The Elementary Linear Algebra Companion

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Contents

Introduction

The intention of this material is to complement a linear algebra text: no effort has (yet) been made to provide exact statements of definitions and theorems. Rather, the text seeks to clarify the structure of the mathematical theory and the ensuing computations.

Textbooks

The material presented here was written to complement a course based on Gilbert Strang's Linear Algebra text.[1]

Acknowledgements

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Work in Progress

This text is far from complete! Besides missing chapters and sections, the reader will find annotations used to remind the author of missing information, and/or the need to rewrite parts of the text. The format of such annotations is as follows:

FIX Reminder: add or rewrite text here! FIX

Chapter 1

Vector and Matrix Operations

In this chapter, we introduce the basic definitions and operations for scalars, vectors and matrices. Unless otherwise noted, scalars will be denoted by Greek letters, vectors by bold face lower case Latin letters, and matrices by upper case Latin letters respectively.

The special case of row and column vectors, i.e., matrices of size $1 \times N$ and $N \times 1$ respectively, uses lower case Latin letters. A second exception are the variables x, y, z and w traditionally representing the unknowns in systems of linear equations.

1.1   Scalars and Vectors

Scalars are a set of numbers $\mathbb{F}$ with addition, subtraction, multiplication and division operations. Examples of scalars are

1. real numbers, i.e., members of the set $\mathbb{R}$ such as $-3, \sqrt{17}, \pi$, etc.

2. complex numbers, i.e., members of the set $\mathbb{C}$ such as $5, \sqrt{3} - i, 2 + 5i$, etc.

3. rational numbers, i.e., members of the set $\mathbb{Q}$ such as $-5, \frac{3}{4}$, etc.

4. integers modulo two, i.e., members of the set $\mathbb{Z}_2$.

Most of the computations in this text will be carried out with rationals, i.e., we will restrict scalars to $\mathbb{Q}$ rather than the real numbers $\mathbb{R}$.

Vectors in a space $\mathbb{F}^N$ consist of N ordered scalars (called **entries** or **components**) numbered from 1 to N: thus $\mathbf{u} = (u_1, u_2, \ldots u_N)$.

Example 1.1.1. *A vector with 4 entries*
The vector
$$\mathbf{u} = (12, sin(\frac{\pi}{3}), -3, 4.3) \tag{1.1}$$

is a vector in $\mathbb{R}^4$ whose third entry is $u_3 = -3$.

☞ Vectors are usually represented by a row or by a column of entries bracketed with parentheses or angular brackets. The orientation of the entries has no significance for vectors.

1.1.1   Equality, Vector Addition, Scalar Multiplication

Vectors are equal if and only if they have the same number of entries, and all of their entries are equal. This definition allows us to switch from a vector equation to a corresponding set of scalar equations.

Vector addition is defined for vectors with the same number of entries by adding their respective entries: Given the vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_N \end{pmatrix}, \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \cdots \\ u_N + v_N \end{pmatrix}. \tag{1.2}$$

Scalar multiplication of a scalar α with a vector u is defined by

$$\alpha \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \cdots \\ \alpha u_N \end{pmatrix}. \tag{1.3}$$

Given a set of n vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, and a set of scalars $\alpha_1, \alpha_2, \cdots \alpha_n$, we define the **linear combination** $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots \alpha_n \mathbf{u}_n$. Linear combinations play a central role in linear algebra.

The **difference of two vectors** is defined using addition and scalar multiplication: given two vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}. \tag{1.4}$$

Example 1.1.2. *Vector equality*
The vector equation

$$\begin{pmatrix} 4x + 3y \\ 2x + 7y \end{pmatrix} = \begin{pmatrix} 12 \\ 14 \end{pmatrix}, \tag{1.5}$$

represents the set of simultaneous linear equations

$$4x + 3y = 12, \quad 2x + 7y = 14. \tag{1.6}$$

Example 1.1.3. *Vector addition*
The left hand side in equation 1.5 above can be rewritten

$$\begin{pmatrix} 4x + 3y \\ 2x + 7y \end{pmatrix} = \begin{pmatrix} 4x \\ 2x \end{pmatrix} + \begin{pmatrix} 3y \\ 7y \end{pmatrix} \tag{1.7}$$
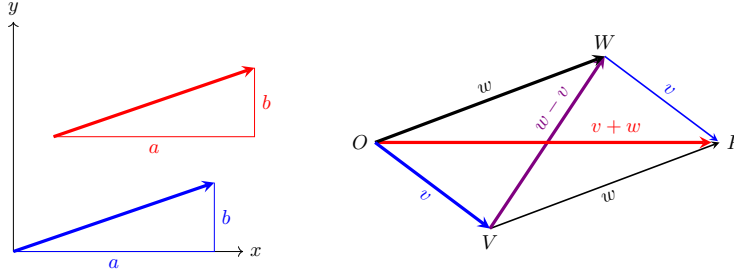
Figure 1.1: Sum and difference of two vectors. The left hand figure shows two graphical representations of the vector $(a\ b)$ in $\mathbb{R}^2$, with two different starting points: from the origin, and from a point in the first quadrant. Ponts in the plane can be defined unambiguously by introducing the convention that vectors start at the origin. The right hand figure illustrates the graphical representation of the sum of two vectors $\mathbf{v}$ and $\mathbf{w}$. Since the starting point of a vector is arbitrary, $\mathbf{v}$ and $\mathbf{w}$ can be used to define a parallelogram. Their sum (the vector from O to P), is then easily seen to be reached in two different ways: either by by running along two sides of the parallelogram (connecting the endpoint of the first vector to the starting point of the second, or by moving along the diagonal from the common starting point of the two vectors at O. The difference of two vectors is then easily obtained by considering the sum of the two vectors $\mathbf{w} + (\mathbf{v} - \mathbf{w})$ drawn by connecting the endpoint of $\mathbf{w}$ to the starting point of $\mathbf{v} - \mathbf{w}$: it lies along the other diagonal.

Example 1.1.4. *Linear combination*
Continuing with example 1.1.2,1.1.3 above, we now use the definition of scalar multiplication in Eq 1.7 to rewrite our system of linear equations in the form

$$x \begin{pmatrix} 4 \\ 2 \end{pmatrix} + y \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 12 \\ 14 \end{pmatrix}. \tag{1.8}$$

1.1.2 Geometrical Interpretation

As illustrated in Fig (1.1), vectors can be represented graphically by arrows, where the entries are equal to the change in magnitude of the coordinates ("endpoint minus origin"). The sum $\mathbf{v} + \mathbf{w}$ and difference $\mathbf{v} - \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ can then be interpreted as the diagonals of a parallelogram defined by $\mathbf{v}$ and $\mathbf{w}$.

Multiplication of a vector by a scalar changes its length. Given some vector $\mathbf{v}$, we can define the set of vectors $V = \{\mathbf{w} \mid \mathbf{w} = c\mathbf{v}, \forall c \in \mathbb{R}\}$. If each vector in the set is drawn starting from a common point, e.g., the origin of some system of coordinates, the set of points described by the ending point of these vectors traces out a line along the vector $\mathbf{v}$. If the parameter c is restricted to some smaller set, we get a subset of the line, as indicated in Fig (1.2).
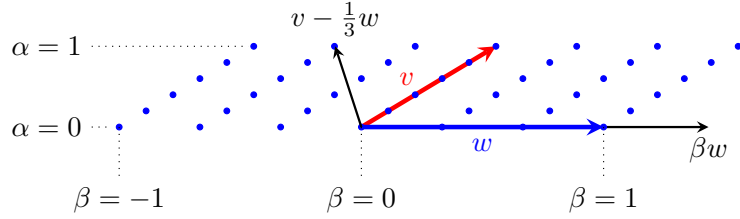
Figure 1.2: Linear combination of vectors. Given two vectors v and w, the figure shows the set of points $P = \{u \mid u = \alpha v + \beta w\}$, where α and β take on equispaced values in $[0, 1]$, and $[-1, 1]$ respectively. As the number of values assumed by α and β increases, the points in P become denser, eventually filling a parallelogram. If we allow α and β to take on any values in $\mathbb{R}$, the set P fills out the plane containing both v and w. Choosing $\alpha \in \mathbb{R}, \beta = 0$, we get the line through the origin with direction vector v.

☞ In general, all possible linear combinations of a given set of vectors describe **hyperplanes** passing through the origin.

For example, given the vectors $\mathbf{u} = (1, -3, 0, 0)$ and $\mathbf{v} = (0, 0, 2, 0)$, the set $P = \{\mathbf{w} \mid \mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}, \forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}\}$ forms a plane.

The two forms Eqs (1.6,1.8) of the same system of equations represent two different ways of looking at the same problem: in the first form, which we shall call the **row view**[1], the solution (x, y) is the intersection of the two lines $3x + 8y = 24$ and $7x + 4y = 21$ represented by the two equations. In the second form, which we shall call the **column view**, the scalars x and y are the scale factors that must be applied to the vectors $v_1 = (3, 7)$ and $v_2 = (8, 4)$ to produce the vector $(24, 21)$. The two representations are illustrated further in Figure 1.3.

The following properties of vector addition and scalar multiplication in $\mathbb{R}^N$ allow us to carry out algebraic manipulations:[2] Let $\mathbf{0}$ be the vector with all N entries equal to zero, let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be *any vectors* in $\mathbb{R}^N$, and let α and β be *any scalars*.

- **Vector Addition**

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \qquad \text{Commutativity} \quad (1.9\text{a})$$
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \qquad \text{Associativity} \quad (1.9\text{b})$$
$$\mathbf{u} + \mathbf{0} = \mathbf{u} \qquad \text{Zero Vector} \quad (1.9\text{c})$$
$$\mathbf{u} + (-1)\mathbf{u} = \mathbf{0} \qquad \text{Inverse} \quad (1.9\text{d})$$

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[1]The row and column view terminology derives from the matrix representation of the system of equations and will become obvious in later sections.

[2]The exact definition of the notion of vector is left to chapter 3. Eq(1.10e) in particular is not part of the definition.
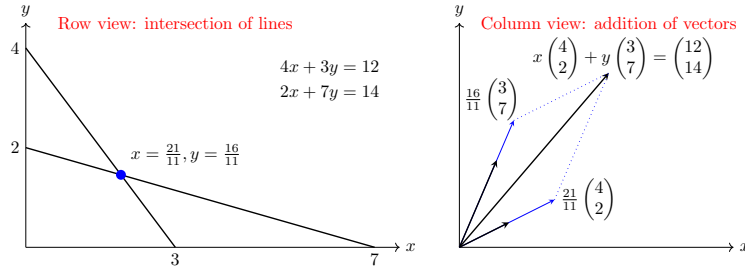
Figure 1.3: The row view Eq (1.6) (intersection of lines) and the column
view Eq (1.8) (vector sum) of a system of simultaneous linear equations. In
the row view, the solution is the intersection of the lines. In the column view,
the solution is a set of scale factors applied to the vectors.

- **Scalar Multiplication**

$$\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u} \qquad\qquad\qquad \text{Associativity} \quad (1.10a)$$
$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} \qquad\qquad\qquad \text{Distributivity} \quad (1.10b)$$
$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u} \qquad\qquad\qquad \text{Distributivity} \quad (1.10c)$$
$$1\mathbf{u} = \mathbf{u} \qquad\qquad\qquad \text{Unit} \quad (1.10d)$$
$$0\mathbf{u} = \mathbf{0} \qquad\qquad\qquad \text{Zero} \quad (1.10e)$$

While not defined in the above, we will allow $\mathbf{u}\alpha = \alpha\mathbf{u}$.

☞ There are two different addition operations that appear in this table:
i) addition of scalars $\alpha + \beta$, e.g., $3 + 2$, and
ii) addition of vectors $\mathbf{u} + \mathbf{v}$, e.g., $(1\ 2) + (3\ 4)$. Both operations use the
same symbol "+".

☞ There are two different multiplications that appear in this table:
i) multiplication of scalars $\alpha\beta$, e.g., 3×2, and
ii) multiplication of a scalar and a vector $\beta\mathbf{u}$, e.g., $3(1, 2)$. Note that there
is no multiplication symbol used.

Example 1.1.5. *Algebraic manipulation of vector equations*
Simplify $3\mathbf{u} + \mathbf{c} = 8\mathbf{u}$, where $\mathbf{c} = (1, 5, 2)$. We note that $\mathbf{u} \in \mathbb{R}^3$ for the addition to be defined. Remember also that the zero vector $\mathbf{0} = (0, 0, 0)$ in this context.

$$
\begin{array}{lll}
(\xi) \Leftrightarrow & 3\mathbf{u} + \mathbf{c} = 8\mathbf{u} & \textit{the equation to be solved for } \mathbf{u} \\
\Leftrightarrow & (3\mathbf{u} + \mathbf{c}) - 8\mathbf{u} = 8\mathbf{u} - 8\mathbf{u} & \textit{subtracting } -8\mathbf{u} \textit{ from both sides} \\
\Leftrightarrow & (3\mathbf{u} - 8\mathbf{u}) + \mathbf{c} = 8\mathbf{u} - 8\mathbf{u} & \textit{repeated use of Eqs(1.9a,1.9b)} \\
\Leftrightarrow & -5\mathbf{u} + \mathbf{c} = \mathbf{0} & \textit{by Eqs(1.10c,1.10e)} \\
\Leftrightarrow & (-5\mathbf{u} + \mathbf{c}) - \mathbf{c} = 0 - \mathbf{c} & \textit{subtracting } \mathbf{c} \textit{ from both sides} \\
\Leftrightarrow & -5\mathbf{u} = -\mathbf{c} & \textit{repeated use of Eqs(1.9a,1.9b)} \\
\Leftrightarrow & \mathbf{u} = \dfrac{1}{5}\mathbf{c} & \textit{multiplying by } -\dfrac{1}{5} \textit{ and using Eq(1.10a)}
\end{array}
$$

The properties of vector addition and scalar multiplication given above are easily established: the method is to rewrite the equations to explicitly show their entries, and to use the properties of addition and multiplication in algebra.

Example 1.1.6. *Proof of* $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$
Let $\mathbf{v} = \mathbf{u} + (-1)\mathbf{u}$. We wish to establish that $\mathbf{v} = \mathbf{0}$. To exhibit the entries, we need to name the components of $\mathbf{u}$ and $\mathbf{v}$: let $\mathbf{u} = (u_1, u_2, \cdots u_N)$ and let $\mathbf{v} = (v_1, v_2, \cdots v_N)$.
We have

$$
\begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_N \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix} + (-1)\begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix} \qquad \textit{explicitly show the entries}
$$

$$
= \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix} + \begin{pmatrix} -u_1 \\ -u_2 \\ \cdots \\ -u_N \end{pmatrix} \qquad \textit{by definition 1.3 of scalar multiplication}
$$

$$
= \begin{pmatrix} u_1 - u_1 \\ u_2 - u_2 \\ \cdots \\ u_N - u_N \end{pmatrix} \qquad \textit{by definition 1.2 of vector addition}
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \qquad \textit{by scalar algebra}
$$

Note that we carefully justified each step by referring to properties that have been established previously.

1.1.3 Dot Products, Lengths and Distances

The **dot product** of two vectors $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ of the same size n is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n. \tag{1.11}$$

It can be thought of as a weighted sum.

Example 1.1.7. *Dot product*
Consider exam grades $\mathbf{g} = (98, 86, 92, 81)$ *weighted by the factors*
$\mathbf{w} = (.30, .25, .25, .20)$. *The resulting combined grade is*

$$\mathbf{w} \cdot \mathbf{g} = .30 \cdot 98 + .25 \cdot 86 + .25 \cdot 92 + .20 \cdot 81 = 90.1, \tag{1.12}$$

the dot product of the weight and grade vectors.

The special case $u = v$ is of considerable interest. For vectors with entries in $\mathbb{R}$, we have

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots u_n^2 \geq 0. \tag{1.13}$$

- **Properties of the dot product** given any three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^N$, and any scalar α,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \tag{1.14a}$$

$$\mathbf{u} \cdot (\alpha \mathbf{v}) = \alpha \mathbf{u} \cdot \mathbf{v} \tag{1.14b}$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \tag{1.14c}$$

$$\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0} \quad \text{equality holds if and only if } \mathbf{u} = \mathbf{0} \tag{1.14d}$$

Note that Eq(1.14d) does not hold for for scalars in $\mathbb{Z}_2$ for example.

When Eq(1.14d) holds, we can define the **length of a vector u** in $\mathbb{R}^N$ by

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}. \tag{1.15}$$

The length of the weight vector $\mathbf{w}$ applied to the grades in Example 1.1.7 is seen to be 1. (The cumulative grade therefore cannot exceed 100!).

The **angle θ between non-zero vectors u** and **v** is given by

$$cos\,\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||} \tag{1.16}$$

which is known as the "cosine formula".

Two *very important uses* of the dot product are

☞ Given a non-zero vector $\mathbf{u}$, the vector $\hat{\mathbf{u}} = \frac{1}{||\mathbf{u}||}\mathbf{u}$ is a **vector of unit length** in the same direction as $\mathbf{u}$.

☞ Two vectors **u** and **v** are **orthogonal** or **perpendicular** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. (To simplify the statements of many theorems, the zero vector is defined to be orthogonal to any other vector.)

Note that a trivial choice for a vector **v** orthogonal to a given 2D vector $\mathbf{u} = (\alpha, \beta)$ is $\mathbf{v} = (\beta, -\alpha)$.

For 3D vectors $\mathbf{u} = (\alpha, \beta, \gamma)$, we have several obvious choices, e.g., $\mathbf{v_1} = (\beta, -\alpha, 0)$ and $\mathbf{v_2} = (\gamma, 0, -\alpha)$. In fact, any linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ is orthogonal to **u**.

Example 1.1.8. *No inverse for dot products*
The algebraic rules for the dot product present one significant difference from scalar multiplication, namely

☞ *There is no inverse operation for the dot product: given the grades in Example 1.1.7 we can find any number of weight vectors that yield the same final grade.*

*We therefore **cannot cancel** **w** in an equation such as* $\mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{v}$ *(from which we might then conclude that* $\mathbf{u} = \mathbf{v}$*)?! Rearranging the equation by subtracting* $\mathbf{w} \cdot \mathbf{v}$ *from both sides yields* $\mathbf{w} \cdot (\mathbf{u} - \mathbf{v}) = 0$*. Thus, the best we can say is that* $\mathbf{u} - \mathbf{v}$ *must be perpendicular to* **w**.

Example 1.1.9. *Angle between vectors*
Given the two vectors $\mathbf{v} = (1, 1, 1, 1)$ *and* $\mathbf{w} = (2, 0, 2, 0)$*, the cosine of the angle between them is given by Eq(1.16), yielding* $\cos\theta = \frac{\sqrt{2}}{2}$*. Both solutions* $\theta = \frac{\pi}{4}$ *and* $\theta = \frac{7\pi}{4}$ *are correct, since the angle between the vectors is not uniquely defined. As seen in Fig(1.4) for example, the angle can be measured along either the short or the long arc of a circle in the plane defined by* **v** *and* **w**.

Example 1.1.10. *Perpendicular unit vector*
To find a unit vector perpendicular to $\mathbf{u} = (4, 3)$*, we need to find a vector* $\mathbf{v} = (\alpha, \beta)$ *such that* $\mathbf{u} \cdot \mathbf{v} = 0$ *and* $||\mathbf{v}|| = 1$*. Substituting from the definitions, we get two equations* $4\alpha + 3\beta = 0$*, and* $\alpha^2 + \beta^2 = 1$*. There are two solutions to this system,* $\mathbf{v} = \frac{1}{5}(3, -4)$ *and* $\mathbf{v} = -\frac{1}{5}(3, -4)$ *pointing in opposite directions.*

Example 1.1.11. *Equation of a plane through the origin*
In three dimensions, the equation of a plane through the origin of a system of coordinates has the form $ax + by + cz = 0$ *which can be interpreted as the dot product of the vector* $\mathbf{v} = (x\ y\ z)$ *and a vector* $\mathbf{n} = (a\ b\ c)$*, i.e.,* $\mathbf{n} \cdot \mathbf{v} = 0$*. Given the results above, this equation can be interpreted geometrically as the set of all vectors* **v** *orthogonal to the given vector* **n**.

The length of a vector can be used to define the **distance** d **between 2 points** represented as the endpoints of two vectors **u** and **v** sharing the same origin by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||. \tag{1.17}$$

Example 1.1.12. *Equation of a circle*
Given a point $\mathbf{c} = (x_0, y_0, z_0)$ *the circle of radius* R *centered on* $\mathbf{c}$ *is the set of points* $\{\mathbf{w} = (x, y, z) \mid d(\mathbf{w}, \mathbf{c}) = R\}$. *Substituting the definition 1.15 and squaring, we get the familiar equation*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

For completeness, we mention two important inequalities associated with the dot product:

- **The triangle inequality**:

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}|| \tag{1.18}$$

  A geometrical interpretation of the triangle inequality states that the path connecting two points by a straight line segment is the shortest possible.

- **The Cauchy-Schwartz inequality**:

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \, ||\mathbf{v}||. \tag{1.19}$$

  A geometrical interpretation of the Cauchy-Schwartz inequality states that the length of the orthogonal projection of a vector onto some given direction cannot exceed the length of the original vector.

☞ Rather than writing square-roots it is usually more convenient to carry out computations with the square of the length as shown in the next three examples. Note that since $f(x) = x^2$ is monotone for $x > 0$, inequalities are preserved when taking square roots, i.e., $x_1^2 \leq x_2^2 \Leftrightarrow |x_1| \leq |x_2|$.

Example 1.1.13. *Proof of the Cauchy-Schwartz inequality*
Given two vectors $\mathbf{u}$ *and* $\mathbf{v}$, *the Cauchy-Schwartz inequality is trivially satisfied if* $\mathbf{v} = \mathbf{0}$. *We therefore look at the case where* $\mathbf{v} \neq \mathbf{0}$. *We proceed by investigating the length of the vector* $\mathbf{u} + t\mathbf{v}$ *for some scalar* $t \in \mathbb{R}$. *From Eq(1.15) and Eqs(1.9c-1.10e)*

$$0 \leq ||\mathbf{u} + t\mathbf{v}||^2$$
$$= (\mathbf{u} + t\mathbf{v}) \cdot (\mathbf{u} + t\mathbf{v})$$
$$= (\mathbf{u} \cdot \mathbf{u})^2 + 2t\mathbf{u} \cdot \mathbf{v} + t^2(\mathbf{v} \cdot \mathbf{v})^2 \quad \text{a quadratic polynomial in } t.$$

Since this equation holds for all t, *it must hold in particular for* $t = -\frac{\mathbf{u} \cdot \mathbf{v}}{(\mathbf{v} \cdot \mathbf{v})^2}$ *for which the polynomial is minimal. Substituting, we find*

$$0 \leq (\mathbf{u} \cdot \mathbf{u})^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{(\mathbf{v} \cdot \mathbf{v})^2}.$$

Rearranging this equation and taking square roots yields the desired inequality. Note that if $||\mathbf{u}|| \neq 0$, *we can divide by* $||\mathbf{u}||^2$ *and take square roots, rewriting the inequality in the form*

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||} \leq 1,$$

which may be used to justify the definition of the angle between vectors of any dimension.

Example 1.1.14. *Proof of the triangle inequality*
Given two vectors $\mathbf{u}$ *and* $\mathbf{v}$, *consider From Eq(1.15) and Eqs(1.9c-1.10e)*

$$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$$
$$= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}$$
$$\leq ||\mathbf{u}||^2 + 2|\mathbf{u} \cdot \mathbf{v}| + ||\mathbf{v}||^2$$
$$\leq ||\mathbf{u}||^2 + 2||\mathbf{u}|| \, ||\mathbf{v}|| + ||\mathbf{v}||^2$$
$$= (||\mathbf{u}|| + ||\mathbf{v}||)^2,$$

where we have used the Cauchy-Schwartz inequality. Taking square roots yields the triangle inequality.

☞ Problems involving vectors can frequently be solved using dot products. Remember that

- **lengths are defined in terms of the dot product**
- **Requiring vectors to be orthogonal is equivalent to requiring the dot product of the vectors to be zero** (the special cases $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ of the cosine formula).

The next two examples are typical for this kind of problem.
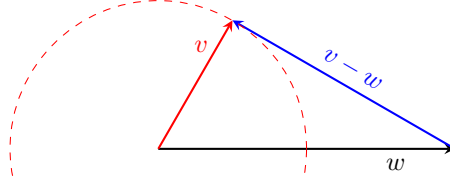
Figure 1.4: Given two vectors **v** and **w** of constant length all possible points described by **v** − **w** is obtained by allowing **v** to achieve all possible angles with respect to **w**, i.e., the endpoint of **v** will trace out a circle centered on the start point of **w**. The length of **v** − **w** is seen to be bounded by the values achieved when **v** and **w** are collinear (two cases).

Example 1.1.15. *Vector inequalities*

Given two vectors **v** *and* **w** *of length* $||\mathbf{v}|| = 3$ *and* $||\mathbf{w}|| = 5$*, find an upper and a lower bound on* $||\mathbf{v} - \mathbf{w}||$*. The geometry is shown in Fig (1.4). Using the definition of the length of a vector Eq (1.15) and the cosine formula Eq (1.16), we find*

$$
\begin{aligned}
||\mathbf{v} - \mathbf{w}||^2 &= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\
&= \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w} \\
&= ||\mathbf{v}||^2 + ||\mathbf{w}||^2 - 2||\mathbf{v}||\,||\mathbf{w}||\,\cos\theta.
\end{aligned}
$$

Using the bounds $-1 \le \cos\theta \le 1$ *(Note that the extrema correspond to the two possible cases when* **v** *and* **w** *are coplanar), and recognizing the perfect squares, we obtain*

$$
(||\mathbf{v}|| - ||\mathbf{w}||)^2 \le ||\mathbf{v} - \mathbf{w}||^2 \le (||\mathbf{v}|| + ||\mathbf{w}||)^2\,.
$$

Taking the square root, we obtain

$$
|\,||\mathbf{v}|| - ||\mathbf{w}||\,| \le ||\mathbf{v} - \mathbf{w}|| \le ||\mathbf{v}|| + ||\mathbf{w}||.
$$

Substituting the given values for the length of **v** *and* **w***, we find* $2 \le ||\mathbf{v} - \mathbf{w}|| \le 8$*.*

Example 1.1.16. *Orthogonal vectors*
Given two unit vectors $\mathbf{v}$ and $\mathbf{w}$, show that the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are orthogonal.

To prove the vectors are orthogonal, it is sufficient to prove that their dot product is 0. Let

$$(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v}$$
$$= ||\mathbf{v}||^2 - ||\mathbf{w}||^2$$
$$= 0$$

since $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ (the dot product is commutative), and since $||\mathbf{v}|| = 1, ||\mathbf{w}|| = 1$ (the vectors are of unit length).

1.2 Matrices

A **matrix of size** $M \times N$ is set of scalars arranged in a rectangular array of M rows and N columns. By convention, the scalar **entries** in the matrix are specified with two indices, with the first index specifying the row and the second index specifying the column of the position of the entry in the matrix.

We will use the notation $A = \left(a_{ij}\right)$ to represent the matrix A made up of elements a_{ij}, where i is the row index with $1 \leq i \leq M$, and j is the column index with $1 \leq j \leq N$. If we wish to emphasize the dimensions of the matrix, the notation will be extended to $A = \left(a_{ij}\right)_{M \times N}$

Example 1.2.1. *Example matrices*
The following are matrices of size 1×1, 2×2 and 2×3 respectively:

$$A = \begin{pmatrix} 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 4 \\ 4 & \pi + 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Patterns in the entries of matrices are of great interest and will be given special names. Matrices A and B are examples of square matrices (i.e., matrices having the same number of rows and columns). Matrix C is the zero matrix of size 2×3. Note that A is a matrix, not the scalar 3.

☞ **Abuse of notation:** unless otherwise noted, we will allow the symbol 0 to represent both the scalar 0 and **an appropriately sized zero vector or matrix**. The exact definition of 0 (e.g., matrix C in the example above), must be determined from the context.

Matrices of size $1 \times N$ and $M \times 1$ are referred to as **row vectors** and **column vectors** respectively. *These are matrices, not vectors: vectors do not have a concept of a horizontal or vertical layout.*

☞ **Abuse of notation:** Given a vector $\mathbf{u}$, we will frequently define a row vector or a column vector with the same entries. Many texts do not

use special notation to make the distinction, but use the same letter u to represent the matrix as well as the vector[3]. We use bold face **u** and commas between entries for vectors, e.g., the row vector matrix $u = (1\,2)$, and a vector with the same entries $\mathbf{u} = (1, 2)$.

1.2.1 Equality, Transpose, Matrix Addition, Scalar Multiplication

Matrix equality, matrix addition, and multiplication of a matrix by a scalar are straight forward generalizations of the corresponding vector operations. Consider two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size $M \times N$.

Matrices are equal if and only if they have the same size and all of their entries are equal.

Matrix addition is defined for matrices of the same size by adding their respective entries.

Scalar matrix multiplication is defined by multiplying each entry in a given matrix by the given scalar.

Given the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ a_{M1} & a_{M2} & \ldots & a_{MN} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1N} \\ b_{21} & b_{22} & \ldots & b_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ b_{M1} & b_{M2} & \ldots & b_{MN} \end{pmatrix},$$

their sum is

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1N} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} & \ldots & a_{2N} + b_{22} \\ \ldots & \ldots & \ldots & \ldots \\ a_{M1} + b_{M1} & a_{M2} + b_{M2} & \ldots & a_{MN} + b_{MN} \end{pmatrix} \tag{1.20}$$

while the product of matrix A with the scalar α is

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \ldots & \alpha a_{1N} \\ \alpha a_{21} & \alpha a_{22} & \ldots & \alpha a_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha a_{M1} & \alpha a_{M2} & \ldots & \alpha a_{MN} \end{pmatrix}. \tag{1.21}$$

The **Matrix transpose** of a matrix A denoted A^t, is obtained by changing each row of A to a column

$$A^t = \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{M1} \\ a_{12} & a_{22} & \ldots & a_{M2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1N} & a_{2N} & \ldots & a_{MN} \end{pmatrix}. \tag{1.22}$$

The i, j element of the transpose is the element in the original matrix with the indices reversed: $a_{ij}^t = a_{ji}$.

---

[3]The interpretation intended will usually be clear from the context.

Example 1.2.2. *Matrix transpose*
The transpose of a matrix of size 4×3 is a matrix of size 3×4:

$$A = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 7 & 3 \\ 1 & 5 & 9 \\ 3 & 2 & 6 \end{pmatrix}, \quad A^t = \begin{pmatrix} 3 & 2 & 1 & 3 \\ 4 & 7 & 5 & 2 \\ 8 & 3 & 9 & 6 \end{pmatrix}.$$

   A very useful special case is the conversion of row vectors to column vectors, and vice versa:

$$\begin{pmatrix} 3 & 4 & 8 \end{pmatrix}^t = \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix}$$

This property is used frequently to conserve space in print: $u = (3\,4\,8)^t$ is a column vector written on a single line to save space.

1.2.2   Matrix Multiplication

Matrix multiplication generalizes the dot product. Given a matrix $A = (a_{ik})$ of size $M \times K$ and a matrix $B = (b_{kj})$ of size $K \times N$, their **product** $C = (c_{ij})$ is a matrix of size $M \times N$ is defined by

$$C_{ij} = \sum_{k=1}^{K} a_{ik} b_{kj}, \tag{1.23}$$

i.e., the C_{ij} entry of the resulting matrix C is the dot product of the ith row of A and the jth column of B. The matrix C is thus made up from all possible dot products of the rows of A with columns of B. In matrix notation, $C = AB$. Take careful note of the location of the index of summation in each of the two terms.

Example 1.2.3. *Matrix product*
Consider the following three examples:
The product of two matrices of size 2×2 yields a matrix of size 2×2.

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.24}$$

If we reverse the order of the matrices, we get a different result!

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.25}$$

The product of a matrix of size 2×3 with a matrix of size 3×2 yields a matrix of size 2×2.

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -4 & 6 \end{pmatrix} \tag{1.26}$$

The first two products in this example show that matrix multiplication is not commutative. It is however easy to show from the definition Eq(1.23) that it is associative. Assuming for example that the matrix product $A = (A_1 A_2) A_3$ exists, we can verify that $A = A_1 (A_2 A_3)$. We may therefore drop the parentheses and write $A = A_1 A_2 A_3$.

The three examples 1.2.3 illustrate the following points:

☞ The **order of the matrices in a matrix product is important**. If we interchange the two matrices in the product:

- the resulting matrices may have different entries.

- the resulting matrices may not even be the same size.

- if the sizes are inconsistent, the product does not exist.

☞ **The product of non-zero matrices may be the zero matrix**. Given two matrices A and B with $AB = 0$, we cannot conclude that either $A = 0$ or $B = 0$.[4]

Matrices such that $AB = BA$ are said to **commute**.

---

[4]The corresponding theorem from scalar algebra requires dividing by a factor, i.e., multiplying by the inverse of the factor. For matrices, we shall see that we do not have matrix inverses in general.

Example 1.2.4. *Computational layout for the matrix product*
Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & -1 & -4 & 1 \\ 2 & 5 & -2 & -5 & 0 \\ 3 & 6 & -3 & -6 & 1 \end{pmatrix}, \tag{1.27}$$

and consider the product $C = AB$.

Lay out the computation by writing B to the right and above A, and write the product C of the matrices to the right of A and below B, such that the entries in the rows and columns are aligned. Observe that a given entry in C lies at the intersection of the row of A and the column of B from which it is computed!

The asterisks and dotted lines were introduced to clearly show the data dependency of the value $-32 = 4(-1) + 5(-2) + 6(-3)$: it lies at the intersection of the second row and third column, and is therefore the dot product of the second row of A with the third column of B. The reader should compute the entries in each of the fields denoted by asterisks, carefully noting the row and column of the entries in A and B respectively from which they are computed.

☞ The **size of the matrix** C can be read directly off the layout: it has the height of A and the width of B.

The computational layout can be readily extended for products involving more than two matrices. Multiplication of a number of matrices can be laid out by associating pairs of matrices either from left to right or from right to left. Writing

$$C = (((A_1 A_2) A_3) \dots A_n) \tag{1.28}$$

we can lay out the computation in two rows of matrices:

$$\begin{pmatrix} A_2 \end{pmatrix} \begin{pmatrix} A_3 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} A_n \end{pmatrix}$$
$$\begin{pmatrix} A_1 \end{pmatrix} \begin{pmatrix} C_2 \end{pmatrix} \begin{pmatrix} C_3 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} C_n \end{pmatrix}$$

where for each index i in $1 \le i \le n$, we compute $C_i = A_1 A_2 \dots A_i$ in turn: $C_2 = A_1 A2$, followed by $C_3 = C_2 A_3$, etc.

Similarly, grouping from left to right,

$$C = A_1(A_2(A_3 \dots (A_{n-1}A_n))) \tag{1.29}$$

we can also lay out the required computations in two columns of matrices:

$$\begin{pmatrix} A_n \end{pmatrix}$$

$$\begin{pmatrix} A_{n-1} \end{pmatrix} \begin{pmatrix} \mathbf{B_{n-1}} \end{pmatrix}$$

$$\dots \qquad \dots$$

$$\begin{pmatrix} A_1 \end{pmatrix} \begin{pmatrix} \mathbf{B_1} \end{pmatrix}$$

Here for each index i, we compute the intermediate result $B_i = A_i A_{i+1} \dots A_n$ and finally $C = B_1$.

Example 1.2.5. ***Product of several matrices (horizontal layout)***
A product of 4 matrices laid out horizontally. The matrix products are computed front to back.

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 12 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5 & -5 & 5 \\ 1 & -1 & 1 \\ -3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} -5 & 60 & -5 \\ -1 & 12 & -1 \\ 3 & -36 & 3 \end{pmatrix}$$

Example 1.2.6. ***Product of several matrices (vertical layout)***
The product of the same 4 matrices as in Example 1.2.5 laid out horizontally. The matrix products are computed back to front, yielding the same result as before. The intermediate products are different, however.

$$\begin{pmatrix} -1 & 12 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 12 & -1 \\ -1 & 12 & -1 \\ -1 & 12 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 12 & -1 \\ -2 & 24 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -5 & 60 & -5 \\ -1 & 12 & -1 \\ 3 & -36 & 3 \end{pmatrix}$$

Example 1.2.7. *Inner and outer matrix products*
Given two vectors $\mathbf{u} = (2, -1, 3, 1)$ *and* $\mathbf{v} = (1, 0, -4, -2)$, *we can form two matrix products. Represent the vectors as matrices of the same name using row vectors.*

The inner product uv^t *is*

$$\begin{pmatrix} 1 \\ 0 \\ -4 \\ -2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} -8 \end{pmatrix}$$

The result is a matrix of size 1×1 *with a single entry equal to the dot product of the two vectors:* $uv^t = (u \cdot v)$. *Note the inner product* $<u, v> = -8$ *results in the entry itself.*

The outer product $u^t v$ *is a matrix of size* 4×4

$$\begin{pmatrix} 1 & 0 & -4 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 2 \\ -1 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -8 & -4 \\ -1 & 0 & 4 & 2 \\ 3 & 0 & -12 & -6 \\ 1 & 0 & -4 & -2 \end{pmatrix}$$

Example 1.2.8. *A system of linear equations*
A system of linear equations can be written concisely in matrix notation in the form $Ax = b$, *where* A *is a given matrix consisting of the coefficients of the unknown variables, b is a given vector consisting of the right hand sides of the equations, and x is a vector of the unknowns:*

$$\left. \begin{array}{rcrcrcr} 3x_1 & + & 8x_2 & +9x_3 & = & 24 \\ 7x_1 & + & 4x_2 & +x_3 & = & 21 \end{array} \right\} \quad \Leftrightarrow \quad Ax = b,$$

where we have set given vector consisting of the right hand sides of the equations, and x is a vector of the unknowns:

$$A = \begin{pmatrix} 3 & 8 & 9 \\ 7 & 4 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad and \quad b = \begin{pmatrix} 24 \\ 21 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 3 & 8 & 9 \\ 7 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 24 \\ 21 \end{pmatrix}.$$

See any of the following sections for additional examples.

A seemingly trivial observation will prove extremely useful to understand the workings of the matrix product. Refer back to the definition Eq(1.23)

$$C_{ij} = \sum_{k=1}^{K} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{K_1} a_{ik} b_{kj} + \sum_{k=K_1+1}^{K} a_{ik} b_{kj}$$

for any subset of rows i and columns j of C, and any $1 \leq K_1 \leq K$. An interpretation of these equations is that we can think of a matrix product as being composed of sums of products of subsets of the matrices.

Example 1.2.9. *Partitioning the matrix product*
To clarify this observation, consider the following example:

$$\left(\begin{array}{ccc|c} 7 & 6 & 9 & 7 \\ \hline 7 & 1 & 3 & 2 \\ 2 & 1 & 5 & 5 \\ 6 & 4 & 2 & 6 \end{array} \right)$$

$$\left(\begin{array}{c|ccc} 8 & 1 & 8 & 1 \\ \hline 9 & 2 & 2 & 2 \\ \hline 5 & 8 & 9 & 6 \\ 1 & 2 & 3 & 4 \end{array} \right) \quad \left(\begin{array}{ccc|c} 85 & 61 & 117 & 104 \\ 93 & 66 & 101 & 89 \\ \hline 145 & 71 & 126 & 132 \\ 51 & 27 & 38 & 50 \end{array} \right)$$

Checking the dependencies of the product entries on the entries in the two product matrices, we see that they are

$$\left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array} \right)$$

$$\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \right) \quad \left(\begin{array}{c|c} A_1 B_1 + A_2 B_3 & A_1 B_2 + A_2 B_4 \\ \hline A_3 B_1 + A_4 B_3 & A_3 B_2 + A_4 B_4 \end{array} \right)$$

where the A_i, B_i and C_i are the matrices outlined by the partitioning lines above. E.g.,

$$A_2 = \left(\begin{array}{ccc} 1 & 8 & 1 \\ 2 & 2 & 2 \end{array} \right).$$

☞ The matrix product can be decomposed into products of submatrices by introducing any number of horizontal and vertical partitions in a matrix, as long as the matrix sizes are kept consistent.

This requirement means that partitioning lines must extend across two matrices. In particular, a vertical partition of matrix A immediately to the right of column i has a corresponding partition of matrix B immediately below row i.

☞ Special care must be taken to keep the order of the submatrices in the products consistent: since A is to the left of B in $C = AB$, the submatrices A_i are to the left of the submatrices B_i in the product.

Example 1.2.10. *Row view of the matrix product*
Given the matrix product $C = AB$, consider partitioning A into each of its individual entries. This results in a corresponding partition of B and C into their constituent rows B_k and C_i

$$\left(\begin{array}{c} \hline B_1 \\ \hline B_2 \\ \hline \cdots \\ \hline B_K \end{array}\right)$$

$$\left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1K} \\ \hline a_{21} & a_{22} & \cdots & a_{2K} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a_{N1} & a_{N2} & \cdots & a_{NK} \end{array}\right) \left(\begin{array}{c} a_{11}B_1 + a_{12}B_2 \cdots + a_{1K}B_K \\ \hline a_{21}B_1 + a_{22}B_2 \cdots + a_{2K}B_K \\ \hline \cdots \\ \hline a_{N1}B_1 + a_{N2}B_2 \cdots + a_{NK}B_K \end{array}\right).$$

As we see, we can compute each row of C as a linear combination of the rows of B. In the following example,

$$\left(\begin{array}{c} \hline B_1 \\ \hline B_2 \\ \hline B_3 \end{array}\right)$$

$$\left(\begin{array}{c|c|c} \cdots & \cdots & \cdots \\ \hline 2 & 3 & 5 \\ \hline \cdots & \cdots & \cdots \end{array}\right) \left(\begin{array}{c} \cdots \\ \hline 2B_1 + 3B_2 + 5B_3 \\ \hline \cdots \end{array}\right)$$

the row of A shown can be read as an instruction to compute the corresponding row of the product as "2 times the first row of B + 3 times the second row + 5 times the third row".

☞ The fact that data dependencies in a matrix product are made explicit by partitioning can prove useful when tracking errors in a computation. An error in the row of $A_k = (2\ 3\ 5)$ shown explicitly in the previous example only affects the corresponding row $B_k = (2B_1 + 3B_2 + 5B_3)$ in the product.

Example 1.2.11. *Column view of the matrix product*
Similarly, if we partition B into each of its individual entries, we find that the columns of the product $C = AB$ can be computed as linear combinations of the columns of A. In the following example,

$$
\left(A_1 \,\middle|\, A_2 \,\middle|\, A_3 \right)
\left(
\begin{array}{c|c|c}
\cdots & 2 & \cdots \\
\hline
\cdots & 3 & \cdots \\
\hline
\cdots & 5 & \cdots
\end{array}
\right)
=
\left(
\begin{array}{ccc}
\cdots & \left| 2A_1 + 3A_2 + 5A_3 \right| & \cdots
\end{array}
\right)
$$

the column of B shown can be read as an instruction to compute the corresponding column of the product as "2 times the first column of A + 3 times the second column + 5 times the third column."

As an immediate application, consider the **identity matrix** I a square matrix of some given size $N \times N$ with entries on the **main diagonal**, i.e., entries $I_{ii} = 1$ for $i \in 1, 2, \cdots N$ and all other entries equal to zero.

$$
I = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
$$

Multiplication of a matrix A with I from the left, $C = IA$ is seen to result in copying the rows of A into the corresponding rows of C, i.e., $IA = A$ (the first row of I for example reads "1 times the first row of A plus zero"). Similarly the product $AI = A$, since the columns of I specify recopying the columns of A into the corresponding column of the product.

Interchanging rows or columns in the identity matrix results in **permutation matrices**. These matrices have the useful property that they interchange rows of a given matrix when multiplied in from the left, or columns of a given matrix when multiplied in from the right.

Example 1.2.12. *Permutation matrices*
Consider the permutation matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
3 & 7 & 4 \\
2 & 0 & 1 \\
9 & 1 & 8 \\
8 & 6 & 2
\end{pmatrix}
=
\begin{pmatrix}
3 & 7 & 4 \\
9 & 1 & 8 \\
2 & 0 & 1 \\
8 & 6 & 2
\end{pmatrix}
$$

☞ To avoid having to discuss special cases, it is useful to allow partitions of width (or height) 0. These submatrices can simply be removed from the resulting equations without affecting their validity.

Example 1.2.13. *Multiplication of partitioned matrices*
We wish to investigate the multiplication of matrices with a distinct pattern of ones and zeros.[a] Specifically, let

$$
L = \left(\begin{array}{cc|c|cc}
\mathbf{7} & \mathbf{0} & 0 & 0 & 0 \\
\mathbf{2} & \mathbf{9} & 0 & 0 & 0 \\
\hline
\mathbf{3} & \mathbf{4} & \mathbf{1} & 0 & 0 \\
\mathbf{2} & \mathbf{-1} & \mathbf{0} & 1 & 0 \\
\mathbf{-2} & \mathbf{5} & \mathbf{0} & 0 & 1
\end{array} \right) , E = \left(\begin{array}{cc|c|cc}
\mathbf{1} & \mathbf{0} & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{1} & 0 & 0 & 0 \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{8} & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{6} & 1 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{-1} & 0 & 1
\end{array} \right)
$$

consisting of the respective submatrices L_i and E_i corresponding to the partitions indicated, i.e., we want to investigate products LE of the form

$$
LE = \left(\begin{array}{c|c|c}
\mathbf{L_1} & 0 & 0 \\
\hline
\mathbf{L_2} & 1 & 0 \\
\hline
\mathbf{L_3} & \mathbf{0} & I
\end{array} \right) \left(\begin{array}{c|c|c}
\mathbf{I} & 0 & 0 \\
\hline
\mathbf{0} & \mathbf{E_2} & 0 \\
\hline
\mathbf{0} & \mathbf{E_3} & I
\end{array} \right) = \left(\begin{array}{c|c|c}
\mathbf{L_1} & 0 & 0 \\
\hline
\mathbf{L_2} & \mathbf{E_2} & 0 \\
\hline
\mathbf{L_3} & \mathbf{E_3} & I
\end{array} \right) .
$$

where we have used entries 0 and I for those submatrices that are equal to zero or an identity matrix respectively. The reader should set up and carry out this product at this point.

On inspection, we see that the net result of this computation is to replace the third column of matrix L with the third column of matrix E ("merging" the off-diagonal elements), yielding a matrix with a similar pattern of zeros, i.e.,

$$
LE = \left(\begin{array}{c|c|c}
\tilde{L}_1 & 0 & 0 \\
\hline
\tilde{L}_2 & 1 & 0 \\
\hline
\tilde{L}_3 & 0 & I
\end{array} \right) = \left(\begin{array}{ccc|c|c}
7 & 0 & 0 & 0 & 0 \\
2 & 9 & 0 & 0 & 0 \\
3 & 4 & 8 & 0 & 0 \\
\hline
2 & -1 & 6 & 1 & 0 \\
\hline
-2 & 5 & -1 & 0 & 1
\end{array} \right)
$$

with the appropriate definitions of $\tilde{L}_i$ as show with the partitions on the right (isolating the next diagonal element). Note that this is the same pattern of zeros that we started with. An example of the repeated application of this pattern is shown in Eq (2.31). It is instructive to compute the product EL in the same way: one finds that while the resulting matrix has a similar structure, the submatrix in the lower left corner is a combination of the submatrices in L_3 and E_3, breaking the pattern.

---

[a]This is the pattern of matrices that must be multiplied together to compute L in the LU decomposition. It shows that L can be obtained by simply combining the non-zero columns of each of the elimination matrices, provided that no row interchanges were used.

1.2.3   Matrix Inverses

Division by a matrix is not defined: we cannot "divide through" by a matrix in general. Suppose that given some matrix A, we can find a matrix B such that

$BA = I$. Then in any expression such as $AC = D$, for example, we can multiply by B from the left, to get $BAC = BD \Leftrightarrow C = BD$ (since matrix multiplication is associative). We have achieved the effect we want!

The following example shows that such matrices can exist, although this will prove not to be the case in general.

Example 1.2.14. *Inverse of matrices*
Consider the matrices A and B and their products:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -6 \\ 3 & 4 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 2 & -3 \\ -6 & -3 & 4 \\ -1 & -1 & 1 \end{pmatrix},$$

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \text{ and } BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

If we remove the last column of A and the last row of B, we find

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 6 & 2 & -3 \\ -6 & -3 & 4 \end{pmatrix},$$

$$\tilde{B}\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ but } \tilde{A}\tilde{B} = \begin{pmatrix} 0 & -1 & 1 \\ -6 & -5 & 6 \\ -6 & -6 & 7 \end{pmatrix}.$$

Given matrices A and B, then B is a **left inverse of** A if $BA = I$, and a **right inverse of** A if $AB = I$. If B is both a left inverse and a right inverse of A, it is an **inverse** of A.

☞ A left inverse L of a matrix A **cannot have a row of zeros**. This is trivial to see since if row i of L is zero, then row i of the product LA would also be zero, but $LA = I$ does not have zero rows.

☞ A right inverse R of a matrix A **cannot have a column of zeros**.

☞ **Only square matrices can have an inverse**.
FIX This fact will follow trivially from the two previous observations and the Gaussian Elimination solution method for matrix equations of the form $AX = I$. FIX

☞ If a matrix A has both a right and a left inverse R and L respectively, then $R = L$ is an inverse of A. Furthermore, if A has an inverse, it is unique.

Proof. Let A be a matrix of size $M \times N$ that has both a left inverse L and a right inverse R, i.e., $LA = I_l$ and $AR = I_r$. Note that the identity matrices I_l and I_r are square, possibly of different sizes. For the products LA and AR to exist and be square, we see that both L and R have size $N \times M$. Since by associativity $AR = I_r \Rightarrow L(AR) = LI_r \Rightarrow (LA)R = L \Rightarrow I_l R = L$, we obtain $R = L$.

To show uniqueness, let B and C be inverses of A. Then $B = BI = B(AC) = (BA)C = IC = C$. □

If it exists, the inverse of a matrix A will be denoted A^{-1}.

Look again at the above example: we have $B = A^{-1}$, (and therefore $A = B^{-1}$). Further, $\tilde{B}$ is a left inverse of $\tilde{A}$, and $\tilde{A}$ is a right inverse of $\tilde{B}$. **FIX Note that since $\tilde{A}$ has a left inverse but is not square, it cannot have a right inverse. A similar observation holds for $\tilde{B}$. FIX**

1.2.4   Substitution

Consider the system of M linear equations in N unknowns

$$Ax = b \Leftrightarrow \sum_{j=1}^{N} \alpha_{ij} x_j = b_i, \ i = 1, 2, \cdots M,$$

and ask how the system changes when we substitute a different set of variables, e.g.,

$$\left.\begin{array}{l} x_1 = \beta_{11}\tilde{x}_1 + \beta_{12}\tilde{x}_2 + \cdots \beta_1 N \tilde{x}_N \\ x_2 = \beta_{21}\tilde{x}_1 + \beta_{22}\tilde{x}_2 + \cdots \beta_{2N} \tilde{x}_N \\ \cdots \\ x_N = \beta_{N1}\tilde{x}_1 + \beta_{N2}\tilde{x}_2 + \cdots \beta_{NN} \tilde{x}_N \end{array}\right\} \Leftrightarrow \quad x_j = \sum_{k=1}^{N} \beta_{jk} \tilde{x}_k, \qquad j = 1, 2, \cdots N,$$

$$\Leftrightarrow \quad x = B\tilde{x} \qquad \qquad \text{where } \tilde{x} = (x_j),$$

where the array $B = (\beta_{ij})$.

Carrying out the substitution, we find

$$Ax = b \Leftrightarrow \sum_{j=1}^{N} \alpha_{ij} x_j = b_i, \qquad\qquad\qquad\qquad i = 1, 2, \cdots M$$

$$\Rightarrow \sum_{j=1}^{N} \alpha_{ij} \sum_{k=1}^{N} \beta_{jk} \tilde{x}_k = b_i, \qquad\qquad\qquad i = 1, 2, \cdots M$$

$$\Leftrightarrow \sum_{k=1}^{N} \left(\sum_{j=1}^{N} \alpha_{ij} \beta_{jk} \right) \tilde{x}_k = b_i, \qquad\qquad\qquad i = 1, 2, \cdots M$$

$$\Leftrightarrow (AB)\, \tilde{x} = b.$$

We see that substitution $x = B\tilde{x}$ yields $Ax = A(B\tilde{x}) = (AB)\tilde{x}$, i.e., the coefficients of $\tilde{x}$ are obtained by the product of the matrices AB.

Example 1.2.15. *Substitution in a linear system*
Consider the system of linear equations from Example (1.2.8),

$$
\begin{aligned}
3x_1 + 8x_2 + 9x_3 &= 24 \\
7x_1 + 4x_2 + x_3 &= 21
\end{aligned}
$$

In matrix form, this is

$$
Ax = b \Leftrightarrow \begin{pmatrix} 3 & 8 & 9 \\ 7 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 24 \\ 21 \end{pmatrix}.
$$

We now substitute $x = B\tilde{x}$, where

$$
\left.\begin{aligned}
x_1 &= 3\tilde{x}_1 + 2\tilde{x}_2 - \tilde{x}_3 \\
x_2 &= \tilde{x}_1 + 6\tilde{x}_2 - 2\tilde{x}_3 \\
x_3 &= 2\tilde{x}_2 + 3\tilde{x}_3
\end{aligned}\right\} \Leftrightarrow \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & -2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_1 \\ \tilde{x}_1 \end{pmatrix}
$$

Since the product of the matrices A and B is given by

$$
\begin{pmatrix} 3 & 8 & 9 \\ 7 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & -2 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 17 & 72 & 8 \\ 25 & 40 & -12 \end{pmatrix},
$$

we obtain

$$
AB\tilde{x} = b \Leftrightarrow \begin{cases} 17\tilde{x}_1 & + & 72\tilde{x}_2 & +8\tilde{x}_3 & = & 24 \\ 25\tilde{x}_1 & + & 40\tilde{x}_2 & -12\tilde{x}_3 & = & 21 \end{cases}
$$

1.2.5   Properties of Matrix Operations

The following properties of matrix addition, matrix multiplication and scalar multiplication in $\mathbb{R}^N$ allow us to carry out algebraic manipulations: Let A, B and C be matrices, α, β and γ be scalars, let 0 be the zero matrix and I the identity matrix with matrix sizes such that the operations below are well defined.

- **Matrix Addition**

$$A + B = B + A \qquad\qquad \text{Commutativity} \quad (1.30\text{a})$$
$$A + (B + C) = (A + B) + C \qquad\qquad \text{Associativity} \quad (1.30\text{b})$$
$$A + 0 = A \qquad\qquad \text{Zero Matrix} \quad (1.30\text{c})$$
$$A + (-A) = 0 \qquad\qquad \text{Inverse} \quad (1.30\text{d})$$

- **Scalar Multiplication**

$$\alpha(\beta A) = (\alpha\beta)A \qquad\qquad \text{Associativity} \quad (1.31\text{a})$$
$$\alpha(A + B) = (\alpha A) + (\alpha B) \qquad\qquad \text{Distributivity} \quad (1.31\text{b})$$
$$(\alpha + \beta)A = (\alpha A) + (\beta A) \qquad\qquad \text{Distributivity} \quad (1.31\text{c})$$
$$1A = A \qquad\qquad \text{Unit} \quad (1.31\text{d})$$
$$0A = 0 \qquad\qquad \text{Zero} \quad (1.31\text{e})$$

- **Matrix Multiplication**

$$(AB)C = A(BC) \qquad\qquad \text{Associativity} \quad (1.32\text{a})$$
$$A(B + C) = AB + AC \qquad\qquad \text{Distributivity} \quad (1.32\text{b})$$
$$(A + B)C = AC + BC \qquad\qquad \text{Distributivity} \quad (1.32\text{c})$$
$$IA = A, AI = A \qquad\qquad \text{Identity Matrix} \quad (1.32\text{d})$$
$$0A = 0, A0 = 0 \qquad\qquad \text{Zero Matrix} \quad (1.32\text{e})$$

Comparison with the corresponding properties of vector operations in Eqs (1.9) shows addition and scalar multiplications have similar properties[5]. Vector dot products and matrix multiplications are different, however. Their relationship is explored in the next section. Assuming the inverses A^{-1} and B^{-1} exist, we also have

---

[5]It is this similarity that lets us use the same notation for vectors, row vectors and column vectors (at the risk of some confusion!)

- **Matrix Inverse and Transpose**

$$(A^t)^t = A \tag{1.33a}$$

$$(A + B)^t = A^t + B^t \tag{1.33b}$$

$$(AB)^t = B^t A^t \tag{1.33c}$$

$$(\alpha A)^t = \alpha A^t \tag{1.33d}$$

$$(A^{-1})^{-1} = A \tag{1.33e}$$

$$(AB)^{-1} = B^{-1} A^{-1} \tag{1.33f}$$

$$(A^t)^{-1} = (A^{-1})^t \tag{1.33g}$$

In summary, algebraic operations involving matrices are similar to operations in scalar algebra, except for the following:

☞ **Matrix multiplication is not commutative**. Be very careful not to change the order of the matrices appearing in an expression.

Matrices A and B that yield the same product regardless of their order, i.e., such that $AB = BA$ are said to **commute**.

☞ Since matrix multiplications are not commutative **patterns from algebra do not hold in general**. For example

$$(A + 3B)(A - 3B) = A^2 - 9B^2 - 3AB + 3BA, \tag{1.34}$$

i.e., the AB and BA terms do not cancel.

☞ **Terms cannot be canceled**. Since most matrices do not have inverses, we cannot simply cancel common terms. $AB = 0$ for example **does not let us conclude that either $A = 0$ or $B = 0$**, nor does $AB = AC$ imply that $B = C$.

If A is known to be invertible however, these results do follow: they are obtained by multiplying with A^{-1} from the left.

1.3   Scalars, Vectors and Matrices

As we have seen in example 1.2.7, dot products of vectors and products of matrices are closely related. Similarly, addition and scalar products of vectors and the corresponding row or column vector addition and scalar multiplications yield results with identical numerical entries.

To switch from a vector to a matrix in a computation, we need to specify whether we wish to represent a given vector as a row vector or as a column vector. In a later chapter, we will formalize this decision by setting up an invertible function $f(\mathbf{u}) = u$ to map vectors to column vectors, for example.

The Dot Product

Let us first consider the relationship between the vector dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ and two column vectors u and v having the same numerical entries as the vectors of the same name: the dot product $\mathbf{u} \cdot \mathbf{v}$ is the $(1,1)$ entry of the matrix product $u^t v$:

$$u^t v = (\mathbf{u} \cdot \mathbf{v}), \tag{1.35}$$

where the right hand side is a matrix of size 1×1 (rather than a scalar quantity in parentheses). If we choose to represent both vectors as row vectors instead, we similarly find $uv^t = (\mathbf{u} \cdot \mathbf{v})$.

We shall frequently require the scalar entry $\mathbf{u} \cdot \mathbf{v}$ rather than the matrix $u^t v$. We therefore introduce the **inner product**

$$<u, v> = \mathbf{u} \cdot \mathbf{v}. \tag{1.36}$$

The Scalar Product

The scalar product for vectors may be expressed as a matrix product as well. Consider the scalar α, the matrix $(\alpha) = \alpha I_{1 \times 1}$ and the vector $\mathbf{u}$ represented as a column vector $u = (u_1 \, u_2 \, \cdots \, u_N)^t$. We find

$$\alpha \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix} (\alpha). \tag{1.37}$$

If the vector $\mathbf{u}$ is represented as a row vector instead, the scalar α may be replaced with the matrix $\alpha I_{N \times N}$. We have

$$\alpha \begin{pmatrix} u_1 & u_2 & \cdots & u_N \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_N \end{pmatrix} \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha \end{pmatrix}. \tag{1.38}$$

As always, pay special attention to the order of the matrices in the product.

Example 1.3.1. *Exploiting the vector and column vector representations*

Consider the matrix product[a] $p = u(u^t u)^{-1} u^t v$ for given column vectors $u \neq 0$ and v of the same size, and the corresponding vectors $\mathbf{u}$, $\mathbf{v}$.

Even though the column vector u cannot be inverted, the product matrix $u^t u = (||\mathbf{u}||^2)$ is invertible for $u \neq 0$: $(u^t u)^{-1} = (||\mathbf{u}||^{-2}) = \left(\frac{1}{\mathbf{u} \cdot \mathbf{u}}\right) = \frac{1}{\mathbf{u} \cdot \mathbf{u}} I$, a matrix of size 1×1. Note that parentheses enclosing matrices are used for grouping, while parentheses enclosing scalars are used to denote matrices of size 1×1.

$$
\begin{aligned}
u(u^t u)^{-1} u^t v &= u \frac{1}{\mathbf{u} \cdot \mathbf{u}} I u^t v \\
&= \frac{1}{\mathbf{u} \cdot \mathbf{u}} u u^t v && \text{by Eq(1.37)} \\
&= \frac{1}{\mathbf{u} \cdot \mathbf{u}} u \, (\mathbf{u} \cdot \mathbf{v} I) && \text{by Eq(1.35)} \\
&= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} u && \text{by Eq(1.37)}
\end{aligned}
$$

As a concrete example, let $\mathbf{u} = (3, 4, 5)$ and $\mathbf{v} = (2, 7, 6)$ and let u and v be the corresponding column vectors with the same entries. Multiplying out all terms involving u in the original expression, we have

$$
u(u^t u)^{-1} u^t = \frac{1}{50} \begin{pmatrix} 9 & 12 & 15 \\ 12 & 16 & 20 \\ 15 & 20 & 25 \end{pmatrix} \Rightarrow u(u^t u)^{-1} u^t v = \frac{1}{50} \begin{pmatrix} 192 \\ 256 \\ 320 \end{pmatrix},
$$

while the same result is obtained more directly from the equivalent expression

$$
\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} u = \frac{64}{50} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 192 \\ 256 \\ 320 \end{pmatrix}.
$$

---

[a]This expression will arise in the study of orthogonal projections.

1.4 Special Matrices

Whenever there is a pattern to the entries in a matrix, we will find ways to exploit it. Matrices exhibiting such patterns are given names. We list some of the most common matrix types here. Each of these will occur repeatedly in this text.

In the absence of an explicit statement to the contrary, the matrices described below are square. In particular let A be some *square matrix*. We will consider

Diagonal matrices have any scalars (including 0) on the main diagonal, but zeros above and below: $A_{ij} = 0$ if $i \neq j$. Diagonal matrices are

traditionally denoted D. Note that D is invertible if and only if all diagonal entries $D_{ii} \neq 0$. The inverse is a diagonal matrix with entries $D_{ii}^{-1} = \frac{1}{D_{ii}}$. The product of diagonal matrices is diagonal.

Example 1.4.1. *Diagonal matrix*

$$
D = \left(\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{array} \right), \quad D^{-1} = \left(\begin{array}{ccc} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{array} \right).
$$

Triangular matrices have two types: i) upper triangular matrices have zeros below the main diagonal, i.e., $A_{ij} = 0$ if $i > j$, and ii) lower triangular matrices have zeros above the main diagonal, i.e., $A_{ij} = 0$ if $i < j$. Triangular matrices are traditionally denoted U (upper triangular) and L (lower triangular). They are invertible if and only if all their diagonal entries are non-zero. The product of two upper triangular matrices is upper triangular, and the product of two lower triangular matrices is lower triangular. The product of an upper and a lower triangular matrix is a full matrix in general.

Unit triangular matrices are triangular matrices with all diagonal entries equal to 1. The product of two unit upper triangular matrices is unit upper triangular, and the product of two unit lower triangular matrices is unit lower triangular. Unit triangular matrices are invertible: the inverse of a unit lower triangular matrix is unit lower triangular, and the inverse of a unit upper triangular matrix is unit upper triangular.

Example 1.4.2. *Upper triangular matrix*

$$
U = \left(\begin{array}{ccc} 2 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{array} \right), \quad U^{-1} = \left(\begin{array}{ccc} \frac{1}{2} & 0 & -\frac{1}{6} \\ 0 & -1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{array} \right).
$$

Permutation matrices are matrices obtained from an identity matrix by an arbitrary number of row (or equivalently column) interchanges. Permutation matrices are typically denoted P. Permutation matrices are invertible with inverse $P^{-1} = P^t$. The product of two permutation matrices is a permutation matrix.

Example 1.4.3. *Permutation matrix*

$$
P = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \quad P^{-1} = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right).
$$

Permutation matrices are extremely useful in practice: they allow rearranging the order of the rows and/or columns of a given matrix A by multiplying with the desired permutation matrix P from the left for a row exchange, or from the right for a column exchange.

Symmetric matrices satisfy $A^t = A$. Given any matrix M, the products $M^t M$ and MM^t are defined (although not necessarily of the same size), and are symmetric. If it exists, the inverse of a symmetric matrix is symmetric. The product of symmetric matrices is not symmetric in general.

Example 1.4.4. *Symmetric matrix*

$$
A = \begin{pmatrix} 2 & -3 & 2 \\ -3 & 3 & -2 \\ 2 & -2 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}.
$$

Skew-symmetric matrices satisfy $A^t = -A$.

Hermitian and Skew-Heritian matrices generalize the notion of symmetric and skew symmetric matrices to matrices with complex entries. The hermitian transpose of a matrix A is denoted A^H, and is obtained from A^t by taking the complex conjugate of each of the entries.

A matrix A is hermitian if $A^H = A$, and skew-hermitian if $A^H = -A$. Note that if we split a hermitian matrix into its real and imaginary parts $A = A_1 + iA_2$, then A_1 is symmetric and A_2 is antisymmetric. An analogous property holds for skew-hermitian matrices.

Example 1.4.5. *Hermitian matrix*

$$
A = \begin{pmatrix} 2 & -3+2i & 2 \\ -3+2i & 3 & -2+5i \\ 2 & -2+5i & 1 \end{pmatrix},
$$

$$
A_1 = \begin{pmatrix} 2 & -3 & 2 \\ -3 & 3 & -2 \\ 2 & -2 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}.
$$

For systems of linear equations $Ax = b$, two special patterns of A are of interest:

Matrices in row echelon form are matrices of arbitrary size $M \times N$ such that the first **non-zero** coefficient (*pivot*) in each row, if any, appears to the right of the first non-zero coefficient in the row directly above. Rows consisting entirely of zeros are at the bottom of such a matrix. Upper

triangular matrices with non-zero coefficients on the main diagonal are a special case. If one connects vertical partitioning lines placed immediately to the left of each pivot, one obtains a "staircase" dividing line such that all entries beneath are zero.

