

January 22, 2025

Problem 1. [12] Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two independent sequences of independent random variables, such that $X_n \sim \text{Bin}(1, p)$ and $Y_n \sim \mathcal{U}(0, 1)$ for any n .

(i) For any n define $Z_n = X_n + Y_n$. Compute the distribution and expectation of Z_n ;

(ii) Compute the limit in probability of the sequence $\frac{Z_1 + Z_2 + \dots + Z_n}{n}$, as $n \rightarrow +\infty$;

(iii) For any n define $V_n = X_n \cdot Y_n$. Compute the distribution, expectation and variance of V_n ;

(iv) Compute the limit in probability of the sequence $\frac{V_1 + V_2 + \dots + V_n}{n}$, as $n \rightarrow +\infty$.

$$\begin{aligned}
 \text{(i)} \quad F_{Z_n}(z) &= \mathbb{P}[X_n + Y_n \leq z] = \mathbb{P}[X_n + Y_n \leq z | X_n = 0] \cdot \mathbb{P}[X_n = 0] \\
 &\quad + \mathbb{P}[X_n + Y_n \leq z] \cdot \mathbb{P}[X_n = 1] = p F_{Y_n}(z) + (1-p) F_{Y_n}(z-1) \\
 &= \begin{cases} 0 & z < 0 \\ pz & 0 \leq z < 1 \\ p + (1-p)(z-1) & 1 \leq z < 2 \\ 1 & z \geq 2 \end{cases} ; \quad \begin{aligned} \mathbb{E}[Z_n] &= \mathbb{E}[X_n] + \mathbb{E}[Y_n] \\ &= p + \frac{1}{2} \end{aligned}
 \end{aligned}$$

$$\text{(ii)} \quad \text{By the WLLN, } \frac{Z_1 + \dots + Z_n}{n} \xrightarrow[n \rightarrow \infty]{P} \mathbb{E}[Z_1] = p + \frac{1}{2}$$

$$\begin{aligned}
 \text{(iii)} \quad F_{V_n}(v) &= \mathbb{P}[X_n \cdot Y_n \leq v] = \mathbb{P}[X_n Y_n \leq v | X_n = 0] \cdot \mathbb{P}[X_n = 0] + \\
 &\quad + \mathbb{P}[X_n \cdot Y_n \leq v | X_n = 1] \cdot \mathbb{P}[X_n = 1] = \\
 &= \begin{cases} 0 & v < 0 \\ (1-p) + pv & 0 \leq v < 1 \\ 1 & v \geq 1 \end{cases} \quad \begin{aligned} \mathbb{E}[V_n] &= p \cdot \frac{1}{2} \\ \text{Var}[V_n] &= \mathbb{E}[X_n^2 \cdot Y_n^2] \\ &\quad - (\mathbb{E}[X_n Y_n])^2 = \\ &= \mathbb{E}[X_n^2] \cdot \mathbb{E}[Y_n^2] - p^2 \frac{1}{4} \end{aligned}
 \end{aligned}$$

$$\text{(iv)} \quad \frac{V_1 + \dots + V_n}{n} \xrightarrow[n \rightarrow \infty]{P} \mathbb{E}[V_1] = \frac{p}{2}$$



Problem 2. [13] We have two rows of 3 balls: in both rows, there are 2 white balls and 1 black ball. We move the balls according to the following scheme: we randomly choose one of the two rows, then we randomly select a ball from that row and move it to the last position in the same row. We repeat this process in the same manner.

- Define a Markov chain that describes the positions of the two black balls in their respective rows;
- Classify the states of the Markov chain;
- Compute the invariant distribution;
- If initially both black balls are in the first position, on average, how many turns will it take for them to return in that position?

(i)

$$S = \{ (i, j) : i, j \in \{1, 2, 3\} \}$$

$$P[X_i = (i, j) | X_0 = (1, 1)] =$$



$$\frac{2}{3} = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3}$$



and similarly for the other cases

P =

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	2/3	0	1/6	0	0	0	1/6	0	0
(1,2)	1/6	1/2	1/6	0	0	0	0	1/6	0
(1,3)	0	1/3	1/2	0	0	0	0	0	1/6
(2,1)	1/6	0	0	1/2	0	1/6	1/6	0	0
(2,2)	0	1/6	0	1/6	1/3	1/6	0	1/6	0
(2,3)	0	0	1/6	0	1/3	1/3	0	0	1/6
(3,1)	0	0	0	1/3	0	0	1/2	0	1/6
(3,2)	0	0	0	0	1/3	0	1/6	1/3	1/6
(3,3)	0	0	0	0	0	1/3	0	1/3	1/3

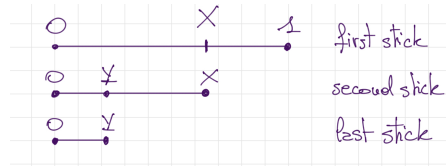
(ii) the MC is irreducible and aperiodic

(iii) Since the matrix P is doubly stochastic, the

invariant distribution is $(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$

(iv)
$$\pi_{(1,1)} = \frac{1}{\pi_{(1,1)}} = \frac{1}{1/9} = 9$$

Problem 3. [9] A stick of length one is broken at a random point X , uniformly distributed over the stick. This remaining stick is broken once more at a random point Y .



- Determine the expected length of the first broken stick (the second stick in the figure above);
- Compute the conditional expected length of the last stick, given that $X = x$.
- Compute the expected length of the last stick.

(i) The (random) length is X ; $\mathbb{E}[X] = \boxed{\frac{1}{2}}$

(ii) The (random) length is Y : $\mathbb{E}[Y|X=x] = \boxed{\frac{x}{2}}$

(iii) $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \int_0^1 \frac{x}{2} \cdot 1 dx$

$$= \left[\frac{x^2}{4} \right]_0^1 = \boxed{\frac{1}{4}}$$