Basic Prob

$1 = P(A \mid B) + P(A^c \mid B)$
$(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$
$P(A \cap B) = P(A \mid B)P(B)$
$P(A^c) = 1 - P(A)$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
$P(A) = P(A \mid B) \cdot P(B) + P(A \mid B^c) \cdot P(B^c)$
$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$
A, B independent :
$P(A \cap B) = P(A)P(B)$
$P(A \mid B) = P(A)$

Characteristic and moment generating function

$\varphi_X(u) = E[e^{iuX}] = E(\cos(uX)) + iE(\sin(uX))$
$E(X^k) = \varphi_X^{(k)}(0)$
$\frac{d^n}{d^n t}\varphi_X(t)\mid_{t=0} = E(X^n)$
$\varphi_{aX+bY}(u) = \varphi_X(au)\varphi_Y(bu)$
$m_X(t) = E[e^{tX}]$
$\frac{d^n}{dt^n}m_X(t)\mid_{t=0} = E(X^n)$
$m_X(t) = \varphi_X(-it)$
$m_{aX+bY}(t) = m_X(at)m_Y(bt)$

Others

if X, Y independent: $P(X = x, Y = y) = P(X = x)P(Y = y)$
$E(X \mid A) = \sum x \cdot P(X = x \mid A)$ or $E(X \mid A) = \int x \cdot P(X = x \mid A)$
$Y = aX + b \Rightarrow m_Y(t) = E(e^{t(aX+b)}) = e^{bt}m_X(at)$
$m_{X+Y}(t) = m_X(t) \cdot m_Y(t)$
$X, Y \sim U(a, b)$, $P(X < 2Y) = \int_a^b \int_a^{\min(2y,1)} 1$ (do the graph)
$P(|X| \geq t) \leq \frac{E(|X|)}{t}$
$-\ln(U(0,1)) = Exp(1)$
$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$
$P(X < Y) = \sum P(X < y \mid Y = y)P(Y = y)$
$E(X \mid X < a) = \int_0^\infty P(X > x \mid X < a) = \frac{\int_0^a x f_X(x)}{P(X < a)}$
Heoffding ineq: $Sn = Z_1 + ... + Z_n$ where Z_i bounded in $[a, b]$
$P(S_n - E(S_n) \geq t) = e^{\frac{-2t^2}{n(b-a)^2}}$

Random Var

$F_X(x) = P(X \leq x)$
$P(X = x) = F_X(x) - F_X(x^-)$

Discrete

$p(x, y) = P(X = x \cap Y = y)$
$p_X(x) = P(X = x)$
$P_X(x) = P(X \leq x)$
$E(X) = \sum x \cdot p_X(x)$
if $X > 0 \Rightarrow E(X) = \sum_{n=1}^{\infty} P(X \geq n)$
$E(X) = \sum_{n=0}^{+\infty} P(X \geq n)$
$E\left(\sum X_i\right) = \sum E(X_i)$
$E(f(X)) = \sum f(X) \cdot p_X(x)$
$E(|X|) \leq +\infty \Rightarrow X$ is integrable
$E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$
$E(X \cdot Y) = E(X) \cdot E(Y)$
$Var(X) = E(X^2) - E(X)^2 = E\left((X - E(X))^2\right)$
$Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y)$ if X, Y indipendent
$Var(X + Y) = E(X) + E(Y) + 2Cov(X, Y)$
$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$
$Cov(X, Y) = 0$ (uncorrelated) $\Leftrightarrow X, Y$ are indipendent
$Cov(X, X) = Var(X)$
$Cov(X, Y) = Cov(Y, X)$
$Cov(\alpha X + \beta Y, Z) = \alpha Cov(X, Z) + \beta Cov(Y, Z)$