Example 1.4.6. *Row echelon form*

$$A = \begin{pmatrix} \mathbf{2} & -3 & 2 & 1 \\ \mathbf{0} & \mathbf{3} & -2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The colorized red entries are the pivots. The entries in black are unconstrained: they can have any value, including zero.

Matrices in reduced row echelon form are matrices in row echelon form with the additional restrictions that the pivots are equal to 1, and the values above the pivots are 0. Note that the successive pivot columns are successive columns in an identity matrix.

Example 1.4.7. *Reduced row echelon form*

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{0} & 2 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & -2 & 4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The colorized red entries are the pivots. The entries in black are unconstrained: they can have any value, including zero. The colorized blue entries above, below and to the left of the pivot entries must be zero. Removing all columns that do not contain pivots, that is the third and fourth column, leaves the first three columns of an identity matrix of size 4×4.

1.5 Exercises

Exercise 1.1. *1. Does Eq(1.14d) hold for vectors in $\mathbb{Q}^N$?*

2. Does Eq(1.14d) hold for vectors in $\mathbb{C}^N$?

1.5.1 Sage Exercises

In practice, linear algebra problems of interest are very large: the google search engine uses matrices with *billions of entries*. It is therefore important to familiarize yourself with computer tools.

Download and install the "SAGE Mathematical Software", (a powerful free open source mathematical softare system) on your computer. It is available from http://www.sagemath.org. The simplest way to use Sage is from a "notebook" in your browser. From a Sage command line, you may simply type the line "notebook()" and follow the instructions.

Start by reading the help section, in particular the tutorial. Carry out the commands in each of the following exercises to start exploring the capabilities of the system.

Exercise 1.2. *Use Sage to enter the following definition of a vector (execute the lines in a notebook one line at a time, using <Shift-Enter>)*

```
sage: a=vector([1,2,4])
sage: a
sage: b = a/3
sage: b
sage: b.n()
sage: print type(b)
sage: print parent(b)
sage: print type(1/3)
sage: print parent(pi.n()), '\n', parent(pi.n(10))
```

As you see, Sage can carry out computations with integers, rationals, and real numbers (and much more!). In an introductory linear algebra course, we are mainly interested in computations with rational numbers.[6] To force Sage to use rational numbers, we can use the parameter QQ (used to represent the field of rationals $\mathbb{Q}$) as follows:

```
sage: a=vector(QQ,[1, -2, 5, 6.5 ])
sage: a
```

Compare the output from the previous line with the output of the following

---

[6]Computations by hand are easier, and we avoid issues arising from inexact number representations in computer arithmetic.

```
sage: a=vector(QQ,[1, -2, 5, 6.2 ])
sage: a
```

What happens when you add a and b?

```
sage: a+b
```

Exercise 1.3. *Matrices can be constructed in Sage using the Matrix command:*

```
sage: A = matrix( QQ, [[1,2,3,8],[4,3,2,1.2],[-1,9,6,4/7]])
sage: A
```

constructs a matrix of size 3×4. *Output in the notebook can be made pretty by using the "show" command:*

```
sage: show(A)
```

This text was written using the free open source document preparation system LATEX. *The latex representation of the matrix A may be obtained in Sage using the "latex" command:*

```
sage: latex(A)
```

Matrix dimensions can be given explicitely, matrices can be extended and subdivided:

```
sage: A.augment( matrix(3, 1, [1, 2, 3.2]))
sage: B=_
sage: B.subdivide(None, 4); B
```

Exercise 1.4. *Many operations we will encounter in this text are implemented for vectors and matrices. Given a vector or a matrix a, use the sage command*

```
sage: a.<Tab>
```

to get a list of available functions. To get information on how to use any of these functions, append a question mark to the function name and hit enter. Given a matrix B for example, the command

```
sage: B.column?
```

will provide explanations and examples. Explore functions available for vectors and matrices. Which do you recognize?

Exercise 1.5. *To enter equations in Sage, we need to define the variables we wish to use:*

```
sage: x, y, z = var('x, y, z')
```

Here are some more examples of expressions to try:

```
sage: f  = 5*x+3*y+z
sage: show(f)
sage: print 'f(3,2,z) =', f(x=3,y=2)
sage: eq1=3*x+2/3*y-5*z==0
sage: show(eq1)
sage: v1 = vector( x, y, z )
sage: v2 = vector( 3, 2/3, - 5)
show( v1.dot_product(v2) )
```

Chapter 2

Systems of Linear Equations

The key to solving linear systems of equations is to replace a given system by a new, equivalent system that is easier to solve. Since the two systems are equivalent, and since therefore each can be derived from the other, they will have the same (possibly empty) set of solutions. Gaussian Elimination uses three operations that are reversible to obtain such a simpler system.

2.1 Introduction to Gaussian Elimination and Backsubstitution

We will illustrate the operations with two simple examples.

2.1.1 An Equation for Every Unknown

Consider the following system

$$(\xi) \Leftrightarrow \begin{cases} x & + & 2y & + & z & = & 1 & (\xi_1) \\ 2x & + & 4y & + & 4z & = & 6 & (\xi_2) \\ & & y & + & z & = & 3 & (\xi_3) \end{cases} \tag{2.1}$$

Row Combination operation: We may replace any one equation by adding some scalar multiple of another equation to it. In the above system, replace Eq (ξ_2) with a new equation $(\xi_4) = (\xi_2) - 2(\xi_1)$. The resulting system is

$$(\xi) \Leftrightarrow \begin{cases} x & + & 2y & + & z & = & 1 & (\xi_1) \\ & & & & 2z & = & 4 & (\xi_4) \\ & & y & + & z & = & 3 & (\xi_3) \end{cases} \tag{2.2}$$

The systems are equivalent: we may recover equation $(\xi_2) = (\xi_4) + 2(\xi_1)$. The net effect of the operation was to eliminate the variable x from all but the first

equation. The remaining equations again form a system of linear equations, but with one less variable than before.

Equation interchange operation: the order in which the equations are listed has no effect on the solutions. We can reorder the equations above to obtain

$$(\xi) \Leftrightarrow \begin{cases} x & + & 2y & + & z & = & 1 & & (\xi_1) \\ & & y & + & z & = & 3 & & (\xi_3) \\ & & & & 2z & = & 4 & & (\xi_4) \end{cases} \tag{2.3}$$

In this form, the system of equations has the following characteristics: Each equation has a different leading variable. Further, the leading variable no longer appears in any one equation further down. This type of system is very easy to solve. Before doing so, we will however introduce one more operation.

Scaling operation: Any equation may be scaled by a non-zero scalar. In the system above, we scale the last Eq (ξ_4) by a factor $\frac{1}{2}$, thereby ensuring the coefficient of each of the leading variables to be equal to 1.

$$(\xi) \Leftrightarrow \begin{cases} x & + & 2y & + & z & = & 1 & & (\xi_1) \\ & & y & + & z & = & 3 & & (\xi_3) \\ & & & & z & = & 2 & & (\xi_5) \end{cases} \tag{2.4}$$

Backsubstitution Method: The solution of this system is trivial - start at the last equation, and work up, substituting values for each of the variables obtained up to then.

$$\left.\begin{array}{r} (\xi_5) \Rightarrow z = 2 \\ (\xi_3) \end{array}\right\} \Rightarrow y = 1 \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} \Rightarrow x = -3 \tag{2.5}$$

$$(\xi_1)$$

2.1.2   Fewer Equations than Unknowns

Consider the following system which apparently consists of 4 equations in 4 unknowns.

$$(\zeta) \Leftrightarrow \begin{cases} 5x & + & y & & & - & 7w & = & 7 & & (\zeta_1) \\ -10x & - & 2y & + & 2z & + & 14w & = & -12 & & (\zeta_2) \\ 20x & + & 4y & - & 9z & - & 28w & = & 19 & & (\zeta_3) \\ 15x & + & 3y & + & 9z & - & 21w & = & 30 & & (\zeta_4) \end{cases} \tag{2.6}$$

In the above system, use Eq (ζ_1) to eliminate x from the remaining equations. The resulting system

$$(\zeta) \Leftrightarrow \begin{cases} 5x & + & y & + & & - & 7w & = & 7 & & (\zeta_5) \\ & & & & 2z & & & = & 2 & & (\zeta_6) \\ & & & - & 9z & & & = & -9 & & (\zeta_7) \\ & & & & 9z & & & = & 9 & & (\zeta_8) \end{cases} \tag{2.7}$$

has a first equation that will be used to solve for x, and the remaining Eqs $(\zeta_6, \zeta_7, \zeta_8)$ should form a new system of 3 equations in the remaining variables. However, the y and w variables both have dropped out. Fortunately, the remaining system of 3 equations in 1 unknown really consists of only one equation, rewritten 3 times with different scale factors applied. We can use Eq (ζ_6) to eliminate z from the remaining equations, and scale Eq (ζ_6) by $\frac{1}{2}$ to get

$$(\zeta) \Leftrightarrow \begin{cases} 5x & + & y & + & & - & 7w & = & 7 & (\zeta_9) \\ & & & + & z & & & = & 1, & (\zeta_{10}) \end{cases} \qquad (2.8)$$

an *equivalent system of only 2 equations in 4 unknowns.* Eq (ζ_9) can be used to solve for x, while Eq (ζ_{10}) fixes z, but we have no equations to constrain y or w: we are free to choose any values for them. Variables that are unconstrained in this fashion are called **free variables**, while variables we can solve for are called **basic variables**.. In this example, the basic variables are x and z, while y and w are free. To write down a solution for each of the variables, introduce parameters $y = \alpha$, $w = \beta$, to get

$$(\zeta) \Leftrightarrow \begin{cases} x & = \frac{7}{5} - \frac{1}{5}\alpha + \frac{7}{5}\beta \\ y & = \alpha \\ z & = 1 \\ w & = \beta. \end{cases} \qquad (2.9)$$

2.2 Elementary Operations in Matrix Form

We now take the crucial step of reformulating the problem and the elementary operations in matrix form.

We have seen in example 1.2.8 that a **system of linear equations** can be represented in the form

$$Ax = b,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_M \end{pmatrix}.$$

Each of the elementary operations can be accomplished by multiplying the equation $Ax = b$ with a suitably chosen matrix E from the left.

The existence of the inverse E^{-1} of the matrix E is required for the systems of equations to be equivalent. The application of E to a system of equations can be "undone" by applying E^{-1} to the result

$$\begin{aligned} Ax = b \Rightarrow & \quad EAx = Eb \\ \Rightarrow & \quad E^{-1}EAx = E^{-1}Eb \\ \Rightarrow & \quad Ax = b. \end{aligned} \qquad (2.10)$$

2.2.1   Row Combination Matrix

The matrix E that replaces row i of a matrix A of size $M \times N$ with the sum of row i and some other row j scaled by a constant α is obtained from the identity matrix I of size $M \times M$ by replacing entry I_{ij} by α.[1] The inverse of E is obtained by replacing α with $-\alpha$ in E. The general form of the matrix is

$$
E = \begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & \alpha & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & I
\end{pmatrix}
\tag{2.11}
$$

with row i shown in color.

Example 2.2.1. 4×4 *Row combination matrix*
The matrix E

$$
E = \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hline
0 & 3 & 0 & 1
\end{array} \right),
\tag{2.12}
$$

applied to a matrix A of size $4 \times N$ replaces the last row of A with 3 times the second row added to the last row. Its inverse is obtained by changing the sign of the off-diagonal element.

$$
E^{-1} = \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hline
0 & -3 & 0 & 1
\end{array} \right).
\tag{2.13}
$$

The reader should choose first compute the product $E^{-1}E$, and then choose some matrix A of size 4×5 and compute the product EA, observing the effect of the colorized entries in E in each case.

Transcribing Eq (2.2) for the simple example above to matrix form, we see that Row Combination matrices can be used to **introduce zeros in some column** k at A_{ik} by using a non-zero value entry A_{jk} in the same column. A value A_{jk} used in such a fashion is called a **pivot**).

---

[1] Elementary row combination matrices have determinant $|E| = 1$.

Example 2.2.2. *Elimination matrix*
Consider the matrix

$$
A = \left(\begin{array}{cccc|c|c}
x & x & x & x & x & x \\
x & x & x & x & \mathbf{\color{red}2} & x \\
x & x & x & x & x & x \\
x & x & x & x & \mathbf{\color{blue}6} & x
\end{array} \right),
\tag{2.14}
$$

where the entries x are any arbitrary scalars, and suppose we want to zero out A_{45} (the value 6), using the "pivot" entry A_{25} (the entry 2). The elimination matrix that accomplishes this is

$$
E = \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\ \hline
0 & -\frac{\mathbf{\color{blue}6}}{\mathbf{\color{red}2}} & 0 & 1
\end{array} \right),
\tag{2.15}
$$

where the fourth row describes the operations on the rows of A required to obtain a new fourth row of A. Note that the pivot is in row 2. Accordingly, the non-zero off-diagonal entries in the elimination matrix appear in column 2.

The fraction has not been simplified to clearly show how it is obtained: the entry $-\frac{6}{2}$ multiplies the pivot row. In particular, it will multiply the pivot value $\mathbf{\color{red}2}$, resulting in the negative of the entry 6 we want to eliminate. The entry 1 multiplies the last row, and hence the entry 6 beneath the pivot. The addition of the two rows therefore cancels the entry 6.[a] The reader should compute the product EA and carefully note the interaction of the colorized numerical values.

☞ *Note that E is a square matrix: it describes the coefficients to be applied to each row in A to produce the same number of rows. The matrix A however does not need to be square.*

---

[a]Note also that all entries in the fourth row are affected by the addition. If all entries to the left of the pivot in the pivot row happen to be zero, the corresponding entries in the fourth row remain the same. In Gaussian elimination, we will arrange that these two sets of entries are 0: once zeroed out, the elimination cannot accidentally reintroduce non-zero values.

2.2.2 Generalized Elimination Matrix

Elimination matrices are frequently combined to zero additional entries in a given column of A using the same pivot. By introducing matrix partitions to clearly identify the components of the column of A of interest, let

$$
A = \left(\begin{array}{c|c|c}
A_{11} & u & A_{13} \\ \hline
A_{21} & p & A_{23} \\ \hline
A_{31} & l & A_{33}
\end{array} \right),
\tag{2.16}
$$

where p is the pivot (a 1×1 submatrix), u is the column vector above the pivot and l is the column vector below the pivot. The elimination matrix

$$
E = \left(
\begin{array}{c|c|c}
I & -\dfrac{1}{p}u & 0 \\
0 & 1 & 0 \\
0 & -\dfrac{1}{p}l & I
\end{array}
\right)
\tag{2.17}
$$

replaces u and l in matrix A with zeros.[2] This generalized elimination matrix is the product of individual elimination matrices taken in any order, since these simple matrices do commute.

- ☞ Note that we can choose to eliminate any subset of the entries above and/or below the pivot value by setting the unneeded entries in u and l to zero.

  The generalized elimination matrix for a given column is the product of the constituent elementary row combination matrices in any order.

- ☞ The elimination matrix E is a square matrix with the same number of rows as the matrix A we wish to apply it to. The matrix A does not need to be square.

- ☞ To construct E for a pivot located in the ith row of A, start with the identity matrix, and replace the ith column of E with the elimination values for the pivot column in A.

- ☞ Generalized elimination matrices are frequently combined with generalized scaling matrices defined below in order to save steps or to clear the pivot from the denominator in hand calculations.

Note also that Eq (2.17) is readily adjusted if we wish to introduce zeros in the first or last column of A: simply omit the first or last column of E respectively.

Since the elimination matrix was constructed to produce an equivalent system of equations, it must be invertible. The inverse is

$$
E^{-1} = \left(
\begin{array}{c|c|c}
I & \dfrac{1}{p}u & 0 \\
0 & 1 & 0 \\
0 & \dfrac{1}{p}l & I
\end{array}
\right)
\tag{2.18}
$$

- ☞ The inverse E^{-1} of a generalized elimination matrix E is obtained by simply changing the signs of the off-diagonal elements.

---

[2] The determinant of a generalized elimination matrix E is $|E| = 1$.

Example 2.2.3. *Generalized elimination matrix*
Consider the matrix

$$A = \begin{pmatrix} x & x & x & x & \textcolor{cyan}{\mathbf{x}} & x \\ x & x & x & x & \textcolor{red}{\mathbf{2}} & x \\ x & x & x & x & \textcolor{cyan}{\mathbf{10}} & x \\ x & x & x & x & \textcolor{cyan}{\mathbf{6}} & x \end{pmatrix}, \tag{2.19}$$

where the entries x are any arbitrary scalars. The generalized elimination matrix that introduces zeros in the A_{35}, A_{45} positions, using entry A_{25} (the pivot entry $\textcolor{red}{\mathbf{2}}$), is

$$E = \begin{pmatrix} 1 & \textcolor{cyan}{\mathbf{0}} & 0 & 0 \\ 0 & \textcolor{cyan}{\mathbf{1}} & 0 & 0 \\ 0 & \textcolor{cyan}{\mathbf{-5}} & 1 & 0 \\ 0 & \textcolor{cyan}{\mathbf{-3}} & 0 & 1 \end{pmatrix}. \tag{2.20}$$

Its inverse is obtained by changing the signs of the off-diagonal terms,

$$E^{-1} = \begin{pmatrix} 1 & \textcolor{cyan}{\mathbf{0}} & 0 & 0 \\ 0 & \textcolor{cyan}{\mathbf{1}} & 0 & 0 \\ 0 & \textcolor{cyan}{\mathbf{5}} & 1 & 0 \\ 0 & \textcolor{cyan}{\mathbf{2}} & 0 & 1 \end{pmatrix}. \tag{2.21}$$

Note again that the width (i.e., the number of columns) of E and A need not be the same since E is a square matrix, but A need not be: the pivot in row $i = 2$ of matrix A determines the location of the non-zero column $j = 2$ in the elimination matrix E.

The reader should compute the products $B = EA$ and $C = E^{-1}B$ and observe the action of the colorized entries.

2.2.3 Row Exchange

The matrix that exchanges two given rows of some matrix A is a special case of a **Permutation Matrix**. To interchange row i and row j in a matrix A of size M by N, form the identity matrix of size $M \times M$, and interchange the rows i and j. *The row exchange matrix we have constructed is symmetric*, and hence **this matrix is its own inverse**.[3] This is not surprising: suppose we use E to interchange two rows in A. If we interchange the same two rows in the resulting matrix once again (which can be accomplished by multiplying with E), we recover the original matrix A: $E(EA) = A$. The two consecutive applications of E cancel out. To obtain the formula we want, choose $A = I$, which yields $E^2 = I$.

---

[3]More generally, the inverse of a permutation matrix P is $P^{-1} = P^t$. The determinant of a permutation matrix that only exchanges two rows is $|E| = -1$.

Example 2.2.4. *Row exchange permutation matrix*
Assume we are given a matrix A of size $5 \times N$. To interchange rows 2 and 4 of A, we compute EA, where

$$E = \left(\begin{array}{c|ccc|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right), \tag{2.22}$$

is the 5x5 identity matrix with rows 2 and 4 interchanged. As expected $E^2 = I$. Since $E^{-1} = E$, multiplication of a system $Ax = b$ yields an equivalent system of equations, as shown in Eq (2.10).

2.2.4 Scaling

The matrix E that applies a constant multiplier α to row i of a matrix A of size $M \times N$ is obtained from the identity matrix I by replacing entry I_{ii} with α.

Example 2.2.5. *Elementary scaling matrix*
Assume we are given a matrix A of size $3 \times N$. To scale the third row of A by -6, we compute EA, where

$$E = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{array} \right). \tag{2.23}$$

This is a special case of a diagonal matrix, with all but one of the diagonal entries equal to 1. We often combine scaling matrices for different rows. Scaling matrices are diagonal.[4] The inverse of a diagonal matrix is obtained by inverting each of the diagonal entries.

Example 2.2.6. *Generalized scaling matrix*
The matrix

$$E = \left(\begin{array}{ccc} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -6 \end{array} \right) \tag{2.24}$$

applied to a matrix A from the left scales the first row of A by 2, the second row by -1, and the third row by -6. The inverse of E is

$$E^{-1} = \left(\begin{array}{ccc} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{6} \end{array} \right). \tag{2.25}$$

Multiplication of a system $Ax = b$ with E yields an equivalent system.

---

[4]The determinant of a diagonal matrix is the product of the diagonal entries.

2.3 Computational Layout

The simple example in section 2.1 consists of an elimination, row interchange and scaling step. Transcribing the example to matrix form, the original system Eq (2.1) is given by

$$(\xi) \Leftrightarrow Ax = b, \quad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 4 \\ 0 & 1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix},$$

with the original variables x, y, z renamed x_1, x_2, x_3 to avoid confusion with the column vector x in $Ax = b$.

To simplify the system, we chose the pivot A_{11} in Eq (2.2), i.e., we decided to use the first equation to solve for the variable x_1, and used it to eliminate the x_1 variable from the remaining equations.

The required elimination matrix E_1 is applied to the system to obtain a new, equivalent system

$$\begin{aligned} (\xi) \Leftrightarrow & E_1 Ax = E_1 b \\ \Leftrightarrow & A_1 x = b_1, \quad \text{where we have set } A_1 = E_1 A, \quad b_1 = E_1 b. \end{aligned}$$

The elimination matrix E and the resulting matrices A_1 and b_1 are given by

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

The row interchange matrix E_2 and scaling matrix E_3 in turn yield

$$\begin{aligned} (\xi) \Leftrightarrow & \quad E_2 A_1 x = E_2 b_1 \\ \Leftrightarrow & \quad A_2 x = b_2, \quad \text{with } A_2 = E_2 A_1, \quad b_2 = E_2 b_1 \\ \Leftrightarrow & \quad E_3 A_2 x = E_3 b_2 \\ \Leftrightarrow & \quad A_3 x = b_3, \quad \text{with } A_3 = E_3 A_2, \quad b_3 = E_3 b_2. \end{aligned}$$

The resulting matrices are given by

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

This final equivalent system is transcribed back into equation form and solved by back-substitution.

☞ Note that if we define a new matrix $(A|b)$ by appending a column b to the original matrix A, the product $E_1(A|b) = (E_1A|E_1b)$ produces the required submatrices A_1 and b_1 in $A_1x = b_1$.

The same observation holds for each succeeding step.

The computations required can therefore be laid out by augmenting A with the right hand side b, and multiplying from the left by each of the E_k matrices in turn in a vertical layout. The following matrix multiplications

$$
\begin{array}{ll}
& (\qquad\quad A|\qquad\quad b\,) \\
(\,E_1\,) & (\qquad E_1A|\qquad E_1b\,) \\
(\,E_2\,) & (\quad E_2E_1A|\quad E_2E_1b\,) \\
(\,E_3\,) & (\,E_3E_2E_1A|\,E_3E_2E_1b\,)
\end{array}
\tag{2.26}
$$

represent the derivation

$$
\begin{array}{llll}
Ax = b & \Leftrightarrow & E_1Ax = E_1b, & A_1 = E_1A,\ b_1 = E_1b, & \mathcal{E}_1 = E_1 \\
& \Leftrightarrow & E_2E_1Ax = E_2E_1b, & A_2 = E_2A_1, b_2 = E_2b_1, & \mathcal{E}_2 = E_1E_2 \\
& \Leftrightarrow & E_3E_2E_1Ax = E_3E_2E_1b, & A_3 = E_3A_2, b_3 = E_3b_2, & \mathcal{E}_3 = E_3E_2E_1
\end{array}
$$

with each of the elementary operation matrices appearing in the column on the left, and the resulting equivalent system of equations in the augmented matrix on the right[5] **We will maintain the naming and indexing convention for the E_k, $\mathcal{E}_k$ and A_k matrices throughout this text.**

☞ Note that we can run the problem backwards: to go from level k to level $k-1$, multiply by E_k^{-1}, since $A_{k-1} = E_k^{-1}A_k$.

☞ Note the partitioning of the augmented matrix. The elimination matrices E_k depend only on the A_k matrices, not on the b_k If we repeat the problem for a new right hand side, the E_k matrices do not change: we will carry out the exact same computations.

Rather than rewriting the whole layout, we may simply augment the system with the new right hand side. This in effect replaces the vector b with a matrix B (Note this is also the setup for the problem $AX = B$).

☞ By rewriting the matrix problem $AX = B$ in the vertical layout form

$$
\left.\begin{array}{c}
(\,X\,) \\
(\,A\,)\,(\,B\,)
\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}
(\,X_1\ X_2\ \cdots\,) \\
(\,A\,)\,(\,B_1\ B_2\ \cdots\,)
\end{array}\right.
\tag{2.27}
$$

and partitioning B into columns, we see that the columns of X are the solutions of the $Ax = b$ problems for the corresponding columns of B.

The special case $B = I$ sets up the computation for the right inverse of A. The rightmost column in the layout reduces to the $\mathcal{E}_k = E_kE_{k-1}\cdots E_2E_1$. This is the matrix that takes us from A to A_k directly: $A_k = \mathcal{E}_kA$.

---

[5]If the matrices E_k were not invertible, the derivation would still hold, but we would lose equivalence (see Eq (2.10).

Example 2.3.1. *Simple Gaussian Elimination example in layout form*
Summarizing the discussion of the previous pages, the simple example 2.1 there-fore looks as follows

$$
\left(\begin{array}{ccc|c} \mathbf{1} & 2 & 1 & 1 \\ 2 & 4 & 4 & 6 \\ 0 & 1 & 1 & 3 \end{array} \right)
$$

Step 0: *Begin by aug-menting the matrix A with b. Choose the first pivot*

$$
\left(\begin{array}{ccc} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ \mathbf{0} & 0 & 2 & 4 \\ \mathbf{0} & 1 & 1 & 3 \end{array} \right)
$$

Step 1: *There is no pivot for the second variable on the second row. Interchange rows.*

$$
\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 \end{array} \right) \left(\begin{array}{ccc|c} \mathbf{1} & 2 & 1 & 1 \\ \mathbf{0} & 1 & 1 & 3 \\ \mathbf{0} & 0 & \mathbf{2} & 4 \end{array} \right)
$$

Step 2: *We now have a pivot in every row. Next, scale the last pivot to 1.*

$$
\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{array} \right) \left(\begin{array}{ccc|c} \mathbf{1} & 2 & 1 & 1 \\ \mathbf{0} & 1 & 1 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 2 \end{array} \right)
$$

The system can now be solved by back-substitution.

Transcribing to equation form, we get Eq (2.4) and the solution Eq (2.5) as before.

☞ *Computations by hand are carried out efficiently row by row, e.g., the second row of A_1 is minus 2 times the first row of A plus the second row.*

2.4 The Gaussian Elimination Method

In the example above, we have used elementary operation matrices to reduce the system of equations to an equivalent system such that each equation has a different leading variable.

The solution method is know as the **Gaussian Elimination Algorithm**:

1 Choose an equation with a non-zero coefficient for the leading variable. If need be, interchange the order of the equations to bring the chosen equation to the top of the system.

2 Use the pivot value (i.e., the coefficient of the leading variable of the cho-sen equation), to eliminate the variable from all equations below. The equation containing the variable in question is now dedicated to the com-putation of its value. Variables associated with a pivot are called **basic variables**.

3 The remaining equations form a new system of equations with one less variable than before. Proceed as before by returning to step 1 applied to this new, smaller system.

The final system can then easily be solved by back-substitution.

Rephrased in terms of the augmented matrix $(A|b)$ for A of size $M \times N$ the **Gaussian Elimination algorithm** in pseudo-code is

for row = 1 to M, column = 1 to N :
    find a pivot (any non-zero entry on or below the current row/column)
    if there is a pivot on or below the current row:
        interchange the pivot row and the current row if necessary
        eliminate all entries in the current column below the pivot
        go to the next row and next column.
    otherwise: go to the next column

☞ The algorithm results in a final matrix A_k in **row echelon form** (see section 1.4) after a finite number of steps k. If the matrix A is square and if there is a pivot in every column, A_k is **triangular**.

☞ The pivots are chosen from A only. The entries in b are not involved

☞ The order of the operations in the algorithm is carefully chosen to guarantee that later operations cannot reintroduce non-zero values for entries that have been zeroed.

☞ The algorithm does not require elementary scale operations: they can be used for convenience or for numerical reasons.

Example 2.4.1. *Gaussian Elimination with free variables, no solution*
Consider the simple example (2.6) with a different right hand side

$$
\begin{pmatrix}
5 & 1 & 0 & -7 & | & 1 \\
-10 & -2 & 2 & 14 & | & 2 \\
20 & 4 & -9 & -28 & | & 1 \\
15 & 3 & 9 & -21 & | & 0
\end{pmatrix}
$$

Step 0: *Represent* $Ax = b$ *in augmented form* $(A|b)$; *identify the first pivot* **5**.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
-4 & 0 & 1 & 0 \\
-3 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
5 & 1 & 0 & -7 & | & 1 \\
0 & 0 & 2 & 0 & | & 4 \\
0 & 0 & -9 & 0 & | & -3 \\
0 & 0 & 9 & 0 & | & -3
\end{pmatrix}
$$

Step 1: GE *step, zero first column below pivot* **5**; *find* E_1 $(A_1|b_1)$; *next pivot* **2**

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \frac{9}{2} & 1 & 0 \\
0 & -\frac{9}{2} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
5 & 1 & 0 & -7 & | & 1 \\
0 & 0 & 2 & 0 & | & 4 \\
0 & 0 & 0 & 0 & | & 15 \\
0 & 0 & 0 & 0 & | & -21
\end{pmatrix}
$$

Step 2: GE *step, zero third column below pivot* **2**; *Contradiction* $0 = 15$: *no solution*

In this example, the second (as well as the fourth) column have no pivots. The third and the fourth equation in the final system, $0 = 15$ *and* $0 = -21$ *are contradictions: since the initial and final systems are equivalent, they have no solution.*

Note that the last row in the above example need not have been computed: once any one row results in a contradiction, the system is shown not to have any solutions, and further computations are unnecessary.

Example 2.4.2. *Gaussian Elimination with free variables, infinite number of solutions*

Transcribing the simple example (2.6) which differs from the previous example only by the right hand side b, we have

$$
\left(\begin{array}{cccc|c}
5 & 1 & 0 & -7 & 7 \\
-10 & -2 & 2 & 14 & -12 \\
20 & 4 & -9 & -28 & 19 \\
15 & 3 & 9 & -21 & 30
\end{array}\right)
$$

Step 0: *Represent* $Ax = b$ *in augmented form* $(A|b)$*; identify the first pivot* **5***.*

$$
\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
-4 & 0 & 1 & 0 \\
-3 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
5 & 1 & 0 & -7 & 7 \\
0 & 0 & 2 & 0 & 2 \\
0 & 0 & -9 & 0 & -9 \\
0 & 0 & 9 & 0 & 9
\end{array}\right)
$$

Step 1: GE *step, zero first column below pivot* **5***; find* E_1 $(A_1|b_1)$*; next pivot* **2**

$$
\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \frac{9}{2} & 1 & 0 \\
0 & -\frac{9}{2} & 0 & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
5 & 1 & 0 & -7 & 7 \\
0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Step 2: GE *step, zero third column below pivot* **2***; In row echelon form*

Note that none of the elimination matrices change compared to the previous example, since they are independent of the right hand side.

Transcribing to equation form and solving for the basic variables, we find $x_1 = \frac{1}{5}(7 - x_2 + 7x_4)$, $x_3 = 1$, *with no equation for the free variables* x_2 *and* x_3*: they are unconstrained and can take on any value. Using parameters* α, β *for the free variables, i.e., setting* $x_2 = \alpha$, $x_4 = \beta$*, we rewrite the solution in matrix form*

$$
\left(\begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{array}\right)
=
\left(\begin{array}{c|cc}
\frac{7}{5} & -\frac{1}{5} & \frac{7}{5} \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right)
\left(\begin{array}{c}
1 \\ \hline
\alpha \\
\beta
\end{array}\right)
$$

$$
=
\left(\begin{array}{c}
\frac{7}{5} \\
0 \\
1 \\
0
\end{array}\right)
+ \alpha
\left(\begin{array}{c}
-\frac{1}{5} \\
1 \\
0 \\
0
\end{array}\right)
+ \beta
\left(\begin{array}{c}
\frac{7}{5} \\
0 \\
0 \\
1
\end{array}\right)
$$

Each of the free variables contributes one vector multiplied by the corresponding parameter as shown explicitly in the customary column view presentation of the solution. The matrix representation is partitioned to separate out the free variable terms. The reader should verify the solution by multiplying from the left by A and checking what happens to the vectors scaled by α *and* β*.*

Observe that the free variable entries in the particular solution are equal to zero, while the free variable entries in the homogeneous solution matrix are equal to I.

2.4.1 Existence of solutions

Consideration of the previous examples shows that the number of solutions of a given system depends on the pivot locations in a row echelon form:

☞ If the system has a pivot in every row as in Example 2.3.1, we can solve for each of the basic variables. The number of solutions then depends on the number of free variables: if there are none, the solution is unique, otherwise we have an infinite number of solutions.

☞ If the system has one or more rows without a pivot, the existence of solutions is determined by the corresponding right hand sides. Examples 2.4.1 and Example 2.4.2 show such cases:

   – if any one such right hand side is not zero (example 2.4.1) the solution method reveals a contradiction, and there is no solution.

   – if all such right hand sides are zero (example 2.4.2) then solutions do exist: the original system is seen to have been made up of redundant equations since Gaussian Elimination yields an equivalent system with fewer equations. Since this equivalent system has a pivot in every row, it has a unique solution or an infinite number of solutions depending on the existence of free variables.

We see that systems that reduce to a row echelon form that has a pivot in every row can be solved for any right hand side vector. Conversely, if a system reduces to a row echelon form that has one or more rows without a pivot, there are right hand sides of the original system that make the system inconsistent. To see this, choose an inconsistent right hand side for the row echelon form of the coefficient matrix, and recover the corresponding right hand side of the original by running the Gaussian Elimination algorithm in reverse.

Example 2.4.3. *Gaussian Elimination, matrix with arbitrary right hand side*

Consider the system

$$
(\zeta) \Leftrightarrow
\begin{cases}
x & & + z & & = b_2 \\
5x & + y & & - 7w & = b_1 \\
11x & + 2y & + z & - 14w & = b_3 \\
-4x & - y & + z & + 7w & = b_4
\end{cases}
$$

Transcribing and reducing the system by Gaussian Elimination, we get

$$
\left(
\begin{array}{cccc|c}
\mathbf{1} & 0 & 1 & 0 & b_1 \\
5 & 1 & 0 & -7 & b_2 \\
11 & 2 & 1 & -14 & b_3 \\
-4 & -1 & 1 & 7 & b_4
\end{array}
\right)
$$

$$
\left(
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\mathbf{-5} & 1 & 0 & 0 \\
\mathbf{-11} & 0 & 1 & 0 \\
\mathbf{4} & 0 & 0 & 1
\end{array}
\right)
\left(
\begin{array}{cccc|c}
\mathbf{1} & 0 & 1 & 0 & b_1 \\
\mathbf{0} & 1 & -5 & -7 & -5b_1 + b_2 \\
\mathbf{0} & 2 & -10 & -14 & -11b_1 + b_3 \\
\mathbf{0} & -1 & 5 & 7 & 4b_1 + b_4
\end{array}
\right)
$$

$$
\left(
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \mathbf{-2} & 1 & 0 \\
0 & \mathbf{1} & 0 & 1
\end{array}
\right)
\left(
\begin{array}{cccc|c}
\mathbf{1} & 0 & 1 & 0 & b_1 \\
0 & \mathbf{1} & -5 & -7 & -5b_1 + b_2 \\
0 & 0 & 0 & 0 & -b_1 - 2b_2 + b_3 \\
0 & 0 & 0 & 0 & -b_1 + b_2 + b_4
\end{array}
\right)
$$

The last two equations in the reduced system are constraints on the right hand side that must be met in order for the system to be consistent. The first two equations allow us to solve for the basic variables x and y in terms of the free variables z and w and the parameters b_1 and b_2.

We obtain the infinite number of solutions

$$
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}
=
\begin{pmatrix} b_1 \\ -5b_1 + b_2 \\ 0 \\ 0 \end{pmatrix}
+ \alpha
\begin{pmatrix} -1 \\ 5 \\ 1 \\ 0 \end{pmatrix}
+ \beta
\begin{pmatrix} 0 \\ 7 \\ 0 \\ 1 \end{pmatrix}
$$

when the right hand sides satisfy $b_3 = b_1 + 2b_2$ and $b_4 = b_1 - b_2$, and no solution otherwise.

Example 2.4.4. *Gaussian Elimination, matrix with parameters*
Gaussian Elimination requires pivots (non-zero entries). If the matrix contains parameters, we must ensure that no pivots that we select are equal to zero. This leads to a discussion of cases for the parameter values that must be investigated separately. Consider the following problem with two parameters k and α:

$$\begin{pmatrix} \mathbf{k} & 2 & \vline & 1 \\ 2 & 1 & \vline & \alpha \end{pmatrix}$$

Step 0: *Identify the first pivot* $\mathbf{k}$*. This assumes $k \neq 0$!*

$$\begin{pmatrix} 1 & 0 \\ -\dfrac{\mathbf{2}}{\mathbf{k}} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{k} & 2 & \vline & 1 \\ \mathbf{0} & \dfrac{\mathbf{k}-\mathbf{4}}{\mathbf{k}} & \vline & \dfrac{k\alpha - 2}{k} \end{pmatrix}$$

Step 1: GE *step, zero first column using pivot* $\mathbf{k}$*; find $E_1\,(A_1|b_1)$*

We have completed Gaussian Elimination under the assumption that $k \neq 0$. We must therefore consider cases:

- *Case $k \neq 0$: For the problem to have a unique solution, we require a pivot for every variable: thus, we further require the entry $\frac{k-4}{k} \neq 0$, i.e., $k \neq 4$. We therefore have two possibilities:*

  - *Subcase $k \neq 4$, unique solution.*
  - *Subcase $k = 4$. The second equation has zeros on the left hand side. If the right hand side is zero as well (i.e., if $\alpha = \frac{1}{2}$, we have an infinite number of solutions. For all other values of α, we have a contradiction, i.e., no solution.*

- *Case $k = 0$: substituting $k = 0$ into the original set of equations yields a system that can be shown to have a unique solution for all values of α.*

Summarizing, we have found the following set of possibilities:

$k = 4$	$\alpha = \frac{1}{2}$	infinite number of solutions
$k = 4$	$\alpha \neq \frac{1}{2}$	no solution
$k \neq 4$	any α	unique solution

Note we could have saved ourselves some work if we had first interchanged the order of the equations: the case $k = 0$ would not have arisen as a case to be discussed separately.

Note also that we first look at the pivots, not the right hand side!

2.5   The Gauss-Jordan Elimination Method

A modification of the Gaussian Elimination method is to try and avoid back-substitution by simplifying the system further: use elimination matrices to zero out all entries above as well as below the pivot in a given column.

The elimination above the pivot may be carried out at the same time as the elimination below, or may follow after the Gaussian Elimination algorithm

completes.

The last step in the Gauss-Jordan algorithm is to scale all pivots to 1. The final matrix A_k is in reduced row echelon form. For square matrices with pivots in every row, we obtain $(A_k|b_k) = (I|b_k)$. Transcribing to equation form, we see that the solution of such a system may be obtained directly: $x = b_k$.

Example 2.5.1. *Gauss-Jordan elimination, unique solution*
*In the following example, we **immediately zero elements above and below each pivot**. Given the system*

$$Ax = b \Leftrightarrow \begin{cases} x & + & 3y & + & 2z & = & 13 \\ 2x & + & 7y & + & 5z & = & 30 \\ x & & 4y & + & 4z & = & 18 \end{cases}$$

Gauss-Jordan elimination yields

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 2 & 7 & 5 & 30 \\ 1 & 4 & 4 & 18 \end{array} \right)$$ **Step 0:** *set up the augmented matrix for the system. Choose the first pivot* **1**.

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{array} \right) \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \end{array} \right)$$ **Step 1:** *Eliminate below the first pivot. Choose the pivot* **1** *in the second row.*

$$\left(\begin{array}{ccc} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{array} \right) \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right)$$ **Step 2:** *Eliminate in the second column. Pick pivot* **1** *in the third column.*

$$\left(\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right) \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array} \right)$$ **Step 3:** *Eliminate above the third pivot* **1**. *Since all pivots equal 1, no scaling is necessary.*

There are no free variables. Transcribing the final system to equation form, we immediately get the unique solution

$$\left(\begin{array}{c} x \\ y \\ z \end{array} \right) = \left(\begin{array}{c} 2 \\ 3 \\ 1 \end{array} \right)$$

Since there are no free variables, Gauss-Jordan elimination has reduced the matrix A to the identity. The elementary matrix product $\mathcal{E}_k = E_k E_{k-1} \cdots E_1$ where k is the number of steps required by the algorithm, thus builds up a left inverse of the matrix A. In equation form, the above computations are $Ax = b \Rightarrow \mathcal{E}_k A x = \mathcal{E}_k b \Leftrightarrow x = \mathcal{E}_k b$ since $\mathcal{E}_k A = I$.

Example 2.5.2. *Gauss-Jordan elimination, unique solution*
In the following example, we **start with Gaussian Elimination, followed by Jordan elimination** *above each pivot.*

$$
\left(\begin{array}{cccc|c}
\mathbf{2} & 2 & -1 & 1 & 10 \\
-4 & -5 & 2 & -1 & -18 \\
4 & 5 & 1 & 1 & 15 \\
-6 & -10 & -6 & -1 & -19
\end{array}\right)
$$

Step 0: *Represent* $Ax = b$ *in augmented form* $(A|b)$*; identify the first pivot* **2**.

$$
\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\mathbf{2} & 1 & 0 & 0 \\
-\mathbf{2} & 0 & 1 & 0 \\
\mathbf{3} & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
\mathbf{2} & 2 & -1 & 1 & 10 \\
\mathbf{0} & -\mathbf{1} & 0 & 1 & 2 \\
\mathbf{0} & 1 & 3 & -1 & -5 \\
\mathbf{0} & -4 & -9 & 2 & 11
\end{array}\right)
$$

Step 1: GE *step, zero first column using pivot* **2**; *find* E_1 $(A_1|b_1)$

$$
\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \mathbf{1} & 1 & 0 \\
0 & -\mathbf{4} & 0 & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
2 & 2 & -1 & 1 & 10 \\
0 & -\mathbf{1} & 0 & 1 & 2 \\
0 & \mathbf{0} & 3 & 0 & -3 \\
0 & \mathbf{0} & -9 & -2 & 3
\end{array}\right)
$$

Step 2: GE *step, zero second column using pivot* **-1**; *find* E_2 $(A_2|b_2)$

$$
\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \mathbf{3} & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
2 & 2 & -1 & 1 & 10 \\
0 & -1 & 0 & 1 & 2 \\
0 & 0 & \mathbf{3} & 0 & -3 \\
0 & 0 & \mathbf{0} & -\mathbf{2} & -6
\end{array}\right)
$$

Step 3: GE *step, zero third column using pivot* **3**; *find* E_3 $(A_3|b_3)$; **in row echelon form**

$$
\left(\begin{array}{cccc}
1 & \mathbf{2} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
\mathbf{2} & \mathbf{0} & -1 & 3 & 14 \\
0 & -\mathbf{1} & 0 & 1 & 2 \\
0 & 0 & 3 & 0 & -3 \\
0 & 0 & 0 & -2 & -6
\end{array}\right)
$$

Step 4: GJ *step, zero column above second pivot* **-1**; *find* E_4 $(A_4|b_4)$

$$
\left(\begin{array}{cccc}
1 & 0 & \frac{1}{3} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
\mathbf{2} & 0 & \mathbf{0} & 3 & 13 \\
0 & -\mathbf{1} & \mathbf{0} & 1 & 2 \\
0 & 0 & \mathbf{3} & 0 & -3 \\
0 & 0 & 0 & -\mathbf{2} & -6
\end{array}\right)
$$

Step 5: GJ *step, zero column above third pivot* **3**; *find* E_5 $(A_5|b_5)$

$$
\left(\begin{array}{cccc}
1 & 0 & 0 & \frac{3}{2} \\
0 & 1 & 0 & \frac{1}{2} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{cccc|c}
\mathbf{2} & 0 & 0 & \mathbf{0} & 4 \\
0 & -\mathbf{1} & 0 & \mathbf{0} & -1 \\
0 & 0 & \mathbf{3} & \mathbf{0} & -3 \\
0 & 0 & 0 & -\mathbf{2} & -6
\end{array}\right)
$$

Step 6: GJ *step, zero column above fourth pivot* **-2**; *find* E_6 $(A_6|b_6)$

$$
\left(\begin{array}{cccc}
\frac{1}{2} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & -\frac{1}{2}
\end{array}\right)
\left(\begin{array}{cccc|c}
\mathbf{1} & 0 & 0 & 0 & 2 \\
0 & \mathbf{1} & 0 & 0 & 1 \\
0 & 0 & \mathbf{1} & 0 & -1 \\
0 & 0 & 0 & \mathbf{1} & 3
\end{array}\right)
$$

Step 7: GJ *step, scale the pivots; find* E_7 $(A_7|b_7)$; **in reduced row echelon form.**

As before, we immediately read out the unique solution $x = (2\ 1\ -1\ 3)^t$.

Example 2.5.3. *Gauss-Jordan elimination, infinite number of solutions*

Apply Gauss-Jordan elimination to the following augmented system

$$
\left(
\begin{array}{ccccc|c}
\mathbf{3} & -3 & 1 & 4 & 2 & 9 \\
6 & -6 & 1 & 7 & 5 & 15 \\
-3 & 3 & -2 & -5 & 0 & -13
\end{array}
\right)
$$

Step 0: *set up the augmented matrix* $(A|b)$ *for the system. Choose the first pivot* **3**.

$$
\left(
\begin{array}{ccc}
1 & 0 & 0 \\
-\mathbf{2} & 1 & 0 \\
\mathbf{1} & 0 & 1
\end{array}
\right)
\left(
\begin{array}{ccccc|c}
\mathbf{3} & -3 & 1 & 4 & 2 & 9 \\
\mathbf{0} & 0 & -\mathbf{1} & -1 & 1 & -3 \\
\mathbf{0} & 0 & -1 & -1 & 2 & -4
\end{array}
\right)
$$

Step 1: *Eliminate below the first pivot. Choose the pivot* $-\mathbf{1}$ *in the* 2nd *row.*

$$
\left(
\begin{array}{ccc}
1 & \mathbf{1} & 0 \\
0 & 1 & 0 \\
0 & -\mathbf{1} & 1
\end{array}
\right)
\left(
\begin{array}{ccccc|c}
3 & -3 & \mathbf{0} & 3 & 3 & 6 \\
0 & 0 & -\mathbf{1} & -1 & 1 & -3 \\
0 & 0 & \mathbf{0} & 0 & 1 & -1
\end{array}
\right)
$$

Step 2: *Eliminate above and below the* 2nd *pivot. Choose pivot* **1** *in the* 3rd *row.*

$$
\left(
\begin{array}{ccc}
1 & 0 & -\mathbf{3} \\
0 & 1 & -\mathbf{1} \\
0 & 0 & 1
\end{array}
\right)
\left(
\begin{array}{ccccc|c}
3 & -3 & 0 & 3 & \mathbf{0} & 9 \\
0 & 0 & -1 & -1 & \mathbf{0} & -2 \\
0 & 0 & 0 & 0 & \mathbf{1} & -1
\end{array}
\right)
$$

Step 3: *Eliminate above* 3rd *pivot.*

$$
\left(
\begin{array}{ccc}
\tfrac{1}{3} & 0 & 0 \\
0 & -\mathbf{1} & 0 \\
0 & 1 & 1
\end{array}
\right)
\left(
\begin{array}{ccccc|c}
\mathbf{1} & -1 & 0 & 1 & 0 & 3 \\
0 & 0 & \mathbf{1} & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & \mathbf{1} & -1
\end{array}
\right)
$$

Step 4: *Scale pivots to 1 to obtain the reduced row-echelon form.*

Setting the free variables $x_2 = \alpha, x_4 = \beta$ *and solving, we obtain*

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}
=
\left(
\begin{array}{c|cc}
3 & 1 & -1 \\
0 & 1 & 0 \\
2 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{array}
\right)
\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}
$$

$$
=
\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}
+ \alpha
\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \beta
\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}
$$

The reader should verify the solution by multiplying it by A from the left. Observe in particular that the vectors multiplied by α and β drop out: they are solutions of $Ax = 0$. The solution can be obtained in three steps that are easy to express in computer code. First, set all free variables to zero and solve $Ax = b$ for the first term in the solution. Next, set each of the free variables to 1, one at a time, and solve the homogeneous system $Ax = 0$ to obtain the remaining terms. I.e., setting $x_2 = \alpha = 0, x_4 = \beta = 0$ yields the first vector, setting $x_2 = 1, x_4 = 0$ yields the term scaled by α, and $x_2 = 0, x_4 = 1$ yields the term scaled by β.

When the Gauss-Jordan elimination method results in a unique solution, we can read the result right out of the layout. This raises the question whether we can do the same when we find an infinite number of solutions. To see how this may work, consider reordering the variables x_i in the system of equations $Ax = b$ such that all free variables occur after the basic variables. This is achieved by applying a suitably chosen permutation matrix P to rewrite the system as $Ax = b \Leftrightarrow (AP)(P^t x) = b$.[6] The systems of equations are equivalent since $PP^t = I$. Introducing a partition to separate the basic variables from the free variables, we see that after applying the Gauss-Jordan elementary operation matrices $\mathcal{E}_k = E_k E_{k-1} \cdots E_1$ to obtain the final equivalent system where $\tilde{A} = \mathcal{E}_k A$, $\tilde{b} = \mathcal{E}_k b$ after k steps

$$
\begin{aligned}
Ax = b \quad \Leftrightarrow \quad & \tilde{A}x = \tilde{b} && \text{upon completion of Gauss-Jordan elimination} \\
\Leftrightarrow \quad & \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_p \\ x_f \end{pmatrix} = \begin{pmatrix} \tilde{b}_p \\ 0 \end{pmatrix} && \text{after reordering and partitioning} \\
\Leftrightarrow \quad & x_p + Fx_f = \tilde{b}_p && \text{carrying out the multiplications} \\
\Leftrightarrow \quad & x_p = \tilde{b}_p - Fc && \text{setting } x_f = c \\
\Leftrightarrow \quad & \begin{pmatrix} x_p \\ x_f \end{pmatrix} = \begin{pmatrix} \tilde{b}_p \\ 0 \end{pmatrix} + \begin{pmatrix} -F \\ I \end{pmatrix} c && \text{rewriting the system in standard form}
\end{aligned}
$$

where c is a set of arbitrary constants. All that remains is to undo the reordering of the equations by multiplying from the left with P. This is best understood by considering the previous example. Gauss-Jordan elimination was complete after $k = 4$ steps, with two free variables x_2 and x_4. A permutation matrix that reorders the variables such that the free variables occur after the basic variables is given by

$$
P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} .
$$

Note we chose to keep the basic variables in the original order. The partition separates the pivot columns from the free variable columns.

We will colorize the entries corresponding to F and $-F$ for easy reference. Reordering the equations in the final system using this permutation results in

$$
\begin{pmatrix}
1 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_2 \\ x_4 \end{pmatrix}
=
\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}
$$

There were no zero rows in the example. The matrix of coefficients is partitioned into $(I \; F)$.

We now set the free variables equal to constants $x_2 = \alpha_1$, $x_4 = \alpha_2$, and solve

---

[6]The product AP interchanges the columns of A, while $P^t x$ interchanges the corresponding variables (rows) in x.

for the basic variables to obtain

$$
\begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}
$$

The final step is to add the definitions of the variables $(x_2\ x_4)^t = (\alpha_1\ \alpha_2)^t$ and to undo the reordering of the variables to obtain the solution in standard form:

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.
$$

Note in particular that the choice of values for the free variables (colorized red), result in zero entries for the particular solution, and in I for the homogeneous solutions.

Example 2.5.4. *Gauss-Jordan elimination, infinite number of solutions*

Consider an $Ax = b$ problem solved by Gauss-Jordan elimination, such that at the final step the system has been reduced to

$$
\begin{pmatrix} 1 & 1 & 7 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 14 \\ 27 \\ 44 \\ 0 \\ 0 \end{pmatrix}
$$

where we have colorized the pivots and basic variables red and the free variables and free variable entries in the pivot columns in blue.

The solution of the system is therefore given by

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \\ 0 \\ 27 \\ 44 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -7 & -8 & -9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.
$$

2.6 The Structure of the Solution

2.6.1 The Number of Solutions

Let us look one more time at the solutions of the two simple examples in section (2.1). They appear strikingly different: the first system has a unique solution $x = x_p$, while the second system has an infinite number of solutions of the form $x = x_p + \alpha x_{h1} + \beta x_{h2}$, where x_p, x_{h1} and x_{h2} are constant vectors, and α, β are arbitrary parameters.

The number of solutions can be made plausible by considering systems with two or three variables. Consider a system of 2 equations in 2 unknowns, e.g., the top set of graphs in Fig 2.1, which we can represent geometrically as two lines in the x, y plane. The solutions of the system are the intersections of the two lines: if the lines cross, we have a single point of intersection (a unique solution). If the lines are parallel there are two possibilities: if the lines coincide, any point on the line is a solution (an infinite number of solutions), otherwise, there are no solutions. Similar observations hold for more variables.

Algebraically, the no solution case occurs when Gaussian Elimination of the system $Ax = b$ results in one or more contradictions. Such systems will be called **inconsistent**, to distinguish them from **consistent** systems which have one or more solutions.
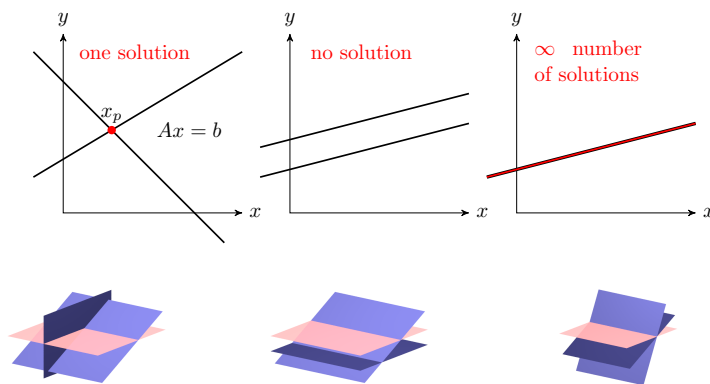
Figure 2.1: Two linear equations in 2 unknowns can be represented by straight lines on a 2D graph. The lines can intersect in a single point, yielding a unique solution. If the lines are parallel, there are 2 cases: distinct lines never intersect, so no solution. If the lines are identical, however, all points are on both lines, yielding an infinite number of solutions.

Three linear equations in 3 unknowns can be represented by planes on a 3D graph. The set of intersection points can be empty (no solution), a single point (unique solution), a line or a plane (infinite number of solutions).

Looking back at the simple system Eqs (2.7), we see that the surface $z = 1$ occurred three times, leading to an infinite number of solutions. Changing the right hand side of the original system Eq(2.7), we easily obtain parallel surfaces $z = z_0$ for three different values of z_0 after the elimination step, with no point in common, and hence no solution.

When we have more than one solution, a second useful question is to ask how they differ? Consider the system $Ax = b$ with more than one solution, say a solution x_1 (and hence $Ax_1 = b$), and a solution x_2, (and hence $Ax_2 = b$). Looking at the difference $x_1 - x_2$, we find

$$
\begin{aligned}
A(x_1 - x_2) =& Ax_1 - Ax_2 \\
=& b - b \\
=& 0.
\end{aligned}
$$

Thus, any two solutions differ by a solution of the **homogeneous system** $Ax = 0$, (the special case $b = 0$ for $Ax = b$).

The complete set of solutions of the system $Ax = b$ therefore has the form

$$
x = x_p + x_h, \tag{2.28}
$$

where x_p is any one **particular solution** of $Ax = b$, and x_h are all possible **homogeneous solutions** of $Ax = 0$.

Further, given a homogeneous solution x_h, then for any constant α we have $A(\alpha x_h) = \alpha A x_h = 0$. Thus, we see that any homogeneous solutions can be scaled by arbitrary parameters and still remain solutions. The reader should look at each of the examples above and identify x_p and x_h.

Consider example (2.5.3). The row echelon form of the matrix contained free variables. Since there were no contradictions, we found the complete set of solutions of the form $x = x_p + \alpha x_{h1} + \beta x_{h2}$ for arbitrary parameters α and β *corresponding to each of the two free variables*. Forming the difference of this solution for arbitrary α and β with the particular solution $x = x_p$ obtained by choosing $\alpha = \beta = 0$, we see that $x_h = \alpha x_{h1} + \beta x_{h2}$ are the homogeneous solutions.

In examples (2.5.1) and (2.5.2), there were no free variables: in each case, the original equation $Ax = b$ was reduced by a sequence of elementary row operations $\mathcal{E}_i = E_i E_{i-1} \cdots E_1$, such that after a total k steps we obtained the identity matrix $\mathcal{E}_k A = I$. Thus

$$
\begin{aligned}
Ax = b \Leftrightarrow& \mathcal{E}_k Ax = \mathcal{E}_k b \\
\Leftrightarrow& x = \mathcal{E}_k b,
\end{aligned}
$$

yields a unique solution. For $b = 0$, i.e., for the homogeneous system, this solution is the trivial solution $x = 0$. Identifying the components of the solution in Eq (2.28), we have $x_p = \mathcal{E}_k b$ and $x_h = 0$.

☞ If the system $Ax = b$ has a solution, it is of the form

$$
x = x_p + \sum_{k=1}^{K} \alpha_k x_{hk}, \tag{2.29}
$$

where K is the number of free variables, x_p is any one particular solution of $Ax = b$, and where the x_{hk} are a set of homogeneous solutions (one for each of the variables) of the system scaled by arbitrary parameters α_k.

Geometrically, the linear combination of homogeneous solutions forms a hyperplane through the origin. The addition of a particular solution pushes this hyperplane away from the origin.

☞ Hyperplanes may be represented graphically by viewing them "edge-on". If we consider a plane in three dimensions, we may choose a viewpoint such that we see the plane as a line. (See Fig 2.2) Similarly, we may look down the length of a line so that we only see it as a point.

Extending this idea to more than three dimensions, we may consider the drawing of a plane, a line, or the origin to represent a hyperplane with the additional dimensions hidden from view.
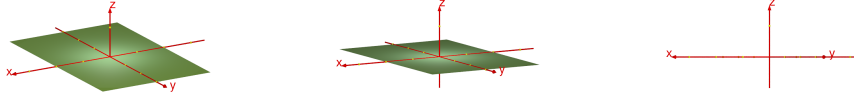
Figure 2.2: Moving the viewpoint inside a given plane leads to an "edge-on" representation. Generalizing to more than 3 dimensions, we may represent hyperplanes by planes, lines, and even the origin.

Checking the solution Eq (2.9), we find that it has two such homogeneous solutions, one scaled by α, the other by β. Checking the solution Eq (2.5), we find the homogeneous solution to be $x_h = 0$.

Look again at the Gauss-Jordan elimination method solutions of examples (2.5.1) and (2.5.2), where we had no free variables: in each case, the original equation $Ax = b$ was reduced to $x = \mathcal{E}_k b$ after k steps. Checking the homogeneous solutions by setting $b = 0$, we see that $x_h = \mathcal{E}_k 0 = 0$, consistent with the form of the solution Eq (2.28).

In example (2.5.3), however, there were free variables. Since there were no contradictions, we found a set of solutions of the form $x = x_p + \alpha x_{h1} + \beta x_{h2}$ for arbitrary parameters α and β corresponding to each of the two free variables. It is easy to check that setting the free variables equal to zero, i.e., setting $\alpha = \beta = 0$, the solution of $Ax = b$ reduces to the particular solution x_p. Looking at differences between solutions, we see that x_{h1} and x_{h2} are solutions of the homogeneous equation: $Ax_{h1} = 0$ corresponds to the solution obtained by choosing the free variables $\alpha = 1$, $\beta = 0$, and similarly $Ax_{h2} = 0$ corresponds to the solution obtained by choosing the free variables $\alpha = 0$, $\beta = 1$.

2.6.2 Row-equivalence and Linear Independence

As we have seen, a sequence of elementary row operations transforms a given matrix A into matrices A_k, where the resulting matrices can have rows of zeros. To discuss this further, we need some vocabulary and some technical results expressed as lemmas.[7]

Row-equivalence

Two given matrices A and B are **row-equivalent** if $A = \mathcal{E}B$, where $\mathcal{E}$ is the matrix representation of some sequence of elementary operations. Since $\mathcal{E}$ is invertible, we similarly have $B = \mathcal{E}^{-1}A$.

☞ The matrices A_k in the layout 2.26 of the Gaussian and Gauss-Jordan algorithms are row equivalent, as are the matrices b_k.

☞ Any two row-echelon form matrices R_1 and R_2 derived from a same matrix A are row-equivalent: since $A = \mathcal{E}_1 R_1$ and $A = \mathcal{E}_2 R_2$ for some sequences of elementary row operations[8] $\mathcal{E}_1$ and $\mathcal{E}_2$, we obtain $R_2 = \mathcal{E}_2^{-1}\mathcal{E}_1 R_1$.

A quick check establishes that row-equivalence is an equivalence relation:

i) **reflexivity:** A is row-equivalent to itself since $A = IA$, the trivial permutation of the rows of A.

ii) **symmetry:** If $A = \mathcal{E}B$ for some sequence of elementary operations $\mathcal{E}$ then $B = \mathcal{E}^{-1}A$ for the sequence of elementary operations $\mathcal{E}^{-1}$.

iii) **transitivity:** If $A = \mathcal{E}_1 B$ for the sequence of elementary operations $\mathcal{E}_1$ and $B = \mathcal{E}_2 C$ for the sequence of elementary operations $\mathcal{E}_2$, then $A = \mathcal{E}_1\mathcal{E}_2 C$ for the sequence of elementary operations $\mathcal{E}_1\mathcal{E}_2$.

☞ Since row-equivalence is an equivalence relation, it partitions the set of all matrices of a given size $M \times N$ into distinct classes.

Linear independence

Next, partition the matrices $\mathcal{E}_k$ used in the Gaussian Elimination algorithm into rows. We see that each of the rows in $A_k = \mathcal{E}_k A$ is a linear combination of the rows in A. This interpretation holds in general:

☞ Consider two matrices A and B, related by $B = CA$ for some given matrix C. Each row of B is linear combination of the rows of A using the coefficients from the corresponding row in C.

---

[7]A lemma is a technical result that is useful to establish theorems.

[8]Strictly speaking the matrices $\mathcal{E}$ represent a sequence of elementary row operations. We will omit the qualification "that represent" for the sake of brevity from here on.

Replacement of the ith row of A of size $M \times N$ is achieved by the matrix C of size $M \times M$ of the form

$$C = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline \mathbf{L} & \mathbf{c_{ii}} & \mathbf{R} \\ \hline 0 & 0 & I \end{array} \right) ,$$

where C has been partitioned to isolate the element on the main diagonal in the ith row. The entries of the row vector $(L\, c_{ii}\, R)$ with $L = (c_{i1} \cdots c_{ii-1})$ and $R = (c_{ii+1} \cdots c_{iM})$ are the coefficients of the desired linear combination. Let a_i for $i = 1, 2, \cdots M$ be the rows of the matrix A. The ith row b_i of $B = CA$ is given by the linear combination

$$b_i = c_{i1}a_1 + c_{i2}a_2 + \cdots c_{iM}a_M.$$

The next example works out this relation in detail, and shows once again how partitioning matrices can be used to exploit patterns of values in the entries of a matrix.

Example 2.6.1. *Linear combinations and row operations*
The following lemma is not really required in the development of the subject presented here. We include it to provide yet another example of proving a result by investigating a matrix product using partitioning of the matrices involved.

Lemma 2.6.1. *The replacement of a given row in a matrix A by a linear combination of the rows of A can be expressed as a sequence of elementary operations.*

Proof. To prove this result, we must decompose C into a product of elementary operation matrices. A moment's thought leads us to the idea that we should obtain the desired result by applying the appropriate scale factor to the ith row of A, followed by additions of scaled versions of the other rows, one row at a time. To verify this, we express C as a product

$$C = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline \mathbf{L} & 1 & 0 \\ \hline 0 & 0 & I \end{array} \right) \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & 1 & \mathbf{R} \\ \hline 0 & 0 & I \end{array} \right) \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & \mathbf{c_{ii}} & 0 \\ \hline 0 & 0 & I \end{array} \right)$$

which is easily established by carrying out the matrix multiplications. Using the result of example 1.2.13, for the matrix containing L and an analogous computation for the matrix containing R, we further expand

$$C = C_1 C_2 \cdots C_{i-1} C_{i+1} C_{i+2} \cdots C_M C_i,$$

where the matrices C_k are the elementary row and scaling operations

$$C_k = \left(\begin{array}{c|c|c|c|c} I & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 \\ \hline 0 & \mathbf{c_{ik}} & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array} \right) \quad k = 1, 2, \cdots i-1,$$

$$C_k = \left(\begin{array}{c|c|c|c|c} I & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \mathbf{c_{ik}} & 0 \\ \hline 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array} \right) \quad k = i+1, i+2, \cdots M,$$

$$C_i = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & \mathbf{c_{ii}} & 0 \\ \hline 0 & 0 & I \end{array} \right).$$

Each of the matrices C_k above was partitioned to isolate the entry c_{ik}, i.e., there is a different partition for every index k. $\square$

We now turn to the fact that a linear combination of rows obtained in Gaussian Elimination at times yields a row of zeros. A **zero row** in a matrix is a row with all entries equal to 0. All other rows are **non-zero rows**.

