

DATA SCIENCE Stochastic Methods

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Problem 1. [10] Let X_1, X_2, X_3 be independent, binomial $(2, p)$ random variables. Define $Y_1 = \min\{X_1, X_2, X_3\}$, $Y_3 = \max\{X_1, X_2, X_3\}$ and $Y_2 = X_1 + X_2 + X_3 - Y_1 - Y_3$.

- (i) Compute $P[Y_1 = 2]$ and $P[Y_3 = 0]$;
- (ii) Compute $P[Y_1 = Y_2 = Y_3]$;
- (iii) Compute $P[Y_2 > n]$ for $n = 0, 1$;
- (iv) Compute $E[Y_2]$.

Problem 2. [10] Let $(X_n)_{1 \leq i \leq n}$ be a family of i.i.d. uniform random variables on $[-1, 1]$. Define $S_n = \sum_{i=1}^n X_i$ and $Z_n = S_n^2 - n/3$.

- (i) Compute the expectation and the variance of S_n and the expectation of Z_n ;
- (ii) Compute the conditional expectation of S_{n+1}^2 given X_1, X_2, \dots, X_n ;
- (iii) Is Z_n a martingale?

Problem 3. [12] Let $(X_n)_{n \geq 0}$ be a Markov chain on $\{1, 2, 3, 4\}$ with transition probabilities given by

$$p_{i,1} = \frac{i}{i+1}, \quad p_{i,i+1} = \frac{1}{i+1}, \quad 1 \leq i \leq 3 \quad \text{and} \quad p_{4,2} = 1$$

- (i) Is the Markov chain irreducible?
- (ii) Is the Markov chain aperiodic?
- (iii) Compute $E[X_3 | X_0 = 1]$;
- (iv) Determine the invariant distribution.