Lecture 11

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Exercise

$X \sim Ber(p) \qquad Y \sim Exp(\lambda) \qquad , p \in (0,1), \quad \lambda > 0$

$X \perp\!\!\!\perp Y$ (independent) $\qquad Z = XY$

Compute expectation, variance and distribution of Z.

$F_Z(z)$

- Expectation : $E[Z] = E[XY]$

$$= E[X] \cdot E[Y] = p \cdot \frac{1}{\lambda}$$

- Variance : $Var[Z] = E[Z^2] - (E[Z])^2$

$$= E[X^2 Y^2] - (E[XY])^2$$

$$\overset{!}{=} E[X^2] \cdot E[Y^2] - (E[X] \cdot E[Y])^2$$

$Var(X) = p(1-p) \qquad Var(Y) = \frac{1}{\lambda^2}$

$E[X^2] = Var(X) + (E(X))^2 = p$

$E[Y^2] = Var(Y) + (E[Y])^2 = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}$

$$= p \frac{2}{\lambda^2} - \left(\frac{p}{\lambda}\right)^2 = 2\frac{p}{\lambda^2} - \frac{p^2}{\lambda^2} = \frac{p}{\lambda^2}(2-p)$$

Fact $\quad X \perp\!\!\!\perp Y \implies g_1(X) \perp\!\!\!\perp g_2(Y)$

for any measurable functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$

- Compute $F_Z(z)$:

$X \in \{0,1\} \qquad P(X=0) = 1-p \qquad P(X=1) = p$

$Y \geq 0 \qquad Y$ abs cont. $\qquad f_Y(y) = \lambda e^{-\lambda y} \mathbb{1}_{[0,+\infty)}(y)$

- $XY \geq 0 \rightarrow F_Z(z) = P(Z \leq z) = 0$ if $z < 0$
- Compute $F_Z(0) = P(Z \leq 0) = P(Z=0) + \underbrace{P(Z<0)}_{=0}$

$P(Z=0) = P(XY=0)$

$$= P(XY=0 \mid X=0)P(X=0) + P(XY=0 \mid X=1)P(X=1)$$

$P(Z=0) = P(XY=0)$

$$= \underbrace{P(XY=0 \mid X=0)}_{=1} \underbrace{P(X=0)}_{=1-p} + \underbrace{P(XY=0 \mid X=1)}_{=0} \underbrace{P(X=1)}_{p}$$

(FORMULA TOTAL PROBABILITY)

$$P(XY=0 \mid X=1) = P(Y=0 \mid X=1) \underset{\text{INDEPENDENCE}}{=} P(Y=0)$$

$P(Y=0) = 0 \rightarrow$ as Y is absolutely continuous

[recall Y ABS CONT $\Rightarrow P(Y=c)=0 \quad \forall c \in \mathbb{R}$]

$\rightarrow F_Z(0) = 1-p$

- $z>0 \rightarrow F_Z(z) = P(XY \leq z)$

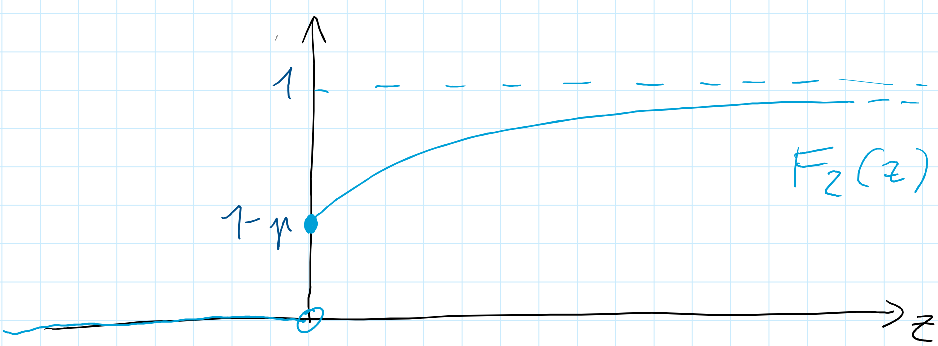
$$= P(XY \leq z \mid X=0) P(X=0) + P(XY \leq z \mid X=1) P(X=1)$$

$$= \underbrace{P(0 \leq z \mid X=0)}_{=1} \underbrace{P(X=0)}_{1-p} + \underbrace{P(Y \leq z \mid X=1)}_{\substack{= P(Y \leq z) \\ \text{INDEPENDENCE}}} \underbrace{P(X=1)}_{=p}$$

$$= 1-p + p\, F_Y(z)$$

$$= 1-p + p(1-e^{-\lambda z}) = 1-pe^{-\lambda z}$$

$$F_Z(z) = \begin{cases} 0 & \text{if } z<0 \\ 1-p & \text{if } z=0 \\ 1-pe^{-\lambda z} & \text{if } z>0 \end{cases}$$