Let A have L non-zero rows a_i, for $i = 1, 2, \cdots L$, and consider a linear combination of these non-zero rows such that yields zero, i.e.,

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots \alpha_i a_i + \cdots + \alpha_L a_L = 0 \tag{2.30}$$

for some set of scalar coefficients $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_L$. The trivial solution is $\alpha_i = 0$ for all $0 \le i \le L$. We can always zero out a row in a matrix by multiplying with zeros. If we can find a non-trivial solution, i.e., a solution with at least one coefficient $\alpha_l \ne 0$ for some $0 \le l \le L$, we see that we can solve this equation for a_l in terms of the remaining rows. We say that **a given set of vectors is linearly independent** iff Eq (2.30) has the trivial solution as its only solution. In this case, none of the vectors a_i can be written as a linear combination of the remaining vectors. If there is a non-trivial solution, the vectors are said to be **linearly dependent**. Any vector a_k with a non-zero coefficient α_k can be written as a linear combination of the other vectors.

☞ Gaussian Elimination results in zeroing out linearly dependent rows of a matrix A by forming suitable linear combinations of the rows of A.

In fact, Gaussian Elimination removes all linearly dependent rows of a matrix, as shown by the following

Lemma 2.6.2. *If A is a matrix in row echelon form, then a non-zero row of A cannot be expressed as a linear combination of the other rows.*

Proof. For the lemma to hold, it is sufficient to show that Eq (2.30) cannot have any non-trivial solution. The proof is simple: we express Eq (2.30) in matrix form $(\alpha_1 \ \alpha_2 \ \cdots \alpha_L \ 0 \cdots 0)A = 0$ and consider the pivot columns starting from the first pivot and working to the right. Assuming the pivots are located in columns $l_1, l_2, \cdots l_L$, we have

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \cdots) \begin{pmatrix} 0 & \mathbf{a_{1l_1}} & \cdots & a_{1l_2} & \cdots & a_{1l_3} & \cdots \\ 0 & 0 & 0 & \mathbf{a_{2l_2}} & \cdots & a_{2l_3} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \mathbf{a_{3l_3}} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \end{pmatrix} = 0.$$

The equations corresponding to the pivot columns are seen to be

$$\alpha_1 \mathbf{a_{1l_1}} = 0$$
$$\alpha_1 a_{1l_2} + \alpha_2 \mathbf{a_{2l_2}} = 0$$
$$\alpha_1 a_{1l_3} + \alpha_2 a_{2l_3} + \alpha_3 \mathbf{a_{3l_3}} = 0$$
$$\cdots$$

Since the a_{il_i} are the pivots and therefore are not equal to zero, we see by forward-substitution that the only solution of the system is $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, ..., as required. □

We next consider the relationship between row-equivalent matrices to obtain a more general result:

Lemma 2.6.3. *(Row-equivalent matrix shape) If A and B are row-equivalent matrices in row echelon form, then they have the same number of pivots in the same rows and columns.*

Proof. The approach to proving this Lemma is similar to the previous one. If A and B are row-equivalent, then there exists some invertible matrix $\mathcal{E}$ representing some sequence of elementary operations such that $B = \mathcal{E}A$ and $A = \mathcal{E}^{-1}B$. We set up these multiplications and compare the results with the left hand sides of these equations. Let r_A be the number of pivots in A, and let $l_1, l_2, ..l_{r_A}$ be the column locations of these pivots. Similarly, let r_B be the number of pivots in B, and let $k_1, k_2, ..k_{r_B}$ be their column locations.

$$
A = \begin{pmatrix}
0 & \boxed{\mathbf{a_{1l_1}}} & \cdots & a_{1l_2} & \cdots & a_{1l_3} & \cdots \\
0 & 0 & 0 & \boxed{\mathbf{a_{2l_2}}} & \cdots & a_{2l_3} & \cdots \\
0 & 0 & 0 & 0 & 0 & \boxed{\mathbf{a_{3l_3}}} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
0 & \boxed{\mathbf{b_{1k_1}}} & \cdots & b_{1k_2} & \cdots & b_{1k_3} & \cdots \\
0 & 0 & 0 & \boxed{\mathbf{b_{2k_2}}} & \cdots & b_{2k_3} & \cdots \\
0 & 0 & 0 & 0 & 0 & \boxed{\mathbf{b_{3k_3}}} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots
\end{pmatrix}.
$$

We need to show that that these matrices have the same number of pivots $r_A = r_B$, and that these pivots occur in the same locations $l_1 = k_1$, ..., $l_{r_A} = k_{r_B}$. The argument proceeds by induction starting with the first row.

- Base Step (row 1) Partition the matrix A to isolate the pivot in the first row, partition matrix $\mathcal{E}$ to isolate the first diagonal element.

$$
\mathcal{E}A = \begin{pmatrix} \epsilon_{11} & \cdots \\ \mathcal{E}_1 & \cdots \end{pmatrix} \begin{pmatrix} 0 & a_{1l_1} & \cdots \\ 0 & 0 & \cdots \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_{11}a_{1l_1} & \cdots \\ 0 & \mathcal{E}_1 a_{1l_1} & \cdots \end{pmatrix}
$$

  and compare the result to matrix B.

  - If A does not have a pivot in the first row, then the product $\mathcal{E}A$ has no non-zero elements in the first row, and hence B cannot have a pivot. In this case, $r_A = r_B = 0$, and the lemma does hold.
  - If A does have a pivot in the first row, then $a_{1l_1} \neq 0$ and $\mathcal{E}_1 = 0$ since by hypothesis B is in row echelon form. Since ϵ_{11} might be zero, we conclude $l_1 \leq k_1$.

    The same argument applied to $\mathcal{E}^{-1}B = A$ results in $l_1 \geq k_1$. Therefore $l_1 = k_1$, and $\epsilon_{11} \neq 0$.

- Induction Step (row i) Assume that we have established that $l_p = k_p$ for $1 \le p < i$, and that $\mathcal{E}$ has the form $\begin{pmatrix} \tilde{\mathcal{E}} & \cdots \\ 0 & \cdots \end{pmatrix}$, where $\tilde{\mathcal{E}}$ is an upper diagonal matrix of size $(i-1) \times (i-1)$ with non-zero diagonal elements, and that $\mathcal{E}^{-1}$ has a similar structure. Partition the matrix A to isolate the pivot in the ith row, partition matrix $\mathcal{E}$ to isolate the ith diagonal element.

$$\mathcal{E}A = \begin{pmatrix} \tilde{\mathcal{E}} & \mathcal{E}_2 & \cdots \\ 0 & \epsilon_{ii} & \cdots \\ 0 & \mathcal{E}_3 & \cdots \end{pmatrix} \begin{pmatrix} A_1 & A_2 & \cdots \\ 0 & a_{il_i} & \cdots \\ 0 & 0 & \cdots \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{E}}A_1 & \tilde{\mathcal{E}}A_2 + \mathcal{E}_2 a_{il_i} & \cdots \\ 0 & \epsilon_{ii} a_{il2} & \cdots \\ 0 & \mathcal{E}_3 a_{il_i} & \cdots \end{pmatrix}.$$

  and compare the pivot column of the product to the pivot column in matrix B.

  - If A does not have a pivot in the ith row, then the ith row of the product $\mathcal{E}A$ has no non-zero elements and hence B cannot have a pivot in this row. In this case, $r_A = r_B = i - 1$, and the lemma does hold.

  - If A does have a pivot in the ith row, then $a_{il_i} \neq 0$ and $\mathcal{E}_3 = 0$ since by hypothesis B is in row echelon form. Because ϵ_{ii} might be zero, we conclude $l_i \le k_i$.

    The same argument applied to $\mathcal{E}^{-1}B = A$ results in $l_i \ge k_i$. Therefore $l_i = k_i$, and $\epsilon_{ii} \neq 0$. Since the matrix $\mathcal{E}$ and its inverse have the required form, the induction hypothesis holds, completing the proof.

    $\square$

- ☞ The reduction of A to row-echelon form therefore always results in the same number of pivots r appearing in the same columns and hence the same leading variables in a system of linear equations $Ax = b$, and the same free variables.

If we partition A and B such that $\tilde{A}$ and $\tilde{B}$ contain the r pivot rows, we see that row-equivalent row echelon form matrices are related by an invertible matrix of the form

$$\begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{E}}_1 & \tilde{\mathcal{E}}_2 \\ 0 & \tilde{\mathcal{E}}_3 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ 0 \end{pmatrix}$$

such that $\tilde{\mathcal{E}}_1$ is an upper triangular matrix of size $r \times r$ with non-zero diagonal entries.

The number of pivots r in a given matrix A is **the rank of the matrix** and is denoted $r = rank(A)$. Assuming that A is of size $M \times N$, we see that the rank $r \le M$ and $r \le N$, since the number of pivots cannot exceed the number of rows or the number of columns in a matrix.

Looking back at the proof of the row-equivalent matrix shape lemma 2.6.3 we see that the shape argument can be further enhanced by making the additional assumption that the matrices A and B are in reduced row-echelon form to

establish the important result that **the reduced row-echelon form of a matrix is unique.**

If A and B are row-equivalent matrices in reduced row-echelon form, they can only differ in columns that do not contain pivots, since the shape lemma guarantees that the pivot columns are the same. As before, let $B = \mathcal{E}A$ for some product of elementary matrices $\mathcal{E}$, and assume that A and B differ in column k. Define the matrices $\tilde{A}$ and $\tilde{B}$ by deleting all non-pivot columns with index less than k and all columns with index greater than k from A and B respectively. The matrices then have the form

$$\tilde{A} = \begin{pmatrix} I & l_A \\ 0 & 0 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} I & l_B \\ 0 & 0 \end{pmatrix},$$

where l_A and l_B are the column vectors from the original colmn k, and so $l_A \neq l_B$. The relationship $\tilde{A} = \mathcal{E}\tilde{B}$ still holds, since we only deleted corresponding columns in A and B. Partitioning $\mathcal{E}$, we see that

$$\begin{pmatrix} I & l_A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} \begin{pmatrix} I & l_B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{11}l_B \\ \mathcal{E}_{21} & \mathcal{E}_{21}l_B \end{pmatrix}.$$

comparison of the first row establishes that $\mathcal{E}_{11} = I$, and $l_A = \mathcal{E}_{11}l_B = l_B$, contradicting our assumption that the matrices A and B could differ.

- ☞ The reduction of A to reduced row-echelon form always results in the same matrix.

- ☞ The reduced row echelon form of a matrix A may be used as the canonical form of the equivalence class of A under row-equivalence. Further, each of the matrices A_k obtained as intermediate steps belong to this equivalence class.

- ☞ For square matrices, the equivalence class containing the identitiy matrix will prove to be important: it consists of the invertible matrices.

The last lemma we present will prove useful when we discuss inverses of a matrix.

Lemma 2.6.4. *If A and B are row-equivalent, and if B is in reduced row echelon form, then the number of zero rows in A cannot exceed the number of zero rows in B.*

Proof. Let A and B have size $M \times N$, and let K_A and K_B be the number of zero rows in A and B respectively.

The proof follows from an analysis of the reduction of A to row echelon form by the Gaussian algorithm applied to the matrix PA, where the matrix P represents a sequence of row interchanges that move all zero rows (if any), below the non-zero rows of A. Partitioning the resulting matrix into the non-zero and zero rows and checking each of the possible elementary operations selected by the Gaussian Elimination algorithm, it is straightforward to show

that the number of zero rows in A_k does not decrease when carrying out the Gauss-Jordan algorithm: the zero rows are always left "as is".

Since all row-equivalent row-echelon form matrices have the number of zero rows by Lemma 2.6.3, the result follows. $\square$

2.7 Matrix Equations

We now turn to considering the problem of solving $Ax = b$ repeatedly with different right-hand sides, a problem that occurs frequently in practice. As discussed previously in Eqs (2.26,2.27), solutions $Ax_i = b_i, i = 1, 2, \cdots l$ can be assembled to yield the solution $X = (x_1 x_2 \cdots x_l)$ of the matrix problem $AX = B$, where $B = (b_1\ b_2 \cdots b_l)$.

2.7.1 Multiple Right-hand Sides

An efficient computational layout for the problems $Ax_i = b_i, i = 1, 2, \cdots l$ is obtained by augmenting the matrix A with each of the right hand sides $(A|b_1 b_2 \cdots b_l)$ and proceeding as before.

Example 2.7.1. *Multiple right-hand sides, unique solution*
Consider the following $Ax = b$ problem with two different right hand sides $b = b_1$
and $b = b_2$, where

$$A = \begin{pmatrix} 2 & 4 & 14 \\ 4 & 9 & 31 \\ -6 & -8 & -29 \end{pmatrix}, \quad b_1 = \begin{pmatrix} -22 \\ -47 \\ 51 \end{pmatrix} \quad and \ b_2 = \begin{pmatrix} -30 \\ -68 \\ 54 \end{pmatrix}.$$

Set up the problem by augmenting A with $B = (b_1 \ b_2)$, i.e., with both b_1 and
b_2. In this example, we use Gauss-Jordan elimination.

$$\begin{pmatrix} 2 & 4 & 14 & -22 & -30 \\ 4 & 9 & 31 & -47 & -68 \\ -6 & -8 & -29 & 51 & 54 \end{pmatrix}$$
*Augment A with both b_1
and b_2. Choose the first
pivot 2.*

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 14 & -22 & -30 \\ 0 & 1 & 3 & -3 & -8 \\ 0 & 4 & 13 & -15 & -36 \end{pmatrix}$$
Choose the second pivot 1.

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 & -10 & 2 \\ 0 & 1 & 3 & -3 & -8 \\ 0 & 0 & 1 & -3 & -4 \end{pmatrix}$$
*The matrix A_3 is in row
echelon form. Choose the
third pivot.*

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & -4 & 10 \\ 0 & 1 & 0 & 6 & 4 \\ 0 & 0 & 1 & -3 & -4 \end{pmatrix}$$
*Finally, scale each of the
pivots to 1.*

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 & 5 \\ 0 & 1 & 0 & 6 & 4 \\ 0 & 0 & 1 & -3 & -4 \end{pmatrix}$$
*There are no free vari-
ables. We have obtained a
unique solution.*

*The solutions for $Ax_1 = b_1$ and $Ax_2 = b_2$ are unique. Transcribing each of the
two problems, we find $x_1 = (-2 \ 6 \ -3)^t$ and $x_2 = (5 \ 4 \ -4)^t$ respectively.*
*   The reader should set up and solve each of the two problems $Ax_1 = b_1$ and
$Ax_2 = b_2$ separately and compare the resulting computations with the layout
above. Next, she/he should verify that the matrix $(x_1 x_2)$ solves the matrix prob-
lem $AX = B$, where $B = (b_1 \ b_2)$.*

    When A does not have a pivot in every row, we have free variables. The
solution of $Ax = b$ now has non-zero homogeneous solutions. If a particular
solution exists for a given right-hand side b, the complete solution is given in
Eq (2.29), $x = x_p + \sum_{i=1}^{L} \alpha_i x_{hi}$.

    If we repeat the problem with a new right hand side, the form of the solutions,
if they exist, are the same: we may reuse the same homogeneous solutions, but
with a new set of arbitrary constants.

Example 2.7.2. *Multiple right-hand sides, infinite number of solutions*
Consider the following $Ax = b$ problem with three different right hand sides
$b = b_1, b = b_2$ *and* $b = b_3$, *where*

$$A = \begin{pmatrix} 1 & 2 & 10 & 1 \\ 1 & 3 & 13 & 1 \\ 3 & 4 & 24 & 3 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 11 \\ 15 \\ 25 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 23 \\ 31 \\ 53 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 39 \\ 48 \\ 99 \end{pmatrix}.$$

Setting up the problem by augmenting A with $B = (b_1 \ b_2 \ b_3)$ and using Gauss-Jordan elimination, we find

$$\left(\begin{array}{cccc|ccc} \mathbf{1} & 2 & 10 & 1 & 11 & 23 & 39 \\ 1 & 3 & 13 & 1 & 15 & 31 & 48 \\ 3 & 4 & 24 & 3 & 25 & 53 & 99 \end{array} \right) \quad \begin{array}{l} \textit{Choose the first} \\ \textit{pivot} \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \left(\begin{array}{cccc|ccc} \mathbf{1} & 2 & 10 & 1 & 11 & 23 & 39 \\ 0 & \mathbf{1} & 3 & 0 & 4 & 8 & 9 \\ 0 & -2 & -6 & 0 & -8 & -16 & -18 \end{array} \right) \quad \begin{array}{l} \textit{Choose the second} \\ \textit{pivot} \end{array}$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \left(\begin{array}{cccc|ccc} \mathbf{1} & 0 & 4 & 1 & 3 & 7 & 21 \\ 0 & \mathbf{1} & 3 & 0 & 4 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \textit{In reduced row-} \\ \textit{echelon form} \end{array}$$

All three right-hand sides are consistent. Solving for each of the right-hand sides separately, we find particular solutions
$x_1 = (3\ 4\ 0\ 0)^t, x_2 = (7\ 8\ 0\ 0)^t, x_3 = (21\ 9\ 0\ 0)^t$, *and the homogeneous solutions*
$x_h = \alpha(-4\ -3\ 1\ 0)^t + \beta(-1\ 0\ 0\ 1)^t$ *with arbitrary constants α, and β.*
   Combining these in matrix form, and remembering to use different arbitrary constants for the homogeneous solutions for each, we obtain the solution of the matrix equation $AX = B$

$$X = \left(\begin{array}{ccc|cc} 3 & 7 & 21 & -4 & -1 \\ 4 & 8 & 9 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}.$$

As usual, the reader should verify the solution by computing $AX - B$ to check that the result is indeed zero.

☞ The solution of the matrix problem $AX = B$ has the form

$$X = X_p + X_h C$$

$$= \begin{pmatrix} X_p & X_h \end{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix},$$

where X_p is any one particular solution of the problem, typically chosen by setting all free variables to zero. The homogeneous solution X_h is the

matrix of homogeneous solutions of $AX = 0$ we found previously, with one
column corresponding to each of the L free variables, respectively. Finally
C is a square matrix of arbitrary constants of size $L \times L$.

The solution of the example above may be rewritten as

$$
X = \begin{pmatrix} 3 & 7 & 21 \\ 4 & 8 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -4 & -1 \\ -3 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}.
$$

☞ If any one of the problems $Ax_i = b_i$ is inconsistent, the matrix problem
$AX = B$ for $B = (b_1 \, b_2 \cdots)$ has no solution.

Test for Solvability

If we have to solve $Ax = b$ repeatedly for different right hand sides b, we may
want some way of checking whether a solution exists prior to attempting a
full solution. We know that we can obtain a solution by Gaussian Elimination
provided that the equivalent system does not lead to a contradiction, i.e., for
each row in A that reduces to zero the corresponding row in b must also equal
zero. Consider the derivation Eq (2.26) that reduces A to row-echelon form

$$
Ax = b \Leftrightarrow \mathcal{E}Ax = \mathcal{E}b
$$
$$
\Leftrightarrow \begin{pmatrix} R \\ 0 \end{pmatrix} x = \begin{pmatrix} \mathcal{E}_R \\ \mathcal{E}_0 \end{pmatrix} b
$$

where we have introduced a partition such that the submatrix R consists of
those rows of $\mathcal{E}A$ that have pivots. Consistency therefore requires that the
corresponding rows $\mathcal{E}_0 b = 0$.

☞ Since $\mathcal{E}_0 b = 0$ is a homogeneous system for b, it follows that the set of
vectors b for which $Ax = b$ has solutions forms a hyperplane.

☞ The equations $\mathcal{E}_0 b = 0$ describe this hyperplane implicitely as a set of
constraints that b must satisfy.

In the following example, we will solve the problem twice, first by apply-
ing Gaussian Elimination directly to $Ax = b$ for an arbitrary right hand-side,
followed by an explicit computation of $\mathcal{E}_2$.

Example 2.7.3. *Characterize right hand sides of solvable $Ax = b$ problems*

Consider the problem $Ax = b$ for $b = (b_1\ b_2\ b_3\ b_4\ b_5)^t$.

$$
\left(\begin{array}{ccccc|c}
1 & -2 & 8 & 0 & 0 & b_1 \\
1 & -3 & 8 & 7 & 3 & b_2 \\
-3 & 4 & -25 & 16 & 7 & b_3 \\
1 & -3 & 7 & 9 & 4 & b_4 \\
-1 & 4 & -7 & -16 & -7 & b_5
\end{array}\right)
$$

$$
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{ccccc|c}
1 & -2 & 8 & 0 & 0 & b_1 \\
0 & -1 & 0 & 7 & 3 & -b_1 + b_2 \\
0 & -2 & -1 & 16 & 7 & 3b_1 + b_3 \\
0 & -1 & -1 & 9 & 4 & -b_1 + b_4 \\
0 & 2 & 1 & -16 & -7 & b_1 + b_5
\end{array}\right)
$$

$$
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{ccccc|c}
1 & -2 & 8 & 0 & 0 & b_1 \\
0 & -1 & 0 & 7 & 3 & -b_1 + b_2 \\
0 & 0 & -1 & 2 & 1 & 5b_1 - 2b_2 + b_3 \\
0 & 0 & -1 & 2 & 1 & -b_2 + b_4 \\
0 & 0 & 1 & -2 & -1 & -b_1 + 2b_2 + b_5
\end{array}\right)
$$

$$
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{array}\right)
\left(\begin{array}{ccccc|c}
1 & -2 & 8 & 0 & 0 & b_1 \\
0 & -1 & 0 & 7 & 3 & -b_1 + b_2 \\
0 & 0 & -1 & 2 & 1 & 5b_1 - 2b_2 + b_3 \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-5b_1 + b_2 - b_3 + b_4} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{4b_1 + b_3 + b_5}
\end{array}\right)
$$

where we have partitioned the row-echelon form A_3 matrix into $A_3 = \begin{pmatrix} R \\ 0 \end{pmatrix}$.

The resulting system is seen to have solutions if and only if

$$
\left.\begin{array}{r}
-5b_1 + b_2 - b_3 + b_4 = 0 \\
4b_1 + b_3 + b_5 = 0
\end{array}\right\}
\Leftrightarrow
\begin{pmatrix}
-5 & 1 & -1 & 1 & 0 \\
4 & 0 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5
\end{pmatrix}
= 0
$$

Solving this system for the all possible values of b, we find that b must be of the form[a]

$$
b = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 5 \\ -4 \end{pmatrix}
+ \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}
+ \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}
$$

for $Ax = b$ to have solutions. The values of α, β and γ are any constants.

---

[a]To reduce the amount of computation required, note that we can immediately solve for b_4, b_5 in terms of b_1, b_2, b_3, i.e., b_1, b_2, b_3 can be used as the free variables.

Example 2.7.4. *Characterize right hand sides of solvable $Ax = b$ problems by explicit computation of $\mathcal{E}$*

Consider the previous example one more time: the computations in the layout produced $Ax = b \Leftrightarrow \mathcal{E}Ax = \mathcal{E}b$ where $\mathcal{E} = E_3 E_2 E_1$ was applied in three steps.

Let us compute $\mathcal{E}$ explicitly by augmenting A with I as explained in Eq (2.26):

$$
\left(\begin{array}{ccccc|ccccc}
1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -3 & 8 & 7 & 3 & 0 & 1 & 0 & 0 & 0 \\
-3 & 4 & -25 & 16 & 7 & 0 & 0 & 1 & 0 & 0 \\
1 & -3 & 7 & 9 & 4 & 0 & 0 & 0 & 1 & 0 \\
-1 & 4 & -7 & -16 & -7 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{ccccc|ccccc}
1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\
0 & -2 & -1 & 16 & 7 & 3 & 0 & 1 & 0 & 0 \\
0 & -1 & -1 & 9 & 4 & -1 & 0 & 0 & 1 & 0 \\
0 & 2 & 1 & -16 & -7 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 1
\end{array}\right)
\left(\begin{array}{ccccc|ccccc}
1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 1 & 5 & -2 & 1 & 0 & 0 \\
0 & 0 & -1 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -2 & -1 & -1 & 2 & 0 & 0 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{array}\right)
\left(\begin{array}{ccccc|ccccc}
1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 1 & 5 & -2 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -5 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1
\end{array}\right)
$$

where we have again introduced the partition of A into pivot and zero rows, with the corresponding partition of $\mathcal{E}$ into $\mathcal{E}_R$ and $\mathcal{E}_0$.

For the system to have a solution, we again see that

$$
\mathcal{E}_0 b = \begin{pmatrix} -5 & 1 & -1 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = 0.
$$

2.7.2 Computation of the Matrix Inverse

We first discussed matrix inverses in section 1.2.3, where we introduced the concept of left and right inverses of a matrix.

To compute a right inverse for some matrix A, we need to solve the matrix equation $AX = I$, while the computation of the left inverse requires solution of the matrix equation $XA = I$. Taking the transpose if this equation, we find

$A^t X^t = I$. Consequently, a left inverse (if it exists), can be obtained by finding a right inverse of A^t, and taking the transpose of the solution.

The following result lets us simplify the problem by showing that a matrix A that is not square cannot have both a left inverse and a right inverse, and furthermore lets us decide which inverse to look for.

☞ **If A has size $M \times N$ with $M < N$ (more columns than rows) then A cannot have a left inverse.**

The proof proceeds by contradiction: since A has more columns than rows, some columns cannot have any pivots. Therefore, the system $Ax = 0$ has an infinite number of solutions. Now suppose A has a left inverse L, i.e., $LA = I$. Therefore $Ax = 0 \Rightarrow LAx = 0 \Rightarrow x = 0$, which states that the homogeneous problem only has the trivial solution.

☞ **If A has size $M \times N$ with $M > N$ (more rows than columns) then A cannot have a right inverse.**

This follows trivially from the previous statement by taking the transpose.

☞ Only square matrices A can have both a left inverse L and a right inverse R, in which case $L = R$, i.e., A has an inverse. Furthermore, this inverse is unique.

The requirement that A must be square is a direct consequence of the previous two observations constraining the sizes of A: if A has size $M \times N$, then L can exist only if $M \leq N$, and R can exist only if $N \leq M$.

If these two matrices exist, then by associativity $AR = I \Rightarrow L(AR) = LI \Rightarrow (LA)R = L \Rightarrow IR = L$.

Given any two inverses B and C of A, we similarly have $B = B(AC) = (BA)C = C$, so that an inverse, if it exists, must be unique.

Given a matrix A of size $M \times N$, we therefore have the following cases:

- If $M < N$ (more columns than rows), we attempt to solve $AX = I$ to find a right inverse $R = X$.

- If $M > N$ (more rows than columns), we attempt to solve $A^t X = I$ to find a left inverse $L = X^t$

- If $M = N$ (square matrix), we attempt to solve $AX = I$ to find the right inverse $A^{-1} = X$.

Computation of a Right Inverse

Let us start by trying to find right inverses: a right inverse X of a matrix A must satisfy $AX = I$, i.e., a right inverse is a solution of the matrix equation $AX = B$ for the special case $B = I$. This is the problem studied in the previous section: we find right inverses by solving the $Ax = b$ problem, with b chosen from each of the columns of I. In the first example, we show a matrix that does not have a right inverse. This is the usual case.

Example 2.7.5. *Right inverse, no solution*
*In the following example, **we immediately zero elements above and below each pivot**.*

We wish to find a right inverse of the matrix

$$A = \begin{pmatrix} 3 & -3 & 1 & 5 \\ 4 & -4 & 2 & 8 \\ -2 & 2 & -1 & -4 \end{pmatrix},$$

i.e., we wish to solve $AX = I$.

Gauss-Jordan elimination (with some modifications to simplify hand calculations) yields

$$\left(\begin{array}{cccc|ccc} 3 & -3 & 1 & 5 & 1 & 0 & 0 \\ 4 & -4 & 2 & 8 & 0 & 1 & 0 \\ -2 & 2 & -1 & -4 & 0 & 0 & 1 \end{array} \right)$$
Step 0: *set up the augmented matrix for the system. Choose the first pivot* **3**.

$$\begin{pmatrix} 1 & 0 & 0 \\ -4 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \left(\begin{array}{cccc|ccc} 3 & -3 & 1 & 5 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 & -4 & 3 & 0 \\ 0 & 0 & -1 & -2 & 2 & 0 & 3 \end{array} \right)$$
Step 1: *Eliminate below the first pivot. Choose the pivot* -1 *in the third row.*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{array}{cccc|ccc} 3 & -3 & 1 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 & -3 \\ 0 & 0 & 2 & 4 & -4 & 3 & 0 \end{array} \right)$$
Step 2: *Interchange rows and scale pivot to* **1**. *Note the E_2 matrix is the product of the corresponding elementary operations.*

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \left(\begin{array}{cccc|ccc} 3 & -3 & 0 & 3 & 3 & 0 & 3 \\ 0 & 0 & 1 & 2 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 6 \end{array} \right)$$
Step 3: *The third equation is seen to be inconsistent.*

Transcribing the final system to equation form for each of the columns of the right hand side matrix, we see that the third equation reads $0 = 3$, a contradiction. Matrix A therefore does not have a right inverse.

Carefully consideration of the $\mathcal{E}_i$ submatrices yields an important result:

☞ The right-most matrices keep track of the product of Gaussian elementary operation matrices $\mathcal{E}_i = E_i E_{i-1} \cdots E_1 I$.

☞ The right-most matrices $\mathcal{E}_i$ are row-equivalent to I.

By Lemma 2.6.4, these matrices cannot have any rows of zeros. If A reduces to a matrix with one or more rows of zeros, the equations $AX = I$ cannot be consistent, and A cannot have a right inverse.

☞ When $AX = I$ is consistent, i.e., when A has a pivot in every row, we have the following two cases: i) for $M < N$, the matrix A has more columns than rows, and therefore there must be columns without pivots.

The solution cannot be unique. ii) for square matrices every column will have a pivot, resulting in a unique solution.

The next example shows what happens when one attempts to find a right inverse for a matrix A of size $M \times N$ with $M > N$, a case which we know does not have a solution.

Example 2.7.6. *Right inverse, no solution*
Consider the matrix

$$A = \begin{pmatrix} 3 & 4 & -2 \\ -3 & -4 & 2 \\ 1 & 2 & -1 \\ 5 & 8 & -4 \end{pmatrix}.$$

A has size 4×3, and therefore has at most three pivots. When we try to find a right inverse of A, i.e., a solution of $AX = I$, we get

$$\left(\begin{array}{ccc|cccc} 3 & 4 & -2 & 1 & 0 & 0 & 0 \\ -3 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 5 & 8 & -4 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ -\frac{5}{3} & 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|cccc} 3 & 4 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & 0 \\ 0 & \frac{4}{3} & -\frac{2}{3} & -\frac{5}{3} & 0 & 0 & 1 \end{array} \right)$$

The second equation is seen to be inconsistent: the system has no solution.

This must happen whenever matrix A has more rows than columns: the A_i matrices eventually must have one or more rows of zeros. The $\mathcal{E}_i$ matrices however are row equivalent to I and can never have a row of zeros: the system $AX = I$ must necessarily be inconsistent.

Example 2.7.7. *Right inverse, solution exists*
*In the following example, **we immediately zero elements above and be-
low each pivot, while scaling the elimination matrix rows to avoid
denominators**.*
*We wish to find a right inverse of a matrix that differs from example 2.7.5
in the last column only. Consider*

$$
A = \begin{pmatrix} 3 & -3 & 1 & 1 \\ 4 & -4 & 2 & -1 \\ -2 & 2 & -1 & 1 \end{pmatrix},
$$

i.e., we wish to solve $AX = I$. We obtain the following vertical layout

$$
\left(\begin{array}{cccc|ccc} \mathbf{3} & -3 & 1 & 1 & 1 & 0 & 0 \\ 4 & -4 & 2 & -1 & 0 & 1 & 0 \\ -2 & 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right)
$$

Step 0: *set up the aug-
mented matrix for the
system. Choose the first
pivot* **3**.

$$
\begin{pmatrix} 1 & 0 & 0 \\ -4 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \left(\begin{array}{cccc|ccc} 3 & -3 & 1 & 1 & 1 & 0 & 0 \\ \mathbf{0} & 0 & 2 & -7 & -4 & 3 & 0 \\ \mathbf{0} & 0 & -1 & 5 & 2 & 0 & 3 \end{array} \right)
$$

Step 1: *Eliminate below
the first pivot. Choose the
pivot* **2** *in the second row.*

$$
\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \left(\begin{array}{cccc|ccc} 6 & -6 & \mathbf{0} & 9 & 6 & -3 & 0 \\ 0 & 0 & \mathbf{2} & -7 & -4 & 3 & 0 \\ 0 & 0 & \mathbf{0} & 3 & 0 & 3 & 6 \end{array} \right)
$$

Step 2: *eliminate above
and below the pivot and
scale.*

$$
\begin{pmatrix} 1 & 0 & -\mathbf{3} \\ 0 & \mathbf{3} & \mathbf{7} \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \left(\begin{array}{cccc|ccc} 6 & -6 & 0 & 0 & 6 & -12 & -18 \\ 0 & 0 & 6 & 0 & -12 & 30 & 42 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 \end{array} \right)
$$

Step 3: *Eliminate above
the third pivot* **3** *and
scale.*

$$
\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cccc|ccc} 1 & -1 & 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 0 & -2 & 5 & 7 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 \end{array} \right)
$$

Step 4: *Apply the final
scale factors.*

*Transcribe the final system to equation form for each of the columns of the right
hand side matrix. Remember that we need to use different arbitrary constants
$\alpha = (\alpha_1\, \alpha_2\, \alpha_3)$ for each of the equations. We find*

$$
X = \left(\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 1 \\ -2 & 5 & 7 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \left(\frac{I}{\alpha} \right)
$$

$$
= \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ -2 & 5 & 7 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} (\alpha_1\ \alpha_2\ \alpha_3)
$$

the sum of a particular and a homogeneous solution.

Computation of a Left Inverse

The left inverse X of a matrix A satisfies $XA = I$. Since the transpose of a product of matrices interchanges the operands, we may reduce this problem to the computation of the right inverse of A^t:
just take the transpose $(XA)^t = I^t \Leftrightarrow A^t X^t = I$ and solve for X^t.

Example 2.7.8. *Left inverse, solution exists*
We wish to find a left inverse X of a matrix

$$A = \begin{pmatrix} 3 & 4 & -2 \\ -3 & -4 & 2 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

To solve this problem, we find a right inverse of the matrix

$$A^t = \begin{pmatrix} 3 & -3 & 1 & 1 \\ 4 & -4 & 2 & -1 \\ -2 & 2 & -1 & 1 \end{pmatrix}$$

by following the steps in example 2.7.7. We obtain

$$X^t = \left(\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 1 \\ -2 & 5 & 7 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \left(\frac{I}{\alpha} \right).$$

All that remains is to take the transpose of this solution. The left inverses of A are given by

$$X = (\ I \mid \alpha\ ) \left(\begin{array}{cccc} 1 & 0 & -2 & 0 \\ -2 & 0 & 5 & 1 \\ -3 & 0 & 7 & 2 \\ \hline 1 & 1 & 0 & 0 \end{array} \right).$$

As always, the solution should be verified by explicitely computing XA.

The observations for the right inverse also apply to the left inverse:

☞ **If A has size $M \times N$ with $M < N$ (fewer rows then columns) then A cannot have a left inverse.**

☞ A matrix A of size $M \times N$ with $M > N$ (more rows then columns) will either have no left inverse or an infinite number of left inverses.

The matrix A in example 2.7.5 and its transpose example 2.7.6 has neither a left nor a right inverse.

Computation of the Inverse

We now turn to square matrices which we know may have a (necessarily unique) inverse. We begin by setting up the computation for a right inverse. Consider

the following example:

Example 2.7.9. *Matrix inverse computation*
Consider the following example of a right inverse computation for a square matrix written in layout form.

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 3 & 0 & 0 & 1 \end{pmatrix}$$ *Begin by augmenting the matrix A with I. Choose the first pivot*

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{pmatrix}$$ *The matrix is now augmented by the elimination matrix E_1. Choose the next pivot*

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & -1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{pmatrix}$$ *The matrix is now augmented by the product $E_2 E_1$. Choose the next pivot*

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & -3 & -3 & 1 \\ 0 & 1 & 1 & 3 & 2 & -1 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{pmatrix}$$ *The matrix is now augmented by the product $E_3 E_2 E_1$.*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & 3 & 2 & -1 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{pmatrix}$$ *Since we reached I, the matrix is now augmented by A^{-1}*

The comments assume that as usual, the successive elementary operation matrices are named E_1, E_2 and E_3 respectively. **Check by verifying that the product AA^{-1} does indeed yield the identity.**

The comments state that the computation yields the inverse. To see why this must be so, consider what is happening in the layout: we are computing a right inverse, i.e., a solution of $AX = I$ by reducing A to the identity matrix. This is carried out by multiplying A from the left by a sequence of elementary operation matrices E_i, such that at step i, we have $A_i = E_i E_{i-1} \cdots E_1 A = \mathcal{E}_i A$. If after some k steps, we succeed in reducing A to the identity, we will have obtained $\mathcal{E}_k A_k = I$, a unique left inverse of A. The net effect is that we solve $AX = I \Leftrightarrow \mathcal{E}_k A X = \mathcal{E}_k I \Leftrightarrow X = \mathcal{E}_k$ by obtaining a left inverse of A, which turns out to be the right inverse solution we were looking for.

- ☞ As we proceed down the stack in the vertical layout of the Gauss-Jordan algorithm in k steps, the matrices A_i move from A to I, while the matrices $\mathcal{E}_i$ move from I to A^{-1} as i increases from 1 to k.

- ☞ The square matrix A has an inverse if and only if has a pivot in every column.

This result follows from the observations that i) if A has a pivot in every column, then the Gauss-Jordan algorithms succeeds, thereby constructing an inverse. ii) Conversely, if A does not have a pivot in every column, it must have at least one row of zeros.But we know from Lemma 2.6.4 that the $\mathcal{E}_i$ matrices will never have a row of zeros, showing that A cannot have a right inverse, and hence no inverse.

☞ The square matrix A has a left inverse if and only if it has a right inverse.

  If A has a left inverse L, then we can compute the right inverse by $AX = I \Leftrightarrow LAX = L \Leftrightarrow X = L$. Conversely, if A has a right inverse X, it is the solution of $AX = I$. The Gauss-Jordan algorithm computes this solution by constructing a left inverse.

☞ To check whether a given matrix B is an inverse of a square matrix A, it is sufficient to check that B is square, and that either $AB = I$ or $BA = I$ hold.

2.8 The LU decomposition

So far, we have solved matrix equations of the form $AX = B$ where X has more than one column by using Gauss-Jordan elimination. We now turn to investigate the solution using Gaussian Elimination and back-substitution.

  The naive approach would be to assemble the solution from individual $Ax = b$ problems by decomposing B into columns, as we did before. We can however do better, using the pattern of the product of unit lower triangular matrices we saw in example 1.2.13, and an extension of the backsubstitution idea.

2.8.1 Forward/Backward substitution

Suppose we want to solve a problem of the form $LUx = b$ where L and U are invertible lower and upper triangular matrices respectively.

  We note we can easily solve this problem for x in two steps, as follows:

1. Define a new vector $y = Ux$. Since x is unknown, so is y. Substituting into the equation we want to solve, we see that $Ly = b$. Since L is lower triangular, this is easy to solve by forward substitution, yielding $y = L^{-1}b$.

2. Now that we have y, we can solve $Ux = y$ by backward substitution.

Example 2.8.1. *Solving $LUx = b$ by forward/backward substitution*
Consider the following problem:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -3 & 1 \end{pmatrix}
\begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
=
\begin{pmatrix} 10 \\ -18 \\ 15 \\ -19 \end{pmatrix}
$$

We note that the two coefficient matrices are lower and upper triangular respectively.

Step 1: Substitute $Ux = y$

$$
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}
=
\begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
$$

and solve the resulting problem $Ly = b$ by forward substitution:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -3 & 1 \end{pmatrix}
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}
=
\begin{pmatrix} 10 \\ -18 \\ 15 \\ -19 \end{pmatrix}
\Leftrightarrow
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}
=
\begin{pmatrix} 10 \\ 2 \\ -3 \\ -6 \end{pmatrix}
$$

Step 2: Insert this solution back into the equation $Ux = y$ and solve the resulting problem by backsubstitution.

$$
\begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
=
\begin{pmatrix} 10 \\ 2 \\ -3 \\ -6 \end{pmatrix}
\Leftrightarrow
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
=
\begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}
$$

The requirement that the matrices L and U be invertible guarantees that the problem has a unique solution. This is not strictly necessary however. In particular, the method is easily extended to the case where U is a matrix, not necessarily square, in row-echelon form.

A further trivial generalization of interest is to allow for permutations: if P is a permutation matrix of suitable size, the problem $PLUx = b$ only requires an extra permutation step, since $P^{-1} = P^t$. Carrying out the computations in the above example, the reader may have been struck by a difference in the two sets of computations: The forward substitution required no divisions, since L was unit triangular. For the backward substitution however, the solution for each variable x_i required a division by the pivot. If this problem is solved repeatedly for different right hand sides b, it is tempting to precompute the inverses of the pivots. The result is a problem of the form shown in the following example:

Example 2.8.2. *Solving $PLDUx = b$*
Consider the following problem

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 110 \\ 120 \\ -235 \end{pmatrix},$$

*where the matrices multiplying the unknown vector x from left to right are
P (a permutation matrix), L (a unit lower triangular matrix), D (a diagonal
matrix), and U (a row-echelon form matrix with pivots scaled to 1). We proceed
as before, inverting one matrix at a time from left to right:*

> *Step 1: substitute $y = LDUx$.   Solve $Py = b$ for y.*
> *Step 2: substitute $w = DUx$.   Solve $Lw = y$ for w.*
> *Step 3: substitute $v = Ux$.   Solve $Dv = w$ for v.*
> *Step 4:   Solve $Ux = v$ for x.*

We obtain

$$Py = b \quad \Rightarrow y = \begin{pmatrix} 110 & -235 & 120 \end{pmatrix}^t \quad \text{by multiplying with } P^t$$

$$Lw = y \quad \Rightarrow w = \begin{pmatrix} 110 & 75 & 10 \end{pmatrix}^t \quad \text{by forward substitution}$$

$$Dv = w \quad \Rightarrow v = \begin{pmatrix} 11 & 15 & 5 \end{pmatrix}^t \quad \text{by scaling with } D^{-1}$$

Finally, solving $Ux = v$ using back-substitution, we obtain the solution

$$x = \begin{pmatrix} -4 & 0 & 0 & 5 \end{pmatrix}^t + \alpha \begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}^t.$$

*If we repeat the problem multiple times with different right-hand sides b, we may
choose to precompute D^{-1}.*

2.8.2 Gaussian Elimination and the LU decomposition

Look again at the derivation Eq (2.26) As we carry out the Gaussian Elimination
algorithm, we compute $A_k = E_k E_{k-1} \cdots E_1 A$ for some number of steps k to
yield a row-echelon form matrix A_k. Since the elementary operation matrices E_i
are invertible, we can rewrite this equation $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A_k = \mathcal{E}_k^{-1} A_k$.

Notice that if we restrict ourselves to matrices A that do not require row
interchanges, then all the E_i^{-1} matrices are lower triangular, and so is their
product $\mathcal{E}_k^{-1}$.

Let us consider this case first.

Case Gaussian Elimination without Row Interchanges

We have seen the $Ax = b$ system in example 2.8.1 for the Forward/Backward
substitution previously in example 2.5.2. Let us look at it in detail. Three
Gaussian Elimination steps led from matrix A to the row-echelon form matrix

A_3, where

$$A = \begin{pmatrix} 2 & 2 & -1 & 1 \\ -4 & -5 & 2 & -1 \\ 4 & 5 & 1 & 1 \\ -6 & -10 & -6 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Since we started with a square matrix A and found pivots in every row, A_3 is upper triangular (with entries colorized red for emphasis) and invertible.

The Gaussian Elimination matrices used were

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

Using Eq (2.18) these matrices are trivial to invert: All we need to do is to change the signs of the offdiagonal entries (colorized blue). What remains is to multiply these matrices to obtain $\mathcal{E}_3^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$.

$$\mathcal{E}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -3 & 1 \end{pmatrix}. \tag{2.31}$$

The product is made up of the non-zero columns of the individual inverses. As expected, this matrix is lower triangular. In summary, we have obtained

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 4 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

the decomposition of the matrix A we used in the forward/backward substitution example 2.5.2.

Since the non-zero off-diagonal entries in the E_i Gaussian Elimination matrix at step i are in column i, the matrix product $\mathcal{E}_k^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-k}$ will always have the pattern of entries analyzed in example 1.2.13: at each step $i = 1, 2, \cdots k$ of the Gaussian algorithm, we obtain the next column in the matrix $L = \mathcal{E}_k^{-1}$, allowing us to read the decomposition matrices $A = LU$ right out of the computational layout.

☞ **Do not use any scaling matrices**: if any of the E_i matrices is not unit lower triangular, its inverse requires a scaling operation in addition to the sign changes, disrupting the pattern.

Example 2.8.3. *LU decompositon of a matrix requiring no row interchanges*

Consider the following Gaussian Elimination computational layout for a matrix A (we are only interested in decomposing A, so we do not need to augment the matrix.:

$$
\begin{pmatrix}
1 & 2 & -1 & 1 & 1 \\
6 & 11 & 0 & 8 & 3 \\
3 & 10 & -26 & -3 & 13 \\
2 & 6 & -11 & 5 & 4 \\
-1 & 0 & -12 & -10 & 3
\end{pmatrix}
$$

The matrix A. Select the 1st pivot, and hence the 1st row of U.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-6 & 1 & 0 & 0 & 0 \\
-3 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 & 1 & 1 \\
0 & -1 & 6 & 2 & -3 \\
0 & 4 & -23 & -6 & 10 \\
0 & 2 & -9 & 3 & 2 \\
0 & 2 & -13 & -9 & 4
\end{pmatrix}
$$

We have obtained the 1st column of L. Select the 2nd pivot, and thus the 2nd row of U.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 & 1 & 1 \\
0 & -1 & 6 & 2 & -3 \\
0 & 0 & 1 & 2 & -2 \\
0 & 0 & 3 & 7 & -4 \\
0 & 0 & -1 & -5 & -2
\end{pmatrix}
$$

We have obtained the 2nd column of L. Select the 3rd pivot and thus the 3rd row of U.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -3 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 & 1 & 1 \\
0 & -1 & 6 & 2 & -3 \\
0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & -3 & -4
\end{pmatrix}
$$

We have obtained the 3rd column of L. Select the 4th pivot and thus the 4th row of U.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 & 1 & 1 \\
0 & -1 & 6 & 2 & -3 \\
0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

We have obtained the 4th (and last unknown) column of L, as well as $U = A_4$.

Check the layout to verify that i) we have only used elimination matrices, ii) no row interchanges and iii) no scaling. Since these requirements are satisfied, the original matrix A decomposes into $A = LU$, where L is a unit lower triangular matrix formed from the columns of the $E_i, i = 1, \cdots 4$ matrices with sign changes below the diagonal, and U is the row echelon form matrix A_4, i.e.,

$$
\begin{pmatrix}
1 & 2 & -1 & 1 & 1 \\
6 & 11 & 0 & 8 & 3 \\
3 & 10 & -26 & -3 & 13 \\
2 & 6 & -11 & 5 & 4 \\
-1 & 0 & -12 & -10 & 3
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
6 & 1 & 0 & 0 & 0 \\
3 & -4 & 1 & 0 & 0 \\
2 & -2 & 3 & 1 & 0 \\
-1 & -2 & -1 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 & 1 & 1 \\
0 & -1 & 6 & 2 & -3 \\
0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
$$

$$(2.32)$$

It may be desirable to scale the pivots of the U matrix to 1. Let S be the

required scaling matrix and let $\tilde{U} = SU$. We find

$$LUx = LS^{-1}SUx = LD\tilde{U}x,$$

where we have renamed $S^{-1} = D$. Note that the pivots of U appear in the corresponding diagonal entries of D. This form of the decomposition is known as the LDU decomposition (dropping the tilde.)

Example 2.8.4. LDU *decomposition*
Continuing the previous example 2.8.3, we use the scaling matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

to reduce the pivots of U to 1. Rewriting $U = S^{-1}(SU)$ we get

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & -1 & 6 & 2 & -3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -6 & -2 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Substituting this product for the matrix U in the LU decomposition Eq (2.32) obtained above, we obtain the LDU decomposition

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 \\ 2 & -2 & 3 & 1 & 0 \\ -1 & -2 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -6 & -2 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Case Gaussian Elimination with Row Interchanges

If Gaussian Elimination of a matrix A requires row interchanges, the pattern of example 1.2.13 is no longer applicable: in general, we have a product of the form $B = P_1^{-1}E_1^{-1} \cdots P_i^{-1}E_i^{-1} \cdots P_k^{-1}E_k^{-1}$, composed of Gaussian Elimination matrices E_j and row interchange matrices P_j (such that at least one of the P_j is not the identity matrix). This matrix is no longer lower triangular, breaking the forward/backward substitution scheme that made the LU decomposition attractive.

If the rows of matrix A had been suitably arranged, we would expect not to require row interchanges. This leads to the question of whether we can move the row interchange up the layout computation stack toward A, while preserving the structure of the elimination matrices: i.e., given a matrix product EP, we

wish to find a matrix $\tilde{E}$ such that $EP = \tilde{E}P$ and investigate its structure. If $\tilde{E}$ is lower unit triangular, replacing the matrix product PE in the layout by $\tilde{E}P$ will accomplish this purpose. If P is a row interchange matrix, we know $P^{-1} = P^t = P$, so that we can readily solve this equation to get $\tilde{E} = PEP$.

Since row interchanges in the Gaussian Elimination algorithm are used to introduce a pivot in the current row i of a matrix by moving up a suitable row j with $j > i$, we are led to consider the following structure of entries in the matrix product PEP show in vertical layout form:

$$\begin{pmatrix} I & 0 & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 \\ 0 & 1 & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 \\ 0 & 0 & I & \mathbf{0} & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{0} & I & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & \mathbf{0} & I \end{pmatrix} = P$$

$$E = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{l_1} & I & 0 & 0 & 0 & 0 \\ 0 & \mathbf{l_2} & 0 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{l_3} & 0 & 0 & I & 0 & 0 \\ 0 & \mathbf{l_4} & 0 & 0 & 0 & 1 & 0 \\ 0 & \mathbf{l_5} & 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{l_1} & I & 0 & 0 & 0 & 0 \\ 0 & \mathbf{l_2} & 0 & 0 & 0 & 1 & 0 \\ 0 & \mathbf{l_3} & 0 & 0 & I & 0 & 0 \\ 0 & \mathbf{l_4} & 0 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{l_5} & 0 & 0 & 0 & 0 & I \end{pmatrix} = EP$$

$$P = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{l_1} & I & 0 & 0 & 0 & 0 \\ 0 & \mathbf{l_4} & 0 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{l_3} & 0 & 0 & I & 0 & 0 \\ 0 & \mathbf{l_2} & 0 & 0 & 0 & 1 & 0 \\ 0 & \mathbf{l_5} & 0 & 0 & 0 & 0 & I \end{pmatrix} = PEP$$

Thus, pushing the row interchange matrix P for rows i and j to the left of an elimination matrix E with a pivot in some previous row i' with $i' < i < j$ results in the interchange of the off-diagonal entries in rows i and j: the pattern of non-zero entries in E remains unchanged. Remember that the column vector entries l_1, l_3, and l_5 can have any height (including zero). We now apply this operation repeatedly to the matrix B defined at the start of this discussion: push each permutation matrix to the left one at a time, starting from the left-most:

$$\begin{aligned} B =& P_1^{-1}\mathbf{E_1^{-1}}\mathbf{P_2^{-1}}E_2^{-1}P_3^{-1} \cdots P_i^{-1}E_i^{-1} \cdots P_k^{-1}E_k^{-1} \\ =& P_1^{-1}\mathbf{P_2^{-1}}\mathbf{\tilde{E}_1^{-1}}E_2^{-1}P_3^{-1} \cdots P_k^{-1}E_k^{-1} \\ =& \cdots \\ =& (P_1^{-1}P_2^{-1} \cdots P_k^{-1})(\tilde{E}_1^{-1}\tilde{E}_2^{-1} \cdots \tilde{E}_k^{-1}). \end{aligned}$$

The final matrix product has been grouped into two matrices, a permutation

matrix P and a unit lower triangular matrix L consisting of the repeatedly transformed elimination matrices (as indicated by the double tilde).

Example 2.8.5. PLU *decomposition*
Consider the following Gaussian Elimination example which requires 2 row interchanges at step 3 and step 6:

$$
\begin{pmatrix}
4 & 1 & 1 & -1 & -1 & 1 \\
12 & 6 & 4 & 0 & -1 & 6 \\
16 & -2 & 2 & -9 & -10 & -1 \\
8 & 5 & 5 & 2 & 3 & 3 \\
12 & 12 & 10 & 11 & 3 & 9 \\
4 & -5 & 3 & -7 & 9 & -8
\end{pmatrix}
\quad \text{Step 0}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-3 & 1 & 0 & 0 & 0 & 0 \\
-4 & 0 & 1 & 0 & 0 & 0 \\
-2 & 0 & 0 & 1 & 0 & 0 \\
-3 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
4 & 1 & 1 & -1 & -1 & 1 \\
0 & 3 & 1 & 3 & 2 & 3 \\
0 & -6 & -2 & -5 & -6 & -5 \\
0 & 3 & 3 & 4 & 5 & 1 \\
0 & 9 & 7 & 14 & 6 & 6 \\
0 & -6 & 2 & -6 & 10 & -9
\end{pmatrix}
\quad \text{Step 1}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & -3 & 0 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
4 & 1 & 1 & -1 & -1 & 1 \\
0 & 3 & 1 & 3 & 2 & 3 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 2 & 1 & 3 & -2 \\
0 & 0 & 4 & 5 & 0 & -3 \\
0 & 0 & 4 & 0 & 14 & -3
\end{pmatrix}
\quad
\begin{array}{l}
\text{Step 2:} \\
\text{interchange} \\
\text{rows 3 and 4} \\
\text{by } P_{34}
\end{array}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
4 & 1 & 1 & -1 & -1 & 1 \\
0 & 3 & 1 & 3 & 2 & 3 \\
0 & 0 & 2 & 1 & 3 & -2 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 4 & 5 & 0 & -3 \\
0 & 0 & 4 & 0 & 14 & -3
\end{pmatrix}
\quad \text{Step 3}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & -2 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
4 & 1 & 1 & -1 & -1 & 1 \\
0 & 3 & 1 & 3 & 2 & 3 \\
0 & 0 & 2 & 1 & 3 & -2 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 3 & -6 & 1 \\
0 & 0 & 0 & -2 & 8 & 1
\end{pmatrix}
\quad \text{Step 4}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -3 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
4 & 1 & 1 & -1 & -1 & 1 \\
0 & 3 & 1 & 3 & 2 & 3 \\
0 & 0 & 2 & 1 & 3 & -2 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 4 & 3
\end{pmatrix}
\quad
\begin{array}{l}
\text{Step 5:} \\
\text{Interchange} \\
\text{rows 5 and 6} \\
\text{by } P_{56}
\end{array}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
4 & 1 & 1 & -1 & -1 & 1 \\
0 & 3 & 1 & 3 & 2 & 3 \\
0 & 0 & 2 & 1 & 3 & -2 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 4 & 3 \\
0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}
\quad
\begin{array}{l}
\text{In row eche-} \\
\text{lon form}
\end{array}
$$

Example 2.8.6. PLU *decomposition (continued)*
We have obtained the Gaussian decomposition

$$A_6 = P_6 E_5 E_4 P_3 E_2 E_1 A \Leftrightarrow A = E_1^{-1} E_2^{-1} P_3^{-1} E_4^{-1} E_5^{-1} P_6^{-1} A_6,$$

where $E_3 = P_3$ and $E_6 = P_6$ are row interchange matrices acting on rows 3 and 4 and on rows 5 and 6 respectively. Carrying out the matrix multiplications not involving row interchanges (by combining columns with the appropriate sign changes as before) and using $P_3^{-1} = P_3$ and $P_6^{-1} = P_6$, we obtain

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 3 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} P_3 \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 1 \end{pmatrix} P_1 A_6$$

$$= P_3 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & -2 & 0 & 1 & 0 & 0 \\ 3 & 3 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 1 \end{pmatrix} P_1 A_6$$

$$= P_3 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & -2 & 0 & 1 & 0 & 0 \\ 3 & 3 & 2 & 3 & 1 & 0 \\ 1 & -2 & 2 & -2 & 0 & 1 \end{pmatrix} P_1 A_6$$

$$= P_3 P_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & -2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 2 & -2 & 1 & 0 \\ 3 & 3 & 2 & 3 & 0 & 1 \end{pmatrix} A_6,$$

where we moved P_3 left in the first step, carried out the matrix multiplication of the elimination matrices using the pattern of example 1.2.13 in the second step, and moved P_1 left in the last step. The entries colored red are intended to emphasize matrix entries used in an operation, while the blue entries emphasize the results once the operation has been carried out.

　　Computing the product of the permutation matrices and substituting A_6, we find the PLU decomposition (colorized to emphasize the structure)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & -2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 2 & -2 & 1 & 0 \\ 3 & 3 & 2 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & 1 & 3 & 2 & 3 \\ 0 & 0 & 2 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

In the example above, we streamlined the computation by moving each row interchange matrix left through a product of generalized elimination matrices, a trivial extension to the pattern.

If the LU decomposition of a matrix A exists, is it unique? If A is invertible, the answer is yes:

☞ Given a square matrix A which has an LU decomposition $A = LU$, where L is unit lower triangular and U is upper triangular. If A is invertible, then the decomposition is unique.

Proof. This is easy to see using an argument about the shape of the matrices: let A be an invertible matrix and let $A = L_1 U_1 = L_2 U_2$ be two LU factorizations of A. We first note that the inverses of L_i where $i = 1, 2$, exist. Solving for $U_i = L_i^{-1} A$, we see that the inverse $U_i^{-1} = A^{-1} L_i$ exists. We therefore can write

$$L_1 U_1 = L_2 U_2 \Leftrightarrow U_2 U_1^{-1} = L_2 L_1^{-1}$$
$$\Rightarrow U_2 U_1^{-1} = L_2 L_1^{-1} = I$$

since the first matrix product is upper triangular and the second matrix product is unit lower trangular

$$\Rightarrow U_2 = U_1 \quad \text{and} \quad L_2 = L_1.$$

$\square$

The uniqueness result easily extends to the LDU decomposition.

The requirement that A be invertible is necessary: the reader should be able to easily construct a counterexample starting from a product of the form $A = LU$.

The Symmetric Case

Consider a symmetric invertible matrix and assume it has the LDU decomposition $A = LDU$, i.e., assume we require no row interchanges. Since the matrix is invertible, it must be square and it must have a pivot in every column. The matrix U therefore will be unit upper triangular. Let us take the transpose and look at the shape of the result

$$A = LDU \Leftrightarrow A^t = (LDU)^t$$
$$= U^t D^t L^t.$$

The matrix U^t is unit lower triangular, $D^t = D$ is diagonal, and L^t is unit upper diagonal. We have obtained the LDU decomposition of $A^t = A$. Since this decomposition is unique, we must have $U = L^t$.

☞ The LDU decomposition of the symmetric invertible matrix A, if it exists, has the special form $A = LDL^t$.

Example 2.8.7. LDL^t **decomposition for symmetric matrices**
Observe that the matrix A in the following layout is symmetric. We carry out Gaussian Elimination.

$$\begin{pmatrix} 2 & 18 & -4 & -4 \\ 18 & 165 & -33 & -45 \\ -4 & -33 & 10 & -3 \\ -4 & -45 & -3 & 35 \end{pmatrix}$$

Select the 1st pivot in A. This determines the 1st column of L and the 1st diagonal entry in D.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -9 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 18 & -4 & -4 \\ 0 & 3 & 3 & -9 \\ 0 & 3 & 2 & -11 \\ 0 & -9 & -11 & 27 \end{pmatrix}$$

Select the 2nd pivot. This determines the 2nd column of L and the 2nd diagonal entry in D.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 18 & -4 & -4 \\ 0 & 3 & 3 & -9 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

Select the 3rd pivot. This determines the 3rd column of L and the 3rd diagonal entry in D.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 18 & -4 & -4 \\ 0 & 3 & 3 & -9 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Select the last pivot, i.e., the last diagonal entry in D.

We read of the LDL^t decomposition

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -2 & -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 9 & -2 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We note that not all symmetric matrices have an LDL^t decomposition, as the reader may verify with the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

2.9 Summary

The Gaussian Elimination algorithm starts with a system $Ax = b$ and yields an equivalent system $\mathcal{E}_k Ax = \mathcal{E}_k b$ in a finite number of steps k. The number of solutions of the system can be characterized by the rank of the matrix A, i.e., the number of pivots in A.

The results may be summarized by the theorems below.

Theorem 2.9.1. *Same number of equations as unknowns:* *Given a square matrix A of size $N \times N$, the following statements are equivalent:*

(a) *A is invertible.*

(b) *A has a right inverse.*

(c) *A has a left inverse.*

(d) *The equation $Ax = b$ has a unique solution x for every b.*

(e) *The equation $Ax = 0$ has only the zero solution $x = 0$.*

(f) *$rank(A) = N$.*

(g) *A is row-equivalent to I.*

(h) *A is a product of elementary row operation matrices.*

Proof. To prove equivalence, we show two circular chains of implications:

$(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (a)$ and
$(a) \Rightarrow (b) \Rightarrow (c)$.

Case (a) $\Rightarrow$ (c) If A has an inverse B then $BA = I$, and hence B is a left inverse.

Case (c) $\Rightarrow$ (d) If A has a left inverse B then $BA = I$. Thus we obtain a solution $Ax = b \Rightarrow (BA)x = Bb \Rightarrow x = Bb$. To show that this solution is unique, assume there are two solutions x_1 and x_2 such that $Ax_1 = b$ and $Ax_2 = b$. Eliminating b from these equations and multiplying by B from the left as before, we have $Ax_1 = Ax_2 \Rightarrow (BA)x_1 = (BA)x_2 \Rightarrow x_1 = x_2$.

Case (d) $\Rightarrow$ (e) This follows trivially from the special case $b = 0$.

Case (e) $\Rightarrow$ (f) Proof by contradiction: Suppose $rank(A) \neq N$. Since the rank of A cannot exceed the number of columns, $rank(A) < N$, and therefore A is row-equivalent to a matrix with at least one row of zeros. This means that A has free variables, and therefore $Ax = 0$ has an infinite number of solutions, contradicting the hypothesis (e).

Case (f) $\Rightarrow$ (g) The matrix A has N pivots, one in each of N columns. The Gauss-Jordan algorithm therefore reduces it to I.

Case (g) $\Rightarrow$ **(h)** This is a restatement of the definition of row equivalence.

Case (h) $\Rightarrow$ **(a)** $A = E_k E_{k-1} \cdots E_1$ for some set of k elementary row operation matrices. Since each of the E_i have inverses, we find

$$\begin{aligned} A^{-1} &= (E_k E_{k-1} \cdots E_1)^{-1} \\ &= E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}, \end{aligned}$$

completing the first chain and therefore establishing equivalence of each of its statements.

Case (a) $\Rightarrow$ **(b)** If A has an inverse B then $AB = I$, and hence B is a right inverse.

Case (b) $\Rightarrow$ **(a)** If A has a right inverse B then $AB = I$. Taking transposes, we have $B^t A^t = I$, so that B^t is a left inverse of A^t. Since $(c) \Rightarrow (a)$, we conclude that $B^t = (A^t)^{-1} = (A^{-1})^t$, and hence $A^{-1} = \left((A^t)^{-1}\right)^t = (B^t)^t = B$.

$\square$

We found a number of different ways to solve matrix problems of the form $AX = B$, which reduce to matrix-vector problems $Ax = b$ for column-vectors $B = b$. They are i) Gaussian Elimination with back-substitution ii) Gauss-Jordan elimination iii) LU decomposition with forward/backward substitution iv) direct computation of A^{-1} with $X = A^{-1}B$ provided the inverse exists. Which should we choose?

One criterion is to look at the number of operations required. Note that current commercial web-search applications deal with matrices with close to 3 billion columns: even minor differences in the number of operations will make a huge difference! Computer chips have grown very complex, with many optimizations to speed up computations, with the result that operation counts have become very involved. To simplify the problem, we will simply count the maximum number of multiplications required.

Consider a square matrix A of size $N \times N$, and a right hand side matrix B of size $N \times L$. We obtain the following counts

- $(N-1)^2 + (N-2)^2 + \cdots 1^2 = \frac{1}{3}N^3 - \frac{1}{2}N^2 + \frac{1}{6}N$ to eliminate terms either above or below the pivots in A

- $((N-1) + (N-2) + \cdots 1)L = \frac{1}{2}(N-1)NL$ multiplications for either forward or back substitution for pivots scaled to 1

- NL multiplications for scaling by pivots

- $N^2 L$ multiplications to compute the product $A^{-1}B$.

Comparing these results, we see that i) multiplication by the inverse matrix requires the same number of multiplications as forward/back substitution and ii) the cost of elimination dominates when $L << N$. Adding the multiplication counts for each of the algorithms and keeping only the leading term, we find

- $\frac{1}{3}N^3 + N^2L$ multiplications for Gaussian Elimination with back-substitution

- $\frac{2}{3}N^3 + N^2L$ multiplications for Gauss-Jordan elimination

- $\frac{1}{3}N^3 + N^2L$ multiplications using the LU decomposition

- $\frac{2}{3}N^3 + N^2L$ direct computation of the inverse

Clearly, Gaussian Elimination with back-substitution and its reformulation as the LU decomposition are more efficient than Gauss-Jordan elimination or direct computation using the matrix inverse.

