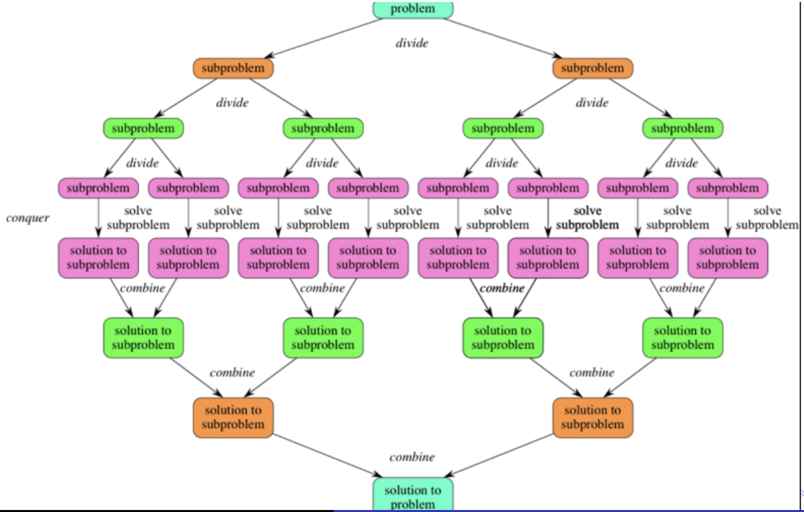
Divide-and-conquer: Selection



Selection

From 9.3 in CLRS

Selection Problem: Given an array A of n unordered distinct keys, and $i \in \{1, \ldots, n\}$, select the ith-smallest element in A, that is the key that is larger than exactly $i - 1$ other keys in A.

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Notice that i can be any rank value, in particular when:

1. $i = 1$, the MINIMUM element
2. $i = n$, the MAXIMUM element
3. $i = \lfloor \frac{n+1}{2} \rfloor$, the MEDIAN
4. $i = \lfloor 0.25\, n \rfloor \Rightarrow$ *order statistics*

A first algorithm

Sort A in $(O(n \lg n))$ steps, then the i-th smallest key is $A[i]$.

Can we do it faster? in linear time?

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Can we do it faster? in linear time?

Yes, selection is easier than sorting

The algorithm: High level

- Chose a split element x.
- Let k be the rank of x, if $k = i$, we found the i-th element. Otherwise,
- Use x to determine a partition of A, smaller than x to the left and larger to the right.
- Compute recursively the i-th element in the left part, when $i < k$, or the $i - k$-th element in the right part, when $i > k$.

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The time depends on the quality of the splitting element to divide fairly the elements

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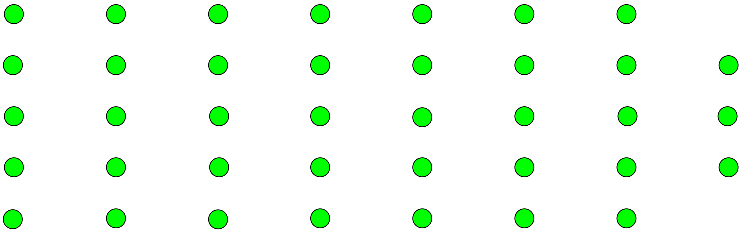

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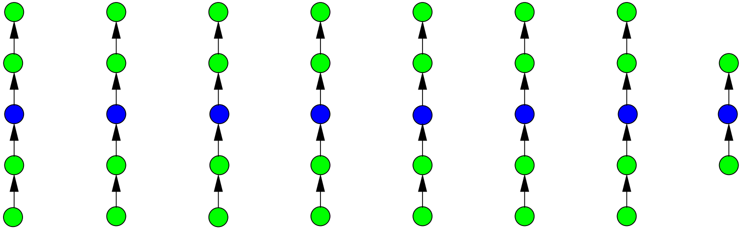

The splitting element x is the median of the set of medians, $\{x_j \mid 1 \leq j \leq \lceil n/5 \rceil\}$.

