CENTRAL LIMIT THEOREM

Let $(X_n)_n$ be a sequence of i.i.d. random variables, with expectation μ and variance σ^2. Let

$$Z_n = \sqrt{n}\,\frac{\overline{X}_n - \mu}{\sigma} = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i - n\mu}{\sigma}$$

Then $Z_n \xrightarrow[n\to\infty]{D} Z$ $\qquad Z \sim N(0,1)$

Proof we prove that

$$\varphi_{Z_n}(u) \xrightarrow{n\to\infty} \varphi_Z(u) \qquad \forall u$$

$$\varphi_Z(u) = e^{-\frac{u^2}{2}}$$

$$Z_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i \qquad Y_i = \frac{X_i - \mu}{\sigma}$$

$$\varphi_{Z_n}(u) = E[e^{iuZ_n}] = E\left[e^{i\frac{u}{\sqrt{n}}\sum_{i=1}^{n} Y_i}\right]$$

$$\underset{\text{INDEPENDENCE}}{=} \prod_{i=1}^{n} E\left[e^{i\frac{u}{\sqrt{n}}Y_i}\right]$$

$$= \prod_{i=1}^{n} \varphi_{Y_i}\left(\frac{u}{\sqrt{n}}\right) = \left(\varphi_{Y_1}\left(\frac{u}{\sqrt{n}}\right)\right)^n$$

Y_i i.i.d $\qquad E[Y_i] = \frac{E[X_i] - \mu}{\sigma} = 0$

$$\mathrm{Var}\, Y_i = E[Y_i^2] = E\left[\frac{(X_i - \mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2}\mathrm{Var}(X_i) = \frac{\sigma^2}{\sigma^2} = 1$$

Use TAYLOR EXPANSION

$$\varphi_{Y_1}(x) = \varphi_{Y_1}(0) + \varphi'_{Y_1}(0)\,x + \varphi''_{Y_1}(0)\,\frac{x^2}{2} + o(x^2)$$

$\varphi_{Y_1}(0) = 1 \qquad \varphi'_{Y_1}(0) = i\,E[Y_1] = 0$

$\varphi''_{Y_1}(0) = i^2 E[Y_1^2] = -1$

$$\varphi_{Y_1}(x) = 1 - \frac{x^2}{2} + o(x^2) \qquad \text{as } x \to 0$$

$$\varphi_{Y_1}\left(\frac{u}{\sqrt{n}}\right) = 1 - \frac{1}{2}\frac{u^2}{n} + o\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty$$

$$\left(\varphi_{Y_1}\left(\frac{u}{\sqrt{n}}\right)\right)^n = \left(1 - \frac{1}{2}\frac{u^2}{n} + o\left(\frac{1}{n}\right)\right)^n$$

Recall $e^v = \lim_{n\to\infty}\left(1 + \frac{v}{n}\right)^n$

$$\lim_{n\to\infty} \varphi_{Z_n}(u) = e^{-\frac{1}{2}u^2} = \varphi_Z(u) \qquad \boxtimes$$

STOCHASTIC PROCESSES

random variable $X : \Omega \to \mathbb{R}$
$\omega \mapsto X(\omega)$

Collection of random variables evolving in time is a STOCHASTIC PROCESS

T is the set of TIMES

$$\underset{\text{TIME}}{T} \times \underset{\text{PROB. SPACE}}{\Omega} \to \mathbb{R} \qquad \text{STOCHASTIC PROCESS}$$

$$(t,\omega) \mapsto X(t,\omega)$$

- $T = \mathbb{N}$ DISCRETE TIME STOCHASTIC PROCESS

  $X_0(\omega) \to X_1(\omega) \to X_2(\omega) \to \ldots$

  $(X_1, X_2, X_3, \ldots)$ collection of r.v.

- $T = [0, +\infty)$ CONTINUOUS TIME STOCHASTIC PROCESS

  $(t,\omega) \to X_t(\omega)$

Examples - Price of stock in the market
- Portfolio value
- weather conditions
- ....

