \dots, x_d) = P(X_1 = x_1, \dots, X_d = x_d)$$

- If $X : \Omega \to \mathbb{R}^d$ is discrete then any component $X_1, \dots, X_d$ is discrete, and we can compute marginal densities

$$p_{X_1}(x_i) = P(X_1 = x_1)$$

$$= P(X_1 = x_1, X_2 \in \mathbb{R}, \dots, X_d \in \mathbb{R})$$

$$\{X_2 \in \mathbb{R}\} = \bigcup_{x_2 \in \mathbb{R}} \{X_2 = x_2\} \quad \text{disjoint union}$$

$$= \sum_{x_2, \dots, x_d \in \mathbb{R}} P(X_1 = x_1, X_2 = x_2, \dots, X_d = x_d)$$

$$p_{X_i}(x_i) = \sum p_{X_1, \dots, X_d}(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_d)$$

$$p_{X_i}(x_i) = \sum_{\substack{x_1,\dots,x_{i-1},\\ x_{i+1},\dots,x_d \in \mathbb{R}}} p_{X_1,\dots,X_d}(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_d)$$

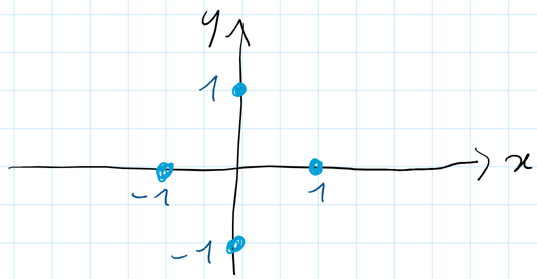
Proposition Let $X : \Omega \to \mathbb{R}^d$ discrete.
Then its components $X_1, \dots, X_d$ are discrete and
INDEPENDENT $\iff$ $p_{X_1,\dots,X_d}(x_1,\dots,x_d) = \prod_{i=1}^{d} p_{X_i}(x_i)$ $\forall\, (x_1,\dots,x_d) \in \mathbb{R}^d$

Example 1

$d = 2$ $\qquad |N| = 4$

(X,Y) is discrete $\qquad N = \{(1,0),(0,1),(-1,0),(0,-1)\}$

density of (X,Y) is uniform



JOINT DENSITY:

$p_{X,Y}(x,y) = 0 \quad \forall\, (x,y) \notin N$

$p_{X,Y}(x,y) = \frac{1}{4} \quad \forall\, (x,y) \in N$

$$p_{X,Y}(1,0) = p_{X,Y}(0,1) = p_{X,Y}(-1,0) = p_{X,Y}(0,-1) = \frac{1}{4}$$

$$p_{X,Y}(1,0) = P(X=1, Y=0)$$

- Compute marginal distributions

$$P(X=1) = P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=-1) = \frac{1}{4} + 0$$

$$P(X=-1) = P(X=-1, Y=0) + P(X=-1, Y=-1) + P(X=-1, Y=1) = \frac{1}{4} + 0$$

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=-1) = 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = 1, -1 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x \neq 0, -1, 1 \end{cases}$$

$$p_Y(y) = \sum_{x \in \mathbb{R}} p_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & \text{if } x = 1, -1 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x \neq 0, 1, -1 \end{cases}$$

- Are X and Y independent?

Means $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \quad \forall (x,y) \in \mathbb{R}^d$

$$\underset{\underset{0}{\shortparallel}}{p_{X,Y}(0,0)} \overset{?}{=} \underset{\underset{\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}}{\shortparallel}}{p_X(0) \cdot p_Y(0)}$$

$$0 \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$$

$\Rightarrow$ X and Y are <u>NOT</u> independent

in point $(1,0)$ $\quad p_{X,Y}(1,0) = p_X(1) \cdot p_Y(0)$

$$\frac{1}{4} \neq \frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2}$$

ABSOLUTELY CONTINUOUS RANDOM VECTORS

<u>Def</u> We say that a random vector $X : \Omega \to \mathbb{R}^d$ is ABSOLUTELY CONTINUOUS if there exists a DENSITY $f_X = f_{X_1,\dots,X_d} : \mathbb{R}^d \to [0,+\infty)$ s.t. $\forall B \in \mathcal{B}(\mathbb{R}^d)$

$$P(X \in B) = \mu_X(B) = \int \cdots \int_B f_X(t_1,\dots,t_d)\, dt_1 \cdots dt_d$$