Random Vector

$X = (X_1, .., X_n)$
$p_{X_i}(x_i) = \sum_{x_1, x_2 ..., x_n} p(x_1, x_2, ..., x_i, ..., x_n)$
$p_{X_i}(x_i) = \int f_X(x_1, x_2, ..., x_i, ..., x_n)$
indipendent iif
Discrete : $p(x_1, x_2, ..., x_i, ..., x_n) = p_{X_1}(x1) \cdot ... \cdot p_{X_n}(x_n)$
Abs cont $f_X(x_1 ... x_d) = \prod f_{X_i}(x_i)$
also $F_{(X,Y)}(x, y) = F_X(x) \cdot F_Y(y)$
$\varphi_X(u) = E[e^{i\langle u,X\rangle}]$ and $m_X(u) = E[e^{\langle u,X\rangle}]$
if X, Y independent, then
$\varphi_{X,Y}(u, v) = \varphi_X(u) \cdot \varphi_Y(v)$
$m_{X,Y}(u, v) = m_X(u) \cdot m_Y(v)$
$\varphi_{X+Y}(u) = \varphi_X(u) \cdot \varphi_Y(u)$
$E(g(X, Y)) = \sum_x \sum_y g(x, y)P(X = x, Y = y)$
$E(g(X, Y)) = \int \int g(x, y)f_{X,Y}(x, y)$

Bernoulli:

$X \sim Be(p)$, $E(X) = p$, $Var(X) = p(1 - p)$
$\varphi_X(u) = 1 - p + pe^{iu}$, $m_X(t) = 1 - p + pe^t$

Binomial:

$X \sim \text{Bin}(n, p)$, $p_X(k) = (n \ \text{ choose } \ k)p^k(1 - p)^{n-k}$
$E(X) = np$, $Var(X) = np(1 - p)$
$\varphi_X(u) = [1 - p + pe^{iu}]^n$, $m_X(t) = [1 - p + pe^t]^n$
Upper=

Geometric

$X \sim Geo(p), p_X(n) = (1 - p)^{n-1}p$
$E(X) = \frac{1}{p}$, $Var(X) = \frac{1 - p}{p^2}$
$\varphi_X(u) = \frac{p}{e^{-iu} - 1 + p}$, $m_X(t) = \frac{p}{e^{-u} - 1 + p}$

Negative Bin (r-th success in n-th position)

$X \sim \text{NegBin}(r, p)$, $p_X(n) = (n - 1 \ \text{ choose } \ r - 1)p^r(1 - p)^{n-r}$
$E(X) = \frac{r}{p}$, $Var(X) = r\frac{1 - p}{p^2}$
$\varphi_X(u) = \left[\frac{p}{e^{-iu} - 1 + p}\right]^r$, $m_X(t) = \left[\frac{p}{e^{-u} - 1 + p}\right]^r$
$NB(r_1, p) + NB(r_2, p) = NB(r_1 + r_2, p)$

Poisson ($Poi(np)$ approx of $B \in (n, p)$)

$X \sim Poi(\lambda)$, $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k}!$
$E(X) = \lambda$, $Var(X) = \lambda$
$Poi(\lambda) + Poi(\mu) = Poi(\lambda + \mu)$
$\varphi_X(u) = e^{\lambda(e^{iu}-1)}$, $m_X(t) = e^{\lambda(e^t-1)}$

Hypergeometric (red ball in n balls drawn in urn with m red and N-m blue without repl)

$X \sim \text{HypGeo(n, N, m)}$, $p_X(k) = \frac{(n \ \text{ choose } \ p)(N - m \ \text{ choose } \ n - k)}{(N \ \text{ choose } \ m)}$
$E(X) = Var(X) = \frac{nm(N - m)(N - n)}{N^2(N - 1)}$

$cx' = c;$
$x^{n\,\prime} = nx^{n-1}$
$k' = 0$
$\log x' = \frac{1}{x}$
$e^{x\,\prime} = e^x$
$e^{kx} = ke^{kx}$
$a^{x\,\prime} = a^x \log a$
$\sin x' = \cos x$
$\cos x' = -\sin x$

$g(x)^{n\,\prime} = g(x)^n g'(x)$
$e^{g(x)\,\prime} = e^{g(x)}g'(x)$
$\ln(g(x))' = \frac{g'(x)}{g(x)}$
$(fg)' = f'g + fg'$
$f(g(x))' = f'(g(x))g'(x)$
$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$(a + b)^n = \sum_{k=0}^{n} (n \ \text{ choose } \ k)a^k b^{n-k}$
$e^{iux} = i\sin(ux) + \cos(ux)$

