DATA SCIENCE **Stochastic Methods**

December 16, 2023

Problem 1. [9] Two friends arrange to have dinner together. Each will arrive independently at some point between 8 and 9 in the evening, wait for a maximum of 10 minutes (but not beyond 9), and if the other has not arrived within this time, they will leave. Describe the arrival times by two independent r.v.'s X and Y both $U(0,60)$, and therefore their joint density is constant on the square of vertex $(0,0)$, $(60,0)$, $(60,60)$ and $(0,60)$.

(i) Compute $\mathrm{P}[X<Y]$;

(ii) What is the probability that at least one of the two friends arrives after 8:30?

(iii) What is the probability that the two friends will have dinner together?

Problem 2. [9] Let X be a Geometric random variable of parameter $1/2$ and Y be a Binomial random variable of parameters $(X,1/2)$, i.e. $Y|X=n \sim Bin(n,1/2)$.

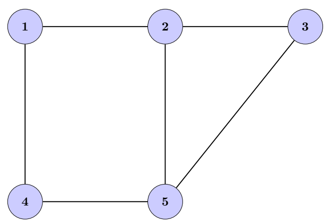
(i) Compute $\mathrm{P}[Y=k|X=n]$ for any $k \leq n$;

(ii) Compute $h(n)=\mathrm{E}[Y|X=n]$ for any n;

(iii) Compute $\mathrm{E}[\mathrm{E}[Y|X]]$;

(iv) Describe the support and the discrete density of the random variable $\mathrm{E}[Y|X]$.

Problem 3. [9] Define a simple Random Walk $\{X_n, n \geq 0\}$ on the undirected graph:



(i) Compute the probability to go from 3 to 4 in three steps.

(ii) Is the chain aperiodic?

(iii) Find the invariant distribution.

(iv) Starting from state 3, what is the probability of visiting any state before visiting a state more than once?

Problem 4. [9] Let $(X_i)_{1\leq i\leq 2n}$ be a family of i.i.d. $Exp(\lambda)$ r.v.'s and let $Y_k = X_{2k-1}+X_{2k}$, for any $k \leq n$.

(i) Compute the moment generating function of Y_1;

(ii) Defined $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$, determine an exponential decay for the "upper tail" of $\overline{Y}_n - \mathrm{E}[\overline{Y}_n]$.
(Hint: use the Chernoff bound proved for exponential random variables.)