EXPECTATION of RANDOM VARIABLES

INDICATOR FUNCTION of a set $A \in \mathcal{A}$

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$\mathbb{1}_A$ is a random variable $\mathbb{1}_A : \Omega \to \{0,1\}$

$\mathbb{1}_A$ is discrete, takes values 0 or 1

Compute expectation:

$$E[\mathbb{1}_A] = 0\, P(\mathbb{1}_A = 0) + 1\, P(\mathbb{1}_A = 1)$$
$$= P(\mathbb{1}_A = 1) = P(A)$$

$$\{\mathbb{1}_A = 1\} = \{\omega : \mathbb{1}_A(\omega) = 1\} = A$$

$$E[\mathbb{1}_A] = P(A)$$

EXPECTATION of a general random variable

$X : \Omega \to \mathbb{R}$ r.v, $(\Omega, \mathcal{A}, P)$ probability space

Consider first $X \geq 0$

→ We approximate from below X by a sequence of positive DISCRETE and increasing random variables:

$(X_n)_n$ X_n discrete, $X_n(\omega) \leq X_{n+1}(\omega)$ $\forall \omega$

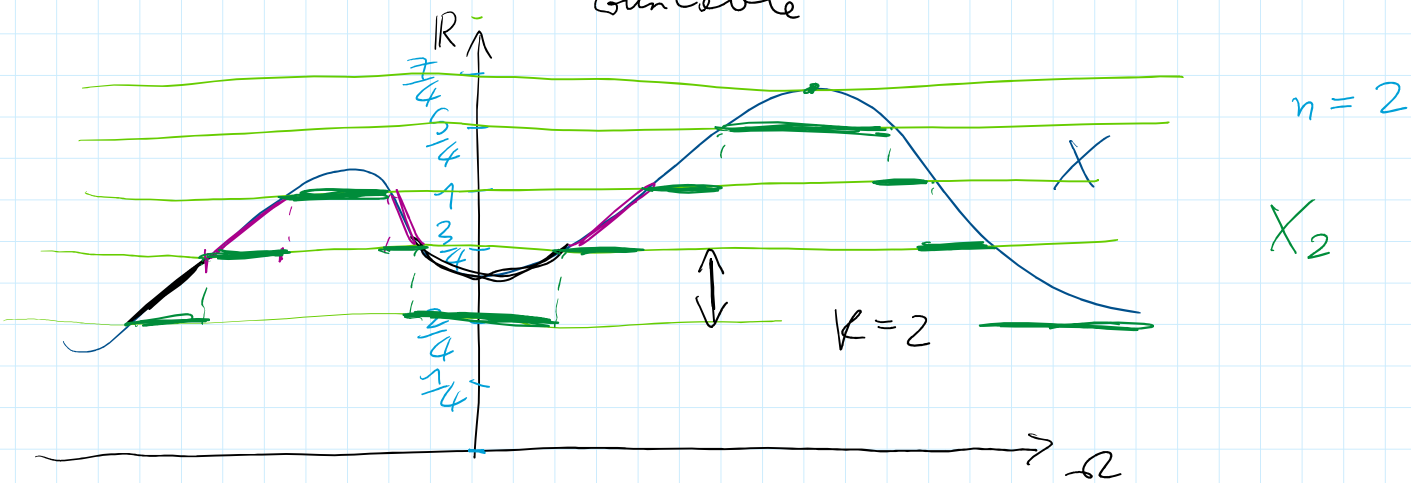
then we want to define $E[X] = \lim_n E[X_n]$

Define, $\forall n \in \mathbb{N}$, $X_n(\omega) = \frac{k}{2^n}$ if $\boxed{\frac{k}{2^n} \leq X(\omega) < \frac{k+1}{2^n}}$

X_n discrete $X_n \in \left\{ \frac{0}{2^n}, \frac{1}{2^n}, \dots, \frac{2^n}{2^n}, \dots, \frac{4 \cdot 2^n}{2^n}, \dots \right\}$

