**Proposition**

Let $X, Y : \Omega \to \mathbb{R}$ r.v., $X \perp\!\!\!\perp Y$. Then

$$\varphi_{X+Y}(u) = \varphi_X(u) \cdot \varphi_Y(u)$$

$$m_{X+Y}(t) = m_X(t) \cdot m_Y(t)$$

**Proof**

$$\varphi_{X+Y}(u) = E[e^{iu(X+Y)}] = E[e^{iuX} e^{iuY}]$$
$$= E[e^{iuX}] \cdot E[e^{iuY}]$$
$$= \varphi_X(u) \cdot \varphi_Y(u) \qquad \blacksquare$$

**Proposition** [From Theorems of lecture 13]

Let $X, Y : \Omega \to \mathbb{R}$ r.v. Then

$$X \perp\!\!\!\perp Y \iff \varphi_{X,Y}(u,v) = \varphi_X(u) \cdot \varphi_Y(v)$$

**Example**

$X \sim \mathrm{Poi}(\lambda) \qquad \lambda > 0$

$$p_X(n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

Compute $\varphi_X(u)$

$$\varphi_X(u) = E[e^{iuX}] = \sum_{n=0}^{\infty} e^{iun} p_X(n)$$
$$= \sum_{n=0}^{\infty} e^{iun} e^{-\lambda} \frac{\lambda^n}{n!}$$
$$= \sum_{n=0}^{\infty} e^{-\lambda} \frac{(\lambda e^{iu})^n}{n!} \qquad e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}$$
$$= e^{-\lambda} \exp(\lambda e^{iu})$$
$$= \exp(\lambda e^{iu} - \lambda) = e^{\lambda(e^{iu}-1)}$$

$$\varphi_X(u) = e^{\lambda(e^{iu}-1)}$$

$$\varphi_X(u) = e^{\lambda(e^{iu}-1)}$$

Example $X \sim Poi(\lambda)$ $Y \sim Poi(\mu)$ $\lambda, \mu > 0$

$X \perp\!\!\!\perp Y$. What is the distribution of $X+Y$?

We compute characteristic function of $X+Y$, using proposition above

$$\varphi_{X+Y}(u) = \varphi_X(u)\,\varphi_Y(u)$$
$$= e^{\lambda(e^{iu}-1)} \cdot e^{\mu(e^{iu}-1)}$$
$$= \exp\left((\lambda+\mu)(e^{iu}-1)\right)$$

$\Rightarrow$ $X+Y$ has characteristic function of $Poi(\lambda+\mu)$

Since characteristic function uniquely characterizes the distribution, we conclude that

$X+Y \sim Poi(\lambda+\mu)$

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In general, there is a way to compute the density of a sum of independent discrete r.v.

$X, Y$ DISCRETE r.v. with values in $\mathbb{N}$, $X \perp\!\!\!\perp Y$

Compute density of $X+Y$:

$$P(X+Y=n) = P\left(\bigcup_{k\in\mathbb{N}} \{X=k, Y=n-k\}\right)$$
$$\overset{\text{DISJOINT}}{=} \sum_{k=0}^{\infty} P(X=k, Y=n-k)$$
$$\overset{\text{INDEPENDENCE}}{=} \sum_{k=0}^{\infty} P(X=k)\,P(Y=n-k)$$
$$= \sum_{k=0}^{\infty} p_X(k)\,p_Y(n-k)$$

$$\Rightarrow \quad p_{X+Y}(n) = \sum_{k=0}^{\infty} p_X(k)\,p_Y(n-k)$$

(DISCRETE) CONVOLUTION

$$\Rightarrow \quad p_{X+Y}(n) = \sum_{k=0}^{\infty} p_X(k)\, p_Y(n-k)$$ (DISCRETE) CONVOLUTION

If $X, Y$ absolutely continuous, $X \perp\!\!\!\perp Y$
$X$ density $f_X$, $Y$ density $f_Y$.
Then $X+Y$ is absolutely continuous with density

$$f_{X+Y}(z) = \int_{\mathbb{R}} f_X(x)\, f_Y(z-x)\, dx$$

$$= f_X * f_Y(z)$$ CONVOLUTION

Example

$X \sim Geo(p) \qquad p \in (0,1)$

$p_X(n) = (1-p)^{n-1} p \qquad n \in \{1, 2, \ldots\}$

Compute $m_X$

$$m_X(t) = E[e^{tX}] = \sum_{n=1}^{\infty} e^{tn} p_X(n)$$

$$= \sum_{n=1}^{\infty} e^{tn} p (1-p)^{n-1} = \sum_{n=1}^{\infty} p\, e^{tn} \frac{(1-p)^n}{1-p}$$

