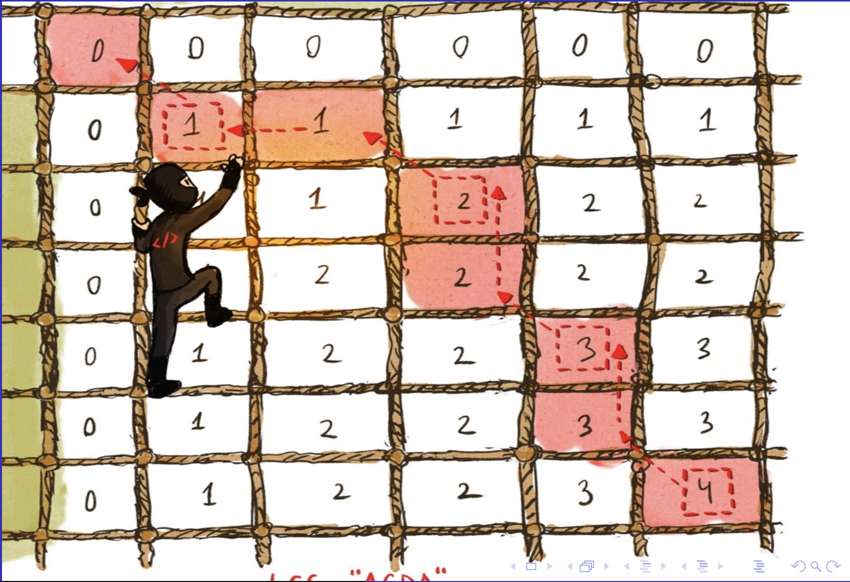
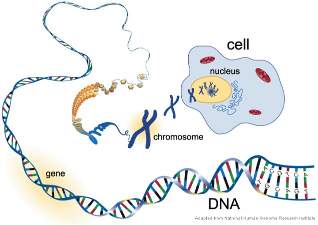
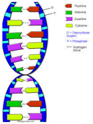
Dynamic Programming II



Matching DNA sequences



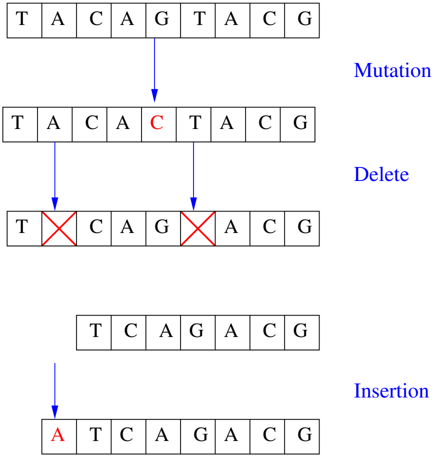


- DNA, is the hereditary material in almost all living organisms. They can reproduce by themselves.
- Its function is like a program unique to each individual organism that rules the working and evolution of the organism.
- Model as a string of 3×10^9 characters over $\{A, T, G, C\}$.

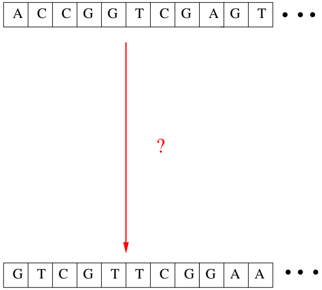
Computational genomics: Some questions

- When a new gene is discovered, one way to gain insight into its working, is to find well known genes (not necessarily in the same species) which match it closely. Biologists suggest a generalization of edit distance as a definition of approximately match.
- GenBank (`https://www.ncbi.nlm.nih.gov/genbank/`) has a collection of $> 10^{10}$ well studied genes, BLAST is a software to do fast searching for similarities between a gene an those in a DB of genes.
- Sequencing DNA: consists in the determination of the order of DNA bases, in a short sequence of 500-700 characters of DNA. To get the global picture of the whole DNA chain, we generate a large amount of DNA sequences and try to assembled them into a coherent DNA sequence. This last part is usually a difficult one, as the position of each sequence is the global DNA chain is not know before hand.

Evolution DNA

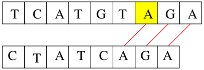


How to compare sequences?

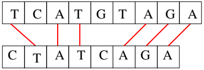


Three problems

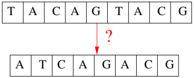
Longest common substring: Substring = consecutive characters in the string.