X_n discrete $\quad X_n \in \left\{ \frac{0}{2^n}, \frac{1}{2^n}, \dots, \frac{2^n}{2^n}, \dots, \frac{4 \cdot 2^n}{2^n}, \dots \right\}$

countable



We approximate with stepwise functions

Compute $E[X_n]$

$$X_n(\omega) = \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbb{1}_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}(\omega)$$

$$E[X_n] = \sum_k \frac{k}{2^n} P\left(X_n = \frac{k}{2^n}\right) = \sum_k \frac{k}{2^n} P\left(\frac{k}{2^n} \le X < \frac{k+1}{2^n}\right)$$

$$\left\{ X_n = \frac{k}{2^n} \right\} = \left\{ \frac{k}{2^n} \le X < \frac{k+1}{2^n} \right\}$$

$X_n \nearrow X \quad$ increasing

$X_n \le X_{n+1} \le X \to E[X_n] \le E[X_{n+1}]$

$\Rightarrow$ the sequence $(E[X_n])_n$ is increasing

$\Rightarrow$ it admits a limit

We define, for $X \ge 0$

$$E[X] = \lim_{n \to \infty} E[X_n] = \sup_{n \in \mathbb{N}} E[X_n]$$

if $X : \Omega \to \mathbb{R}$ r.v.

$X^+(\omega) = \max\{X(\omega), 0\} \qquad$ POSITIVE PART

$X^+(\omega) = \max\{X(\omega), 0\}$ POSITIVE PART

$X^-(\omega) = -\min\{X(\omega), 0\}$ NEGATIVE PART

$$\left[x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases} \qquad x^- = \begin{cases} -x & x \leq 0 \\ 0 & x \geq 0 \end{cases} \right]$$

$X^+ \geq 0 \qquad X^- \geq 0$

$X = X^+ - X^- \qquad |X| = X^+ + X^-$

Def We say that $X : \Omega \to \mathbb{R}$ r.v admits EXPECTATION if at least one of $E[X^+]$ and $E[X^-]$ is finite, and we define $E[X] = E[X^+] - E[X^-]$

If $E[X^+]$ and $E[X^-]$ are both finite then we say that X is integrable, we write $X \in L^1(\Omega)$, and we have $E[X] = E[X^+] - E[X^-]$ is FINITE.

Note X is integrable if and only if $|X|$ is integrable

$$L^1(\Omega) = \{X : \Omega \to \mathbb{R} \mid E|X| < \infty\}$$

ABSOLUTELY CONTINUOUS RANDOM VARIABLES

$X \geq 0 \qquad X$ admits density f_X

$$E[X] = \lim_{n\to\infty} E[X_n]$$

$$= \lim_n \sum_{k=0}^{\infty} \frac{k}{2^n} \underbrace{P\left(\frac{k}{2^n} \leq X < \frac{k+1}{2^n}\right)}_{= \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f_X(t)\, dt}$$

$$= \sum^{\infty} \int^{\frac{k+1}{2^n}} k$$

$$= \lim_{n \to \infty} \sum_{k=0}^{\infty} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \frac{k}{2^n} f_X(t)\, dt$$ = approximation of the integral

$$\boxed{E[X] = \int_0^{+\infty} t\, f_X(t)\, dt}$$ if $X \geq 0$

for a general absolutely continuous r.v $X : \Omega \to \mathbb{R}$

$$E[X] = E[X^+] - E[X^-]$$

$$\Omega \xrightarrow{X} E \xrightarrow{g} \mathbb{R}$$ $(E, \mathcal{E})$ general measurable space

$$g \circ X = g(X) : \Omega \to \mathbb{R}$$

- if X is discrete

  (only when $|g\, p_X|$ is summable)

  $$E[g(X)] = \sum_{x \in E} g(x) \cdot p_X(x)$$

- if X is absolutely continuous

  $$E[g(X)] = \int_{\mathbb{R}} g(x) \cdot f_X(x)\, dx$$

  (only when $|g\, f_X|$ is integrable)