DISCRETE RANDOM VARIABLES

BERNOULLI $\quad X \sim Be(p) \quad p \in [0,1]$

$X \sim Bi(1, p) = Be(p)$

BERNOULLI $X \sim Ber(p)$ $p \in [0,1]$

$X \sim Bin(1, p) = Ber(p)$

$\Omega = \{b_1, b_2\}$ $\mathcal{A} = 2^{\Omega}$

$P(\{b_1\}) = p$ $P(\{b_2\}) = 1 - p$

Define Bernoulli random variable $X : \Omega \to \mathbb{R}$

$X(b_1) = 1$ $X(b_2) = 0$ $N = \{0, 1\}$

$P(X = 1) = P(\{b_1\}) = p = p_X(1)$

$P(X = 0) = P(\{b_2\}) = 1 - p = p_X(0)$

Consider a transformation of X

$Y : \Omega \to \mathbb{R}$ $Y(b_1) = 1$, $Y(b_2) = -1$

$Y \not\sim Ber(p)$, but we have $Y = f(X)$

f linear transformation $\to$ $Y = aX + b$
(affine)

mean $Y(\omega) = a X(\omega) + b \quad \forall \omega$

$\omega = b_1$ or b_2 $<$ $Y(b_1) = aX(b_1) + b \to 1 = a \cdot 1 + b$
$Y(b_2) = aX(b_2) + b \to -1 = a \cdot 0 + b$

$$\begin{cases} 1 = a + b \\ -1 = b \end{cases} \longrightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$$

$\Rightarrow$ $Y = 2X - 1$ linear transformation of X

$\to$ easy to compute expectations

$E[Y] = 2E[X] - 1 = 2p - 1$

alternatively, $E[Y] = 1\,P(Y = 1) + (-1)\,P(Y = -1)$

$= 1 \cdot p - 1(1 - p) = 2p - 1$

BINOMIAL DISTRIBUTION $\quad X \sim Bin(n, p)$

$$\Omega = \{b_1, b_2\}^n$$
$$= \{\omega = (\omega_1, .., \omega_n) : \omega_i \in \{b_1, b_2\} \quad i = 1, .., n\}$$

Sample space of n independent trials

$\mathcal{A} = 2^\Omega \qquad P$ on $\mathcal{A}$?

$P(\{\omega\}) = p^k (1-p)^{n-k}$ where $K(\omega)$ is the number of b_1 in ω
($n-K$ is number of b_2)

- $\omega = (b_1, b_1, b_2, b_1, b_2) \quad n = 5$
  $P(\omega) = p^3 (1-p)^2$

Binomial random variable $\quad X : \Omega \to \mathbb{R}$

$X(\omega) :=$ the number of outcomes equal to b_1

$= \# \{ i : \omega_i = b_1 \}$

$X : \Omega \to \mathbb{R} \qquad N = \{0, 1, .., n\} \qquad |N| = n$

$X((b_1, b_2, ..., b_2)) = 1$

$X((b_1, b_1, b_2, .., b_2)) = 2$

$X((b_2, .., b_2, b_1, b_1, b_1)) = 3$

Distribution of X ? $\qquad k = 0, 1, .., n$

$P(X = k) = P(\omega : k \; b_1$ in the vector $\omega)$

$= p^k (1-p)^{n-k} \cdot \# \{\omega$ has $k \; b_1\}$

example
$k=2$, $n=4$: $(b_1, b_1, b_2, b_2), \; (b_1, b_2, b_1, b_2), \; (b_1, b_2, b_2, b_1), \ldots$

$\# \{\omega$ has $k \; b_1\} = \binom{n}{k} = \#$ subsets of cardinality k on a set of cardinality n

$$\Rightarrow p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$X \sim Bin(n, p)$

$\Rightarrow p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ $X \sim Bin(n,p)$

$E[X] = np$, $Var(X)$?

$$E[X^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} = \dots \text{ DIFFICULT}$$

BINOMIAL is SUM of INDEPENDENT BERNOULLI

$X_1 : \Omega \to \mathbb{R}$

$$X_1(\omega) = \begin{cases} 1 & \text{if } \omega_1 = b_1 \\ 0 & \text{if } \omega_1 = b_2 \end{cases}$$

$X_1(b_1, \omega_2, \dots, \omega_n) = 1$ no matter the values of $\omega_2, \dots, \omega_n$