We end this chapter with an admonition:

☞ **Do not use the inverse directly in numerical computations.**

2.10 Exercises

Exercise 2.1. *Consider the following Gauss-Jordan elimination example for the matrix $(A \mid I)$:*

$$\left(\begin{array}{ccccc|ccccc} 1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 8 & 7 & 3 & 0 & 1 & 0 & 0 & 0 \\ -3 & 4 & -25 & 16 & 7 & 0 & 0 & 1 & 0 & 0 \\ 1 & -3 & 7 & 9 & 4 & 0 & 0 & 0 & 1 & 0 \\ -1 & 4 & -7 & -16 & -7 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccccc|ccccc} 1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 16 & 7 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 9 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -16 & -7 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccccc|ccccc} 1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 5 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \left(\begin{array}{ccccc|ccccc} 1 & -2 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 5 & -2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -5 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccccc|ccccc} 1 & 0 & 8 & -14 & -6 & 3 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 5 & -2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -5 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 2 & 2 & 43 & -18 & 8 & 0 & 0 \\ 0 & -1 & 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 5 & -2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -5 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 2 & 2 & 43 & -18 & 8 & 0 & 0 \\ 0 & 1 & 0 & -7 & -3 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & -5 & 2 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -5 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1 \end{array}\right)$$

Questions:

1. *Is this matrix invertible? At what point in the computations do you know for sure?*

2. *Which column vectors of A are independent? When do you know for sure?*

What linear relationships between a given dependent column vector and the independent column vectors can you trivially read from the layout? What are their relative positions?

3. *What is the range of $y = Ax$ expressed as an implicit set of equations relating the entries of x?*

4. *Express the range of $y = Ax$ as a linear combination of vectors.*

5. *Does the LU decomposition of A exist? If so, what is it?*

Exercise 2.2. *Consider the problem $Ax = b$ and its solution by Gaussian Elimination and backsubstitution.*

1. *Explain why the free variables in the particular solution obtained by backsubstitution are equal to zero.*

2. *Explain why the free variables in the homogeneous solution obtained by backsubstitution are the entries of an identity matrix of size $K \times K$, where K is the number of free variables.*

3. *Consider the column view of $Ax = b$ and set the free variables equal to zero. How does the computational layout for this reduced problem compare to the original layout?*

4. *Consider the homogeneous problem $Ax = 0$ in column view. Obtain the computational layout resulting from setting all but one of the free variables equal to zero, with the remaining variable set equal to one. How does this compare with the layout in the previous question?*

5. *Consider the problem from Example 2.5.3. Obtain the particular and homogeneous solution vectors by using the reduced systems. Note that no computations are needed beyond those shown in the example.*

Exercise 2.3. *Consider the Gaussian Elimination example 2.4.1.*

1. *Choose a right hand side vector $\tilde{b}$ for the row echelon form matrix that makes the system inconsistent. Reverse the gaussian elimination steps to obtain a right hand side vector b for which the system is inconsistent.*

2. *Find all possible right hand side vectors for which the original system is inconsistent.*

3. *Explain how you can obtain these vectors from the elementary operation matrices. (See Eq (2.27).*

Exercise 2.4. *Obtain the $PLDU$ decomposition of the matrices*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Is $U = L^t$ in both cases?

Chapter 3

Vector Spaces and Vector Decompositions

3.1 Introduction

Our investigations so far have yielded a solution method for systems of linear equations $Ax = b$. When there are solutions, we saw in Eq (2.29) that they are of the form $x = x_p + \sum_{k=1}^{K} \alpha_k x_k$, where x_p is any one solution of the system, and the x_k solve the system $Ax = 0$. The number K of arbitrary parameters α_k is the number of free variables that appeared in the Gaussian Elimination algorithm.

We now focus on the column view of the system of equations to gain a better understanding of the free variables. This modest goal will however lead us to a new understanding of hyperplanes and systems of coordinates of hyperplanes.

3.1.1 A Simple Example

Closure under Addition and Scalar Multiplication

Consider the set of vectors $L = \{w \mid w = \alpha\,(2\ 4\ 3)^t\}$ that describes a line through the origin with direction vector $(2\ 4\ 3)^t$. The set L has a striking feature: start with any two vectors in L, say $w_1 = \alpha_1\,(2\ 4\ 3)^t$ and $w_2 = \alpha_2\,(2\ 4\ 3)^t$ and look at $w_1 + w_2 = (\alpha_1 + \alpha_2)\,(2\ 4\ 3)^t$. We see that this vector is a member of L with $\alpha = \alpha_1 + \alpha_2$. Thus, adding any two vectors on the line L yields a vector on the line. This property is refered to as **closure under addition**.

Further, $\beta w_1 = (\beta \alpha_1)\,(2\ 4\ 3)^t$ is a member of L with $\alpha = \beta \alpha_1$, i.e., scaling any vector on the line by some scalar again yields a vector on the line: This property is refered to as **closure under scalar multiplication**. This observation can be generalized as follows:

- ☞ Any linear combination of vectors from a given a set of vectors V that is closed under addition and closed under scalar multiplication results in a

<div align="center">99</div>

vector that is still in V.

Span of Vectors

Sets of vectors closed under addition and scalar multiplication are easy to construct: given some vectors $v_1, v_2, \ldots v_n$, just form the set of all possible linear combinations

$$S = \{w \mid w = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n \mid \alpha_i \in \mathbb{R}, i = 1, 2, \ldots n\}.$$

This set is called the **span of the vectors** $v_1, v_2, \cdots v_n$, denoted by $S = \mathrm{span}\{v_1, v_2, \cdots v_n\}$.

Consider the following linear combination of vectors in 3D:

$$\mathbf{u} = 2 \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + 4 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$$

Adding the vectors, we see that $\mathbf{u} = (11\ 4\ 6)^t$ which identifies a point in space. If we allow the coefficients of this linear combination to vary, we find that the span of these three vectors is a set of points in 3D of the form

$$\mathbf{u} = \alpha \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix} \tag{3.1}$$

or equivalently

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 0 & 4 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \tag{3.2}$$

where we have set $\mathbf{u} = (x\ y\ z)$ and transcribed the problem to matrix form.

The reader is invited to check that the set

$$P = \{u \mid u \text{ satisfies Eq 3.1}, (\alpha\ \beta\ \gamma) \in \mathbb{R}^3\}$$

is closed under addition and scalar multiplication. From the second representation Eq (3.2), we readily recognize that the set of points P is the set of all possible right hand sides of an $Ax = b$ problem where A is a matrix with columns made up from the vectors in the linear combination, and the right hand side is the column vector representation of u. Augmenting A with u and reducing to row echelon form, we get

$$(A \mid u) \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \left. \begin{matrix} \frac{1}{4}y \\ \frac{1}{3}x - \frac{1}{6}y \\ -\frac{1}{3}x - \frac{7}{12}y + z \end{matrix} \right)$$

Looking at this result, we make a number of important observations:

☞ The system 3.2 is consistent provided u lies in the plane P given by $z = \frac{1}{3}x + \frac{7}{12}y$. (This is the idea first presented in Example 2.7.3).

The span of the vectors

$$P = span\left((2\,4\,3)^t, (3\,0\,1)^t, (5\,4\,4)^t\right)$$

is a plane in 3D that contains the origin.

☞ Any point in the plane P can be described as a linear combination of the vectors in the span. For the three vectors used in the definition of P, this linear combination is not unique: any solution of Eq (3.2) will describe this point in the form Eq (3.1).

☞ Instead of using the coordinates $(x\,y\,z)$ directly, we may alternatively use the coefficients $(\alpha\,\beta\,\gamma)$ to identify a point in this plane. To recover the original coordinate vector, we only need to compute the linear combination Eq (3.2).

However, the third variable γ is free, and can be chosen arbitrarily. The choice $\gamma = 0$ effectively removes the third vector from 3.2, leaving us with the reduced system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

A point u in the plane b is identified uniquely by the **the coordinate vector** $(\alpha\,\beta)^t$ with respect to the **basis vectors** $(2\,4\,3)^t$ and $(3\,0\,1)^t$. In terms of x, y, z, the coordinate vector is specified by $\alpha = \frac{1}{4}y$, and $\beta = \frac{1}{3}x - \frac{1}{6}y$.

☞ The remaining two vectors still determine the same plane, and any point in the plane is uniquely identified with just 2 variables $(\alpha\,\beta)^t$ rather than the 3 coordinates $(\alpha\,\beta\,\gamma)$ we started with.

These two vectors therefore can be used to define coordinate axes for this plane: the axis $\{\alpha(2\,4\,3)^t \mid \alpha \in \mathbb{R}\}$ and the axis $\{\beta(3\,0\,1)^t \mid \beta \in \mathbb{R}\}$.

The number of variables required (the number of axes) is **the dimension of the plane**: it is 2, even though the plane is embedded in $\mathbb{R}^3$, i.e., the vectors used in the definition of P reside in $\mathbb{R}^3$.

We rephrase this result as follows:

The span P can equally well be described with a reduced set of vectors, i.e.,

$$P = span\left((2\,4\,3)^t, (3\,0\,1)^t\right).$$

Any point in P can be described as a linear combination of this reduced set of vectors. This description is unique: the reduced set of vectors in

the definition of the span were chosen by simply removing all vectors corresponding to free variables in Eq (3.2).

The number of vectors required to uniquely describe elements in P is the dimension of P. (We will be able to show that this number is unique).

3.2 Vector Spaces

To avoid having to discuss combinations of rows with M entries separately from combinations of rows of N entries, we will generalize the notion of vector to be an otherwise unspecified element of a set V. We will need an addition operation for these vectors, i.e., some function that given two vectors $u \in V$ and $v \in V$ produces a new vector $w \in V$, which we will denote by $w = u + v$. Further, we will need an operation called scalar multiplication, which given any scalar $\alpha \in \mathbb{F}$ and any vector $u \in V$ produces a new vector $w \in V$. We will denote this operation by $w = \alpha u$.

☞ Take special note of the part of the definition that specifies that for *any two vectors* u and v in V, the vectors can be added, *and the result is again a vector* w in V. This property is usually referred to as V **is closed under addition**.

☞ Similarly, the product of any vector u in V with any scalar yields some vector w in V, i.e., V **is closed under scalar multiplication**.

☞ More generally, the notion of scalar may be abstracted as well. Computer programs such as SAGE allow one to specify that scalars should be real numbers in $\mathbb{R}$, or rationals in $\mathbb{Q}$, etc. For simplicity, we will avoid this further generalization, since it would require analyzing the properties required for scalars.

3.2.1 Definition of a Vector Space

Given the set of vectors V, the set of scalars $\mathbb{F}$ together with vector addition and scalar multiplication, we next specify the axioms that these operations must satisfy to manipulate linear combinations. The required properties fall into two groups:

- **Properties of Vector Addition** For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, the following axioms must hold

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \qquad \text{Commutativity} \quad (3.3a)$$
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \qquad \text{Associativity} \quad (3.3b)$$
$$\exists \mathbf{0} \in V \ s.t. \ \mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \qquad \text{Existence of a Zero Vector} \quad (3.3c)$$
$$\exists -\mathbf{u} \in V \ s.t. \ \mathbf{u} + -\mathbf{u} = \mathbf{0} \qquad \text{Existence of an Additive Inverse} \quad (3.3d)$$

- **Properties of Scalar Multiplication** For any vectors $\mathbf{u}, \mathbf{v}$ in V and any scalars α, β in $\mathbb{F}$, the following axioms must hold

$$\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u} \qquad\qquad\qquad\text{Associativity} \quad (3.4\text{a})$$
$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} \qquad\qquad\text{Distributivity of vector sums} \quad (3.4\text{b})$$
$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u} \qquad\qquad\text{Distributivity of scalar sums} \quad (3.4\text{c})$$
$$1\mathbf{u} = \mathbf{u} \qquad\qquad\qquad\qquad\qquad\text{Unit} \quad (3.4\text{d})$$

Comparing with the properties of vector addition Eqs(1.9), we see they are almost identical. The differences are in Eq(3.3c), (which states that there must be some vector denoted by $\mathbf{0}$ in V that acts the way we would expect for "zero"), and in Eq(3.3d) (which states that for any vector $\mathbf{u} \in V$, there must exist some vector denoted by $-\mathbf{u}$ in V, that acts as an additive inverse). Finally we note the omission of the property 1.10e, which can be derived from the properties above. In particular, we have

- **Additional Properties of Vector Spaces** For any vector $\mathbf{u} \in V$ and any scalar $\alpha \in \mathbb{F}$, the axioms imply

$$\mathbf{0} \text{ is unique} \qquad\qquad\qquad\text{Uniqueness of the zero vector} \quad (3.5\text{a})$$
$$-\mathbf{u} \text{ is unique} \qquad\qquad\text{Uniqueness of the additive inverse} \quad (3.5\text{b})$$
$$-1\mathbf{u} = -\mathbf{u} \qquad\qquad\text{Scaling by -1: the additive inverse} \quad (3.5\text{c})$$
$$0\mathbf{u} = \mathbf{0} \qquad\qquad\qquad\text{Scaling by 0: the zero vector} \quad (3.5\text{d})$$
$$\alpha\mathbf{0} = \mathbf{0} \qquad\qquad\qquad\qquad\text{Scaling the } \mathbf{0} \text{ vector} \quad (3.5\text{e})$$
$$\alpha\mathbf{u} = \mathbf{0} \Rightarrow \alpha = 0 \text{ or } \mathbf{u} = \mathbf{0} \qquad\qquad\text{Factors of the } \mathbf{0} \text{ vector.} \quad (3.5\text{f})$$

Note the last statement in particular: it is quite different than the statement for matrices A, B: given $AB = 0$, the conclusion $A = 0$ or $B = 0$ is not valid.

The complete specification of a vector space thus requires the following:

- the set of vectors V
- the set of scalars $\mathbb{F}$, (typically $\mathbb{R}$)
- the addition operation $\mathbf{u} + \mathbf{v}$ for vectors $\mathbf{u}$ and $\mathbf{v}$
- the scalar multiplication operation $\alpha\mathbf{u}$
  (typically represented by a small space between the scalar and the vector).

A vector space $\mathcal{V}$ specification is a 4-tuple

$$\mathcal{V} = (V, \mathbb{F}, \text{ addition, scalar multiplication}).$$

It is common practice to identify a vector space with the name of the set of vectors V rather than the tuple $\mathcal{V}$ when the remaining specifications are obvious from the context.

We may generalize matrices to allow enties that are vectors from a vector space. This allows us to express linear combinations of vectors as matrix products. E.g.,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11}v_1 + \alpha_{21}v_2 \\ \alpha_{21}v_1 + \alpha_{22}v_2 \end{pmatrix},$$

where the entries v_i, wi are vectors in a vector space $\mathcal{V}$ and the α_{ij} are scalars. Note however that only one matrix with vector entries may appear in a product of matrices, since we have not defined a product of vectors.

3.2.2   Definition of a Vector Subspace

Motivated by the simple example at the beginning of the chapter, where we found that lines and planes formed subsets of $\mathbb{R}^3$ closed under addition and scalar multiplication, we consider a **non-empty subset** U of the vectors V in some vector space $\mathcal{V} = (V, \mathbb{F}, \text{addition, scalar multiplication})$, and check whether $\mathcal{U} = (U, \mathbb{F}, \text{addition, scalar multiplication})$ satisfies the requirements for a vector space. If so, we say that $\mathcal{U}$ is a **subspace** of $\mathcal{V}$.

☞ **It is not necessary to check that $\mathcal{U}$ satisfies all of the axioms** 3.3 and 3.4 since most will be inherited from $\mathcal{V}$. We only need to verify that U is not empty, and that it is closed under both vector addition and scalar multiplication, i.e., we only need to check that

– **For any vectors u and v in U, $u + v \in U$**

– **For any vector $u \in U$, and any scalar $\alpha \in \mathbb{F}$, $\alpha u \in U$.**

☞ Note that **the zero vector must be in the subspace** U: this follows from the choice $\alpha = 0$ in the closure under scalar multiplication requirement.

☞ Note that for any vector $u \in U$, **the additive inverse must also be in the subspace** U: this follows from the choice $\alpha = -1$ in the closure under scalar multiplication requirement.

☞ The **smallest subspace of a vector space** consists of only the zero vector $U = \{0\}$.

We generalize the span of a set of vectors to any type of vector: given a set of n vectors $v_1, v_2, \cdots v_n$ in V drawn from a vector space $\mathcal{V} = (V, \mathbb{F}, +,)$ we define

$$\text{span}(v_1, v_2, \cdots v_n) = \{w \mid w = \sum_{i=1}^{n} \alpha_i v_i, \ \alpha_i \in \mathbb{F}, \ i \in 1, 2, \cdots n\}$$

The check that **the span$(v_1, v_2, \cdots v_n)$ is a subspace of** $\mathcal{V}$ is a trivial exercise left to the reader.

3.2.3   Vector Space Examples

The following examples all satisfy the requirements for a vector space.

The Vector Space $\mathbb{F}^N$ and Matrices of Fixed Size

- The space of vectors $\mathbb{F}^N$ with a fixed number of entries N and their representation as matrices, the space of row vectors $\mathbb{F}^{1 \times N}$ and the space of column vectors $\mathbb{F}^{N \times 1}$. The scalars are the elements of $\mathbb{F}$, the addition and scalar multiplication operations are vector and matrix addition and scalar multiplication respectively. The spaces in question are

  - $\mathbb{F}^N = \left(\mathbb{F}^N, \mathbb{F}, \text{ vector addition, scalar multiplication} \right)$
  - $\mathbb{F}^{1 \times N} = \left(\mathbb{F}^{1 \times N}, \mathbb{F}, \text{ matrix addition, scalar multiplication} \right)$
  - $\mathbb{F}^{N \times 1} = \left(\mathbb{F}^{N \times 1}, \mathbb{F}, \text{ matrix addition, scalar multiplication} \right)$

  These spaces served as the prototype for the generalization to vector spaces.

- Matrices of fixed size $M \times N$, (e.g., all matrices of size 2×2) with vector addition and scalar multiplication by scalars in $\mathbb{F}$ also satisfy the requirements of a vector space.

  - $\mathcal{M}_{MN} = \left(\mathbb{F}^{M \times N}, \mathbb{F}, \text{ matrix addition, scalar multiplication} \right)$

- The set of lower triangular matrices of fixed size $N \times N$ with matrix addition and scalar multiplication by scalars in $\mathbb{F}$ form a subspace of $\mathcal{M}_{NN}$.

  In this text, the ususal choice for $\mathbb{F}$ are the reals $\mathbb{R}$ or the rationals $\mathbb{Q}$..

- The set of symmetric matrices of fixed size $N \times N$ is a subspace of $\mathcal{M}_{NN}$.

Function Spaces

Consider the set of functions $F = \{ f \mid f \text{ is a function with } f : \mathbb{R} \to \mathbb{R} \}$, with scalars in $\mathbb{R}$, with the usual definitions for the addition of functions (denoted by $+$), and for multiplication of a function by a scalar (denoted by a small space between the scalar and the function).

- $\mathbb{F}$ is a vector space.

- $\mathbb{F}$ has many subspaces of interest that can be obtained by restricting the functions in F. In particular, consider

  - $\mathcal{C} = (C, \mathbb{R}, +,)$ where C are the continuous functions contained in F
  - $\mathcal{C}^n = (C^n, \mathbb{R}, +,)$ where C^n are the functions contained in F that have n continuous derivatives.

    Note that we obtain a hierarchy of subspaces since $\mathcal{C}^{n+1}$ is a subspace of $\mathcal{C}^n$. Of particular interset is $\mathcal{C}^\infty$, the subspace of infinitely differentiable functions.

- Consider the subset $Y = \{ y \mid \frac{d^2}{dt^2} y - 9y = 0 \}$ of $\mathcal{C}^\infty$, i.e., the set of solutions of the differential equation.

       – $(Y, \mathbb{R}, +,)$ is a subspace of $\mathbb{C}^\infty$.

Considering the set of functions F over domains and codomains other than $\mathbb{R}$, e.g., by considering the set of functions over an interval $[a, b]$ of $\mathbb{R}$, $F[a, b] = \{f \mid f$ is a function with $f : [a, b] \to \mathbb{R}\}$, yields examples of vector spaces similar to those considered above.

☞ Vector spaces are common mathematical objects. Subspaces of a given vector space are easy to construct: it suffices to pick some vectors in the space and form their span.

☞ Subspaces can contain other subspaces: consider vectors $v_1, v_2, \cdots v_n$ in some vector space, and consider the subspaces $S_1 = span(v_1)$, $S_2 = span(v_1, v_2)$, ..., $S_n = span(v_1, v_2, \cdots v_n)$. Each of the S_k for k in $1, 2, \cdots n$ contains the S_i for $i < k$. For example in R^3, the origin, a line through the origin, a plane containing such a line, and all of R^3 form such a nested set of subspaces.

☞ Subspaces can be thought of as hyperplanes: note in particular that given two vectors u and v in a subspace, all points w on the line from u to v, i.e., all points $w = u + t(v - u)$ lie in the subspace.

Example 3.2.1. *The vector space of polynomials of degree ≤ 2*
Consider the set of polynomials
$\mathcal{P}_2(-1,1) = \{p(x) \mid p(x) = \alpha + \beta x + \gamma x^2, \alpha, \beta, \gamma \in \mathbb{R}, -1 < x < 1\}$

If we draw $y = p(x)$ in the $x-y$ plane, we see lines and parabolae that are the vectors in this vector space. Note that we can think of this as the representation of the entries $p(x)$ of the vector indexed by x, i.e., a vector with an infinite number of entries. The zero vector $p_0(x)$ is obtained by taking any one vector, i.e., $p_1(x) = 1 + x$, and multiplying it by the scalar zero: $p_0(x) = 0(1+x) = 0$. The zero vector is the x-axis. The additive inverse of a vector is obtained by multiplying it by the scalar -1. So $-p_1(x) = -1(1+x)$.
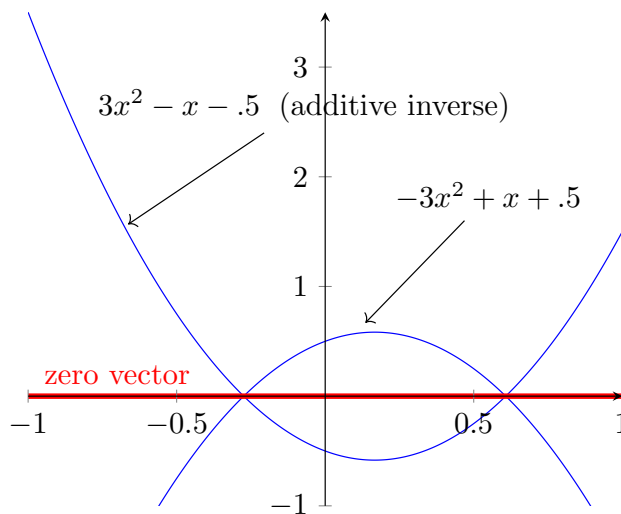
Figure 3.1: Three example vectors in $\mathcal{P}_2(-1,1)$, the vector space of polynomials of degree ≤ 2, $x \in (-1,1)$: the zero vector (the x-axis $p_0(x) = 0$), the vector $p(x) = -3x^2 + x + 0.5$ and its additive inverse $-p(x) = 3x^2 - x - 0.5$. The graphical representation may be thought of as a display of the entries $p(x)$ of a vector indexed by x.

Example 3.2.2. *Checking for subspaces*

Consider $S = \{p(x) \mid p(x) = \alpha(1 + x^2) + \beta(1 + x^3), \text{ for any } \alpha, \beta \text{ in } \mathbb{R}\}$. Does this set of vectors form a subspace of $\mathcal{P}_3$ the set polynomials of degree less than or equal to 3?

Since $\mathcal{P}_3$ is known to form a vector space under addition and scalar multiplication of functions, we only need to check that S is not empty, and that it is closed under addition and scalar multiplication.

S is not empty: To verify this, it is sufficient to exhibit an element of S. Since we know that 0 must be a member of any subspace, it is often advantageous to check whether it is present in S. Choosing $\alpha = 0, \beta = 0$ in the definition of elements of S, we obtain $p(x) = 0$, a member of S.

*S is closed under addition: Starting with any two elements of S,
$p_1(x) = \alpha_1(1 + x^2) + \beta_1(1 + x^3)$ and $p_2(x) = \alpha_2(1 + x^2) + \beta_2(1 + x^3)$ for any choice of $\alpha_1, \beta_1, \alpha_2$ and β_2, we must verify that $p_1(x) + p_2(x)$ is in S, i.e., we must show that there are values α and β such that $p_1(x) + p_2(x) = \alpha(1 + x^2) + \beta(1 + x^3)$.*

Since

$$p_1(x) + p_2(x) = \alpha_1(1 + x^2) + \beta_1(1 + x^3) + \alpha_2(1 + x^2) + \beta_2(1 + x^3)$$
$$= (\alpha_1 + \alpha_2)(1 + x^2) + (\beta_1 + \beta_2)(1 + x^3)$$

we find that the choice $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$ satisfies this requirement. S is indeed closed under addition.

Since the closure requirements are satisfied, the set of vectors S is a subspace of $\mathcal{P}_3$.

Subspaces are formed from spans of vectors. Any set of vectors with a definition that cannot be written as a span of vectors will not be a subspace. To prove that a set is not a subspace, try for a counterexample: frequently, we can show that $\mathbf{0}$ is not in the set, or that for some given $\mathbf{u}$, the vector $-\mathbf{u}$ is not in the set. More generally, we start with parameterized versions u and v of the vectors in the subset and a scalar parameter α (we need to show closure of addition and scalar multiplication for all vectors and scalars), and compute $u + v$ and αu and compare the results to the definition of the subset.

Example 3.2.3. *Checking for subspaces*
Consider $S = \{w \in \mathbb{R}^3 \mid w = \alpha(1\ 2\ 5) + \beta(0\ 1\ 2) + \gamma(1\ 0\ -1), \alpha, \beta, \gamma \in \mathbb{R}, \gamma \geq 0\}$.
Does S form a subspace of $\mathbb{R}^3$?

Note that the definition of S would be a span of vectors (and therefore a subspace), were it not for the restriction $\gamma \geq 0$. Suspecting that S is not a subspace, we look for a counterexample to the closure requirements.

Is $\mathbf{0} = (0\ 0\ 0)$ a member of S? Writing

$$\mathbf{0} = \alpha(1\ 2\ 5) + \beta(0\ 1\ 2) + \gamma(1\ 0\ -1)$$

we see that the choice $\alpha = 0, \beta = 0, \gamma = 0$ is allowed: $\mathbf{0}$ is in S.

Is S closed under scalar multiplication? Since $\gamma \geq 0$, let us check what happens if we try for a vector constructed by violating this restriction: for $\alpha = 0, \beta = 0, \gamma = 1$, we obtain the vector $(1\ 0\ -1)$ in S. Closure under scalar multiplication requires that its additive inverse $(-1\ 0\ 1)$ must be a member of S.

This vector could be constructed with $\alpha = 0, \beta = 0, \gamma = -1$, which is not allowed by the definition of S, but are these the only possible choices for α, β and γ? We need to solve

$$\alpha(1\ 2\ 5) + \beta(0\ 1\ 2) + \gamma(1\ 0\ -1) = (-1\ 0\ 1).$$

Rewriting this equation in matrix form (by writing the equivalent set of scalar equations, or more directly by taking the transpose), we are looking for solutions of

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 5 & 2 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Solving, we find the unique solution $\alpha = 0, \beta = 0, \gamma = -1$, which the definition of S does not allow.

We conclude that S is not a subspace of $\mathbb{R}^3$.

The Fundamental Vector Spaces of a Matrix

Given a matrix A of size $M \times N$ with entries in $\mathbb{F}$, we may define a number of vector spaces that will prove very useful:

- Consider first the columns of A. These are column vectors of size $M \times 1$, i.e., vectors drawn from the vector space $\mathbb{F}^{M \times 1}$. The span formed from these columns is a subspace $\mathcal{C}(A)$ of this vector space, **the column space of A**.

- Consider the rows of A. These are row vectors of size $1 \times N$, i.e., vectors

drawn from the vector space $\mathbb{F}^{1 \times N}$. The span formed from these rows is a subspace $\mathcal{R}(A)$ of this vector space, **the row space of** A.

- Consider the homogeneous solutions of $Ax = 0$. These are column vectors of size $N \times 1$, i.e., vectors drawn from the vector space $\mathbb{F}^{N \times 1}$. The span formed from these homogeneous solutions is a subspace $\mathcal{N}(A)$ of this vector space, **the null space of** A, also known as the **kernel of** A.

Note that all right hand sides b for which $Ax = b$ has a solution lie in the column space of A, which is obvious when $Ax = b$ is rewritten in column view form.

☞ The column space of a matrix A is the range of the function $y = Ax$.

A similar set of spaces can be defined for the transpose A^t. Since rows/columns of A are columns/rows of A^t respectively, this leaves a set of 4 distinct spaces. Given a matrix A of size $M \times N$, these are $\mathcal{R}(A)$ and $\mathcal{N}(A)$ (vectors in $\mathbb{R}^N$, the domain of the function $y = Ax$), and $\mathcal{C}(A)$ and $\mathcal{N}(A^t)$ (vectors in $\mathbb{R}^M$, the codomain of the function $y = Ax$).

(There are many other spaces of interest associated with matrices that will not be discussed here.)

3.3 Bases

The exposition in this section is carefully written to apply to any kind of vector, although vectors in $\mathbb{R}^N$ serve as a model. In Eq (2.30) we first encountered the notion of linearly independent vectors by looking at the rows of a matrix A. We now look at homogeneous solutions $Ax = 0$. In particular, we are interested in the case where $Ax = b$ has a unique solution, i.e., the case when the homogeneous equation has only the trivial solution $x = 0$. Rewriting this equation in column view, and generalizing to any type of vector yields the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_N v_N = 0. \tag{3.6}$$

This suggests the definition that a set of vectors is **linearly independent** if and only if this equation only has the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$. The vectors are said to be **linearly dependent** otherwise.

☞ If the vectors are linearly dependent, this means that some α_i must be non-zero, so that we can solve for the vector

$$v_i = - \sum_{j=1, j \neq i}^{N} \frac{\alpha_j}{\alpha_i} v_j,$$

i.e., we can express the vector v_i as a linear combination of the remaining vectors.

Example 3.3.1. *Linearly dependent vectors*
*Consider the following Gauss-Jordan Elimination problem for solving $Ax = 0$,
where A consists of five columns $A = (a_1\ a_2\ a_3\ a_4\ a_5)$:*

$$\begin{pmatrix} 1 & 2 & 10 & 1 & 11 \\ 1 & 3 & 13 & 1 & 15 \\ 3 & 4 & 24 & 3 & 25 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 10 & 1 & 11 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & -2 & -6 & 0 & -8 \end{pmatrix} \quad \text{Multiply by } E_1$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 & 1 & 3 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Multiply by } E_2$$

It is easy to see that each non-pivot column in the reduced row-echelon form
matrix is a linear combination of the preceding pivot columns. E.g., columns 3,
1 and 2 are related by

$$\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

*Multiplying this equation by E_2^{-1} and E_1^{-1}, we see that this equation must hold
at every level of the layout. In particular, for the original matrix A, columns 3,
1 and 2 are related by*

$$\begin{pmatrix} 10 \\ 13 \\ 24 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

We also observe that the pivot columns are linearly independent, since they
reduce to the columns of I.

Example 3.3.2. *Linearly independent functions*
Consider the set of polynomials of degree less than or equal to 2 over some interval, e.g. $(-5, 10)$:

$$\mathcal{P}_2(-5, 10) = \{p(x) \mid p(x) = \alpha + \beta x + \gamma x^2, \alpha, \beta, \gamma \in \mathbb{R}, -5 < x < 10\}.$$

We see that this space can be described as

$$\mathcal{P}_2(-5, 10) = span\{p_0(x), p_1(x), p_2(x)\},$$
$$p_0(x) = 1, -5 < x < 10,$$
$$p_1(x) = x, -5 < x < 10,$$
$$p_2(x) = x^2, -5 < x < 10.$$

To show that these three functions are linearly independent, we have to show that $\alpha p_0(x) + \beta p_1(x) + \gamma p_2(x) = 0 \Rightarrow \alpha = \beta = \gamma = 0$.
   One way to establish this is to substitute some values for x in the domain $(-5, 10)$. E.g., for $x \in \{-1, 0, 1\}$ we obtain the equations

$$\alpha - \beta + \gamma = 0, \ for \ x = -1,$$
$$\alpha = 0, \ for \ x = 0,$$
$$\alpha + \beta + \gamma = 0, \ for \ x = 1,$$

which yields the unique solution $\alpha = \beta = \gamma = 0$.
   Another way to establish the same result is to note that the functions $p_0(x), p_1(x)$ and $p_2(x)$ have derivatives:
let $p(x) = \alpha p_0(x) + \beta p_1(x) + \gamma p_2(x)$, and compute $p(x) = 0$, $\frac{dp}{dx} = 0$ and $\frac{d^2p}{dx^2} = 0$.
   Evaluating at $x = 0$, this yields the set of equations

$$p(0) = 0 \quad \Rightarrow \quad \alpha = 0,$$
$$\frac{dp}{dx}(0) = 0 \quad \Rightarrow \quad \beta = 0,$$
$$\frac{d^2p}{dx^2}(0) = 0 \quad \Rightarrow \quad 2\gamma = 0,$$

again establishing that the functions $p_0(x), p_1(x)$ and $p_2(x)$ are linearly independent.

   We make the following observations about adding and removing vectors from a set of linearly independent vectors:

☞ Let $S = \{v_1, v_2, \cdots v_n\}$ be a set of n linearly independent vectors in a vector space V.

    – Given a vector $v \in V$. If $v \notin S$, then $\{v\} \bigcup S$ is a set of linearly independent vectors.

    – Remove any one vector v_i from S. The remaining set $S - \{v_i\}$ is linearly independent.

☞ Let $S = \{v_1, v_2, \cdots v_n\}$ be a set of n linearly dependent vectors

- Given a vector $v \in V$. The set $\{v\} \bigcup S$ is linearly dependent.

- There is a vector $v_i \in S$ such that span$\{S\} = $ span$\{S - \{v_i\}\}$ is linearly independent.

FIX Did I already cover the following in the text? FIX

- Theorem: Spans of vectors form a vector space

- Definition: Spanning set

- Theorem: Can throw out linear dependent vectors in a spanning set

- Theorem: Linearly independent spanning sets for the same vector space have the same (finite) number of vectors

- Definition: the dimension of a vector space is the number of vectors in a basis

row echelon form matrix

- rows containing pivots are linearly independent

- columns containing pivots are linearly independent

Theorems, concepts, examples:

- a set of k vectors in $\mathbb{F}^N$, where $k > N$ must be linearly dependent

- a basis is a set of vectors that i) is linearly independent and ii) spans the space

- any two bases for a given vector space have the same number of vectors, called the dimension of the space

- In a subspace of $\mathbb{F}^N$, any two bases have the same number of vectors k, where k is the dimension of the subspace. Given l vectors in the subspace, if $l < k$, they cannot span the space, and if $l > k$ they cannot be linearly independent

- extend a basis for a subspace to a basis for a vector space

Example 3.3.3. ***Extend a subspace basis to a basis for*** $\mathbb{Z}_2^4$
Given two linearly independent vectors $v_1 = (1\ 0\ 1\ 0)^t, v_2 = (1\ 1\ 0\ 1)^t$ *in* $\mathbb{Z}_2^4$,
add vectors to obtain a basis for $\mathbb{Z}_2^4$*. One way to proceed is to add vectors from*
a known basis, and remove vectors we do not need. We will use the columns
of I for the known basis for $\mathbb{Z}_2^4$*, and check for pivots to decide which vectors to*
retain:

$$\begin{pmatrix} \mathbf{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 \end{pmatrix}$$

where we have used $1 + 1 = 0$*, and* $-1 = 1$ *in modulo two arithmetic.*
    Since each of the first four columns has a pivot, the desired basis is

$$\{(1\ 0\ 1\ 0)^t,\ (1\ 0\ 0\ 0)^t,\ (0\ 1\ 0\ 0)^t,\ (1\ 1\ 0\ 1)^t\}$$

- meet, join, dual

- the four fundamental subspaces of a matrix

- column rank = row rank = rank

- Fundamental Theorem of linear algebra

- Orthogonality of vectors and subspaces

Theorem:

- Nullspace(A B) $\supseteq$ nullspace(B)

- Col space(A B) $\subseteq$ col space(A)

- Left nullspace(A B) $\supseteq$ left nullspace(A)

- row space(A B) $\subseteq$ row space(B)

Finding bases for each of the spaces:

Example 3.3.4. *Finding bases for the fundamental matrix subspaces*
Consider the following row reduction layout for a matrix A of size $M = 3 \times N = 4$ augmented by the identity.

$$
\begin{pmatrix}
-5 & 9 & 3 & 6 & 1 & 0 & 0 \\
2 & -3 & -3 & -3 & 0 & 1 & 0 \\
1 & -1 & -3 & -2 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 \\
\frac{2}{5} & 1 & 0 \\
\frac{1}{5} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-5 & 9 & 3 & 6 & 1 & 0 & 0 \\
0 & \frac{3}{5} & -\frac{9}{5} & -\frac{3}{5} & \frac{2}{5} & 1 & 0 \\
0 & \frac{4}{5} & -\frac{12}{5} & -\frac{4}{5} & \frac{1}{5} & 0 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\frac{4}{3} & 1
\end{pmatrix}
\left(
\begin{array}{cccc|ccc}
-5 & 9 & 3 & 6 & 1 & 0 & 0 \\
0 & \frac{3}{5} & -\frac{9}{5} & -\frac{3}{5} & \frac{2}{5} & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{4}{3} & 1
\end{array}
\right)
$$

Next, identify the pivots. The rank is the number of pivots: $r = rank(A) = 2$.

- *Basis for $\mathcal{C}(A)$: Pick the pivot columns in A*

$$
\mathbf{v_1} = \begin{pmatrix} -5 & 2 & 1 \end{pmatrix},
$$
$$
\mathbf{v_2} = \begin{pmatrix} 9 & -3 & -1 \end{pmatrix}.
$$

  This subspace has dimension $r = 2$.

- *Basis for $\mathcal{R}(A)$: Pick the pivot rows in a row echelon form matrix U*

$$
\mathbf{u_1} = \begin{pmatrix} -5 & 9 & 3 & 6 \end{pmatrix},
$$
$$
\mathbf{u_2} = \begin{pmatrix} 0 & 3 & -9 & -3 \end{pmatrix},
$$

  where we have chosen to apply a scale factor 5 to the second vector to avoid the fraction. We can choose any non-zero scale factor, since the desired subspace is a span, i.e. $\mathcal{R}(A) = span\{\mathbf{u_1}, \mathbf{u_2}\}$. This subspace has dimension $r = 2$.

- *Basis for $\mathcal{N}(A)$: Solve $Ax = 0$ and pick the constant vectors to get*

$$
\mathbf{s_1} = \begin{pmatrix} 6 & 3 & 1 & 0 \end{pmatrix},
$$
$$
\mathbf{s_2} = \begin{pmatrix} 3 & 1 & 0 & 1 \end{pmatrix},
$$

  This subspace has dimension $N - r = 2$.

- *Basis for $\mathcal{N}(A^t)$: Pick the non-pivot rows in $\mathcal{E}$ **next to a row echelon form:***

$$
\mathbf{t_1} = \begin{pmatrix} -1 & -4 & 3 \end{pmatrix},
$$

  where we have chosen to apply a scale factor 3 to the single non-pivot row. This subspace has dimension $M - r = 1$.

Augmenting Bases to find a full basis for $\mathbb{R}^N$ Reference to finding eigenspace bases.

3.3.1 Sums of Vector Spaces

Given subsets V_1 and V_2 of vectors in a vector space V, we define the **sum** $V_1 + V_2$ **of two sets of vectors** by $V_1 + V_2 = \{x_1 + x_2 \mid x_1 \in V_1, x_2 \in V_2\}$.

- ☞ $V_1 \cup V_2$ is different from $V_1 + V_2$. For example, if V_1 and V_2 are the respective x and y-axes in $\mathbb{R}^2$, their union is the two axes, while their sum is all of $\mathbb{R}^2$.

- ☞ $V_1 + V_2$ is not necessarily a vector space. Consider for example $V_1 = \{x_1 | x_1 = (\alpha_1, 1), \alpha_1 \in \mathbb{R}\}$ and $V_2 = \{x_2 | x_2 = (\alpha_2, 0), \alpha_2 \in \mathbb{R}\}$, and note that $V_1 + V_2 = V_1$, which is not a subspace of $\mathbb{R}^2$.

- ☞ $V_1 + V_2$ can be a vector space, even if V_1 and V_2 are not vector spaces. Let $V_1 = \{x_1 | x_1 = (\alpha_1, 1), \alpha_1 \in \mathbb{R}\}$ and $V_2 = \{x_2 | x_2 = (1, \alpha_2), \alpha_2 \in \mathbb{R}\}$. We have $V_1 + V_2 = \{x | x = (\alpha_1 + 1, \alpha_2 + 1), \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}\} = \mathbb{R}^2$.

- ☞ If V_1 and V_2 are both subspaces of V, then $V_1 + V_2$ is a subspace of V.

- ☞ If V_1 and V_2 are subspaces of V, then by definition any vector in $W = V_1 + V_2$ can be written as the sum of a vector in V_1 and a vector in V_2. Such a decomposition is not necessarily unique.

  Consider for example $V_1 = \{x \in \mathbb{R}^3 \mid x = (\alpha\ \beta\ 0), \alpha \in \mathbb{R}, \beta \in \mathbb{R}\}$, and $V_2 = \{x \in \mathbb{R}^3 \mid x = (\alpha\ 0\ \beta), \alpha \in \mathbb{R}, \beta \in \mathbb{R}\}$, i.e., the xy and xz coordinate planes in $\mathbb{R}^3$, and note that a vector on the x-axis (the intersection of the two coordinate planes) can be written in an infinite number of ways.

- ☞ If V_1 and V_2 are subspaces of V, it is possible to have $W = V_1 + V_2$ provide a unique decomposition.

  The sum of the x-y plane $V_1 = \{x \in \mathbb{R}^3 \mid x = (\alpha\ \beta\ 0), \alpha \in \mathbb{R}, \beta \in \mathbb{R}\}$, and the z-axis $V_2 = \{x \in \mathbb{R}^3 \mid x = (0\ 0\ \alpha), \alpha \in \mathbb{R}\}$ provide a simple example.

The intersection of the two spaces V_1 and V_2 of the two previous examples differ. A vector space V is the **internal direct sum** of two subspaces V_1 and V_2 iff $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$, i.e., V_1 and V_2 have only the zero vector in common. The internal direct sum is denoted by

$$V = V_1 \oplus V_2. \tag{3.7}$$

For example, let $V_1 = \{x \in \mathbb{R}^3 \mid x = (\alpha\ 0\ 0), \alpha \in \mathbb{R}\}$ and $V_2 = \{x \in \mathbb{R}^3 \mid x = (\alpha\ \alpha\ 0), \alpha \in \mathbb{R}\}$, then $V = V_1 \oplus V_2$, where V is the xy-plane (a vector space contained in $\mathbb{R}^3$).

☞ Given a vector space V with two subspaces V_1 and V_2, then
$V = V_1 \oplus V_2 \Leftrightarrow$ every vector $v \in V$ can be written uniquely as $v = v_1 + v_2$,
where $v_1 \in V_1$, and $v_2 \in V_2$.

☞ Let $V = V_1 \oplus V_2$ such that the subspaces V_1 and V_2 have finite-dimensional
bases $\{u_1, u_2, \cdots u_k\}$ and $\{v_1, v_2, \cdots v_l\}$ respectively. Then V is finite-
dimensional, and $\{u_1, u_2, \cdots u_k, v_1, v_2, \cdots v_l\}$ is a basis for V. So the com-
bined bases form a basis for the internal direct sum, and
$dim(V) = dim(V_1) + dim(V_2)$.

Given a vector x in a vector space V with a set of basis vectors $\{v_1, v_2, \cdots v_N\}$,
we can express the vector $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_N v_N$. where the scalar coef-
ficients $\alpha_1, \alpha_2, \cdots \alpha_N$ are unique. If we assume the vectors are in $\mathbb{R}^N$, we see
that the decomposition can be interpreted as a system of linear equations for
unknown scalar coefficients written in column view

$$x = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_N \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_N v_N. \qquad (3.8)$$

Since the v_i are assumed to form a basis, the solution must exist and be
unique. Thus, if we set $V = (v_1 \, v_2 \, \cdots \, v_N)$, then the **coefficient vector**
$\alpha = (\alpha_1 \, \alpha_2 \, \cdots \, \alpha_N)^t$ is given by $\alpha = V^{-1} x$.

Example 3.3.5. *Express a vector in a new basis*
Express the vector $\mathbf{v} = (2, 1, -1)$ *in the basis* $\{v_1, v_2, v_3\}$ *where*
$\mathbf{v_1} = (-1, 1, -1), \mathbf{v_2} = (2, -1, 3)$ *and* $\mathbf{v_3} = (1, 0, 3)$.
The inverse of the matrix $V = (\mathbf{v_1 \, v_2 \, v_3})$ *was obtained in Example (2.7.9)*

$$V = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & 3 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 3 & 3 & -1 \\ 3 & 2 & -1 \\ -2 & -1 & 1 \end{pmatrix}.$$

We obtain the coefficient vector

$$\alpha = \begin{pmatrix} 3 & 3 & -1 \\ 3 & 2 & -1 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ -6 \end{pmatrix}.$$

The required vector decomposition is $\mathbf{v} = 10\mathbf{v_1} + 9\mathbf{v_2} - 6\mathbf{v_3}$.

Looking at the preceding example, we see that terms in the expression for v
may be grouped, e.g., $\mathbf{v} = \mathbf{w_1} + \mathbf{w_2}$, where $\mathbf{w_1} = 10\mathbf{v_1} - 6\mathbf{v_3} = (-16, 10, -8)$
and $\mathbf{w_2} = 9\mathbf{v_2} = (18, -9, 7)$. The component vectors $\mathbf{w_1} \in span\{\mathbf{v_1}, \mathbf{v_3}\}$ and
$\mathbf{w_2} \in span\{\mathbf{v_2}\}$. With this reading, we have interpreted the decomposition of

v into two vectors $\mathbf{w_1} \in W_1$ and $\mathbf{w_2} \in W_2$ where the two subspaces $W_1 = span\{\mathbf{v_1}, \mathbf{v_3}\}$ and $W_2 = span\{\mathbf{v_2}\}$ satisfy $\mathbb{R}^3 = W_1 \oplus W_2$.

We again consider an invertible matrix $V \in \mathbb{R}^{N \times N}$. Its columns $\{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_N}\}$ are a basis for $\mathbb{R}^N$. The rows of V^{-1} also form a basis of $\mathbb{R}^N$ that is of considerable practical interest: it is known as the **dual basis**. Setting $U = (V^{-1})^t$, the dual basis consists of the columns of U, i.e., $\{\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_N}\}$.

Since $V^{-1}V = I$, these two bases satisfy

$$\mathbf{u_i} \cdot \mathbf{v_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

Refering back to Eq (3.8), we see that the coefficient α_i for a given index i in $1, 2, \cdots N$ can be computed by taking the dot product of the equation with $\mathbf{u_i}$ or equivalently by multiplication with the row vector u_i^t from the left

$$\mathbf{u_i} \cdot \mathbf{x} = \mathbf{u_i} \cdot (\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots \alpha_N \mathbf{v_N}) = \alpha_i \Leftrightarrow$$
$$u_i^t x = u_i^t (\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_N v_N) = (\alpha_i).$$

We recover the component $\alpha_i v_i$ of v along the basis vector v_i by multiplying this equation with v_i, i.e.,

$$\alpha_i \mathbf{v_i} = \mathbf{u_i} \cdot \mathbf{v_i} \mathbf{v_i} \Leftrightarrow \alpha_i v_i = v_i u_i^t v_i. \tag{3.10}$$

☞ The matrix $P_i = v_i u_i^t$ appearing in Eq (3.10) projects a given vector x onto the subspace $span\{v_i\}$.

☞ Since $I = VV^{-1} = VU^t$, we have

$$I = P_1 + P_2 + \cdots P_N = \sum_{i=1}^{N} v_i u_i^t. \tag{3.11}$$

☞ Let $W = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_N}\}$ be a basis of $\mathbb{R}^N$, and let $x = \alpha_1 v_1 + alpha_2 v_2 + \cdots \alpha_N v_N$.

We can split the x vector into two components by reordering and grouping the right hand side expression into two parts. E.g., let $W_1 = \{\mathbf{v_{i_1}}, \mathbf{v_{i_2}}, \cdots \mathbf{v_{i_k}}\}$ and $W_2 = \{\mathbf{v_{i_{k+1}}}, \mathbf{v_{i_{k+2}}}, \cdots \mathbf{v_{i_N}}\}$ be two distinct subsets of $W = W_1 \cup W_2$, then $x = w_1 + w_2$, with $w_1 \in span\{W_1\}$ and $w_2 \in span\{W_2\}$, with $\mathbb{R}^N = W_1 \oplus W_2$. We obtain

$$\mathbf{w_1} = \mathbf{u_{i_1}} \cdot \mathbf{v_{i_1}} \mathbf{v_{i_1}} + \mathbf{u_{i_2}} \cdot \mathbf{v_{i_2}} \mathbf{v_{i_2}} + \cdots \mathbf{u_{i_k}} \cdot \mathbf{v_{i_k}} \mathbf{v_{i_k}} \Leftrightarrow$$
$$w_1 = v_{i_1} u_{i_1}^t v_{i_1} + v_{i_2} u_{i_2}^t v_{i_2} + \cdots v_{i_k} u_{i_k}^t v_{i_k}$$

The w_2 term can be computed similarly, or more simply by noting that $w_2 = x - w_1$.

Setting $P = P_{i_1} + P_{i_2} + \cdots P_{i_k}$, the projection matrix form of these equations is given by $w_1 = Px$ and $w_2 = (I - P)x$.

Example 3.3.6. *Decomposition of a vector*
Given the basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ where
$\mathbf{v_1} = (-1, 1, -1), \mathbf{v_2} = (2, -1, 3)$ *and* $\mathbf{v_3} = (1, 0, 3)$ *and define*
$W_1 = span\{\mathbf{v_1}, \mathbf{v_3}\}$, *and* $W_2 = span\{\mathbf{v_2}\}$, *so that* $\mathbb{R}^3 = W_1 \oplus W_2$. *Decompose the vector* $\mathbf{v} = (2, 1, -1)$ *into two components* $\mathbf{v} = \mathbf{w_1} + \mathbf{w_2}$, *where* $\mathbf{w_1} \in W_1$
and $\mathbf{w_2} \in W_2$.

This is Example 3.3.5 and its continuation in the text below. From the inverse of the matrix $V = (\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3})$

$$V = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & 3 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 3 & 3 & -1 \\ 3 & 2 & -1 \\ -2 & -1 & 1 \end{pmatrix}.$$

we find the dual basis $\mathbf{u_1} = (3, 3, -1), \mathbf{u_2} = (3, 2, -1)$ *and* $\mathbf{u_3} = (-2, -1, 1)$ *and the components* $\alpha_1 = \mathbf{u_1} \cdot \mathbf{v} = 10, \alpha_3 = \mathbf{u_3} \cdot \mathbf{v} = -6$.
We therefore have

$$\begin{aligned} \mathbf{w_1} &= \alpha_1 \mathbf{v_1} + \alpha_3 \mathbf{v_3} \\ &= \mathbf{u_1} \cdot \mathbf{v}\mathbf{v_1} + \mathbf{u_3} \cdot \mathbf{v}\mathbf{v_3} \\ &= 10\mathbf{v_1} - 6\mathbf{v_3} \\ &= (-16, 10, -8) \\ \mathbf{w_2} &= \mathbf{v} - \mathbf{w_1} \\ &= (18, -9, 7). \end{aligned}$$

The same computation may be carried out with projection matrices. Setting $P = v_1 u_1^t + v_3 u_3^t$, *we have*

$$P = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} -2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -4 & 2 \\ 3 & 3 & -1 \\ -3 & 0 & 2 \end{pmatrix},$$

$$I - P = \begin{pmatrix} 6 & 4 & -2 \\ -3 & -2 & 1 \\ 3 & 0 & -1 \end{pmatrix},$$

we obtain $w_1 = Pv = (-16\ 10\ -8)^t$ *and* $w_2 = (I - P)v = (18\ -9\ 7)^t$ *as before.*

3.4 The Fundamental Theorem of Linear Algebra (Part 1)

Fundamental theorem of Linear Algebra

3.5 Exercises

In the following exercises, let $v_1 = (3\,6\,-3)^t$, $v_2 = (-3\,-6\,3)^t$, $v_3 = (1\,1\,-2)^t$, $v_4 = (4\,7\,-5)^t$, and $v_5 = (2\,5\,0)^t$.

Exercise 3.1. *Describe $S = span\,(v_1, v_2, v_3, v_4)$ geometrically. What is the dimension of S?*

Exercise 3.2. *Describe $S = span\,(v_1, v_2, v_3, v_4, v_5)$ geometrically. What is the dimension of S?*

Exercise 3.3. *Select vectors in $S = span\,(v_1, v_2, v_3, v_4)$ to uniquely describe $u = (9\,15\,h)$ as a linear combination of those vectors. What are the values of h for which this is possible? Obtain the coordinate vector of u with respect to the selected vectors.*

Exercise 3.4. *Select vectors in $S = span\,(v_1, v_2, v_3, v_4, v_5)$ to uniquely describe $u = (9\,15\,-13)$ as a linear combination of those vectors. Obtain the coordinate vector of u with respect to the selected vectors.*

Exercise 3.5. *Redo the previous exercise with $S = span\,(v_1, v_3, v_2, v_4, v_5)$, i.e., with the vectors in the definition of S listed in a different order.*

The following tongue in cheek example is designed to emphasize the meaning of the definitions for vector spaces.

Exercise 3.6. *A game designer decides to use vector spaces to model the behaviour of her characters: little blue men (LBM). The LBM are vectors. Collision of any two LBMs u and v on the screen results in a single LBM w, denoted by $w = u + v$. The game action involves magic potions labeled by scalars. There are two mixing methods for these magic potions: given two potions α and β, they can be mixed to yield potions $\alpha + \beta$ and $\alpha\beta$. When an LBM u quaffs a magic potion α, a new LBM v results. The transformation is denoted by $v = \alpha u$.*
Describe the action on the screen for each of the properties 3.3, 3.4 and 3.5. Note the special roles played by the potions labeled 0, 1 and -1.

Direct inner sums

Exercise 3.7. *Let S_1 and S_2 be two subspaces of V. Show that $W = S_1 + S_2$ is a subspace of V. Also show that $S_1 \subset W$ and $S_2 \subset W$.*

Some properties of the rank of matrix products:

Exercise 3.8. $rank(AB) \leq min\,(rank(A), rank(B))$:

- *Given matrices A and B show that $rank(AB) \leq rank(A)$ by showing that every vector in the column space of AB is in the column space of A.*

- *Show that $rank(AB) \leq rank(B)$ by considering the rank of $(AB)^t$.*

Exercise 3.9. *If A is invertible, $rank(AB) = rank(B)$. If B is invertible, $rank(AB) = rank(A)$.*

- *Given an invertible matrix A, show that $rank(AB) = rank(B)$ by applying the previous exercise to AB and to $A^{-1}(AB)$.*

- *Given an invertible matrix B, show that $rank(AB) = rank(A)$ by considering the $rank((AB)^t)$.*

Exercise 3.10. *Let A be size $M \times K$ and B be size $K \times N$. If $AB = 0$, show that $rank(A) + rank(B) \leq K$. Hint: one of the four subspaces $\mathcal{N}(A), \mathcal{N}(B), \mathcal{C}(A), \mathcal{C}(B)$ is contained in one of the other three subspaces.*

Exercise 3.11. *Use the existence of rank factorizations of A and B to prove that $rank(A + B) \leq rank(A) + rank(B)$ by writing $A + B$ as the product of two partitioned matrices.*

Chapter 4

Linear Transformations

4.1 Definition of Linear Transformations

Let U and V be two vector spaces over the same field $\mathbb{F}$. A transformation $\mathbf{T} : U \to V$ is linear, if and only if the following two conditions hold:

$$\forall u, v \in U, \quad \mathbf{T}(u + v) = \mathbf{T}u + \mathbf{T}v \tag{4.1}$$

$$\forall u \in U, \forall \alpha \in \mathbb{F}, \quad \mathbf{T}(\alpha u) = \alpha \mathbf{T}u. \tag{4.2}$$

☞ Either test may succeed or fail: both tests must succeed to conclude the transformation is linear.

   If either test fails, the transformation is non-linear.

☞ The tests can be combined into a single test

$$\forall u, v \in U, \ \forall \alpha, \beta \in \mathbb{R}, \quad \mathbf{T}(\alpha u + \beta v) = \alpha \mathbf{T}u + \beta \mathbf{T}v.$$

☞ The transformation $y = Ax$, where A is a matrix of size $M \times N$, $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$ is linear.

☞ A linear transformation preserves linear combinations: vectors in U may be scaled and added, and the resulting vector may be mapped into V. Equivalently, the vectors may be mapeed into V, and the resulting vectors may be scaled and added in V. The resulting vector is the same.

☞ Given two linear transformations $\mathbf{T_1} : U \to V$ and $\mathbf{T_2} : V \to W$ where U, V, W are vector spaces over the same field $\mathbb{F}$, the transformation defined by $\mathbf{T}(u) = \mathbf{T_2}(\mathbf{T_1}(u))$ is a linear transformation $\mathbf{T} : U \to W$.

   The combined transformation is denoted by $\mathbf{T} = \mathbf{T_2} \circ \mathbf{T_1}$

FIX add a figure showing a paralellogram projected into the x-y plane FIX

Example 4.1.1. *Linear transformation of vectors in a plane*
Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -y \end{pmatrix}$$

To check the first condition, pick any $u = (x_1, y_1), v = (x_2, y_2)$ in U, and consider

$$\xi = \mathbf{T}(u + v) - \mathbf{T}u - \mathbf{T}v \qquad \qquad \textit{should be zero}$$

$$= \mathbf{T}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} - \mathbf{T}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \mathbf{T}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \qquad \textit{by definition of u and v}$$

$$= \begin{pmatrix} (x_1 + x_2) + 2(y_1 + y_2) \\ -(y_1 + y_2) \end{pmatrix} - \begin{pmatrix} x_1 + 2y_1 \\ -y_1 \end{pmatrix} - \begin{pmatrix} x_2 + 2y_2 \\ -y_1 \end{pmatrix} \qquad \textit{by definition of T}$$

$$= \begin{pmatrix} (x_1 + x_2 + 2y_1 + 2y_2) - (x_1 + 2y_1) - (x_2 + 2y_2) \\ -(y_1 + y_2) - y_1 - y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

satisfying the first condition. Next, consider any $\alpha \in \mathbb{R}$, and any $u = (x, y) \in U$, and let

$$\xi = \mathbf{T}(\alpha u) - \alpha \mathbf{T}u \qquad \qquad \textit{this should be zero if $\mathbf{T}$ is linear}$$

$$= \mathbf{T}\left(\alpha \begin{pmatrix} x \\ y \end{pmatrix}\right) - \alpha \mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \qquad \textit{by definition of u}$$

$$= \mathbf{T}\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} - \alpha \mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \qquad \textit{property of matrix multiplication}$$

$$= \begin{pmatrix} \alpha x + 2\alpha y \\ -\alpha y \end{pmatrix} - \alpha \begin{pmatrix} x + 2y \\ -y \end{pmatrix} \qquad \qquad \textit{by definition of T}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since both conditions are satisfied, we conclude that this transformation is linear.
A simpler proof consists of simply rewriting the definition of T in the form

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

T *is a linear transformation since matrix multiplication is linear.*

Example 4.1.2. *Nonlinear transformation of vectors in a plane*
Consider the transformation defined by

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ -y \end{pmatrix}$$

To check the first condition, pick any $u = (x_1, y_1), v = (x_2, y_2)$ in U and consider

$$\xi = \mathbf{T}(u + v) - \mathbf{T}u - \mathbf{T}v \qquad\qquad \text{this should be zero if linear}$$

$$= \mathbf{T}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} - \mathbf{T}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \mathbf{T}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \qquad\qquad \text{by definition of u and v}$$

$$= \begin{pmatrix} (x_1 + x_2)(y_1 + y_2) \\ -(y_1 + y_2) \end{pmatrix} - \begin{pmatrix} x_1 y_1 \\ -y_1 \end{pmatrix} - \begin{pmatrix} x_2 y_2 \\ -y_2 \end{pmatrix} \qquad\qquad \text{by application of } \mathbf{T}$$

$$= \begin{pmatrix} (x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2) - x_1 y_1 - x_2 y_2 \\ -(y_1 + y_2) - y_1 - y_2 \end{pmatrix} \qquad \text{using properties of vector addition}$$

$$= \begin{pmatrix} x_2 y_1 + x_1 y_2 \\ 0 \end{pmatrix}.$$

Since this is not zero in general (e.g., the choice $x_1 = x_2 = y_1 = y_2 = 1$ provides a counterexample), this transformation is not linear.

If instead we had started with the second condition, we would similarly consider any $\alpha \in \mathbb{R}$, and any $u = (x, y) \in U$, and let

$$\xi = T(\alpha u) - \alpha \mathbf{T}u \qquad\qquad \text{this should be zero if } \mathbf{T} \text{ is linear}$$

$$= \mathbf{T}\left(\alpha \begin{pmatrix} x \\ y \end{pmatrix} \right) - \alpha \mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} \qquad\qquad \text{by definition of u}$$

$$= \mathbf{T}\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} - \alpha \mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} \qquad\qquad \text{property of matrix multiplication}$$

$$= \begin{pmatrix} \alpha x \alpha y \\ -\alpha y \end{pmatrix} - \alpha \begin{pmatrix} xy \\ -y \end{pmatrix} \qquad\qquad \text{by definition of } \mathbf{T}$$

$$= \begin{pmatrix} \alpha^2 xy \\ 0 \end{pmatrix} \qquad\qquad \text{by properties of matrix operations.}$$

Since this is not zero in general (e.g., $\alpha = x = y = 1$ provides a counterexample), we again conclude that this transformation is not linear.

Note that in this example, we cannot rewrite $\mathbf{T}$ in the form of a constant matrix acting on a vector.

Example 4.1.3. *Linear transformation of polynomials*
Consider the transformation $\mathbf{T} = x\frac{d}{dx} + 1$ *applied to polynomials in* $\mathcal{P}_2(-\infty, \infty)$,
resulting in polynomials in $\mathcal{P}_2(-\infty, \infty)$.
   To check the first condition, pick any $p_1(x) = a_1 + b_1 x + c_1 x^2$,
$p_2(x) = a_2 + b_2 x + c_2 x^2$ *in* $\mathcal{P}_2(-\infty, \infty)$, *and consider*

$$
\begin{aligned}
\xi &= \mathbf{T}(p_1(x) + p_2(x)) - \mathbf{T}p_1(x) - \mathbf{T}p_2(x) \\
&= \mathbf{T}\left((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2\right) \\
&\quad - \mathbf{T}\left(a_1 + b_1 x + c_1 x^2\right) - \mathbf{T}\left(a_2 + b_2 x + c_2 x^2\right) \\
&= x\frac{d}{dx}\left((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2\right) + \left((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2\right) \\
&\quad - x\frac{d}{dx}\left(a_1 + b_1 x + c_1 x^2\right) - \left(a_1 + b_1 x + c_1 x^2\right) \\
&\quad - x\frac{d}{dx}\left(a_2 + b_2 x + c_2 x^2\right) - \left(a_2 + b_2 x + c_2 x^2\right) \\
&= 0,
\end{aligned}
$$

satisfying the first condition.
   Next, consider any $\alpha \in \mathbb{R}$, *and any* $p(x) = a + bx + cx^2 \in \mathcal{P}_2(-\infty, \infty)$. *Then*

$$
\begin{aligned}
\xi &= \mathbf{T}(\alpha p(x)) - \alpha \mathbf{T}p(x) \\
&= x\frac{d}{dx}\left(\alpha\left(a + bx + cx^2\right) + \alpha\left(a + bx + cx^2\right)\right) \\
&\quad - \alpha\left[x\frac{d}{dx}\left(a + bx + cx^2\right) + \left(a + bx + cx^2\right)\right] \\
&= 0.
\end{aligned}
$$

Since both conditions are satisfied, we conclude that this transformation is linear.

4.2 Matrix Representation of Linear Transformations

Given a basis $B = \{u_1, u_2, \cdots u_n\}$ of a vector space U over a field $\mathbb{F}$, a linear transformation $\mathbf{T} : U \to V$ is uniquely defined by specifying $\{\mathbf{T}u_1, \mathbf{T}u_2, \cdots \mathbf{T}u_N\}$, since any vector $u \in U$ can be expressed as a unique linear combination of the basis vectors, i.e., $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_N u_N\}$, $\alpha_i \in \mathbb{F}, i = 1, 2, \cdots, N$. Computing $\mathbf{T}u$ and using the linearity properties, we have

$$
\mathbf{T}u = \alpha_1 \mathbf{T}u_1 + \alpha_2 \mathbf{T}u_2 + \cdots + \alpha_N \mathbf{T}u_n.
$$

If $\mathbf{T}u_i$ is a vector in $\mathbb{F}^M$, we may write these vectors into a matrix as columns to obtain the equivalent matrix representation

$$\mathbf{T}u = \begin{pmatrix} \mathbf{T}u_1 & \mathbf{T}u_2 & \cdots & \mathbf{T}u_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_N \end{pmatrix}. \qquad (4.3)$$

The matrix $A_{\mathbf{T}}$ made up of the columns of $\mathbf{T}u_i$ is called the **matrix representation** of the transformation $\mathbf{T}$, and the vector made up from the coefficients α_i is the **coefficient vector** representing the vector u with respect to the basis B.

4.2.1   Geometric Transformations

The following transformations can readily be shown to be linear using arguments in geometry:

- a stretch in a single coordinate direction

- an orthogonal projection onto a hyperplane: a line, a plane, etc

- a reflection through a hyperplane: a point, a line, a plane, etc

- a rotation by some angle in a hyperplane

- a shear along a coordinate axis.

All linear transformations of $\mathbb{R}^N$, such as, e.g., rotations can be built up as a composition of these basic types.

Example 4.2.1. *Orthogonal projection into the xy-plane*
Consider the orthogonal projection of a vector in 3D onto the xy-plane. Let us use the standard basis in 3D. The projections of the standard basis vectors are readily seen to be

$$(1\ 0\ 0) \xrightarrow{\mathbf{T}} (1\ 0\ 0)$$

$$(0\ 1\ 0) \xrightarrow{\mathbf{T}} (0\ 1\ 0)$$

$$(0\ 0\ 1) \xrightarrow{\mathbf{T}} (0\ 0\ 0),$$

since e_1 and e_2 lie in the xy-plane and therefore remain unchanged, while e_3 lies on the z-axis and is therefore projected into the origin.
 *We assemble our result in matrix form*

$$\mathbf{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\ \mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\ \mathbf{T}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\ \mathbf{T}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where column i in the matrix is the vector $\mathbf{T}e_i$. This is the matrix representation of the transformation $\mathbf{T}$. The vector $(x\ y\ z)^t$ is the coefficient vector with respect to the basis $\{e_1, e_2, e_3\}$.

