Lecture 20

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DISTRIBUTION of MARKOV CHAIN

$X_0, X_1, X_2, \dots$ discrete time M.C.

$X_n \in S$ finite or countable state space

$P = (p_{xy})_{x,y \in S}$ STOCHASTIC MATRIX

$\nu^{(0)}$ initial distribution : $\nu^{(0)}_x = P(X_0 = x)$

$\nu^{(n)}$ distribution at time n $\nu^{(n)}_x = P(X_n = x)$

$\nu^{(n)} = \mu_{X_n}$

$\nu^{(n)} \in \mathbb{R}^{|S|}$ if S FINITE , $\nu^{(n)} \in \mathbb{R}^{\infty}$ if $|S| = +\infty$

distribution $\nu^{(n)}$ has properties :

- $\nu^{(n)}_x \geq 0$
- $\sum_{x \in S} \nu^{(n)}_x = 1$

How to compute $\nu^{(n)}$ knowing $\nu^{(0)}$ and P

$$\nu^{(1)}_x = \mathbb{P}(X_1 = x) = \sum_y \mathbb{P}(X_1 = x \mid X_0 = y)\, \mathbb{P}(X_0 = y)$$

FORMULA TOTAL PROBABILITY

$$= \sum_{y \in S} p_{yx}\, \nu^{(0)}_y$$

sum can be infinite

$$\Rightarrow \quad \nu^{(1)} = \nu^{(0)} P$$

"matrix product", DISTRIBUTIONS as ROW VECTORS

In the same way, we find

$$\nu^{(n+1)} = \nu^{(n)} P \qquad \forall n$$

From this, we can compute the distribution at time n:

$$\nu^{(n)} = \nu^{(n-1)} P = (\nu^{(n-2)} P) \cdot P = \dots$$

$$= \nu^{(0)} P^n$$

Denote $p^n_{xy} = P(X_n = y \mid X_0 = x)$

$$\Rightarrow \quad p^n_{xy} = (P^n)_{xy}$$

↳ entry (x,y) of n-PRODUCT P^n

$\Rightarrow$ $p^{n}_{xy} = (P^{n})_{xy}$

↳ component (x,y) of n - PRODUCT P^n

[note $p^n_{xy} \neq (p_{xy})^n$]

Product of matrices usually done if $|S| < \infty$

Otherwise, we prove using MARKOV PROPERTY that

$$\nu^{(n+2)} = \nu^{(n)} P^2$$

where $(P^2)_{yx} = \sum_{z \in S} p_{yz}\, p_{zx}$

- $P(X_{n+2} = x \mid X_n = y) = p^2_{yx}$

TOTAL PROB. $= \sum_{z\in S} P(X_{n+2} = x, X_{n+1} = z \mid X_n = y)$

$P(A \cap B \mid C)$ $= P(A \mid B \cap C) \cdot P(B \mid C)$

$= \sum_{z\in S} P(X_{n+2} = x \mid X_{n+1} = z, \cancel{X_n = y})\, P(X_{n+1} = z \mid X_n = y)$

MARKOV $\rightarrow$ $= \sum_{z\in S} P(X_{n+2} = x \mid X_{n+1} = z)\, P(X_{n+1} = z \mid X_n = y)$

$= \sum_{z\in S} p_{zx}\, p_{yz}$

$\Rightarrow$ $p^2_{yx} = (P^2)_{yx}$

- $p^n_{yx} = P(X_{m+n} = x \mid X_m = y)$ $\quad \forall n, \forall m$

$= (P^n)_{yx}$

$\nu^{(m+n)} = \nu^{(m)} P^n$ $\quad \forall n, m$

$\rightarrow$ CHAPMAN - KOLMOGOROV EQUATION

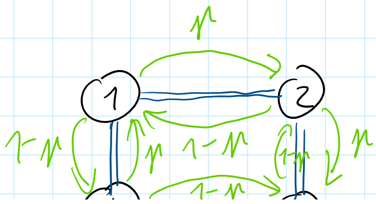
$$P^{n+m} = P^n \cdot P^m$$

where $(P^n)_{xy} = p^n_{xy}$

Example (SENTINEL)

$S = \{1, 2, 3, 4\}$

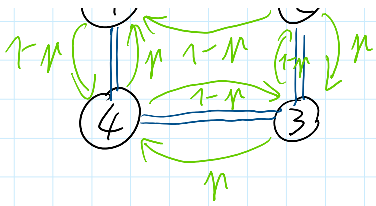
TOWERS of a CASTLE



$S = \{1,2,3,4\}$

TOWERS of a CASTLE

a SENTINEL MOVES



X_n = position of sentinel at time n

SENTINEL MOVES TO $<$ NEXT TOWER — prob. p

PREVIOUS TOWER — prob. $1-p$

(CLOCKWISE ORDER)

