Lecture 4

venerdì 10 ottobre 2025 08:28

Independence of 3 events

Def We say that A, B, C are INDEPENDENT if

$$P(A \cap B) = P(A) \cdot P(B)$$
$$P(A \cap C) = P(A) \cdot P(C)$$
$$P(B \cap C) = P(B) \cdot P(C)$$

PAIRWISE INDEPENDENT

$$P(A \cap B \cap C) = P(A) \cdot P(A) \cdot P(C)$$

The last property is rewritten as

$$P(A \mid B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A) \cdot \cancel{P(B)} \cdot \cancel{P(C)}}{\cancel{P(B)} \cdot \cancel{P(C)}} = P(A)$$

Thus the last property means that the occurrence of B AND C does not influence the probability of A

Independence of n events

$A_1, \dots, A_n$ are INDEPENDENT if

$$P\Big(\bigcap_{j \in J} A_j\Big) = \prod_{j \in J} P(A_j) \qquad \forall\ J \subseteq \{1, \dots, n\}$$

Thm if A and B are independent then A^c and B are independent.

Proof

We have to show that $P(A^c \cap B) = P(A^c) \cdot P(B)$

$$P(A^c \cap B) = P(B) - P(B \cap A) \underset{\text{INDEPENDENCE}}{=} P(B) - P(A) \cdot P(B)$$

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

$$= P(B)\,\underbrace{(1 - P(A))}_{= P(A^c)} = P(B) \cdot P(A^c)$$

∎

Def Given $\mathcal{C} \subseteq 2^{\Omega}$, the σ-algebra GENERATED by $\mathcal{C}$ is the smallest σ-algebra which contains $\mathcal{C}$.

$\rightarrow \mathcal{A}$ is the σ-algebra generated by $\mathcal{C}$ if

- $\mathcal{A}$ is a σ-algebra, $\mathcal{C} \subseteq \mathcal{A}$
- $\mathcal{A} \subseteq \mathcal{B}$ for any σ-algebra $\mathcal{B}$ such that $\mathcal{C} \subseteq \mathcal{B}$.

Given $A \subseteq \Omega$, let

$\sigma(A)$ the σ-algebra generated by A

$\mathcal{C} = \{A\}$

$\sigma(A) = \{\emptyset, \Omega, A, A^c\}$

$\sigma(B) = \{\emptyset, \Omega, B, B^c\}$

Def Two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are INDEPENDENT if $A \perp\!\!\!\perp B$ for any $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$.

Def The σ-algebras $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are INDEPENDENT if $P\left(\bigcap_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} P(A_k)$ for any $A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n$.

Note if two events A and B are independent then $\sigma(A)$ and $\sigma(B)$ are independent.

Example of BAYES formula

$$P(A|B) = P(B|A) \frac{P(A)}{P(B)}$$

- $P(A)$ PRIOR PROBABILITY

- $P(B|A)$ LIKELIHOOD
- $P(B)$ MARGINAL PROBABILITY
- $P(A|B) =$ POSTERIOR PROBABILITY

Example MEDICAL TEST H.I.V - E.L.I.S.A

(Healty, Ill) → each individual

Result of Test → (Positive, Negative)

How accurate is a test?

$P(I \mid Pos) = ?$ $\qquad$ $P(H \mid Neg) = ?$

$$P(I \mid Pos) = \frac{P(Pos \mid I) \cdot P(I)}{\underbrace{P(Pos \mid I) \cdot P(I) + P(Pos \mid H)\, P(H)}_{= P(Pos)}}$$

BAYES FORMULA

Formula total probability

Suppose we know percentage of ill people

$P(I) = 1 - P(H) = 0,000025$

$P(H) = 0,999975$

Suppose we know

$P(Pos \mid I) = 0,993$

by a study on a small sample of the population.

$P(Neg \mid I) = 1 - P(Pos \mid I) = 0,007$

$P(Neg \mid H) = 0,9999$ "SPECIFICITY of the TEST"

$P(Pos \mid H) = 0,0001$

$\Rightarrow P(I \mid Pos) \cong 0,1988 \simeq 20\%$

LOW → if the test is positive then the individual is ill with probability $\frac{1}{5}$

is ill with probability $\frac{1}{5}$

- $P(I)$ small influences accuracy of the test.
- The test is accurate on a part of the population with larger $P(I)$.