The matrix representation of a geometrical transformation is most easily obtained by analyzing the geometry using congruent triangles, finding angles and lengths, and then translating positions to position vectors. In two dimensions the position vector $(x\ y) = r(\cos\theta\ \sin\theta)$, where r is the length of the vector and θ its angle with the x-axis. As an example, consider the transformation $\mathbf{T}$ from a given point A in a plane resulting in the symmetric point with respect to a given line L through the origin. This is a linear transformation.

It is most readily understood by considering the angles of the vectors $\mathbf{OA}$ and its transform $\mathbf{OB}$ and the angle of the line (L) with respect to the x-axis. Consider the figure to the right: the given point A is transformed into its mirror image B with respect to the line (L). The triangles $\mathbf{OAC}$ and $\mathbf{OBC}$ are easily seen to be congruent since by construction, the side $\mathbf{OC}$ is in common, $||\mathbf{AC}|| = ||\mathbf{BC}||$ and $\widehat{OCA}$, $\widehat{OCB}$ are right angles.
 It follows that $||\mathbf{OA}|| = ||\mathbf{OB}||$ and $\widehat{AOC} = \widehat{COB}$,

Figure 4.1: Projection and reflection
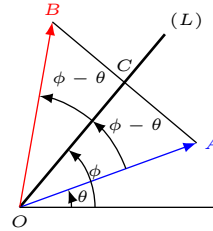
so that the angle of **OB** with respect to the x-axis is
given by $\theta + 2(\phi - \theta) = 2\phi - \theta$

To obtain the matrix representation, we look at
the transforms of the unit vectors along the x and y-axis respectively.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\mathbf{T}} \begin{pmatrix} \cos(2\phi) \\ \sin(2\phi) \end{pmatrix} \qquad\qquad\qquad \text{since } \theta = 0$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\mathbf{T}} \begin{pmatrix} \cos(-\frac{\pi}{2} + 2\phi) \\ \sin(-\frac{\pi}{2} + 2\phi) \end{pmatrix} = \begin{pmatrix} \sin(2\phi) \\ -\cos(2\phi) \end{pmatrix} \qquad \text{since } \theta = \frac{\pi}{2}.$$

Assembling the matrix, we obtain the matrix representation A_T of T:

$$A_{\mathbf{T}} = \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}. \tag{4.4}$$

.

Example 4.2.2. *Symmetry with respect to a line*
Consider the linear transformation T that computes the mirror image of a point
with respect to the line $y = x \, \tan\left(\frac{\pi}{7}\right)$. This line makes an angle of $\phi = \frac{\pi}{7}$ with
respect to the x-axis.

Using Eq(4.4), we find

$$A_T = \begin{pmatrix} \cos(\frac{2\pi}{7}) & \sin(\frac{2\pi}{7}) \\ \sin(\frac{2\pi}{7}) & -\cos(\frac{2\pi}{7}) \end{pmatrix}.$$

4.2.2   Linear transformations of polynomials

An invertible linear transformation from a vector space U of dimension N onto
a vector space V of dimension N is called an **isomorphism**. Given a vector
space U of dimension N over a field $\mathbb{F}$, let $B = b_1, b_2, \cdots b_N$ be a basis of U.

The following provides an important example. Define a linear transformation
D from U to $\mathbb{F}^N$ by expressing u as a linear combination of the basis vectors
and setting

$$u = \sum_{i=1}^{N} \alpha_i b_i,$$

$$D\left(\sum_{i=1}^{N} \alpha_i b_i\right) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_N \end{pmatrix}$$

Since the coefficients $\alpha_i, i = 1, 2, \cdots N$ uniquely define a given vector u, this
mapping is one-to-one and onto, and hence invertible. In essence, this mapping
sets up a dictionary between the vectors of U and the **coefficient vectors** of
$\mathbb{F}^N$. It is easy to check that D is a linear transformation.

Example 4.2.3. *Isomorphism from a subspace of polynomials to a plane*

Let $S = \{p(x)|\ p(x) = \alpha + \beta x + \gamma x^2,\ \alpha, \beta, \gamma \in \mathbb{R},\ -2 \leq x \leq 1,\ p(1) = 0\}$. Note that S is a subspace of $\mathcal{P}_2[-2, 1]$ of dimension 2, since S contains the zero vector $p(x) = 0$, and is easily shown to be closed under addition and scalar multiplication.

To obtain a basis for S, observe that the constraint $p(1) = 0$ implies $\alpha + \beta + \gamma = 0$, so that

$$p(x) = \alpha + \beta x + \gamma x^2$$
$$= \beta(x - 1) + \gamma(x^2 - 1).$$

The functions $p_1(x) = x - 1$ and $p_2(x) = x^2 - 1$ therefore span S. To see that these functions are linearly independent, we investigate

$$\alpha(x - 1) + \beta(x^2 - 1) = 0 \Rightarrow \begin{cases} -\alpha - \beta = 0 & at\ x = 0 \\ -2\alpha = 0 & at\ x = -1 \end{cases}$$
$$\Rightarrow \quad \alpha = \beta = 0.$$

This establishes that $p_1(x), p_2(x)$ are a basis for S, and therefore $dim(S) = 2$. We can now define the isomorphism

$$D\left(\alpha(x - 1) + \beta(x^2 - 1)\right) = (\alpha \quad \beta)^t.$$

As we see, the two-dimensional subspace S of the three-dimensional space $\mathcal{P}_2[-2, 1]$ can be represented as a two-dimensional plane such that every point in the plane uniquely identifies a polynomial in S, and every polynomial in S uniquely identifies a point in the plane.

Coefficient vectors may be used to represent a linear transformation T from an N-dimensional vector space U over the field $\mathbb{F}$ into an M-dimensional vector space V over the field $\mathbb{F}$ as a matrix A_T of size $M \times N$.

Let D_u be an isomorphism from U to $\mathbb{F}^N$ and D_v be an isomorphism from V to $\mathbb{F}^M$ and consider the diagram on the right. Starting with a vector x in $\mathbb{F}^N$, we use the inverse D_u^{-1} to obtain a corresponding vector in U, apply T to obtain the resulting vector in V, then use D_v to obtain the resulting vector y in $\mathbb{F}^M$, i.e., $y = D_v(T(D_u^{-1}(x)))$.

$$
\begin{array}{ccc}
U & \xrightarrow{\ T\ } & V \\
D_u^{-1} \big\uparrow\big\downarrow D_u & & D_v^{-1}\big\uparrow\big\downarrow D_v \\
\mathbb{F}^N & \xrightarrow{\ A_T\ } & \mathbb{F}^M
\end{array}
$$

Starting with $x = e_i, i = 1, 2, \cdots N$, the unit vectors in $\mathbb{F}^N$ yields the columns of the matrix A_T. This matrix may be used to compute $v = Tu$ for any $u \in U$, $v \in V$ by applying D_u to u, applying A_T to the resulting vector, then using D_v^{-1} to translate back to the resulting vector in V, i.e., $v = D_v^{-1}(A_T(D_u(u)))$.

Example 4.2.4. *Matrix representation of a linear transformation*
Consider the transformation $\mathbf{T} = x\frac{d}{dx} + 1$ *applied to polynomials in* $S = \{p(x)|\, p(x) = \alpha + \beta x + \gamma x^2,\ \alpha, \beta, \gamma \in \mathbb{R},\ -2 \leq x \leq 1,\ p(1) = 0\}$. *We may choose the codomain of* S *to be* $\mathcal{P}_2[-2, 1]$.

In Example 4.2.3 we established that S *is a two-dimensional subspace of* $\mathcal{P}_2[-2, 1]$ *with basis* $p_1(x) = x - 1$ *and* $p_2(x) = x^2 - 1$, *and we defined the coefficient vector representation*

$$D_u(\alpha(x-1) + \beta(x^2 - 1)) = (\alpha\ \ \beta)^t.$$

We know that the codomain $\mathcal{P}_2[-2, 1]$ *has dimension 3, and has a basis* $\{1,\ x,\ x^2\}$. *We similarly define the coefficient vector representation*

$$D_v(\alpha + \beta x + \gamma x^2) = (\alpha\ \ \beta\ \ \gamma)^t.$$

We establish that T *is a linear transformation from* S *to* $\mathcal{P}_2[-2, 1]$ *similarly to the procedure shown in Example 4.1.3.*

To obtain the matrix A_T, *consider*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{D_u^{-1}} (x-1) \xrightarrow{T} x\frac{d}{dx}(x-1) + (x-1) = 2x - 1 \xrightarrow{D_v} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{D_u^{-1}} (x^2-1) \xrightarrow{T} x\frac{d}{dx}(x^2-1) + (x^2-1) = 3x^2 - 1 \xrightarrow{D_v} \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

Assembling the matrix, we obtain the matrix representation A_T *of* $\mathbf{T}$:

$$A_T = \begin{pmatrix} -1 & -1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

4.2.3   Change of Coordinates

"Dictionaries" can also be used to effectuate a **change of coordinate system**. The column view of $Ax = b \Leftrightarrow b = x_1 a_1 + x_2 a_2 + \cdots + x_N a_N$ shows that this equation expresses the vector $b \in \mathcal{C}(A)$ as a linear combination of the columns of A. If the columns of A are linearly independent, the unique solution vector x is the *coordinatevector* representing b with respect to the basis $a_i, i = 1, 2, \cdots N$.

Change the notation to

$$x = \tilde{x}_1 a_1 + \tilde{x}_2 a_2 + \cdots + \tilde{x}_N a_N = S\tilde{x}, \tag{4.5}$$

and consider that as long as we agree on the set of basis vectors $\{a_i, i = 1, 2, \cdots N\}$ and their ordering, the coordinate vector $\tilde{x} \in \mathbb{R}^N$ uniquely iden-

tifies the original vector x. To retrieve the original vector from $\tilde{x}$, we apply the linear transformation D defined by

$$(1\ 0\ 0\ \cdots\ 0) \xrightarrow{D} a_1$$
$$(0\ 1\ 0\ \cdots\ 0) \xrightarrow{D} a_2$$
$$(0\ 0\ 0\ \cdots\ 1) \xrightarrow{D} a_N,$$

which therefore has matrix representation $S = (a_1\ a_2\ \cdots\ a_N)$.

Example 4.2.5. ***Change of coordinates, subspace***
Consider the linear combination of vectors vectors

$$a_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad x = 3a_1 + 2a_2 = \begin{pmatrix} 5 \\ 16 \\ 14 \end{pmatrix}.$$

The vector x can be uniquely specified by the coordinate vector

$$\tilde{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The original vector x is recovered from the coordinate vector $\tilde{x}$ by the linear transform

$$x = S\tilde{x} = \begin{pmatrix} 1 & 1 \\ 4 & 2 \\ 2 & 4 \end{pmatrix} \tilde{x}.$$

Note that this coordinate vector is a vector in the $\mathbb{R}^2$ plane representing the original vector c that lies in the 2 dimensional plane $span\{a_1, a_2\} \subset \mathbb{R}^3$.

The inverse transformation, i.e., solving for $\tilde{x}$ in terms of x, can be accomplished only if x lies in $span\{a_1, a_2\}$. Solving the previous equation, we find the solution

$$\tilde{x} = \tilde{S}x = \frac{1}{2} \begin{pmatrix} -2 & 1 & 0 \\ 4 & -1 & 0 \end{pmatrix} x$$

subject to the constraint $x_3 = 6x_1 - x_2$. Note that $\tilde{S}$ is a left inverse of S.

For the special case where $x \in \mathbb{R}^N$ is expressed in terms of a basis $a_1, a_2, \cdots a_N$, i.e., a basis of $\mathbb{R}^N$, S becomes an invertible matrix and $S^{-1} = \tilde{S}$. We then have no restrictions on x and/or $\tilde{x}$ since the transformation is one-to-one onto. Figure 4.2 illustrates a two-dimensional example.

☞ The change of coordinates may be viewed as a substitution $x = S\tilde{x}$. This substitution will be one-to-one and onto if and only if S is invertible.

☞ The change of coordinates $x = S\tilde{x} = (a_1 a_2 \cdots a_N)\tilde{x}$ is a linear transformation mapping the new standard coordinate vector $\tilde{e}_i$ in the $\tilde{x}$ coordinate system to the vector a_i in the x coordinate system.
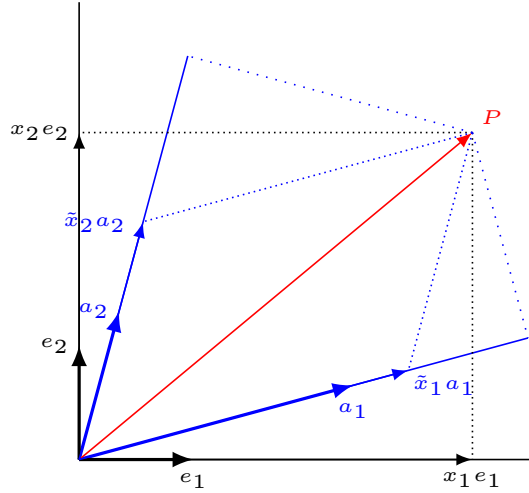
Figure 4.2: Change of coordinates: the point P is specified in two different systems of coordinates. $\mathbf{OP} = x_1 e_1 + x_2 e_2$ and $\mathbf{OP} = \tilde{x}_1 a_1 + \tilde{x}_2 a_2$. The coordinate vectors are related by $x = S\tilde{x}$, where $S = (a_1\ a_2)$. P has coordinates (x_1, x_2) with respect to the $\{e_1, e_2\}$ basis, and $(\tilde{x}_1, \tilde{x}_2)$ with respect to the $\{a_1, a_2\}$ basis.

☞ If the matrix S is square, this mapping is one-to-one and onto, and will frequently be depicted in a single figure.

Example 4.2.6. *Change of coordinates, full space*
Given the system of coordinates with direction vectors $a_1 = (1\ 1\ -1)^t, a_2 = (1\ -1\ 0)^t, a_3 = (0\ 1\ -1)^t$, the new coordinates $\tilde{x}$ of a point are related to the original coordinates x by $x = \tilde{x}_1 a_1 + \tilde{x}_2 a_2 + \tilde{x}_3 a_3 = S\tilde{x}$, i.e.,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = S\tilde{x}.$$

Since the S matrix is full rank it is invertible. We find

$$\tilde{x} = S^{-1}x, \quad with \quad S^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$$

The vector $x = (5\ 1\ 3)^t$ with respect to the new system of coordinates $\tilde{x}$ is given by $\tilde{x} = S^{-1}x = (2\ 1\ -2)^t$.

We now return to the consideration of a linear transformation $y = Ax$, that transforms a vector $x \in \mathbb{R}^N$ into a vector $y \in \mathbb{R}^N$ by multiplication with a

matrix A. An example might be a rotation in $3D$, i.e., with $N = 3$. The coordinate vectors x and y are typically expressed with respect to the basis for $\mathbb{R}^N$, i.e., the same system of coordinates. The question that arises is what form the matrix for this transformation would take if we were to use a new basis, say $\{a_1, a_2, \cdots a_N\}$? The transformation matrix that carries out this change of coordinates is $S = (a_1 \ a_2 \ \cdots \ a_N)$. If we use the same new coordinate system for both the domain and the codomain of the transformation, we have $x = S\tilde{x}$ and $y = S\tilde{y}$. Substituting, we find $y = Ax \Leftrightarrow S\tilde{y} = AS\tilde{x} \Leftrightarrow \tilde{y} = S^{-1}AS\tilde{x}$. We see that the matrix representation of the transformation A with respect to the new system of coordinates is given by $\tilde{A} = S^{-1}AS$, which is known as a **similarity transform** with respect to S.

Example 4.2.7. *Similarity transform of a projection matrix*
Consider the matrix A representing an orthogonal projection **T** *onto the line* $y = x \ \tan\theta$ *in two dimensions. Apply the similarity transform to obtain the matrix representation of this transformation with respect to the basis* $a = (1 \ 1)^t$, $b = (1 \ -1)^t$.

Refer back to Figure 4.2.1. To obtain the matrix representation, we look at the transforms of the unit vectors along the x and y-axis respectively. The length of the projection of the vector **OA** *onto* (L) *is* $\cos(\phi - \theta)$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\textbf{T}} \cos\phi \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \qquad\qquad since \ \theta = 0$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\textbf{T}} \sin\phi \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \qquad\qquad since \ \theta = \frac{\pi}{2}.$$

With respect to the original coordinate system, the linear transformation therefore has matrix representation

$$A = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}.$$

The change of coordinates matrix S and its inverse are

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

With respect to the new coordinate system, the linear transform therefore has matrix representation

$$P = S^{-1}AS = \frac{1}{2}\begin{pmatrix} (\sin\phi + \cos\phi)^2 & \cos^2\phi - \sin^2\phi \\ \cos^2\phi - \sin^2\phi & (\sin\phi - \cos\phi)^2 \end{pmatrix},$$

which reduces to

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

at $\phi = \frac{\pi}{4}$, *as expected.*

4.2.4   Composition of Linear Transformations

Let $\mathbf{S}$ and $\mathbf{T}$ be linear transformations such that $\mathbf{T} \circ \mathbf{S}$ is defined. Note the order: assuming both compositions are defined, we have $\mathbf{S} \circ \mathbf{T} \neq \mathbf{T} \circ \mathbf{S}$ in general: for example, the compositions an orthogonal projection of a vector onto the xaxis followed by an orthogonal projection onto the line $y = x$ results in a point on this line. Reversing the order of the transformations however results in a point on the xaxis.

While the matrix representation may be obtained directly, it is usually simpler to use the matrix representations of the individual transformations and multiplying the results: starting with some vector x, we obtain $x' = A_{\mathbf{T}}x$. If we now apply transormation $\mathbf{S}$ to x', we obtain $x'' = A_{\mathbf{S}}x'$. Substituting for x' in terms of x, we see that the matrices are multiplied together.

☞ Let $A_{\mathbf{S}}$ and $A_{\mathbf{T}}$ be the matrix representation of the linear transformations $\mathbf{S}$ and $\mathbf{T}$ respectively, and let $A_{\mathbf{T} \circ \mathbf{S}}$ be the matrix representation of $\mathbf{T} \circ \mathbf{S}$.

$$A_{\mathbf{T} \circ \mathbf{S}} = A_{\mathbf{T}} A_{\mathbf{S}}. \tag{4.6}$$

Example 4.2.8. *Composition of linear transformations*
Consider the 2D transformation consisting of a rotation $\mathbf{R}$ by an angle α followed by an orthogonal projection $\mathbf{P}$ onto the line $y = x \, \tan \phi$.

The rotation matrix $A_{\mathbf{R}}$ is easily obtained by considering
$(1\ 0) \to (\cos \alpha \ \sin \alpha), \ (0\ 1) \to \left(\cos(\frac{\pi}{2} + \alpha) \ \sin(\frac{\pi}{2} + \alpha)\right).$
The projection matris $A_{\mathbf{P}}$ was obtained in Example 4.2.7 above. We have

$$A_{\mathbf{R}} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \qquad A_{\mathbf{P}} = \begin{pmatrix} \cos^2 \phi & \frac{1}{2} \sin(2\phi) \\ \frac{1}{2} \sin(2\phi) & \sin^2 \phi \end{pmatrix},$$

The matrix of the transformation $\mathbf{P} \circ \mathbf{R}$ is given by

$A_{\mathbf{P}} A_{\mathbf{R}}$
$$= \begin{pmatrix} \cos(\alpha)\cos(\phi)^2 + \frac{1}{2}\sin(\alpha)\sin(2\phi) & -\sin(\alpha)\cos(\phi)^2 + \frac{1}{2}\cos(\alpha)\sin(2\phi) \\ \sin(\alpha)\sin(\phi)^2 + \frac{1}{2}\cos(\alpha)\sin(2\phi) & \cos(\alpha)\sin(\phi)^2 - \frac{1}{2}\sin(\alpha)\sin(2\phi) \end{pmatrix}$$

For the special case $\alpha = \frac{\pi}{6}$, $\phi = \frac{\pi}{4}$, we find

$$A_{\mathbf{R}} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \qquad A_{\mathbf{P}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

so that the matrix representation of a rotation by and angle $\alpha = \frac{\pi}{6}$ followed by an orthogonal projection onto the line $y = x$ is given by

$$A_{\mathbf{P}} A_{\mathbf{R}} = \frac{1}{4} \begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}+1 & \sqrt{3}-1 \end{pmatrix}.$$

Chapter 5

Orthogonal Vectors

5.1 The Fundamental Theorem of Linear Algebra (Part 2)

☞ **The material in this section requires the dot product property**

$$\mathbf{u} \cdot \mathbf{u} = \begin{cases} \alpha > 0 & \text{if } \mathbf{u} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for every vector $\mathbf{u}$. This holds true if the entries of $\mathbf{u}$ are in $\mathbb{Q}$ or $\mathbb{R}$, but fails for $\mathbb{C}$ and $\mathbb{Z}_2$. For complex numbers in $\mathbb{C}$, we have the similar property

$$\bar{\mathbf{u}} \cdot \mathbf{u} = \begin{cases} \alpha > 0 & \text{if } \mathbf{u} \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

where the bar in $\bar{\mathbf{u}}$ denotes complex conjugation. All formulas in this chapter can also be established for complex scalars provided that the transpose A^t is replaced with the Hermitian transpose A^H, i.e., the transpose of A with all entries replaced by their complex conjugates.

FIX write this.... FIX

The Fundamental Theorem of Linear Algebra for matrices of size $M \times N$

The domain $\mathbb{R}^N$ of the transformation $y = Ax$ decomposes into two orthogonal subspaces: the *row space* $\mathcal{R}(A)$ of dimension $rank(A)$ and the *null space* $\mathcal{N}(A)$ of dimension $N - rank(A)$. The combined bases for these two spaces form a basis for $\mathbb{R}^N$.

The codomain $\mathbb{R}^M$ of the transformation $y = Ax$ similarly decomposes into two orthogonal subspaces: the *column space* $\mathcal{C}(A)$ of dimension $rank(A)$ and
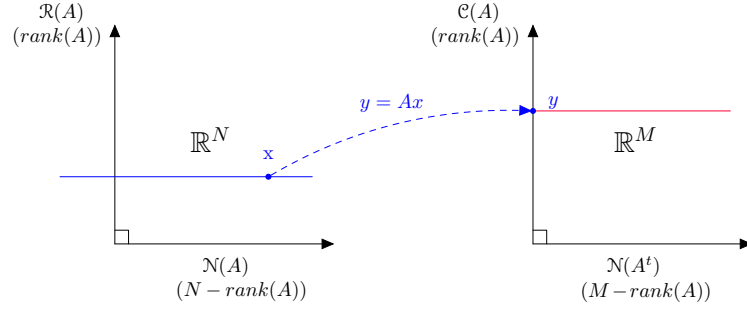
Figure 5.1: The transformation $y = Ax$ from $\mathbb{R}^N$ into $\mathbb{R}^M$ decomposes the domain and codomain into orthogonal complementary subspaces: $\mathbb{R}^N = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and $\mathbb{R}^M = \mathcal{C}(A) \oplus \mathcal{N}(A^t)$.

the *null space* $\mathcal{N}(A^t)$ of dimension $M - rank(A)$. The combined bases for these two spaces form a basis for $\mathbb{R}^M$.

   A maps any vector in the *null space* $\mathcal{N}(A)$ into the zero vector of $\mathbb{R}^M$. If a given vector $x \in \mathbb{R}^N$ is mapped onto a given vector $y \in \mathcal{C}(A)$, any vector that differs from x by a vector in the *null space* maps to the same vector y. Such vectors x lie on a hyperplane parallel to the *null space* as shown in blue above.

   Note also that vectors in $\mathbb{R}^M$ not in the *column space* $\mathcal{C}(A)$ cannot be reached. Further, the action of the transformation $y' = A^t x'$ results in a similar figure, with A^t mapping any point in $\mathbb{R}^M$ onto the *row space* of A. In particular, for the choice $x' = y$ in the figure, $A^t y$ maps back to the vector $y' = A^t A x$ in the *row space* $\mathcal{R}(A)$.

5.2   The Normal Equations

For a matrix of size $M \times N$, the **normal equations** are obtained from $Ax = b$ written in column view by taking the dot product of the equation with each of the column vectors $a_i, i = 1, 2, \cdots M$.

$$A^t A x = A^t b \qquad\qquad \Leftrightarrow \qquad (5.1a)$$

$$a_i \cdot (x_1 a_1) + a_i \cdot (x_2 a_2) + \cdots a_i \cdot (x_N a_N) = a_i \cdot b, i = 1, 2, \cdots M. \qquad (5.1b)$$
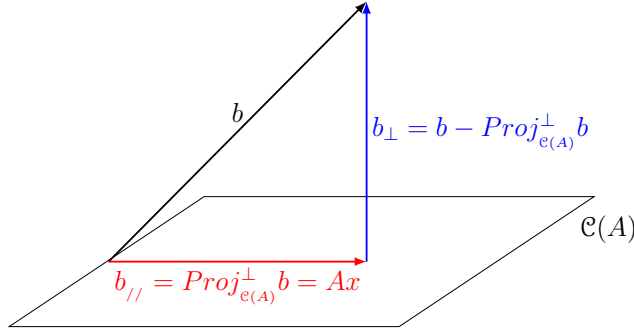
☞ If $Ax = b$ has a solution, the normal equations 5.1 have the same solution.

   Since any solution of $Ax = b$ satisfies the normal equation, the only difference in the set of solutions must be a null space solution. We thus only need to show that $\mathcal{N}(A) = \mathcal{N}(A^t A)$ as follows:
   If $x \in \mathcal{N}(A)$, then $Ax = 0 \Rightarrow A^t A x = 0$, so $x \in \mathcal{N}(A^t A)$.
   Conversely, if $x \in \mathcal{N}(A^t A)$, then
   $A^t A x = 0 \Rightarrow x^t A^t A x = 0 \Rightarrow (Ax)^t (Ax) = 0 \Rightarrow (\mathbf{A}\mathbf{x}) \cdot (\mathbf{A}\mathbf{x}) = 0 \Rightarrow x \in \mathcal{N}(A)$.

Figure 5.2: Any one solution x of the **normal equations** $A^t Ax = A^t b$ is the coordinate vector of the **orthogonal projection** $b_{/\!/} = Proj^{\perp}_{\mathcal{C}(A)} b$ of b onto the column space of A. The vector $b_{\perp}$ from the projection point to the vector b is the shortest vector from any point in $\mathcal{C}(A)$ to b and lies in $\mathcal{N}(A^t)$. If $b_{\perp} \neq 0$, the equation $Ax = b$ does not have a solution, but $Ax = \mathbf{b} - b_{\perp}$ does. The solution of the normal equations therefore lets us compute the decomposition $b = b_{/\!/} + b_{\perp}$.

FIX rewrite: Fundamental Theorem part 2 FIX

The orthogonal complement of the column space $\mathcal{C}(A)$ is the null space of $\mathcal{N}(A^t)$. Therefore any vector b in the codomain of $y = Ax$ can be written as a unique combination[1] $b = b_{/\!/} + b_{\perp}$, where $b_{/\!/} \in \mathcal{C}(A)$ and $v_{\perp} \in \mathcal{N}(A^t)$. The result of this split is shwon in Figure 5.2.

We therefore have $Ax = b \Rightarrow A^t Ax = A^t b = A^t(b_{/\!/} + b_{\perp}) = A^t b_{/\!/}$. We see that $Ax = b_{/\!/}$ has a solution, since the right hand side is in $\mathcal{C}(A)$, and therefore the equation $A^t Ax = A^t b$ has the same solution(s).

The reader should compare the algebraic manipulations in the preceding paragraph to their geometric interpretation by referencing Figure 5.2: it shows the codomain of the transformation $y = Ax$..

Given a matrix A of size $M \times N$, a vector $b \in \mathbb{R}^N$, and a solution x of the normal equations $A^t Ax = A^t b$, the following observations hold:

☞ The equation $Ax = Proj^{\perp}_{\mathcal{C}(A)} b$ has the same solution(s) as the normal equation $A^t Ax = A^t b$.

☞ A vector b may be decomposed into two orthogonal components $b = b_{/\!/} +$

---

[1]The union of bases for $\mathcal{C}(A)$ and $\mathcal{N}(A^t)$ forms a basis for the whole space. Expressing any vector in this basis therefore splits the vector into a component in the column space $\mathcal{C}(A)$ and the null space $\mathcal{N}(A^t)$. These vectors are orthogonal.

$b_\perp$, where $b_\parallel \in \mathcal{C}(A)$, i.e.,

$$b = x_1 a_1 + x_2 a_2 + \cdots x_N a_N + b_\perp. \tag{5.2}$$

Here $b, a_1, a_2, \cdots a_N$ are known vectors, while the vector $b_\perp$ and the scalars $x_1, x_2, \cdots x_N$ are unknown.

The solution is carried out in three steps:

- solving the normal equation $A^t A x = A^t b$
- computing $b_\parallel = Proj_{\mathcal{C}(A)}^\perp b = Ax$, where x is a solution of the normal equation.
- computing $b_\perp = b - b_\parallel$.

☞ Given a decomposition $b = b_\parallel + \tilde{b}$ where $b_\parallel \in \mathcal{C}(A)$ is any one vector in the column space of A, then $\|b_\perp\| \leq \|\tilde{b}\|$, i.e., $b_\perp$ is the shortest vector from the column space of A to b. Therefore, a solution of the normal equation yields the solution of

$$\|b_\perp\| = \min_x \|b - Ax\| \tag{5.3}$$

known as the **least squares** problem.

The normal equation gives rise to a number of different formulae. If the columns of A are linearly independent, then $A^t A$ has a pivot in every column and is therefore invertible. We find

$$A^t A x = A^t b \Rightarrow Proj_{\mathcal{C}(A)}^\perp b = Pb, \tag{5.4}$$

where

$$P = A(A^t A)^{-1} A^t \tag{5.5}$$

is the **orthogonal projection matrix** onto the column space $\mathcal{C}(A)$.

Since $b_\perp = b - Pb = (I - P)b$, we see that the matrix $I - P$ projects b onto the null space $\mathcal{N}(A^t)$.

Observe that orthogonal projection matrices P are symmetric, i.e., $P^t = P$, and have the property that $P^2 = P$. The converse is true as well: a symmetric matrix P such that $P^2 = P$ is a projection matrix.
FIX add this to the exercises FIX

An important special case of Eq (5.2) occurs when the columns of A are **non-zero and mutually orthogonal**, i.e.,

$$a_i \cdot a_j = \begin{cases} 0 & \text{if } i \neq j \\ a_i \cdot a_i > 0 & \text{otherwise} \end{cases} \tag{5.6}$$

In this case $A^t A$ reduces to a diagonal matrix with i^{th} diagonal element equal to $\|a_i\|^2$, and the projection matrix Eq (5.5) simplifies to

$$P = \frac{1}{\|a_1\|^2} a_1 a_1^t + \frac{1}{\|a_2\|^2} a_2 a_2^t + \cdots + \frac{1}{\|a_N\|^2} a_N a_N^t, \tag{5.7}$$

a sum of orthogonal projection matrices onto each of the orthogonal axes $a_1, a_2, \cdots a_N$. We first encountered these matrices in Example 1.3.1, where we showed that Pb could be rewritten as

$$Proj^{\perp}_{\mathbb{e}(A)} b = Pb = \frac{\mathbf{a_1} \cdot \mathbf{b}}{\mathbf{a_1} \cdot \mathbf{a_1}} \mathbf{a_1} + \frac{\mathbf{a_2} \cdot \mathbf{b}}{\mathbf{a_2} \cdot \mathbf{a_2}} \mathbf{a_2} + \cdots \frac{\mathbf{a_N} \cdot \mathbf{b}}{\mathbf{a_N} \cdot \mathbf{a_N}} \mathbf{a_N}. \tag{5.8}$$

We can obtain this result directly from the column view form of the normal equations instead.

☞ Note that the a_i vectors in this expression can be scaled by any non-zero constants without affecting the result: we could replace a_i by $\alpha_i a_i, \alpha_i \neq 0$, and the α_i would cancel out: what matters is the direction of the line onto which we project, not the length of the direction vector for the line.

☞ The orthogonal components can be shifted between $b_{/\!/}$ and $b_{\perp}$,e.g.,

$$b = \frac{\mathbf{a_1} \cdot \mathbf{b}}{\mathbf{a_1} \cdot \mathbf{a_1}} \mathbf{a_1} + \frac{\mathbf{a_2} \cdot \mathbf{b}}{\mathbf{a_2} \cdot \mathbf{a_2}} \mathbf{a_2} + b_{\perp} = \frac{\mathbf{a_1} \cdot \mathbf{b}}{\mathbf{a_1} \cdot \mathbf{a_1}} \mathbf{a_1} + \tilde{b}_{\perp}.$$

☞ The projections can be added, e.g., the sum of the orthogonal projections onto the vectors a_1 and a_2 respectively is the orthogonal projection onto the plane $span\{a_1, a_2\}$. Figure 4.2 shows an example where this is not true when a_1 and a_2 are not perpendicular.

When the orthogonal basis vectors are unit vectors, Eqs (5.8) simplify even more,

$$P = \hat{a}_1 \hat{a}_1^t + \hat{a}_2 \hat{a}_2^t + \cdots + \hat{a}_N \hat{a}_N^t, \tag{5.9a}$$

$$Proj^{\perp}_{\mathbb{e}(A)} b = Pb = \hat{a}_1 \cdot b \, \hat{a}_1 + \hat{a}_2 \cdot b \, \hat{a}_2 + \cdots + \hat{a}_N \cdot b \, \hat{a}_N, \tag{5.9b}$$

where $\hat{a}_i \cdot \hat{a}_j = 0$ when $i \neq j$, and $\|a_i\| = 1$. Note that in this case, $A^t A$ reduces to the identity matrix.

☞ A matrix P is an orthogonal projection matrix if and only if $P^t = P$ and $P^2 = P$.

The forward direction follows trivially from Eq (5.5). To establish the only if part of the theorem, consider the column space of P. For any vector x, we have $Px \in \mathcal{C}(P)$ by definition. Given a vector b, and consider the vector $b - Pb$. We have $(b - Pb)^t Px = b^t (P - I)^t Px = b^t (P - P^2)x = 0$, from which we conclude that the vector $Pb \in \mathcal{C}(P)$ and $(b - Pb) \perp (Pb)$.

Example 5.2.1. *Solving the normal equations*
Solve the normal equations and find the projections $b_{/\!/}$ and $b_\perp$ for

$$Ax = b \Leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 8 \end{pmatrix}$$

   Set up the $Ax = b$ problem, multiply through by A^t and solve the resulting normal equation using Gaussian elimination:

$$\left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -3 & -2 \\ 8 & -2 & 8 \end{array} \right) \quad \text{The equation } Ax = b$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 4 \end{pmatrix} \left(\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 29 & 54 \end{array} \right) \quad \text{The normal equations}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \left(\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 27 & 54 \end{array} \right) \quad \text{Gaussian elimination}$$

   The solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \quad \text{Thus, } b_{/\!/} = Ax = \begin{pmatrix} 6 \\ -6 \\ 6 \end{pmatrix}, \quad b_\perp = b - b_{/\!/} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}.$$

Example 5.2.2. *Orthogonal projection onto lines and planes*
Let $a_1 = (1\ 2\ 0)^t, a_2 = (0\ 1\ 1)^t, b = (2\ -2\ -2)^t$. Find the orthogonal projections of b onto the lines $span\{a_1\}$, $span\{a_2\}$ and onto the plane $span\{a_1, a_2\}$.
   For the orthogonal projection onto a single vector, we can use the special case Eq (5.8) of the normal equations Eq (5.5).

$$Proj^\perp_{\mathfrak{e}(a_1)} b = \frac{a_1 \cdot b}{a_1 \cdot a_1} a_1 = -\frac{2}{5}(1\ 2\ 0)^t$$

$$Proj^\perp_{\mathfrak{e}(a_2)} b = \frac{a_2 \cdot b}{a_2 \cdot a_2} a_2 = -4(0\ 1\ 1)^t$$

The orthogonal projection onto the plane defined by $span\{a_1, a_2\}$ is given by Ax, where x is the solution of the normal equations Eq (5.1) with $A = (a_1 a_2)$. We find

$$A^t Ax = A^t b \Rightarrow \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Therefore $Proj^\perp_{\mathfrak{e}(A)} b = Ax = (1\ 3\ 1)^t$. Note $Proj^\perp_{\mathfrak{e}(a_1)} b + Proj^\perp_{\mathfrak{e}(a_2)} b \neq Proj^\perp_{\mathfrak{e}(A)} b$.

Example 5.2.3. *Orthogonal projection onto lines and planes (cont)*
Continuing the previous example, let $a_1 = (1\ 2\ 0)^t, a_3 = (-2\ 1\ 5)^t, b = (2\ -2\ -2)^t$. Find the orthogonal projections of b onto the lines span$\{a_1\}$, span$\{a_3\}$ and onto the plane span$\{a_1, a_3\}$.

Note that the vectors $a_3 = 5a_2 - 2a_1$, i.e., the plane span$\{a_1, a_3\}$ is the same as the plane of the previous example. Note further that $a_1 \cdot a_3 = 0$, thus $\{a_1, a_3\}$ is an orthogonal basis for this plane.

Using the special case Eq (5.8) of the normal equations Eq (5.5).

$$Proj^{\perp}_{e(a_1)} b = \frac{a_1 \cdot b}{a_1 \cdot a_1} a_1 = -\frac{2}{5}(1\ 2\ 0)^t$$

$$Proj^{\perp}_{e(a_3)} b = \frac{a_3 \cdot b}{a_3 \cdot a_3} a_3 = \frac{-8}{15}(-2\ 1\ 5)^t$$

Setting $A = (a_1\ a_3)$, we see that $A^t A$ is diagonal, since the vectors a_1 and a_2 are orthogonal. The normal equations decompose, and we obtain $Proj^{\perp}_{e(A)} b = Proj^{\perp}_{e(a_1)} b + Proj^{\perp}_{e(a_3)} b = \frac{2}{3}(1\ -2\ -4)^t$.
FIX check! FIX

Example 5.2.4. *Projection matrix onto the row and null spaces*
Find the orthogonal projection matrix P onto the row space and the orthogonal projectin matrix $\tilde{P}$ onto the nullspace for the following matrix:

$$\begin{pmatrix} 1 & 4 & -4 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Since the row and null space of a matrix are orthogonal complements, the projection matrices are related by $P + \tilde{P} = I$. We begin by finding bases for the spaces in question. Using Gaussian elimination, we obtain

$$\begin{pmatrix} 1 & 4 & -4 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \sim \quad \begin{pmatrix} 1 & 4 & -4 \\ 0 & -6 & 9 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the row space is therefore given by $\{(1\ 4\ -4)^t, (0\ -2\ 3)^t\}$. Using back substitution, we obtain the null space $x_H = \alpha(-4\ 3\ 2)^t$. The projection matrix $\tilde{P}$ onto the null space is easier to obtain since the null space has dimension 1: Using Eq (5.8) with $N = 1$, we find

$$\tilde{P} = \frac{1}{\|x_H\|^2} x_H x_H^t = \frac{1}{29} \begin{pmatrix} 16 & -12 & -8 \\ -12 & 9 & 6 \\ -8 & 6 & 4 \end{pmatrix},$$

and so

$$P = I - \tilde{P} = \frac{1}{29} \begin{pmatrix} 13 & 12 & 8 \\ 12 & 20 & -6 \\ 8 & -6 & 25 \end{pmatrix}.$$

To compute the matrix P directly using the normal equation, we construct a matrix with columns that form a basis for the row space of the example matrix:

$$A = \begin{pmatrix} 1 & 0 \\ 4 & -2 \\ -4 & 3 \end{pmatrix},$$

and compute the solution of the normal equation $P = A(A^t A)^{-1} A^t$, we find

$$P = \begin{pmatrix} 1 & 0 \\ 4 & -2 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 33 & -20 \\ -20 & 13 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 & -4 \\ 0 & -2 & 3 \end{pmatrix},$$

yielding the same result.

A matrix Q with orthonormal columns is an **orthogonal matrix**.. Therefore $Q^t Q = I$. If Q is symmetric, we know this is sufficient to show that $Q^{-1} = Q^t$. If not, we necessarily have that $QQ^t \neq I$.

Orthogonal matrices have nice properties. In particular, for $Qx = b$, the

normal equations are $x = Q^t b$, i.e., finding solutions to the normal equations is reduced to a matrix multiplication! Orthogonal matrices have nice numerical properties, and are therefore very useful in practice.

Let u, v be vectors in $\mathbb{R}^N$, and let Q be an orthogonal matrix of size $M \times N$. If we compare the vectors u, v and their transforms Qu, Qv, we find

☞ $\|Qu\| = \|u\|$, i.e., the lengths stay the same.

☞ $(Qu) \cdot (Qv) = u \cdot v$, i.e., the angles between the vectors stays the same.

☞ $Qu \perp Qv \Leftrightarrow u \perp v$, in particular, orthogonality between vectors is preserved.

The proofs rely on the observation that $(Qu)^t(Qv) = u^t Q^t Q v = u^t v$.

5.3 Least Squares

An important application of the normal equations is the least squares method (also known as regression).

Consider an experiment consisting of a set of measurements y_{meas} for a given set of values x, and a **model** that describes the relationship between y_{meas} and x. As an example, consider synthetic data generated by adding a Gaussian distributed error with $\sigma = 0.8$ to points on the line $y = 1 + 2x$. The data are shown graphically in Figure 5.3, and given in Table 5.3.

In practice, we only have the x and y_{meas} values. Suppose we conjecture a model for the data, e.g., we may suppose the data lie on a line (an equation of the form $y = a + bx$) with unknown **model parameters** a and b. When we substitute the data into this model equation we get a number of linear equations for these unknowns. When y_{meas} contains errors (or when the model is itself erroneous), these equations will not have a solution in general.

To address this, we may consider somehow changing the right hand side (i.e., the y_{meas} values which we assume to be in error) to some new y_{est} values) in such a way that the system does have a solution. Refering back to Eq 5.3, we see that the normal equation produces a solution for the model parameters that minimizes the norm $\|y_{meas} - y_{est}\|$. To clarify this even more, let us compare the two representations Figure 5.3 and Figure 5.2. The x vector may be thought of as an index for the entries of the y_{meas} vector, which corresponds to the b vectore in Figure 5.2. So $b_\perp = y_{meas} - y_{est}$. Note that we are not interested in $b_\parallel$ in this problem, but rather the solution of the normal equation, since it yields the model parameters.

Solving the normal equations is simple. Transcribe the $a + bx = y_{meas}$ equations to matrix form, and multiply by the transpose of the coefficient matrix to yield the normal equation. This yields

$$\begin{pmatrix} 20.000 & 30.000 \\ 30.000 & 46.837 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 83.750 \\ 128.420 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} \approx \begin{pmatrix} 1.906 \\ 1.521 \end{pmatrix},$$

x	y_{meas}	$a + bx = y_{meas}$	$a + bx + cx^2 = y_{meas}$
1.00	4.34	$a + b1.00 = 4.34$	$a + b1.00 + c1.00^2 = 4.34$
1.05	3.81	$a + b1.05 = 3.81$	$a + b1.05 + c1.05^2 = 3.81$
1.11	3.55	$a + b1.11 = 3.55$	$a + b1.11 + c1.11^2 = 3.55$
1.16	3.80	$a + b1.16 = 3.80$	$a + b1.16 + c1.16^2 = 3.80$
1.21	3.08	$a + b1.21 = 3.08$	$a + b1.21 + c1.21^2 = 3.08$
1.26	4.57	$a + b1.26 = 4.57$	$a + b1.26 + c1.26^2 = 4.57$
1.32	3.51	$a + b1.32 = 3.51$	$a + b1.32 + c1.32^2 = 3.51$
1.37	2.45	$a + b1.37 = 2.45$	$a + b1.37 + c1.37^2 = 2.45$
1.42	4.88	$a + b1.42 = 4.88$	$a + b1.42 + c1.42^2 = 4.88$
1.47	3.95	$a + b1.47 = 3.95$	$a + b1.47 + c1.47^2 = 3.95$
1.53	3.65	$a + b1.53 = 3.65$	$a + b1.53 + c1.53^2 = 3.65$
1.58	4.81	$a + b1.58 = 4.81$	$a + b1.58 + c1.58^2 = 4.81$
1.63	2.76	$a + b1.63 = 2.76$	$a + b1.63 + c1.63^2 = 2.76$
1.68	4.15	$a + b1.68 = 4.15$	$a + b1.68 + c1.68^2 = 4.15$
1.74	4.57	$a + b1.74 = 4.57$	$a + b1.74 + c1.74^2 = 4.57$
1.79	5.53	$a + b1.79 = 5.53$	$a + b1.79 + c1.79^2 = 5.53$
1.84	5.10	$a + b1.84 = 5.10$	$a + b1.84 + c1.84^2 = 5.10$
1.89	5.61	$a + b1.89 = 5.61$	$a + b1.89 + c1.89^2 = 5.61$
1.95	4.66	$a + b1.95 = 4.66$	$a + b1.95 + c1.95^2 = 4.66$
2.00	4.97	$a + b2.00 = 4.97$	$a + b2.00 + c2.00^2 = 4.97$

Table 5.1: Table of measurements y_{meas} as a function of x. The data is substituted into a linear model $y = a + bx$ and a quadratic model $y = a + bx + cx^2$.

which seems quite different from the equation $y = 1 + 2x$ that the data were generated with. Note however that the x vector does not contain the value 0, i.e., the x-axis intercept lies outside of the data covered by our experiment! Within the range of values x, the line $y_{est} = 1.906 + 1.521x$ lies quite close to the actual equation $y = 1 + 2x$, as can be seen in Figure 5.3. The norm of the error is $\|b_\perp\| \approx 3.196$.
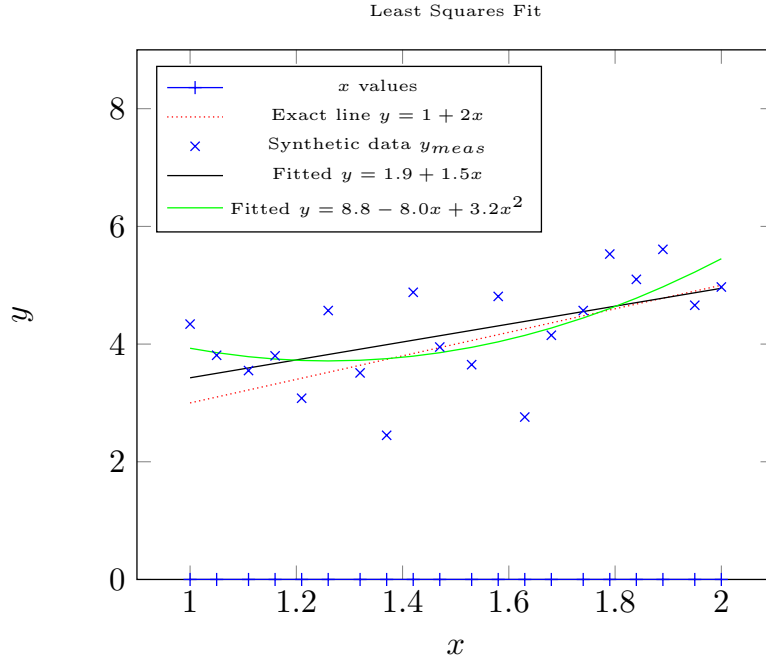
Figure 5.3: The least squares fit of a line and a parabola to the synthetic example data. The data y_{meas} was generated by adding Gaussian noise with $\sigma = 0.8$ to the line $y = 1 + 2x$.

We may choose other models, e.g., $y = a + bx + cx^2$. The only restriction for the method is that upon substitution, we should find a set of linear equations for the model parameters, and that the errors should be confined to the right hand side vector. The substitution is shown in Table 5.3. Solving the associated normal equation for the model parameters yields

$$
\begin{pmatrix} 20.000 & 30.000 & 46.837 \\ 30.000 & 46.837 & 75.766 \\ 46.837 & 75.766 & 126.353 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 83.750 \\ 128.419 \\ 204.940 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \approx \begin{pmatrix} 8.757 \\ -8.002 \\ 3.174 \end{pmatrix},
$$

so that $y_est = 8.757 - 8.002x + 3.174x^2$. The error norm is $b_\perp \approx 2.976$.

Example 5.3.1. *Least squares fit of an exponential*

Consider again the data in Table 5.3. If we choose the model $y = ae^{bx}$, the resulting equations are not linear in the model parameters a, b. We may however take the logarithm of the model of interest, i.e., $ln(y) = ln(a) + bx$, and solve for the parameters $ln(a)$ and b by least squares.

Computing the normal equations, we find the same A matrix that we had for the linear model, with a different right hand side, i.e.,

$$\begin{pmatrix} 20.000 & 30.000 \\ 30.000 & 46.837 \end{pmatrix} \begin{pmatrix} ln(a) \\ b \end{pmatrix} = \begin{pmatrix} 83.750 \\ 128.420 \end{pmatrix} \Rightarrow \begin{pmatrix} ln(a) \\ b \end{pmatrix} \approx \begin{pmatrix} 0.880 \\ 0.353 \end{pmatrix}.$$

Computing a and substituting in our model, we obtain $y_{est} = 2.41e^{0.35x}$. The error norm is $b_{\perp} \approx 3.180$.

5.4   The Gram-Schmidt Process

FIX Expository Material here FIX

A direct transcription of the Gram-Schmidt idea leads to the following set of equations:

$$\begin{aligned}
\mathbf{w_1} &= \mathbf{v_1} \\
\mathbf{w_2} &= \mathbf{v_2} - \frac{\mathbf{v_2} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1} \\
\mathbf{w_3} &= \mathbf{v_3} - \frac{\mathbf{v_3} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1} - \frac{\mathbf{v_3} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \mathbf{w_2} \\
\mathbf{w_4} &= \mathbf{v_4} - \frac{\mathbf{v_4} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1} - \frac{\mathbf{v_4} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \mathbf{w_2} - \frac{\mathbf{v_4} \cdot \mathbf{w_3}}{\mathbf{w_3} \cdot \mathbf{w_3}} \mathbf{w_3} \\
&\quad \cdots \\
\mathbf{w_K} &= \mathbf{v_K} - \frac{\mathbf{v_K} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1} - \frac{\mathbf{v_K} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \mathbf{w_2} \cdots - \frac{\mathbf{v_K} \cdot \mathbf{w_{K-1}}}{\mathbf{w_{K-1}} \cdot \mathbf{w_{K-1}}} \mathbf{w_{K-1}}
\end{aligned} \tag{5.10}$$

which result in a set of orthogonal vectors $\mathbf{w_1}, \mathbf{w_2}, \cdots \mathbf{w_K}$ that span the same subspace as the original vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_K}$.

The computation requires many dot products and is error prone. A good computational layout is obtained by writing the vectors $\mathbf{v_i}$ and $\mathbf{w_i}$ as column vectors into two matrices

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \quad \text{and } W = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix},$$

and setting up the matrix products

$$\begin{matrix} & \begin{pmatrix} A & | & W \end{pmatrix} \\ \begin{pmatrix} W^t \end{pmatrix} & \begin{pmatrix} W^t A & | & W^t W \end{pmatrix} \end{matrix} \tag{5.11}$$

where we fill in the columns of W (and corresponding rows of W^t) and carry out the products as soon as they become available. The entries in $W^t A$ are the

numerators in the Gram-Schmidt equations, while the diagonal entries in W^tW are the denominators. The correctness of the computations can be judged by noting the following properties of the matrices in the layout:

☞ the matrix W^tW is **diagonal** (by construction).

☞ the matrix W^tA is **upper triangular**, as we will show in the next section 5.5.

   To obtain unit length vectors, an additional step is required: each of the vectors $\mathbf{w_i}$ must be normalized to unit length, i.e.,

$$\mathbf{q_i} = \frac{1}{||\mathbf{w_i}||}\mathbf{w_i}, i = 1, 2, \cdots K.$$

Since the diagonal entries of W^tW are precisely the entries $\mathbf{w_i}\cdot\mathbf{w_i}$, we define the scaling matrix $S = diag\left(\frac{1}{\sqrt{\mathbf{w_i}\cdot\mathbf{w_i}}}\right)$, and add a final computation to the layout:

$$
\begin{array}{ccc}
 & \left(\begin{array}{c|c} A & W \end{array}\right) & \\
\left(\ W^t\ \right) & \left(\begin{array}{c|c} W^tA & W^tW \end{array}\right) & \text{(5.12)} \\
\left(\ S\ \right)\left(\ SW^t\ \right) & \left(\begin{array}{c|c} SW^tA & SW^tW \end{array}\right) &
\end{array}
$$

although the bottom right matrix SW^tW is not required and may be omitted.

   To complete the computation, it remains to copy out the results: The transpose of the matrix SW^t, traditionally denoted Q, has orthonormal columns q_i for $i = 1, 2, \cdots K$, i.e., $Q^tQ = I$. The upper triangular matrix SW^tA is traditionally denoted R. Expressed in terms of Q, we have $R = Q^tA$. Finally, the third matrix $SW^tW = S^{-1}$ is diagonal. The entries on the diagonal are the lengths of the vectors w_i. Note that by construction Q^t is a left inverse of Q. If Q is square, then $Q^t = Q^{-1}$ and thus $Q^tQ = QQ^t = I$, while $QQ^t \neq I$ otherwise. This observation is important[2]:

☞ Given $A = QR$ with $Q^tQ = I$, we can multiply from the left to obtain $R = Q^tA$.

---

[2]We will show below that the Gram-Schmidt process produces the factorization $A = QR$. The matrix R converts the columns of Q to the columns of A.

Example 5.4.1. *Gram-Schmidt orthogonalization*
Use the Gram-Scmidt algorithm to orthogonalize the vectors

$$\mathbf{v_1} = \left(\begin{array}{cccc} 1 & -1 & 0 & 0 \end{array}\right)$$
$$\mathbf{v_2} = \left(\begin{array}{cccc} 0 & 1 & -1 & 0 \end{array}\right)$$
$$\mathbf{v_3} = \left(\begin{array}{cccc} 0 & 0 & 1 & -1 \end{array}\right)$$

Set up the computational layout and carry out the initial multiplication for w_1. Carefully label the rows and columns with the name of the corresponding vector.
   Step 1: $\mathbf{w_1} = \mathbf{v_1}$ *is trivial. It replaces the original set of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ with a new set $\mathbf{w_1}, \mathbf{v_2}, \mathbf{v_3}$ (the columns with labels shown in blue). The matrix multiplications (in blue) compute $\mathbf{v_i} \cdot \mathbf{w_1}$ for i in $1, 2, 3$, and the product $\mathbf{w_1} \cdot \mathbf{w_1}$, the exact quantities required for step 2.*

	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$
	1	0	0	**1**		
	−1	1	0	**−1**		
	0	−1	1	**0**		
	0	0	−1	**0**		
$\mathbf{w_1}$	**1 −1 0 0**	**2**	**−1**	**0**	**2**	
$\mathbf{w_2}$						
$\mathbf{w_3}$						

   Step 2:

$$\mathbf{w_2} = \mathbf{v_2} - \frac{\mathbf{v_2} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1}$$
$$= (0 \ 1 \ -1 \ 0) - \frac{-1}{2}(1 \ -1 \ 0 \ 0)$$
$$= (\tfrac{1}{2} \ \tfrac{1}{2} \ -1 \ 0).$$

The resulting set of basis vectors is now $\mathbf{w_1}, \mathbf{w_2}, \mathbf{v_3}$. Inserting $\mathbf{w_2}$ in the computational layout yields

	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$
	1	0	0	1	$\tfrac{1}{2}$	
	−1	1	0	−1	$\tfrac{1}{2}$	
	0	−1	1	0	**−1**	
	0	0	−1	0	**0**	
$\mathbf{w_1}$	1 −1 0 0	2	−1	0	2	**0**
$\mathbf{w_2}$	$\tfrac{1}{2}$ $\tfrac{1}{2}$ **−1** 0	**0**	$\tfrac{3}{2}$	**−1**	**0**	$\tfrac{3}{2}$
$\mathbf{w_3}$						

*Note the pattern of zeros that are developing: the first product is **upper diagonal**, while the second is **diagonal**.*

Example 5.4.2. *Gram-Schmidt orthogonalization (continued)*
Continuing the previous example, we now compute
  Step 3:

$$\mathbf{w_3} = \mathbf{v_3} - \frac{\mathbf{v_3} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1} - \frac{\mathbf{v_3} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \mathbf{w_2}$$

$$= (0 \ 0 \ 1 \ -1) - \tfrac{0}{2}(1 \ -1 \ 0 \ 0) - \tfrac{-1 \cdot 2}{3}(\tfrac{1}{2} \ \tfrac{1}{2} \ -1 \ 0)$$

$$= (\tfrac{1}{3} \ \tfrac{1}{3} \ \tfrac{1}{3} \ -1).$$

Inserting this result in the computational layout and carrying out the computations, we obtain

				$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$
				1	0	0	1	$\tfrac{1}{2}$	$\tfrac{1}{3}$
				-1	1	0	-1	$\tfrac{1}{2}$	$\tfrac{1}{3}$
				0	-1	1	0	-1	$\tfrac{1}{3}$
				0	0	-1	0	0	-1
$\mathbf{w_1^t}$	1	-1	0	0	2	-1	0	2	0	0
$\mathbf{w_2^t}$	$\tfrac{1}{2}$	$\tfrac{1}{2}$	-1	0	0	$\tfrac{3}{2}$	-1	0	$\tfrac{3}{2}$	0
$\mathbf{w_3^t}$	$\tfrac{1}{3}$	$\tfrac{1}{3}$	$\tfrac{1}{3}$	-1	0	0	$\tfrac{4}{3}$	0	0	$\tfrac{4}{3}$

The final step is to rescale the vectors $\mathbf{w_i}$ to unit length. The squares of the lengths are the diagonal entries in the bottom right matrix in the layout, so we need to scale the entries of $\mathbf{w_i}$ by the inverse square root of those entries. Since the squares of the lengths of the $\mathbf{w_i}$ appear on the diagonal of the matrix at the bottom right, we read out the results to obtain a set of orthonormal vectors

$$\mathbf{q_1} = \frac{1}{\sqrt{\mathbf{w_1} \cdot \mathbf{w_1}}} \mathbf{w_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{q_2} = \frac{1}{\sqrt{\mathbf{w_2} \cdot \mathbf{w_2}}} \mathbf{w_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & -2 & 0 \end{pmatrix}$$

$$\mathbf{q_3} = \frac{1}{\sqrt{\mathbf{w_3} \cdot \mathbf{w_3}}} \mathbf{w_3} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & -3 \end{pmatrix}.$$

that span the same subspace as span $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

When the Gram-Schmidt algorithm is applied to a set of linearly dependent vectors, one or more of the vectors $\mathbf{w_k}$ will equal 0. These vectors correspond to vectors $\mathbf{v_k}$ that are linear combinations of the preceding $\mathbf{v_i}$ for $i = 1, 2, \cdots, k-1$. Since these $\mathbf{v_k}$ can be omitted without affecting the *span* $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_K}\}$, we simply omit the vector from the set of $\mathbf{w_i}$.

Example 5.4.3. *Gram-Schmidt orthogonalization with linearly dependent vectors*

Obtain an orthogonal basis for the column space $\mathcal{C}(A)$, where the matrix A is given by

$$A = \begin{pmatrix} 2 & 3 & 7 & 1 \\ 1 & 0 & 2 & 1 \\ 4 & 2 & 10 & 1 \\ 2 & 1 & 5 & 1 \end{pmatrix}$$

The completed computational layout is as follows

$$
\begin{array}{c}
\begin{array}{ccccccc} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} & \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_4} \end{array} \\
\begin{pmatrix}
2 & 3 & 7 & 1 & 2 & \frac{43}{25} & \frac{8}{47} \\
1 & 0 & 2 & 1 & 1 & -\frac{16}{25} & \frac{32}{47} \\
4 & 2 & 10 & 1 & 4 & -\frac{14}{25} & -\frac{19}{47} \\
2 & 1 & 5 & 1 & 2 & -\frac{7}{25} & \frac{14}{47}
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{c}
\mathbf{w_1} \\ \mathbf{w_2} \\ \mathbf{w_4}
\end{array}
\begin{pmatrix}
2 & 1 & 4 & 2 \\
\frac{43}{25} & -\frac{16}{25} & -\frac{14}{25} & -\frac{7}{25} \\
\frac{8}{47} & \frac{32}{47} & -\frac{19}{47} & \frac{14}{47}
\end{pmatrix}
\begin{pmatrix}
25 & 16 & 66 & 9 & 25 & 0 & 0 \\
0 & \frac{94}{25} & \frac{94}{25} & \frac{6}{25} & 0 & \frac{94}{25} & 0 \\
0 & 0 & 0 & \frac{35}{47} & 0 & 0 & \frac{35}{47}
\end{pmatrix}.
$$

Note that since vector $\mathbf{w_3}$ is omitted, the matrix $W^t A$ now has zero entries on the main diagonal. It is however in row echelon form. Completion of the layout is driven by the set of computations

$$\mathbf{w_1} = \mathbf{v_1} = \begin{pmatrix} 2 & 1 & 4 & 2 \end{pmatrix},$$

$$\mathbf{w_2} = \mathbf{v_2} - \frac{\mathbf{v_2} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1}$$

$$= \begin{pmatrix} 3 & 0 & 2 & 1 \end{pmatrix} - \frac{16}{25} \begin{pmatrix} 3 & 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{43}{25} & -\frac{16}{25} & -\frac{14}{25} & -\frac{7}{25} \end{pmatrix}$$

$$\mathbf{w_3} = \mathbf{v_3} - \frac{\mathbf{v_3} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1} - \frac{\mathbf{v_3} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \mathbf{w_2}$$

$$= \begin{pmatrix} 7 & 2 & 10 & 5 \end{pmatrix} - \frac{66}{25} \begin{pmatrix} 2 & 1 & 4 & 2 \end{pmatrix} - \frac{94 \cdot 25}{25 \cdot 94} \begin{pmatrix} \frac{43}{25} & -\frac{16}{25} & -\frac{14}{25} & -\frac{7}{25} \end{pmatrix} = 0$$

$$\mathbf{w_4} = \mathbf{v_4} - \frac{\mathbf{v_4} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \mathbf{w_1} - \frac{\mathbf{v_4} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \mathbf{w_2}, \quad omit\ \mathbf{w_3}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} - \frac{9}{25} \begin{pmatrix} 2 & 1 & 4 & 2 \end{pmatrix} - \frac{6 \cdot 25}{25 \cdot 94} \begin{pmatrix} \frac{43}{25} & -\frac{16}{25} & -\frac{14}{25} & -\frac{7}{25} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{47} & \frac{32}{47} & -\frac{19}{47} & \frac{14}{47} \end{pmatrix}$$

Since $\mathbf{w_3} = 0$, we see that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent. (In fact $\mathbf{v_3} = 2\mathbf{v_1} + \mathbf{v_2}$.) An orthonormal basis for $\mathcal{C}(A)$ is obtained by scaling the $\mathbf{w_i}$ by the inverse of their length. Reading out the results, we obtain a set of orthonormal vectors

$$\mathbf{q_1} = \frac{1}{\sqrt{\mathbf{w_1} \cdot \mathbf{w_1}}} \mathbf{w_1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{4}{5} & \frac{2}{5} \end{pmatrix}$$

$$\mathbf{q_2} = \frac{1}{\sqrt{\mathbf{w_2} \cdot \mathbf{w_2}}} \mathbf{w_2} = \begin{pmatrix} \frac{43}{5\sqrt{94}} & \frac{-16}{5\sqrt{94}} & \frac{-14}{5\sqrt{94}} & \frac{-7}{5\sqrt{94}} \end{pmatrix}$$

$$\mathbf{q_4} = \frac{1}{\sqrt{\mathbf{w_4} \cdot \mathbf{w_4}}} \mathbf{w_4} = \begin{pmatrix} \frac{8}{\sqrt{1645}} & \frac{32}{\sqrt{1645}} & \frac{-19}{\sqrt{1645}} & \frac{14}{\sqrt{1645}} \end{pmatrix}$$

5.5   The QR Decomposition

We next address the question why the matrix $W^t A$ in Eq 5.11 is upper triangular. The transcription of the Gram-Schmidt process Eq 5.10 to matrix form is revealing: starting with the original set of vectors written as columns in a matrix
$A = \begin{pmatrix} v_1 & v_2 & \cdots v_K \end{pmatrix}$, we form a matrix from the set of basis vectors at step k

$$W_k = \begin{pmatrix} w_1 & w_2 & \cdots & w_{k-1} & w_k & v_{k+1} & \cdots & v_n \end{pmatrix}$$

and the matrix N_k of coefficients required to compute W_{k+1}

$$N_k = \begin{pmatrix} I & -n_k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix} \text{ where } n_k = \begin{pmatrix} \frac{\mathbf{v_k} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \\ \frac{\mathbf{v_k} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \\ \cdots \\ \frac{\mathbf{v_k} \cdot \mathbf{w_k}}{\mathbf{w_k} \cdot \mathbf{w_k}} \end{pmatrix}.$$

We now see that the Gram-Schmidt equations take the form

$$\begin{aligned} W_1 &= A \\ W_2 &= W_1 N_1 = A\left(N_1\right) \\ W_3 &= W_2 N_2 = A\left(N_1 N_2\right) \\ &\cdots \\ W_K &= W_{K-1} N_{K-1} = A\left(N_1 N_2 \cdots N_{K-1}\right), \end{aligned}$$

where the product of coefficient matrices $N_1 N_2 \cdots N_k$ is upper unit triangular. It has a pattern that may be investigated similarly to Example 1.2.13. In particular, the inverse is unit upper triangular, and takes the form

$$\left(N_1 N_2 \cdots N_{k-1}\right)^{-1} = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1 & n_1 & n_2 & \cdots & n_K & 0 \\ \hline 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & & 1 & \\ & & & & 0 & I \end{array} \end{pmatrix}$$

Scaling the columns of W_K to unit length by multiplying from the right by a diagonal matrix of scale factors S, we find $A = QR$, where $Q = W_K S$ is orthogonal (i.e., $Q^t Q = I$), and where $R = \left(N_1 N_2 \cdots N_{K-1} S\right)^{-1}$ is upper triangular. This decomposition of A is known as the QR decomposition of A.

