PROBLEMS - SET 5

Problem 1. Let (X_n) be a i.i.d. sequence of $Bin(1,p)$. Prove the following Chernoff bounds:

$$\mathrm{P}(\overline{X}_n \geq (1+\delta)p) \leq e^{-np\frac{\delta^2}{2+\delta}},$$

and

$$\mathrm{P}(\overline{X}_n \leq (1-\delta)p) \leq e^{-np\frac{\delta^2}{2}}.$$

Problem 2. Let (X_n) be a i.i.d. sequence of $N(0,1)$.

(a) Prove the following Chernoff bound:

$$\mathrm{P}(\overline{X}_n \geq \varepsilon) \leq e^{-n\frac{\varepsilon^2}{2}}$$

(b) Obtain a sharper upper tail estimate, proving first the inequality

$$\mathrm{P}(X_1 \geq x) \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

and deriving then that

$$\mathrm{P}(\overline{X}_n \geq \varepsilon) \leq \frac{1}{\varepsilon\sqrt{n}}\frac{1}{\sqrt{2\pi}}e^{-n\frac{\varepsilon^2}{2}}$$

(c) Prove the further inequality

$$\mathrm{P}(X_1 \geq x) > \left(\frac{1}{x} - \frac{1}{x^3}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Problem 3. Let (X_n) be a i.i.d. sequence of Pois(λ). Prove the following Chernoff bounds:

$$\mathrm{P}(\overline{X}_n \geq \lambda(1+\varepsilon)) \leq e^{-n\lambda a(\varepsilon)},$$

and, for $0 < \varepsilon < 1$

$$\mathrm{P}(\overline{X}_n \leq \lambda(1-\varepsilon)) \leq e^{-n\lambda b(\varepsilon)},$$

where

$$a(\varepsilon) = \frac{\varepsilon^2}{2+\varepsilon}$$

and

$$b(\varepsilon) = \frac{\varepsilon^2}{2}.$$

Hint: prove (and use) the following inequalities

$$(1+\varepsilon)\log(1+\varepsilon) - \varepsilon := f(\varepsilon) \geq a(\varepsilon) = \frac{\varepsilon^2}{2+\varepsilon}, \quad \forall \varepsilon > 0, \tag{0.1}$$

$$\varepsilon + (1-\varepsilon)\log(1-\varepsilon) := k(\varepsilon) \geq b(\varepsilon) = \frac{\varepsilon^2}{2} \quad \forall\, 0 < \varepsilon < 1. \tag{0.2}$$

Note that if F, G are C^2 functions in $[0,a)$, $F(0) = G(0)$, $F'(0) = G'(0)$ and $F''(x) \geq G''(x)$ for all $x \in [0,a)$, then $F(x) \geq G(x)$ for $x \in [0,a)$.

Problem 4. Let (X_n) be a i.i.d. sequence with distribution Unif$(-1,1)$

(a) Show that $m_{X_n}(t) = \frac{\sinh(t)}{t}$.

(b) Show that there exists $k > 0$ such that for all $0 \leq t \leq k$

$$\frac{\sinh(t)}{t} \leq 1 + \frac{t^2}{2}$$

(numerically $k \simeq 4.75$.)

(c) Prove the Chernoff bound: for all $0 < \varepsilon \leq k$

$$\mathrm{P}(\overline{X}_n \geq \varepsilon) \leq e^{-n\frac{\varepsilon^2}{2}}.$$

(d) Without any further computation, explain why the bound for the lower tail

$$\mathrm{P}(\overline{X}_n \leq -\varepsilon) \leq e^{-n\frac{\varepsilon^2}{2}}.$$

also holds for all $0 < \varepsilon \leq k$.

Problem 5. Let (X_n) be a i.i.d. sequence such that

$$\varphi_{X_n}(u) = \frac{1}{1+u^2}.$$

For every $\varepsilon > 0$ find $a(\varepsilon) > 0$ (no "nice" form is necessary) such that

$$\mathrm{P}(\overline{X}_n \geq \mathrm{E}(X_1) + \varepsilon) \leq e^{-na(\varepsilon)}$$
$$\mathrm{P}(\overline{X}_n \leq \mathrm{E}(X_1) - \varepsilon) \leq e^{-na(\varepsilon)}.$$

Then extend to the case in which

$$\varphi_{X_n}(u) = \frac{e^{iua}}{1+u^2}.$$

(Try to make no further calculations).

Problem 6. Let $Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n$ be independent random variables such that $Y_i \sim \mathrm{Exp}(1)$ and $Z_i \sim N(0,1)$. Set $X_i := Y_i + Z_i$.

(a) Compute mean, variance and moment generating function of X_i.

(b) For $0 < \varepsilon < 1$, determine $a(\varepsilon) > 0$ such that the following *lower tail Chernoff bound* holds:

$$\mathrm{P}(\overline{X}_n \leq 1 - \varepsilon) \leq e^{-na(\varepsilon)}.$$

Hint: use the inequality $\log(1+t) \geq t - \frac{1}{2}t^2$ for $t \geq 0$.

Problem 7. (a) Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with distribution $\mathrm{N}(0, \sigma^2)$. Show that for every $\varepsilon > 0$

$$\mathrm{P}(\overline{X}_n > \varepsilon) \leq e^{-n\varepsilon^2/2\sigma^2}.$$

(b) A random variable X with $\mathrm{E}(X) = 0$ is said to be *Subgaussian* for the parameter $\sigma > 0$ if its moment generating function $m_X(t)$ is such that

$$m_X(t) \leq e^{t^2\sigma^2/2}$$

for all $t \in \mathbb{R}$. Show that the inequality in (a) holds if $X_1, X_2, \ldots, X_n$ are i.i.d random variables, and Subgaussian for the parameter $\sigma > 0$.