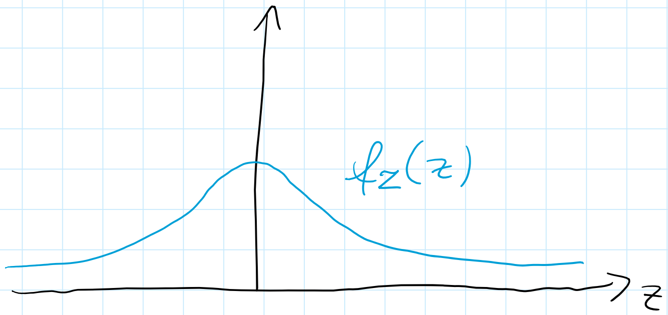
NORMAL (GAUSSIAN) DISTRIBUTION

A real-valued random variable Z is called STANDARD NORMAL if it is absolutely continuous with density $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$



- $\int_{\mathbb{R}} f_Z(z)\,dz = 1 \iff \sqrt{2\pi} = \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}\,dx$ ✓
- $Z \in L^1(\Omega)$?

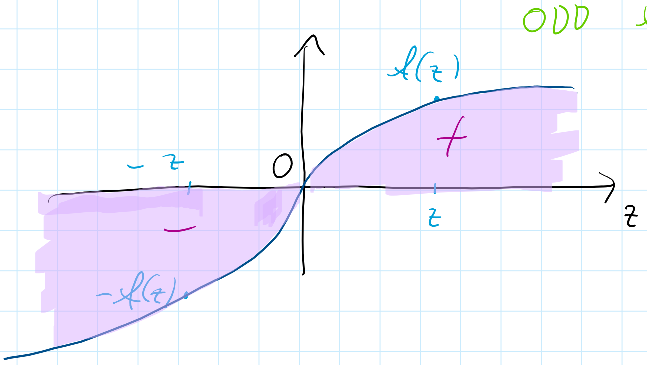
$$\iff E|Z| = \int_{-\infty}^{+\infty} |z| f_Z(z)\,dx < \infty$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |z| e^{-\frac{z^2}{2}}\,dx = \int_{-\infty}^{+\infty} \frac{|z|}{e^{\frac{z^2}{2}}}\,dz < \infty$$

- $E[Z] = \int_{-\infty}^{+\infty} z f_Z(z)\,dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{z\, e^{-\frac{z^2}{2}}}\,dz = 0$$

ODD function $f(-z) = -f(z)$



graph symmetric respect to the origin

" integrals of odd functions on symmetric intervals are zero "

- $E[Z^2] = Var(Z) = \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$

- $E[Z^2] = Var(Z) = \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{z}_{\frac{d}{dz}=1} \cdot \underbrace{z e^{-\frac{z^2}{2}}}_{\int z e^{-\frac{z^2}{2}} dz = -e^{-\frac{z^2}{2}}} dz = \text{INTEGRATION BY PARTS}$$

$$= \left[-\frac{1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}} \right]_{z\to-\infty}^{z\to+\infty} - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot 1 \left(-e^{-\frac{z^2}{2}}\right) dz$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz = 1$$

Z STANDARD NORMAL $\quad E[Z]=0, \quad Var(Z)=1$

$Z \sim N(0,1)$

Consider $\mu \in \mathbb{R}$ $\quad \sigma > 0$

Let $X = \mu + \sigma Z$

Compute density of X $\qquad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz$

Let $\Phi(z) = P(Z \leq z)$ distribution function of $Z \sim N(0,1)$

$\Phi(z) = \int_{-\infty}^{z} f_Z(t) dz \qquad f_Z(t) = \Phi'(z) \quad \forall z$

$$F_X(x) = P(X \leq x) = P(\mu + \sigma Z \leq x)$$
$$= P\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

density $f_X(x) = \frac{d}{dx} F_X(x) = \Phi'\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}\right) \cdot \frac{1}{\sigma}$$

$X \sim N(\mu, \sigma^2)$

NORMAL (GAUSSIAN) r.v

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\sigma\sqrt{2\pi} \qquad 2\sigma^2$$

$E[X] = E[\mu + \sigma Z] = \mu + \sigma E[Z] = \mu$

$Var(X) = Var(\mu + \sigma Z)$

μ constant $\quad Var(\mu) = E[\mu^2] - (E[\mu])^2 = \mu^2 - \mu^2 = 0$

μ constant is independent of any other r.v.

