1. $m(a+bX) = \frac{1}{N}\sum_{i=1}^{N}(a+bx_i)$

$= \frac{1}{N}\left(\sum_{i=1}^{N} a + \sum_{i=1}^{N} bx_i\right)$

$= \frac{1}{N}a\sum_{i=1}^{N} 1 + \frac{1}{N}b\sum_{i=1}^{N} x_i$

$= \frac{1}{N}\left(aN + b\sum_{i=1}^{N} x_i\right)$

$= a + b\cdot\frac{1}{N}\sum_{i=1}^{N} x_i$

$= a + bm(X)$

2. $cov(X, a+bY) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(a+bY - m(a+bY))$

$= m[(X-m(X))((a+bY) - m(a+bY))]$

$\hookrightarrow m(a+bY) = a + bm(Y)$

$\hookrightarrow (a+bY) - a + bm(Y)$

$= b(Y - m(Y))$

$= m[(X-m(X))(b(Y-m(Y)))]$

$= b\cdot m[(X-m(X))\cdot(Y-m(Y))]$

$= b\, cov(X, Y)$

3. $cov(a+bX, a+bX) = \frac{1}{N}\sum_{i=1}^{N}(a+bx - \underbrace{m(a+bx)}_{= a+bm(x)})(a+bx - m(a+bx))$

$= \frac{1}{N}\sum_{i=1}^{N}[(a+bx - (a+bm(x)))(a+bx - (a+bm(x)))]$

$= \frac{1}{N}\sum_{i=1}^{N}[b(X-m(X))b(X-m(X))]$

$= b^2 cov(X, X)$

$\hookrightarrow = b\cdot\frac{1}{N}\sum_{i=1}^{N}[(x-m(x))(x-m(x))]$

$= cov(X, X) = s^2$

4. Does median$(g(X)) = g($median$(X))$?

if $X = [0, 1, 2]$ then $g(X) = [2, 7, 12]$ for the function $2 + 5X$

median$(X) = 1$

Median$(g(X)) = 7 = g(1)$ YES, the median is the median of the transformed variables.

$IQR(X) = Q_{.75}(X) - Q_{.25}(X)$

$= 2 - 0$

$= 2$

$IQR(g(x)) = g(Q_{.75}(X)) - g(Q_{.25}(X))$

$= g(2) - g(0)$

$= 12 - 2$

$= 10 \neq g(2)$ NOT applicable to IQR

$Range(x) = X_{max} - X_{min}$

$= 2 - 0$

$= 2$

$Range(g(X)) = g(X_{max}) - g(X_{min})$

$= 12 - 2$

$= 10 \neq g(2)$ NOT applicable to range

5. Does $m(g(X)) = g(m(X))$?

$g(x) = 2 + 5x$

$$m(g(X)) = \frac{1}{N}\sum_{i=1}^{N}(2+5x_i)$$
$$= \frac{1}{N}\left(\sum_{i=1}^{N} 2 + \sum_{i=1}^{N} 5x_i\right)$$
$$= \frac{1}{N}\left(2N + 5\sum_{i=1}^{N} x_i\right)$$
$$= 2 + 5\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$
$$= 2 + 5(m(X))$$
$$= g(m(X))$$ Holds if $g(x)$ is linear (affine)

However if monotone not always true:

If $g(x) = x^3$ and $X = [0, 1, 2]$

$$m(X) = \frac{0+1+2}{3} = 1$$
$$g(m(X)) = g(1) = 1^3 = 1$$
$$m(g(X)) = \frac{0^3+1^3+2^3}{3} = \frac{9}{3} = 3$$
$\Big\}\ 1 \neq 3$

Jensen's Inequality:

if $g(X)$ is convex:

$$m(g(x)) \geq g(m(X))$$ (reverse is true if concave)

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y)$$

if $N = 2$ (equal weighting)

$$m(X) = \frac{N-1}{N} m(X_{-N}) + \frac{1}{N} x_N$$
$$g(m(X)) \leq \frac{N-1}{N} g(m(X_{-N})) + \frac{1}{N} g(x_N)$$
$$g(m(X_{-N})) \leq \frac{1}{N-1}\sum_{i=1}^{N-1} g(x_i)$$
$$g(m(X)) \leq \frac{N-1}{N}\cdot\frac{1}{N-1}\sum_{i=1}^{N-1} g(x_i) + \frac{1}{N} g(x_N)$$
$$\leq \frac{1}{N}\sum_{i=1}^{N-1} g(x_i) + \frac{1}{N} g(x_N)$$
$$\leq \frac{1}{N}\sum_{i=1}^{N} g(x_i)$$