DATA SCIENCE **Stochastic Methods**

January 22, 2025

Problem 1. [12] Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two independent sequences of independent random variables, such that $X_n \sim \mathrm{Bin}(1,p)$ and $Y_n \sim \mathscr{U}(0,1)$ for any n.

(i) For any n define $Z_n = X_n + Y_n$. Compute the distribution and expectation of Z_n;

(ii) Compute the limit in probability of the sequence $\frac{Z_1+Z_2+\ldots+Z_n}{n}$, as $n \to +\infty$;

(iii) For any n define $V_n = X_n \cdot Y_n$. Compute the distribution, expectation and variance of V_n;

(iv) Compute the limit in probability of the sequence $\frac{V_1+V_2+\ldots+V_n}{n}$, as $n \to +\infty$.

Problem 2. [13] We have two rows of 3 balls: in both rows, there are 2 white balls and 1 black ball. We move the balls according to the following scheme: we randomly choose one of the two rows, then we randomly select a ball from that row and move it to the last position in the same row. We repeat this process in the same manner.

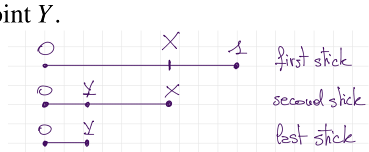
(i) Define a Markov chain that describes the positions of the two black balls in their respective rows;

(ii) Classify the states of the Markov chain;

(iii) Compute the invariant distribution;

(iv) If initially both black balls are in the first position, on average, how many turns will it take for them to return in that position?

Problem 3. [9] A stick of length one is broken at a random point X, uniformly distributed over the stick. This remaining stick is broken once more at a random point Y.



(i) Determine the expected length of the first broken stick (the second stick in the figure above);

(ii) Compute the conditional expected length of the last stick, given that $X = x$.

(iii) Compute the expected length of the last stick.