DATA SCIENCE **Stochastic Methods**

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Problem 1. [12] Let $X_1, X_2, \dots, X_n$ be independent, absolutely continuous uniform $[0,4]$ random variables. Define $Y_k = |X_k - 2|$ for any $k = 1, \dots, n$.

(i) Prove that $\mathrm{P}[Y_1 \leq 2] = 1$ and compute $\mathrm{P}[Y_1 \leq y]$ for $y \in \mathbb{R}$;

(ii) Compute $\mathrm{E}[Y_1]$;

(iii) Compute $m(t) = \mathrm{E}[e^{tY_1}]$;

(iv) Use the Heoffding's inequality to prove a Chernoff's Bound Upper tail estimate for $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

Problem 2. [10] Let X be a Binomial random variable of parameters $(2, 0, 5)$ and be Y be a Geometric random variable of parameter $(X+1)/3$, i.e. $Y|X = n \sim Geom((n+1)/3)$.

(i) Compute $\mathrm{P}[Y = k|X = n]$ for any $k \in \mathbb{N}, n = 0, 1, 2$;

(ii) Compute $h(n) = \mathrm{E}[Y|X = n]$ for any n;

(iii) Compute $\mathrm{E}[\mathrm{E}[Y|X]]$.

Problem 3. [10] The pattern of sunny and rainy days is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.7. Every rainy day is followed by another rainy day with probability 0.8.

(i) Classify the states of this Markov Chain;

(ii) Today is sunny: what is the chance of rain the day after tomorrow?

(iii) Compute approximately the probability that November 1st next year is rainy.

(iv) If today is a rainy day, on average, how long will it take to have another rainy day?