Lecture 22

lunedì 24 novembre 2025 10:27

- RECURRENCE / TRANSIENCE of STATES of MC
- ERGODIC THEOREM for infinite state space

• RECURRENCE and TRANSIENCE

Let $f_i = P[\, X_n = i \text{ for some } n \geq 1 \mid X_0 = i \,]$

- $i \in S$ is RECURRENT if $f_i = 1$
- $i \in S$ is TRANSIENT if $f_i < 1$

Let $N_i(\omega) = |\{ n \geq 1 : X_n(\omega) = i \}|$

- if i is RECURRENT then $P(N_i = +\infty \mid X_0 = i) = 1$
- if i is TRANSIENT then $P(N_i = +\infty \mid X_0 = i) = 0$

  In this case, $N_i \mid X_0 = i \sim Geo(1 - f_i)$

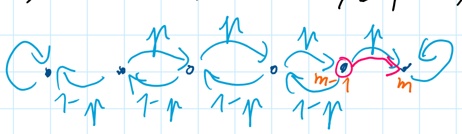
<u>Proposition</u> if $i \in S$ is recurrent (or transient) and i communicates with j, then j is recurrent (or transient)

$\Rightarrow$ being recurrent (or transient) is a property of the communication class

<u>Def</u> A communication class C is CLOSED if, for any $i \in C$ and j accessible from i $(i \to j)$, we have $j \in C$

<u>Example</u> GAMBLER'S RUIN model

3 classes $\{0\}$, $\{m\}$, $\{1, 2, \dots, m-1\}$



$\{0\}$ and $\{m\}$ are closed classes

$\{1, 2, \dots, m-1\}$ is NOT closed

because m and 0 are accessible from any state because m and 0 are accessible from any state in $\{1, \dots, m-1\}$, but m and 0 do not belong to the class $\{1, \dots, m-1\}$, as

$$P(X_n = j \mid X_0 = 0) = 0 \quad \forall j = 1, \dots, m-1$$

Property 1 Every recurrent class is closed

Property 2 Every finite closed class is recurrent.

Consequence of 2 for an irreducible MC with S finite, every state is recurrent

(irreducible $\Rightarrow$ ONE class $\Rightarrow$ closed)

GAMBLER'S RUIN

→ states 0 and m are RECURRENT
(as $\{0\}$ and $\{m\}$ are closed classes)

→ the other states $1, 2, \dots, m-1$ are transient
(by property 1)

S countable

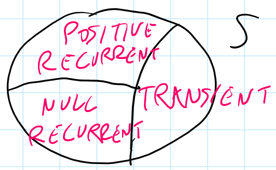
Let $T_i = \inf\{n \geq 1 : X_n = i\}$

and $m_i = E[T_i \mid X_0 = i]$

m_i is the expected time to return to state i, when MC starts from state i

Def A state $i \in S$ is POSITIVE RECURRENT if $m_i < +\infty$

A state $i \in S$ is NULL RECURRENT if $m_i = +\infty$



- if S is finite then there can not be null recurrent states

ERGODIC THEOREM III

if S is countable, P is irreducible, aperiodic and positive recurrent, then there exists a unique invariant distribution π such that

$$\pi_j = \lim_{n\to\infty} p_{ij}^n \qquad \forall j \in S$$

and is independent of i.

In particular, for any initial distribution $\nu^{(0)}$ we have

$$\lim_{n\to\infty} \nu^{(n)} = \lim_{n\to\infty} \nu^0 P^n = \pi$$

Moreover, the following properties hold:

i) $\pi = (\pi_0, \pi_1, \dots, \pi_n, \dots)$ is the unique solution to

$$\begin{cases} \pi = \pi P \\ \sum_{i\in S} \pi_i = 1 \end{cases}$$

ii) $\pi_j = \frac{1}{m_j} \quad \forall j \in S \longrightarrow$ in particular $\pi_j > 0 \ \forall j$

(holds true also for S finite)

iii) $\pi_j = \lim_{n\to\infty} \frac{n^\circ \text{ visits to states } j \text{ by time } n}{n}$

= long run proportion of time the MC spends in state j.

iv) MC is irreducible and there exists an invariant distribution then any state is positive recurrent

v) MC is irreducible, aperiodic and exists invariant distribution then $\lim_n p_{ij}^n = \pi_j$

vi) if S countable, MC irreducible and recurrent, then there exists a solution to $\pi = \pi P$, but π might not be a distribution (possible $\pi_i < 0$)