For example, when $B = [a_1,b_1] \times \dots \times [a_d,b_d]$

$$P(X \in B) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_d}^{b_d} f_X(t_1,\dots,t_d)\, dt_1 \cdots dt_d$$

f_X is called the JOINT DENSITY of the vector X

If X is absolutely continuous then the marginals $X_1, \dots, X_d$ are absolutely continuous real r.v and

$$f_{X_i}(x_i) = \int\cdots\int_{\mathbb{R}^{d-1}} f(t_1, \dots, t_{i-1}, x_i, t_{i+1}, \dots, t_d)\, dt_1 \cdots dt_{i-1}\, dt_{i+1} \cdots dt_d$$

MARGINAL DENSITY of $X_i : \Omega \to \mathbb{R}$

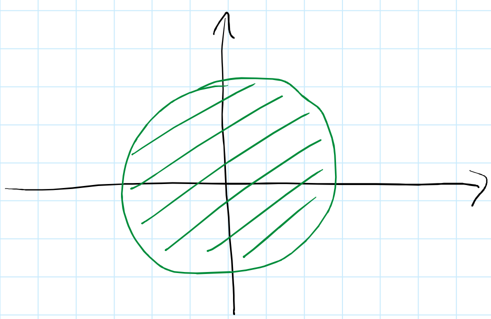
<u>Prop</u> Let $X_1, \dots, X_d$ be absolutely continuous random variables. They are independent if and only if $X = (X_1, \dots, X_d)$ is absolutely continuous and

$$f_{X_1, \dots, X_d}(x_1, \dots, x_d) = \prod_{i=1}^{d} f_{X_i}(x_i) \quad \text{for (almost) every } (x_1, \dots, x_d) \in \mathbb{R}^d$$

<u>Example</u> (X, Y) absolutely continuous random vector in $\mathbb{R}^2$.

$$f_{X,Y}(x,y) = \begin{cases} K > 0 & \text{if } x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

"uniform density on the circle"



$f_{X,Y}$ is a density if $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\, dx\, dy = 1$

This determines K

$CIRCLE = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$

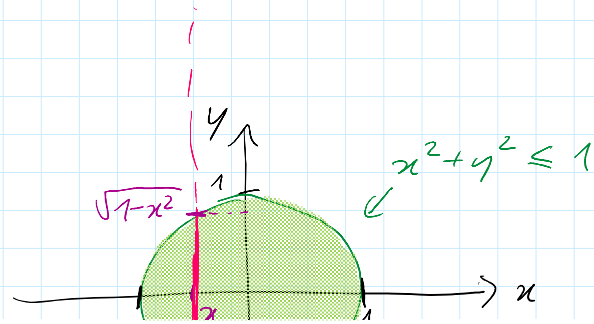
$$\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\, dx\, dy = \iint_{CIRCLE} K \, dx\, dy = K \, \text{Area}(CIRCLE) = K \cdot \pi = 1$$

$\Rightarrow K = \frac{1}{\pi}$

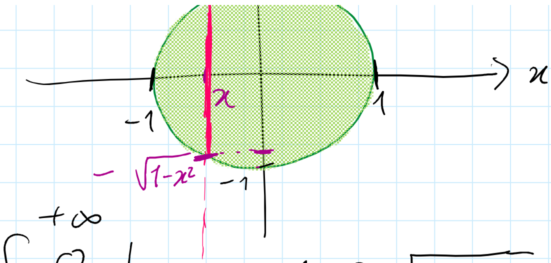
$$f_{X,Y}(x,y) = \frac{1}{\pi} \mathbb{1}_{CIRCLE}(x,y)$$

$\rightarrow$ MARGINAL DENSITIES ?

