Lecture 1

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STOCHASTIC METHODS

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LECTURES : MONDAY 10:30 - 12:30 AULA B

$\boxed{48\ h}$ THURSDAY 16:30 - 18:30 LUF1

FRIDAY 8:30 - 10:30 P150

NO CLASS THURSDAYS 2 and 9 OCTOBER

$\boxed{\text{MOODLE}}$ → DATA SCIENCE

- notes of lectures
- VIDEO of recorded lectures
- exercise sheets

TUTOR ?

OFFICE HOURS : by appointment

CONTENT

1) PROBABILITY REVIEWS : probability space, σ - algebra , conditional probability , independence random variables, conditional expectation, law large numbers , Chernoff bounds, approximation of probability distributions.
2) DISCRETE-TIME MARKOV CHAINS, definition and classification of states, stationary distribution.
3) DISCRETE-TIME MARTINGALES : properties of conditional expectation, SUB- SUPER - martingales

3) DISCRETE-TIME MARTINGALES: properties of conditional expectation, SUB-SUPER-martingales, stopping times.

PREREQUISITES: basic calculus and linear algebra.

MORE ADVANCED COURSE, for students who have a background in probability theory:
- HIGH DIMENSIONAL PROBABILITY (ALBERTO CHIARINI)
  THURSDAY - FRIDAY 8:30 1BC 45 TA

MATERIAL: notes of PAOLO DAI PRA
notes of MICHELE PAVON
other book JACOD-PROTTER

EXAM WRITTEN, 3/4 exercises
- DECEMBER ($\sim$ 17 DEC)
- JANUARY, FEBRUARY, JUNE, JULY, SEPTEMBER

Lecture 1

PROBABILISTIC MODEL

experiment $\rightarrow$ result $\rightarrow$ RANDOM OUTCOME
(not deterministic)

3 STEPS TO describe outcome of random experiment
(= probabilistic model)

1) Ω = SAMPLE SPACE

Ω is a SET and CONTAINS all the

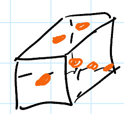
Ω is a SET and CONTAINS all the POSSIBLE OUTCOMES of a random experiment.

Example 1 FLIP a coin $\rightarrow$ $\Omega = \{\text{Head}, \text{Tail}\}$
$= \{H, T\}$

- Ω is "just" a set $\rightarrow$ pure mathematical object
operations permitted on a set : $\cup, \cap, \setminus, {}^c, \Delta$

Example 2 Flip 2 coins :
$$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$$

Example 3 TOSS a DIE



$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- TOSS 2 DICE

$$\Omega = \{ (1,1), (1,2), (1,3), \ldots \\ (2,1), (2,2), \ldots \}$$

$6 \cdot 6 = 36$ elements

Example 4 PRICE of a STOCK in the market
$\Omega = \mathbb{R}^+$ (positive real numbers)

Want to associate to each outcome a number
$\hookrightarrow$ PROBABILITY

Step 2 $\boxed{\text{EVENTS}}$

- not satisfactory to associate number = probability

- not satisfactory to associate number = probability just to outcome (elements of sample space)

- if Ω is FINITE, the set of EVENTS is

$$2^{\Omega} = \mathcal{P}(\Omega) = \{\text{set of all subsets of } \Omega\}$$

POWER SET of Ω

Ex 1 $\quad \Omega = \{H, T\}$

$$2^{\Omega} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$$

Ex 2 $\quad \Omega = \{1, 2, 3, 4, 5, 6\}$

$$2^{\Omega} = \{\emptyset, \{1\}, \{2\}, \dots$$
$$\{1,2\}, \{1,3\}, \dots \{2,3\}, \{2,4\}, \dots$$
$$\{1,2,3\}, \{1,2,4\}, \dots$$
$$\{1,2,3,4\} \dots$$
$$\dots \}$$

example of event : $\{2, 4, 6\}$ EVEN NUMBERS

AIM: ASSIGN number = probability to each event

the larger the event, the higher should be the probability

$|\Omega| = 6 \quad$ CARDINALITY OF Ω

= NUMBER of elements of Ω FINITE

$$\boxed{|2^{\Omega}| = 2^{|\Omega|}}$$

ex $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$2 \times 2 \times 2 \times 2 \times 2 \times 2$$

toss 2 DICE $|\Omega| = 6 \cdot 6 = 36$

$|2^\Omega| = 2^{|\Omega|} = 2^{36} = 68.719.476$

$\Rightarrow$ FINITE $\neq$ SIMPLE

Step 1 Ω = sample space

Step 2 SET of EVENTS

is a subset $\mathcal{A}$ of 2^Ω

$\mathcal{A}$ is a σ-ALGEBRA (or σ-FIELD)

if it has 3 properties

i) $\Omega \in \mathcal{A}$

ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

iii) if $(A_n)_{n \in \mathbb{N}}$ is an infinite sequence of events $(A_n \in \mathcal{A} \ \forall n)$ then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$$

In general $\mathcal{A} \subseteq 2^\Omega$ (could be $\mathcal{A} \subsetneq 2^\Omega$)

Example $\Omega = \{H, T\}$

$2^\Omega = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$

- Consider $\mathcal{A} \subseteq 2^\Omega$, $\mathcal{A} = \{\emptyset, \{H\}\}$

  Is $\mathcal{A}$ a σ-algebra?

