Lecture 3

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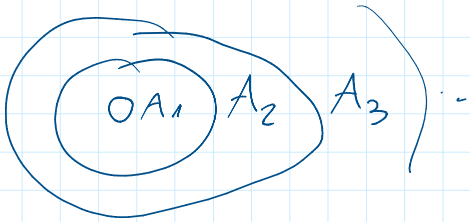
Property 9) CONTINUITY of PROBABILITY

$P : \mathcal{A} \to [0,1]$

- $(A_n)_{n\in\mathbb{N}}$ increasing sequence of events

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \subseteq A_n \subseteq \dots$



$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

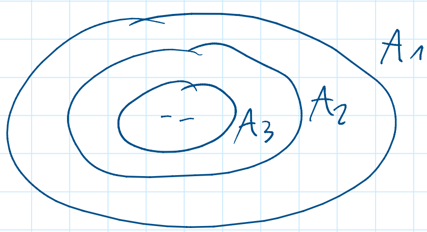
Then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) = \sup_{n\in\mathbb{N}} P(A_n)$$

[Recall, a function $f:\mathbb{R}\to\mathbb{R}$ is continuous if $f(\underbrace{\lim_n x_n}_{=x}) = \lim_n f(x_n)$ for any sequence $x_n \to x$]

- Decreasing sequence of events $(A_n)_{n\in\mathbb{N}}$

$A_1 \supseteq A_2 \supseteq A_3 \dots \supseteq A_n \supseteq A_{n+1} \supseteq \dots$



$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Then

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) = \inf_{n\in\mathbb{N}} P(A_n)$$

Note $P(A_{n+1}) \leq P(A_n)$ by monotonicity

Thus the sequence $(P(A_n))_{n\in\mathbb{N}}$ is decreasing and hence there exists the limit

CONDITIONAL PROBABILITY

$P(A)$ represents how likely the output of random experiment belongs to A

$P(A|B)$ express how the information concerning the occurrence of the event B modifies our belief that A will occur

<u>Def</u> $(\Omega, \mathcal{A}, P)$. Let $A, B \in \mathcal{A}$ with $P(B) > 0$. We define the CONDITIONAL PROBABILITY of A given B as

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

<u>Property 1</u> Given B with $P(B) > 0$, the function

$$\begin{aligned} \mathcal{A} &\longrightarrow [0,1] \\ A &\longmapsto P(A|B) \end{aligned}$$

is a probability on $\mathcal{A}$

<u>Proof</u> i) $P(\Omega|B) = 1$?

$P(\underbrace{\Omega \cap B}_{=B})$, so $P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

ii) σ-additivity ? exercise

<u>Example</u>

Toss 2 dice: what is the probability that the first die is less or equal to 2, given that the sum of the 2 dice is equal to 4?

<u>Sol</u>

Sol

$\Omega = \{1,2,3,4,5,6\}^2 = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$

$\mathcal{A} = 2^\Omega \qquad |\Omega| = 6 \cdot 6 = 36$

P uniform probability : $P(\{(i,j)\}) = \frac{1}{|\Omega|} = \frac{1}{36}$

$B = \{\text{the sum is } 4\} = \{(1,3),(2,2),(3,1)\}$

$|B| = 3 \qquad P(B) = \frac{|B|}{|\Omega|} = \frac{3}{36} = \frac{1}{12}$

$A = \{\text{first die} \leq 2\}$

$= \{(1,j),(2,j) : j \in 1,\dots,6\}$

$= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\}$

$|A| = 6+6 = 12 \qquad P(A) = \frac{|A|}{|\Omega|} = \frac{12}{36} = \frac{1}{3}$