$$= \frac{p}{1-p} \sum_{n=1}^{\infty} \left(e^t (1-p)\right)^n$$

$$\left[\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \quad \text{if} \quad |q| < 1\right]$$

$$\sum_{n=1}^{\infty} q^n = \frac{1}{1-q} - 1 = \frac{q}{1-q}$$

$$= \cancel{\frac{p}{1-p}} \frac{e^t \cancel{(1-p)}}{1-e^t(1-p)}$$

$$= \frac{p e^t}{1-e^t(1-p)} \qquad \text{IF} \quad |e^t(1-p)| < 1$$

$$e^t (1-p) < 1 \qquad e^t < \frac{1}{1-p}$$

$$t < \log\left(\frac{1}{\phantom{x}}\right) = -\log(1-p)$$

$$t < \log\left(\frac{1}{1-p}\right) = -\log(1-p)$$

$$m_X(t) = \begin{cases} \dfrac{pe^t}{1-e^t(1-p)} & \text{if } t < \overbrace{-\log(1-p)}^{>0} \\ +\infty & \text{if } t \geq -\log(1-p) \end{cases}$$

$X$ admits moment generating function

$\Rightarrow$ We can compute $E[X^n] = \frac{d^n}{dt^n} m_X(t)\big|_{t=0} \quad \forall n$

$\left[\; E[X] = \frac{1}{p} \;\right]$

## ABSOLUTELY CONTINUOUS R.V.

$X \sim \mathcal{U}(a,b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

$$m_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f_X(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_{x=a}^{x=b}$$

$$m_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)} \qquad m_X(0) = 1$$

IS $m_X$ continuous in $t=0$ ?

$$\lim_{t\to 0} \frac{e^{tb}-e^{ta}}{t(b-a)} = \lim_{t\to 0} \frac{\frac{d}{dt}(e^{tb}-e^{ta})}{\frac{d}{dt}\, t(b-a)}$$

$$= \lim_{t\to 0} \frac{be^{tb} - ae^{ta}}{b-a} = \frac{b-a}{b-a} = 1 \quad \checkmark$$

- Compute $E[X] = \frac{d}{dt} m_X(t)\big|_{t=0}$

$(t \neq 0)$

$$m_X'(t) = \frac{(be^{tb} - ae^{ta})\,t(b-a) - (e^{tb}-e^{ta})(b-a)}{t^2(b-a)^2}$$

$$= \ldots\ldots$$

Exercise Compute $\lim_{t\to 0} m_X'(t)$

Instead, we compute $m_X'(0)$ by definition

$$m_X'(0) = \lim_{h\to 0} \frac{m_X(h) - m_X(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{e^{hb} - e^{ha}}{h(b-a)} - 1}{h}$$

$$= \lim_{h\to 0} \frac{e^{hb} - e^{ha} - h(b-a)}{h^2(b-a)} = \frac{0}{0}$$

$$\overset{(H)}{=} \lim_{h\to 0} \frac{b\, e^{hb} - a\, e^{ha} - (b-a)}{2h\,(b-a)} = \frac{0}{0}$$

$$\overset{(H)}{=} \lim_{h\to 0} \frac{b^2 e^{hb} - a^2 e^{ha}}{2(b-a)} = \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2}$$

NORMAL DISTRIBUTION

$X \sim N(\mu, \sigma^2)$ $\qquad Z \sim N(0,1)$

$Z$ standard normal $\qquad f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \; x \in \mathbb{R}$

$$m_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

complete square $e^{-\frac{x^2}{2} + tx}$

$$-\frac{x^2}{2} + tx - \frac{t^2}{2} + \frac{t^2}{2} = -\frac{(x-t)^2}{2} + \frac{t^2}{2}$$

$$= \int_{-\infty}^{+\infty} e^{\frac{t^2}{2} - \frac{(x-t)^2}{2}} \frac{1}{\sqrt{2\pi}} dx = e^{\frac{t^2}{2}} \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}}}_{\text{density of } N(t,1)} dx$$

$$= e^{\frac{t^2}{2}} \cdot 1 \qquad \int \text{density} = 1$$

$$m_Z(t) = e^{\frac{t^2}{2}}$$

Similarly, we find $\quad \left[\ \varphi_X(u) = E[e^{iuX}] \simeq m_X(iu)\ \right]$

$$\varphi_Z(u) = e^{\frac{(iu)^2}{2}} = e^{-\frac{u^2}{2}}$$

$$Z \sim N(0,1) \iff \varphi_Z(u) = e^{-\frac{u^2}{2}}$$

<u>Note</u> $\varphi_Z(u) \in \mathbb{R}$, NO imaginary part

<u>Lemma</u> If $X$ is symmetric r.v., i.e. $X$ and $-X$ have the same distribution. Then $\varphi_X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R}$.

