Lecture 21

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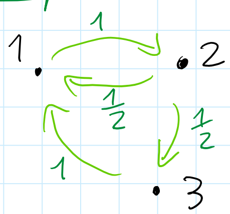
PERIODICITY of the STATES

- if $\{n \geq 1 : p_{ii}^{n} > 0\}$ has NO common divisor other than 1, then we say that i is APERIODIC
- if $\{n \geq 1 : p_{ii}^{n} > 0\}$ has common divisor $d(i) > 0$ then we say that i is PERIODIC with PERIOD $d(i)$

Remark 1 if $p_{ii} > 0$, then i is aperiodic

Remark 2 if i, j belong to the same communication class and i is aperiodic (or has period d) then the same holds for the state j.

Example 1 $S = \{1, 2, 3\}$



$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

- STATE 1 $p_{11}^{1} = 0$, $p_{11}^{2} > 0$, $p_{11}^{3} > 0$

  $\{n \geq 1 : p_{11}^{n} > 0\} \supseteq \{2, 3\}$

  $\Rightarrow$ 1 is aperiodic

$\Rightarrow$ 2 and 3 are also aperiodic because the MC is irreducible (Remark 2)

Example 2 "SENTINEL"

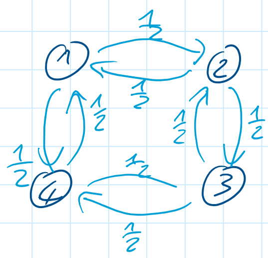
$S = \{1, 2, 3, 4\}$

$p = \frac{1}{2}$

$p_{ii} = 0$

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$P^2 = P \cdot P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$



$P^3 = P$ and $P^4 = P^2$

$\Rightarrow P^0 = I \qquad \begin{cases} P^{2n} = P^2 & \forall n \\ P^{2n+1} = P & \forall n \end{cases}$

state $1 \to \{n \geq 1 : p_{11}^n > 0\} = \{2, 4, 6, \dots\}$

1 is PERIODIC with period 2

Then all the states are periodic with period 2 because the MC is irreducible

ERGODIC THEOREM I

If S is finite, P stochastic matrix irreducible and aperiodic, then $\exists$ unique invariant distribution π and

$$\lim_{n\to\infty} p_{ij}^n = \pi_j \qquad \forall i, j \in S$$

In particular $\lim_{n\to\infty} v^{(0)} P^n = \lim_{n\to\infty} v^{(n)} = \pi$ for any $v^{(0)}$

Example GOOGLE ALGORITHM

$p_{ij} \approx$ probability of going from page i to page j

algorithm finds the invariant distribution of P

Remark If P is not aperiodic then we can modify a bit P to make it aperiodic

Given P and $\lambda > 0$ small, the matrix

$P_\lambda = (1-\lambda)P + \lambda \mathbb{1}$ is the transition matrix of an given P and $\lambda > 0$ small, the matrix

$P_\lambda = (1-\lambda)P + \lambda \mathbb{1}$ is the transition matrix of an aperiodic MC, as $(P_\lambda)_{ii} \geq \lambda > 0$

$$\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\hookrightarrow$ invariant distribution of P_λ and P is the same.

- example of sentinel : $\lim_n P^n$ does not exist

$$P^n = \begin{cases} P & \text{if } n \text{ odd} \\ P^2 & \text{if } n \text{ even} \end{cases}$$

ERGODIC THEOREM II

if S is finite and P irreducible then $\exists!$ invariant π

and $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \nu^{(k)} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \nu^{(0)} P^k = \pi$

for any $\nu^{(0)}$

- **MARKOV CHAIN MONTE CARLO ALGORITHM (MCMC)**

finite S and distribution π on S,

How to generate a sequence of r.v $(X_n)_{n\geq 1}$

s.t. the following LLN holds?

$$\text{(☆)} \quad \lim_{n\to\infty} P\left(\left| \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \pi(f) \right| > \varepsilon \right) = 0 \quad \forall \varepsilon$$

$$\pi(f) = \sum_{x \in S} f(x) \pi_x \longrightarrow = E[f(X)] \text{ if } X \sim \pi$$

- This holds true (by weak LLN) if (X_n) are INDEPENDENT

$\hookrightarrow$ can be costly to generate independent samplings.