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, dy$$



$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy$$



$= 0$ if $x < -1$ or $x > 1$

if $|x| \le 1$

$$= \int_{-\infty}^{-\sqrt{1-x^2}} 0\,dy + \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy + \int_{\sqrt{1-x^2}}^{+\infty} 0\,dy = \frac{1}{\pi} 2\sqrt{1-x^2}$$

$$f_X(x) = \frac{2}{\pi}\sqrt{1-x^2}\,\mathbb{1}_{[-1,1]}(x)$$

$$f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}\,\mathbb{1}_{[-1,1]}(y)$$

→ Are X and Y independent ?

$$f_{X,Y}(x,y) \overset{?}{=} f_X(x)\, f_Y(y) \qquad \forall (x,y) \in \mathbb{R}^2$$

$x = y = 0$ $\qquad f_{X,Y}(0,0) \qquad f_X(0)\cdot f_Y(0)$

$$\frac{1}{\pi} \neq \frac{2}{\pi}\cdot\frac{2}{\pi}$$

$\Rightarrow$ X and Y are NOT independent

$E[XY] = ?$

$X, Y \in L^2 \Rightarrow XY \in L^1$

$X : \Omega \to \mathbb{R}^d$ random vector $\qquad g : \mathbb{R}^d \to \mathbb{R}$ measurable

$Y = g(X) : \Omega \to \mathbb{R}$

i) if X is discrete $\qquad E[g(X)] = \sum_{x \in \mathbb{R}^d} g(x)\, p_X(x)$

ii) if X is absolutely continuous

$$E[g(X)] = \int\!\!\cdots\!\!\int_{\mathbb{R}^d} g(x)\, f_X(x)\, dx_1 \cdots dx_d$$

- $E[XY] = \sum_{(x,y)\in\mathbb{R}^2} xy\, p_{X,Y}(x,y)$

$g(x,y) = xy \qquad g : \mathbb{R}^2 \to \mathbb{R}$

Example 1 (X,Y) $|N| = 4$ $N = \{(-1,0),(1,0),(0,1),(0,-1)\}$

$$E[XY] = (-1)\cdot 0 \; p_{X,Y}(-1,0) + 1\cdot 0 \; p_{X,Y}(1,0) + 0\cdot(-1) \; p_{X,Y}(0,-1) + 0\cdot 1 \; p_{X,Y}(0,1) = 0$$

$$E[X] = 0\cdot p_X(0) + 1 \; p_X(1) - 1 \; p_X(-1) = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{4} - 1\cdot\frac{1}{4} = 0$$

$E[Y] = 0$

Here we have $E[XY] = E[X]\,E[Y]$

Proposition Let $X, Y \in L^1(\Omega)$.
If X and Y are INDEPENDENT then $XY \in L^1(\Omega)$ and $E[XY] = E[X]\cdot E[Y]$ $\rightarrow$ X and Y are UNCORRELATED

Note opposite is NOT true.
$\hookrightarrow$ $E[XY] = E[X]\cdot E[Y]$ does NOT imply that X and Y are independent (as in Example 1)

Corollary if X and Y are independent then

$$\mathrm{Cov}(X,Y) = 0 \rightarrow \text{UNCORRELATED}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]\cdot E[Y]$$

- $\mathrm{Var}(X) = \mathrm{Cov}(X,X) = E[X^2] - (E[X])^2$

  $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$

- if X and Y are independent then

  $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

  (enough to require $\mathrm{Cov}(X,Y) = 0 \leftrightarrow X$ and Y uncorrelated)

- $E[X+Y] = E[X] + E[Y]$ for any r.v. X and Y
- $E[cX] = c\,E[X]$ for X r.v and c constant

$$\mathrm{Var}(cX) = E[(cX)^2] - (E[cX])^2$$
$$= c^2 E[X^2] - c^2(E[X]) = c^2(E[X^2] - (E[X])^2)$$

$$\mathrm{Var}(cX) = c^2\,\mathrm{Var}(X)$$

- $\mathrm{Var}(-X) = \mathrm{Var}(X)$
- $\mathrm{Var}(X+Y) = \mathrm{Var}(X-Y)$ if X and Y independent.