Properties

1) MONOTONICITY $X, Y : \Omega \to \mathbb{R}$

if $X \leq Y$ then $E[X] \leq E[Y]$

(means $X(\omega) \leq Y(\omega)$ $\forall \omega$)

2) LINEARITY $X, Y : \Omega \to \mathbb{R}$ $a, b \in \mathbb{R}$

$$E[aX + bY] = a\,E[X] + b\,E[Y]$$

$\to$ $E[X+Y] = E[X] + E[Y]$

$$\to \quad E[X+Y] = E[X] + E[Y]$$
$$E[aX] = a\,E[X]$$

3) fundamental inequality

$$|E[X]| \le E[|X|]$$

4) In measure theory, what we have defined is called LEBESGUE INTEGRAL, and is denoted

$$E[g(X)] = \int_\Omega g(X)\,dP = \int_E g(x)\,d\mu_X$$

$X : \Omega \to E$
$g : E \to \mathbb{R}$
$(\Omega, \mathcal{A}, P)$

$X : \Omega \to \mathbb{R}$

$$E[X] = \int_{\mathbb{R}} x \, dF_X(x)$$

$L^1(\Omega)$ = space of integrable r.v. $X : \Omega \to \mathbb{R}$

Example of a NON-INTEGRABLE r.v.

discrete r.v. $\quad X : \Omega \to \{1, 2, \dots\}$

$$P(X = k) = p_X(k) = \frac{6}{\pi^2}\frac{1}{k^2} \qquad \text{for } k \in \mathbb{N},\ k \neq 0$$

- p_X is a density:

$$1 \overset{?}{=} \sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} \frac{6}{\pi^2}\frac{1}{k^2} = \frac{6}{\pi^2} \underbrace{\sum_{k=1}^{\infty}\frac{1}{k^2}}_{=\frac{\pi^2}{6}} = 1$$

- X is not integrable, i.e. $E[X] = +\infty$

note $X \geq 0$

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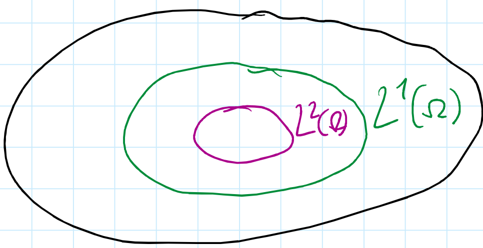
$$E[X] = \sum_{k=1}^{\infty} k \, p_X(k) = \sum_{k=1}^{\infty} k \cdot \frac{6}{\pi^2} \frac{1}{k^2} = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

SPACE of SQUARE-INTEGRABLE r.v.

$X : \Omega \to \mathbb{R}$ consider $X^2 \geq 0$

$$L^2(\Omega) = \{ X : \Omega \to \mathbb{R} \mid E[X^2] < \infty \}$$

- X discrete $\to$ $E[X^2] = \sum_{x \in \mathbb{R}} x^2 p_X(x)$
- X absolutely continuous $\to$ $E[X^2] = \int_0^{+\infty} x^2 f_X(x) \, dx$



$L^0(\Omega) \to$ space of r.v $X : \Omega \to \mathbb{R}$

Lemma $L^2(\Omega) \subseteq L^1(\Omega)$ and $E|X| \leq \sqrt{E[X^2]}$

i.e. $E[X^2] < \infty \Rightarrow E[|X|] < \infty$

Proof

We use inequality $|x| \leq 1 + x^2$

$$\begin{aligned} |x| &= |x| \mathbb{1}_{\{|x| \leq 1\}} + |x| \mathbb{1}_{\{|x| \geq 1\}} \\ &\leq 1 \, \mathbb{1}_{\{|x| \leq 1\}} + x^2 \mathbb{1}_{\{|x| \geq 1\}} \\ &\leq 1 + x^2 \end{aligned}$$

$$\overset{\text{(MONOTONICITY)}}{\Rightarrow} \quad E|X| \leq E[1 + X^2] \overset{\text{(LINEARITY)}}{=} E[1] + E[X^2] < \infty$$

$$\Rightarrow \quad E|X| \le E[1+X^2] = \underbrace{E[1]}_{=1} + \underbrace{E[X^2]}_{<\infty} < \infty$$