$\int k = kx$
$\int x^n = \frac{x^{n+1}}{n + 1}$
$\int \frac{1}{x} = \ln x$
$\int \log x = x\log x - x$
$\int e^{kx} = \frac{e^{kx}}{k}$
$\int a^x = a^x \log a$

$\sum_{i=0}^{N} a^i = a \cdot \frac{a^N - 1}{a - 1}$

$\int f'g = fg - \int fg'$
$\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q}$ if $|q| < 1$
$\sum_{i=o}^{n} i = \frac{(n + 1)n}{2}$
$\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$
$\sum_{i=0}^{\infty} i\frac{x^i}{i!} = xe^x$

Abs Continuous $\Leftrightarrow F_X(x) = \int_{-\infty}^{x} f_X(t)dt$

if $X > 0 \Rightarrow E(X) = \int_{0}^{\infty} P(X > x)$

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)dx$$

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)dx$$

$$E(X) = -\int_{-\infty}^{0} F_X(x) + \int_{0}^{+\infty} 1 - F_X(x)dx$$

if $X \in L^1$ then $E(X) = \int_{0}^{+\infty} 1 - F_X(x)dx - \int_{-\infty}^{0} F_X(x)dx$

Uniform

$X \sim \text{Uni}(a, b)$,

$$f_X(x) = 1_{\langle a,b \rangle} \frac{1}{b-a} \text{ , } F_X(x) = \frac{x-a}{b-a} 1_{\langle a,b \rangle}$$

$$E(X) = \frac{a+b}{2} \text{ , } Var(X) = \frac{(b-a)^2}{12}$$

$m_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$, $\varphi_X(t) = \frac{e^{itb} - e^{ita}}{it(b-a)}$ if $t \neq 0$ else 1

Exponential

$X \sim Exp(\lambda)$

"Memoryless" : $P(X > T + S \mid X > T) = P(X > S)$

$$f_X(x) = \lambda e^{-\lambda x} 1_{0,+\infty}(x) \text{ , } F_X(x) = 1 - e^{-\lambda x}$$

$$E(X) = \frac{1}{\lambda} \text{ , } Var(X) = \frac{1}{\lambda^2}$$

$m_X(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$, $\varphi_X(t) = \frac{\lambda}{\lambda - it}$

$$\lambda E(\lambda) = E(1)$$

Normal

$X \sim N\left(\mu, \sigma^2\right)$

$$f_x(x) = \frac{e^{-\left(\frac{x-\mu}{\sigma}\right)^2 \frac{1}{2}}}{\sigma\sqrt{2\pi}}$$

$$E(x) = \mu \text{ , } Var(X) = \sigma^2$$

$$m_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \text{ , } \varphi_X(t) = e^{\mu i t - \frac{\sigma^2 t^2}{2}}$$

$$aX + b \sim N\left(a\mu + b, s^2\sigma^2\right)$$

$$N\left(\mu_1, \sigma_1^2\right) + N\left(\mu_2, \sigma_2^2\right) = N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right)$$

Upper $\leq e^{-n\frac{\varepsilon^2}{2}}$

Chi-squared

$X \sim \mathcal{X}^2(k)$

$$f_X(x) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \text{ , } F_X(x) =$$

$$E(X) = k \text{ , } Var(X) = 2k$$

$$m_X(t) = (1-2t)^{-\frac{k}{2}} \text{ , } \varphi_X(t) = (1-2it)^{-\frac{k}{2}}$$

CLT, Chebischev inequality, Chernoff bounds

CLT : $P(|X - u| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$ and also $\frac{S_n - n\mu}{\sigma\sqrt{n}} = \sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} \sim N(0,1)$

Chebischev: $P\left(\left|\overline{X}_n - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n \cdot \varepsilon^2}$

Chernoff bounds:

$P(X \geq a) = e^{-at} \cdot m_X(t)$ for any t

If we have X_n a sequence of iid rand var, then:

$P\left(\overline{X}_n \geq \mu + \varepsilon\right) \leq e^{-g\left(t^*\right) \cdot n}$ where $g\left(t^*\right) = t^*(\mu + \varepsilon) - \log\left(m\left(t^*\right)\right)$

$P\left(\overline{X}_n \leq \mu - \varepsilon\right) \leq e^{-h\left(t^*\right) \cdot n}$ where $h\left(t^*\right) = -t^*(\mu - \varepsilon) - \log\left(m\left(-t^*\right)\right)$

Inequality

$$\log(1+x) \geq \frac{2x}{2+x} \forall x \geq 0$$

$$-\log(1+x) \geq \frac{x^2}{2(1+x)} - x$$

$$-\log(1-x) \geq x + \frac{x^2}{2}$$

$$-\log(1+x) \geq -x$$

$$-\log(1-x) \geq x - \frac{x^2}{2}$$

$$\log(1+x) \geq x - \frac{x^2}{2}$$

$$(1-x)\log(1-x) + x \geq \frac{x^2}{2}$$

$e^x \geq 1 + x$ (for x small)

$$F_X(x) = F_Y(y) \Leftrightarrow \varphi_X(t) = \varphi_Y(t)$$

$(1+x)\log(1+x) - x \geq \frac{x^2}{2+x}$ for $x > 0$

$$-\frac{e^t - 1}{t} \geq -\frac{t}{2} - \frac{t^2}{24}$$

Markov Chain

$$P^{n+m} = P^n \cdot P^m$$

$P(X_n = x) = \left[\varphi \cdot P^n\right]_x$ where φ is the initial distribution

State **communication** $\Rightarrow i \to j \Rightarrow \exists n: P_{ij}^n > 0$

State i **aperiodic** $\Rightarrow n > 0 P_{ii}^n > 0$ has common divisor $= 1$

State i is **periodic** otherwise (and only come back in a even number of steps)

class properties

let $f_i = lm_{n \to \infty} P(X_n = i \mid X_0 = i)$

State i is **recurrent** if $f_i = 1$ (will visit state i infinite number of times)

State i is **transient** (will visit i s finite number of times)

Let $T_i = \inf\{n \geq 1 \mid X_n = i\}$ (time to visit the first time the state i)

and $m_i = E(Ti \mid X_0 = i)$

State i is **positive recurrent** if $m_i < +\infty$

State i is **null recurrent** if $m_i = +\infty$

$$m_i = E_j\left(T_j\right) = E\left(T_j \mid X_0 = j\right)$$

State i **aperiodic** + **positive recurrent** $\Rightarrow$ **ergodic**

if all states of the MC are ergodic, then the MC is ergodic

Class C is composed by communicating states $S = C_1 \cup C_2 \cup ...$

Class C is **closed** if $i \to j, i \in C$ for $\forall j \in C$

(you can't go out of the class and not come back)

if S is composed by only 1 class, is called **irreducible**

if MC irreducible and ergodic

- $\Pi_j = lm_{n \to +\infty} P_{ij}^n \forall j \in S$
- $\sum \Pi_i = 1$ (is a distribution)
- $\forall \Pi_i \geq 0$

Π_i is called the **limit distribution**

$\Pi = (\Pi_1, \Pi_2, ..., \Pi_n)$ is called the (**unique** if S finite) **distribution**

$\Pi = \left[(\Pi \cdot P) \& \left(\sum \Pi_i = 1\right)\right]$ (if so, Π is called **invariant**)

$$\Pi_j = \frac{1}{m_j}$$

$$\Pi_j = lm_{n \to +\infty} \frac{\text{\#visit to j by time n}}{n}$$

if (detailed balance condition) $\Pi_i P_{ij} = \Pi_j P_{ji} \forall i, j \in S$ then Π is **reversible**

if Π is **reversible**, than it's **invariant**

if MC is **irreducible** and **aperiodic** than it is **regular**

For undirected graph: $\Pi_i = \frac{\text{\# neighbors of state i}}{\text{total number of neighbors}}$