$A \cap B = \{(1,3),(2,2)\}$

$P(A\cap B) = \frac{|A\cap B|}{|\Omega|} = \frac{2}{36} = \frac{1}{18}$

Then $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|} = \frac{2}{3}$

consider $C = \{\text{sum equal to } 12\} = \{(6,6)\}$

$P(A|C) = 0$ as $A \cap C = \emptyset$

$= \frac{P(A\cap C)}{P(C)} = 0$

PROPERTIES OF CONDITIONAL PROBABILITY

1) $A, B \in \mathcal{A}$ FORMULA OF TOTAL PROBABILITY

$$P(A) = P(A|B)\,P(B) + P(A|B^c)\,P(B^c)$$

1 bis) $(B_n)_{n\in\mathbb{N}} \in \mathcal{A}$ DISJOINT, i.e. $B_i \cap B_j = \emptyset$ if $j \neq i$,

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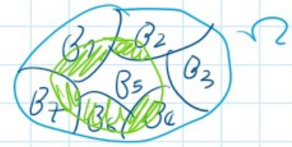
and $\bigcup_{n=1}^{\infty} B_n = \Omega$

$$P(A) = \sum_{n=1}^{\infty} P(A|B_n)\,P(B_n)$$



2) $$\boxed{P(A|B) = P(B|A)\,\frac{P(A)}{P(B)}}$$ BAYES FORMULA

Proof

1 (BIS) $A = \bigcup_{n=1}^{\infty} A\cap B_n$ and $(A\cap B_n)\cap(A\cap B_m) = \emptyset$

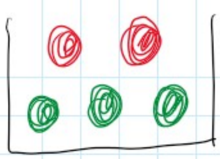
$\overset{\shortparallel}{A} \cap \underbrace{B_n \cap B_m}_{=\emptyset}$

$P(A) = P\left(\bigcup_{n=1}^{\infty} A\cap B_n\right) \leftarrow$ DISJOINT UNION

$\overset{\sigma\text{-ADDITIVITY}}{=} \sum_{n=1}^{\infty} P(A\cap B_n) = \sum_{n=1}^{\infty} P(A|B_n)\,P(B_n)$

2) note $P(A\cap B) = P(A|B)\,P(B) = P(B|A)\cdot P(A)$.

Exercise Consider a box with 2 RED and 3 GREEN balls. Withdraw 2 balls without replacement.



$A_1 = \{$ the first ball is green $\}$

$A_2 = \{$ the second ball is green $\}$

Compute $P(A_1)$ and $P(A_2)$.

Solution

$P(A_1) = \frac{3}{5}$

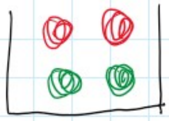
$P(A_2)$? We don't know color of first ball.

We use formula of total probability:

$P(A_2) = P(A_2|A_1)\,P(A_1) + P(A_2|A_1^c)\,P(A_1^c)$

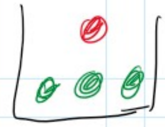
$P(A_1^c) = 1 - P(A_1) = \frac{2}{5}$

$P(A_2 \mid A_1)$ = probability that 2^{nd} ball is green knowing that first ball is green $\rightarrow$ "event A_1 occurs"



$$P(A_2 \mid A_1) = \frac{2}{4} = \frac{1}{2}$$

$$P(A_2 \mid A_1^c) = \frac{3}{4}$$



$A_1^c = \{$ first ball is red $\}$

then $P(A_2) = \frac{1}{2} \cdot \frac{3}{5} + \frac{3}{4} \cdot \frac{2}{5} = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$

Note $P(A_2) = P(A_1)$

Exercise: $P(A_3) = \frac{3}{5}$

INDEPENDENCE

$A \perp\!\!\!\perp B$ A and B are independent

Def $A, B \in \mathcal{A}$ with $P(B) > 0$

We say that A and B are INDEPENDENT if

$$P(A \mid B) = P(A)$$

"Means that the information that B has happened gives no information about the occurrence of A"

Note $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(A)$

Def $A, B \in \mathcal{A}$ (without requiring $P(B) > 0$)

A and B are INDEPENDENT if

$$P(A \cap B) = P(A) \cdot P(B)$$

$P(A \cap B) = P(A) \cdot P(B)$

<u>Remark 1</u> if $P(B) = 0$ then B is independent of any other event.

