Lecture 15

giovedì 6 novembre 2025 16:31

- FINAL EXAM (for students DATA SCIENCE)

  FIXED 18 DECEMBER 16:30 - 19:30 P300 & LUF1
- TUTOR DS - meeting 10 NOV 14:30 1BC45

  moodle TUTOR DS, and CF
- NOTES on double integrals (in Italian)

EXERCISES

Let $X_1, X_2, \dots, X_n$ independent r.v, each uniformly distributed on $[0,1]$.

Set $Z = X_1 + \dots + X_n$

1) Show that $F_Z(z) = P(Z \le z) = \frac{z^n}{n!}$ for $0 \le z \le 1$

- $X_1 \sim \mathcal{U}(0,1)$ $(n=1)$

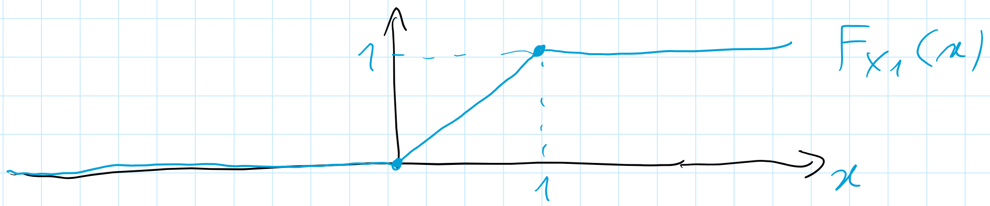
  X_1 abs. continuous r.v. density

  $$f_{X_1}(x) = 1\, \mathbb{1}_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

  Compute distribution function

  $$F_{X_1}(x) = P(X_1 \le x) = \int_{-\infty}^{x} f_{X_1}(t)\,dt$$

  - $= 0$ if $x < 0$
  - $= \int_{-\infty}^{0} 0\,dt + \int_0^x 1\,dt = x$ if $0 \le x \le 1$
  - $= \int_0^1 1\,dt + \int_1^x 0\,dx = 1$ if $x > 1$



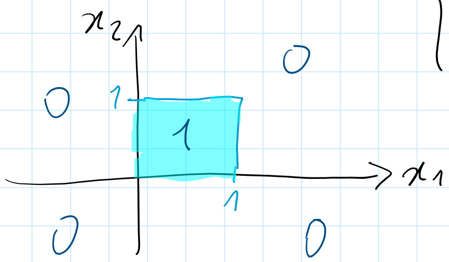
- $n=2 \qquad P(X_1+X_2 \le z) = ? \qquad z \in [0,1]$

$X_1 \perp\!\!\!\perp X_2 \qquad$ consider joint distribution

distribution of vector (X_1,X_2) on $\mathbb{R}^2$

X_1, X_2 ABS. CONT $\Rightarrow (X_1,X_2)$ is ABS. CONT

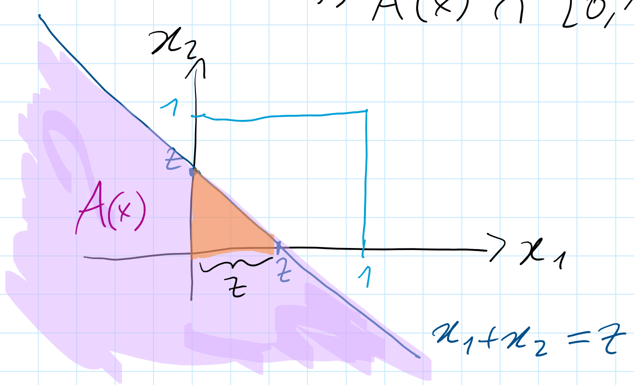
$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$
$$= 1\, \mathbb{1}_{[0,1]}(x_1)\ 1\ \mathbb{1}_{[0,1]}(x_2)$$
$$= \begin{cases} 1 & \text{if } (x_1,x_2) \in [0,1]^2 \\ & (\text{if } x_1 \text{ and } x_2 \in [0,1]) \\ 0 & \text{otherwise} \end{cases}$$



Compute $P(X_1+X_2\le z)$ through joint density

$$\{X_1+X_2 \le z\} = \{(X_1,X_2) \in A(z)\}$$
$$A(z) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1+x_2 \le z\}$$

$$P(X_1+X_2 \le z) = P((X_1,X_2)\in A(z))$$
$$= \iint_{A(x)} f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$
$$= \iint_{A(x)} \mathbb{1}_{[0,1]^2}(x_1,x_2)\, dx_1\, dx_2$$
$$= \iint_{A(x)\cap[0,1]^2} 1\, dx_1\, dx_2 = \text{Area}\left(A(x)\cap[0,1]^2\right)$$



$$= \text{Area (Triangle)}$$
$$= \frac{z^2}{2}$$

- We prove the claim $\forall n$ by induction

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Assume claim true for n

Prove claim true for $n+1$

We assume $P(\underbrace{X_1 + \dots X_n}_{=Z} \leq z) = \frac{z^n}{n!}$ for $0 \leq z \leq 1$

