Chapter 2

Radiative Transfer

2.1 Definition of Standard Quantities

The light emitted from or passing through objects in space is almost the only way that we have to probe the vast majority of the universe we live in. The most distant object object to which we have travelled and brought back samples, besides the moon, is a single asteroid. Collecting solar wind gives us some insight into the most tenuous outer layers of our nearby star, and meteorites on earth provide insight into planets as far away as Mars, but these are the only things from space that we can study in laboratories on earth. Beyond this, we have sent unmanned missions to land on Venus, Mars, another asteroid and a comet (kind of). To study anything else in space we have to interpret the radiation we get from that source. As a result, understanding the properties of radiation, including the variables and quantities it depends on and how it behaves as it moves through space, is then key to interpreting almost all of the fundamental observations we make as astronomers.

2.1.1 Energy

To begin to define the properties of radiation from astronomical objects, we will start with the energy that we receive from an emitting source somewhere in space. Consider a source of radiation in the vacuum of space (for familiarity, you can think of the sun). At some point in space away from our source of radiation we want to understand the amount of energy dE that is received from this source. What is this energy proportional to?

As shown in Figure 2.1, our source of radiation has an intensity I_0 (we will get come back to this in a moment) over an apparent angular size (solid angle) of $d\Omega$. Though it may give off radiation over a wide range of frequencies, as is often the case in astronomy we only concern ourselves with the energy emitted in a specific frequency range $\nu + d\nu$ (think of using a filter to restrict the colors of light you see, or even just looking at something with your eyeball, which only detects radiation in the visible range). At the location of detection, the radiation passes through some area dA in space (an area perhaps like a spot on the surface of earth) at an angle θ away from the normal to that surface. The last property of the radiation that we might want to consider is that we are detecting it over a given window of time (and many astronomical sources are time-variable). You might be wondering why the distance between our detector and the source is not being mentioned yet: we will get to this.

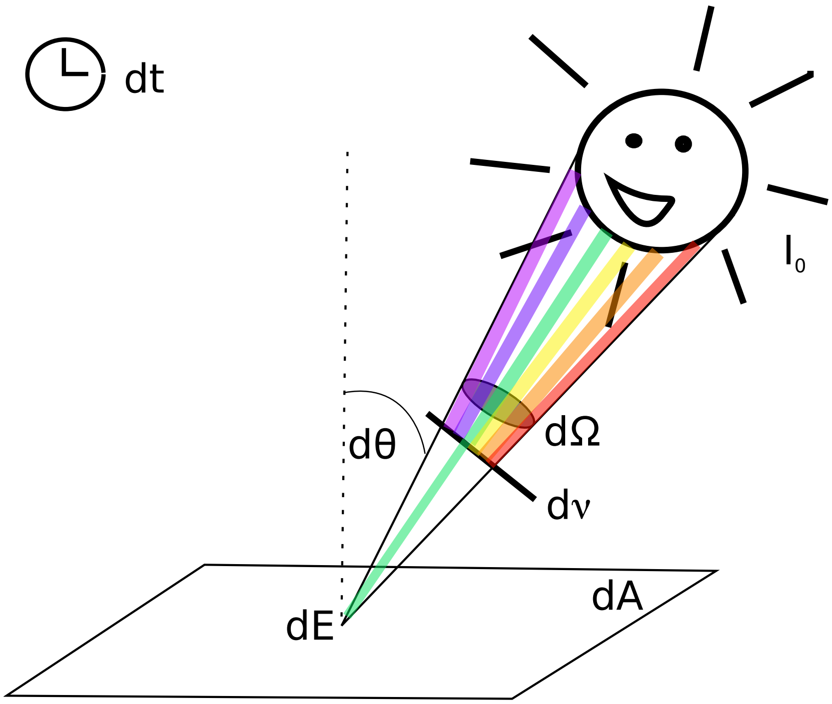

Figure 2.1: *Description of the energy detected at a location in space for a period of time dt over an area dA arriving at an angle θ from an object with intensity I_0, an angular size $d\Omega$, through a frequency range $\nu, \nu + d\nu$*

Considering these variables, the amount of energy that we detect will be proportional to the apparent angular size of our object, the range of frequencies over which we are sensitive, the time over which we collect the radiation, and the area over which we do this collection. The constant of proportionality is the specific intensity of our source: I_0. Technically, as this is the intensity just over a limited frequency range, we will write this as $I_{0,\nu}$.