<u>Proof</u> $A \cap B \subseteq B$

$$0 \le P(A \cap B) \underset{\substack{\uparrow \\ \text{MONOTONICITY}}}{\le} P(B) = 0$$

$$0 = P(A \cap B) = P(B) \cdot P(A) = 0$$

<u>Remark 2</u>

$$\text{DISJOINT} \overset{\text{FAR}}{\neq} \text{INDEPENDENCE}$$

Take A, B disjoint : $A \cap B = \emptyset$

$$P(A \cap B) = 0 \overset{?}{=} P(A)\,P(B)$$

$\Longrightarrow$ Two disjoint events are independent only if $P(A) = 0$ or $P(B) = 0$

<u>Note</u> that A and A^c are disjoint and never independent (except when $P(A) = 0$ or $P(A) = 1$)

<u>Example</u> Toss two dice

$A = \{$ First die gives $1 \}$

$B = \{$ Second die gives $6 \}$ $\quad A \perp\!\!\!\perp B$?

$A = \{ (1, j) : j \in \{1, \dots, 6\} \}$ $\qquad |A| = 6$

$B = \{ (i, 6) : i \in \{1, \dots, 6\} \}$ $\qquad |B| = 6$

$$P(A) = \frac{6}{36} = P(B)$$

$A \cap B = \{(1, 6)\}$ $\qquad P(A \cap B) = \dfrac{|A \cap B|}{|\Omega|} = \dfrac{1}{36}$

$A \cap B = \{(1,6)\}$ $\qquad$ $P(A \cap B) = \dfrac{|A \cap B|}{|\Omega|} = \dfrac{1}{36}$

So $\quad P(A \cap B) = \dfrac{1}{36} = \dfrac{1}{6} \cdot \dfrac{1}{6} = P(A) \cdot P(B)$

$\Rightarrow$ A and B are independent

$C = \{$ the sum of the 2 dice is equal to 7 $\}$

$A \perp\!\!\!\perp C$? $\quad B \perp\!\!\!\perp C$?

$C = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$P(C) = \dfrac{6}{36} = \dfrac{1}{6}$

$A \cap C = \{(1,6)\}$ $\qquad$ $B \cap C = \{(1,6)\}$

$\left[P(A|C) = \dfrac{1}{6} = P(B|C) \right]$ and $P(A|C) = P(A)$

$P(A \cap C) = \dfrac{1}{36}$ $=$ $P(A) \cdot P(C) = \dfrac{1}{6} \cdot \dfrac{1}{6} = 36$

$\Rightarrow$ $A \perp\!\!\!\perp C$ and $B \perp\!\!\!\perp C$

Means that the information that the sum is 7 does not influence the probability that the first die gives 1 (or second gives 6)

$D = \{$ the sum is equal to 6 $\}$, $A \perp\!\!\!\perp D$?

$D = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$P(D) = \dfrac{5}{36}$

$A \cap D = \{(1,5)\}$ $\qquad$ $P(A \cap D) = \dfrac{1}{36}$

$P(A \cap D) = \dfrac{1}{36} \neq \dfrac{1}{6} \cdot \dfrac{1}{5} = P(A) \cdot P(D)$

$\Rightarrow$ A and D are not independent

$B \cap D = \emptyset$ $\qquad$ $P(A \cap D) = 0$

$B \cap D = \emptyset \qquad P(A \cap D) = 0$

B and D are disjoint, but not independent,

as $P(B) \neq 0$ and $P(D) \neq 0$

We could also show that the event

$\{$sum equal to 2$\}$ or 3, 4, 5, 6, 8, 9, 10, 11, 12

is NOT independent of $\{$first die gives 1$\}$

(OR 2, 3, 4, 5, 6)

They are independent only in case sum is 7.