Longest common subsequence: Subsequence = ordered chain of characters (might have gaps).

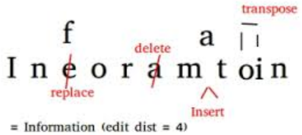


Edit distance: Convert one string into another one using a given set of operations.



The EDIT DISTANCE problem

(Section 6.3 in Dasgupta, Papadimritriou, Vazirani's book.)



The edit distance between strings $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ is defined to be the minimum number of *edit operations* needed to transform X into Y.

All the operations are done on X

Edit distance: Applications

- Computational genomics: evolution between generations, i.e. between strings on $\{A, T, G, C, -\}$.
- Natural Language Processing: distance, between strings on the alphabet.
- Text processor, suggested corrections

EDIT DISTANCE: Levenshtein distance

In the Levenshtein distance the set of operations are

- insert$(X, i, a) = x_1 \cdots x_i a x_{i+1} \cdots x_n$.
- delete$(X, i) = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$
- modify$(X, i, a) = x_1 \cdots x_{i-1} a x_{i+1} \cdots x_n$.

the cost of modify is 2, and the cost of insert/delete is 1.

To simplify, in the following we assume that *the cost of each operation is 1.*

For other operations and costs the structure of the DP will be similar.

Exemple-1

$X = aabab$ and $Y = babb$

$aabab = X$

$X' =$insert$(X, 0, b)$ $\quad baabab$

$X'' =$delete$(X', 2)$ $\quad babab$

$X'' =$delete$(X'', 4)$ $\quad babb$

$X = aabab \to Y = babb$

Exemple-1

$X = aabab$ and $Y = babb$
$aabab = X$
$X' = $insert$(X, 0, b)$ $baabab$
$X'' = $delete$(X', 2)$ $babab$
$X'' = $delete$(X'', 4)$ $babb$
$X = aabab \rightarrow Y = babb$

A shortest edit distance

$aabab = X$
$X' = $modify$(X, 1, b)$ $babab$
$Y = $delete$(X', 4)$ $babb$

Use dynamic programming.

The structure of an optimal solution

- In a solution O with minimum edit distance from $X = x_1 \cdots x_n$ to $Y = y_1 \cdots y_m$, we have three possible alignments for the last terms

(1)	(2)	(3)
x_n	$-$	x_n
$-$	y_m	y_m

- In (1), O performs delete x_n, and it transforms optimally, $x_1 \cdots x_{n-1}$ into $y_1 \cdots y_m$.
- In (2), O performs insert y_m at the end of x, and it transforms optimally, $x_1 \cdots x_n$ into $y_1 \cdots y_{m-1}$.
- In (3), if $x_n \neq y_m$, O performs modify x_n by y_m, otherwise O, aligns them without cost. Furthermore O transforms optimally $x_1 \cdots x_{n-1}$ into $y_1 \cdots y_{m-1}$.

The recurrence

Let $X[i] = x_1 \cdots x_i$, $Y[j] = y_1 \cdots y_j$.
$E[i,j]$ = edit distance from $X[i]$ to $Y[j]$ is the maximum of

- I *put y_j at the end of x*: $E[i, j-1] + 1$
- D *delete x_i*: $E[i-1, j] + 1$
- if $x_i \neq y_j$, M *change x_i into y_j*: $E[i-1, j-1] + 1$, otherwise $E[i-1, j-1]$

Edit distance: Recurrence

Adding the base cases, we have the recurrence

$$E[i,j] = \begin{cases} j & \text{if } i = 0 \text{ (converting } \lambda \to Y[j]) \\ i & \text{if } j = 0 \text{ (converting } X[i] \to \lambda) \\ \min \begin{cases} E[i-1,j]+1 & \text{if D} \\ E[i,j-1]+1, & \text{if I} \\ E[i-1,j-1]+\delta(x_i,y_j) & \text{otherwise} \end{cases} & \end{cases}$$

where

$$\delta(x_i,y_j) = \begin{cases} 0 & \text{if } x_i = y_j \\ 1 & \text{otherwise} \end{cases}$$

Computing the optimal costs and pointers

Edit(X, Y)
for $i = 0$ **to** n **do**
    $E[i,0] = i$
for $j = 0$ **to** m **do**
    $E[0,j] = j$
for $i = 1$ **to** n **do**
    for $j = 1$ **to** m **do**
        $\delta = 0$
        if $x_i \neq y_j$ **then**
            $\delta = 1$
        $E[i,j] = E[i,j-1] + 1$ $b[i,j] = \uparrow$
        if $E[i-1,j-1] + \delta < E[i,j]$ **then**
            $E[i,j] = E[i-1,j-1] + \delta,\ b[i,j] := \nwarrow$
        if $E[i-1,j] + 1 < E[i,j]$ **then**
            $E[i,j] = E[i-1,j] + 1,\ b[i,j] := \leftarrow$

Space and time complexity: $O(nm)$.