In equation form, we can write all of this as:

$$dE_\nu \, d\nu = I_{0,\nu} \, cos\theta \, dA \, d\Omega \, d\nu \, dt \tag{2.1}$$

Here, the cos θ dA term accounts for the fact that the area that matters is actually the area "seen" from the emitting source. If the radiation is coming straight down toward our unit of area dA at an angle of 0, it "sees" an area equal to that of the full dA (cos 0 = 1). However, if the radiation comes in at a different angle θ, then it "sees" our area dA as being tilted: as a result, the apparent area is smaller (cos $\theta < 1$). You can test this for yourself by thinking of the area dA as a sheet of paper, and observing how its apparent size changes as you tilt it toward or away from you.

2.1.2 Intensity

Looking at Equation 2.1, we can figure out the units that the specific intensity must have: energy per time per frequency per area per solid angle. In SI units, this would be W Hz^{-1} m^{-2} sr^{-1}. Specific intensity is also sometimes referred to as surface brightness, as this quantity refers to the brightness over a fixed angular size on the source. Technically, the specific intensity is a

7-dimensional quantity: it depends on position (3 space coordinates), direction (two more coordinates), time (in general, we won't worry too much about this) as well as frequency (it's freaking complicated).

2.1.3 Flux

The flux from a source is defined as the total energy of radiation received from all directions at a point in space, per unit area and per unit time. Given this definition, we can modify equation 2.1 to give the flux at a frequency ν:

$$F_\nu = \int_\Omega \frac{dE_\nu d\nu}{dA dt} = \int_\Omega I_\nu cos\theta d\Omega \tag{2.2}$$

The total flux at all frequencies (the bolometric flux) is then:

$$F = \int_\nu F_\nu \, d\nu \tag{2.3}$$

As expected, the SI units of flux are W m^{-2}

The last, related property that one should consider (particularly for spatially well-defined objects like stars) is the Luminosity. The luminosity of a source is the total energy emitted per unit time. The SI unit of luminosity is W.

Luminosity can be determined from the flux of an object by integrating over its entire surface:

$$L = \int F \, dA \tag{2.4}$$

Having defined these quantities, we now ask how the flux you detect from a source varies as you increase the distance to the source. Looking at Figure 2.2, we take the example of our happy sun, and imagine two shells or bubbles around the sun: one at a distance R_1, and one at a distance R_2. The amount of energy passing through each of these shells per unit time is the same: in each case, it is equal to the luminosity of the sun. However, as $R_2 > R_1$, the surface area of the second shell is greater than the first shell. Thus, the energy is spread thinner over this larger area, and the flux (which by definition is the energy per unit area) must be smaller for the second shell. Comparing the equations for surface area, we see that flux decreases proportional to $1/d^2$.

We have showed that the flux obeys an inverse square law with distance from a source. How does the specific intensity change with distance? The specific intensity can be described as the flux divided by the angular size of the source. We have just shown that the flux decreases with distance, proportional to $1/d^2$. What about the angular source size? It happens that the source size also decreases with distance, proportional to $1/d^2$. As a result, the specific intensity is a quantity that is independent of distance.

2.2 The Equation of Radiative Transfer

We can use the fact that the specific intensity does not change with distance to begin deriving the radiative transfer equation. For light traveling in a vacuum along a path length s, we say that the intensity is a constant. As a result,