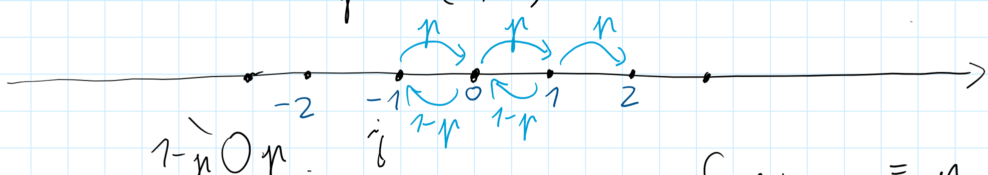
there exists a solution to $\pi = \pi P$,

but π might not be a distribution (possible $\pi_i < 0$)

→ have to require positive recurrent

RANDOM WALK

$S = \mathbb{Z}$ $\qquad p = (0,1)$



$$P = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & 1-p & 0 & p & \\ & & 1-p & 0 & p \\ & & & 1-p & 0 \quad p \\ & & & & \ddots \end{pmatrix}$$

$$\begin{cases} p_{i,i+1} = p \\ p_{i,i-1} = 1-p \\ p_{i,j} = 0 \qquad \forall j \neq i \pm 1 \end{cases}$$

$(S_n)_{n \geq 0}$ M.C. → random walk

$$\begin{cases} S_0 = 0 \\ S_1 = S_0 + X_1 \\ S_2 = S_2 + X_2 = S_0 + X_1 + X_2 \\ \vdots \\ S_{n+1} = S_n + X_{n+1} = X_1 + \dots + X_n + X_{n+1} \end{cases}$$

$(X_n)_{n \geq 1}$ INDEPENDENT identically distributed r.v.

$X_n \in \{1, -1\}$ $\qquad P(X_n = 1) = p$, $P(X_n = -1) = 1-p$

→ $(S_n)_n$ is IRREDUCIBLE

Is state 0 recurrent?

$(X_n)_{n \geq 1}$ i.i.d. $\qquad E[X_n] = 1\, P(X_n = 1) - 1\, P(X_n = -1)$

$\qquad = 1 \cdot p - (1-p) = 2p - 1$

STRONG LAW LARGE NUMBERS:

$$\frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} \longrightarrow E[X_1] = 2p - 1$$

for almost every ω

$$P\left(\omega : \lim_{n \to \infty} \frac{S_n(\omega)}{n} = 2p - 1\right) = 1$$

• if $p > \frac{1}{2} \rightarrow 2p-1 > 0 \Rightarrow \frac{S_n(\omega)}{n} \rightarrow 2p-1 > 0$

$\Rightarrow \lim_{n\to\infty} S_n(\omega) = +\infty$ for almost every ω

$\rightarrow$ 0 is TRANSIENT if $p > \frac{1}{2}$

• if $p < \frac{1}{2} \rightarrow 2p-1 < 0$

$\rightarrow \lim_{n\to\infty} S_n(\omega) = -\infty$ for almost every ω

$\rightarrow$ 0 is TRANSIENT

• $p = \frac{1}{2}$ $\quad 2p-1 = 0 \quad \frac{S_n(\omega)}{n} \rightarrow 0 \not\Rightarrow \lim_{n\to\infty} S_n = ?$

SYMMETRIC RANDOM WALK

$N_0 = |\{n \geq 0 : S_n = 0\}|$ $\rightarrow$ Consider $S_0 = 0$

0 is recurrent $\Leftrightarrow P(N_0 = +\infty) = 1$

0 is transient $\Leftrightarrow P(N_0 = +\infty) = 0$

N_0 r.v. $N_0 \in \{0, 1, \dots, \}$

$[N_0 \sim Geo(1 - f_0)]$

$$E[N_0] = \begin{cases} +\infty & \rightarrow 0 \text{ is recurrent} \\ \frac{1}{1-f_0} & \rightarrow 0 \text{ is transient} \end{cases}$$

$$N_0 = \sum_{n=0}^{+\infty} I_n \qquad I_n = \begin{cases} 1 & \text{if } S_n = 0 \\ 0 & \text{if } S_n \neq 0 \end{cases}$$

$$E[N_0] = E\left[\sum_{n=0}^{+\infty} I_n\right] = \sum_{n=0}^{+\infty} E[I_n]$$