$\leftarrow$ is a I operation,
$\uparrow$ is a D operation, and
$\nwarrow$ is either a M or a no-operation.

Computing the optimal costs: Example

X=aabab; Y=babb. Therefore, $n = 5, m = 4$

		0	1	2	3	4
		λ	b	a	b	b
0	λ	0	$\leftarrow 1$	$\leftarrow 2$	$\leftarrow 3$	$\leftarrow 4$
1	a	$\uparrow 1$	$\nwarrow 1$	$\nwarrow 1$	$\leftarrow 2$	$\leftarrow 3$
2	a	$\uparrow 2$	$\nwarrow 2$	$\nwarrow 1$	$\leftarrow 2$	$\leftarrow 3$
3	b	$\uparrow 3$	$\nwarrow 2$	$\uparrow 2$	$\nwarrow 1$	$\nwarrow 2$
4	a	$\uparrow 4$	$\uparrow 3$	$\nwarrow 2$	$\uparrow 2$	$\nwarrow 2$
5	b	$\uparrow 5$	$\nwarrow 4$	$\uparrow 3$	$\uparrow 2$	$\nwarrow 2$

$\leftarrow$ is a I operation, $\uparrow$ is a D operation, and
$\nwarrow$ is either a M or a no-operation.

Obtain Y in edit distance from X

Uses as input the arrays E and b.
The first call to the algorithm is **con-Edit** (n, m)

con-Edit(i, j)
if $i = 0$ or $j = 0$ **then**
  return
  if $b[i, j] = \nwarrow$ and $x_i = y_j$ **then**
    change$(X, i, y_j))$; **con-Edit**$(i-1, j-1)$
  if $b[i, j] = \uparrow$ **then**
    delete(X, i); **con-Edit**$(i-1, j)$
  if $b[i, j] = \leftarrow$ **then**
    insert(X, i, y_j), **con-Edit**$(i, j-1)$

This algorithm has time complexity $O(nm)$.

The Longest Common Subsequence

(Section 15.4 in CormenLRS' book.)

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- $Z = z_1 \cdots z_k$ is a subsequence of X if there is a subsequence of integers $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $z_j = x_{i_j}$.

  TTT is a subsequence of $ATATAT$.

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LCS Given sequences $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$, compute the longest common subsequence Z.

DP approach: Characterization of optimal solution

Let $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ and let Z be a longest common subsequence (lcs). Then,

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- There are no i, j, with $i > i_k$ and $j > j_k$, s.t. $x_i = y_j$. Otherwise, Z will not be optimal.

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- There are no i, j, with $i > i_k$ and $j > j_k$, s.t. $x_i = y_j$. Otherwise, Z will not be optimal.
- $a = x_{i_k}$ might appear after i_k in X, but not after j_k in Y, or viceversa.
- There is an optimal solution in which i_k and j_k are the last occurrence of a in X and Y respectively.

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Let $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ and let $Z = x_{i_1} \ldots x_{i_k} = y_{j_1} \ldots y_{j_k}$ a lcs s.t. the index of the final common symbol in Z is its last occurrence in both X and Y.

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Let $X^- = x_1 \cdots x_{n-1}$ and $Y^- = y_1 \cdots y_{m-1}$

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Let $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ and let $Z = x_{i_1} \ldots x_{i_k} = y_{j_1} \ldots y_{j_k}$ a lcs s.t. the index of the final common symbol in Z is its last occurrence in both X and Y.

Let $X^- = x_1 \cdots x_{n-1}$ and $Y^- = y_1 \cdots y_{m-1}$

- Let us look at x_n and y_m.
- If $x_n = y_m$, $i_k = n$ and $j_k = m$ so, $x_{i_1} \ldots x_{i_{k-1}}$ is a lcs of X^- and Y^-.