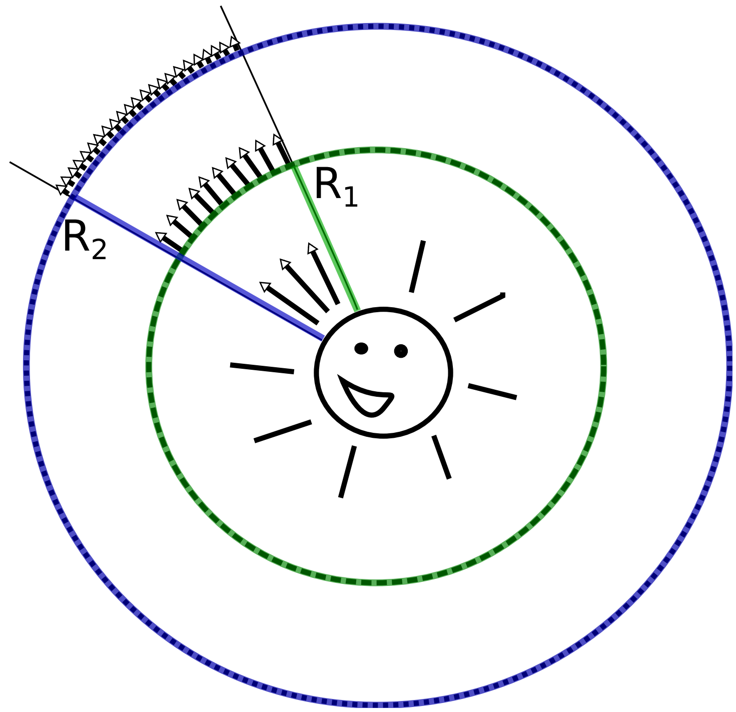

Figure 2.2: *A depiction of the flux detected from our sun as a function of distance from the sun. Imagining shells that fully enclose the sun, we know that the energy passing through each shell per unit time must be the same (equal to the total luminosity of the sun). As a result, the flux must be less in the larger outer shell: reduced proportional to $1/d^2$*

$$\frac{dI_\nu}{ds} = 0\ (for\ radiation\ traveling\ through\ a\ vacuum) \tag{2.5}$$

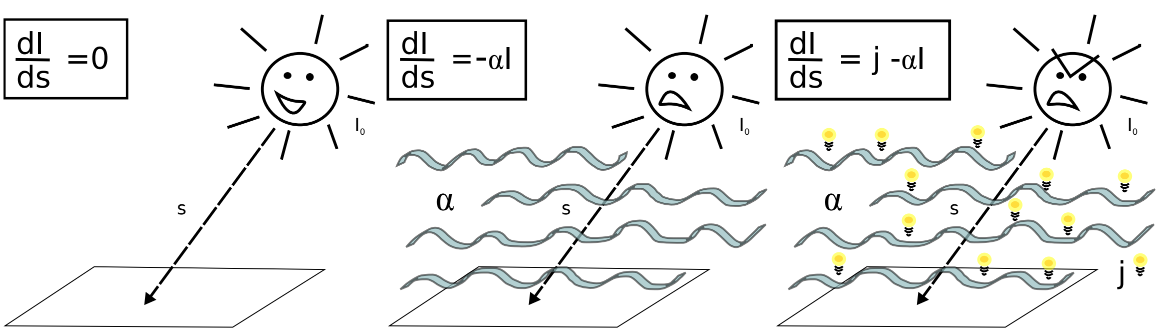

Figure 2.3: *The radiative transfer equation, for the progressively more complicated situations of: (left) radiation traveling through a vacuum; (center) radiation traveling through a purely absorbing medium; (right) radiation traveling through an absorbing and emitting medium.*

This case is illustrated in the first panel of Figure 2.3. However, space (particularly objects in space, like the atmospheres of stars) is not a vacuum everywhere. What about the case when there is some junk between our detector and the source of radiation? This possibility is shown in the second panel of Figure 2.3. One quickly sees that the intensity you detect will be less than it was at the source. You can define an extinction coefficient α for the space junk, with units of extinction (or fractional depletion of intensity) per distance (path length) traveled, or m^{-1} in SI units. For our purposes right now, we will assume that this extinction is uniform and frequency-independent (but in real life of course, it never is). Now, our equation of radiative transfer has been modified to be:

$$\frac{dI_\nu}{ds} = -\alpha I_\nu \; (\textit{when there is some junk between us and our source}) \tag{2.6}$$

As is often the case when simplifying differential equations, we then find it convenient to try to get rid of some of these pesky units by defining a new unitless constant: τ, or optical depth. If α is the fractional depletion of intensity per path length, τ is just the fractional depletion. We then can define

$$d\tau = \alpha ds \tag{2.7}$$

and re-write our equation of radiative transfer as:

$$\frac{dI_\nu}{d\tau} = -I_\nu \tag{2.8}$$

Remembering our basic calculus, we see that this has a solution of the type

$$I = I_0 e^{-\tau} \tag{2.9}$$

So, at an optical depth of 1 (the point at which something begins to be considered optically thick), your initial source intensity I_0 has decreased by a factor of e.