In general, $\quad \varphi_{-X}(u) = \overline{\varphi_X(u)}$

complex conjugate : $z = a + ib$, $\bar{z} = a - ib$

$$z = \bar{z} \iff z \in \mathbb{R} \quad (b = 0)$$

<u>Proof</u>

$$\varphi_{-X}(u) = E[e^{iu(-X)}] = E[e^{iX(-u)}]$$
$$= \varphi_X(-u)$$

$$\varphi_X(-u) = \cos(-uX) + i\sin(-uX)$$
$$= \cos(uX) - i\sin(uX)$$
$$= \overline{\cos(uX) + i\sin(uX)} = \overline{\varphi_X(u)}$$

From this, if $X \sim -X$ then $\varphi_X(u) = \varphi_{-X}(u)$

$$\varphi_X(u) = \varphi_{-X}(u) = \overline{\varphi_X(u)}$$
$$\implies \varphi_X(u) \in \mathbb{R}$$

$X \sim N(\mu, \sigma^2)$, $\quad Z \sim N(0,1)$

$$X = \mu + \sigma Z$$

$\varphi_X(u) = E[e^{iuX}] = E[e^{iu(\mu+\sigma Z)}]$

$$X = \mu + \sigma Z$$

$$m_X(t) = E[e^{tX}] = E[e^{t(\mu+\sigma Z)}]$$
$$= E[e^{t\mu} \cdot e^{t\sigma Z}]$$
$$= e^{t\mu} E[e^{t\sigma Z}] = e^{t\mu} m_Z(t\sigma)$$
$$= e^{t\mu} e^{\frac{(t\sigma)^2}{2}} \ .$$

$$m_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

$$\varphi_X(u) = e^{t\mu} \varphi_Z(u\sigma) = e^{u\mu - \frac{u^2\sigma^2}{2}}$$

**Proposition** If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ $X \perp\!\!\!\perp Y$. Then $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

**Proof** $\varphi_{X+Y}(u) = \varphi_X(u) \cdot \varphi_Y(u)$ (INDEPENDENCE)

$$= \exp\left(u\mu_1 - u^2\frac{\sigma_1^2}{2}\right) \cdot \exp\left(u\mu_2 - u^2\frac{\sigma_2^2}{2}\right)$$
$$= \exp\left(u(\mu_1+\mu_2) - \frac{u^2}{2}(\sigma_1^2+\sigma_2^2)\right)$$
$$\Rightarrow X+Y \sim N(\mu_1+\mu_1, \sigma_2^2+\sigma_2^2)$$

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LINEAR TRASFORMATION

$X$ d-dimensional random vector

$A \in \mathbb{R}^{n\times d}$ MATRIX $b \in \mathbb{R}^n$

$Y = AX + b$ $Y \in \mathbb{R}^n$

$Y$ n-dimensional random vector

$$m_Y(t) = E[e^{\langle t, Y\rangle}] \qquad t \in \mathbb{R}^n$$
$$= E[e^{\langle t, AX+b\rangle}] = E[e^{\langle t,b\rangle} e^{\langle t, AX\rangle}]$$
$$= e^{\langle t,b\rangle} E[e^{\langle t, AX\rangle}]$$

$$\langle t, Ax\rangle = t^T Ax = (t^T A x)^T$$

$$\langle t, Ax \rangle = t^T A x = (t^T A x)^T$$
$$= x^T A^T t = \langle x, A^T t \rangle$$

$$= e^{\langle t,b \rangle} E\left[ e^{\langle A^T t, X \rangle} \right]$$
$$= e^{\langle t,b \rangle} m_X (A^T t)$$

$$\varphi_Y(u) = e^{i\langle u,b \rangle} \varphi_X(A^T u) \qquad u \in \mathbb{R}^n$$

Transformation of densities

$X \sim f_X$ absolutely continuous

$X \in \mathbb{R}^n$ $\qquad \varphi : \mathbb{R}^n \to \mathbb{R}^m$, $Y = \varphi(X)$

$\varphi$ $C^1$ with inverse $\varphi^{-1}$ $C^1$, $Y \in \mathbb{R}^m$

$$f_Y(y) = |\det(D\varphi^{-1}(y))| \, f_X(\varphi^{-1}(y))$$

In particular, for $\varphi(x) = Ax + b$

$$f_Y(y) = \frac{1}{\det(A)} f_X(A^{-1}(y-b))$$