Let $W = X_1 + \dots + X_n + X_{n+1} = Z + X_{n+1}$

$Z \in [0,n] \rightarrow (Z, X_{n+1}) \in [0,n] \times [0,1]$

Density of joint vector (Z, X_{n+1})

$$F_Z(z) = \frac{z^n}{n!} \rightarrow f_Z(z) = F_Z'(z) = \frac{\cancel{n}\, z^{n-1}}{\cancel{n}(n-1)!}$$

$$f_Z(z) = \frac{z^{n-1}}{(n-1)!} \quad \text{if} \quad 0 \leq z \leq 1$$

$$f_{Z,X_{n+1}}(z,x) = f_Z(z) \cdot f_{X_{n+1}}(x)$$

($Z = X_1 + \dots + X_n$ INDEPENDENT of X_{n+1})

$$= \frac{z^{n-1}}{(n-1)!}\, 1 \quad \text{if} \quad \begin{matrix} 0 \leq z \leq 1 \\ 0 \leq x \leq 1 \end{matrix}$$

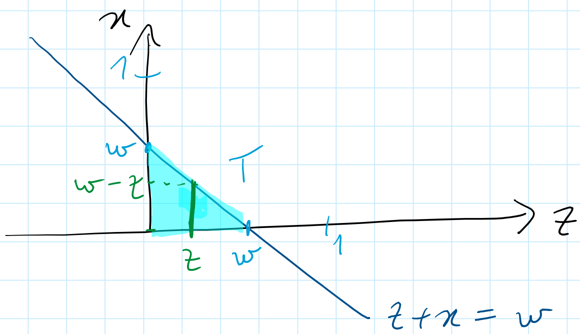
$$(= 0 \quad \text{if} \quad z < 0)$$

$$F_W(w) = P(W \leq w) = P(Z + X_{n+1} \leq w)$$

$$= P((Z, X_{n+1}) \in B(w))$$

$$B(w) = \{(z,x) : z + x \leq w\}$$

$$= \iint_{B(x)} f_{Z,X_{n+1}}(z,x)\, dx\, dz$$



$$\iint_T \frac{z^{n-1}}{(n-1)!}\, 1\, dx\, dz$$

$=$ iterative formula

$$T = \begin{cases} 0 \leq z \leq w & \leftarrow \text{projection of } T \text{ over } z \text{ axes} \\ 0 \leq x \leq w - z & \leftarrow \text{for } z \text{ fixed} \end{cases}$$

$= \int^{w} \int^{w-z}$

$(0 \le x \le w-z$ for z fixed

$$= \int_0^w \left(\int_0^{w-z} \frac{z^{n-1}}{(n-1)!} dx \right) dz$$

$$= \int_0^w \frac{z^{n-1}}{(n-1)!} (w-z) \, dz$$

$$= w \int_0^w \frac{z^{n-1}}{(n-1)!} dz - \int_0^w \frac{z^n}{(n-1)!} dz$$

$$= \frac{w}{(n-1)!} \left[\frac{1}{n} z^n \right]_{z=0}^{z=w} - \left[\frac{1}{n+1} \frac{z^{n+1}}{(n-1)!} \right]_{z=0}^{z=w}$$

$$= \frac{w^{n+1}}{(n-1)!\,n} - \frac{w^{n+1}}{(n-1)!(n+1)} = \frac{w^{n+1}}{(n-1)!} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \frac{w^{n+1}}{(n-1)!} \frac{1}{n(n+1)} = \frac{w^{n+1}}{(n+1)!}$$

The claim is true for $n+1$ ☒

2) Determine the expected number of independent $U(0,1)$ r.v.-s that need to be summed to exceed 1

- $X_1 \le 1$ $\qquad\qquad$ $P(X \in [0,1]) = 1$

  $P(X_1 > 1) = 0$

- X_1, X_2 $\qquad$ event $\{\omega \in \Omega : X_1(\omega) + X_2(\omega) > 1\}$

  $N(\omega) = 2$ if $\omega \in$

Define discrete r.v. $N(\omega)$

$N(\omega) := \min\{n \in \mathbb{N} : X_1(\omega) + X_2(\omega) + \dots + X_n(\omega) > 1\}$

Compute $E[N]$ $\qquad\qquad$ $N : \Omega \to \mathbb{N}$

Density? $\qquad P(N=0) = 0 \qquad\qquad P(N=1) = 0$

$P(N=2) = P(X_1 + X_2 > 1)$

How to compute expectation?

$E[N] = \sum_{n=0}^{\infty} n \, P(N=n)$

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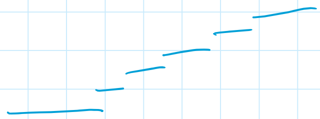
$$= \sum_{n=0}^{\infty} P(N>n)$$

$\left[\text{recall} \quad E[X] = \int_0^{+\infty} P(X>x)\,dx \quad \text{if } X \geq 0 \right]$

if X discrete $\quad P(X>x) = P(X>n)$ if $n \leq x < n+1$

F_N is piecewise constant



We have $\{N>m\} = \{X_1 + \dots + X_m \leq 1\}$

Question: Compute $E[N]$

$$E[N] = \sum_{n=0}^{\infty} P(N>n) = \sum_{n=0}^{\infty} P(X_1 + \dots + X_n \leq 1)$$

From point (1) $\quad P(X_1 + \dots + X_n \leq z) = \dfrac{z^n}{n!}$ for $z \in [0,1]$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

$\left[\text{recall} \quad e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} \right]$