However, radiation traveling through a medium does not ALWAYS result in a net decrease. It is also possible for the radiation from our original source to pass through a medium or substance that is not just absorbing the incident radiation but is also emitting radiation of its own, adding to the initial radiation field. To account for this, we define another coefficient: j_ν. This emissivity coefficient has units of energy per time per volume per frequency per solid angle. Note that these units (in SI: W m^{-3} Hz^{-1} sr^{-1}) are slightly different than the units of specific intensity. Including this coefficient in our radiative transfer equation we have:

$$\frac{dI_\nu}{ds} = j - \alpha I_\nu \tag{2.10}$$

or, putting it in terms of the dimensionless variable τ, we have:

$$\frac{dI_\nu}{d\tau} = j/\alpha - I_\nu \tag{2.11}$$

The quantity j/α is typically defined as the Source function S. Making this substitution, we arrive at the final form of the radiative transfer equation:

$$\frac{dI_\nu}{d\tau} = S_\nu - I_\nu \tag{2.12}$$

What is the solution of this equation? For now, we will again take the simplest case, and assume that the medium through which the radiation is passing is uniform. In this case,

$$I_\nu = I_{\nu,0}\, e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) \tag{2.13}$$

What happens to this equation when τ becomes large? In this case, the term $e^{-\tau_\nu}$ becomes negligible, and you arrive at the result

$$I_\nu = S_\nu \ (for\, an\, optically - thick\, source) \tag{2.14}$$

This makes some sense: travel far enough through an absorbing medium, and the background radiation I_0 is totally blocked, and the only radiation that makes it out is from the emission of the medium itself. So, what the hell is this source function thing?

For a source in thermodynamic equilibrium, you can show that $S_\nu = B_\nu$, the blackbody function.

$$B_\nu = \frac{2h\nu^3}{c^2} \left[e^{\frac{h\nu}{kT}} - 1 \right]^{-1} \tag{2.15}$$

For an optically-thick source (say, a star like our sun) we can use Equation 2.14 to then say that $I_\nu = B_\nu$.

This gives us the ability to define key properties of stars – like their flux and luminosity – as a function of their temperature. Using equations 2.2 and 2.3, we can integrate the blackbody function to determine the flux of a star (or other blackbody) as a function of temperature:

$$F = \sigma T^4 \tag{2.16}$$

This fundamental result is known as the Stefan-Boltzmann law.

Another classic result, the peak frequency (or wavelength) at which a star (or other blackbody) radiates, based on its temperature, can be found by differentiating the blackbody equation with respect to frequency (or wavelength). The result must be found numerically, but the peak wavelength can be expressed as

$$\lambda_{peak} = \frac{2.898 \times 10^{-3}}{T} \tag{2.17}$$

This relation is known as Wien's law.

2.3 Brightness Temperature

Radio and millimeter astronomers use two main units to talk about specific intensity.

The first is the Jansky:

$$1\mathrm{Jy} = 10^{-26}\mathrm{W m^{-2} Hz^{-1}}$$

You should note that a Jansky alone is not a unit of specific intensity, as there is no dependence on source solid angle. Radio astronomers typically get around this by image specific intensity in terms of Jy/beam, where the beam has units of solid angle (In radio astronomy, a beam is roughly equivalent to the point spread function: it represents the resolution limit of the telescope. It is generally specified as an ellipse, with major and minor axis length, and a position angle)

Radio astronomers also often describe the specific intensity of a source using a quantity known as Brightness temperature. This quantity relates the specific intensity I_ν of an observed source to that of a blackbody function (B_ν) at a temperature T.

As Draine correctly notes, this is only a linear relation in the Rayleigh-jeans limit $(h\nu << kT)$. You can see this visually if you plot the Planck function as a function of frequency, in log-space: the specific intensity at a given temperature is proportional to ν^2 which leads to a constant (linear) slope of +2 in the plot.