The algorithm

Select(A, i)
Divide A into $m = \lceil n/5 \rceil$ groups, all but at most one with 5 elements
$X[j] =$ median of group j, $j = 1, \ldots, m$
$x =$ **Select**$(X, \lfloor (m+1)/2 \rfloor)$ i.e. the median of X
Let k be the rank of x in A
if $i = k$ **then**
  return x
else
  $L =$ the elements of A smaller than x
  $R =$ the elements of A bigger than x
  if $i < k$ **then**
    return **Select**(L, i)
  else
    return **Select**$(R, i - k)$

Example: Find the median

Let $n = 15$, we want to get the 5-th element on the following input:

$$A = \quad 3\ 13\ 9\ 4\ 5 \quad 1\ 15\ 12\ 10\ 2 \quad 6\ 14\ 8\ 17\ 11$$

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The median of $X = (5, 10, 11)$ is 10 which has rank 9
As $5 < 9$, recursively ask for the 5-th element in the left part with respect to $x = 10$, i.e., $(3, 9, 4, 5, 1, 2, 6, 8)$

Example: Find the median

In the next call $n = 8$, we look for the 5-th element in the following input:

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The median of $X = (4, 6)$ is 4 which has rank 4.
As $5 > 4$ the algorithm looks for the 1st element in the right part $(5, 6, 8, 9)$, which is 5.

Selection algorithm: Cost

Select(A, i)
Divide A into $m = \lceil n/5 \rceil$ groups, all but at most one with 5 elements $O(n)$
$X[j]$ = median of group j, $j = 1, \ldots, m$ $O(n)$
$x =$ **Select**$(X, \lfloor (m+1)/2 \rfloor)$ i.e. the median of X $T(n/5)$
Let k be the rank of x in A
if $i = k$ **then**
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else
  $L =$ the elements of A smaller than x $O(n)$
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  if $i < k$ **then**
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Selection algorithm: elements bigger than x

At least $3(\frac{1}{2}(\lceil n/5 \rceil - 2)) \geq \frac{3n}{10} - 6$ of the elements are $< x$.

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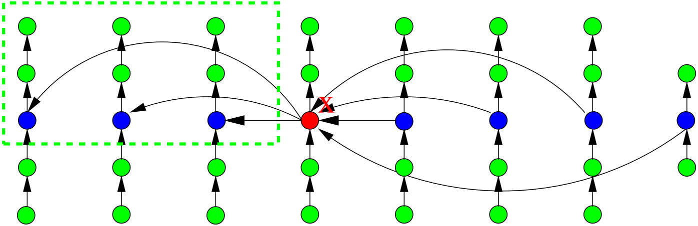

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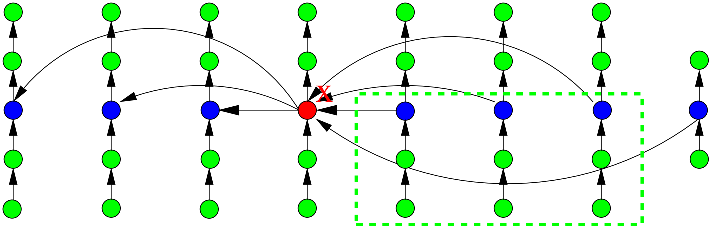

Selection algorithm: the recurrence

- As at least $\geq \frac{3n}{10} - 6$ of the elements are $> x$ ($< x$), at most $n - (\frac{3n}{10} - 6) = 6 + 7n/10$ elements are $\leq x$ ($\geq x$).
- In the worst case, **Select** recursively calls on a vector with size $\leq 6 + 7n/10$. So, step 5 takes time $\leq T(6 + 7n/10)$.

Therefore, selecting 50 as the size to stop the recursion, we have

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 50, \\ T(\lceil n/5 \rceil) + T(6 + 7n/10) + \Theta(n) & \text{if } n > 50. \end{cases}$$

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Solving we get $T(n) = \Theta(n)$ How?

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- Assume that $T(n) \leq c\, n$, for some constant c and $n \leq 50$. Note that $6 + 7n/10 < n$, for $n > 12$.
- Prove that $T(n) \leq c\, n$ by induction. As usual we replace a $\Theta(n)$ term by $d\, n$, for an adequate constant d.

$$\begin{aligned} T(n) &\leq T(\lceil n/5 \rceil) + T(6 + 7n/10) + d\, n \\ &\leq c\lceil n/5 \rceil + c(6 + 7n/10) + d\, n \\ &\leq c(n/5 + 1) + c(6 + 7n/10) + d\, n \\ &\leq 9\, c\, n/10 + 7c + d\, n \leq cn \end{aligned}$$

Taking $c = 10d$, for large n, the inequality holds.

Remarks on the cardinality of the groups

Notice:

- If we make groups of 7, the number of elements $\geq x$ is $\frac{2n}{7}$, which yield $T(n) \leq T(n/7) + T(5n/7) + O(n)$ with solution $T(n) = O(n)$.
- However, if we make groups of 3, then $T(n) \leq T(n/3) + T(2n/3) + O(n)$, which has a solution $T(n) = O(n \ln n)$.