In the example of last time (two dice)
A, B, C are pairwise independent : $A \perp\!\!\!\perp B$, $A \perp\!\!\!\perp C$, $B \perp\!\!\!\perp C$,
but NOT independent, as
$A \cap B \cap C = \{(1,6)\}$
$P(A \cap B \cap C) = \frac{1}{36} \neq P(A) \cdot P(B) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$

When $P(A) = 1$ we say that A occurs ALMOST SURELY
We see how to construct events with PROB. 1

- if $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ then $\lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n$
  (increasing sequence) $\to$ $P(\bigcup_n A_n) = \lim_n P(A_n)$
- if $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ then $\lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n$
  (decreasing sequence) $\to$ $P(\bigcap_n A_n) = \lim_n P(A_n)$

In general, limit is not defined
We can define limsup and liminf
Given $(A_n)_{n \in \mathbb{N}}$

$$\limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$$

$$\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right)$$

Call $B_n = \bigcup^{\infty} A_k$, note $B_1 = B_2 \cup A_1, \ldots$

Call $B_n = \bigcup_{k=n}^{\infty} A_k$, note $B_1 = B_2 \cup A_1, \dots$

$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \supseteq B_n \supseteq \dots$ → decreasing sequence

$$\limsup_{n\to\infty} A_n = \lim_n B_n = \bigcap_{n=1}^{\infty} B_n$$

$$\omega \in \limsup_n A_n \iff \omega \in B_n \quad \forall n$$

$$\forall n, \ \omega \in B_n = \bigcup_{k=n}^{\infty} A_k \iff \exists k_n \geq n \text{ s.t. } \omega \in A_{k_n}$$

$$\iff \forall n \ \exists k_n \geq n \text{ s.t. } \omega \in A_{n_k}$$

$$\limsup_{n\to\infty} A_n = \{ \omega \in A_n \text{ infinitely often} \}$$

$$= \{ \omega \in A_n \text{ for infinitely many } n \}$$

BOREL - CANTELLI LEMMA

Let $(A_n)_n$ be a sequence of events

i) if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(\limsup_{n\to\infty} A_n) = 0$

ii) if $\sum_{n=1}^{\infty} P(A_n) = +\infty$ AND $(A_n)_n$ are independent

then $P(\limsup_{n\to\infty} A_n) = 1$

Proof

i) $P(\limsup_n A_n) \underset{\downarrow}{=} \lim_n P(B_n) = \lim_n P(\bigcup_{k=n}^{\infty} A_k) \leq 0$

PROPERTY 9 → decreasing limit

$$P(\bigcup_{k=n}^{\infty} A_k) \leq \sum_{k=n}^{\infty} P(A_k) = S - S_n \xrightarrow{n\to\infty} 0$$

where $S_n = \sum_{k=1}^{n} P(A_k)$ ↳ reduced sum $\qquad S = \sum_{n=1}^{\infty} P(A_n) < \infty$

$\lim_{n\to\infty} S_n = S \qquad$ Then $P(\limsup_n A_n) = 0$

$$\lim_{n\to\infty} s_n = s \qquad \text{Then } P(\limsup_n A_n) = 0$$

ii) $$\left(\limsup_n A_n\right)^C = \left(\bigcap_{n=1}^{\infty} B_n\right)^C = \bigcup_{n=1}^{\infty} B_n^C$$

We have to show that $P\left(\left(\limsup_n A_n\right)^C\right) = 0$

It is enough to prove that $P(B_n^C) = 0 \quad \forall n$

since then $$P\left(\bigcup_{n=1}^{\infty} B_n^C\right) \leq \sum_{n=1}^{\infty} P(B_n^C) = \sum_{n=1}^{\infty} 0 = 0$$

$$B_n^C = \left(\bigcup_{k=n}^{\infty} A_k\right)^C = \bigcap_{k=n}^{\infty} A_k^C$$

$$P\left(\bigcap_{k=n}^{\infty} A_n^C\right) = \lim_{N\to\infty} P\left(\bigcap_{k=n}^{N} A_k^C\right)$$

$$\overset{\text{INDEPENDENCE}}{=} \lim_{N\to\infty} \prod_{k=n}^{N} P(A_k^C) = \lim_{N\to\infty} \prod_{k=n}^{N} \left(1 - P(A_k)\right)$$

Since the inequality $1 - x \leq e^{-x}$ holds $\forall x \geq 0$

$$P\left(\bigcap_{k=n}^{\infty} A_n^C\right) \leq \lim_{N\to\infty} \prod_{k=n}^{N} e^{-P(A_k)} = \lim_{N\to\infty} \exp\left\{-\sum_{k=n}^{N} P(A_k)\right\}$$

$$= \exp\Big\{-\underbrace{\lim_{N\to\infty} \sum_{k=n}^{N} P(A_k)}_{= \sum_{k=n}^{\infty} P(A_k) = +\infty}\Big\} = e^{-\infty} = 0$$

$\blacksquare$