$$Var(X) \overset{\text{INDEP.}}{=} Var(\mu) + Var(\sigma Z)$$
$$= 0 + \sigma^2 Var(Z) = \sigma^2$$

- $X \sim N(\mu, \sigma^2) \Rightarrow E[X] = \mu \;, \; \sigma^2 = Var(X)$

  Given $Z \sim N(0,1) \Rightarrow X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

- $X \sim N(\mu, \sigma^2) \Rightarrow Z = \dfrac{X - \mu}{\sigma} \sim N(0,1)$

- If $X \sim N(\mu, \sigma^2) \quad a, b \in \mathbb{R} \quad b \neq 0$

  then $a + bX \sim N(a + b\mu, b^2\sigma^2)$

MOMENT GENERATING FUNCTION

SCALAR CASE $\qquad X : \Omega \to \mathbb{R}$

Define $\quad m_X : \mathbb{R} \to [0, +\infty] = [0, +\infty) \cup \{+\infty\}$

$$m_X(t) = E[e^{tX}]$$

- $E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$ if X absolutely cont.

m_X is called MOMENT GENERATING FUNCTION

- fix $\omega \qquad \dfrac{d^n}{dt^n} e^{tX(\omega)} = X^n(\omega)\, e^{tX(\omega)}$

$$\frac{d^n}{dt^n} m_X(t) = \frac{d^n}{dt^n} E[e^{tX(\omega)}] = E\left[\frac{d^n}{dt^n} e^{tX}\right]$$

$$\frac{d^n}{dt^n} m_X(t) = \frac{d^n}{dt^n} E[e^{tX(\omega)}] = E\left[\frac{d^n}{dt^n} e^{tX}\right]$$

$$= E[X^n e^{tX}]$$

$$\frac{d^n}{dt^n} m_X(t)\Big|_{t=0} = E[X^n]$$

It can be proved that this identity holds true whenever $\exists\, \varepsilon > 0$ s.t.

$$\boxed{m_X(t) < \infty} \qquad \forall -\varepsilon < t < \varepsilon$$

- recall $L^r(\Omega) = \{ X : E[|X|^r] < \infty \}$

$$m_X(t) < \infty \implies X \in L^n(\Omega) \quad \forall\, n \in \mathbb{N}$$

X admits MGF $\iff E[e^{tX}] < \infty$ for $|t| < \varepsilon$, $\exists\, \varepsilon$
In this case $E[|X|^n] < \infty \quad \forall\, n \in \mathbb{N}$

Example $X \sim Exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$m_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\, dx$$

$$= \int_0^{+\infty} e^{tx} \lambda e^{-\lambda x}\, dx = \lambda \int_0^{+\infty} e^{-(\lambda - t)x}\, dx$$

$$= \left[-\frac{\lambda}{\lambda - t} e^{-(\lambda - t)x} \right]_{x=0}^{x \to +\infty}$$

$$= \frac{\lambda}{\lambda - t} - \underbrace{\lim_{x \to +\infty} \frac{\lambda}{\lambda - t} e^{-(\lambda - t)x}}_{= \begin{cases} 0 & \text{if } \lambda - t > 0 \\ +\infty & \text{if } \lambda - t \leq 0 \end{cases}}$$

$$m_X(t) = \begin{cases} \dfrac{\lambda}{\lambda - t} & \text{if } t < \lambda \\ +\infty & \text{if } t \geq \lambda \end{cases}$$

$+\infty$ if $\lambda - t \leq 0$

$m_X(t) < +\infty$ if $t < \lambda \Rightarrow X$ admits MFG

We can compute moments of X:

$$E[X] = \frac{d}{dt} m_X(t) \Big|_{t=0} = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$\frac{d}{dt} m_X(t) = \frac{\lambda}{\lambda - t} = -\frac{\lambda}{(\lambda-t)^2}(-1) = \frac{\lambda}{(\lambda-t)^2}$$

$$\frac{d^2}{dt^2} m_X(t) = -\frac{2\lambda}{(\lambda-t)^3}(-1) = \frac{2\lambda}{(\lambda-t)^3}$$

$$E[X^2] = \frac{d^2}{dt^2} m_X(t)\Big|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