DP approach: Characterization of optimal solution

Let $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ and let $Z = x_{i_1} \ldots x_{i_k} = y_{j_1} \ldots y_{j_k}$ a lcs s.t. the index of the final common symbol in Z is its last occurrence in X and Y.

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Let $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ and let $Z = x_{i_1} \ldots x_{i_k} = y_{j_1} \ldots y_{j_k}$ a lcs s.t. the index of the final common symbol in Z is its last occurrence in X and Y.

Let $X^- = x_1 \cdots x_{n-1}$ and $Y^- = y_1 \cdots y_{m-1}$

- Let us look at x_n and y_m.
- If $x_n \neq y_m$,
  - If $i_k < n$ and $j_k < m$, Z is a lcs of X^- and Y^-.
  - If $i_k = n$ and $j_k < m$, Z is a lcs of X and Y^-.
  - If $i_k <$ and $j_k = m$, Z is a lcs of X^- and Y.
  - The last two include the first one!

DP approach: Supproblems

Subproblems = lcs of pairs of prefixes of the initial strings.

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Notation:

- $X[i] = x_1 \dots x_i$, for $0 \leq i \leq n$
- $Y[j] = y_1 \dots y_j$, for $0 \leq j \leq m$
- $c[i,j]$ = length of the LCS of $X[i]$ and $Y[j]$.
- Want $c[n,m]$ i.e. length of the LCS for X and Y.

DP approach: Recursion

Therefore, given X and Y

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if } x_i = y_j \\ \max(c[i,j-1], c[i-1,j]) & \text{otherwise} \end{cases}$$

The recursive algorithm

LCS(X, Y)
$n = X.size(); \ m = Y.size()$
if $n = 0$ or $m = 0$ **then**
  return 0
else if $x_n = y_m$ **then**
  return $1+$**LCS**(X^-, Y^-)
else
  return $\max\{$**LCS**$(X, Y^-),$ **LCS**$(X^-, Y)\}$

The recursive algorithm

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  return $1+$**LCS**(X^-, Y^-)
else
  return $\max\{$**LCS**$(X, Y^-),$ **LCS**$(X^-, Y)\}$

The algorithm makes 1 or 2 recursive calls and explores a tree of depth $O(n+m)$, therefore the time complexity is $2^{O(n+m)}$.

DP: tabulating

We need to find the correct traversal of the table holding the $c[i,j]$ values.

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- Base case is $c[0,j]=0$, for $0 \leq j \leq m$, and $c[i,0]=0$, for $0 \leq i \leq n$.
- To compute $c[i,j]$, we have to access

$c[i-1,j-1]$	$c[i-1,j]$
$c[i,j-1]$	$c[i,j]$

  A row traversal provides a correct ordering.
- To being able to recover a solution we use a table b, to indicate which one of the three options provided the value $c[i,j]$.

Tabulating

LCS(X, Y)
for $i = 0$ **to** n **do**
  $c[i, 0] = 0$
for $j = 1$ **to** m **do**
  $c[0, j] = 0$
for $i = 1$ **to** n **do**
  for $j = 1$ **to** m **do**
    if $x_i = y_j$ **then**
      $c[i, j] = c[i-1, j-1] + 1,\ b[i.j] = \nwarrow$
    else if $c[i-1, j] \geq c[i, j-1]$ **then**
      $c[i, j] = c[i-1, j],\ b[i, j] = \leftarrow$
    else
      $c[i, j] = c[i, j-1],\ b[i, j] = \uparrow.$

complexity:
$T = O(nm)$.

Example.

X=(ATCTGAT); Y=(TGCATA). Therefore, $m = 6, n = 7$

		0	1 T	2 G	3 C	4 A	5 T	6 A
0		0	0	0	0	0	0	0
1	A	0	$\uparrow 0$	$\uparrow 0$	$\uparrow 0$	$\nwarrow 1$	$\leftarrow 1$	$\nwarrow 1$
2	T	0	$\nwarrow 1$	$\leftarrow 1$	$\leftarrow 1$	$\uparrow 1$	$\nwarrow 2$	$\leftarrow 2$
3	C	0	$\uparrow 1$	$\uparrow 1$	$\nwarrow 2$	$\leftarrow 2$	$\uparrow 2$	$\uparrow 2$
4	T	0	$\nwarrow 1$	$\uparrow 1$	$\uparrow 2$	$\uparrow 2$	$\nwarrow 3$	$\leftarrow 3$
5	G	0	$\uparrow 1$	$\nwarrow 2$	$\uparrow 2$	$\uparrow 2$	$\uparrow 3$	$\uparrow 3$
6	A	0	$\uparrow 1$	$\uparrow 2$	$\uparrow 2$	$\nwarrow 3$	$\uparrow 3$	$\nwarrow 4$
7	T	0	$\nwarrow 1$	$\uparrow 2$	$\uparrow 2$	$\uparrow 3$	$\nwarrow 4$	$\uparrow 4$

Following the arrows: TCTA

Construct the solution

Access the tables c and d.