- Instead, one can construct a MC with transition matrix P for which π is the invariant measure

Then (☆) holds by the ergodic theorem II

How to compute invariant distribution π?

Solving $\pi P = \pi$ can be difficult if $|S|$ is large.

We have a sufficient condition to determine whether a given distribution π is invariant for P

Def Let P be a stochastic matrix.
A distribution π is called REVERSIBLE for P if

$$\pi_x P_{xy} = \pi_y P_{yx} \qquad \forall x, y \in S$$

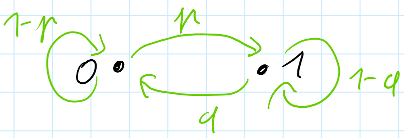
$\rightarrow$ DETAILED BALANCE CONDITION

Proposition if π is reversible for P then π is invariant for P, i.e. $\pi P = \pi$

Proof
$$(\pi P)_x = \sum_{y \in S} \pi_y P_{yx} = \sum_{y \in S} \pi_x P_{xy}$$
$$= \pi_x \underbrace{\sum_{y \in S} P_{xy}}_{=1} = \pi_x \qquad \forall x \qquad \boxtimes$$

Example 1 $|S| = 2$ $\quad S = \{0, 1\}$

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \qquad p, q \in [0,1]$$



P is irreducible $\iff$ $p \neq 0$ and $q \neq 0$

In fact, if $p = 0$, 0 does not communicate with 1

if $p = q = 0$, $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

0 and 1 are absorbing states

Invariant distribution π? Solve $\pi P = \pi$

- $p = 0 = q$ $\rightarrow$ $\pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pi$ $\qquad \begin{cases} \pi_1 = \pi_1 \\ \pi_2 = \pi_2 \end{cases}$

- $p = 0 = q \quad \to \pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pi \qquad \begin{cases} \pi_1 = \pi_1 \\ \pi_2 = \pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$

  Any distribution is invariant

- $q \neq 0$ (at least one of p and q is not 0)

  $\to$ look for distributions solving the detailed balance equation

  $\pi_x P_{xy} = \pi_y \pi_{yx} \qquad x = 0, y = 1$

  $\to \pi_0 P_{01} = \pi_1 P_{10} \to \pi_0 p = \pi_1 q$

  $\Rightarrow \begin{cases} \pi_1 = \frac{p}{q} \pi_0 \\ \pi_1 + \pi_0 = 1 \end{cases} \qquad \pi_0 + \frac{p}{q}\pi_0 = \left(1 + \frac{p}{q}\right)\pi_0 = 1 \qquad \pi_0 = \frac{q}{p+q}$

  $\Rightarrow (\pi_0, \pi_1) = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$

  is the invariant distribution

  - unique and $\pi_1, \pi_0 > 0$ if $p, q > 0$, that is, if MC is irreducible

Example 2 SENTINEL $(p = \frac{1}{2}, |S| = 4)$

MC is irreducible $\Rightarrow \exists!$ invariant distribution

detailed balance equation : $\pi_x P_{xy} = \pi_y P_{yx} \quad \forall x \neq y$

$x = 1, y = 2 \qquad \pi_1 \frac{1}{2} = \pi_2 \frac{1}{2} \to \pi_1 = \pi_2$

$x = 1, y = 4 \qquad \pi_1 \frac{1}{2} = \pi_4 \frac{1}{2} \qquad \pi_1 = \pi_4$

$- - - \ldots$

$\Rightarrow \pi_1 = \pi_2 = \pi_3 = \pi_4$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

$\Rightarrow \pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is the invariant distribution

Def A DOUBLY STOCHASTIC MATRIX is P such that

$$\sum_{x \in S} P_{xy} = 1 \quad \forall y \in S \qquad \text{and} \qquad \sum_{y \in S} P_{xy} = 1$$

Proposition if S is finite and P is doubly stochastic

Proposition if S is finite and P is doubly stochastic then the UNIFORM DISTRIBUTION $\pi = \left(\frac{1}{|S|}, \dots, \frac{1}{|S|} \right)$ is invariant for P.

Proof $\pi P = \pi$?

$$(\pi P)_x = \sum_{y \in S} \underbrace{\pi_y}_{= \frac{1}{|S|}} P_{yx} = \frac{1}{|S|} \underbrace{\sum_{y \in S} P_{yx}}_{= 1} = \frac{1}{|S|} = \pi_x$$

$\boxtimes$