$I_n = \mathbb{1}_{\{S_n = 0\}}$ $\qquad E[I_n] = P(S_n = 0)$

$$= \sum_{n=0}^{+\infty} \underbrace{P(S_n = 0 \mid S_0 = 0)}_{p_{00}^n} = \sum_{n=0}^{+\infty} p_{00}^n$$

i is RECURRENT $\Leftrightarrow \sum_{n=0}^{\infty} p_{ii}^n = +\infty$

i is TRANSIENT $\Leftrightarrow \sum_{n=0}^{\infty} p_{ii}^n < +\infty$

i is TRANSIENT $\Longleftrightarrow \sum_{n=0} p^n_{ii} < +\infty$

$$P(S_n = 0 \mid S_0 = 0) = \begin{cases} 0 & \text{if } n \text{ odd} \\ ? & \text{if } n \text{ even} \end{cases}$$

$\to S_n$ is PERIODIC with period 2

let $n = 2k \qquad k \in \mathbb{N}$

$$p^{2k}_{00} = P(S_n = 2k \mid S_0 = 0) \approx \text{BINOMIAL}$$
$$= \binom{2k}{k} p^k (1-p)^k$$

$$p = \frac{1}{2} \to p^{2k}_{00} = \binom{2k}{k} \frac{1}{2^{2k}} = \frac{(2k)!}{k!\,k!} \frac{1}{2^{2k}}$$

$$\sum_{k=0}^{+\infty} p^{2k}_{00} = ?$$

STIRLING FORMULA : $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

$$p^{2k}_{00} \sim \frac{\frac{(2k)^{2k}}{e^{2k}} \cancel{\sqrt{2\pi\, 2k}}}{\frac{k^k}{e^k} \frac{k^k}{e^k} \cancel{\sqrt{2\pi k}} \sqrt{2\pi k} \cdot 2^{2k}}$$

$$= \frac{\frac{\cancel{2^{2k}} \cancel{(k^k)^2}}{\cancel{(e^k)^2}}}{\frac{\cancel{(k^k)^2}}{\cancel{(e^k)^2}} \cancel{2^{2k}} \sqrt{\pi k}} = \frac{1}{\sqrt{\pi k}}$$

$$\Rightarrow \sum_{k=1}^{+\infty} p^{2k}_{00} < \infty \Longleftrightarrow \sum_{k=1}^{+\infty} \frac{1}{\sqrt{\pi k}} < \infty$$

$\left[\text{Recall } \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty \Longleftrightarrow \alpha > 1\right]$

$$\sum_{k=1}^{+\infty} \frac{1}{\sqrt{k} \cdot \sqrt{\pi}} = +\infty$$

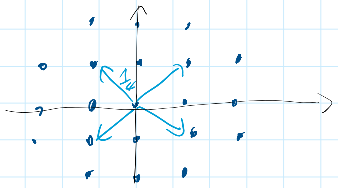
$\Rightarrow \sum_{k=1}^{+\infty} p^{2k}_{00} = +\infty \Rightarrow$ 0 is RECURRENT STATE

Then any state is recurrent as M.C. is irreducible

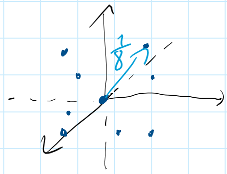
BUT one can show that any state is NULL RECURRENT and invariant distribution does not exist

- RANDOM WALK on PLANE or SPACE

$S = \mathbb{Z} \times \mathbb{Z}$



or $S = \mathbb{Z}^3$



- RANDOM WALK IS RECURRENT in $d=2$
  as $\sum_{k=0}^{+\infty} p_{00}^{2k} \sim \sum_{k=0}^{+\infty} \frac{1}{k} = +\infty$
- RANDOM WALK IS TRANSIENT in $d=3$
  as $\sum_{k=0}^{+\infty} p_{00}^{2k} \sim \sum_{k=0}^{+\infty} \frac{1}{k^{\frac{3}{2}}} < +\infty$
- in dimension d $\sum_{k=0}^{+\infty} p_{00}^{2k} \sim \sum_{k=0}^{+\infty} \frac{1}{k^{\frac{d}{2}}}$