The first call to the algorithm is **sol-LCS**(n, m)

sol-LCS(i, j)
if $i = 0$ or $j = 0$ **then**
  STOP.
else if $b[i, j] = \nwarrow$ **then**
  sol-LCS$(i - 1, j - 1)$
  return x_i
else if $b[i, j] = \uparrow$ **then**
  sol-LCS$(i - 1, j)$
else
  sol-LCS$(i, j - 1)$

The algorithm has time complexity $O(n + m)$.

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- LCSt: Given two strings $X = x_1 \dots x_n$ and $Y = y_1 \dots y_m$, compute their longest common substring Z, i.e., the largest k for which there are indices i and j with $x_i x_{i+1} \dots x_{i+k} = y_j y_{j+1} \dots y_{j+k}$.

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- For example:
  X : DEADBEEF
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- For example:
  X : DEADBBEEF
  Y : EATBEEF
  Z : BEEF pick the longest substring

Characterization of optimal solution

- Let $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ and let Z be a longest common substring.
  - $Z = x_i \ldots x_{i+k} = y_j \ldots y_{j+k}$

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- We can consider the subproblems $LCStf(i,j)$: compute the longest common suffix of $X(i)$ and $Y(j)$.
- The $LCSf(X, Y)$ is the longest of such common suffixes.

Computing the LC Suffixes

- To solve $LCSf(i,j)$ it is enough to go backward from position i in X and j in Y until we find two different characters.
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- Can we do it faster?

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- To solve $LCSf(i,j)$ it is enough to go backward from position i in X and j in Y until we find two different characters.
- This has cost $O(n+m)$ per subproblem.
- We get a $O(nm(n+m))$ algorithm for LCSt
- Can we do it faster? Let us use DP!

A recursive solution for LC Suffixes

Notation:

- $X[i] = x_1 \dots x_i$, for $0 \leq i \leq n$
- $Y[j] = y_1 \dots y_j$, for $0 \leq j \leq m$
- $s[i,j]$ = the length of the LC Suffix of $X[i]$ and $Y[j]$.
- Want $\max_{i,j} s[i,j]$ i.e., the length of the LCSt of X, Y.

DP approach: Recursion

Therefore, given X and Y

$$s[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ 0 & \text{if } x_i \neq y_j \\ s[i-1,j-1]+1 & \text{if } x_i = y_j \end{cases}$$

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Using the recurrence the cost per recursive call (or per element in the table) is constant

Tabulating

LCSf(X, Y)
for $i = 0$ **to** n **do**
  $s[i, 0] = 0$
for $j = 1$ **to** m **do**
  $s[0, j] = 0$
for $i = 1$ **to** n **do**
  for $j = 1$ **to** m **do**
    s[i,j]=0
    if $x_i = y_j$ **then**
      $s[i, j] = s[i - 1, j - 1] + 1$

complexity: $O(nm)$.

Which gives an algorithm with cost $O(nm)$ for LCSt

Multiplying a Sequence of Matrices

(This example is from Section 15.2 in CormenLRS' book.)

MULTIPLICATION OF n MATRICES Given as input a sequence of n matrices $(A_1 \times A_2 \times \cdots \times A_n)$. Minimize the number of operation in the computation $A_1 \times A_2 \times \cdots \times A_n$

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Recall that Given matrices A_1, A_2 with $\dim(A_1) = p_0 \times p_1$ and $\dim(A_2) = p_1 \times p_2$, the basic algorithm to $A_1 \times A_2$ takes time at most $p_0 p_1 p_2$.

Example:

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 13 & 18 & 23 \\ 18 & 25 & 32 \\ 23 & 32 & 41 \end{bmatrix}$$

Multiplying a Sequence of Matrices

- Matrix multiplication is NOT commutative, so we can not permute the order of the matrices without changing the result.
- It is associative, so we can put parenthesis as we wish.
- How to multiply is equivalent to the problem of how to parenthesize.
- We want to find the way to put parenthesis so that the product requires the minimum total number of operations. And use it to compute the product.

