Lecture 16

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MULTIVARIATE NORMAL DISTRIBUTION

$X : \Omega \to \mathbb{R}$ $\qquad X \sim N(\mu, \sigma^2)$

Let $Z_1, \dots, Z_d \sim N(0,1)$ INDEPENDENT

Consider the vector $Z = (Z_1, \dots, Z_d)$

$u \in \mathbb{R}^d$, $\quad |u| = \sqrt{\sum_{i=1}^{d} u_i^2} = \sqrt{\langle u, u \rangle}$

Characteristic function of Z:

$$\varphi_Z(u) = \varphi_Z(u_1, \dots, u_d) = \prod_{i=1}^{d} \varphi_{Z_i}(u_i)$$

$$= \prod_{i=1}^{d} e^{-\frac{u_i^2}{2}} = e^{-\frac{\sum_{i=1}^{d} u_i^2}{2}} = e^{-\frac{|u|^2}{2}}$$

Let $A \in \mathbb{R}^{n \times d}$, $\quad \mu \in \mathbb{R}^n$

$X = AZ + \mu$ $\quad$ random vector in $\mathbb{R}^n$

Compute $\varphi_X(v)$ $\quad v \in \mathbb{R}^n$

$$\varphi_X(v) = e^{i\langle v, \mu \rangle} \cdot \varphi_Z(A^T v)$$

$$= e^{i\langle v, \mu \rangle} e^{-\frac{|A^T v|^2}{2}}$$

$$|A^T v|^2 = \langle A^T v, A^T v \rangle = (A^T v)^T A^T v$$

$$= v^T \underbrace{(A^T)^T}_{=A} A^T v = v^T (A A^T v) = \langle v, A A^T v \rangle$$

Denote $\Sigma = A A^T$ $\qquad \Sigma \in \mathbb{R}^{n \times n}$

Properties

- Σ is symmetric

$$\Sigma^T = (A A^T)^T = (A^T)^T A^T = A A^T = \Sigma$$

- Σ is POSITIVE SEMIDEFINITE,

  i.e. $\langle v, \Sigma v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n$

  in fact $\langle v, \Sigma v \rangle = |A^T v|^2 \geq 0$

$$\varphi_X(v) = e^{-i\langle v, \mu \rangle} e^{-\frac{1}{2}|v|^2}$$

- Expectation of X: $\qquad A = (a_{ij})_{i=1,\dots,n}^{j=1,\dots,d}$

- Expectation of X: $\quad A = (a_{ij})_{i=1,..,n}^{j}$

$$E[X] = E[AZ+\mu] = E\left[\begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix}\right] = \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_d] \end{pmatrix}$$

$$E[X_i] = E\left[\sum_{j=1}^{d} a_{ij} Z_j + \mu_i\right]$$

$$= \sum_{j=1}^{d} a_{ij} \underbrace{E[Z_j]}_{=0} + E[\mu_i] = \mu_i$$

$$\Rightarrow E[X] = \mu$$

- Compute Covariances

$$Cov(X_i, X_j) = E[(X_i - \underbrace{E[X_i]}_{=\mu_i})(X_j - \underbrace{E[X_j]}_{=\mu_j})]$$

$$= E\left[\sum_{k=1}^{d} a_{ik} Z_k \cdot \sum_{\ell=1}^{d} a_{j\ell} Z_\ell\right]$$

$$= \sum_{k=1}^{d} a_{ik} \sum_{\ell=1}^{d} a_{j\ell} E[Z_k Z_\ell]$$

$$E[Z_k Z_\ell] = \begin{cases} E[Z_k^2] = 1 & \text{if } k = \ell \\ E[Z_k] \cdot E[Z_\ell] = 0 & \text{if } k \neq \ell \end{cases}$$

$$= \sum_{k=1}^{d} a_{ik} a_{jk} = (AA^T)_{ij} = \Sigma_{ij}$$

$$\Rightarrow Cov(X_i, X_j) = \Sigma_{ij}$$

Σ is called the COVARIANCE MATRIX

Def A random vector X is called NORMAL with mean μ and covariance matrix Σ (Σ symmetric and positive semidefinite) if its characteristic function is equal to

$$\varphi_X(v) = e^{i\langle v, \mu\rangle} e^{-\frac{1}{2}\langle \Sigma v, v\rangle}$$

and we denote $X \sim N(\mu, \Sigma)$

- Z standard if $Z \sim N(0, Id)$

Proposition 1

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Let $X \sim N(\mu, \Sigma)$ and set $Y = BX + b$

then $Y \sim N(B\mu + b, B \Sigma B^T)$

Proposition 2

If (X,Y) is a 2-dimensional normal vector. Then

$X \perp\!\!\!\perp Y \iff Cov(X,Y) = 0$

Proof

$(\Longrightarrow)$ always true

$(\Longleftarrow)$ We have to show that

X and Y uncorrelated $\Rightarrow X \perp\!\!\!\perp Y$

Covariance matrix $\Sigma = \begin{pmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(X,Y) & Cov(Y,Y) \end{pmatrix}$

