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FINAL EXAM IN DECEMBER (PREAPPELLO)
JUST for students in DATA SCIENCE
NOT for students in COMPUTATIONAL FINANCE

RANDOM VARIABLES

$(\Omega, \mathcal{A}, P)$ probability space

$X : \Omega \to \mathbb{R}$

s.t. $X^{-1}(B) \in \mathcal{A} \quad \forall B \in \mathcal{B}(\mathbb{R})$

μ_X distribution (or law) of X

$\mu_X(B) = P(X^{-1}(B)) = P(X \in B)$

We can prove that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$
is a PROBABILITY SPACE

$\mu_X : \mathcal{B}(\mathbb{R}) \to [0,1]$ is a PROBABILITY

DISCRETE r.v.

$X : \Omega \to \mathbb{R}$ is DISCRETE is $\exists N \subseteq \mathbb{R}$ DISCRETE
s.t. $P(X \in N) = 1$

We can define the DENSITY (DISCRETE)
$p : \mathbb{R} \to [0,1]$ given by $p(x) = P(X = x)$
has 3 properties :

i) $p(x) = 0 \quad \forall x \in \mathbb{R} \setminus N$

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ii) $p(x) \geq 0 \quad \forall x \in N$

iii) $\sum_{x \in \mathbb{R}} p(x) = \sum_{x \in N} p(x) = 1$

On the converse, any function p with properties i) ii) iii) is the density of a discrete r.v.

Example

$N = \{0, 1, \dots, n\} \qquad |N| = n+1$

Consider the function $p : \mathbb{R} \to [0,1]$

- $p(x) = 0$ if $x \notin N$
- $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in N$

where $p \in [0,1]$

Is it the density of a discrete r.v. ?

$(0! = 1)$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{k(k-1)\cdots 1\cdot (n-k)(n-k-1)\cdots 1}$$

$\hookrightarrow$ represents the number of subsets of cardinality k in a set of cardinality n

We check that p satisfies i) ii) iii)

i) ✓

ii) $p(k) \geq 0 \qquad p(k) = \underbrace{\binom{n}{k}}_{\geq 0} \underbrace{p^k}_{\geq 0} \underbrace{(1-p)^{n-k}}_{\geq 0} \geq 0$ ✓

iii) $\sum_{k=0}^{n} p(k) = 1$

We use BINOMIAL FORMULA

$(a+b)^n = \sum^{n} \binom{n}{k} a^k b^{n-k}$

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$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Apply with $a = p$, $b = 1-p$

$$(p+1-p)^n = 1^n = 1 = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \quad \checkmark$$

$\Rightarrow$ p is the density of a discrete random variable

$p \sim Bin(n,p)$ BINOMIAL of parameters n and p

Remark $a = b = 1$

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

"total number of subsets of a set of cardinality n"

$= |2^\Omega|$ where $|\Omega| = n$

$= 2^n$

$X : \Omega \to \mathbb{R}$

$\mu_X : \mathcal{B}(\mathbb{R}) \to [0,1]$ is a probability

In particular we have

- $A, B \subseteq \mathbb{R}$ $\quad A \subseteq B \Rightarrow \mu_X(A) \leq \mu_X(B)$

(MONOTONICITY)

- $\lim_n A_n = A \Rightarrow \lim_n P(A_n) = P(A)$

(CONTINUITY)

if $(A_n)_{n\in\mathbb{N}}$ is increasing or decreasing

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$

We define

$F_X : \mathbb{R} \to [0,1]$ DISTRIBUTION FUNCTION of X

$x \longmapsto F_X(x) = P(X \leq x)$

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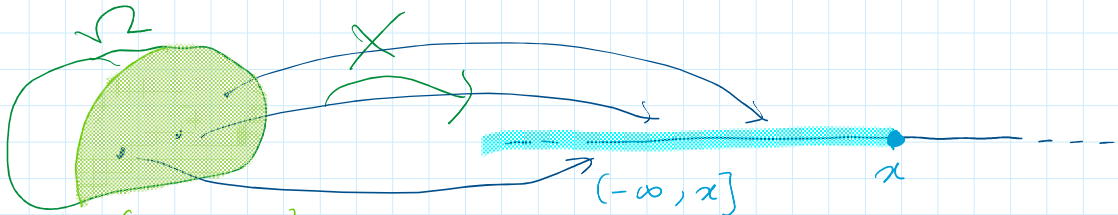
$$= P(X \in (-\infty, x])$$
$$= \mu_X((-\infty, x])$$