Example Consider $A_1 \times A_2 \times A_3$, where dim $(A_1) = 10 \times 100$ dim $(A_2) = 100 \times 5$ and dim $(A_3) = 5 \times 50$.

- $((A_1 A_2) A_3)$ takes $(10 \times 100 \times 5) + (10 \times 5 \times 50) =$ *7500* operations, Example Consider $A_1 \times A_2 \times A_3$, where dim $(A_1) = 10 \times 100$ dim $(A_2) = 100 \times 5$ and dim $(A_3) = 5 \times 50$.

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- $(A_1(A_2A_3))$ takes $(100 \times 5 \times 50) + (10 \times 100 \times 50) =$ 75000 operations.

The order in which we make the computation of products of two matrices makes a big difference in the total computation's time.

How to parenthesize $(A_1 \times \ldots \times A_n)$?

- If $n = 1$ we do not need parenthesis.

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- Then, combine any way to parenthesize $(A_1 \times \cdots \times A_k)$ with any way to parenthesize $(A_{k+1} \times \cdots \times A_n)$.

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Using this structure, we can count the number of ways to parenthesize $(A_1 \times \cdots \times A_n)$ as well as to define a backtracking algorithm that goes over all those ways to parenthesize and eventually to a brute force recursive algorithm to solve the problem of computing efficiently the product.

How many ways to parenthesize $(A_1 \times \cdots \times A_n)$?

Let $P(n)$ be the number of ways to paranthesize $(A_1 \times \cdots \times A_n)$. Then,

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{si } n \geq 2 \end{cases}$$

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The Catalan numbers.

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The Catalan numbers.

Brute force will take too long!

Structure of an optimal solution

- We want to compute $(A_1 \times \cdots \times A_n)$ efficiently.
- In an optimal solution the last matrix product must correspond to a break at some position k, $((A_1 \times \cdots \times A_k)(A_{k+1} \times \cdots \times A_n))$ Let $A_{i-j} = (A_i A_{i+1} \cdots A_j)$.

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- The parenthesization of the subchains $(A_1 \times \cdots \times A_k)$ and $(A_{k+1} \times \cdots \times A_n)$ within the optimal parenthesization must be an optimal paranthesization of $(A_1 \times \cdots \times A_k)$, $(A_{k+1} \times \cdots \times A_n)$. So,

$$\begin{aligned} \text{cost}(A_1 \ldots A_n) = & \text{cost}(A_1 \ldots A_k) \\ & + \text{cost}(A_{k+1} \ldots A_n) + p_0 p_k p_n. \end{aligned}$$

Structure of an optimal solution

- An optimal solution decomposes in optimal solutions of the same problem on subchains.
- Subproblems: compute the product $A_i \times A_{i+1} \times \cdots \times A_j$, for $1 \leq i \leq j \leq n$

Structure of an optimal solution

- An optimal solution decomposes in optimal solutions of the same problem on subchains.
- Subproblems: compute the product $A_i \times A_{i+1} \times \cdots \times A_j$, for $1 \leq i \leq j \leq n$
- Let us call $B_i^j = A_i \times A_{i+1} \times \cdots \times A_j$.

Cost Recurrence

- Let $m[i,j]$ be the minimum cost of computing $B_i^j = (A_i \times \ldots \times A_j)$, for $1 \leq i \leq j \leq n$.
- $m[i,j]$ is defined by the value k, $i \leq k \leq j$ that minimizes

$$m[i,k] + m[k+1,j] + \text{ cost } (B_i^k, B_{k+1}^j).$$

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- That is,

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j}\{m[i,k] + m[k+1,j] + p_{i-1}p_kp_j\} & \text{otherwise} \end{cases}$$

Computing the cost of an optimal solution: Rec

Assume that vector P holds the values $(p_0, p_1, \ldots, p_n)$.

MCR(i,j)
if $i = j$ **then**
  return 0
$m[i,j] = \infty$
for $k = i$ **to** $j-1$ **do**
  $q = \mathrm{MCR}(i,k) + \mathrm{MCR}(k+1,j) + P[i-1] * P[k] * P[j]$
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return $(m[i, j])$

Cost: $T(n) \geq 2 \sum_{i=1}^{n-1} T(i) + n \sim \Omega(2^n)$.