$\Sigma = \begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{pmatrix}$ $\qquad \mu_X = E[X], \quad \mu_Y = E[Y]$

$\qquad \sigma_X^2 = Var(X), \quad \sigma_Y^2 = Var(Y)$

$$\varphi_{X,Y}(u,v) = \exp\left(i \langle (u,v), (\mu_X, \mu_Y) \rangle\right) \exp\left(-\frac{1}{2} \langle \Sigma \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle\right)$$

$$= \exp\left(i (u\mu_X + v\mu_Y)\right) \exp\left(-\frac{1}{2}\left(\sigma_X^2 u^2 + \sigma_Y^2 v^2\right)\right)$$

$$\left| \langle \Sigma \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \langle \begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \langle \begin{pmatrix} \sigma_X^2 u \\ \sigma_Y^2 v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle \right.$$

$$= e^{i u \mu_X} e^{i v \mu_Y} e^{-\frac{1}{2}\sigma_X^2 u^2} e^{-\frac{1}{2}\sigma_Y^2 v^2}$$

$$= e^{i u \mu_X - \frac{1}{2}\sigma_X^2 u^2} e^{i v \mu_Y - \frac{1}{2}\sigma_Y^2 v^2}$$

$$= \varphi_X(u) \, \varphi_Y(v)$$

Then $X \perp\!\!\!\perp Y$, as $\varphi_{X,Y}(u,v) = \varphi_X(u)\varphi_Y(v)$ $\square$

Remark

- In scalar case $\mu \in \mathbb{R}$, $\Sigma = \sigma^2 \in \mathbb{R}$

  $N(\mu, \sigma^2)$ is usual normal distribution on $\mathbb{R}$

- Σ is covariance matrix in the sense that

  $\Sigma_{ij} = Cov(X_i, X_j)$ if $X \sim N(\mu, \Sigma)$

$$\Sigma_{ij} = Cov(X_i, X_j) \quad \text{if} \quad X \sim N(\mu, \Sigma)$$

<u>Note</u> The vector $X \sim N(\mu, \Sigma)$ might not be absolutely continuous

<u>Result</u> $X \sim N(\mu, \Sigma)$ multivariate normal random vector in $\mathbb{R}^n$

X is absolutely continuous $\iff det(\Sigma) \neq 0$

(Σ is invertible)

In this case the density of X is

$$f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}} |det A|} \exp\left(-\frac{1}{2} < x, \Sigma^{-1} x >\right)$$

A such that $X = AZ + \mu$

<u>Exercise</u>

$X_1 \sim U(0,1)$

$F_{X_1}(x) = x \quad \text{if} \quad 0 \leq x \leq 1$

$Z = MAX(X_1, \dots, X_n)$ $\qquad X_1, \dots, X_n$ INDEPENDENT

$$F_Z(z) = \prod_{i=1}^{n} F_{X_i}(z) = \prod_{i=1}^{n} z = z^n \qquad 0 \leq z \leq 1$$

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ 1 & z \geq 1 \end{cases}$$

$$E[Z] = \int_0^{+\infty} (1 - F_Z(z))\, dz$$

so $\quad F_Z' = f_Z \qquad E[Z] = \int_0^{+\infty} z\, f_Z(z)\, dz$

- $$\begin{aligned} E[Z] &= \int_0^1 (1 - F_Z(z))\, dz + \int_1^{+\infty} (1 - F_Z(z))\, dz \\ &= \int_0^1 (1 - z^n)\, dz + 0 \\ &= 1 - \left[\frac{z^{n+1}}{n+1}\right]_0^1 = 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

- Alternatively

$$f_Z(z) = F_Z'(z) = n z^{n-1} \mathbb{1}_{[0,1]}(z)$$

$$\begin{aligned} E[Z] &= \int_0^{+\infty} z\, f_Z(z) = \int_0^1 z\, n z^{n-1}\, dz = n \int_0^1 z^n\, dz \\ &= n \left[\frac{z^{n+1}}{n+1}\right]_0^1 = \frac{n}{n+1} \end{aligned}$$

$$= n\left[\frac{z^{n+1}}{n+1}\right]_0^1 = \frac{n}{n+1}$$

$$W = \max(X_1, \dots, X_n)$$

$$F_W(w) = 1 - \prod_{i=1}^{n}\left(1 - F_{X_i}(w)\right)$$

$$= 1 - (1-w)^n \quad \text{if} \quad 0 \leq w \leq 0$$

$$= \begin{cases} 0 & \text{if} \quad w \leq 0 \\ 1 & \text{if} \quad w \geq 1 \end{cases}$$

$$E[W] = \int_0^{+\infty}\left(1 - F_W(w)\right) dw$$

$$= \int_0^1 (1-w)^n \, dw = \left[-\frac{(1-w)^{n+1}}{n+1}\right]_{w=0}^{w=0}$$

$$= \frac{1}{n+1}$$