Remark

$X = c \in \mathbb{R}$ constant random variable : $X(\omega) = c \quad \forall \omega$

$E[X] = c$

X discrete $\quad X(\omega) = c \quad \forall \omega \qquad P(X=c) = 1$

$E[X] = c \cdot P(X=c) = c$

Example of $X \in L^1(\Omega) \setminus L^2(\Omega)$

a r.v. integrable, but not square integrable

$X : \Omega \to \{1, 2, \dots\}$ discrete

$P(X=k) = \frac{c}{k^3} \qquad \forall k \in \mathbb{N} \setminus \{0\}$

- $\sum_{k=1}^{\infty} P(X=k) = c \sum_{k=1}^{\infty} \frac{1}{k^3} = 1$ if $c = \frac{1}{\sum_{k=1}^{\infty} \frac{1}{k^3}}$

- $X \in L^1(\Omega)$: $E[X] = \sum_{k=1}^{\infty} k \cdot P(X=k)$
$$= \sum_{k=1}^{\infty} k \cdot \frac{c}{k^3} = c \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

- $X \notin L^2(\Omega)$: $E[X^2] = \sum_{k=1}^{\infty} k^2 P(X=k)$
$$= \sum_{k=1}^{\infty} k^2 \cdot \frac{c}{k^3} = c \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

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given $p > 0$

$L^p(\Omega) = \{ X : \Omega \to \mathbb{R} \mid E[|X|^p] < \infty \}$

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$$L^p(\Omega) = \{ X : \Omega \to \mathbb{R} \mid E[|X|^p] < \infty \}$$

given $p, q > 0$ $p < q$, we have

$$L^q(\Omega) \subseteq L^p(\Omega) \quad \text{and} \quad \left(E[|X|^p]\right)^{\frac{1}{p}} \leq \left(E[|X|^q]\right)^{\frac{1}{q}}$$

Example of ABSOLUTELY CONTINUOUS r.v.

$X : \Omega \to \mathbb{R}$

1) density $f_X(x) = \frac{1}{2\sqrt{x}} \mathbb{1}_{(0,1]}(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x > 1 \\ & \phantom{\text{if }} x \leq 0 \end{cases}$

- f_X is a density: $\int_{-\infty}^{+\infty} f_X(x)\,dx \overset{?}{=} 1$

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = \left[\sqrt{x}\right]_{x=0}^{x=1} = 1$$

- $X \in L^1(\Omega)$

$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\,dx = \int_0^1 x \frac{1}{2\sqrt{x}}\,dx = \int_0^1 \frac{\sqrt{x}}{2}\,dx$$

$$= \frac{2}{3} \cdot \frac{1}{2} \cdot \left[x^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}$$

2) $f_X(x) = \frac{1}{x^2} \mathbb{1}_{[1,+\infty)}(x)$

- $\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_1^{+\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{x=1}^{x\to+\infty} = \underbrace{-\frac{1}{+\infty}}_{=0} + 1 = 1$

- $E[X] = \int_{\mathbb{R}} x\, f_X(x)\,dx = \int_1^{+\infty} x \frac{1}{x^2}\,dx$

$$= \int_1^{+\infty} \frac{1}{x}\,dx = \left[\log|x|\right]_{x=0}^{x\to+\infty} = \log(+\infty) - \log 0$$

$= +\infty$

$\Rightarrow$ This r.v. is not integrable

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