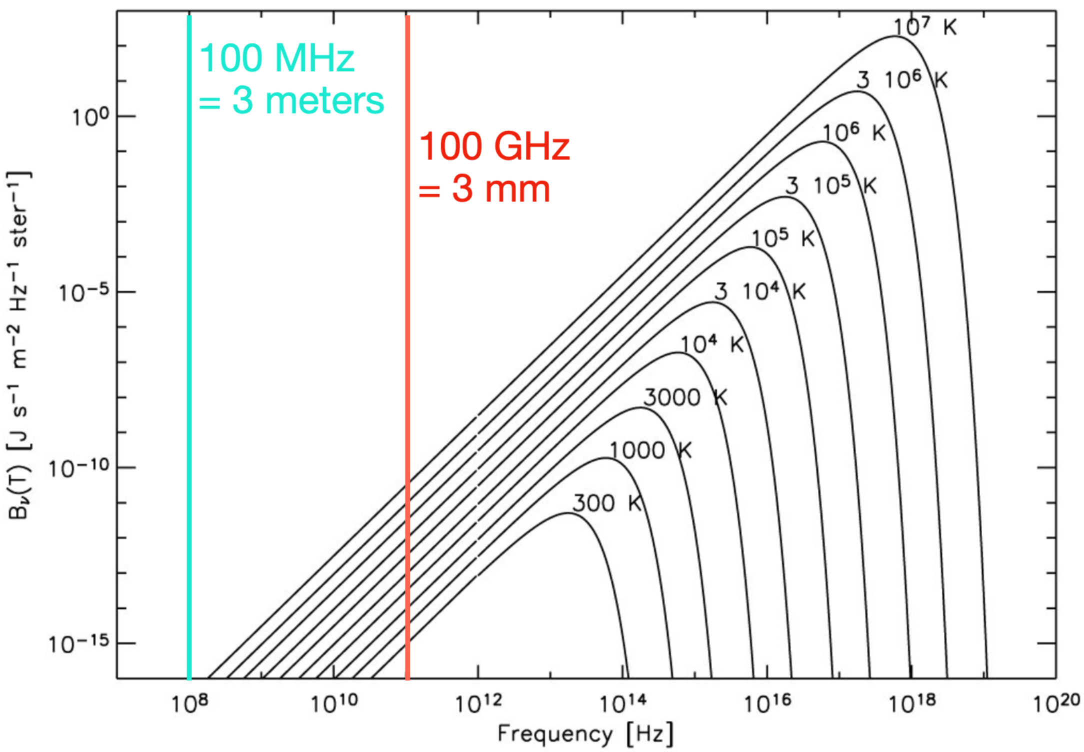

Figure 2.4: *RJ_limit.png*

This quantity is then typically **only** used by radio and millimeter astronomers, for observations at wavelengths where the Rayleigh-Jeans limit applies. In this case:

$$B_\nu = \frac{2\nu^2 k_B T}{c^2} \tag{2.18}$$

and we can define the brightness temperature as

$$T_B = \frac{c^2}{2k_B\nu^2} I_\nu \tag{2.19}$$

IMPORTANT NOTE: This is the same as Draine's definition of Antenna Temperature (T_A), which appears to be totally wrong. In practice, Antenna temperature is an analogous but more observationally/instrumentally-oriented quantity: the temperature of a blackbody (in practice, usually a thermal resistor) that would lead to an equivalent amount of power per unit frequency being detected by the antenna/telescope (after a lot of corrections for the antenna efficiency)

$$T_A = \frac{P_\nu}{k_B} \tag{2.20}$$

Antenna temperature is a convenient quantity because:

1. 1 K of antenna temperature is a conveniently small power per unit bandwidth. T_A =1 K corresponds to $P_\nu = kT_A = 1.38 \times 10^{-23}$ J K^{-1} $\times$ 1 K $= 1.38 \times 10^{-23}$ W Hz^{-1}.
2. It can be calibrated by a direct comparison with matched resistors connected to the antenna receiver input.
3. The units of system noise are also K, so comparing the signal in K with the receiver noise in K makes it easy to compare the signal and noise powers.

(definitions taken from https://www.cv.nrao.edu/ sransom/web/Ch3.html)