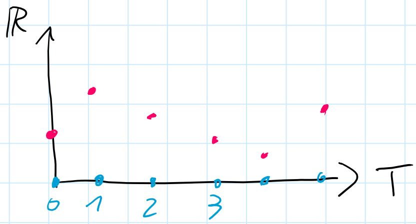
TRAJECTORY of a STOCHASTIC PROCESS

Trajectory is function of time, for fixed ω

fix $\omega \in \Omega$ PATH $T \to \mathbb{R}$

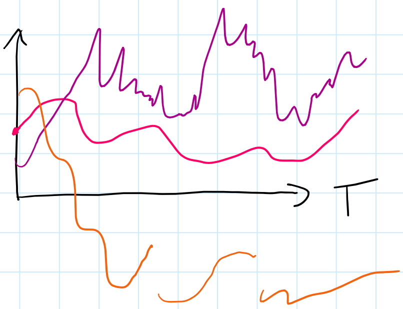
$t \mapsto X(t,\omega)$

- discrete time



function $\mathbb{N} \to \mathbb{R}$
(for fixed ω)
is a SEQUENCE

- continuous time



$[0,+\infty) \to \mathbb{R}$

$t \longmapsto X(t,\omega)$

for fixed ω

$T = \mathbb{N} \to$ DISCRETE TIME STOCHASTIC PROCESSES

$X_0(\omega) \to X_1(\omega) \to X_2(\omega) \to \ldots$

X_1 may depend on X_0

X_2 may depend on X_0, X_1

X_3 may depend on X_0, X_1, X_2

Stochastic processes do not depend on the future

$\to$ The value of X_n does non depend on $X_{n+1}, X_{n+2}, \ldots$

DISCRETE TIME MARKOV CHAINS

$T = \mathbb{N}$ discrete time

MARKOV = the r.v. X_{n+1} depends just on X_n,
does not depend on all the PAST $(X_0, X_1, .., X_{n-1})$

CHAIN = STATE SPACE S is countable or finite

$X : \mathbb{N} \times \Omega \to S$

$(n,\omega) \mapsto X_n(\omega)$

S STATE SPACE

$(n, \omega) \mapsto X_n(\omega)$

$(X_n)_n$ is a collection of discrete r.v.s.

- What is the distribution of the MARKOV CHAIN ?

<u>notation</u> $x_m^n = (x_m, x_{m+1}, \dots, x_{n-1}, x_n)$

MARKOV CHAIN $X_0, X_1, X_2, \dots$

$P(X_i = k) = ? \quad \forall k \in S \quad \forall i \in \mathbb{N}$

What is the distribution of the discrete vector $(X_0, \dots, X_n)$?

$(X_0, \dots, X_n) : \Omega \to S^{n+1}$ random vector

given $(x_0, \dots, x_n) \in S^{n+1}$, we want to compute

$$P((X_0, \dots, X_n) = (x_0, \dots, x_n)) = P(X_0 = x_0, X_1 = x_1, \dots, X_n = x_n)$$

$$= P(X_n = x_n \mid X_0 = x_0, \dots, X_{n-1} = x_{n-1})\, P(X_0 = x_0, \dots, X_{n-1} = x_{n-1})$$

$$= P(X_n = x_n \mid \cancel{X_0 = x_0, \dots,}\ X_{n-1} = x_{n-1}) \to \text{MARKOV PROPERTY}$$

$$\cdot P(X_{n-1} = x_{n-1} \mid \cancel{X_0 = x_0, \dots,}\ \boxed{X_{n-2} = x_{n-2}})$$

$$= P(X_n = x_n \mid X_{n-1} = x_{n-1})$$

$\cdot \ \dots\dots\dots$

$\cdot \ P(X_2 = x_2 \mid X_1 = x_1, \cancel{X_0 = x_0})$

$\cdot \ P(X_1 = x_1 \mid X_0 = x_0)\ P(X_0 = x_0)$

<u>Def</u> A MARKOV CHAIN is a sequence $(X_n)_{n \in \mathbb{N}}$ of random variables, taking values in a finite or countable set S s.t. $\forall x_0^{n+1} \in S^{n+2}$, $x_0^{n+1} = (x_0, \dots, x_n, x_{n+1})$ and any $n \geq 1$ we have

$$P(X_{n+1} = x_{n+1} \mid X_0^n = x_0^n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

$\hookrightarrow$ MARKOV PROPERTY

S is called the STATE SPACE

the conditional probabilities $P(X_{n+1} = x_{n+1} \mid X_n = x_n)$ are called the TRANSITION PROBABILITIES

The complete law of the MARKOV CHAIN can be obtained if we know the transition probabilities and the initial distribution

$$P(X_0^n = x_0^n) = P(X_n = x_n \mid X_{n-1} = x_{n-1}) \quad \text{MAY DEPEND on } n$$
$$\cdot P(X_{n-1} = x_{n-1} \mid X_{n-2} = x_{n-2})$$
$$\cdots\cdots$$
$$\cdot P(X_1 = x_1 \mid X_0 = x_0) \cdot P(X_0 = x_0)$$