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- How many subproblems?
  As $1 \leq i < j \leq n$, we have only $O(n^2)$ subproblems.
- We can use DP!

Dynamic programming: Memoization

MCP(P)
for all $1 \leq i < j \leq n$ **do**
  $m[i,j] = -1$
for $i = 1$ **to** n **do**
  $m[i,i] = 0$
MCR$(1,n)$
return $(m[1,n])$

MCR(i,j)
if $m[i,j] != -1$ **then**
  return $(m[i,j])$
$m[i,j] = \infty$
for $k = i$ **to** $j-1$ **do**
  $q = \text{MCR}(i,k) + \text{MCR}(k+1,j) + P[i-1] * P[k] * P[j]$
  if $q < m[i,j]$ **then**
    $m[i,j] = q$
return $(m[i,j])$

$T(n) = \Theta(n^3)$ additional space $\Theta(n^2)$.

Dynamic programming: Tabulating

To compute the element $m[i,j]$ the base case is when $i = j$, we need to access $m[i,k]$ and $m[k+1,j]$. We can achieve that by filling the (half) table by diagonals.

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MCP(P)
for $i = 1$ **to** n **do**
  $m[i,i] = 0$
for $d = 2$ **to** n **do**
  for $i = 1$ **to** $n - d + 1$ **do**
    $j = i + d - 1$
    $m[i,j] = \infty$
    for $k = i$ **to** $j - 1$ **do**
      $q =$
      $m[i,k] + m[k+1,j] + P[i-1] * P[k] * P[j]$
      if $q < m[i,j]$ **then**
        $m[i,j] = q$
return $(m[1,n])$

$T(n) = \Theta(n^3)$,
space $= \Theta(n^2)$.

Example.

We wish to compute $A_1 \times A_2 \times A_3 \times A_4$ with $P =< 3, 5, 3, 2, 4 >$

$i \setminus j$	1	2	3	4
1				
2				
3				
4				

Example.

We wish to compute $A_1 \times A_2 \times A_3 \times A_4$ with $P =< 3, 5, 3, 2, 4 >$

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We wish to compute $A_1 \times A_2 \times A_3 \times A_4$ with $P =< 3, 5, 3, 2, 4 >$

$i \setminus j$	1	2	3	4
1	0	45		
2		0	30	
3			0	24
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Recording more information about the optimal solution

We have been working with the recurrence

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j}\{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{otherwise} \end{cases}$$

To keep information about the optimal solution the algorithm keep additional information about the value of k that provides the optimal cost as

$$s[i,j] = \begin{cases} i & \text{if } i = j \\ \arg\ \min_{i \le k < j}\{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{otherwise} \end{cases}$$

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return m, s

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if $m[i,j]! = -1$ **then**
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$m[i,j] = \infty$
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  $q = \mathrm{MCR}(i,k) + \mathrm{MCR}(k+1,j) + P[i-1] * P[k] * P[j]$
  if $q < m[i,j]$ **then**
    $m[i,j] = q$; $s[i,j] = k$;
return $(m[i,j])$

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Computing optimally the product

- $s[i,j]$ contains the value of k that decomposes optimally the product as product of two submatrices, i.e.,

$$A_i \times \cdots \times A_j = (A_i \times \cdots \times A_{s[i,j]})(A_{s[i,j]+1} \times \cdots \times A_j).$$

- Therefore,

$$A_1 \times \cdots \times A_n = (A_1 \times \cdots \times A_{s[1,n]})(A_{s[1,n]+1} \times \cdots \times A_n).$$

- We can design a recursive algorithm to perform the product in an optimal way.

The product algorithm

The input is the sequence of matrices $A = A_1, \ldots, A_n$ and the table s computed before.

Product(A, s, i, j)
if $i = j$ **then**
  return (A_i)
$X =$**Product**$(A, s, i, s[i,j])$
$Y =$**Product**$(A, s, s[i,j]+1, j)$
return $(X \times Y)$

The total number operations required to compute the product is $m[1, n]$ and the cost of the complete algorithm is $T(n) = O(n^3 + m[1,n])$

Example.

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The optimal way to minimize the number of operations is

$$(((A_1) \times (A_2 \times A_3)) \times (A_4))$$

Multiplying matrices

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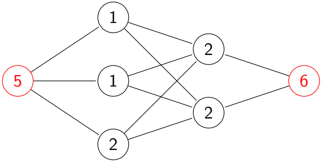
- In order to compute s, we only need the dimensions of the matrices.
- What if we use Strassen algorithm to compute a two matrices product instead of the naive algorithm?