$p_{12} = p \quad , \quad p_{14} = 1-p \quad , \quad p_{13} = 0 \qquad p_{11} = 0$

$$P = \begin{pmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{pmatrix}$$

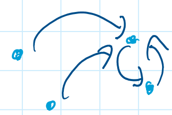
Note 2 steps to return to the same state (with positive probability)

EVEN numbers have property that 2 steps are required to go to another even number

CLASSIFY THE STATES

Note that a MARKOV CHAIN can be represented by a DIRECTED GRAPH

GRAPH = { VERTEX , EDGES }



VERTEX = STATES = S

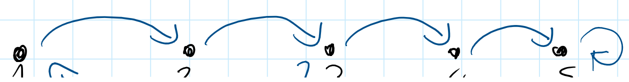
$\exists$ EDGE FROM STATE x to y is $p_{xy} > 0$

COMMUNICATION

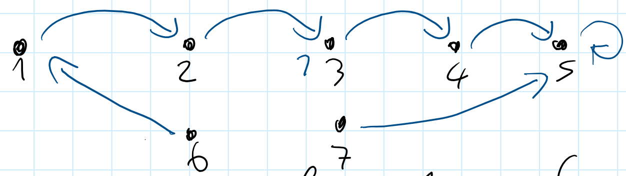
We say that $j \in S$ is ACCESSIBLE from $i \in S$ if

$\exists\ n \geq 0$ s.t. $p^n_{ij} > 0$ $\qquad [\, p^0_{ii} = 1 \,]$

example



example



- 5 is ACCESSIBLE from 1 (in 4 steps) $(p^4_{15} > 0)$
- 1 is NOT ACCESSIBLE FROM 3

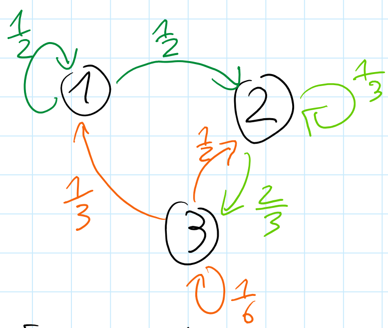
- We say that i and j COMMUNICATE if i is accessible from j and j is accessible from i
- COMMUNICATION is an EQUIVALENT RELATION

→ we can divide the state space in equivalent classes COMMUNICATION CLASSES

→ elements in same communication class share the same properties (PERIODICITY, RECURRENCE, ...)

Example $|S| = 3$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \longrightarrow$$



Just one COMMUNICATION CLASS $\{1,2,3\}$

We say that the MARKOV CHAIN IS IRREDUCIBLE if all states communicate with each other (just one communication class)

Exercise MC SENTINEL is also irreducible

Example GAMBLER'S RUIN MODEL

$S = \{0, 1, \dots, m\}$

Communication classes? 3

$C_1 = \{0\}$

$C_2 = \{5\}$

$C_3 = \{1, 2, 3, 4\}$

When $C = \{i\}$, i is called ABSORBING STATE

STATIONARY DISTRIBUTION

INVARIANT or STATIONARY distribution π

it does not change in time,

i.e. $X_0 \sim \pi \Rightarrow X_1 \sim \pi$

Questions
- existence / uniqueness of invariant distribution?
- properties of π?
- Convergence to the invariant distribution?

$\nu^{(0)} = \pi$, we look for π s.t. $\nu^{(1)} = \pi$

$\pi = \nu^{(1)} = \nu^{(0)} P = \pi P$

- INVARIANT DISTRIBUTION is SOLUTION to equation

$\pi = \pi P$ and $\pi_i \geq 0 \quad \forall i \in S$ and $\sum_{i \in S} \pi_i = 1$

$\pi(P - I) = 0 \rightarrow \pi$ is LEFT EIGENVECTOR of eigenvalue 1 (if $|S| < \infty$)

Thm 1 if S is FINITE and P is IRREDUCIBLE

Then there exists a unique invariant distribution π

AND π is such that $\pi_x > 0 \quad \forall x$

Exercise $|S| = 3$

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

is IRREDUCIBLE

Compute π invariant distribution

$$\begin{cases} \pi = \pi P \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

$$\begin{cases} \pi_1 = \frac{\pi_1}{2} + \frac{\pi_3}{3} \\ \pi_2 = \frac{\pi_1}{2} + \frac{\pi_2}{3} + \frac{\pi_3}{2} \\ \pi_3 = \frac{2}{3}\pi_2 + \frac{\pi_3}{6} \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\Longrightarrow \pi = \left(\frac{8}{35}, \frac{15}{35}, \frac{12}{35} \right)$$

- Convergence? given $\nu^{(0)}$

is it true that $\lim_{n \to \infty} \nu^{(n)} = \pi$ ERGODIC THEOREM

true if MC is APERIODIC (and $|S| < \infty$)