$\{X \leq x\} = \{\omega \in \Omega : X(\omega) \leq x\}$

$F_X(0) = P(X \leq 0)$

$F_X\left(\frac{3}{2}\right) = P\left(X \leq \frac{3}{2}\right)$

...

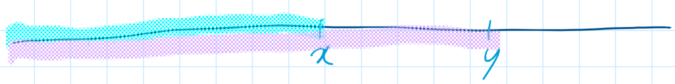
The DISTRIBUTION FUNCTION has 3 properties:

1) F_X is NON-DECREASING:

$$x < y \implies F_X(x) \leq F_Y(y)$$

proof $F_X(x) = \mu_X((-\infty, x])$, $F_X(y) = \mu_X((-\infty, y])$

$(-\infty, x] \subseteq (-\infty, y]$

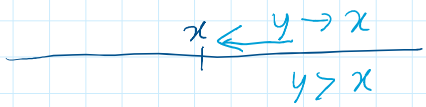


$\implies$ (MONOTONICITY) $\mu_X((-\infty, x]) \leq \mu_X((-\infty, y])$

$$\overset{\shortparallel}{F_X}(x) \leq F_X(y)$$

2) F_X is RIGH-CONTINUOUS:

$$\lim_{y \searrow x} F_X(y) = F_X(x)$$



Equivalently, for any sequence $x_n \searrow x$

i.e. $\lim_n x_n = x$ and $x_n > x$

we have $\lim_n F_X(x_n) = F_X(x)$

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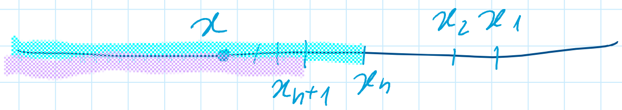
Proof Consider a decreasing sequence converging to x

$$x_1 \geq x_2 \geq \dots \geq x_n \geq x_{n+1} \geq \dots \quad x \quad , \lim_{n\to\infty} x_n = x$$



$$A_n = (-\infty, x_n] \quad \supseteq \quad A_{n+1} = (-\infty, x_{n+1}]$$

The sequence $(A_n)_{n\in\mathbb{N}}$ is DECREASING

$$A_n \supseteq A_{n+1} \quad \forall n$$

by continuity of the probability μ_X

$$\mu_X\Big(\underbrace{\lim_n A_n}\Big) = \lim_n \mu_X(A_n)$$

$$= \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} (-\infty, x_n] = (-\infty, x]$$

$$\Rightarrow \mu_X((-\infty, x]) = F_X(x) = \lim_n \mu_X((-\infty, x_n]) = \lim_n F_X(x_n)$$

$$\Rightarrow F_X(x) = \lim_n F_X(x_n)$$

3) $\lim_{x\to+\infty} F_X(x) = 1$ $\qquad$ $\lim_{x\to-\infty} F_X(x) = 0$

Note $F_X(x) \in [0,1]$

Proof

- Consider $x_n \to +\infty$, claim $\lim_n F_X(x_n) = 1$

  can consider increasing sequence $x_n \leq x_{n+1}$

  Consider sets $A_n = (-\infty, x_n] \subseteq A_{n+1} = (-\infty, x_{n+1})$

  sequence $(A_n)_{n\in\mathbb{N}}$ increasing

  by continuity of μ_X

  $$\mu_X\Big(\underbrace{\lim_n A_n}\Big) = \lim_n \mu_X(A_n)$$

  $$= \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (-\infty, x_n] = (-\infty, +\infty) = \mathbb{R}$$

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$$\mu_X(\mathbb{R}) = 1 = \lim_n \mu_X((-\infty, x_n]) = \lim_n F_X(x_n)$$

- Consider $x_n \to -\infty$, ...