Dynamic Programming in Trees

- Trees are nice graphs easily adapted to recursion.
- Once you root the tree each node can be seen as the root of a subtree .
- We can use Dynamic Programming to give polynomial solutions to "difficult" graph problems when the input is restricted to be a tree, or to have a treee-like structure (small treewidth).
- In this case instead of having a global table, each node in the tree keeps additional information about the associated subproblem.

The Maximum Weight Independent Set (MWIS)

Given as input $G = (V, E)$, together with a weight $w : V \to \mathbb{R}$. Find the heaviest $S \subseteq V$ such that no two vertices in S are connected in G.



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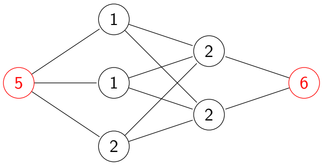
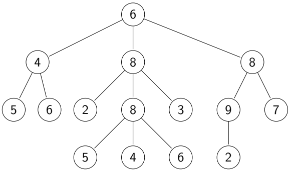


For general graphs, the problem is hard, even for the case in which all vertex have weight 1, i.e. MAXIMUM INDEPENDENT SET is NP-complete.

Maximum Weight Independent Set on Trees

Given a tree $T = (V, E)$ choose a $r \in V$ and root it from r

i.e. Given a rooted tree $T = (V, E, r)$ and weights $w : V \to \mathbb{R}$, find the independent set with maximum weight.



Notation:

- For $v \in V$, let T_v be the subtree rooted at v. $T = T_r$.
- Given $v \in V$ let $C(v)$ be the set of children of v, and $G(v)$ be the set of grandchildren of v.

Characterization of the optimal solution

Key observation: An IS can't contain vertices which are father-son.

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Let S be an optimal solution.

- If $r \in S$: then $C(r) \not\subseteq S_r$. So $S - \{r\}$ contains an optimum solution for each T_v, with $v \in G(r)$.
- If $r \notin S$: S contains an optimum solution for each T_u, with $u \in C(r)$.

Recursive definition of the optimal solution

- To implement DP, tor every node v, we add one value, $v.M$: the value of the optimal solution for T_v
  Following the recursive structure of the solution we have the following recurrence

$$v.M = \begin{cases} w(v) & v \text{ a leaf,} \\ \max\{\sum_{u \in C(v)} u.M, w(v) + \sum_{u \in G(v)} u.M\} & \text{otherwise.} \end{cases}$$

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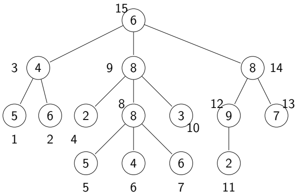
- Notice that for any $v \in T$: we have to compute $\sum_{u \in C(v)} u.M$ and for this we must access to the children of its children
- To avoid this we add another value to the node $v.M'$: the sum of the values of the optimal solutions of their children, i.e., $\sum_{u \in C(v)} u.M$.

Post-order traversal of a rooted tree

To perform the computation, we can follow a DFS, post-order, traversal of the nodes in the tree, computing the additional values at each node.



DP Algorithm to compute the optimal weight

Let $v_1, \ldots, v_n = r$ be the post-order traversal of T_r

WIS T_r

Let $v_1, \ldots, v_n = r$ the post-order traversal of T_r

for $i = 1$ **to** n **do**

  if v_i is a leaf **then**

    $v_i.M = w[v_i], v_i.M' = 0$

  else

    $v_i.M' = \sum_{u \in C(v)} u.M$

    $aux = \sum_{u \in C(v)} u.M'$

    $v_i.M = \max\{aux + w[v_i], v_i.M'\}$

return r.M

Complexity: space $= O(n)$, time $= O(n)$

Top-down traversal to obtain an optimal IS

RWIS(v)
if v is a leaf **then**
  return $(\{v\})$
if $v_i.M = v_i.M' + w[v_i]$ **then**
  $S = S \cup \{v_i\}$
  for $w \in G(v)$ **do**
    $S = S \cup$ **RWIS**(w)
else
  for $w \in N(v)$ **do**
    $S = S \cup$ **RWIS**(w)
return S

RWIS(r)
provides an optimal solution in time $O(n)$

Total cost $O(n)$ and additional space $O(n)$