  NO because $\Omega = \{H, T\} \notin \mathcal{A}$

  $\Rightarrow$ property (i) is not satisfied

- $\mathcal{A}_0 = \{\emptyset, \Omega\}$ is a σ-algebra?

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  $\boxed{\text{YES}} \rightarrow$ TRIVIAL σ-ALGEBRA

  i) $\Omega \in \mathcal{A}_0$? YES

  ii) $A \in \mathcal{A}_0 \Rightarrow A^c \in \mathcal{A}_0$?

  - $\emptyset \rightarrow \emptyset^c = \Omega \in \mathcal{A}_0$
  - $\Omega \rightarrow \Omega^c = \emptyset \in \mathcal{A}_0$

  iii) $A_n \in \mathcal{A}_0 \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_0$?

  if Ω is one of A_n then $\bigcup_n A_n = \Omega \in \mathcal{A}_0$

  if Ω is not one of A_n then $A_n = \emptyset \ \forall n$

  then $\bigcup_n \emptyset = \emptyset \in \mathcal{A}_0$

Step 3 PROBABILITY

$A \in \mathcal{A}$: $P(A) \in [0,1]$, PROBABILITY of an EVENT

Example $\Omega = \{1,2,3,4,5,6\}$

$\mathcal{A} = 2^{\Omega}$

$A, B \in \mathcal{A}$ we want some operations to be allowed in $\mathcal{A}$

- $A \cup B \rightarrow$ one of the events A or B occur
- $A \cap B \rightarrow$ both A and B occur
- $A^c \rightarrow A$ does NOT occur

ie. $A = \{2,4,6\}$ even numbers

$B = \{1,2\}$

$A \cup B = \{1,2,4,6\}$

$A \cap B = \{2\}$

$A \cup B = \{1, 2, 4, 6\}$

$A \cap B = \{2\}$

DEFINE PROBABILISTIC MODEL

1) Ω = sample space (outcomes)
2) $\mathcal{A} \subseteq 2^{\Omega}$ σ - algebra ($\mathcal{A}$ set of EVENTS)
3) a FUNCTION P called PROBABILITY

$$P : \mathcal{A} \longrightarrow [0,1] \qquad A \mapsto P(A)$$

i) $P(\Omega) = 1$

ii) if $(A_n)_{n \in \mathbb{N}}$ is a family of pairwise disjoint events (i.e. $A_n \in \mathcal{A}$ $\forall n$, $A_n \cap A_m = \emptyset$ $\forall n \neq m$)

$$P\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \sum_{n \in \mathbb{N}} P(A_n)$$

Properties

1) $A, B \in \Omega$ and $A \cap B = \emptyset$ then

$P(A \cup B) = P(A) + P(B)$

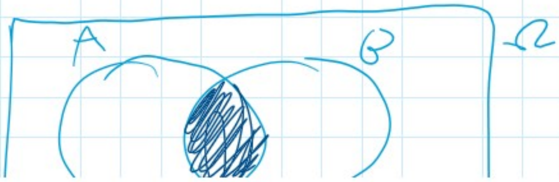
2) $P(A^c) = 1 - P(A)$

$\hookrightarrow$ $P(\emptyset) = 0$ $(= 1 - P(\Omega))$

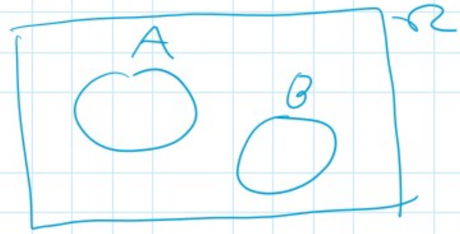
proof $\Omega = A \cup A^c$ and $A \cap A^c = \emptyset$

$P(\Omega) \overset{(i)}{=} 1 = P(A \cup A^c) = P(A) + P(A^c)$

VENN - DIAGRAMS



A and B disjoint



<u>Definition</u> The TRIPLE $(\Omega, \mathcal{A}, P)$ is called a PROBABILITY SPACE