☞ **Since Q need not be square, it is not necessarily invertible**:

   – If Q is square, then $Q^t Q = Q Q^t = I$, i.e., $Q^{-1} = Q^t$.
   – If Q is not square, then $Q^t Q = I$, but $Q Q^t \neq I$.
     The matrix $Q Q^t$ is the projection matrix into the $\mathcal{C}(Q)$.

FIX refine the text to allow for linearly dependent columns in A FIX

Given A and Q, we easily obtain R by observing that $A = QR \Rightarrow Q^t A = Q^t QR \Rightarrow Q^t A = R$. Refering back to the computational layout 5.12, we see that matrix $SW^t = Q^t$, and matrix $SW^t A = R$, i.e., we can read out the QR decomposition directly, taking care to transpose the SW^t matrix.

Example 5.5.1. QR *decomposition*
We return to example 5.4.1 and complete the scaling operation in the layout:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{3} \\
-1 & 1 & 0 & -1 & \frac{1}{2} & \frac{1}{3} \\
0 & -1 & 1 & 0 & -1 & \frac{1}{3} \\
0 & 0 & -1 & 0 & 0 & -1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & -1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & -1 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1
\end{pmatrix}
\begin{pmatrix}
2 & -1 & 0 & \mathbf{2} & 0 & 0 \\
0 & \frac{3}{2} & -1 & 0 & \frac{3}{2} & 0 \\
0 & 0 & \frac{4}{3} & 0 & 0 & \frac{4}{3}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\
0 & 0 & \frac{\sqrt{3}}{2}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 \\
\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{-3}{\sqrt{12}}
\end{pmatrix}
\begin{pmatrix}
\frac{2}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \sqrt{2} & 0 & 0 \\
0 & \frac{3}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\
0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{3}}
\end{pmatrix},
$$

i.e., we chose the scaling matrix by inverting the square roots of the entries of the rightmost matrix in the second row. Reading out the results, we see that $A = QR$, where

$$
A = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
0 & 0 & -1
\end{pmatrix}, \quad
Q = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\
\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\
0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\
0 & 0 & \frac{-3}{\sqrt{12}}
\end{pmatrix}, \quad
R = \begin{pmatrix}
\frac{2}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\
0 & \frac{3}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\
0 & 0 & \frac{2}{\sqrt{3}}
\end{pmatrix}
$$

Note again that since Q is not square, $Q^t Q = I$, but $QQ^t \neq I$.

Example 5.5.2. *Reduced* **QR** *decomposition, linearly dependent vectors*

We return to example 5.4.3 and complete the scaling operation in the layout:

$$
\begin{pmatrix}
2 & 3 & 7 & 1 & 2 & \frac{43}{25} & \frac{8}{47} \\
1 & 0 & 2 & 1 & 1 & \frac{-16}{25} & \frac{32}{47} \\
4 & 2 & 10 & 1 & 4 & \frac{-14}{25} & \frac{-19}{47} \\
2 & 1 & 5 & 1 & 2 & \frac{-7}{25} & \frac{14}{47}
\end{pmatrix}
$$

$$
\begin{pmatrix}
2 & 1 & 4 & 2 \\
\frac{43}{25} & \frac{-16}{25} & \frac{-14}{25} & \frac{-7}{25} \\
\frac{8}{47} & \frac{32}{47} & \frac{-19}{47} & \frac{14}{47}
\end{pmatrix}
\begin{pmatrix}
25 & 16 & 66 & 9 & 25 & 0 & 0 \\
0 & \frac{94}{25} & \frac{94}{25} & \frac{6}{25} & 0 & \frac{94}{25} & 0 \\
0 & 0 & 0 & \frac{35}{47} & 0 & 0 & \frac{35}{47}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{1}{5} & 0 & 0 \\
0 & \frac{5}{\sqrt{94}} & 0 \\
0 & 0 & \frac{\sqrt{47}}{\sqrt{35}}
\end{pmatrix}
\begin{pmatrix}
\frac{2}{5} & \frac{1}{5} & \frac{4}{5} & \frac{2}{5} \\
\frac{43}{5\sqrt{94}} & \frac{-16}{5\sqrt{94}} & \frac{-14}{5\sqrt{94}} & \frac{-7}{5\sqrt{94}} \\
\frac{8}{\sqrt{1645}} & \frac{32}{\sqrt{1645}} & \frac{-19}{\sqrt{1645}} & \frac{14}{\sqrt{1645}}
\end{pmatrix}
\begin{pmatrix}
5 & \frac{16}{5} & \frac{66}{5} & \frac{9}{5} & 5 & 0 & 0 \\
0 & \frac{\sqrt{94}}{5} & \frac{\sqrt{94}}{5} & \frac{6}{5\sqrt{94}} & 0 & \frac{\sqrt{94}}{5} & 0 \\
0 & 0 & 0 & \frac{\sqrt{35}}{\sqrt{47}} & 0 & 0 & \frac{\sqrt{35}}{\sqrt{47}}
\end{pmatrix}
$$

We obtain the reduced QR *decomposition (reduced since we removed one of the* $\mathbf{w_i}$ *from Q:*

$$
A = \begin{pmatrix}
2 & 3 & 7 & 1 \\
1 & 0 & 2 & 1 \\
4 & 2 & 10 & 1 \\
2 & 1 & 5 & 1
\end{pmatrix}
\quad
Q = \begin{pmatrix}
\frac{2}{5} & \frac{43}{5\sqrt{94}} & \frac{8}{\sqrt{1645}} \\
\frac{1}{5} & \frac{-16}{5\sqrt{94}} & \frac{32}{\sqrt{1645}} \\
\frac{4}{5} & \frac{-14}{5\sqrt{94}} & \frac{-19}{\sqrt{1645}} \\
\frac{2}{5} & \frac{-7}{5\sqrt{94}} & \frac{14}{\sqrt{1645}}
\end{pmatrix},
\quad
R = \begin{pmatrix}
5 & \frac{16}{5} & \frac{66}{5} & \frac{9}{5} \\
0 & \frac{\sqrt{94}}{5} & \frac{\sqrt{94}}{5} & \frac{6}{5\sqrt{94}} \\
0 & 0 & 0 & \frac{\sqrt{35}}{\sqrt{47}}
\end{pmatrix}.
$$

The Gram-Schmidt algorithm can be used to extend a set of orthonormal vectors to a full basis for the vector space. In Exercise 5.1 below, we first extend the vectors to a complete basis for the vector space and use the Gram-Schmidt algorithm on the resulting basis. We may also use Gram-Schmidt directly, as show in the following example:

Example 5.5.3. *Extend an orthonormal subspace basis to an orthonormal basis for $\mathbb{R}^4$*

Extend the orthonormal vectors
$$v_1 = \left(\tfrac{1}{\sqrt{2}} \ -\tfrac{1}{\sqrt{2}} \ 0 \ 0\right)^t, \quad v_2 = \left(\tfrac{1}{\sqrt{3}} \ \tfrac{1}{\sqrt{3}} \ \tfrac{1}{\sqrt{3}} \ 0\right)^t$$
to an orthonormal basis for $\mathbb{R}^4$.

We proceed by extending the given vectors with a known basis for $\mathbb{R}^4$, i.e., the columns of I, and applying the Gram-Schmidt algorithm. Vectors found to lie in the hyperplane of the vectors obtained in previous steps are skipped, (i.e., we compute the reduced QR decomposition).

Note that the arithmetic can be simplified by initially removing the square root scale factors from the given vectors v_1, v_2. The computations required are as follows:

Let $e_i, i = 1, \cdots 4$ be the columns of I, and $w_i, i = 1, \cdots 4$ be the orthogonal vectors we are looking for. Applying Gram-Schmidt to $\{v_1, v_2, e_1, e_2, e_3, e_4\}$, we find

$$w_1 = \sqrt{2}v_1$$

$$w_2 = \sqrt{3}v_2$$

$$w_3 = e_1 - \frac{e_1 \cdot w_1}{w_1 \cdot w_1}w_1 - \frac{e_1 \cdot w_2}{w_2 \cdot w_2}w_2 = \frac{1}{6}(1 \ 1 \ -2 \ 0)^t, \quad drop \ \frac{1}{6} \ for \ simplicity$$

$$w_4 = e_2 - \frac{e_2 \cdot w_1}{w_1 \cdot w_1}w_1 - \frac{e_2 \cdot w_2}{w_2 \cdot w_2}w_2 - \frac{e_2 \cdot w_3}{w_3 \cdot w_3}w_3 = 0, \quad thus \ skip \ e_2$$

$$w_4 = e_3 - \frac{e_3 \cdot w_1}{w_1 \cdot w_1}w_1 - \frac{e_3 \cdot w_2}{w_2 \cdot w_2}w_2 - \frac{e_3 \cdot w_3}{w_3 \cdot w_3}w_3 = 0, \quad thus \ skip \ e_3$$

$$w_4 = e_4 - \frac{e_4 \cdot w_1}{w_1 \cdot w_1}w_1 - \frac{e_4 \cdot w_2}{w_2 \cdot w_2}w_2 - \frac{e_4 \cdot w_3}{w_3 \cdot w_3}w_3 = (0 \ 0 \ 0 \ 1)$$

Normalizing the w_i to unit length, we obtain the orthonormal basis
$$q_1 = v_1, q_2 = v_2, q_3 = \tfrac{1}{\sqrt{6}}(1 \ 1 \ -2 \ 0), q_4 = e_4.$$

An alternate method to extend an orthonormal basis uses Gaussian Elimination. Recall that the $\mathcal{N}(A^t)$ of a matrix A can be obtained by inspecting the reduction of the matrix $(A|I)$ to the row echelon form. Let $A = (a_1|a_2|\ldots|a_k)$ be an orthogonal matrix of size $M \times k$ with $k < M$. We obtain a basis for $\mathcal{N}(A^t)$ by selection the non-pivot rows of the matrix E reducing A to a row echelon form R, i.e., $E(A|I) = (R|E)$. These basis vectors are not mutually orthogonal in general, they are however orthogonal to the columns of A.

If we augment A by one of these basis vectors (labeled a_{k+1}) however, we can repeat the process. Since the first k columns of the matrix have not changed, the previous elimination steps still apply: $E(A|a_{k+1}|I) = (R|Ea_{k+1}|E)$, and we can just continue the computation. The computation terminates with a set of mutually orthogonal vectors $\{a_1, a_2, \ldots a_M\}$.

Example 5.5.4. *Extend an orthonormal subspace basis to an orthonormal basis for $\mathbb{R}^4$ using the Fundamental theorem*

Extend the orthonormal vectors

$$v_1 = \left(\tfrac{1}{\sqrt{6}} \ \ 0 \ \ -\tfrac{1}{\sqrt{6}} \ \ \tfrac{2}{\sqrt{6}} \right)^t, \quad v_2 = \left(\tfrac{1}{\sqrt{3}} \ \ \tfrac{1}{\sqrt{3}} \ \ \tfrac{1}{\sqrt{3}} \ \ 0 \right)^t$$

to an orthonormal basis for $\mathbb{R}^4$.

We can simplify the arithmetic by rescaling the v_1 and v_2 vectors to remove the square roots. Set $A = (\sqrt{6}v_1 \mid \sqrt{3}v_2 \mid \mid \mid I)$, leaving room for the two column vectors needed for the basis for $\mathbb{R}^4$, and reduce to row echelon form. Insert each vector found in the derivation in the next empty column and proceed.

$$\left(\begin{array}{rrrr|rrrr} 1 & 1 & \mathbf{1} & \mathbf{-1} & 1 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{-2} & \mathbf{0} & 0 & 1 & 0 & 0 \\ -1 & 1 & \mathbf{1} & \mathbf{1} & 0 & 0 & 1 & 0 \\ 2 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} eliminate \ \ in \ \ the \\ 1^{st} \ column \end{array}$$

$$\left(\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{rrr|rrrr} 1 & 1 & \mathbf{1} & 1 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{-2} & 0 & 1 & 0 & 0 \\ 0 & 2 & \mathbf{2} & 1 & 0 & 1 & 0 \\ 0 & -2 & \mathbf{-2} & -2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} eliminate \ \ in \ \ the \\ 2^{nd} \ column \end{array}$$

$$\left(\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \left(\begin{array}{rrr|rrrr} 1 & 1 & \mathbf{1} & 1 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{-2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{6} & \mathbf{1} & \mathbf{-2} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & \mathbf{-6} & -2 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} update \ 3^{rd} \ column \\ and \ eliminate \end{array}$$

$$\left(\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \left(\begin{array}{rrr|rrrr} 1 & 1 & \mathbf{1} & 1 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{-2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{6} & 1 & -2 & 1 & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{array}\right) \begin{array}{l} We \ now \ have \ the \\ 2^{nd} \ vector \end{array}$$

.

To obtain an orthonormal basis, we rescale each of the columns of A to unit length. The corresponding orthogonal matrix is

$$Q = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

.

Note that we may replace the Gaussian elimination steps in the above computation with alternate matrices (Givens rotations or Householder transformations - see the appendices).

5.6   Exercises

Exercise 5.1. *In Example 3.3.3 we extended a set of linearly dependent vectors in $\mathbb{F}^N$ to a basis for $\mathbb{F}^N$. We now consider the case when part II of the fundamental theorem applies.*

Consider the columns of the matrix over the scalars $\mathbb{R}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

1. *Verify that the columns are linearly independent.*

2. *Since $\mathbb{R}^4 = \mathcal{C}(A) \oplus \mathcal{N}(A^t)$, we can generate a full basis for $\mathbb{R}^4$ by combining the columns of A with a basis for $\mathcal{N}(A^t)$. Compute such a basis for R^4 i) by solving $A^t x = 0$. ii) by augmenting A with I and reducing this matrix to row-echelon form.*

3. *Verify that the basis vectors obtained for $\mathbb{C}(A)$ are orthogonal to the basis vectors obtained for $\mathcal{N}(A^t)$.*

4. *Starting with an extended basis for $\mathbb{R}^4$ found above, use the Gram-Schmidt algorithm to obtain an orthonormal basis for $\mathbb{R}^4$.*

5. *Instead of ii) in part 2, compute an orthonormal basis using the method in Example 5.5.4.*

Chapter 6

Determinants

6.1 The Wedge Product

We now turn to the derivation of a vector product designed to capture the notion of area. We begin with an example in 2D: the area of the parallelogram representing the geometric view of vector addition.

Given two vectors a and b, we wish to define the bivector $a \wedge b$ (read a wedge b) such that this vector represents an area in the plane defined by a and b. Let us begin by considering the properties of this **wedge product**.

We would like the area associated with $a \wedge b$ to be the same size as the area of the parallelogram used in the geometrical interpretation of vector addition.

Consider what happens to this area when one of the vectors is scaled. In Fig 6.1 we see the original wedge product $a \wedge b$. Scaling the length of the vector a by α increases the area by a factor α, shown shaded blue. We therefore postulate

$$(\alpha a) \wedge b = a \wedge (\alpha b) = \alpha(a \wedge b), \tag{6.1}$$

so that we may drop the parentheses in expressions of the form $\alpha a \wedge b$.
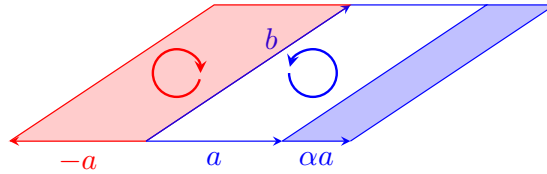
Figure 6.1: Scaling of the wedge product. This 2D figure shows the wedge product $a \wedge b$ in blue. Scaling of the vector a by a scalar α increases the area by this factor: we therefore define $(\alpha a) \wedge b = \alpha(a \wedge b)$. The special case $\alpha = -1$ yields the wedge product $-a \wedge b$ shown in red. The geometric areas of $a \wedge b$ and $(-a) \wedge b$ are the same. The sign difference may be accounted for by the orientation of the perimeters.

Note that the special case $\alpha = -1$ does not change the geometric area, but does change the sign of the wedge product:

$$(-a) \wedge b = -(a \wedge b) = a \wedge (-b).$$

☞ **The wedge product $\alpha a \wedge b$ therefore encodes three pieces of information**:

    – the **hyperplane** defined by the vectors a and b containing the area

    – the **geometric magnitude $\mid \alpha \mid$ of the area** compared to the reference area $a \wedge b$

    – the **orientation of the path enclosing the area**: the same as the reference path $a \wedge b$ if $\alpha > 0$, the opposite if $\alpha < 0$.

☞ The multiplier α in $\alpha a \wedge b$ is a scale factor that expresses the **area measured relative to the reference area** $a \wedge b$. The sign of α encodes the orientation of this area in comparison to the orientation of the reference area.

The previous equation leads us to define the wedge product to be anticommutative: since changing the order of the terms reverses the orientation of the path:

$$a \wedge b = -b \wedge a \tag{6.2}$$

Note that for the special case $b = a$, this implies that

$$a \wedge a = 0, \tag{6.3}$$

so that if a and b are colinear, the area defined by a and b is zero, consistent with our geometric expectations.

We now turn to distributivity of the wedge product over vector addition, i.e., we consider products of the form $a \wedge (b + c)$. Referring to Fig 6.2, we see that this expression can be interpreted as the area of two parallelograms $a \wedge b$ and $b \wedge c$. If a, b and c are colinear, we can use the theorem that the area of a parallelogram is the product of the length of the base of the parallelogram times the height. We are thus led to postulate

$$a \wedge (b + c) = a \wedge b + a \wedge c. \tag{6.4}$$

Note the special case

$$a \wedge (b + \alpha a) = a \wedge b. \tag{6.5}$$

We inferred the distributivity rule by considering Fig 6.2 to be 2-dimensional: the vectors a, b and c in the geometric figure were considered to lie in the same plane. If we consider what happens in three dimensions, the same figure shows that the wedge product can be used to decompose areas: the parallelogram formed from a and $b + c$ can be deformed by moving the endpoint of b anywhere in space!
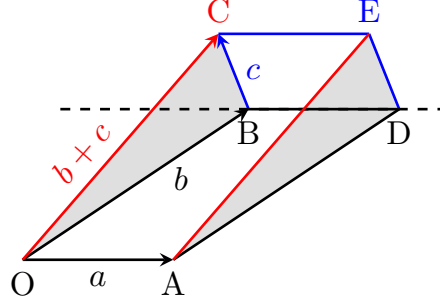
Figure 6.2: Distributivity of the wedge product over addition. This 2D figure shows the deformation of two parallelograms $OADB$ and $BDEC$ resulting when the segment BD moves along the dashed line containing the original segment. Since the sides OA, BD and CE are parallel and of the same length, the shaded triangles OBC and ADE are congruent and therefore have the same area. The relationship of the areas of the parallelograms $area(OADB)+ area(BDEC) = area(OAEC)$ shows that we require $a \wedge b + a \wedge c = a \wedge (b+c)$, where $a = OA$, $b = OB$ and $c = BC$.

The wedge product is readily generalized to allow computations with hypervolumes by introducing the notion of n-vectors of the form $a \wedge b \wedge c$, and more generally, of n-vectors of the form $a_1 \wedge a_2 \wedge \cdots \wedge a_n$. No parentheses are necessary if we postulate the product to be associative:

$$(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \wedge (b_1 \wedge b_2 \wedge \cdots \wedge a_m) = (a_1 \wedge a_2 \wedge \cdots \wedge a_{n-1}) \wedge (a_n \wedge b_1 \wedge b_2 \wedge \cdots \wedge a_m)$$
(6.6)

We note the following useful properties about reordering the vectors appearing in a wedge product:

- moving a vector in a wedge product to the left or to the right changes the sign according to the number of vectors "jumped over":

$$a_1 \wedge \cdots \wedge a_{i-1} \wedge \mathbf{a_i} \wedge a_{i+1} \wedge \cdots a_{i+p} \wedge a_{i+p+1} \cdots \wedge a_n$$
$$= (-1)^p a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots a_{i+p} \wedge \mathbf{a_i} \wedge a_{i+p+1} \cdots \wedge a_n$$

- interchanging two vectors in a wedge product changes the sign (one of the vectors "jumps" over p intervening vectors toward the other vector; the vectors then exchange places; the other vector then "jumps" over the intervening p vectors back to the place the first vector had occupied, yielding an overall sign chane $(-1)^{1+2p} = -1$.)

$$a_1 \wedge \cdots \wedge a_{i-1} \wedge \mathbf{a_i} \wedge a_{i+1} \wedge \cdots \mathbf{a_{i+p}} \wedge a_{i+p+1} \cdots \wedge a_n$$
$$= - a_1 \wedge \cdots \wedge a_{i-1} \wedge \mathbf{a_{i+p}} \wedge a_{i+1} \wedge \cdots a_{i+p-1} \wedge \mathbf{a_i} \wedge a_{i+p+1} \cdots \wedge a_n$$

Also, if a vector appears more than once in a wedge product we get

$$a_1 \wedge \cdots \wedge \mathbf{b} \cdots \wedge \mathbf{b} \wedge \cdots \wedge a_n = 0.$$

The scaling and distribution of n-vectors over addition similarly generalize.

Given a vector space V, we can use the wedge product of n vectors to define a new vector space: $\bigwedge^n V = \{a_1 \wedge a_2 \wedge \cdots \wedge a_n \mid a_i \in V, i = 1, 2, \cdots n\}$.

Consider a basis $\{a_1, a_2, \cdots a_N\}$ for a vector space V, and define the k-vectors

$$a_{i_1 i_2 \cdots i_k} = a_{i_1} \wedge a_{i_2} \cdots \wedge a_{i_k},$$

where the $i_j, j = 1, 2, \cdots k$ are distinct indices in $1, 2, \ldots N, k \leq N$.

We list bases for each of the vector spaces $\bigwedge^k V$:

basis	space
1	$\bigwedge^0 V = \mathcal{F}$
$a_1, a_2, \cdots a_n$	$\bigwedge^1 V = V$
$a_{12}, a_{13}, \cdots a_{1n}, a_{23}, \cdots a_{2n} \cdots a_{n-1n}$	$\bigwedge^2 V$
$a_{123}, a_{124}, \cdots a_{12n}, a_{234}, \cdots a_{23n}, \cdots a_{n-2n-1n}$	$\bigwedge^3 V$
$\cdots$	$\cdots$
$a_{12\cdots n}$	$\bigwedge^n V$

The proof is left as an exercise.

FIX Introduce $\bigwedge V = \mathcal{F} \oplus V \oplus \bigwedge^2 V \oplus \cdots \bigwedge^n V$, or we can make sense of adding scalars and vectors! ... FIX

A consequence of the properties of the wedge product is that we can carry out algebra in a manner similar to the scalar case. The only proviso is that we need to be careful to keep track of sign changes whenever we change the order of vectors in a wedge product.

Example 6.1.1. *Comparison of the wedge and cross products*
Given $u = 3e_1 - 2e_3$, $v = -e_1 + e_2 + 2e_3$, compute the wedge and the cross product of u and v. We will simplify the appearance of the computations by using the basis vectors e_{12}, e_{23} and e_{31}, which describe the three coordinate planes. The ordering of the vectors is motivated by the right hand rule: $e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2$.

We have

$$
\begin{aligned}
u \wedge v =& (3e_1 - 2e_3) \wedge (-e_1 + e_2 + 2e_3) \\
=& 3e_1 \wedge (-e_1 + e_2 + 2e_3) - 2e_3 \wedge (-e_1 + e_2 + 2e_3) \\
=& 3e_{12} + 6e_{13} + 2e_{31} - 2e_{32} \\
=& 3e_{12} + 2e_{23} - 4e_{31},
\end{aligned}
$$

where we have used the properties $e_1 \wedge e_1 = 0$ and $e_3 \wedge e_3 = 0$ The cross product is given by a very similar computation:

$$
\begin{aligned}
u \times v =& (3e_1 - 2e_3) \times (-e_1 + e_2 + 2e_3) \\
=& 3e_1 \times (-e_1 + e_2 + 2e_3) - 2e_3 \times (-e_1 + e_2 + 2e_3) \\
=& 3e_3 - 6e_2 + 2e_2 + 2e_1 \\
=& 3e_3 + 2e_1 - 4e_2.
\end{aligned}
$$

The difference between the products is that the wedge product keeps track of the coordinate planes, while the cross product replaces each of them with the vector orthogonal to the coordinate plane using the right hand rule convention.

FIX comment on the decomposition of an area into areas in the coordinate planes. Contrast this with the next example, which collapses the produt to a single term (the only coordinate plane in $\bigwedge^3$ FIX

Example 6.1.2. *Computation of the wedge product*
Given $u = 3e_1 - 2e_3$, $v = -e_1 + e_2 + 2e_3$, *and* $w = 2e_1 + 4e_2$ *Simplify the following wedge product representing the volume of a parallelepiped with edges specified by u, v and w:*

$$
\begin{aligned}
u \wedge v \wedge w =&(3e_1 - 2e_3) \wedge (-e_1 + e_2 + 2e_3) \wedge (2e_1 + 4e_2) \\
=&3e_1 \wedge (-e_1 + e_2 + 2e_3) \wedge (2e_1 + 4e_2) \\
&- 2e_3 \wedge (-e_1 + e_2 + 2e_3) \wedge (2e_1 + 4e_2) \\
=&(-3e_1 \wedge e_1 + 3e_1 \wedge e_2 + 6e_1 \wedge e_3) \wedge (2e_1 + 4e_2) \\
&+ (2e_3 \wedge e_1 - 2e_3 \wedge e_2 - 4e_3 \wedge e_3) \wedge (2e_1 + 4e_2) \\
=&(3e_1 \wedge e_2 + 6e_1 \wedge e_3) \wedge (2e_1 + 4e_2) \\
&+ (2e_3 \wedge e_1 - 2e_3 \wedge e_2) \wedge (2e_1 + 4e_2) \\
=&(6e_1 \wedge e_2 \wedge e_1 + 12e_1 \wedge e_2 \wedge e_2 + 12e_1 \wedge e_3 \wedge e_1 + 24e_1 \wedge e_3 \wedge e_2) \\
&+ (4e_3 \wedge e_1 \wedge e_1 + 8e_3 \wedge e_1 \wedge e_2 - 4e_3 \wedge e_2 \wedge e_1 - 8e_3 \wedge e_2 \wedge e_2) \\
=&(-6e_1 \wedge e_1 \wedge e_2 + 12e_1 \wedge e_2 \wedge e_2 - 12e_1 \wedge e_1 \wedge e_3 - 24e_1 \wedge e_2 \wedge e_3) \\
&+ (4e_1 \wedge e_1 \wedge e_3 + 8e_1 \wedge e_2 \wedge e_3 + 4e_1 \wedge e_2 \wedge e_3 - 8e_2 \wedge e_2 \wedge e_3) \\
=&(-24e_1 \wedge e_2 \wedge e_3) \\
&+ (8e_1 \wedge e_2 \wedge e_3 + 4e_1 \wedge e_2 \wedge e_3) \\
=&- 12e_1 \wedge e_2 \wedge e_3
\end{aligned}
$$

Thus the volume of the parallelepiped is -12 times the volume of the reference parallelepiped $e_1 \wedge e_2 \wedge e_3$.

Looking at the expansion in Ex (6.1.2), we see that the computations can easily be streamlined. Once we choose any one term in a vector (say the e_2 term in vector v, the remaining vectors can no longer contribute an e_2 term: immediately dropping such terms in the remaining vectors, we could have written

$$
\begin{aligned}
u \wedge v \wedge w =&(3e_1 - 2e_3) \wedge (-e_1 + e_2 + 2e_3) \wedge (2e_1 + 4e_2) \\
=&3e_1 \wedge (e_2 + 2e_3) \wedge (4e_2) - 2e_3 \wedge (-e_1 + e_2) \wedge (2e_1 + 4e_2) \\
=&3e_1 \wedge e_2 \wedge 0 + 6e_1 \wedge e_3 \wedge (4e_2) - 2e_3 \wedge e_1 \wedge (4e_2) - 2e_3 \wedge e_2 \wedge (2e_1) \\
=&24e_1 \wedge e_3 \wedge e_2 - 8e_3 \wedge e_1 \wedge e_2 - 4e_3 \wedge e_2 \wedge e_1 \\
=&- 24e_1 \wedge e_2 \wedge e_3 - 8e_1 \wedge e_2 \wedge e_3 - 4e_1 \wedge e_2 \wedge e_3 \\
=&- 12e_1 \wedge e_2 \wedge e_3.
\end{aligned}
$$

This pattern is even more striking if we consider what happens when the vectors are presented in matrix form:

$$
\wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \wedge \begin{pmatrix} 3 & 0 & -2 \\ -1 & 1 & 2 \\ 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},
$$

where the $\wedge$ operator in front of the matrices indicates that we want to compute the wedge product of the three rows of vectors.

The wedge computation of $u \wedge v \wedge w$ proceeds by first selecting the e_1 term from u. The remaining multipliers appear in the submatrix resulting from dropping the first row (we have selected an entry from this vector) and the first column (no other e_1 term need be included from the remaining columns): using each entry of u in turn, we get

$$
\begin{aligned}
u \wedge v \wedge w = \quad & \mathbf{3e_1} \wedge \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}, \\
& + \mathbf{0e_2} \wedge \begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_3 \end{pmatrix} \\
& -\mathbf{2e_3} \wedge \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\
= \quad & 3e_1 \wedge (2e_3 \wedge 4e_2) - 2e_3 \wedge (-4e_1 \wedge e_2 + 2e_2 \wedge e_1) \\
= \ -& 12e_1 \wedge e_2 \wedge e_3,
\end{aligned}
$$

where we expanded the matrices recursively, being careful to maintain the order of the wedge products.

Note that we need not have started with the u vector, i.e., the first row: starting with the second row, for instance, leads to the computation of $u \wedge v \wedge w = -v \wedge u \wedge w$, where the row change needs to be accounted for by a minus sign.

$$
\begin{aligned}
u \wedge v \wedge w = - \quad & (-1)\mathbf{e_1} \wedge \begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} \\
- \quad & \mathbf{1e_2} \wedge \begin{pmatrix} 3 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_3 \end{pmatrix} \\
- \quad & \mathbf{2e_3} \wedge \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\
= \quad & e_1 \wedge (-8e_3 \wedge e_2) - 1e_2 \wedge (-4e_3 \wedge e_1) - 2e_3 \wedge (12e_1 \wedge e_2) \\
= \ -& 12e_1 \wedge e_2 \wedge e_3.
\end{aligned}
$$

We might also collect terms by going across colums. Starting with the first column, for example, collect all terms containing e_1. We again need to be careful about the interchanges of e_i vectors resulting in sign changes.

$$
\begin{aligned}
u \wedge v \wedge w = \quad & 3e_1 \wedge \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} \\
& + 1e_1 \wedge \begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} \\
& + 2e_1 \wedge \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} \\
= \quad & 3e_1 \wedge (8e_3 \wedge e_2) + 1e_1 \wedge (-8e_3 \wedge e_2) + 2e_1 \wedge (-2e_3 \wedge e_2) \\
= \ -& 12e_1 \wedge e_2 \wedge e_3.
\end{aligned}
$$

☞ Selecting a term in the matrix corresponds to selecting an entry (e.g., the e_i entry) from a given vector. Corresponding entries (e_i) in the other vectors cannot appear in the current term.

☞ Since we can expand by either rows or columns, we see that $\wedge A = \wedge A^t$. The wedge product is a product of vectors. These vectors may be entered in a matrix as either rows or columns. The expansions yield the same result.

These observations lead to the recursive **Laplace expansion** of the wedge product when the matrix of vectors is square of size $N \times N$.

We note first that we do not need to write the $(e_1 \ e_2 \ \cdots e_N)^t$ vectors explicitly, nor write the n-vector term $e_1 \wedge e_2 \cdots \wedge e_N$, since it is the only n-vector that will appear in the result:

$$a_1 \wedge a_2 \cdots \wedge a_N = \alpha \, e_1 \wedge e_2 \cdots \wedge e_N.$$

the constant α is known as the **determinant** of the vectors $a_1, \cdots a_N \in \mathbb{R}^N$, i.e., the hypervolume of the parallelepiped defined by the vectors with respect to the reference hypervolume defined by the standard basis $e_1, e_2 \cdots e_N$. We do however need to keep track of the ordering of the e_i in the intermediate wedge products. We adopt the conventions that the vectors in a wedge product will appear such that their indices are in numerical order. E.g., we will write $e_1 \wedge e_3$ rather than $e_3 \wedge e1$.

We will find it usefult to define the sign associated with a matrix element in row i and column j to be $(-1)^{i+j}$, resulting in a checkerboard pattern, starting with $+1$ in the top left corner:

$$\left((-1)^{i+j} \right) \quad = \quad \begin{pmatrix} 1 & -1 & 1 & \cdots \\ -1 & 1 & -1 & \cdots \\ 1 & -1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Further, we define the **minor** A_{ij} of a matrix A of size $M \times N$ as the submatrix of size $(M-1) \times (N-1)$ obtained from A by deleting the ith row and j^{th} column.

The Laplace expansion is defined recursively, under the convention that the determinants of submatrices at each stage appear with the wedge products of the corresponding vectors in numerical order.

The determinant of the wedge product can then be expressed as follows:

• the determinant of a 1×1 matrix is

$$det(a) = a \tag{6.7}$$

• For a matrix of size $N \times N$, with $N > 1$ pick any row i of A. We then have

$$det(A) = \sum_{j=1}^{N} (-1)^{i+j} a_{ij} det(A_{ij}) \tag{6.8}$$

- Equivalently, for any matrix of size $N \times N$, with $N > 1$ pick any col j of A to get

$$det(A) = \sum_{i=1}^{N} (-1)^{i+j} a_{ij} det(A_{ij}).\tag{6.9}$$

☞ A convenient definition is to combine the sign and the determinant of the minor: the **cofactor** of an entry a_{ij} of a matrix A is

$$C_{ij} = (-1)^{i+j} A_{ij}$$

The Laplace expansion is then given by

$$det(A) = \sum_{j=1}^{N} a_{ij} C_{ij} \qquad\qquad \text{expanding along row } i, \text{ or}$$

$$det(A) = \sum_{j=1}^{N} a_{ij} C_{ij} \qquad\qquad \text{expanding along column } j.$$

The 2×2 case (expanded about the first column) is given by

$$det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = a\, det(d) - b\, det(c) = ad - bc.$$

Example 6.1.3. 3×3 ***Determinant using the Laplace Expansion***
Consider again the wedge product from Example (6.1.2)
Expanding about the first row,

$$A = det \begin{pmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ 0 & 1 & 4 \\ -2 & 2 & 0 \end{pmatrix}$$

$$= \mathbf{3}\, det \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix} - (\mathbf{-1}) \begin{pmatrix} 0 & 4 \\ -2 & 0 \end{pmatrix} + \mathbf{2}\, det \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$$

$$= 3(-2 \times 4) + (-(-2) \times 4) + 2(-(2 \times 1))$$

$$= -12$$

There is a further expansion of the determinant known as the **Leibnitz expansion** that is worth noting. Denoting vectors $a_i = \sum_{j=1}^{N} \alpha_{ij} e_j$, we see that the wedge product consists of a sum of terms such that each vector contributes exactly one entry for distinct components e_i, (that is each term is made up by selecting one entry from each row and each column),

$$a_1 \wedge a_2 \wedge \cdots \wedge a_N = \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_N=1}^{N} \epsilon_{i_1 i_2 \cdots i_N} \alpha_{1 i_1} \alpha_{2 i_2} \cdots \alpha_{N i_N} e_1 \wedge e_2 \cdots \wedge e_N$$

$$\tag{6.10}$$

where the Levi-Civitta symbol

$$\epsilon_{i_1 i_2 \cdots i_N} = \begin{cases} 0 & \text{if any two indices have the same value} \\ 1 & \text{if the indices are an even permutation of } 1, 2, \cdots N \\ -1 & \text{if the indices are an odd permutation of } 1, 2, \cdots N \end{cases}$$

accounts for the overall sign of each term. For example,
$\epsilon_{121} = 0, \epsilon_{312} = -\epsilon_{132} = \epsilon_{123}$.

The rules for the Levi-Civitta symbol capture the fact that every vector contributes entries with a different index, and that the e_i vectors must be re-ordered into a term $e_1 \wedge e_2 \cdots \wedge e_N$ with indices in numerical sequence, thereby introducing sign changes.

We can also choose to order the computations as

$$a_1 \wedge a_2 \wedge \cdots \wedge a_N = \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_N=1}^{N} \epsilon_{i_1 i_2 \cdots i_N} \alpha_{i_1 1} \alpha_{i_2 2} \cdots \alpha_{i_N N} e_1 \wedge e_2 \cdots \wedge e_N, \tag{6.11}$$

where we have chosen to expand one e_i component at a time.

6.2   Computation of Determinants

Given the wedge product definition of determinants, the following useful properties of determinants are easy to see:

☞ Let A be a square matrix with a column (or row) of zeros. We have $det(A) = 0$ since one of the vectors of $\wedge A = a_1 \wedge a_2 \cdots \wedge a_N$ is the zero vector.

☞ If any column (or row) of a matrix A is a linear combination of the remaining columns (or rows), then $det(A) = 0$. This is a direct consequence of Eq (6.5).

☞ Consider the determinant to be a function D of the columns (or the rows) of a matrix A of size $N \times N$, i.e., $det(A) = D(a_1, a_2, \ldots, a_N)$. The function D is linear in each of its arguments:

$$D(a_1, \ldots, a_{i-1}, \alpha b + \beta c, a_{i+1}, \ldots, a_N) = \quad \alpha D(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_N) \tag{6.12}$$
$$+ \quad \beta D(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_N) \tag{6.13}$$

for any scalars α and β, and any vectors b, c in $\mathbb{R}^N$.

In particular, if we choose $b = a_j, beta = 1, c = a_i$, for some $j \neq i$, we have

$$D(a_1, \ldots, a_{i-1}, a_i + \alpha a_j, a_{i+1}, \ldots, a_N) = \quad \alpha D(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_N). \tag{6.14}$$

Thus, the elementary elimination operation used in Gaussian elimination does not change the determinant of a matix.

If instead we choose $c = 0$, we find the effect of the elementary scaling operation

$$D(a_1, \ldots, a_{i-1}, \alpha a_i, a_{i+1}, \ldots, a_N) = \alpha D(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_N).$$
$$(6.15)$$

☞ Interchanging a pair of columns (or rows) of a matrix A changes the sign of the determinant, since this corresponds to interchanging the corresponding vectors in the wedge product.

☞ The determinant of a triangular matrix is the product of of the diagonal elements.

Example 6.2.1. *Simple examples of determinant evaluation*
Consider the following determinants:

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ -1 & 0 & -1 \end{pmatrix} = 0, \quad \text{column of zeros}$$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ -3 & 2 & -1 \end{pmatrix} = 0, \quad \text{linearly dependent columns: } a_3 = a_1 + a_2$$

Example 6.2.2. *Simple examples of determinant evaluation (cont)*
Consider the following determinants:

$$\det \begin{pmatrix} \mathbf{1} & 3 & 1 \\ 0 & \mathbf{4} & 3 \\ 0 & 0 & \mathbf{-1} \end{pmatrix} = \mathbf{1} \times \mathbf{4} \times (\mathbf{-1}) = -4, \qquad \text{Laplace expansion, first column}$$

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -1, \qquad \text{single row exchange}$$

The Leibnitz expansion for the determinant of a matrix of size $N \times N$ shows that the computation requires $N!$ terms of the form $a_{1i_1} a_{2i_2} \cdots a_{Ni_N}$ (N choices of entries from the first row, followed by $N - 1$ choices from the second row, etc. For $N = 15$, this is approximately 1.310^12 terms. Clearly, we need an alternative method to compute determinants. Looking back at the properties of the determinant we found so far, we see that we know the effect of elementary operations performed on a matrix have on the determinant of a matrix. Since Gaussian Elimination on a square matrix produces an upper triangular matrix

with an easy to compute determinant (the product of the diagonal terms), we can use Gaussian Elimination for this task.

  Specifically, we see that

i) for an elementary elimination E, we have $det(EA) = det(A)$. Since E is unit lower triangular, $det(E) = 1$, so that $det(EA) = det(E)det(A)$.

ii) for a row exchange E, we have $det(EA) = -det(A)$. Since $det(E) = -1$, we again have $det(EA) = det(E)det(A)$.

ii) for an elementary scaling operation E, (i.e., scaling a single row by some factor α), $det(EA) = \alpha det(A)$. We again note that $det(E) = \alpha$, so that we again have $det(EA) = det(E)det(A)$.

☞ Since any matrix A can be written as a product $E_k E_{k-1} \cdots E_1 U$, where the $E_i, i = 1, 2, \cdots k$ matrices are elementary operation matrices, we have

$$det(A) = \frac{det(U)}{det(E_1)det(E_2)\cdots det(E_k)}. \tag{6.16}$$

Example 6.2.3. *Computing the determinant using Gaussian Elimination*
We return to the simple example 2.1

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 4 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{array}{l} \textit{The matrix } A \\ \textit{with determinant} \\ det(A) = \Delta. \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{array}{l} A_1 = E_1 A, \\ det(E_1) = 1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} A_2 = E_2 A_1, \\ det(E_2) = -1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{l} A_3 = E_3 A_2, \\ det(E_3) = \frac{1}{2} \end{array}$$

  Therefore

$$det(A) = \frac{det(A_3)}{det(E_1)\,det(E_2)\,det(E_3)} = \frac{1}{1 \times (-1) \times \frac{1}{2}} = -2.$$

☞ $det(A) = 0 \Leftrightarrow A$ is not full column rank, i.e., A^{-1} does not exist.

We have previously established that any matrix A can be written int the form $A = E_k E_{k-1} \cdots E_1 U$ where the $E_i, i = 1, 2, \cdots k$ are elementary operation matrices and the matrix U is in reduced row echelon form. Consider two square matrices A and B of the same size.

If A is not invertible, the reduced row echelon form U has at least one row of zeros (since there is a missing pivot), Since UB has a row of zeros, $det(UB) = 0$, $det(AB) = det(E_k E_{k-1} \cdots E_1(UB))$, we see that $det(AB) = 0$, which we can rewrite as $det(AB) = det(A)det(B)$. If A is invertible, the reduced row echelon form $U = I$, and so
$det(AB) = det(E_k E_{k-1} \cdots E_1 B) = det(E_k E_{k-1} \cdots E_1)det(B) = det(A)det(B)$.
We have established the useful property that

$$det(AB) = det(A)det(B). \tag{6.17}$$

The wedge product makes this property of determinants obvious: if we express vectors $a_i, i = 1, 2, \cdots N$ as linear combinations a set of vectors $b_1, b_2, \cdots b_N$, and similarly express the vectors b_i as linear combinations of the vectors $e_i, i = 1, 2, \cdots N$, we have

$$\begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_N \end{pmatrix} = A \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_N \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_N \end{pmatrix} = B \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_N \end{pmatrix}.$$

Substitution shows that the a_i vectors are related to the e_i vectors by the product of the A and B matrices, i.e., by

$$\begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_N \end{pmatrix} = AB \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_N \end{pmatrix}.$$

Computing the associated hypervolumes, we see that

$$a_1 \wedge a_2 \cdots \wedge a_N = det(A) \quad b_1 \wedge b_2 \cdots \wedge b_N$$
$$b_1 \wedge b_2 \cdots \wedge b_N = det(B) \quad e_1 \wedge e_2 \cdots \wedge e_N$$
$$a_1 \wedge a_2 \cdots \wedge a_N = det(AB) \, e_1 \wedge e_2 \cdots \wedge e_N$$

and thus $det(AB) = det(A)det(B)$.

Let A be a matrix of size $N \times N$. We define the kth power of A as the product $A^k = AA \cdots A$, where the matrix A on the right is repeated k times. For $k = 0$, we define $A^0 = I$. If A is invertible, we further extend the definition to the negative integers by setting $A^{-k} = (A^{-1})^k$.

☞ $det(A^k) = (det(A))^k$ holds for $k = 0, 1, 2, \ldots$.

   If A^{-1} exists, this relationship holds for $k = 0, \pm 1, \pm 2, \ldots$. This follows from the application of Eq (6.17) to $AA^{-1} = I$ and the definition of the power of a matrix.

☞ If A is a matrix of size $N \times N$, $det(\alpha A) = \alpha^N det(A)$.

This relationship is a trivial consequence of the wedge product. It also follows from Eq (6.17) by noting that
$det(\alpha A) = det(\alpha I A) = det(\alpha I) det(A) = \alpha^N det(A)$.

Example 6.2.4. *Determinants of matrix products*
Let A and B be matrices of size 5×5 such that $det(A) = 2, det(B) = 10$.
    Since the matrices have non-zero determinant, they are invertible. We compute

$$det(A^{-1}) = (det(A))^{-1} = \frac{1}{2}$$

$$det(3B^t A^{100}) = det(3B^t) det(A^{100}) = det(3B) \left(det(A)\right)^{100} = 3^5 10 2^{100}$$

Example 6.2.5. *Determinants of block matrices*
Let A and B be square matrices, and consider the following block matrix

$$C = \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right).$$

We can factor C as

$$C = \left(\begin{array}{cc} A & 0 \\ 0 & I \end{array} \right) \left(\begin{array}{cc} I & 0 \\ 0 & B \end{array} \right).$$

Applying the product rule and the Laplace expansion, we therefor have

$$det(C) = det(A) det(B).$$

For example

$$det \left(\begin{array}{ccccc} 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array} \right) = (2 \times 2 - 1 \times 2)(4 \times 1 \times 2) = 16.$$

6.3   Cramer's Rule

We can use the wedge product property $a \wedge a = 0$ to solve $Ax = b$. For simplicity, assume the columns of A are linearly independent. When we take the wedge product of $Ax = b$ with column vector a_i for some i, the x_i variable drops out:

$$Ax = b \Leftrightarrow x_1 a_1 + x_2 a_2 + x_3 a_3 + \cdots x_N a_N = b$$
$$\Rightarrow x_1 a_2 \wedge a_1 + x_2 a_2 \wedge a_2 + x_3 a_2 \wedge a_3 + \cdots x_N a_2 \wedge a_N = a_2 \wedge b$$
$$\Rightarrow x_1 a_2 \wedge a_1 + x_3 a_2 \wedge a_3 \cdots x_N a_2 \wedge a_N = a_2 \wedge b.$$

By taking the wedge product with all but the ith column a_i, all but the ith term drop out, leaving us with

$$Ax = b \Rightarrow x_i a_1 \wedge a_2 \cdots \wedge a_{i-1} \wedge a_{i+1} \cdots \wedge a_N \wedge a_i = a_1 \wedge a_2 \cdots \wedge a_{i-1} \wedge a_{i+1} \cdots \wedge a_N \wedge b.$$

Note that if we rearrange the order of the vectors by increasing indices the resulting sign change, if any, can be absorbed by also rearranging the right hand side as follows

$$x_i a_1 \wedge a_2 \cdots \wedge a_{i-1} \wedge a_i \wedge a_{i+1} \cdots \wedge a_N = a_1 \wedge a_2 \cdots \wedge a_{i-1} \wedge b \wedge a_{i+1} \cdots \wedge a_N.$$

If b is in the column space of A, this can be solved for x_i.

Example 6.3.1. *Using the wedge product to solve a system of linear equations*
Let $A = (a_1 a_2)$ of size 3×2, and consider the two probles $Ax = b_1$ and $Ax = b_2$, where $a_1 = (1\ 0\ 2)^t$, $a_2 = (1\ 2\ 3)^t$, $b_1 = (7\ 10\ 10)^t$, and $b_2 = (7\ 10\ 10)^t$. Using the standard basis, we have $a_1 = e_1 + 2e_3$, $a_2 = e_1 + 2e_2 + 3e_3$, $b_1 = 7e_1 + 10e_2 + 19e_3$ and $b_2 = 7e_1 + 10e_2 + 20e_3$.
The wedge products of interest are

$$
\begin{aligned}
a_1 \wedge a_2 &= \ \ 2e_{12} + \ \ e_{13} - \ \ 4e_{23} \\
a_1 \wedge b_1 &= 10e_{12} + 5e_{13} - 20e_{23} \\
a_1 \wedge b_2 &= 10e_{12} + 6e_{13} - 20e_{23} \\
a_2 \wedge b_1 &= -4e_{12} - 2e_{13} + \ \ 8e_{23} \\
a_2 \wedge b_2 &= -4e_{12} - \ \ e_{13} + 10e_{23},
\end{aligned}
$$

where we have used the bivector notation $e_{ij} = e_i \wedge e_j$. Note that these bivectors are basis vectors for $\bigwedge^2 \mathbb{R}^3$.
Taking the wedge product of $x_1 a_1 + x_2 a_2 = b_1$ with a_1, we obtain

$$x_1 a_1 \wedge a_1 + x_2 a_1 \wedge a_2 = a_1 \wedge b_1 \Rightarrow x_2(2e_{12} + e_{13} - 4e_{23}) = 10e_{12} + 5e_{13} - 20e_{23}.$$

Since the e_{ij} bivectors are linearly independent, we conclude that $2x_2 = 10$, $x_2 = 5$, $-4x_2 = -20$, i.,e, $x_2 = 5$.
Taking the wedge product of $x_1 a_1 + x_2 a_2 = b_1$ with a_2, we obtain

$$x_1 a_2 \wedge a_1 + x_2 a_2 \wedge a_2 = a_2 \wedge b_1 \Rightarrow -x_1(2e_{12} + e_{13} - 4e_{23}) = -4e_{12} - 2e_{13} + 8e_{23}$$

so that $-2x_1 = -4$, $-x_1 = -2$, $4x_1 = 8$, i.,e, $x_1 = 2$.
A similar computation, taking the wedge product of $x_1 a_1 + x_2 a_2 = b_2$ with a_1 yields

$$x_1 a_1 \wedge a_1 + x_2 a_1 \wedge a_2 = a_1 \wedge b_2 \Rightarrow x_2(2e_{12} + e_{13} - 4e_{23}) = 10e_{12} + 6e_{13} - 20e_{23},$$

from which we conclude that $2x_2 = 10, x_2 = 6, -4x_2 = -20$, which is inconsistent: $Ax = b_2$ dowes not have a solution.

Example 6.3.2. *Using the wedge product to solve a system of linear equations*
Solve $x_1 a_1 + x_2 a_2 + x_3 a_3 = b$ using the wedge product, where

$$
a_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad
a_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad
a_3 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad
b = \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix}.
$$

The wedge products of interest are given by

$$
\begin{aligned}
a_1 \wedge a_2 &= e_{12} - e_{13} - 3e_{23} \\
a_1 \wedge a_3 &= -e_{12} + e_{13} + 3e_{23} \\
a_2 \wedge a_3 &= -2e_{12} + 2e_{13} + 6e_{23} \\
a_1 \wedge b &= 3e_{12} - 3e_{13} - 9e_{23} \\
a_2 \wedge b &= e_{12} - e_{13} - 3e_{23} \\
a_3 \wedge b &= 3e_{12} - 3e_{13} - 9e_{23}
\end{aligned}
$$

and $a_1 \wedge a_2 \wedge a_3 = 0$, from which we conclude that $\{a_1, a_2, a_3\}$ are linearly dependent. Since $a_1 \wedge a_2 \neq 0$, the vectors a_1 and a_2 are linearly independent. We therefore conclude that we can treat x_3 as a free variable.

Computing the wedge products of $x_1 a_1 + x_2 a_2 + x_3 a_3 = b$ with a_1 and a_2 yields

$$
\begin{aligned}
(x_2 - x_3 - 1)e_{12} - (x_2 - x_3 - 1)e_{13} \quad & -3(x_2 - x_3 - 1)e_{23} = 0 \\
(-x_1 + 2x_3 + 1)e_{12} - (-x_1 + 2x_3 + 1)e_{13} \quad & -3(-x_1 + 2x_3 + 1)e_{23} = 0,
\end{aligned}
$$

Setting the free variable $x_3 = \alpha$, we obtain

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.
$$

We conclude that we must be careful when $y = Ax$ is not onto (possible inconsistency) or not one-to-one (free variables). The special case when A is square and invertible is much simpler: it reduces to

$$
x_i \, det(a_1 \; a_2 \; \cdots \; a_N) = det(a_1 \; a_2 \cdots a_{i-1} \; b \; a_{i+1} \cdots a_N)
$$

which is known as Cramer's rule. The matrix that gives rise to the determinant on the right side is usually denoted by $A_i(b)$, i.e., the matrix A with the ith column replaced by b. We find

$$
x_i = \frac{det(A_i(b))}{det(A)}, \quad \text{provided } det(A) \neq 0. \tag{6.24}
$$

A geometric interpretation of Cramer's rule in 2 dimensions is given in Figure 6.3.
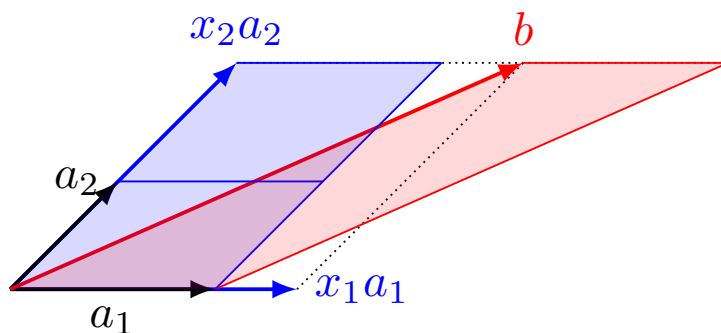
Figure 6.3: The geometric interpretation of Cramer's rule in 2 dimensions. The figure shows the solution of the problem $x_1 a_1 + x_2 a_2 = b$. Taking the wedge product with a_1 results in the equation $x_2 a_1 \wedge a_2 = a_1 \wedge b$, which shows that the area of the blue parallelogram with sides a_1, $x_2 a_2$ is equal to the area of the red parallelogram with sides a_1 and b.

Example 6.3.3. *Cramer's rule*
Consider

$$Ax = b \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \\ 17 \\ 3 \end{pmatrix}.$$

We need to compute the following determinants:

$$det(A) = det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ -1 & 1 & -1 \end{pmatrix} = -2,$$

$$det(A_1(b)) = det \begin{pmatrix} 5 & 1 & 1 \\ 17 & 4 & 3 \\ 3 & 1 & -1 \end{pmatrix} = -4,$$

$$det(A_2(b)) = det \begin{pmatrix} 1 & 5 & 1 \\ 2 & 17 & 3 \\ -1 & 3 & -1 \end{pmatrix} = -8,$$

$$det(A_3(b)) = det \begin{pmatrix} 1 & 1 & 5 \\ 2 & 4 & 17 \\ -1 & 1 & 3 \end{pmatrix} = 2.$$

By Cramer's rule, we then have

$$x_1 = \frac{det(A_1(b))}{det(A)} = \frac{-4}{-2} = 2,$$

$$x_2 = \frac{det(A_2(b))}{det(A)} = \frac{-8}{-2} = 4,$$

$$x_3 = \frac{det(A_3(b))}{det(A)} = \frac{2}{-2} = -1.$$

As we can see, Cramer's rule is impractical for numerical computations, since in addition to the computation of the $N \times N$ size determinant of A, each x_i requires the computation of an additional $N \times N$ size determinant.

Cramer's rule does however provide us with a closed form solution which is of interest algebraically. One application is to obtain a closed form solution for the inverse of a matrix. Since the inverse is the solution of the matrix equation $AX = I$, column x_j of the inverse is the solution of $Ax_j = e_j$, where e_j is the jth column of I. By Cramer's rule Eq 6.3 the ith entry of x_j, which we denote by x_{ij}, is given by

$$x_{ij} = \frac{det(A_i(e_j))}{det(A)} A = \frac{1}{det(A)} C_{ji},$$

where C_{ji} is the cofactor of the a_{ji} entry in A. Using the transpose to inter-

change the indices on the cofactor, we therefore have

$$A^{-1} = \frac{1}{det(A)} C^t,$$

where $C = (C_{ij})$, the matrix of cofactors.

Example 6.3.4. *Matrix inverse using Cramer's rule*
Use Cramer's rule to compute the inverse of a 2×2 matrix

$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right).$$

The required determinant and cofactors are

$$det(A) = ad - bc,$$
$$C_{11} = (-1)^{1+1} d, \quad C_{12} = (-1)^{1+2} c$$
$$C_{21} = (-1)^{2+1} b, \quad C_{22} = (-1)^{2+2} a,$$

and thus

$$A^{-1} = \frac{1}{det(A)} \left(\begin{array}{cc} d & -c \\ -b & a \end{array} \right)^t = \frac{1}{ad - bc} \left(\begin{array}{cc} d & -b \\ -c & a \end{array} \right).$$

6.4   Exercises

Exercise 6.1. *Investigate the Laplace expansion of the determinant for matrices up to size $N = 3$.*

- *Expansion about the first row: for each determinant of a submatrix, order the wedge product of the corresponding basis vectors with indices in numerical order. Explain the sign associated with a given entry by considering the sign resulting from reordering terms such as $e_2 \wedge (e_1 \wedge e_3) = -e_1 \wedge e_2 \wedge e_3$.*

- *Repeat the previous exercise by expanding about the second row.*

Exercise 6.2. *Investigate the Leibnitz expansion for the determinant of a matrix A of size $N \times N$:*

1. *Explain why $\wedge A$ results in terms of the form*
   $\alpha_{1i_1} \alpha_{2i_2} \cdots \alpha_{N i_N} e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_N}$.

2. *Consider a matrix of the form*

$$A = \begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix}$$

, where the entries $a_{ij} \in \mathbb{R}$. Use the Leibnitz expansion to explain why the determinant of this matrix is a polynomial in λ of degree 3 with real coefficients.

Exercise 6.3. Let $\{a_1, a_2, \cdots a_N\}$ be a basis for a vector space V. Prove that $\{a_{i_1} \wedge a_{i_2} \cdots \wedge a_{i_K} \mid i_j, j = 1, 2, \cdots K$ are distinct indices in $1, 2, \ldots N, K \leq N\}$ is a basis of the vector space $\bigwedge^K V$.

Exercise 6.4. Use Cramer's rule for the inverse of a matrix to construct an example of a 3×3 matrix with mostly non-zero integer entries that has an inverse with integer entries. I.e., construct an example such as Example 2.7.9.

Exercise 6.5. Consider the $A = QR$ decomposition resulting from the Gram-Schmidt process applied to a set of vectors $a_1, a_2, \cdots a_N \in \mathbb{R}^M$ to obtain the corresponding orthonormal basis $q_1, q_2, \cdots q_N$ obtained by Show that $a_1 \wedge a_2 \cdots \wedge a_N = \prod_{i=1}^N r_{ii} q_1 \wedge q_2 \cdots \wedge q_N$, where the $r_i i$ are the diagonal entries of the R matrix.

Use this result to obtain the area of the paralellogram defined by the two column vectors in the matrix of Exercise 5.1

Chapter 7

Eigenvalues and Eigenvectors

We have seen that a matrix A can be thought of as a transformation acting on some vector x to yield a new vector $y = Ax$. In this chapter, we will consider this transformation further.

7.1 Simple Examples

Example 1: Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

that represents the orthogonal projection of a vector onto the x_1-x_2 plane. For this transformation, some directions are special

- if the vector x lies in the x_1-x_2 plane, it does not change:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix},$$

  i.e., we have $Ax = x$.

- if the vector x is orthogonal to the x_1-x_2, it maps into the origin:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

  i.e., we have $Ax = 0x$.

Example 2: Next consider a matrix A representing the reflection of a point through some plane: any vector x in this plane is unaffected, $Ax = x$, while any vector x orthogonal to the plane is transformed into its mirror image $Ax = -x$.

Example 3: As a final example, consider a 3D rotation matrix A rotating a vector x in a given plane by an angle θ. There again is a special direction: the axis of rotation. A vector x along this axis does not change, we again have $Ax = x$.

In each of these cases, we have $Ax = \lambda x$, for some constants λ, with a corresponding set of x vectors that lie in either a plane or along a line, i.e., vectors that lie in a subspace.

7.2   Eigenvalues, Eigenvectors and Eigenspaces

Motivated by the simple examples of the previous section, we want to find special vectors x such that $Ax = \lambda x$ for some constant λ. We notice three properties right away:

- Since A maps a vector x onto a scaled version of x, i.e., maps a vector from $\mathbb{R}^N$ to a vector in $\mathbb{R}^N$, the matrix A must necessarily be square of size $N \times N$.

- the vector $x = 0$ would always qualify, but it is uninteresting. For any non-zero vector x that satisfies this equation, a vector αx, i.e., the original vector scaled by some constant α also satisfies the equation.

- In fact, **if $Ax = \lambda x$ has non-zero solutions x for some particular constant λ, the vectors x together with the zero vector form a subspace**, namely the null space $\mathcal{N}(A - \lambda I)$.

  Of course subspaces contain the zero vector, which we had declared uninteresting above! This problem is easily resolved: we will look for basis vectors for this subspace. Basis vectors are necessarily non-zero. Then any non-zero linear combination of these basis vectors is a solution of $Ax = \lambda x$.

Since vectors x and values λ such that $Ax = \lambda x$ are characteristic for a given matrix A, we define such non-zero vectors **characteristic vectors** or **eigenvectors** of A, the corresponding value λ a **characteristic value** or **eigenvalue** of A, and the subspace of vectors associated with an eigenvalue λ the **eigenspace** of A for the eigenvalue λ.

Example 7.2.1. *Testing whether a given vector is an eigenvector*
Consider the matrix
$$A = \begin{pmatrix} -36 & -6 & 14 \\ -102 & -12 & 36 \\ -141 & -21 & 53 \end{pmatrix}.$$

Which of the following vectors are eigenvectors of A?

$$v_1 = (2\ 8\ 9)^t,\, v_2 = (1\ 3\ 4)^t,\, v_3 = (4\ 11\ 15)^t,\, v_4 = (1\ 1\ 1)^t,\, v_5 = (0\ 0\ 0)$$

We immediately reject v_5: the zero vector is not an eigenvector by definition.
For each of the remaining vectors $v_i, i = 1, 2, 3, 4$, we need to check whether there is a corresponding constant value λ_i such that $Av_i = \lambda_i v_i$ " To save space, we combine each of the vectors v_i into a matrix V, and compute AV. We find

	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$	$\mathbf{v_4}$	$\mathbf{v_5}$
	2	1	4	1	0
	8	3	11	1	0
	9	4	15	1	0

$$\begin{pmatrix} -36 & -6 & 14 \\ -102 & -12 & 36 \\ -141 & -21 & 53 \end{pmatrix} \begin{pmatrix} 6 & 2 & 0 & -28 & 0 \\ 24 & 6 & 0 & -78 & 0 \\ 27 & 8 & 0 & -109 & 0 \end{pmatrix}$$

where we have labeled the columns with the corresponding vector for easy reference.

- *Checking the $\mathbf{v_1}$ column, we see that $Av_1 = 3v_1$, so v_1 is an eigenvector with eigenvalue 3.*

- *The $\mathbf{v_2}$ column shows that $Av_2 = 2v_2$, so v_2 is an eigenvector with eigenvalue 2.*

- *The $\mathbf{v_3}$ column shows that $Av_3 = 0v_3$, so the non-zero vector v_3 is an eigenvector with eigenvalue 0.*

- *The $\mathbf{v_4}$ column shows that $Av_4 = \lambda v_4$ cannot be satisfied for any choice of λ. Therefore v_4 is not an eigenvector.*

- *We have added the $\mathbf{v_5}$ column to emphasize once again that **the zero vector is not an eigenvector!***

7.2.1 The Characteristic Polynomial

We next turn to the problem of finding eigenvalues of a matrix A of size $N \times N$. We note that we can rewrite $Ax = \lambda x \Leftrightarrow (A - \lambda I)x = 0$. Therefore eigenvectors x are vectors from the null space of the matrix $A - \lambda I$. Since we are only interested in non-trivial solutions, (i.e., non-zero eigenvectors x), the square matrix $A - \lambda I$ must be singular. An obvious approach to finding eigenvalues

using hand computation[1] is to find values λ such that $det(A - \lambda I) = 0$. Since the determinant of a matrix is made up from products of N entries chosen from distinct rows and columns, it is easy to see that $det(A - \lambda I)$ is a polynomial of degree N with leading term λ^N. This polynomial is the **characteristic polynomial** of the matrix A. Its roots are the only possible eigenvalues of A.

☞ The eigenvalues of a matrix A of size $N \times N$ are the roots λ_i of the characteristic polynomial $p(\lambda) = det(A - \lambda)$ of degree N.

☞ A well known theorem from algebra states that a polynomial

$$p(\lambda) = \lambda^N + \alpha_{N-1}\lambda^{N-1} + \cdots + \alpha_1\lambda + \alpha_0$$

can be factored in the form

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N),$$

where the λ_j are the roots of $p(\lambda)$. The λ_j need not be distinct. Further, some of the roots λ_j may be complex, even though all coefficients α_j of the polynomial are real. Such roots occur in complex conjugate pairs, i.e., if $\lambda = a + ib$ (where a and b are real) is a complex root of $p(\lambda)$, then so is $\lambda = a - ib$.

☞ It is useful to combine terms with the same numerical value for roots λ_j: the factorization of the characteristic polynomial then assumes the form

$$p(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\cdots(\lambda - \lambda_k)^{m_k},$$

where we have renamed the k distinct roots. In this form the λ_j are distinct, and the integral powers m_i, called the **algebraic multiplicities** of the roots λ_j are greater or equal to 1 and sum to $m_1 + m_2 + \cdots + m_k = N$.

---

[1]More practical algorithms use different approaches. In particular, algorithms based on the QR factorization are widely used.