Example LOG-NORMAL DISTRIBUTION

$Y \sim N(\mu, \sigma^2)$ $\qquad X = e^Y$

In this case we have $E[|X|^n] < \infty \quad \forall n$

BUT $m_X(t) = +\infty \quad \forall t > 0$

Exercise Compute $E[X^n]$ $\quad m_X(t)$

$$E[X^n] = E[e^{nY}] = \int_{-\infty}^{+\infty} e^{ny} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = \dots$$

Example $X \sim Ber(p)$

$$m_X(t) = E[e^{tX}] = e^{t\cdot 1} P(X=1) + e^{t \cdot 0} P(X=0)$$
$$= p e^t + 1 - p$$

CHARACTERISTIC FUNCTION

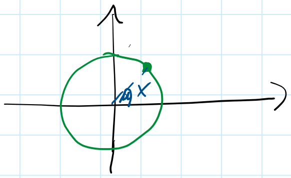
$X : \Omega \to \mathbb{R}$

$\varphi_X : \mathbb{R} \to \mathbb{C}$ complex numbers

$\mathbb{C} = \{ a + ib : a, b \in \mathbb{R} \}$ $\quad i^2 = -1$

$\varphi_X(u) = E[e^{iuX}]$ CHARACTERISTIC FUNCTION

$x \in \mathbb{R} \quad e^{ix} = \cos x + i \sin x \in \mathbb{C}$



$|e^{ix}| = 1$

$|a+ib| = a^2+b^2$

$\varphi_X(u)$ always defined and $|\varphi_X(u)| \leq 1$

Remark If $E|X|^n < \infty$ (X admits moment of order n)

then $\left.\frac{d^n}{du^n} \varphi_X(u)\right|_{u=0} = i^n E[X^n]$

Example $X \sim Ber(p)$

$\varphi_X(u) = E[e^{iuX}] = e^{iu\cdot 0} P(X=0) + e^{iu\cdot 1} P(X=1)$

$= 1\cdot(1-p) + e^{iu} p$

$\frac{d}{du}\varphi_X(u) = 0 + p\, i\, e^{iu}$

$\left.\frac{d}{du}\varphi_X(u)\right|_{u=0} = i E[X] = p\, i\, e^0 = p\, i$

$\Rightarrow E[X] = p$

---

random variables $\longrightarrow$ random vectors

$\mathbb{R} \quad \leftrightarrow \quad m_X(t) = E[e^{tX}]$

$\mathbb{R}^n \quad X = (X_1, \dots, X_d)$

$t = (t_1, \dots, t_n)$

$t \cdot X$ SCALAR PRODUCT

$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

$$x = (x_1, \dots, x_n) \qquad y = (y_1, \dots, y_n)$$

Given $X : \Omega \to \mathbb{R}^n$ random vector, we define

$m_X : \mathbb{R}^n \to [0, +\infty)$ MOMENT GENERATING FUNCTION

$$m_X(t) = E\left[e^{\langle t, X \rangle}\right] = E\left[e^{t_1 X_1 + \dots + t_n X_n}\right]$$

$\varphi_X : \mathbb{R}^n \to \mathbb{C}$ CHARACTERISTIC FUNCTION

$$\varphi_X(u) = E\left[e^{i \langle u, X \rangle}\right]$$

Theorem 1

The characteristic function φ_X characterizes the distribution of the random vector X. This means: if X and Y are random vectors then

$$\varphi_X = \varphi_Y \iff \mu_X = \mu_Y$$

Theorem 2

Let $(X, Y) \in \mathbb{R}^2$ 2-dim random vector with $X \perp Y$ (INDEPENDENT)

Then
$$\varphi_{(X,Y)}(u, v) = \varphi_X(u) \cdot \varphi_Y(v)$$
$$m_{(X,Y)}(s, t) = m_X(s) \cdot m_Y(t)$$

Proof

$$\varphi_{(X,Y)}(u, v) = E\left[e^{i \langle (u,v), (X,Y) \rangle}\right]$$
$$= E\left[e^{iuX + ivY}\right] = E\left[e^{iuX} \cdot e^{ivY}\right]$$
$$\overset{\text{INDEPENDENCE}}{=} E\left[e^{iuX}\right] \cdot E\left[e^{ivY}\right]$$
$$= \varphi_X(u) \cdot \varphi_Y(v) \qquad \blacksquare$$

Recall that $X \perp\!\!\!\perp Y \Rightarrow g_1(X) \perp\!\!\!\perp g_2(Y)$

for any $g_1, g_2 : \mathbb{R} \to \mathbb{R}$