Thm 1 The distribution function F_X characterizes the law μ_X, i.e. if X and Y are two random variables with $F_X = F_Y$ then $\mu_X = \mu_Y$

$$\mu_X : \mathcal{B}(\mathbb{R}) \to [0,1]$$
$$F_X : \mathbb{R} \to [0,1]$$

Thm A function $F : \mathbb{R} \to [0,1]$ is the distribution function of a (unique) random variable if and only if the properties 1-2-3 above are satisfied.

(Give F with the 3 properties, it is possible to construct (in a unique) way a r.v. X such that $F = F_X$ F is the distribution function of X)

Example 1

Distribution function of $X \sim Ber(p) = Bin(1,p)$
discrete r.v X s.t. $N = \{0,1\}$

$P(X=1) = p \qquad P(X=0) = 1-p$

$P(X=x) = 0 \qquad \forall x \in \mathbb{R},\ x \neq 0,1$

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ P(X \leq 0) = P(X=0) \underset{1-p}{=} & \text{if } x = 0 \end{cases}$$

if $0 \le x < 1$

$\{\omega : X(\omega) \le x\}$

$\{\omega : X(\omega) \overset{\shortparallel}{=} 0\}$

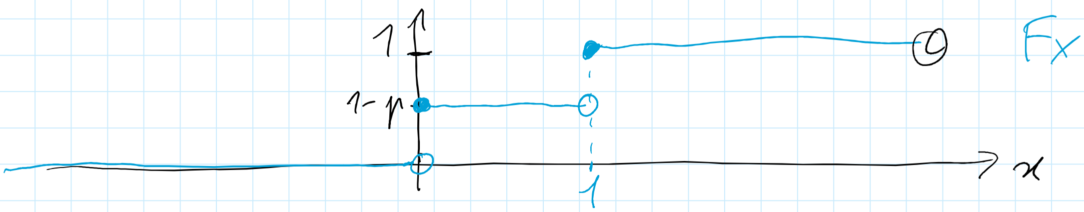
$$\begin{cases} P(X \le x) = P(X = 0) \overset{\shortparallel}{=} 1-p & \text{if } 0 \le x < 1 \\ P(X \le x) & \text{if } x \ge 1 \\ = P(\Omega) = 1 \end{cases}$$

if $x \ge 1$: $\{\omega : X(\omega) \le 1\}$

$= \{\omega : X(\omega) = 0 \text{ or } X(\omega) = 1\}$

$= \{X = 0\} \cup \{X = 1\} = \Omega$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

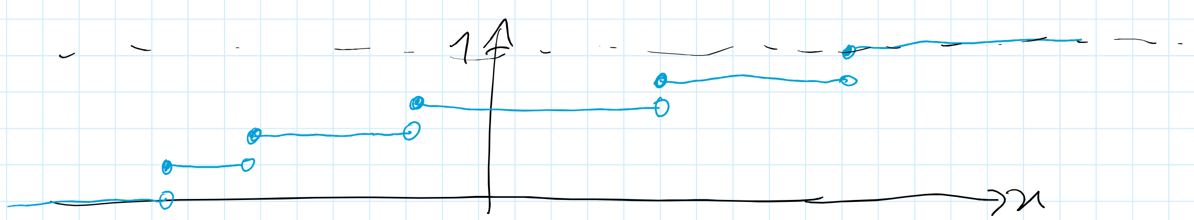


F_X is 1) increasing, 2) right-continuous, 3) $\lim_{x \to +\infty} = 1$, $\lim_{x \to -\infty} = 0$

F_X is PIECEWISE-CONSTANT

F_X is not continuous $\to$ has jumps and is constant between jumps

<u>FACT</u> the distribution function of any DISCRETE r.v. is PIECEWISE CONSTANT



$$F_X(x) = P(X \le x) = \sum_{\substack{y \in N \\ y \le x}} P(X = y)$$

$$F_X(x) = P(X \leq x) = \sum_{\substack{y \in \mathbb{N} \\ y \leq x}} P(X = y)$$