We say that a MC is HOMOGENEOUS (in time) if the transition probabilities $P(X_n = x_n \mid X_{n-1} = x_{n-1})$ do NOT DEPEND on n

$$n \longrightarrow n+1$$
$$x \longrightarrow y \qquad p_{xy} = P(X_{n+1} = y \mid X_n = x)$$

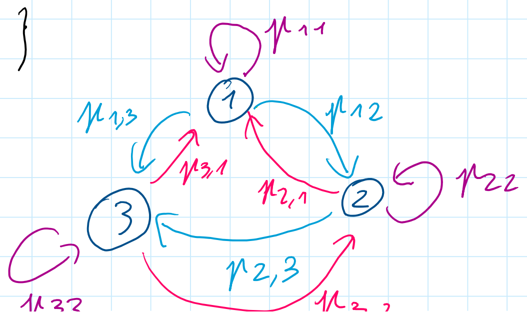
$$P(X_0^n = x_0^n) = p_{x_{n-1},x_n} \cdot p_{x_{n-2},x_{n-1}} \cdots p_{x_0 x_1} \lambda_{x_0}$$
$\lambda_{x_0} = P(X_0 = x_0)$ initial distribution

$P = (p_{xy})_{x,y \in S}$ is called the TRANSITION MATRIX of the MARKOV CHAIN

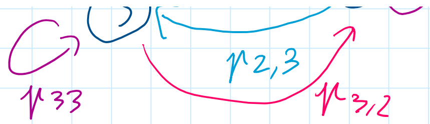
- if $|S| < \infty$ this is a square matrix $|S| \times |S|$
- if $|S| = \infty$ P is a generalized matrix

Example $\quad S = \{1, 2, 3\}$

$$P = (p_{xy})_{x,y \in S}$$



1 2 3



$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \end{array} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{array}{l} \xrightarrow{\text{SUM}} 1 \\ \xrightarrow{\text{SUM}} 1 \\ \xrightarrow{\text{SUM}} 1 \end{array}$$

$= p_{3,2} = P(X_{n+1} = 2 \mid X_n = 3)$

The transition matrix must satisfy some properties

Recall that $\mathcal{A} \longrightarrow [0,1]$
(for any B) $A \longmapsto P(A|B)$ is a probability

$\Rightarrow$ ex. from state 1 :
the sum of transition probabilities must be 1
$p_{11} + p_{12} + p_{13} = 1$

A matrix $P = (p_{xy})_{x,y \in S}$ is a STOCHASTIC MATRIX
if
- $p_{xy} \geq 0 \quad \forall x, y$
- $\sum_{y \in S} p_{xy} = 1 \quad \forall x$

RANDOM WALK

Example GAMBLER'S RUIN MODEL

$S = \{0, 1, \dots, n\}$

Consider IID random variables $U_1, \dots, U_n$

$$U_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

$(X_n)_n$ wealth of player

$X_0 = k$ given

$X_1 = X_0 + U_1 \qquad X_2 = X_1 + U_2 = X_0 + U_1 + U_2$

$X_0 = k$ given

$X_1 = X_0 + U_1$ $\qquad$ $X_2 = X_1 + U_2 = X_0 + U_1 + U_2$

$X_{n+1} = X_n + U_{n+1}$

$$U_{n+1} = \begin{cases} 1 & \to \text{player gains } 1 \\ -1 & \to \text{player loses } 1 \end{cases}$$

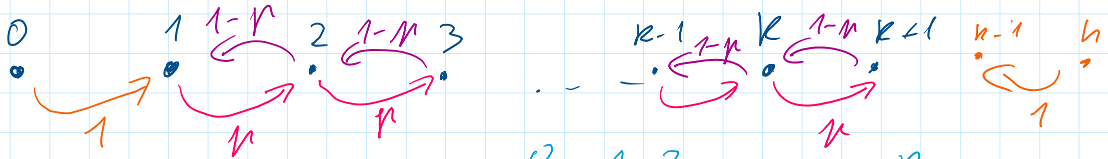
$(X_n)_n$ is a MARKOV CHAIN

What are the transition probabilities?

$P(X_{n+1} = k+1 \mid X_n = k) = P(U_{n+1} = 1) = p$

$P(X_{n+1} = k-1 \mid X_n = k) = P(U_{n+1} = -1) = 1-p$

$P(X_{n+1} = m \mid X_n = k) = 0$ if $m \neq k \pm 1$



Transition MATRIX

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & n \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ 1-p & 0 & p & & & \\ & 1-p & 0 & p & & 0 \\ & & & \ddots & \ddots & \\ & 0 & & 1-p & 0 & p \\ & \cdots & & & 0 & 1 \end{bmatrix} \end{array}$$