Example 7.2.2. *Finding Eigenvalues*
Consider the matrix from the previous example 7.2.1

$$A = \begin{pmatrix} -36 & -6 & 14 \\ -102 & -12 & 36 \\ -141 & -21 & 53 \end{pmatrix}.$$

Forming $A - \lambda I$ and computing the determinant using the Laplace expansion, we obtain the characteristic polynomial

$$p(\lambda) = det \begin{pmatrix} -36 - \lambda & -6 & 14 \\ -102 & -12 - \lambda & 36 \\ -141 & -21 & 53 - \lambda \end{pmatrix}.$$
$$= -\lambda^3 + 5\lambda^2 - 6\lambda$$
$$= -\lambda(\lambda - 2)(\lambda - 3).$$

The matrix A therefore has three distinct eigenvalues, $\lambda = 0, \lambda = 2$, and $\lambda = 3$. Each of the eigenvalues has multiplicity $m = 1$.

Example 7.2.3. *Finding Eigenvalues*
Consider the matrix
$$A = \begin{pmatrix} 0 & -2 & 2 \\ -8 & -6 & 8 \\ -9 & -9 & 11 \end{pmatrix}.$$

Forming $A - \lambda I$ and computing the determinant, we obtain the characteristic polynomial

$$p(\lambda) = det \begin{pmatrix} -\lambda & -2 & 2 \\ -8 & -6 - \lambda & 8 \\ -9 & -9 & 11 - \lambda \end{pmatrix}.$$
$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$
$$= -(\lambda - 1)(\lambda - 2)^2.$$

The matrix A therefore has two distinct eigenvalues, $\lambda = 1$ with multiplicity 1 (the root occurs exactly once), and $\lambda = 2$ with multiplicity 2 (the root occurs twice).

☞ Factoring polynomials can be hard. Practical problems (i.e., problems likely to be on an exam) must factor easily. Do not expand the terms $(a_{ii} - \lambda)$ while computing determinants until you are sure that they do not factor out!

When faced with a polynomial of degree 3 or more, check whether $1, -1, 2, \cdots$ might be roots.

Example 7.2.4. *Finding Eigenvalues*
Consider the matrix

$$A - \lambda I = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 & 0 & 0 \\ 0 & 1 & 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 0 & 0 & -1 & 1-\lambda \end{pmatrix}$$

Forming $A - \lambda I$ and computing the determinant, carefully choosing the rows and columns for the expansion, we obtain the characteristic polynomial

$$p(\lambda) = -\lambda(1-\lambda)(2-\lambda)^2((1-\lambda)^2 + 1)$$
$$= \lambda(\lambda-1)(\lambda-2)^2(\lambda-(1-i))(\lambda-(1+i)).$$

The matrix A therefore has the following set of eigenvalues: $\lambda = 0$ with multiplicity 1, $\lambda = 1$ with multiplicity 1, $\lambda = 2$ with multiplicity 2, and the complex conjugate pair $\lambda = 1 + i, \lambda = 1 - i$, each with multiplicity 1.

7.2.2   Properties of Eigenvalues

It is important to check whether eigenvalues are correct. There are 2 quick tests that are useful in practice: they arise from properties of the eigenvalues that we see by multiplying out the characteristic polynomial

$$det(A - \lambda I) = (-1)^N \prod_{i=1}^{N} (\lambda - \lambda_i).$$

We note that the constant term is easily obtained by setting $\lambda = 0$. We obtain the important theorem $det(A) = \prod_{i=1}^{N} \lambda_i$.

☞ **The product of the N (not necessarily distinct) eigenvalues is equal to the determinant of the matrix.**

☞ An obvious corollary it that **a matrix A is singular if and only if it has a zero eigenvalue.**

The second useful property is to consider the term multiplying λ^{N-1} of the characteristic polynomial. Considering the determinant of the matrix, we see that this term is the sum of the diagonal entries in A, called the **trace** of a matrix A of size $N \times N$:

$$tr(A) = \sum_{i=1}^{N} a_{ii}. \tag{7.1}$$

Multiplying out the polynomial on the right, we find that this term is the sum of the roots of the polynomial.

☞ **The sum of the N (not necessarily distinct) eigenvalues is the equal to the trace of the matrix.**

$$tr(A) = \sum_{i=1}^{N} \lambda_i, \tag{7.2}$$

where λ_i are the N (not necessarily distinct) eigenvalues of the matrix.

Example 7.2.5. *Checking the eigenvalue computation*
In example 7.2.4, we found the characteristic polynomial for the matrix

$$A = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{2} & 1 & 0 & 0 \\ 0 & 1 & 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 & -1 & \mathbf{1} \end{pmatrix}$$

to be
$$p(\lambda) = \lambda \, (\lambda - 1) \, (\lambda - 2)^2 \, (\lambda - (1 - i)) \, (\lambda - (1 + i)) \,.$$

The trace is the sum of the diagonal terms in A (shown in red). The trace test yields

$$tr(A) - \sum_{i=1}^{6} \lambda_i = (0 + 1 + 2 + 2 + 1 + 1) - (0 + 1 + 2 + 2 + (1 - i) + (1 + i)) = 0,$$

while the determinant test yields

$$det(A) - \prod_{i=1}^{6} \lambda_i = (0) - (0 \cdot 1 \cdot 2 \cdot 2 \cdot (1 - i) \cdot (1 + i)) = 0.$$

Observe that each eigenvalue enters as often as it appears in the characteristic polynomial (the same number of times as its algebraic multiplicity).

Observe also how the imaginary parts of the conjugate complex eigenvalue pair cancel in the trace formula and in the formula for the determinant.

A common type of problems results in matrices with the property that every row (or every column) add to the same constant value λ. This constant value must be an eigenvalue, as can be easily seen by considering the vector $v = (1\,1\,\cdots\,1)^t$, where every entry equals 1, and noting that $Av = \lambda v$. Thus, v is an eigenvector for the eigenvalue λ.

Similarly, the sum of the columns of A is given by $v^t A = \lambda v^t \Leftrightarrow A^t v = \lambda v$. Thus, v is an eigenvector of A^t, but not of A in general. Since $det(A - \lambda I) = det(A - \lambda I)^t = det(A^t - \lambda I)$, we see that λ is also an eigenvalue of A.

☞ **If all rows of a square matrix A add up to the same value λ, then λ is an eigenvalue of A, and $(1\,1\,\cdots\,1)$ is a corresponding eigenvector.**

☞ If all columns add up to the same value λ, then $(\lambda, (1\ 1\ \cdots\ 1)^t)$ is an eigenpair of A^t, so λ is an eigenvalue of A.

7.2.3   Computational Shortcuts

The Characteristic Polynomial and Eigenvalues

Triangular matrices yield the simplest eigenvalue problems: the eigenvalues are simply the diagonal entries of the matrix: while not true in general, we see that the non-zero eigenvalues are also the pivots for the matrix.

Example 7.2.6. *Eigenvalues for a triangular matrix*
Consider the upper triangular matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -8 & 12 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The characteristic polynomial is obtained trivially by expanding the determinant based on the first column of each of the matrices in the Laplace expansion:

$$p(\lambda) = det(A - \lambda I) = (-\lambda)(3 - \lambda)^2(-8 - \lambda),$$

so the eigenvalues are given by $\lambda_1 = 0$ with algebraic multiplicity 1, $\lambda_2 = 3$ with algebraic multiplicity 2, and $\lambda_3 = -8$ with algebraic multiplicity 1. Note that λ_2 and λ_3 are pivots of the matrix A.

The trace and determinant formulae relating entries in a given matrix to its eigenvalues can sometimes be used to quickly write down the characteristic polynomial without too much computation. The relations for a matrix A of size $N \times N$ with eigenvalues $\lambda_i, i = 1, 2, \ldots N$ are

$$tr(A) = \sum_{i=1}^{N} \lambda_i \tag{7.3}$$

and

$$det(A) = \prod_{i=1}^{N} \lambda_i \tag{7.4}$$

For a matrix A of size $N \times N$ the characteristic equation has the form

$$p(\lambda) = (-1)^N \prod_{i=1}^{N} (\lambda - \lambda_i) \tag{7.5}$$

$$= (-1)^N \left(\lambda^N - \sum_{i=1}^{N} \lambda_i + \cdots + (-1)^N \prod_{i=1}^{N} \lambda_i \right) \tag{7.6}$$

$$= (-1)^N \left(\lambda^N - tr(A)\,\lambda^{N-1} + \cdots + (-1)^N det(A) \right) \tag{7.7}$$

For the cases $N = 2$ and $N = 3$, the above reduces to

$$p(\lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \qquad \text{Case } N = 2 \tag{7.8a}$$
$$p(\lambda) = -\left[\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \right](\lambda - \lambda_3) \qquad \text{Case } N = 3 \tag{7.8b}$$

where we have multiplied out two of the $(\lambda - \lambda_i)$ terms. The $N = 3$ case in particular can be used whenever we know an eigenvalue $\lambda_3 \neq 0$.

Example 7.2.7. *Eigenvalues for a matrix of size 2×2*
Consider the matrix
$$A = \begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix}.$$

The trace formula for the two eigenvalues yields
$$\lambda_1 + \lambda_2 = tr(A) = 3,$$

while the determinant equation yields
$$\lambda_1\lambda_2 = det(A) = -18.$$

Substituting into Eq (7.8a) for the characteristic equation for $N = 2$, we obtain the characteristic equation
$$p(\lambda) = \lambda^2 - 3\lambda - 18$$

which has roots $\lambda_1 = 6$ and $\lambda_2 = -3$.

Example 7.2.8. *Eigenvalues for a matrix of size* 3×3 *with a non-zero eigenvalue*

Consider the matrix

$$A = \begin{pmatrix} 1 & 5 & 2 \\ 4 & 2 & 2 \\ 1 & 1 & 6 \end{pmatrix}.$$

Since each of the rows sums to 8, A has eigenvector $x = (1 \quad 1 \quad 1)$ with an eigenvalue $\lambda_3 = 8$. The trace and determinant formulae yield

$$\lambda_1 + \lambda_2 + \lambda_3 = 9 \qquad \Rightarrow \lambda_1 + \lambda_2 = 1 \text{ (trace formula)}$$
$$\lambda_1 \lambda_2 \lambda_3 = -96 \qquad \Rightarrow \lambda_1 \lambda_2 = -12 \text{(determinant formula)}.$$

Substituting these into the formula for the characteristic polynomial Eq (7.8b) yields

$$p(\lambda) = -(\lambda^2 - \lambda - 12)(\lambda - 8)$$
$$= -(\lambda + 3)(\lambda - 4)(\lambda - 8)$$

so the eigenvalues of A are $\lambda = -3, 4, 8$.

Example 7.2.9. *Eigenvalues for a matrix of size* 3×3 *with a zero eigenvalue*

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Since the first and second rows are the same, $\det(A) = 0$, and therefore $\lambda = 0$ is an eigenvalue.

We cannot use Eq (7.8b) here since we cannot determine the product of the remaining two eigenvalues. Note that the Laplace expansion would have a factor λ, so that the remaining roots would be easily computed.

For this particular problem however, we see that each of the columns sums to 4, so A^t has eigenvector $x = (1 \quad 1 \quad 1)$ with an eigenvalue $\lambda_3 = 4$. Since A and A^t have the same eigenvalues, A has an eigenvalue $\lambda_3 = 4$.

The trace formula together with $\lambda_1 = 0$ and $\lambda_2 = 4$ yields

$$\lambda_1 + \lambda_2 + \lambda_3 = 4, \quad \lambda_1 = 0, \quad \lambda_2 = 4 \Rightarrow \lambda_3 = 0,$$

i.e., the characteristic polynomial of A is $p(\lambda) = -\lambda^2(\lambda - 4)$.

Sometimes the structure of the matrix A can be used to advantage: consider the matrix with block structure

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},$$

where B and C are square matrices. Since A can be factored in the form

$$A = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}$$

we see that

$$det(A) = det(B)det(C),$$

i.e., the determinant of A is the product of the determinants of the blocks B and C of A. The characteristic equation of A therefore factors

$$p(\lambda) = det(B - \lambda I)det(C - \lambda I).$$

Example 7.2.10. *Eigenvalues for a matrix with two blocks of size 2×2*
Consider the matrix

$$A = \left(\begin{array}{cc|cc} 1 & 3 & 6 & 1 \\ 2 & 6 & 4 & 5 \\ \hline 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & -2 \end{array} \right)$$

which has two non-zero square blocks on the diagonal, namely

$$C = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \quad and \quad D = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$

Since $tr(B) = 7$, $det(B) = 0$, and $tr(C) = 0$, $det(C) = -1$, the characteristic polynomial is

$$p(\lambda) = (\lambda^2 - 7\lambda)(\lambda^2 - 1).$$

Finding the roots, we obtain the eigenvalues $\lambda = 0, 7, -1, 1$.

Eigenvectors

A basis for the null space of a given a non-zero row vector $u = (u_1 \ u_2 \ \ldots u_n)$, can be obtained trivially: let k be the lowest index for which $u_k \neq 0$. A set of $n - 1$ linearly independent vectors orthogonal to u is given by b_i, where $i = 1, 2, \ldots k - 1, k + 1, \ldots k_n$, where

$$b_i = \begin{cases} e_i & \text{the } i^{th} \text{ standard basis vector if } u_i = 0 \\ (v_1, v_2, \ldots, v_n) & \text{where } v_k = u_i, \ v_i = -u_k, \text{ and all other entries are 0 otherwise} \end{cases}.$$

Example 7.2.11. *Null Space Basis, One-dimensional Row Space*
A basis $\mathfrak{B}$ for the null space of the matrix

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is given by

$$\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 0 \\ -2 \end{pmatrix} \right\}$$

The vectors in $\mathfrak{B}$ form a basis for the null space $\mathcal{N}(A)$, since by construction i) they are orthogonal to the rows of A, ii) they are linearly independent and iii) the $\dim N(A) = n - 1$ and there are $n - 1$ vectors.

Note that the matrix A in the previous example is square, the vectors in $\mathfrak{B}$ thus form a basis for the eigenspace E_λ associated with the eigenvalue $\lambda = 0$.

Also note that there are only four eigenvectors in the basis $\mathfrak{B}$, but the eigenvalue $\lambda = 0$ clearly has algebraic multiplicity 5.

7.3   Matrix Diagonalization

The dimension of the eigenspace E_μ for some eigenvalue μ is the called the **geometric multiplicity** of the eigenvalue.

☞ Let $\xi(\mu)$ be the geometric multiplicity and $\zeta(\mu)$ be the algebraic multiplicity of and eigenvalue μ. The following inequality holds. $1 \le \xi(\mu) \le \zeta(\mu)$.

To prove this, consider a square matrix A of size $N \times N$ with eigenvalue μ, and assume the associated eigenspace E_μ has dimension $\xi(\mu) = k$. Let $\mathfrak{B} = \{v_1, v_2, \cdots v_k\}$ be a basis for E_μ made up of eigenvectors (i.e., $\mathfrak{B}$ is a basis for the null space $\mathcal{N}(A)$). If $k < N$, augment $\mathfrak{B}$ with linearly independent vectors $\tilde{\mathfrak{B}} = \{v_{k+1}, \cdots v_N\}$ to obtain a complete basis for $\mathbb{F}^N$, and form the matrix $S = (v_1 \ v_2 \ \cdots v_N)$. Note that by construction, S is invertible. Computation of AS yields $AS = SB$, where B is a matrix of the form

$$B = \begin{pmatrix} \mu I & \cdots \\ 0 & \cdots \end{pmatrix},$$

with I of size $k \times k$. The remaining columns have no special structure in general since the corresponding column entries from $\tilde{\mathfrak{B}}$ are not eigenvectors.

Since B is obtained by a similarity transform of A, the two matrices have the same characteristic polynomial. Due to the special structure of B, this

polynomial has a factor $(\lambda - \mu)^k$, and therefore the algebraic multiplicity of μ is at least k. The lower bound on the geometric multiplicity results from the definition of the characteristic polynomial, which expressed the requirement that the null space $\mathcal{N}(A - \mu I)$ must have non-zero elements.

☞ A matrix A of size $N \times N$ can have at most N linearly independent eigenvectors.

This observation follows trivially from the observation that the characteristic polynomial of A has degree N and therefore has exactly N eigenvalues, not necessarily distinct, and the inequality $1 \leq \xi(\lambda) \leq \zeta(\lambda)$.

☞ Eigenvectors for different eigenvalues of a matrix A are linearly independent.

Let A have eigenpairs (λ_1, v_1) and (λ_2, v_2) with $\lambda_1 \neq \lambda_2$. Set $\alpha v_1 + \beta v_2 = 0$. Multiplying by λ_1, λ_2, and by A from the left results in the three equations

$$\alpha \lambda_1 v_1 + \beta \lambda_1 v_2 = 0$$
$$\alpha \lambda_2 v_1 + \beta \lambda_2 v_2 = 0$$
$$\alpha A v_1 + \beta A v_2 = 0 \Rightarrow \alpha \lambda_1 v_1 + \beta \lambda_2 v_2 = 0$$

Subtracting the second and third equations from each other yields $\alpha(\lambda_2 - \lambda_1)v_1 = 0 \Rightarrow \alpha(\lambda_2 - \lambda_1)|v_1| = 0$. Since $|v_1| \neq 0$ (eigenvectors are non-zero by definition), and since $\lambda_2 - \lambda_1 \neq 0$, we obtain $\alpha = 0$. Similarly, the first and third equations yield $\beta = 0$, establishing the linear independence of the two eigenvectors.

7.3.1 Eigenvector Basis

FIX add text FIX

☞ Diagonalizable matrices A and B share the same eigenvectors if and only if $AB = BA$.

Example 7.3.1. *Diagonalization of a matrix*
Consider the matrix

$$A = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}$$

Step 1: *The characteristic polynomial is given by*

$$
\begin{aligned}
p(\lambda) =& det(A - \lambda I) \\
=& det \begin{pmatrix} 3 - \lambda & -1 \\ -6 & 2 - \lambda \end{pmatrix} \\
=& \lambda(\lambda - 5)
\end{aligned}
$$

The roots of $p(\lambda)$ are the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 5$, each occurring with algebraic multiplicity 1.

Step 2: *Finding bases for the eigenspaces*
Case *$\lambda = 0$: Find a basis for the null space $\mathcal{N}(A) = \left\{ x \in \mathbb{R}^2 : Ax = 0 \right\}$*

$$A - 0I = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \quad basis \ \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

Case *$\lambda = 5$: Find a basis for $\mathcal{N}(A - 5I) = \left\{ x \in \mathbb{R}^2 : (A - 5I)x = 0 \right\}$*

$$A - 5I = \begin{pmatrix} -2 & -1 \\ -6 & -3 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad basis \ \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

Step 3: *Combine the results*

λ	**0**	**5**
multiplicity	*1*	*1*
basis	$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
S	$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
Λ	$\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathbf{5} \end{pmatrix}$

The diagonal form of the matrix A is given by Λ. We have obtained the similarity transform

$$
\begin{aligned}
A =& S\Lambda S^{-1} \\
=& \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}^{-1}
\end{aligned}
$$

Example 7.3.2. *Power of a diagonalizable matrix*
In example 7.3.1, we found the diagonal form of the matrix

$$A = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}$$

in the form $A = S\Lambda S^{-1}$, with S and Λ shown in the table in that example.
   The matrix A^n is thus given by

$$
\begin{aligned}
A =& S\Lambda^n S^{-1} \\
=& \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}^{-1} \\
=& \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix} \frac{1}{5} \\
=& 5^{n-1} A
\end{aligned}
$$

Example 7.3.3. *Diagonalization of a 3×3 matrix*
Consider the matrix

$$A = \begin{pmatrix} 41 & -14 & 6 \\ 100 & -34 & 15 \\ -40 & 14 & -5 \end{pmatrix}$$

Step 1: *The characteristic polynomial is given by $p(\lambda) = det(A - \lambda I)$*

$$p(\lambda) = det \begin{pmatrix} 41 - \lambda & -14 & 6 \\ 100 & -34 - \lambda & 15 \\ -40 & 14 & -5 - \lambda \end{pmatrix}$$
$$= -\lambda^2(\lambda - 1)$$

The roots of $p(\lambda)$ are the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, occurring with algebraic multiplicity 2 and 1 respectively.

Step 2: *Finding bases for the eigenspaces*
Case $\lambda = 0$: *Find a basis for the null space $\mathcal{N}(A) = \left\{ x \in \mathbb{R}^2 : Ax = 0 \right\}$*

$$A - 0I = \begin{pmatrix} 41 & -14 & 6 \\ 100 & -34 & 15 \\ -40 & 14 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix} \quad basis \quad \left\{ \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix} \right\}$$

Case $\lambda = 1$: *Find a basis for $\mathcal{N}(A - 1I) = \left\{ x \in \mathbb{R}^2 : (A - I)x = 0 \right\}$*

$$A - 1I = \begin{pmatrix} 40 & -14 & 6 \\ 100 & -33 & 15 \\ -40 & 14 & -4 \end{pmatrix} \sim \begin{pmatrix} 20 & -7 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad basis \quad \left\{ \begin{pmatrix} 7 \\ 20 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 20 \end{pmatrix} \right\}$$

Step 3: *Combine the results*

λ	**1**		**0**
multiplicity	*2*		*1*
basis	$\begin{pmatrix} 7 \\ 20 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 0 \\ 20 \end{pmatrix}$	$\begin{pmatrix} 20 \\ -7 \\ 3 \end{pmatrix}$
S	$\begin{pmatrix} 7 & -3 & 20 \\ 20 & 0 & -7 \\ 0 & 20 & 3 \end{pmatrix}$		
Λ	$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{0} \end{pmatrix}$		

The diagonal form of the matrix A is given by Λ. We have obtained the similarity transform $A = S\Lambda S^{-1}$ with S and Λ as shown in the table.

Example 7.3.4. *Power of a 3×3 matrix*
In example 7.3.3, we found the diagonalization of the matrix

$$A = \begin{pmatrix} 41 & -14 & 6 \\ 100 & -34 & 15 \\ -40 & 14 & -5 \end{pmatrix}$$

in the form $A = S\Lambda S^{-1}$, with S and Λ shown in the table in that example.
 The matrix A^n is thus given by

$$A = S\Lambda^n S^{-1}$$

$$= \begin{pmatrix} 7 & -3 & -2 \\ 20 & 0 & -5 \\ 0 & 20 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 7 & -3 & -2 \\ 20 & 0 & -5 \\ 0 & 20 & 2 \end{pmatrix}^{-1}$$

$$= A$$

The power of a matrix can be generalized: if A has an inverse, we define $A^{-n} = (A^{-1})^n$. We can further extend the definition to non-integer powers by setting $A^p = S\Lambda^p S^{-1}$ with $\Lambda^p = diag(\lambda_i^p)$. Note that non-integer powers of scalars are not unique.

Example 7.3.5. *Non-integral power of a matrix*
Consider the following matrix A and its diagonalization

$$A = \begin{pmatrix} -29 & 18 \\ -135 & 70 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 25 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

We compute a square root and an inverse cube root

$$A^{\frac{1}{2}} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -15 & 10 \end{pmatrix}$$

$$A^{-\frac{1}{3}} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 25^{-\frac{1}{3}} & 0 \\ 0 & 16^{-\frac{1}{3}} \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} \approx \begin{pmatrix} 0.49896 & 0.80835 \\ -6.06263 & 4.94896 \end{pmatrix}$$

☞ An important property of diagonalizable matrices is related to the rank of the matrix: If A is diagonalizable, the $rank(A)$ is equal to **the number of non-zero eigenvalues**.

7.3.2 Projections into Eigenspaces

When a matrix A of size $N \times N$ is diagonalizable, it provides a basis for $\mathbb{R}^N$ consisting entirely of eigenvectors. Let $v_1, v_2, \cdots v_N$ be such a basis, with associated eigenvalues λ_i such that $Av_i = \lambda_i v_i$. We now consider the dual basis $u_1, u_2, \cdots u_n$.

Let $V = (v_1 v_2 \cdots v_N)$ and $U = (u_1 u_2 \cdots u_N)$. By definition of the dual basis, $U = (V^{-1})^t$. We have $AV = V\Lambda \Leftrightarrow V^{-1}A = \Lambda V^{-1} \Leftrightarrow A^t U = \Lambda U$.

☞ If v_i is an eigenvector of A with eigenvalue u_i, then u_i is an eigenvector of A^t with eigenvalue λ_i.

Since $A^t u_i = \lambda_i u_i \Leftrightarrow u_i^t A^t = \lambda_i u_i^t$, the vectors u_i are frequently referred to as **left eigenvectors** of A.

In the discussion of dual bases, we obtained Eq (3.11) $I = \sum_{i=1}^N v_i u_i^t$. Applying the matrix A to this equation, we obtain

$$A = \lambda_1 v_1 u_1^t + \lambda_2 v_2 u_2^t + \cdots \lambda_N v_N u_N^t, \qquad (7.10)$$

which clearly shows the action of a diagonalizable matrix on some arbitrary vector x: it decomposes the vector into its components along the eigenvectors and scales each by the corresponding eigenvalue. If λ_i happens to be complex, the scaled component also gets rotated. Note that this equation can be seen directly from $A = S\Lambda S^{-1}$.

Example 7.3.6. *Decomposition of a diagonalizable matrix into eigenspace projection matrices*

If possible, decompose the following matrix into projection matrices into its eigenspaces:

$$A = \begin{pmatrix} 12 & 5 & -5 \\ 0 & 2 & 0 \\ 30 & 15 & -13 \end{pmatrix}.$$

By expanding the determinant of $(A - \lambda I)$ starting with the second row, he characteristic equation of A is readily seen to be $p(\lambda) = -(\lambda - 2)^2(\lambda + 3)$.

To find the eigenvector basis, we need to find a basis for $\mathcal{N}(A - \lambda I)$ for each of the eigenvalues. The result is summarized in the following table:

λ	**2**		**-3**
multiplicity	*2*		*1*
basis for E_λ	$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$
V	$\begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & 3 \end{pmatrix}$		
V^{-1}	$\begin{pmatrix} 3 & 3 & -1 \\ 3 & 2 & -1 \\ -2 & -1 & 1 \end{pmatrix}$		

The right and left eigenvectors for $\lambda_1 = 2$ are the first and second columns v_1, v_2 of V and the first and second row u_1^t, u_2^t of V^{-1} respectively. For $\lambda_2 = -3$, the left and right eigenvectors are the third column v_3 of V and the third row u_3^t of V^{-1} respectively. Since A has a complete eigenvector basis, it can be diagonalized. The expansion of A into projection matrices into the two eigenspaces is given by

$$A = \lambda_1(v_1 u_1^t + v_2 u_2^t) + \lambda_2 v_3 u_3^t$$

$$= 2\begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 6 & 3 & -2 \end{pmatrix} - 3\begin{pmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -6 & -3 & 3 \end{pmatrix}.$$

7.4 Schur Decomposition

What if the matrix A is not diagonalizable? One approach is to seek a similarity transform of A that yields an upper triangular matrix.

The Schur decomposition of a real square matrix A with real eigenvalues is a similarity transformation $A = QUQ^{-1}$, where Q is an orthogonal matrix (i.e.,

$Q^{-1} = Q^t$), and U is upper triangular, and thus has the eigenvalues of A on its diagonal. The matrix U is called a **Schur form** of A. If the matrix A is complex or has complex eigenvalues, the orthogonal matrix Q is replaced with a corresponding unitary matrix (i.e., $Q^{-1} = Q^H$).

The existence proof for the Schur decomposition is constructive, and relies on the fact that for an orthogonal (or unitary) matrix W, a matrix of the form

$$\begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix}$$

is orthogonal (or unitary).

Let A be a matrix of size $N \times N$. This matrix necessarily has at least one eigenvalue λ_1 and an associated unit length eigenvector v_1. Extend $\{v_1\}$ to an orthonormal basis $\{v_1, w_2, w_3, \cdots w_N\}$ for $\mathbb{R}^N$. With respect to this basis the matrix $A = Q_1 A_1 Q_1^t$, where $Q_1 = (v_1 \,|\, w_2 \,|\, \cdots \,|\, w_N) = (v_1 \, W_1)$, is an orthogonal matrix constructed from the column vectors $v_1, w_i, i = 2, 3, \cdots N$, and A_1 has the form

$$A_1 = \begin{pmatrix} \lambda_1 & * \\ 0 & A_2 \end{pmatrix}$$

where matrix A_2 has size $(N-1) \times (N-1)$.

Repeating this construction for successive submatrices $A_i, i = 2, 3, \cdots A_N$, we obtain a sequence of eigenpairs $(\lambda_i, v_i), i = 1, 2, \cdots N$, and unitary matrices $W_i = (v_i \,|\, W_i)$, where the vectors v_i have dimension $N - i$ and W_i has size $(N-i) \times (N-i-1)$. Defining square matrices Q_i of size $N \times N$ by

$$Q_i = \begin{pmatrix} I & 0 \\ 0 & W_i \end{pmatrix},$$

we obtain $A = QUQ^t$, where $Q = Q_1 Q_2 \cdots Q_N$, and U has the form

$$U = \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & \lambda_3 & * \\ & & \cdots & \\ 0 & 0 & 0 & \lambda_N \end{pmatrix}$$

Example 7.4.1. *Schur form decomposition of a matrix*
Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \end{pmatrix}$$

which has eigenvalues $\{1, 1, -1, -1\}$. The reader may verify this matrix is not diagonalizable.

For $\lambda_1 = 1$, we find an eigenvector $v_1 = (1\ 1\ -1\ -1)^t$. To complete $\{v_1\}$ to an orthogonal basis, we could use the method in Example 5.5.4. By inspection, we see that $w_2 = (1\ -1\ 0\ 0)^t$, $w_3 = (0\ 0\ 1\ -1)^t$, and $w_4 = (1\ 1\ 1\ 1)^t$ is such a basis. Scaling these vectors to unit length and writing them into a matrix as columns yields a matrix Q_1, where

$$Q_1 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix} \qquad Q_1^t A Q_1 = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{3}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Repeating this process for the matrix W_1 (shown in red), we find eigenvalue $\lambda_2 = -1$ has an eigenvector $v_2 = (1\ 1\ 0)^t$. Again by inspection, we find $\{v_2, ((1\ -1\ 0)^t, (0\ 0\ 1)^t\}$ is an orthogonal basis for $\mathbb{R}^3$. Scaling these vectors to unit length and assembling them into matrix Q_2, we find

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Q_2^t Q_1^t A Q_1 Q_2 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since the matrix W_2 (shown in red) is upper triangular, no further computation is necessary. We have obtained a Schur form $U = Q_2^t Q_1^t A Q_1 Q_2$ using the orthogonal matrix $Q = Q_1 Q_2$ given by

$$U = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

A Schur decomposition $U = Q^t A Q$ of a symmetric matrix A is symmetric since $U = Q^t A Q = Q^t A^t Q = (Q^t A Q)^t = U^t$. Since U is upper diagonal by construction, this shows that for symmetric matrices, a Schur form matrix is diagonal. This proves the fact that

☞ A symmetric matrix A has a complete set of orthonormal eigenvectors, and hence is diagonalizable.

7.5   The Jordan Normal Form

Another way to approach the problem of matrices that are not diagonalizable is to relax the diagonalization requirement: matrices that are "almost" diagonal. The Jordan Normal Form decomposition is an algebraic tool, never used in numerical computations.

A **Jordan Block** matrix has diagonal entries equal to a constant λ and a superdiagonal of entries equal to 1, while all other entries are zero:

$$
J_m(\lambda) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda & 1 & \cdots & 0 & 0 \\
0 & 0 & \lambda & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & \lambda & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda
\end{pmatrix}
\tag{7.11}
$$

is a Jordan block of size $m \times m$. Note that Jordan blocks have a single eigenvalue, with an associated eigenspace of dimension 1.

A **Jordan normal form** matrix is a block diagonal matrix consisting of Jordan blocks on the diagonal, i.e., a matrix of the form

$$
J = \begin{pmatrix}
\mathbf{J_{m_1}(\lambda_1)} & 0 & \cdots & 0 \\
0 & \mathbf{J_{m_2}(\lambda_2)} & \cdots & 0 \\
\cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \mathbf{J_{m_k}(\lambda_k)}
\end{pmatrix},
\tag{7.12}
$$

where the $\lambda_i, i = 1, 2, \cdots m_i$ are not necessarily distinct.

☞ Any square matrix A is similar to a Jordan normal form matrix J, that is, for every A there is an invertible matrix S such that $A = SJS^{-1}$. We omit the proof.

☞ Note that the λ_i in the Jordan blocks $J_{m_i}(\lambda_i)$ in J are the eigenvalues of A, and that A is diagonalizable if and only if each of the Jordan blocks has size $m_i = 1$.

If we restrict S to the columns $(x_1 \, x_2 \cdots x_m)$ corresponding to a Jordan block $J_m(\lambda)$ in $A = SJS^{-1} \Leftrightarrow AS = SJ$, we find

$$
\begin{aligned}
Ax_1 &= \lambda x_1 \\
Ax_2 &= \lambda x_2 + x_1 \\
&\cdots \\
Ax_m &= \lambda x_m + x_{m-1}
\end{aligned}
$$

i.e., x_k is a non-zero solution of

$$
(A - \lambda I)x_k = x_{k-1} \quad \text{for } k = 2, 3, \cdots m.
\tag{7.13}
$$

The sequence of vectors $\{x_1, x_2, \cdots, x_m\}$ is referred to as a **chain of generalized eigenvectors**. Note that this implies that the generalized eigenvectors $x_2, x_3, \cdots x_{m-1}$ are in the column space of $(A - \lambda I)$. Multiplying the equations in Eq (7.13) by $(A - \lambda I)^{k-1}$, we see that x_k is a non-zero solution of

$$(A - \lambda I)^k x_k = 0, \ k = 1, 2, \cdots, m.$$

Using this last equation, it is trivial to show that the vectors in a chain of generalized eigenvectors are linearly independent (see the exercises).

The following two results will prove useful:

$$J_m^k(0) = (J_m(\lambda) - \lambda I)^k = N_k, \quad k = 1, 2, \cdots m - 1, \qquad (7.14)$$

where N_k is a matrix of size $m \times m$ with the kth superdiagonal equal to 1 and all other entries equal to zero. In particular, we have $J_m^m(0) = 0$. For $\lambda \neq 0$,

$$J_m^k(\lambda) = \sum_{i=0}^{m} \binom{k}{i} \lambda^{m-i} N_m^i,$$

which we obtain by expanding $J_m(\lambda) = \lambda I + N_1$, i.e.,

$$J_m^k(\lambda) = \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{m}{2}\lambda^{k-2} & \cdots & \binom{k}{m-1}\lambda^{k-m+1} \\ & \lambda^k & \binom{m}{1}\lambda^{k-1} & \cdots & \binom{k}{m-2}\lambda^{k-m+2} \\ & & \cdots & \cdots & \cdots \\ & & & & \lambda^k \end{pmatrix}, \quad \text{where } \lambda \neq 0.$$

$$(7.15)$$

Note that the rank of $J_m^k(0) = m - k - 1$, for $k = 0, 1, \cdots m - 1$ decreases by 1 as k increases by 1.

Computation of the Jordan Normal Form of a matrix A uses Eq(7.13).

1. Compute the eigenvalues, algebraic multiplicities and eigenvector bases.

2. For each deficient eigenspace E_λ with algebraic multiplicity m, i.e, for each eigenspace with $\dim(E_\lambda) < m$, compute the generalized eigenvector chains for each of the associated eigenvectors.

   The chains end when an equation in Eq(7.13) no longer has a solution.

3. Assemble the Jordan blocks for each of the eigenvalues into matrix J

4. Assemble the eigenvectors and generalized eigenvector chains into matrix S, making sure that the order of the vectors is consistent with the ordering in J.

5. The computation can be checked by verifying that $AS = SJ$.

Example 7.5.1. *Jordan Normal Form Block Structure*
As an example, assume we have a matrix A of size 12×12 for which we have computed the following table

λ	**2**	**3**	**5**	**7**
multiplicity	*2*	*3*	*3*	*4*
basis	x_1, x_2	y_1, y_2	u_1	v_1, v_2

Eigenvalues 3, 5, and 7 are seen to be deficient.

Eigenvalue $\lambda = 2$: *Since the algebraic multiplicity is 2 and we have two eigenvectors in our basis, this eigenvalue contributes*
$\operatorname{diag}(2,2) = \operatorname{diag}(J_1(2), J_1(2))$ to J, and corresponding columns $(x_1 \ x_2)$ to S.

Eigenvalue $\lambda = 3$: *Since the algebraic multiplicity is 3 but we only have two eigenvectors in our basis, either y_1 or y_2 will have an additional vector y_3 in its chain. Let us assume it is y_2, i.e., we have $y_1, y_2 \to y_3$. This results in blocks $\operatorname{diag}(J_1(3), J_2(3))$ in the matrix J, and corresponding columns $(y_1 \ y_2 \ y_3)$ in the matrix S.*

Eigenvalue $\lambda = 5$: *The algebraic multiplicity is 3, but we only have a single vector u_1. It will produce a chain $u_1 \to u_2, u_3$ and contribute a single block $J_3(5)$ to J and corresponding columns $(u_1 \ u_2 \ u_3)$ to S.*

Eigenvalue $\lambda = 7$: *The algebraic multiplicity is 4, but we only have two vector v_1 and v_2. Here we have several possibilities:*

  – *one chain produces both the missing vectors, e.g., $v_2 \to v_3, v_4$, resulting in a block structure contribution $\operatorname{diag}(7, J_3(7))$ to J and corresponding columns $(v_1 \ v_2 \ v_3 \ v_4)$ to S.*

  – *two chains produce one vector each, $v_1 \to v_3, v_2 \to v4$, resulting in a block structure contribution $\operatorname{diag}(J_2(7), J_2(7))$ to J and corresponding columns $(v_1 \ v_3 \ v_2 \ v_4)$ to S.*

We therefore have two possible outcomes of the computations:

 • $J = \operatorname{diag}(J_1(2), J_1(2), J_1(3), J_2(3), J_3(5), J_1(7), J_3(7))$,
   $S = (x_1 \ x_2 \ y_1 \ y_2 \ y_3 \ u_1 \ u_2 \ u_3 \ v_1 \ v_2 \ v_3 \ v_4)$

 • $J = \operatorname{diag}(J_1(2), J_1(2), J_1(3), J_2(3), J_3(5), J_2(7), J_2(7))$,
   $S = (x_1 \ x_2 \ y_1 \ y_2 \ y_3 \ u_1 \ u_2 \ u_3 \ v_1 \ v_3 \ v_2 \ v_4)$.

Note that the order of the blocks can be interchanged, as long as the ordering of the corresponding generalized eigenvectors is kept consistent.

Example 7.5.2. *Jordan Normal Form of a 3×3 matrix*
Find the Jordan Normal Form of

$$\begin{pmatrix} 1 & 2 & -3 \\ -3 & 6 & -4 \\ -1 & 1 & 2 \end{pmatrix}$$

Using the Laplace expansion, the characteristic polynomial
$p(\lambda) = \det(A - \lambda I) = (\lambda - 3)^3$. Computing a basis for the $\mathcal{N}(A - 3I)$, we find
only one vector: $x_1 = (1\,1\,0)$, i.e., the matrix is degenerate. Since there is only
one eigenvector in the basis, there is only one chain of generalized eigenvectors:
the Jordan Normal form of A therefore has a single jordan block of size 3 with
eigenvalue $\lambda = 3$, namely $J = J_3(3)$.

Computing the chain: since we have to repeatedly solve the equation
$(A - 3I)x = b$ for different right hand sides, we set up the computational layout
$(A - \lambda I \,|\, b_1 b_2 \cdots)$, and fill in each of the right hand side vectors as they are
obtained. Here $b_1 = x_1$, the first eigenvector.

$$\left(\begin{array}{ccc|ccc} -2 & 2 & -3 & 1 & 1 & 4 \\ -3 & 3 & -4 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & -3 \end{array}\right)$$ *Initial layout:*
$(A - 3I | x_1 \cdots)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{array}{ccc|ccc} -1 & 1 & -1 & 0 & -1 & -3 \\ -3 & 3 & -4 & 1 & 1 & 0 \\ -2 & 2 & -3 & 1 & 0 & 4 \end{array}\right)$$ *Row exchange to avoid*
fractions.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} -1 & 1 & -1 & 0 & -1 & -3 \\ 0 & 0 & -1 & 1 & 3 & 9 \\ 0 & 0 & -1 & 1 & 3 & 10 \end{array}\right)$$ *Obtain the reduced row*
echelon form for conve-
nience

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 4 & 13 \\ 0 & 0 & 1 & -1 & -3 & -9 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$ *Now add the solution as*
a new right hand side and
repeat the computation

As expected, the third equation no longer has a solution. We summarize the
results in a table.

λ	**3**		
multiplicity	*3*		
basis	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \rightarrow$	$\begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$
S	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}$	$\begin{matrix} -1 \\ 3 \\ 3 \end{matrix}$
J	$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$	$\begin{matrix} 1 \\ 3 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ 1 \\ 3 \end{matrix}$

To check the result, verify that $AS = SJ$.

Example 7.5.3. *Jordan Normal Form of a* 4×4 *matrix*
Find the Jordan Normal Form of

$$
A = \begin{pmatrix} -5 & 2 & -4 & 0 \\ -2 & 1 & -4 & 4 \\ 3 & -1 & 1 & 2 \\ 2 & -1 & 2 & -1 \end{pmatrix}, \quad p(\lambda) = \det(A - \lambda I) = (\lambda + 1)^4
$$

Basis vectors for $\mathcal{N}(A + I)$*:* $x_1 = (0\ 2\ 1\ 0)$ *and* $y_1 = (-2\ -4\ 0\ 1)$*. There are therefore two chains of generalized eigenvectors. The only possible Jordan normal forms are* $J = diag\,(J_3(-1), J_1(-1))$ *or* $J = diag\,(J_2(-1), J_2(-1))$*.*

In the following layout, we have precomputed the product of elementary matrices that reduces $(A + I)$ *to reduced row echelon form.*

$$
\left(\begin{array}{cccc|cccc} -4 & 2 & -4 & 0 & 0 & 1 & -2 & -1 \\ -2 & 2 & -4 & 4 & 2 & 2 & -4 & -3 \\ 3 & -1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 & 0 & 0 & 1 & 0 \end{array} \right)
$$

$$
\left(\begin{array}{cccc} 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 4 \end{array} \right) \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 4 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{array} \right)
$$

The inconsistent right hand sides result in the end of the two chains. We summarize the results in a table.

λ	**-1**			
multiplicity	*4*			
basis	$\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$	,	$\begin{pmatrix} -2 \\ -4 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ -3 \\ 0 \\ 0 \end{pmatrix}$	
S	$\begin{pmatrix} 0 & 1 & -2 & -1 \\ 2 & 2 & -4 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$			
J	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$			

To check the result, verify that $AS = SJ$*.*

Example 7.5.4. *Power of a degenerate matrix*
Given a matrix $A = SJS^{-1}$ with Jordan normal form J, we can compute
$A^k = SJ^kS^{-1}$ using Eq (7.14) and Eq (7.15) and the observation that
$diag\left(J_{m_1}(\lambda_1)\, J_{m_2}(\lambda_2)\, \cdots J_{m_n}(\lambda_n)\right)^k = diag\left(J^k_{m_1}(\lambda_1)\, J^k_{m_2}(\lambda_2)\, \cdots J^k_{m_n}(\lambda_n)\right)$.
For the matrix in Example (7.5.3) with

$$
S = \begin{pmatrix} 0 & 1 & -2 & -1 \\ 2 & 2 & -4 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad
J = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$

Using Eq (7.15) on each of the two Jordan blocks in J, we find

$$
J^k = (-1)^k \begin{pmatrix} 1 & -k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -k & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Computing the inverse of S and multiplying the matrices $A^k = SJ^kS^{-1}$ yields

$$
A^k = (-1)^k \begin{pmatrix} 4\,k+1 & -2\,k & 4\,k & 0 \\ 2\,k & -2\,k+1 & 4\,k & -4\,k \\ -3\,k & k & -2\,k+1 & -2\,k \\ -2\,k & k & -2\,k & 1 \end{pmatrix}
$$

For invertible matrices A, the Jordan normal form can be used to define A^n for negative integers n by noting that $J_k^{-1}(\lambda) = -J_m(-\frac{1}{\lambda})$. We can therefore define A^n for integral values of n. We cannot however define A^p for non-integral values p when A is not diagonalizable.

7.6 Functions of Matrices

7.6.1 Functions of Diagonalizable Matrices

The computation of the power of a matrix in the previous section is a special case of a more general result. We will restrict our attention to diagonalizable matrices. Let $f(x)$ be an analytic function, i.e., $f(x)$ has a locally convergent power series $f(x) = \sum_{i=0}^{\infty} \alpha_i x^i$.

Given a diagonalizable matrix $A = S\Lambda S^{-1}$, we define the function $f(A) = \sum_{i=0}^{\infty} \alpha_i A^i$, where we set $A^0 = I$. Substitution of the diagonalization of A yields

$$f(A) = S\Phi S^{-1}, \text{ where } \Phi = diag(f(\lambda_i)) \tag{7.16}$$

We can compute this function directly in the form given, or use the decomposition Eq (7.10) to factor factor out the $f(\lambda_i)$, which is known as Sylvester's matrix theorem

$$f(A) = f(\lambda_1)v_1 u_1^t + f(\lambda_2)v_2 u_2^t + \cdots f(\lambda_N)v_N u_N^i, \tag{7.17}$$

where the v_i are a complete eigenvector basis for A, and u_i are the associated dual basis. The expansion is valid whenever $f(x)$ is analytic at each of the eigenvalues.

Example 7.6.1. *Functions of a diagonalizable matrix*
We again consider the matrix from Example 7.3.6

$$A = \begin{pmatrix} 12 & 5 & -5 \\ 0 & 2 & 0 \\ 30 & 15 & -13 \end{pmatrix},$$

and it's expansion into eigenspace projection matrices

$$A = \lambda_1(v_1 u_1^t + v_2 u_2^t) + \lambda_2 v_3 u_3^t$$
$$= \lambda_1 \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 6 & 3 & -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -6 & -3 & 3 \end{pmatrix}$$

for the two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$.
The function $f(x) = e^x t$ is analytic at $x = 2$ and $x = -3$, for any parameter t, so that

$$e^{At} = e^{2t} \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 6 & 3 & -2 \end{pmatrix} + e^{-3t} \begin{pmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -6 & -3 & 3 \end{pmatrix}.$$

The function $f(x) = (1-x)^{-1}$ is analytic at $x = 2$ and $x = -3$, so that

$$(I - A)^{-1} = (1-2)^{-1} \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 6 & 3 & -2 \end{pmatrix} + (1+3)^{-1} \begin{pmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -6 & -3 & 3 \end{pmatrix}$$
$$= \frac{1}{4} \begin{pmatrix} -14 & -5 & 5 \\ 0 & -4 & 0 \\ 30 & -15 & 11 \end{pmatrix}.$$

7.6.2 Functions of Non-Diagonalizable Matrices

Power Series of Jordan Normal Form matrices can be computed using Eq (7.11). Given a function $f(x)$, we have

$$f(J_m(\lambda)) = \begin{pmatrix} \frac{f(\lambda)}{0!} & \frac{f^{(1)}(\lambda)}{1!} & \frac{f^{(2)}(\lambda)}{2!} & \cdots & \frac{f^{(m-1)}(\lambda)}{(m-1)!} & \frac{f^{(m)}(\lambda)}{m!} \\ 0 & \frac{f(\lambda)}{0!} & \frac{f^{(1)}(\lambda)}{1!} & \cdots & \frac{f^{(m-2)}(\lambda)}{(m-2)!} & \frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ 0 & 0 & \frac{f(\lambda)}{0!} & \cdots & \frac{f^{(m-3)}(\lambda)}{(m-3)!} & \frac{f^{(m-2)}(\lambda)}{(m-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{f(\lambda)}{0!} & \frac{f^{(1)}(\lambda)}{1!} \\ 0 & 0 & 0 & \cdots & 0 & \frac{f(\lambda)}{0!} \end{pmatrix} \qquad (7.18)$$

Example 7.6.2. *Functions of a non-diagonalizable matrix*
Consider the Jordan block

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Using Eq (7.18), the function $sin(Jx)$ is given by

$$sin(Jx) = \begin{pmatrix} sin(2x) & 2cos(2x) & 0 & 0 \\ 0 & sin(2x) & 0 & 0 \\ 0 & 0 & sin(3x) & 3cos(3x) \\ 0 & 0 & 0 & sin(3x) \end{pmatrix}$$

7.7 Moving Eigenvalues

The following examples show how to transform matrices to move eigenvalues to desired values.

An **eigenpair** (λ, x) for a matrix A is an eigenvalue λ and corresponding eigenvector x of A.

The **spectrum** of a square matrix A denoted $\lambda(A)$ is the set of eigenvalues of A.

☞ If (λ, v) is an eigenpair of a matrix A, then $(\lambda, S^{-1}v)$ is an eigenpair of the similarity transform $S^{-1}AS$

Let A be a square matrix of size $N \times N$, and consider the similarity transform $\tilde{A} = S^{-1}AS$ expressing the matrix with respect to a new basis consisting of the columns of S. Unsurprisingly $\tilde{A}$ has the same eigenvalues as A with the same

eigenvectors expressed in the new basis since

$$p(\lambda) = det(S^{-1}AS - \lambda I)$$
$$= det\left(S^{-1}(A - \lambda I)S\right)$$
$$= det(A - \lambda I)$$

and $Ax = \lambda x \Leftrightarrow \left(S^{-1}AS\right)\left(S^{-1}x\right) = \lambda\left(S^{-1}x\right)$.

☞ Let $B = a_0 I + a_1 A + a_2 A^2 + \cdots + z_k A^k$, be the matrix obtained by formally evaluating a given polynomial $q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^k$, by substituting a matrix A for the variable x, i.e., $B = q(A)$. Let A have spectrum $\lambda(A) = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$.

The spectrum of $\lambda(B) = \{q(\lambda_1), q(\lambda_2), \cdots q(\lambda_n)\}$ with the same algebraic multiplicities as the λ_i.[2] Further, if (λ, v) is an eigenpair of A, then $(q(\lambda), v)$ is an eigenpair of $q(A)$.

Given an eigenpair (λ, v) of A, the straightforward computation $Bv = a_0 v + a_1 \lambda v + \cdots \lambda^k v = p(\lambda)v$ shows that B has eigenvector v and associated eigenvalue $q(\lambda)$.

For the converse, let μ be a (possibly complex) eigenvalue of $B = q(A)$. Factor $q(x) - \mu = a \prod_{i=1}^k (x - x_i)$, where a is the coefficient of the highest power of x in $q(x)$, and x_i are the roots of $q(x) - \mu$. Evaluating this factorization at A yields $B - \mu I = a \prod_{i=1}^k (A - x_i I)$. Since $B - \mu I$ is not invertible, this implies that for some index i, the matrix $A - x_i I$ is not invertible, and therefore x_i is an eigenvalue of A. Evaluating the factorization $q(x) - \mu$ at x_i yields $q(x_i) = \mu$, as required.

To show that the algebraic multiplicity is conserved, let $A = SJS^{-1}$ be a Schur form decomposition of A. We thus have $q(A) = q(SJS^{-1}) = Sq(J)S^{-1}$, where $q(J)$ is a Schur form matrix with eigenvalues $q(\lambda)$ on the diagonal. The algebraic multiplicity of an eigenvalue λ of A is therefore maintained as stated above.

Note that this theorem ensures that if A is diagonalizable, then A and $q(A)$ have the same eigenvectors. This is not necessarily true if A is not diagonalizable however: the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is degenerate. The matrix A^2 however has a complete eigenbasis for $\mathbb{R}^2$. Note also that the theorem may require complex eigenvalues:

The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has spectrum $\lambda(A) = \{i, -i\}$, so that $\lambda(A^2) = \lambda(-I) = \{(\pm i)^2\} = \{-1\}$.

Two special cases of this theorem are

---

[2]This statement is subject to the observation that if distinct eigenvalues $\{\lambda_{i_1}, \lambda_{i_2}, \cdots \lambda_{i_k}\}$ of A of algebraic multilicity $\{m_{i_1}, m_{i_2}, \cdots m_{i_k}\}$ respectively map to the same eigenvalue $q(\lambda_{i_1}) = q(\lambda_{i_2}) = \cdots = q(\lambda_{i_k})$ of $q(A)$, this eigenvalue will have algebraic multiplicity $\sum_{j \in \{i_1, i_2, \cdots i_k\}} m_j$.

- $q(x) = x - \mu$, which shows that if (λ, x) is an eigenpair of a matrix A, then $(\lambda - \mu, x)$ is an eigenpair of the matrix $A - \mu I$,

- $q(x) = \alpha x$, which shows that if (λ, x) is an eigenpair of a matrix A the $(\alpha \lambda, x)$ is an eigenpair of αA.

For invertible matrices, we can extend the previous theorem to negative powers:

☞ Let A be an invertible matrix. If (λ, v) is an eigenpair of A, then $(\frac{1}{\lambda}, v)$ is an eigenpair of A^{-1}.

Since A is invertible, $det(A) \neq 0$, and therefore all eigenvalues of A are nonzero. The result follows by multiplying $Ax = \lambda x$ by A^{-1} from the left.

Example 7.7.1. *Eigenvalues and Eigenvectors of Matrix Polynomials*
Consider a square matrix A with characteristic polynomial

$$p(\lambda) = (\lambda - 2)(\lambda - 1)^3(\lambda + 3)^2.$$

- *The spectrum of A is given by $\lambda(A) = \{2, 1, -3\}$ with respective algebraic multiplicities 1,1, and 2.*

- *Consider the matrix $B = 2(A - 3I)$ which is obtained by formally substituting A in the polynomial $q(x) = 2(x - 3)$. The spectrum of B is therfore given by $\lambda(B) = \{q(2), q(1), q(-3)\} = \{-2, -4, -12\}$. Since the algebraic multiplicities are conserved, we have*
  $det(B) = q(2)q^3(1)q^2(-3) = (-2)(-4)^3(-12)^2 = 18432.$

- *The matrix A^{-1} exists and has eigenvalues $\lambda = \frac{1}{2}, 1, -\frac{1}{3}$.*

- *The matrix $C = 3A^{-2} + A^2$ is obtained by formally substituting A in $q(x) = 3x^{-2} + x^2$. For invertible matrices the original theorem generalizes, so that $\lambda(C) = \{q(2), q(1), q(-3)\} = \{\frac{19}{4}, 4, \frac{28}{3}\}$.*

☞ Let A be a square matrix of size $N \times N$. A and A^t have the same eigenvalues. If A has N linearly independent eigenvectors $\mathfrak{B} = \{v_1, v_2, \cdots v_N\}$ (and therefore $\mathfrak{B}$ is a basis for $\mathbb{F}^N$), then the corresponding dual basis vectors $\{u_1, u_2, \ldots u_N\}$ are eigenvectors of A^t with the same eigenvalues:

If (λ_i, v_i) is an eigenpair of A, then (λ_i, u_i) is an eigenpair of A^t.

Let $S = (v_1\ v_2\ \cdots v_N)$, then $U = (u_1, u_2, \cdots u_N)$ with $U^t = S^{-1}$. The similarity transform $S^{-1}AS = \Lambda \Leftrightarrow S^{-1}A = \Lambda S^{-1}$, where Λ is a diagonal matrix with entries $\Lambda_{ii} = \lambda_i$. Taking the transpose, we obtain $A^t U = U \Lambda$.

Remember that for $i \neq j$, the basis vectors are orthogonal: $u_i \perp v_j$. This statement depends on A having N linearly independent eigenvectors. This requirement can however be relaxed:

☞ Let (λ, v) and let (μ, w) be eigenpairs for A and A^t respectively. If $\lambda \neq \mu$, then $v \perp w$.

Consider $Av = \lambda v \Rightarrow v^t A^t w = \lambda v^t w \Rightarrow (\mu - \lambda)v^t w = 0$. Since $\lambda \neq \mu$, this shows $v^t w = 0$ and establishes the above result.

☞ Let (λ, v) be an eigenpair of a matrix A and consider the matrix $B = A - \sigma v x^t$, where σ is an arbitrary scalar and x is an arbitrary vector.

$(\lambda - \sigma x \cot v, v)$ is an eigenpair of B. The remaining eigenvalues are unchanged.

This follows from the straightforward computation

$$Bv = (A - \sigma v x^t)v = (\lambda - \sigma x \cdot v)v.$$

For any other eigenpair (μ, w) of A^t, we have
$B^t w = A^t w - \lambda x v^t w = A^t w$, since $v^t w = 0$ by the previous example.

Note in particular, if we choose x such that $x \cdot v = 1$, the transformation $B - \lambda v x^t$ shifts the eigenvalue λ to zero, while leaving all other eigenvalues unaffected.

A good choice for the vector x is to choose an appropriately scaled row of the matrix A: let $a_i = (a_{ij})_{j=1,2,\ldots N}$ be the ith row of AA. We obtain $a_i v = \lambda v_i$

FIX add eigenvector info; e.g.09-power_method-1.pdf, wielandt deflation FIX

```
#original example:
A=matrix(QQ,4,4,[11,-6,4,-2, 4,1,0,0, -9,9,-6,5,   -6,6,-6,7])
L,Q=A.right_eigenmatrix()

#modified example
A=Q.inverse()*matrix(QQ,4,4,[5,0,0,0,  0,3,0,0,  0,0,3,0, 0,0,0,1])*Q; print "\nA="; sh
L,Q=A.right_eigenmatrix(); print "eigenvalues =", A.eigenvalues()

#choose an eigenvalue and eigenvector, choose a vector x
l=3; v=Q[:,2]; x=matrix(QQ,1,4,[1,-1,0,1])
#print x*v
#print v.transpose(), (A*v).transpose()

# Compute the transform
B=A-3*v*x
print "B ="; show(B)
print "eigenvalues",B.eigenvalues()

# Now evaluate: new eigenvalues, eigenvectors
PL,P=B.right_eigenmatrix()

#NQ=v
#for i in range(4):
```

```
#    NQ=NQ.augment( (PL[i,i]-l)*P[:,i] - ((l*x * P[:,i])[0,0]) * v
##print "NQ ="; show(NQ[:,1:5]); show(Q)

NQ=v

for i in range(4):
    w=(PL[i,i]-l)*P[:,i] + ((l*x * P[:,i])[0,0]) * v
    #ww = sqrt((w.transpose()*w)[0,0])
    NQ=NQ.augment( w )

print "NQ =";
print(NQ[:,1:])
print "\nQ="; print(Q)
```

We see that only one of the repeated eigenvalues shifts to zero.
The eigenvectors of both the shifted and unshifted eigenvalues do stay the same, however!?!!

Example 7.7.2. *Eigenvalues and Eigenvectors for Matrix Transformations*

Let

$$C = \begin{pmatrix} -34 & 4 & 10 \\ 27 & -3 & -8 \\ -129 & 15 & 38 \end{pmatrix}$$

C has eigenvalues $\lambda = 2, -1, 0$ and corresponding eigenvectors $v_1 = (1 \ -1 \ 4)^t$, $v_2 = (2, -1, 7)^t$, and $v_3 = (1 \ 1 \ 3)^t$. Writing the eigenvector into a matrix as columns, we have $A = S\Lambda S^{-1}$, where

$$S = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 4 & 7 & 3 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -10 & 1 & 3 \\ 7 & -1 & -2 \\ -3 & 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the eigenvector v_1. The infinity norm entry in v_1 is the third entry $|v|_\infty = 4$. Choosing the third row $c_3 = (-129 \ 15 \ 38)$ of C, we set $x = \frac{1}{\lambda_1 |v|_\infty} c_3^t$, we obtain

$$C - 2v_1 x^t = \frac{1}{4} \begin{pmatrix} -7 & 1 & 2 \\ -21 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

which has eigenvalues $\lambda = 0, -1, 0$.

7.8 The Cayley-Hamilton theorem

The Cayley-Hamilton theorem states that every square matrix A satisfies its characteristic polynomial: given the characteristic polynomial

$$\det(A - \lambda I) = (-1)^N \left(\lambda^N - tr(A)\lambda^{N-1} + \cdots + (-1)^N \det(A) \right),$$

where A has size $N \times N$, then

$$A^N - tr(A)A^{N-1} + \cdots + (-1)^N det(A)I = 0, \tag{7.19}$$

Example 7.8.1. *The Cayley-Hamilton theorem*

Consider the matrix

$$A = \begin{pmatrix} -11 & 28 \\ -6 & 15 \end{pmatrix}.$$

Its characteristic polynomial is given by

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$$

Formally substituting the matrix A for λ, we obtain

$$A^2 - 4A + 3I = \begin{pmatrix} -47 & 112 \\ -24 & 57 \end{pmatrix} - 4 \begin{pmatrix} -11 & 28 \\ -6 & 15 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If the matrix A is diagonalizable, the theorem is easy to prove, and left as an exercise. Various proofs exist for degenerate matrices and will be omitted.[3]

The theorem is quite useful, as the next three subsections show.

7.8.1   Inverse of a matrix

One consequence of the Caley-Hamilton theorem yields another formula for the inverse of a matrix. Since invertible matrices A have $\det(A) \neq 0$, we can solve Eq (7.19) for I

$$I = \frac{(-1)^{N+1}}{\det(A)} \left(A^{N-1} - tr(A)A^{N-2} \cdots \right) A.$$

By factoring out the matrix A, the remaining expression is seen to be the inverse A^{-1}.

Example 7.8.2. *Matrix inverse using the Cayley-Hamilton theorem*
Continuing the previous example (7.8.1), consider the matrix

$$A = \begin{pmatrix} -11 & 28 \\ -6 & 15 \end{pmatrix}.$$

The roots of its characteristic polynomial satisfy

$$\lambda^2 - 4\lambda + 3 = 0 \Leftrightarrow 1 = \frac{1}{3}(-\lambda + 4)\lambda,$$

and therefore

$$A^{-1} = \frac{1}{3}(-A + 4I) = \frac{1}{3}\begin{pmatrix} 15 & -28 \\ 6 & -11 \end{pmatrix}.$$

7.8.2   Eigenvectors of a 2×2 matrix

Using the Cayley-Hamilton theorem for a 2×2 matrix A with eigenvalues λ_1 and λ_2, we have

$$\begin{aligned} 0 &= A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I \\ &= (A - \lambda_1 I)(A - \lambda_2 I) \\ &= (A - \lambda_2 I)(A - \lambda_1 I). \end{aligned}$$

This shows that the columns of $(A - \lambda_2 I)$ are in $\mathcal{N}(A - \lambda_1)$, and the columns of $(A - \lambda_1 I)$ are in $\mathcal{N}(A - \lambda_2)$.

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[3]One of the simplest proofs involves obtaining a Schur decompositon of A: by modifying repeated eigenvalues with a term of the form $\alpha\epsilon$ for appropriately chosen constants α to ensure all eigenvalues are distinct, one obtains a continuous function of ϵ that converges to A in the limit as $\epsilon \to 0$. Since the theorem holds for $\epsilon \neq 0$, the result can be shown to hold for $\epsilon = 0$ by appealing to continuity.

Example 7.8.3. *Eigenspaces of a* 2×2 *matrix using the Cayley-Hamilton theorem*

Continuing the previous example (7.8.1), consider the matrix

$$A = \begin{pmatrix} -11 & 28 \\ -6 & 15 \end{pmatrix}$$

which has eigenvalues $\lambda = 1, 3$. Using $\lambda = 1$, we have

$$A - I = \begin{pmatrix} -12 & 28 \\ -6 & 14 \end{pmatrix}.$$

From the first row of this matrix, we see that its null space has basis

$$x_1 = (28 \ 12) = 4 \, (7 \ 3),$$

while the first column yields a basis

$$x_2 = (-12 \ -6) = -6 \, (2 \ 1)$$

for its column space. The result is summarized in the following table:

λ	**1**	**3**
multiplicity	*1*	*1*
basis for E_λ	$\begin{pmatrix} 7 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

7.8.3 Power Series

As a final example of the use of the Cayley-Hamilton theorem, we use the characteristic polynomial of a matrix A of size $N \times N$ to solve for its highest power: Eq (7.19 can thus be rewritten as

$$A^N = tr(A)A^{N-1} - \cdots - (-1)^N det(A)I.$$

This expression can be substituted into any polynomial of A to reduce its degree to at most $N - 1$.

Example 7.8.4. *Reducing the order of a matrix polynomial*
Continuing the previous example (7.8.1), consider the matrix

$$A = \begin{pmatrix} -11 & 28 \\ -6 & 15 \end{pmatrix} \quad \text{with characteristic polynomial } p(\lambda) = \lambda^2 - 4\lambda + 3.$$

The matrix A^2 is therefore given by

$$A^2 = 4A - 3I$$

Repeatedly substituting this expression for A^2 into a polynomial such as $A^4 + 3A^3 - 2A$ results in a matrix polynomial of degree 1.

$$\begin{aligned} A^4 + 3A^3 - 2A &= (4A - 3I)^2 + 3A(4A - 3I) - 2A \\ &= 16A^2 - 24A + 9I + 12A^2 - 9A - 2A \\ &= 28(4A - 3I) - 35A + 9I \\ &= 77A - 75I. \end{aligned}$$

We obtain
$$A^4 + 3A^3 - 2A = 77A - 75I = \begin{pmatrix} -922 & 2156 \\ -462 & 1080 \end{pmatrix}$$

☞ Applying this method to a power series $f(t) = \sum_i \alpha_i t^i A^i$ for a matrix of size $N \times N$ results in $f(t) = \sum_{i=0}^{N-1} p_i(t) A^i$, where the functions $p_i(t)$ are power series in t.

Example 7.8.5. *Power series of a matrix*
Consider the computation of e^{At} for the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is readily found to be

$$p(\lambda) = \lambda^3 - \lambda^2.$$

The Cayley Hamilton theorem states that $A^3 = A^2$. A simple inference argument shows that $A^k = A^2$ for $k = 2, 3, 4, \cdots$. Substituting this expression into the power series for e^{At} yields

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \dots$$

$$= I + tA + \left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) A^2$$

$$= I + tA + \left(e^t - 1 - t \right) A^2.$$

Substituting the matrix A shows

$$e^{At} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ e^t - 1 & -t - e^t + 1 & e^t \end{pmatrix}$$

7.9   Minimal Polynomial

The Cayleigh Hamilton theorem shows the existence of polynomials $p(\lambda)$ that annihilate a given square matrix A, i.e., that satisfy $p(A) = 0$. We now consider the set $\mathcal{M}$ of all such polynomials. Since any non-zero constant α times a polynomial $p(\lambda)$ in this set also satisfies $\alpha p(A) = 0$, we may restrict the set to **monic polynomials**, i.e., polynomials with the coefficient of the highest power equal ot 1: $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 + a_0$. In this set are polynomials of minimal degree. It turns out that there is only one such polynomial.