$X_1(b_2, \omega_2, \dots, \omega_n) = 0$

$X_1 : \Omega \to \{0,1\}$

$$P(X_1 = 1) = P(\{(b_1, \omega_2, \dots, \omega_n) : \omega_2, \dots, \omega_n \in \{b_1, b_2\}\})$$
$$= p \left(\underbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{k-1}}_{=1} \right) = p$$

$P(X_1 = 0) = 1 - p$

$\Rightarrow X_1 \sim Ber(p)$

Define $X_i : \Omega \to \mathbb{R}$ $i = 1, \dots, n$

$$X_i(\omega) = \begin{cases} 1 & \text{if } \omega_i = b_1 \\ 0 & \text{if } \omega_i = b_2 \end{cases}$$

$X_i \sim Ber(p)$

We can show that the random variables $X_1, \dots, X_n$ are independent

Let $X = X_1 + \dots + X_n$

$X(\omega) = \#$ number of outcomes equal to b_1

$\Rightarrow X \sim Bin(n,p)$ is sum of n independent $Ber(p)$

$X(\omega) =$ # number of outcomes equal to b_1

$\Rightarrow X \sim Bin(n, p)$ is sum of n independent $Ber(p)$

$$\Rightarrow \quad E[X] = E[X_1] + \dots + E[X_n] = np$$

$$Var(X) = Var(X_1) + \dots + Var(X_n) = np(1-p)$$

$$Var(X_i) = p(1-p)$$

GEOMETRIC RANDOM VARIABLE parameter p

$$\Omega = \{ \omega = (\omega_1, \omega_2, \dots, \omega_n, \dots), \ \omega_i = \{b_1, b_2\} \}$$

sequences of independent trials

$\Omega = \{b_1, b_2\}^{\mathbb{N}}$ $\qquad |\Omega| = 2^{|\mathbb{N}|} = |\mathbb{R}|$

σ-algebra $\mathcal{A} \neq 2^{\Omega}$ $\qquad \Omega$ more than countable

Consider $\mathcal{A}$ = minimal σ-algebra generated by

CYLINDRICAL SETS $= \{ C^n_a : n \in \mathbb{N}, a \in \{b_1, b_2\}^n \}$

$$C^n_a = \{ \omega : \omega_1 = a_1, \dots, \omega_n = a_n, \ \omega_i \text{ arbitrary for } i > n \}$$

given n and $a = (a_1, \dots, a_n) \in \{b_1, b_2\}^n$

We can define the probability P on $\mathcal{A}$ by

$$P(\{(\omega_1, \dots, \omega_n, \dots)\}) = p^k (1-p)^{n-k} \quad \forall n$$

$$k = \# \, b_1 \text{ in } (\omega_1, \dots, \omega_n)$$

$$X(\omega) := \inf \{ n \geq 1 : \omega_n = b_1 \}$$

= index of trial of first success

$\omega = (b_2, b_2, b_2, b_2, b_1, \dots) \to X(\omega) = 5$

X = index of first b_1

$\omega = (b_1, b_2, \dots) \qquad X(\omega) = 1$

Density of X?

$P(X = n) \to \{X = n\} = \{$ first success at n-th trial

first $(n-1)$ trials are failures$\}$

$P(X=n) \rightarrow \{X=n\} = \{$ first success at n-th trial
first (n-1) trials are failures$\}$

$$\{X=n\} = \{\omega = (\underbrace{b_2, \dots, b_2}_{n-1}, b_1, \underbrace{\omega_{n+1}, \omega_{n+2}, \dots}_{\text{arbitrary}})\}$$

$$P(X=n) = (1-p)^{n-1} p = p_X(n)$$ DENSITY of GEOMETRIC r.v.

$$E[X] = \frac{1}{p}$$ $\mathcal{N} = \{1, 2, \dots\} = \mathbb{N}$

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

Compute $E[X]$:

$$E[X] = \sum_{n=1}^{\infty} n\, p_X(n) = \sum_{n=1}^{\infty} n\, (1-p)^{n-1} p$$

$$= p \sum_{n=1}^{\infty} \left(-\frac{d}{dp} (1-p)^n \right)$$

$$= -p \frac{d}{dp} \left(\sum_{n=0}^{\infty} (1-p)^n - 1 \right)$$

$\left[\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \right.$ geometric series $\left. \right]$

$$= -p \frac{d}{dp} \left(\frac{1}{1-(1-p)} - 1 \right) = -p \frac{d}{dp} \left(\frac{1}{p} - 1 \right)$$

$$= -p \left(-\frac{1}{p^2} \right) = \frac{1}{p}$$