☞ Given a square matrix A, there exists a unique monic polynomial $p_A(\lambda)$ of minimal degree that annihilates A. This polynomial divides all polynomials that annihilate A.

Let $\{\lambda_i, i = 1, 2, \dots k\}$ be the set of distinct eigenvalues of A, and let $\{n_i, i = 1, 2, \dots k\}$ be the corresponding length of the longest generalized eigenvector chain for each of theses eigenvalues. Then

$$p_A(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{n_i}.$$

FIX add proof and example. FIX

7.10 Symmetric Matrices

Symmetric matrices in $\mathbb{R}^{N \times N}$ play a special role in linear algebra: in particular, we have seen the importance of the normal equations $A^t A x = A^t b$. Note that the matrix $A^t A$ is symmetric.

☞ Symmetric matrices turn out to be really nice:

- Symmetric matrices are diagonalizable
- eigenvalues and eigenvectors are real
- eigenvectors for different eigenvalues are orthogonal

Since we are free to select any basis from an eigenspace associated with a particular eigenvalue, we can use QR to make the eigenvectors of that eigenspace orthonormal. Thus, applying QR to the basis vectors of eich of the eigenspaces of a symmetric matrix, we obtain an orthonormal basis of $\mathbb{R}^N$.

A matrix with a complete set of orthonormal eigenvectors is said to be **orthogonally diagonalizable**.

Example 7.10.1. *Orthonormal Eigendecomposition for a symmetric matrix*
Diagonalize the symmetric matrix

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 3 \end{pmatrix}$$

The characteristic polynomial is given by $p(\lambda) = -(\lambda + 2)(\lambda - 7)^2$. For the eigenvalue $\lambda = 7$, we obtain $x_1 = (1\ 2\ 0)^t$ and $x_2 = (1\ 0\ 1)^t$. For the eigenvalue $\lambda = -2$, we find a basis vector $x_3 = (-2\ 1\ 2)^t$,

As expected, we see that $x_1 \cdot x_3 = 0$ and $x_2 \cdot x_3 = 0$. However, the vectors for $\lambda = 7$ are not orthogonal, since $x_1 \cdot x_2 \neq 0$.

At this point, we have obtained the eigendecomposition $A = S\Lambda S^{-1}$ as shown in the table below. We can improve on it however by orthonormalizing the bases for each of of the eigenvalues. For the $\lambda = 7$ case, we obtain $q_1 = \frac{\sqrt{6}}{6}(1\ 1\ -2)^t$ and $q_2 = \frac{\sqrt{5}}{15}(4\ -2\ 5)^t$. For $\lambda = -2$, QR reduces to just scaling x_3 to a unit vector $q_3 = \frac{1}{3}(-2\ 1\ 2)^t$.

λ	**7**		**-2**
multiplicity	*2*		*1*
basis	$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$
Λ	$\begin{pmatrix} \mathbf{7} & 0 & 0 \\ 0 & \mathbf{7} & 0 \\ 0 & 0 & \mathbf{-2} \end{pmatrix}$		
S	$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$		
orthonormal basis	$\begin{pmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ -\frac{2\sqrt{6}}{6} \end{pmatrix}$	$\begin{pmatrix} \frac{4\sqrt{5}}{15} \\ -\frac{2\sqrt{5}}{15} \\ \frac{\sqrt{5}}{3} \end{pmatrix}$	$\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$
Q	$\begin{pmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{2\sqrt{6}}{6} \\ \frac{4\sqrt{5}}{15} & -\frac{2\sqrt{5}}{15} & \frac{\sqrt{5}}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$		

Since $Q^{-1} = Q^t$, we have obtained an alternate symmetric decomposition $A = Q\Lambda Q^t$.

In the present context, note that symmetric matrices satisfy the following condition: A **normal matrix** A is a matrix that commutes with its conjugate transpose $AA^H = A^H A$.

Normal matrices are interesting due to the following theorem

☞ A matrix A has a complete set of orthonormal eigenvectors if and only if A is normal.

Other normal matrices are skew-symmetric matrices $A^t = -A$ and orthogonal square matrices. If we allow complex entries, hermitian, skew-hermitian and unitary matrices are normal as well. This list is not exhaustive. The reader may verify that

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

is normal.

In the exercises, we show that for real symmetric matrices A, the eigenvalues and eigenvectors are real, and further, eigenvectors of A corresponding to different eigenvectors are automatically orthogonal.

7.11   Applications

7.11.1   Ordinary Linear Differential Equations with Constant Coefficients

Example 7.11.1. *System of ODEs with constant coefficients*
Solve the following initial value problem:

$$\frac{d}{dx}u_1(x) = 3u_1(x) - 6u_2(x)$$
$$\frac{d}{dx}u_2(x) = 4u_1(x) - 7u_2(x)$$

with initial values $u_1(0) = -1$ and $u_2(0) = 1$. What happens to the solution as $x \to \infty$?

Let

$$A = \begin{pmatrix} 3 & -6 \\ 4 & -7 \end{pmatrix}, \quad u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad u_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

With these definitions, the problem is seen to be

$$\frac{d}{dx}u = Au, u(0) = u_0,$$

which has solution $u = e^{Ax}u_0$.

To compute the matrix exponential, we diagonalize A. Since the rows of A add up to -3, we immediately have a first eigenvalue $\lambda_1 = -3$ and a corresponding eigenvector $x_1 = (1\ 1)^t$. The second eigenvalue can be obtained from the trace formula $tr(A) = \lambda_1 + \lambda_2$, so $\lambda_2 = -1$. The matrix is not symmetric, so we need to compute an eigenvector corresponding to λ_2: it will not be orthogonal to x_1. The first row of $A - \lambda_2 I$ is $(4\ -6)$, so we can choose the eigenvector $x_2 = (6\ 4)$, or scaling by $\frac{1}{2}$, $x = (3\ 2)$.

The matrix A therefore diagonalizes as follows

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}^{-1}$$

so that

$$e^{Ax}u_0 = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-3x} & 0 \\ 0 & e^{-x} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -2\,e^{-3x} + 3\,e^{-x} & 3\,e^{-3x} - 3e^{-x} \\ -2\,e^{-3x} + 2\,e^{-x} & 3\,e^{-3x} - 2e^{-x} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 5e^{-3x} - 6e^{-x} \\ 5e^{-3x} - 4e^{-x}. \end{pmatrix}$$

Since both eigenvalues are negative and real, $\lim_{x \to \infty} u(x) = (0\ 0)^t$.

7.11.2   Difference Equations

Example 7.11.2. *System of Difference Equations with constant coefficients*

Obtain the explicit formula for the solution of the following linear iterative system:

$$u^{(k+1)} = u^{(k)} + v^{(k)} + 2w^{(k)}$$
$$v^{(k+1)} = u^{(k)} + 2v^{(k)} + w^{(k)}$$
$$w^{(k+1)} = 2u^{(k)} + v^{(k)} + w^{(k)},$$

with $u^{(0)} = 1$, $v^{(0)} = 0$ and $w^{(0)} = 1$.

   Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad U^{(k)} = \begin{pmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{pmatrix}, \quad U^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

so that

$$U^{(k)} = AU^{(k-1)},$$

which has solution $U^{(k)} = A^k U^{(0)}$.

   To compute the kth power of A, we will try to diagonalize the matrix.

   Since the rows add to 4, A has an eigenvalue $\lambda_1 = 4$, and a corresponding eigenvector $x_1 = (1\ 1\ 1)$.

   The trace of the matrix is $tr(A) = 4$, the determinant $det(A) = -4$, and therefore the characteristic polynomial is given by

$$p(\lambda) = -(\lambda - 4)(\lambda^2 - (tr(A) - \lambda_1)\lambda + \tfrac{1}{\lambda_1} det(A)).$$

Substituting, we get $p(\lambda) = -(\lambda - \lambda_1)(\lambda^2 - 1)$. We see that the remaining eigenvalues are $\lambda_2 = 1$ and $\lambda_3 = -1$.

   The corresponding eigenvectors for λ_2 and λ_3 work out to $x_2 = (1\ -2\ 1)^t$ and $x_3 = (-1\ 0\ 1)^t$.

   Assembling the matrices, we get

$$A^k = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^k \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}^{-1}$$

The solution multiplies out to

$$U^{(k)} = \frac{1}{3} \begin{pmatrix} 2\ 4^k + 1 \\ 2\ 4^k - 2 \\ 2\ 4^k + 1 \end{pmatrix},$$

which diverges as k approaches infinity.

7.11.3   Markov Processes

7.11.4   Mathematical Economics

7.11.5   Quadratic Forms

We now turn to investigate **quadratic forms**, which are homogeneous polynomials[4] of degree two in a number of variables. An example of such an expression is $4x^2 + 6xy - 2yz + z^2$.

Quadratic forms can be expressed using matrices. Given

$$q(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} x_i x_j, \text{ where } x = (x_1 \ x_2 \cdots x_N)^t \qquad (7.20)$$

then $q(x)I_{1\times 1} = x^t A x$, with $A = (a_{ij})$.

For the example given above, we have

$$q(x)I_{1\times 1} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 4 & 6 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 4 & 3 & 0 \\ 3 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Observe that the matrix representation is not unique since a term multiplying $x_i x_y$ for $i \neq j$ occurs with coefficient a_{ij} and a_{ji} in the matrix. The second representation is special since it is symmetric. Observe that the representation can always be symmetrized: given a square matrix A, let

$$\begin{aligned} A_+ &= \frac{1}{2}(A + A^t) \\ A_- &= \frac{1}{2}(A - A^t), \end{aligned} \qquad (7.21)$$

then $A = A_+ + A_-$, where $A_+^t = A_+$ is symmetric, $A_-^t = -A_-$ is skew-symmetric, and $x^t A_- x = 0$ for any vector x.

We will always choose the symmetric matrix representation for reasons that will become clear in the following discussion.

There are two problems associated with quadratic forms that we will consider: let $w = q(x)$. The first problem is to consider the extremum values of this function. The second problem is to analyze the shape of the level curves $q(x) = w_0$ for some constant w_0, i.e., the shape of the intersection of $w = q(x)$ with the hyperplane $w = w_0$.

The key to the analysis is to find an appropriate coordinate system in which Eq (7.20) simplifies. Consider a new system of coordinate axes specified by a

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[4]A homogeneous polynomial is a polynomial whose non-zero terms all have the same degree. The polynomial $x^2 + 3xz + 2y$ is not homogeneous due to the presence of the term $2y$.

set of basis vectors $\{s_i, i = 1, 2, \cdots N\}$. The change in coordinates is then given by $x = S\tilde{x}$, where $S = (s_1 \ s_2 \cdots s_N)$ is the invertible matrix with columns equal to s_i, and $\tilde{x}$ is the coordinate vector representing the original vector x in the new coordinate system. Carrying out the substitution, we obtain $x^t A x = (S\tilde{x}^t)AS\tilde{x} = x^t S^t A S\tilde{x}$. Two matrices A and B are related by $B = S^t A S$ for some invertible matrix A, are said to be **congruent**. Congruent matrices represent the same associated quadratic form with respect to different bases.

Example 7.10.1 ...

Extrema of Quadratic Form

We observe that $x = 0$ is the only critical point. A matrix[5] A of size $N \times N$ is

- **positive definite** if $\langle x, Ax \rangle > 0$ for all non-zero $x \in \mathbb{R}^N$
- **negative definite** if $\langle x, Ax \rangle < 0$ for all non-zero $x \in \mathbb{R}^N$
- **positive semi-definite** if $\langle x, Ax \rangle \geq 0$ for all non-zero $x \in \mathbb{R}^N$
- **negative semi-definite** if $\langle x, Ax \rangle \leq 0$ for all non-zero $x \in \mathbb{R}^N$
- **indefinite** if $\langle x, Ax \rangle$ can take on both positive and negative values

Let A be a symmetric matrix. The following are equivalent:

1. A is positive definite
2. The determinant of each leading principal submatrix of A is positive
3. A can be reduced to row echelon form using only elimination matrices (i.e., no row exchange, no scaling), and the pivots are all positive
4. $A = G^t G$, where G is lower triangular with positive diagonal entries (i.e., A has a Cholesky factorization)

7.12 The Singular Value Decomposition

Another way to approach the problem of diagonalization is to generalize the problem to arbitrary size matrices. Looking at $Ax = b$, we see that if A is size $M \times N$, then $x \in \mathbb{R}^N$ and $b \in \mathbb{R}^M$, i.e., we cannot use the same system of coordinates for both. This leads us to try using two orthonormal coordinate systems, say $x = U\tilde{x}$ and $b = V\tilde{b}$, where U is an orthogonal (or, more generally unitary) matrix of size $N \times N$ and V is an orthogonal (or, more generally unitary) matrix of size $M \times M$. Substituting into $Ax = b$, we obtain $V^t A U\tilde{x} = \tilde{b}$. We would like to choose U and V such that the matrix $\Sigma = V^t A U$ is zero everywhere except for entries on the main diagonal. Such a set of coordinates does indeed exist!

For any matrix A of size $M \times N$, there is a matrix decomposition, called the **Singular Value Decomposition** or **SVD** such that $A = U\Sigma V^t \Leftrightarrow \Sigma = U^t A V$

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[5]Most authors require A to be symmetric. This is not however necessary, since $x^t A x = x^t A_+ x$, where A_+ is the symmetric part of the matrix A.

such that U is an orthogonal matrix of size $M \times M$, V is an orthogonal matrix of size $N \times N$ and Σ is a matrix of size $M \times N$ (the same as A) with non-zero entries located on the main diagonal only (i.e., values $\sigma_i = \Sigma_{ii}$). These non-zero values called the **singular values** of A are positive, as shown below, and by convention are ordered in decreasing order along the diagonal: $\sigma_1 >= \sigma_2 >= \cdots \sigma_r > 0$, where r is the number of non-zero values.

☞ **Note that** $r = rank(A)$, since
$rank(A) = rank(U\Sigma V^t) = rank(\Sigma V^t) = rank(\Sigma^t)$, where each step is justified by the fact that multiplication by an invertible matrix does not change the rank of a given matrix.

7.12.1   Computing the SVD

To compute the SVD, we first note that the Σ matrix has the block structure

$$\Sigma = \left(\begin{array}{cc} \Sigma_r & 0 \\ 0 & 0 \end{array} \right)$$

where Σ_r is a diagonal matrix of size $r \times r$ with non-zero diagonal entries. We thus see that the SVD decomposition can be reduced to $A = U_r \Sigma_r V_r^t$, where U_r and V_r are the submatrices of U of size $M \times r$ and of V of size $N \times r$ resulting from deleting all columns $j > r$, respectively, since all other entries are multiplied by zero:

$$A = \left(\begin{array}{cc} U_r & \tilde{U} \end{array} \right) \left(\begin{array}{cc} \Sigma_r & 0 \\ 0 & 0 \end{array} \right) \left(\begin{array}{c} V_r^t \\ \tilde{V}^t \end{array} \right) \tag{7.22}$$

$$= U_r \Sigma_r V_r^t. \tag{7.23}$$

These equations are known as the full SVD and the reduced SVD respectively. Note that the reduced SVD can be written in outer product form

$$A = \sigma_1 u_1 v_1^t + \sigma_2 u_2 v_2^t + \cdots + \sigma_r u_r v_r^t, \tag{7.24}$$

where the u_i, the first r columns of U are known as left singular vectors, while the v_i, the first r columns of V, are known as the right singular vectors of A.

Starting with the reduced SVD, we find

$$A^t A = V_r \Sigma_r^2 V_r^t$$
$$AA^t = U_r \Sigma_r^2 U_r^t$$

which can be computed as the orthonormal diagonalization of the symmetric matrices AA^t and $A^t A$.

☞ **Matrices of the form $A^t A$ and AA^t are positive semi-definite** since for any eigenvector x with corresponding eigenvalue λ, we have

$$(Ax) \cdot (Ax)I = (Ax)^t(Ax) = x^t((A^tA)x) = x^t\lambda x = \lambda x \cdot xI. \tag{7.25}$$

Using unit length eigenvectors $x \cdot x = 1$, we see that $0 <= ||Ax||^2 = (Ax) \cdot (Ax) = \lambda$.

Since A^tA and AA^t are positive semidefinite, their eigenvalues are real non-negative, and the singular values are the positive square roots of these eigenvalues.

Rewriting the equation $A = U\Sigma V^t \Leftrightarrow AV = U\Sigma \Leftrightarrow AV_r = U_r\Sigma_r \Leftrightarrow U_r = AV_r\Sigma_r^{-1}$, we see that the singular vectors forming the columns of U_r are in the column space of A and thus form a basis for $\mathcal{C}(A)$ (since U is invertible, its columns are linearly independent, and we have the required number of vectors for a basis). Once V_r and Σ_r are computed, the above formula allows us to efficiently compute U_r.

- ☞ **Orthogonality of the vectors**: Let $v_i, i = 1, 2, ..r$ be a set of orthonormal eigenvectors obtained from an orthonormal diagonalization of $A^tA = V_r\Sigma_r^2V_r^t$. The vectors $u_i, i = 1, 2, \cdots r$ computed using $U_r = AV_r\Sigma_r^{-1}$ are orthonormal, as desired. For each column u_i in U_r, we have $u_i = \frac{1}{\sigma_i}Av_i$, so that similar to the derivation of Eq (7.25), $\sigma_i\sigma_j u_i \cdot u_j = (Av_i) \cdot (Av_j) = \sigma_i^2 v_i \cdot v_j = 0$ for $i \neq j$.

  A similar argument holds for the computation of left singular vectors $v_i = \frac{1}{\sigma_i}A^tu_i$.

- ☞ principal axes
  FIX Geometrical Interpretation FIX

An identical argument holds for $A^t = V\Sigma U^t \Leftrightarrow A^tU = V\Sigma \Leftrightarrow A^tU_r = V_r\Sigma_r \Leftrightarrow V_r = A^tU_r\Sigma_r^{-1}$, i.e., the singular vectors in the columns of V_r are in the column space of A^t (and hence in the row space of A), and thus form a basis for $\mathcal{R}(A)$.

- ☞ Since the matrix U is orthogonal, its columns are linearly independent and orthogonal to each other. Since the first r columns are in $\mathcal{C}(A)$, it follows that the remaining columns are in $\mathcal{N}(A^t)$.

- ☞ Similarly the first r columns of V are in $\mathcal{R}(A)$, and the remaining columns are in $\mathcal{N}(A)$.

This observation provides the basis for the computation required to complete U and/or V from the matrices U_r and V_r.

In summary, **computation of the SVD** proceeds as follows: Let A be a matrix of size $M \times N$. Our basic choice is whether to base computations on A^tA or on AA^t. We choose the smaller matrix to reduce the number of computations.

- If $M < N$, computation of the SVD may be carried out as follows:

  1. Compute the orthogonal diagonalization of $AA^t = U_r\Sigma^2U_r^t$ to obtain U, U_r and Σ_r.
  2. Compute $V_r = A^tU_r\Sigma_r^{-1}$

3. If we require the full SVD, compute the remaining columns of V by finding an orthonormal basis of $\mathcal{N}(V^t) = \mathcal{N}(A)$ as described in Example 5.5.3 or in Example 5.5.4.

- If $M > N$, computation of the SVD requires

  1. Compute the orthogonal diagonalization of $A^t A = V_r \Sigma_r^2 V_r^t$ to obtain V, V_r, Σ and Σ_r.
  2. Compute $U_r = A V_r \Sigma_r^{-1}$
  3. If we require the full SVD, compute the remaining columns of U by finding an orthonormal basis of $\mathcal{N}(U^t) = \mathcal{N}(A^t)$ as described in Example 5.5.3 or in Example 5.5.4.

- If $M = N$, each path requires the same number of computations.

7.12.2 Pseudo Inverse

We first encountered the matrix product $A^t A$ when we derived the normal equation and it's properties. Let us briefly investigate this equation again in light of the SVD decomposition of A in the explicit form Eq(7.22), which separates out the bases for the fundamental spaces of the matrix A. In the following computation, we will split a solution x of $A^t A x = A^t b$ into $x = x_{/\!/} + x_\perp$, where $x_{/\!/} \in \mathcal{R}(A)$, the row space of A, and $x_\perp \in \mathcal{N}(A)$, the null space of A. In this derivation, we make use of the fact that the columns of V_r form a basis for the row space and the columns of $\tilde{V}_r$ form a basis of the null space of A, so that $V_r^t x_\perp = 0$ and $\tilde{V}_r^t x_{/\!/} = 0$.

$$
\begin{aligned}
(\xi) \Leftrightarrow \quad & A^t A x = A^t b \\
\Leftrightarrow \quad & V \Sigma^t \Sigma V^t x = V \Sigma^t U^t b \\
\Leftrightarrow \quad & \Sigma^t \Sigma V^t x = \Sigma^t U^t b \\
\Leftrightarrow \quad & \begin{pmatrix} \Sigma_r^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_r^t \\ \tilde{V}^t \end{pmatrix} x = \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_r^t \\ \tilde{U}^t \end{pmatrix} b \\
\Leftrightarrow \quad & \Sigma_r^2 V_r^t x = \Sigma_r U_r^t b \\
\Leftrightarrow \quad & V_r^t x = \Sigma_r^{-1} U_r^t b \\
\Leftrightarrow \quad & V_r^t x_{/\!/} = \Sigma_r^{-1} U_r^t b \\
\Leftrightarrow \quad & \begin{pmatrix} V_r^t \\ \tilde{V}^t \end{pmatrix} x_{/\!/} = \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_r^t \\ \tilde{U}^t \end{pmatrix} b \\
\Leftrightarrow & x_{/\!/} = \begin{pmatrix} V_r & \tilde{V} \end{pmatrix} \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_r^t \\ \tilde{U}^t \end{pmatrix} b \\
\Leftrightarrow & x_{/\!/} = V \Sigma^\dagger U^t b = A^\dagger b,
\end{aligned}
$$

where we have defined the pseudo inverse

$$
A^\dagger = V \Sigma^\dagger U^t, \quad \text{where} \quad \Sigma^\dagger = \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.26}
$$

As with the SVD, the structure of the matrices is such that we may equivalently use the reduced form

$$A^\dagger = V_r \Sigma_r^{-1} U_r^t. \tag{7.27}$$

☞ The reader should carefully note the sizes of the matrices involved, as the 0 submatrices are different sizes. In particular, Σ is the same size as A, and $\Sigma^\dagger$ and $A^\dagger$ are the same size as A^t.

☞ The steps in this derivation must take account of these sizes. We have $V^{-1} = V^t$, but V_r is not square in general, and hence not invertible: $V_r^t V_r = I$, but $V_r V_r^t \neq I$. This necessitates the switch between submatrix and full matrix equations as the derivation proceeds.

☞ If A of size $M \times N$ is full column rank, $r = N$, then $A^t A$ is invertible. In this case $x_\perp = 0$, and the reduced pseudo-inverse may be expressed as

$$A^\dagger = (A^t A)^{-1} A^t. \tag{7.28}$$

☞ If A of size $M \times N$ is full row rank, $r = N$, then a similar argument applies to the matrix A^t, so that

$$A^\dagger = (A A^t)^{-1} A. \tag{7.29}$$

☞ If A is full column rank, the pseudo-inverse computes the unique solution $x = x_\parallel$ of the normal equation. Otherwise, the normal equation has an infinite number of solutions, and $x_\parallel = A^\dagger b$ is the particular solution that lies in the row space of A. The triangle inequality shows that this solution is the shortest possible, i.e.,

$$x_\parallel = \operatorname*{argmin}_{A^t A x = A^t b} \| x \| \tag{7.30}$$
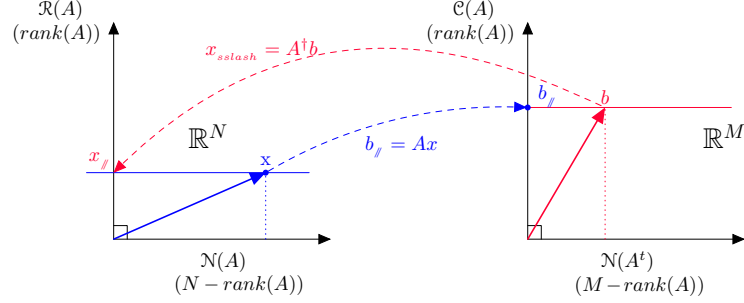
Figure 7.1: The transformation $b_{/\!/} = Ax$ maps a vector $x \in \mathbb{R}^N$ onto the column space $\mathcal{C}(A)$. The transformation $x_{/\!/} = A^\dagger b$ maps a vector $b \in \mathbb{R}^M$ onto the row space $\mathcal{R}(A)$.
The solution $x_{/\!/}$ of $Ax = b_{/\!/}$ is the shortest of all possible solutions.
If we restrict the domain and codomain of $y = Ax$ to the row space and column space of A, the transformation is invertible, and its inverse is given by $x = A^\dagger y$.

Revisiting the figure illustrating the fundamental theorem of linear algebra, we see that the transformations A and $A^\dagger$ perform similar operations

$$Ax = b_{/\!/} \quad and \quad A^\dagger b = x_{/\!/},$$

where $x = x_{/\!/} + x_\perp$ and $b = b_{/\!/} + b_\perp$ have been split into two components in the row and null space of A and the row and null space of A^t respectively. (See Figure 7.1).

Example 7.12.1. *SVD and Pseudo-inverse for a* 1×3 *matrix*

Compute the SVD and the pseudo-inverse $A = U\Sigma V^t$ for the matrix $A = \begin{pmatrix} 4 & -3 & 0 \end{pmatrix}$.

The product $A^t A$ has size 3×3, while the product AA^t has size 1×1. We therefore choose to start with $AA^t = (25)$. The computation thus involves i) Computing the orthogonal eigendecomposition of AA^t, resulting in U and Σ since $AA^t = U\Sigma\Sigma^t U^t = U_r \Sigma_r^2 U_r^t$ ii) Computing the singular vectors in V from the first $r = rank(A)$ column vectors in U using $v_i = \frac{1}{\sigma_i} A^t u_i$ since $A^t U_r = V_r \Sigma_r$ iii) Computing the remaining columns of V by finding an orthonormal basis for the left null space of $(v_1 \; v_2 \; \cdots v_r)$. .

Step i) Compute the eigendecomposition of AA^t to find U and Σ. The eigenvalue of AA^t is $\lambda = 25$, with eigenvector $u = (1)$. Combining these results in a table, we find

λ	**25**
multiplicity	*1*
$\sigma = \sqrt{(\lambda)}$	*5*
basis	*(1)*
orthonormal basis	*(1)*
U	*(1)*
Σ *of size* 1×3	$\begin{pmatrix} 5 & 0 & 0 \end{pmatrix}$

since Σ is the same size as A.

We note that $r = rank(A) = 1$, the total number of non-zero entries in Σ.

Step ii) Computing v_1

$$v_1 = \frac{1}{\sigma_1} A^t u_1 = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} (1) = \frac{1}{5} \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

Step iii) Computing v_2 and v_3. We need to find the and orthonormal basis for the null space of $(4 \; -3 \; 0)$. (The length of this vector will be scaled out in this computation, hence we rescaled the v_1 vector by a factor 5 to remove the fractions. Since this matrix has a single non-zero row, a basis for its null space is trivial to write down as shown in Ex 7.2.11. We find $w_1 = \begin{pmatrix} 3 & 4 & 0 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

We need an orthonormal base however, so we have to use QR on these two vectors, resulting in $v_2^t = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \end{pmatrix}$ and $v_3^t = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

Combining all our results, we obtain the SVD and the reduced SVD

$$A = (1) \begin{pmatrix} 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = (1) \begin{pmatrix} 5 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \end{pmatrix}$$

The pseudo-inverse is given by

$$A^\dagger = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} \end{pmatrix} (1) = \frac{1}{25} \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

Example 7.12.2. *SVD of a 4×2 matrix*
Compute the SVD of $A = U\Sigma V^t$ for the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$$

We first compute $A^t A = V\Sigma^t \Sigma V^t$

$$A^t A = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}.$$

Step i) *Compute the eigendecomposition of $A^t A$ to find V and Σ. The characteristic polynomial is given by $p(\lambda) = (\lambda - 2)(\lambda - 7)$. The eigenvector basis for $\lambda = 2$ is $u_1 = \begin{pmatrix} -2 & 1 \end{pmatrix}$. For $\lambda = 7$ we know the eigenvectors are orthogonal to u_1, so we can choose $u_1 = \begin{pmatrix} 1 & 2 \end{pmatrix}$ by inspection. Combining these results in a table, taking care to **order the eigenvectors by decreasing absolute magnitude**, we find*

λ	**7**	**2**
multiplicity	1	1
$\sigma = \sqrt{\lambda}$	$\sqrt{7}$	$\sqrt{2}$
basis	$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
orthonormal basis	$\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
V	$\frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$	
Σ *of size 4×2*	$\begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$	

since Σ is the same size as A.

We note that the rank $r = rank(A) = 2$, the total number of non-zero values in Σ.

Step ii) *Computing $U_r = AV_r \Sigma_r^{-1}$*

$$U_r = AV_r \Sigma_r^{-1} = \begin{pmatrix} -\frac{4}{\sqrt{35}} & \frac{2}{\sqrt{10}} \\ -\frac{3}{\sqrt{35}} & -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{35}} & \frac{2}{\sqrt{10}} \end{pmatrix}$$

At this point, we have obtained the reduced SVD

$$A = U_r \Sigma_r V_r,$$

where Σ_r is the principal submatrix of size 2×2 of Σ.

Example 7.12.3. *SVD of a* 4×2 *matrix continued*
Step iii) *Completing U: we need to expand U_r to a square matrix by adding a set of orthonormal vectors perpendicular to the columns of U_r, i.e., a set of orthonormal basis vectors for the null space of U_r^t. For the null space of U_r^t, we find*

$$\mathcal{N}(U_r^t) = span\{(1\ 0\ 2\ -2),(0\ 1\ 1\ 0)\}$$

Since we need an orthonormal base, we have to use QR on these two vectors, resulting in

$$U = \begin{pmatrix} -\frac{4}{\sqrt{35}} & \frac{2}{\sqrt{10}} & \frac{1}{3} & -\frac{2}{3\sqrt{14}} \\ -\frac{3}{\sqrt{35}} & -\frac{1}{\sqrt{10}} & 0 & \frac{9}{3\sqrt{14}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & \frac{2}{3} & \frac{5}{3\sqrt{14}} \\ \frac{1}{\sqrt{35}} & \frac{2}{\sqrt{10}} & -\frac{2}{3} & \frac{4}{3\sqrt{14}} \end{pmatrix}$$

Combining all our results, we obtain the SVD and the reduced SVD

$$A = \begin{pmatrix} -\frac{4}{\sqrt{35}} & \frac{2}{\sqrt{10}} & \frac{1}{3} & -\frac{2}{3\sqrt{14}} \\ -\frac{3}{\sqrt{35}} & -\frac{1}{\sqrt{10}} & 0 & \frac{9}{3\sqrt{14}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & \frac{2}{3} & \frac{5}{3\sqrt{14}} \\ \frac{1}{\sqrt{35}} & \frac{2}{\sqrt{10}} & -\frac{2}{3} & \frac{4}{3\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{\sqrt{35}} & \frac{2}{\sqrt{10}} \\ -\frac{3}{\sqrt{35}} & -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{35}} & \frac{2}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

The pseudo-inverse and reduced pseudo-inverse are given by

$$A^\dagger = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{7}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{4}{\sqrt{35}} & -\frac{3}{\sqrt{35}} & \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{35}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ \frac{1}{3} & 0 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3\sqrt{14}} & \frac{9}{3\sqrt{14}} & \frac{5}{3\sqrt{14}} & \frac{4}{3\sqrt{14}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{7}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{4}{\sqrt{35}} & -\frac{3}{\sqrt{35}} & \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{35}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{pmatrix}$$

This multiplies out to

$$A^\dagger = \frac{1}{14} \begin{pmatrix} 6 & 1 & -1 & 2 \\ 4 & -4 & 4 & 6 \end{pmatrix}$$

Since A is full column rank, the matrix $A^t A$ is invertible, and the pseudo-inverse may be computed more directly by $A^\dagger = (A^t A)^{-1} A^t$, which we saw previously when solving the normal equation for full column rank matrices A. Specifically, we found that $A^t A x = A^t b \Leftrightarrow x = (A^t A)^{-1} A^t b = A^\dagger b$.

7.13   Minimum Principles

Another solution method for $Ax = b$ and $Ax = \lambda x$ is to use minimum principles. These lead to the formulation of iterative algorithms such as the **steepest descent** method, which we will not investigate further. Instead, we will confine ourselves to establishing the objective functions that lead to these methods, and look at some of the consequences.

7.13.1   Minimum Principle for $Ax = b$

The minimum of the parabola $p(x) = \frac{1}{2}ax^2 - bx$ is $min(p(x) = -\frac{b^2}{2a}$, and occurs at the critical point x where $ax = b$.

This can be generalized for positive definite matrices A, for which

$$P(x) = \frac{1}{2}<x, Ax> - <x, b> \tag{7.31a}$$

$$\min_x P(x) = -\frac{1}{2}<b, A^{-1}b> \quad \text{for } x = A^{-1}b, \tag{7.31b}$$

i.e., we can solve $Ax = b$ by finding the argmin of $P(x)$.

To prove this let x be the solution of $Ax = b$ and consider
$P(y) - P(x) = \frac{1}{2}<y - x, A(y - x)>$, which is non-negative, and zero only when $y - x = 0$.

- ☞ A minimization problem with the added constraint $Cx = d$ can be solved using **Lagrange multipliers**, resulting in the objective function $L(x, y) = P(x) + y^t(Cx - d)$ which we minimize over x and y.

- ☞ We can now recognize that the least squares problem has this form: it minimizes $E^2(x) = \|Ax - b\| = x^t A^t Ax - 2x^t A^t b + b^t b$. This minimum occurs at x such that $A^t Ax = A^t b$, i.e., at a solution of the normal equations.

7.13.2   Minimum Principle for $Ax = \lambda x$

The eigenvector/eigenvalue problem for a symmetric matrix A of size $N \times N$ can similarly be expressed in terms of extrema of the **Rayleigh coefficient**

$$R(x) = \frac{<x, Ax>}{<x, x>} = <\hat{x}, A\hat{x}> \tag{7.32a}$$

$$\min_x R(x) = \lambda_1, \tag{7.32b}$$

$$\max_x R(x) = \lambda_N, \tag{7.32c}$$

where $\lambda_1 \le \lambda_2 \cdots \le \lambda_N$ are the eigenvalues of A.

To see this, orthogonally diagonalize the matrix $A = Q^t \Lambda Q$, and arrange the eigenvalues in Λ in increasing order: $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. Changing the

system of coordinates to the eigenvector basis $x = Q\tilde{x}$, the Rayleigh coefficient with respect to the coordinate vector $\tilde{x}$ is given by

$$R(\tilde{x}) = \frac{<Q\tilde{x}, AQ\tilde{x}>}{<Q\tilde{x}, Q\tilde{x}>} = \frac{\lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \cdots + \lambda_N \tilde{x}_N^2}{\tilde{x}_1^2 + \tilde{x}_2^2 + \cdots + \tilde{x}_N^2}. \tag{7.33}$$

The result follows by noting that

$$\lambda_1 \left(\tilde{x}_1^2 + \tilde{x}_2^2 + \cdots + \tilde{x}_N^2 \right) \leq \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \cdots + \lambda_N \tilde{x}_N^2 \leq \lambda_N \left(\tilde{x}_1^2 + \tilde{x}_2^2 + \cdots + \tilde{x}_N^2 \right),$$

and substituting the standard basis vectors $\tilde{x} = e_1$ and $\tilde{x} = e_N$ respectively. We further note that if we restrict $x \in span\{q_i, q_{i+1}, \cdots q_j\}$ for some choice of indices $1 \leq i \leq j \leq N$, we have $\lambda_i \leq R(x) \leq \lambda_j$.

The intermediate eigenvectors $q_2, \cdots q_{N-1}$ are saddle points of $R(x)$. We need a constraint on x to guarantee the sign of $R(x) - \lambda_k$ in order to get a minimum principle. The key is to look at two subspaces $U_k = span\{q_1, q_2, \cdots q_k\}$ and $V_k = span{q_k, q_{k+1}, q_N}$, and an arbitrary k-dimensional subspace S_k. Since by a simple dimensional argument $S_k \cap V_k$ is not empty, we have $\max_{x \in S_k} R(x) \geq \lambda_k$. If in particular we choose $S_k = U_k$, we have $\min_{x \in U_k} R(x) \leq \lambda_2$, with $x = q_k$ achieving this bound. It follows that

$$\min_{S_k} \max_{x \in S_k} R(x) = \lambda_k, \tag{7.34}$$

which is known as the **Min-Max theorem**. Note that it allows us to estimate the kth eigenvalue without requiring any knowledge of the other eigenvalues.

☞ By substituting the standard basis vectors $x = e_i$ into the Rayleigh coefficient we find that $R(e_i) = <e_i, Ae_i> = a_{ii}$. Since $\lambda_1 \leq R(x) \leq \lambda_N$, we see that **the diagonal entries of a symmetric matrix satisfy** $\lambda_1 \leq a_{ii} \leq \lambda_N$.

FIX add eigenvalue interleaving FIX

Principal Component Analysis

FIX fix the text FIX

A problem known to statisticians as **principal component analysis** Consider a set of data points $x_i, i = 1, 2, \cdots M$, where each point is an N-dimensional vector, with zero empirical mean (i.e., the mean of the original data points has been subtracted out).

We want to find a new orthogonal coordinate system such that the components of the data points along the first axis have the largest possible variance (i.e, account for as much of the variability in the data as possible). For each of the remaining axes in turn, we again require the vector components along those axes to have the highest variance possible.

Let $q_1, q_2, \cdots q_N$ be the desired orthonormal basis vectors, with $Q = (q_1 \; q_2 \; \cdots q_N)$, and let $x_i = Q\tilde{x}_i$. We want to obtain

$$q_1 = \operatorname*{argmax}_{\|q\|=1} \sum_{i=1}^{M} \tilde{x}_{i1}^2 = \operatorname*{argmax}_{\|q\|=1} \sum_{i=1}^{M} (x_i \cdot q)^2.$$

To rewrite this in matrix notation, we define the matrix $X = (x_1 \; x_2 \; \dots x_M)^t$, i.e. each of the data points appears as a row of the matrix X, and note that $\sum_{i=1}^{M} (x_i \cdot q)^2 = \|Xq\|^2 = \, < q, X^t X q >$. Substituting into the expression for q_1, we get

$$q_1 = \operatorname*{argmax}_{\|q\|=1} \, <q, X^t X q> = \operatorname*{argmax}_{w \neq 0} \frac{<w, X^t X w>}{<w, w>},$$

where the expression on the right is the Rayleigh coefficient for the matrix $X^t X$. It follows that q_1 is an eigenvector for the largest eigenvalue of $X^t X$.

The remaining components work out to a complete set of orthonormal eigenvectors of $X^t X$ ordered such that the associated eigenvalues decrease in magnitude.

Example 7.13.1. *Principal component analysis*
Consider the data points x, y shown in Table 7.1. The data were preprocessed by subtracting out the mean values, so that $mean(x) = 0$, $mean(y) = 0$.
Setting $X = (x \; y)$, we obtain

$$X^t X = \begin{pmatrix} 6.07348 & 8.16922 \\ 8.16922 & 12.898655 \end{pmatrix}.$$

which has eigenvalues and orthonormal eigenvectors

$$\lambda_1 \approx 18.34 \qquad q_1 \approx (0.55 \; 0.83)$$
$$\lambda_2 \approx 0.63 \qquad q_1 \approx (-0.83 \; 0.55)$$

x	y
-0.314	-0.7385
-0.324	-0.4585
-1.024	-0.9985
-0.084	0.5415
0.016	0.3615
-0.274	-0.5185
0.586	0.7415
-0.624	-1.2685
0.746	0.8015
0.756	0.6915
-0.724	-0.7885
0.326	0.5515
0.316	0.5815
0.266	-0.1685
0.726	1.3015
0.206	0.1615
-1.124	-1.7285
0.086	-0.1585
-0.024	0.0215
0.486	1.0715

Table 7.1: A set of x,y data points with zero empirical mean.
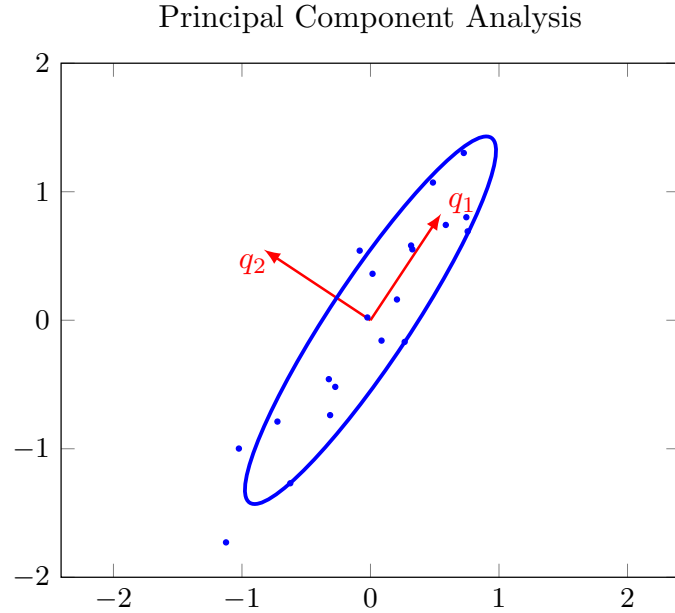
Figure 7.2: Principal Component Analysis. The eigenvectors of the data in Table 7.1 show the directions that maximize the variance of the data. Associated concentration ellipsoids depend on the eigenvalues and the assumptions about the distributions of the data.

7.14   Exercises

Exercise 7.1. *i) Prove the following statement: If A is diagonalizable, the $rank(A)$ is equal to the number of non-zero eigenvalues.*

ii) Show that a non-diagonalizable matrix A can have a different rank than the number of non-zero eigenvalues by giving a 2×2 counterexample.

Exercise 7.2. *Let A be a square matrix, let (λ_l, x_l) and (λ_k, x_k) be eigenpairs of A, and consider the chains of generalized eigenvectors obtained from these eigenpairs. Use Eq (7.13) to show that i) the vectors in a given chain of generalized eigenvectors are linearly independent. ii) any two vectors from the two chains of generalized eigenvectors are linearly independent if $\lambda_l \neq \lambda_k$. iii) any two vectors from the two chains of generalized eigenvectors are linearly independent if x_l and x_k are linearly independent.*

Exercise 7.3. *Let A and B be two matrices in $\mathbb{C}^{N \times N}$. Show that $(AB)^H = B^H A^H$.*

Exercise 7.4. *Let A be a symmetric matrix in $\mathbb{R}^{N \times N}$.*

i) Show that for any complex vector x, the expression $x^H A x$ is real by computing $(x^H A x)^H$.

ii) Use this result to show that the eigenvalues of $Ax = \lambda x$ are real.

iii) Consider two eigenvectors x_1 and x_2 of A with respective eigenvalues λ_1 and λ_2.

Compute $(x_2^t A)x_1 - x_2^t(Ax_1)$ to show that $x_1 \cdot x_2 = 0$ when $\lambda_1 \neq \lambda_2$.

Exercise 7.5. *Consider $p(x_1, x_2) = x^2 + 2xy + y^2$, and its matrix representations $p(x)I_{1 \times 1} = x^t A x = x^t B x$, where*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ and \ B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

i) Sketch the function $z = p(x)$. ii) Show that the matrices A and B do not have the same signature. iii) Verify that the matrices A and B are not congruent.

Exercise 7.6. *Derive Eq (7.28) and Eq (7.29) by direct substitution of the definition of the SVD and its pseudo-inverse.*

Exercise 7.7. *For a given matrix A with SVD $A = U\Sigma V^t$, show that $A^\dagger A = V_r V_r^t$ and $AA^\dagger = U_r U_r^t$, i.e., these matrices are projection matrices into $\mathcal{C}(A)$ and $\mathcal{R}(A)$ respectively.*

Appendix A

Givens Rotations and Householder Transformations

The solution of a system of equation $Ax = b$ studied in Chapter 2 relied on the construction of invertible matrices that were used to systematically eliminate variables in the system of equations. In this appendix, we will explore this idea further.

One way to proceed is to explore the geometric meaning of the elimination matrix E applied to the vector $(a\ b)^t$: it takes the vector and orthogonally projects it onto the vector $(1\ 0)^t$. Instead, one may consider rotating the vector (Given's Rotation) or mirroring the vector about an axis (Householder transform) as shown in Figure A.1.
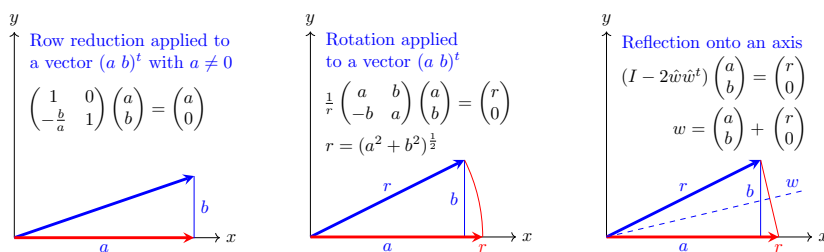
Figure A.1: Three operations that introduce zero entries in a vector:
i) a row reduction projects a vector orthogonally onto a coordinate axis. ii) a rotation and iii) a reflection transform the vector into a vector along the chosen coordinate axis. Note that the row reduction changes the length of the vector, while the rotation and the reflection do not.

A.1 Givens Rotations

FIX add FIX

A.2 Householder Transforms

FIX add FIX

A.3 Systems of Linear Diophantine Equations

Diophantine equations are algebraic equations with two or more variables whose coefficients are integers, for which one wishes to determine all integral solutions.

We will look at systems of linear Diophantine equations: $Ax = b$. Thus the entries of A and b are integers, and we want to find all solutions x whose entries are integers. If we naively try to solve such a system using the Gaussian algorithm, we find solutions in terms of fractions multiplied by arbitrary parameters. The attempt to find parameters that yield integer solutions leads to another system of linear Diophantine equations. We need another method!

A solution can be formulated in terms of an elimination algorithm that only uses integers.

A.3.1 Euclid's Algorithm in Matrix Form

Given two positive numbers $r_1 \geq r_2 > 0$, and the remainder r_3 of the division of r_1 by r_2, we have

$$r_1 = q_1 r_2 + r_3, \text{where } 0 \leq r_3 < r_2$$

for some integer q_1.

If we express the above in matrix form, we have

$$\begin{pmatrix} r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -q_1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \text{ where} \tag{A.1}$$

Note that this matrix is invertible, and that the inverse has integral entries:

$$\begin{pmatrix} 0 & 1 \\ 1 & -q_1 \end{pmatrix}^{-1} = \begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.2}$$

The process can be repeated, yielding a sequence of decreasing non-negative numbers $r_1 \geq r_2 > r_3 \cdots r_k > 0$ that must therefore terminate with 0 after a finite number of steps, say k. Starting from r_1, r_2, we have an iterative system such that at each step $1 \leq i \leq k$, we have

$$r_i = q_i r_{i+1} + r_{i+2}, \text{where } 0 \leq r_{i+3} < r_{i+2}$$

with $r_{k+1} = 0$.

In matrix form, this yields

$$\begin{pmatrix} r_k \\ 0 \end{pmatrix} = E_k \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \quad \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = E_k^{-1} \begin{pmatrix} r_k \\ 0 \end{pmatrix},$$

where the invertible matrix E_k is given by

$$E_k = \begin{pmatrix} 0 & 1 \\ 1 & -q_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -q_{k-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -q_1 \end{pmatrix}$$

$$E_k^{-1} = \begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_k & 1 \\ 1 & 0 \end{pmatrix} \tag{A.3}$$

We have obtained a matrix E_k and its inverse, both with integer entries, that can be used to zero an entry in a vector. To lift the restriction $r_1 \geq r_2 > 0$, it is sufficient to note that we may start by scaling either entry with -1 and permute the rows if necessary.

Eq(A.3) is known as Euclid's algorithm for finding the greatest common divisor of two given numbers r_1 and r_2, namely $gcd(r_1, r_2) = r_k$, the last non-zero remainder in the sequence.

Example A.3.1. *Euclid's algorithm*
Let $v = (-245 \quad 1575)^t$. *To obtain the desired elimination matrix, we take absolute values and apply Euclid's algorithm to* $r_1 = 1575, r_2 = 245$:

$$
\begin{aligned}
1575 &= 6 \cdot \mathbf{245} &+\mathbf{105} \\
\mathbf{245} &= 2 \cdot \mathbf{105} &+\mathbf{35} \\
\mathbf{105} &= 3 \cdot \mathbf{35},
\end{aligned}
$$

where the colorization is intended to show the role of individual r_i *in successive iterations.*
Transcribing to matrix form Eq(A.3), we find

$$
\begin{pmatrix} 35 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -245 \\ 1575 \end{pmatrix}
$$

with the inclusion of a row interchange and multiplication by -1. *Multiplying out the matrices, we find the delimitation matrix E and it's inverse*

$$
\begin{pmatrix} 35 \\ 0 \end{pmatrix} = E \begin{pmatrix} -245 \\ 1575 \end{pmatrix} \text{ where } E = \begin{pmatrix} -13 & -2 \\ 45 & 7 \end{pmatrix} \text{ and } E^{-1} = \begin{pmatrix} -7 & -2 \\ 45 & 13 \end{pmatrix}.
$$

Euclid's algorithm is sufficient to find all solutions of a single linear Diophantine equation. Continuing the previous example,

Example A.3.2. ***A linear Diophantine equation***
Find all integer solutions of $-245x + 1575y = 70$.
    We rewrite the equation in matrix form

$$\begin{pmatrix} -245 & 1575 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 70 \end{pmatrix}$$

and use the elimination matrix $V = E^t$ derived in Example A.3.1 above. Introducing the change of variables $(X \quad Y)^t = V^{-1}(x \quad y)^t$, we obtain

$$\begin{aligned} \begin{pmatrix} 70 \end{pmatrix} &= \begin{pmatrix} -245 & 1575 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} -245 & 1575 \end{pmatrix} V V^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 35 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &= \begin{pmatrix} 35X \end{pmatrix} \end{aligned}$$

which has integral solutions $X = 2, Y = n$ for any integer n. Rewriting in our original set of variables yields the solutions

$$\begin{aligned} x &= -26 + 45n \\ y &= -4 + 7n, \quad n \in \mathbb{Z}_2. \end{aligned}$$

A.3.2   The solution of a linear Diophantine system

It is not obvious that we can use elimination matrices to diagonalize a given matrix A, since the elimination matrices have no zero elements in general as seen in Example A.3.1. The problem is that eliminating terms in a column may reintroduce terms in a row and vice versa as shown in the next example.

It can however be shown that that using the diagonal entry (i, i) as the pivot element and alternatively eliminating off-diagonal terms in row i to the right of the pivot followed by eliminating off-diagonal terms in column i below the pivot will eventually terminate after a finite number of steps with all zero values for these terms.

Example A.3.3. *Zeroing to the right and below a pivot*
Consider the matrix

$$A = \begin{pmatrix} -245 & 75 \\ 1575 & 25 \end{pmatrix}.$$

To zero the entries a_{12} and a_{21}, begin by applying the elimination matrix $E_1 = E$ derived in Example A.3.1 above.

$$E_1 = \begin{pmatrix} -13 & -2 \\ 45 & 7 \end{pmatrix}, \qquad E_1 A = \begin{pmatrix} 35 & -1025 \\ 0 & 3550 \end{pmatrix}.$$

To eliminate the value -1025 in the first row, we use the Euclidean algorithm on the first row to obtain an elimination matrix F_1 and compute

$$F_1 = \begin{pmatrix} 88 & -205 \\ 3 & -7 \end{pmatrix}, \qquad E_1 A F_1 = \begin{pmatrix} 5 & 0 \\ 10650 & -24850 \end{pmatrix}.$$

Switching back to eliminating the entry 10650 in the first column, we obtain an elimination matrix E_2 and compute

$$E_2 = \begin{pmatrix} 1 & 0 \\ -2130 & 1 \end{pmatrix}, \qquad E_2 E_1 A F_1 = \begin{pmatrix} 5 & 0 \\ 0 & -24850 \end{pmatrix}.$$

By construction each of the matrices E_i and F_i have determinant $+1$ or -1, and are therefore invertible. Further, their inverses have integer entries as do their products $U = \prod_{i=n}^{1} E_i$ and $V = \prod_{i=1}^{m} F_i$.

Unimodular matrices are square integer matrices invertible over the integers, i.e., matrices with determinant equal to 1 or -1.[1]

☞ The process outlined in the above example results in a **matrix decomposition** $A = UDV$ where the matrices U and V are unimodular, and D is diagonal with integer entries.

For completeness, we mention that additional unimodular transformations may be used to resize and reorder the nonzero coefficients of the diagonal matrix D of the form $D = diag(d_1, d_2, \cdots d_k, 0, \cdots 0)$ such that the non-zero d_i terms form an increasing sequence $0 < d_1 \leq d_2 \leq \cdots \leq d_k$ such that d_i divides d_{i+1} for each $i < k$, yielding a unique representation known as the **Smith normal form** of the matrix.

Indeed, given an integer 2×2 diagonal matrix D and an elimination matrix E for $(d_{11}, d_{22})^t$ constructed with the Euclidean algorithm,

$$D = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \; E = \begin{pmatrix} x & y \\ \beta & -\alpha \end{pmatrix} \text{ such that } E \begin{pmatrix} d_{11} \\ d_{22} \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$$

---

[1] **FIX reference the formula for the inverse FIX**

we apply the following unimodular matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & -\alpha\beta d \end{pmatrix}.$$

The generalization to matrices larger than 2×2 proceeds in the same way as encountered earlier for Gaussian elimination: the entries of the 2×2 unimodular matrices replace the appropriate entries in an identity matrix such that they multiply the desired entries in the matrix of interest.

Given a Diophantine linear system $Ax = b$, the solution proceeds by successively zeroing off-diagonal entries in each row i and column i, starting from $i = 1$. This process yields two unimodular matrices E and F such that the product $D = EAF$ is diagonal. We choose $U = E^{-1}$ and $V = F^{-1}$ to achieve our decomposition.

To find all possible integer solutions of the system it suffices to rewrite it as follows

$$\begin{aligned} Ax = b \Leftrightarrow \quad EAFF^{-1}x \quad &= Eb \\ \Leftrightarrow \qquad Dy \qquad &= Eb, \end{aligned} \tag{A.4}$$

where we have set $y = F^{-1}x \Leftrightarrow x = Fy$. If there are no integer solutions for y, the Diophantine system is seen to have no solutions. Note that the matrix E multiplies both A and b from the right, suggesting we augment A with b for eliminations in a column. Further, we require the matrix F both for eliminations within a row, and to recover x from the solution y of the diagonal system, suggesting augmenting A with the identity matrix to keep track of the matrix F. The details of a suitable computational layout are left to the reader.

Example A.3.4. *A Diophantine linear system example*
Consider the Diophantine linear system $Ax = b$ with

$$A = \begin{pmatrix} 29 & 66 & 74 \\ 7 & 16 & 18 \end{pmatrix}, \quad b = \begin{pmatrix} 95 \\ 23 \end{pmatrix}.$$

We begin by eliminating in the first column

$$E_1 = \begin{pmatrix} 1 & -4 \\ -7 & 29 \end{pmatrix}, \quad E_1 A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}, \quad E_1 b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Next eliminate in the first row

$$F_1 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_1 A F_1 F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \end{pmatrix},$$

We are done with the first row and column. In the second row and column, we only need an elimination in the second row.

$$F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = E_1 A F_1 F_2 F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Solving the resulting system $Dy = E_1 b$, $x = F_1 F_2 F_3 y$, we finally obtain

$$y = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + n \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad \text{for any integer } n.$$

Appendix B

Schur Components

One question we may ask about Gaussian elimination is whether it might be possible to eliminate entries in more than one column at a time, i.e., partitioning the matrix and eliminating whole submatrices.

B.1   Elimination in a set of columns

Assume a given square matrix V can be partitioned such that

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is invertible.[1] Note that while this implies that A and D are square matrices, B and C need not be.

We now follow the steps that led to the LDU decomposition, using submatrices instead of individual entries. Readers are invited to carefully check the matrix sizes to verify that the multiplications used are well defined. Using the same computational layout as in chapter 2, we have

$$\begin{pmatrix} \mathbf{A} & B \\ C & D \end{pmatrix}$$ Choose A as the pivot matrix. Note A^{-1} must exist.

$$\begin{pmatrix} I & 0 \\ -\mathbf{CA^{-1}} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$ The elimination matrix is trivially invertible.

Rewriting the above equation in standard notation, solving for V and pulling

---

[1]The invertibility requirement of A should not be a surprise: since we intend to use A to eliminate terms in each of the columns underneath, we must have a pivot value in every one of its columns.

out the pivot terms, we get

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & S_A \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & S_A \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \qquad \text{(B.1)}$$

where we have defined the **Schur component**

$$S_A = D - CA^{-1}B. \qquad \text{(B.2)}$$

Inverting Eq(B.1), we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & S_A{}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \qquad \text{(B.3)}$$

provided S_A is invertible.

This matrix decomposition shows how the inverse of a matrix can be built up from its principal submatrices. Two special cases that may be of interest result from setting either $B = 0$ or $D = 0$ in Eq(B.1). The results are

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

which show the structure of the inverse of block triangular matrices.

B.2   A useful identity

If we assume D is invertible, we may proceed as in the previous section to eliminate entries in B in the matrix V:

$$\begin{pmatrix} A & B \\ C & \mathbf{D} \end{pmatrix} \qquad \begin{array}{l}\text{Choose } D \text{ as the pivot} \\ \text{matrix.} \quad \text{Note } D^{-1} \\ \text{must exist.}\end{array}$$

$$\begin{pmatrix} I & \mathbf{-BD^{-1}} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix} \qquad \begin{array}{l}\text{The elimination ma-} \\ \text{trix is trivially invert-} \\ \text{ible.}\end{array}$$

We get equations similar to Eqs(B.1,B.2,B.1):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}$$

with **Schur component**

$$S_D = A - BD^{-1}C$$

and inverse

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} \begin{pmatrix} S_D{}^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \tag{B.4}$$

provided S_D is invertible.

Expanding the right hand sides of Eqs(B.1,B.2) and equating entries yields the matrix inversion lemma

$$S_D^{-1} = A^{-1} + A^{-1}BS_A^{-1}CA^{-1} \tag{B.5}$$

and its corollary

$$DS_A^{-1}C = CS_D^{-1}A.$$

Substituting $A = -I$ and $D = \sigma^2 I$ in Eq(B.5) yields the very useful special case

$$C\left(BC + \sigma^2 I\right)^{-1} = \left(CB + \sigma^2 I\right)^{-1}C \tag{B.6}$$

for any non-zero scalar σ. Assuming the matrix sizes are $M \times N$ for B and $N \times M$ for C, with $M > N$, it allows us to convert the left hand side matrix inverse problem of size $M \times M$ to a matrix inverse problem of size $N \times N$.

Appendix C

The Computational Layout and Latex

The alignments of matrix entries across a multiple matrices in LaTeXcan be very difficult. The computational layouts in the present text were produced with the LaTeXeasymat and easybmat packages, with hand-tuning of the heights and widths to approximate the desired output representation.

To facilitate writing, the latex equation representation of a given computational layout was generated with SAGE[1]

A given computational layout example is represented by a Sage object of class MatFmt with a latex representation in the desired form. It is defined as follows

```
import sage.misc.latex

class MatFmt:
  def __init__(self,w,h,specs,list_of_matrix_pairs):
    self.w     = w
    self.h     = h
    self.l     = list_of_matrix_pairs
    self.specs = specs
  # ----------------------------------------------------------
  def _putMAT(self,w,h,mat,spec):
    r"""
    Return latex MAT representation of matrix mat.
    w and h are the width end height for a cell in cm

    EXAMPLES:
    """
    if mat == None : return "% None"
```

---

[1]SAGE Mathematical Software,

```
nr = mat.nrows()
nc = mat.ncols()

if nr == 0 or nc == 0:
  return \
    "\\left(\\begin{MAT}(b,%scm,%scm){}\\end{MAT}\\right)" \
    %(str(w),str(h))

S    = mat.list()
rows = []
m    = 0

row_divs, col_divs = mat.get_subdivisions()
# compute rows
latex = sage.misc.latex.latex
for r in range( 0, nr):
  s = ""
  for c in range( 0, nc):
    if c == nc-1: sep=""
    else:         sep=" & "
    entry = latex(S[r*nc+c])
    if c == 0: m = max(m, len(entry))
    s = s + entry + sep
  rows.append(s)

# Put brackets around in a single string
tmp = []
for row in rows: tmp.append(str(row))
s = " \\\\\n".join(tmp) + "\\\\\n"

if spec == None or len(spec) < nc:
  tmp = ['r'*(b-a) for a,b in \
          zip([0]+col_divs, col_divs+[nc])]
else:
  tmp = [spec[a:b] for a,b in \
          zip([0]+col_divs, col_divs+[nc])]
format = '0'.join(tmp)
geom   = "(b,%3.2fcm,%3.2fcm)" %(w,h)
return "\\left(\\begin{MAT}%s{%s}\n" %(geom,format) \
        + s +"\\end{MAT}\\right)"

# -----------------------------------------------------------
def _putBMATrow( self,w,h,specs,l ):
  """
  generate a row of the BMAT with
```

```
      MAT entry widths  w=(w1,w2,w3), entry height h
      and tuple l = ( MAT 1, MAT 2, text )
      """
      w1,w2,w3=w
      m1,m2,txt=l
      if txt == None: txt = ""
      if w3 == 0:
        b_fmt  = "{ll}"
        parbox = "\n \\\\"
      else:
        b_fmt  = "{lll}"
        parbox = "\n & \\parbox{%3.2gcm}{%s}\n \\\\ %% ----" \
                  %(w3,txt)

      spec1,spec2 = specs
      return self._putMAT(w1,h,m1,spec1)+"\n & " \
              +self._putMAT(w2,h,m2,spec2)+parbox

   # -------------------------------------------------------------
   def _putBMAT( self,w,h,specs,list_of_pairs ):
     r"""
     Return latex BMAT representation of matrix top
     followed by a list of matrix pairs.

     w=(w1,w2,w3) and h are the widths and height for a cell in cm

     EXAMPLES:
     """
     nc = len(list_of_pairs)
     _,_,w3=w
     if w3 == 0: b_fmt = "{ll}"
     else:       b_fmt = "{lll}"

     b1 = "\\begin{BMAT}" + b_fmt + "{" + 'c'*nc + "}\n"

     l = [ self._putBMATrow(w,h,specs,mr) for mr in list_of_pairs]

     return b1 + "\n".join(l) + "\n\\end{BMAT}"

   # -------------------------------------------------------------
   def _latex_(self):
     """
     Return \LaTeX representation of X.

     EXAMPLE:
       m0=matrix([[1/3,5,9],[x+3,6,-3]])
```

```
    m0.subdivide(1,2)
    e1=matrix([[1,0],[2,3]])
    m1=matrix([[1/3,5,9],[0,(x-5)/(x^2+2),-2]])
    m1.subdivide(1,2)
    e2=matrix([[1/7,0],[2,3*x]])
    m2=matrix([[sin(x),5,9],[0,(x-5)/(x^2+2),-2]])
    m2.subdivide(1,2)
    efmt="cc"
    mfmt="ccc"
    a=MatFmt((0.6,2.2,4),1.0,(efmt,mfmt), \
       [(None,m0,"text 1"),(e1,m1,"text 2"), \
       (e2,m2,"text 3\nin several lines")])
    latex(a)
"""

    return self._putBMAT( self.w, self.h, self.specs, self.l)
```

Example C.0.1. *ᴸᴬTᴇXrepresentation of the computational layout using Sage*

The Gauss-Jordan elimination with multiple right hand sides example 2.7.1 found in this text was produced with the following code:

```
# Define matrix A
lft=matrix( QQ,3,3,[2,0,0,  4,1,0,  -6,4,1])
rgt=matrix( QQ,3,3,[1,2,7,  0,1,3,   0,0,1])
A=lft*rgt
# ================================================
# Define matrix B and augmented matrix a0=[A B]
B=A*matrix(QQ,3,2,[-2,5,6,  4,-3,-4])
a0=A.augment(B)
a0.subdivide( None, A.ncols())
print "a0"; print a0


# ================================================
# Gaussian Elimination matrices
e1=matrix([[1,0,0],[-2,1,0],[3,0,1]])
a1=e1*a0; a1.subdivide( None, A.ncols())
print "e1"; print e1; print "a1"; print a1

e2=matrix([[1,-4,0],[0,1,0],[0,-4,1]])
a2=e2*a1; a2.subdivide( None, A.ncols())
print "e2"; print e2; print "a2"; print a2

e3=matrix([[1,0,-2],[0,1,-3],[0,0,1]])
a3=e3*a2; a3.subdivide( None, A.ncols())
print "e3"; print e3; print "a3"; print a3

e4=matrix([[1/2,0,0],[0,1,0],[0,0,1]])
a4=e4*a3; a4.subdivide( None, A.ncols())
print "e4"; print e4; print "a4"; print a4

# ================================================
# setup for printing
e_fmt="rrr"
ak_fmt="rrrrr"
f=MatFmt((0.6,1.0,4),0.65,(e_fmt,ak_fmt), \
[(None,a0,"Augment $A$ with both $b_1$ and $b_2$.
                  Choose the first pivot 2."), \
(e1,a1,"Choose the second pivot 1."), \
(e2,a2,"Choose the third pivot. ", \
(e3,a3,"Finally, scale each of the pivots to 1."), \
(e4,a4,"There are no free variables. The solution is unique.") \
])
print "\\begin{flalign*}";latex(f);print "\\end{flalign*}"
```

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Bibliography

[1] Gilbert Strang. *Introduction to Linear Algebra, Fourth Edition.* Wellesley Cambridge Press, 2009.