ASTR 792
T/R 9:30 - 10:45 AM
Due September 18

Week #5

Draine 2.3 a b c

Consider a dust grain of radius a, and mass $M >> m_H$, where m_H is the mass of an H atom. Suppose that the grain is initially at rest in a gas of H atoms with number density n_H and temperature T. Assume the grain is large compared to the radius of an H atom. Suppose that the H atoms "stick" to the grain when they collide with it, so that all of their momentum is transferred to the grain, and that they subsequently "evaporate" from the grain with no change in the grain velocity during the evaporation.

(a) What is the mean *speed* $\langle v_H \rangle$ of the H atoms (in terms of m_H, T, and Boltzmann's constant k_B)?

SOLUTION:

To determine the mean speed— the geometric average of the speed (or magnitude of the velocity vector) of the particles– we must integrate the velocity distribution function:

$$\langle v \rangle = \int_0^\infty v f(v) dv$$

where

$$f(v) = \left[\frac{m}{2\pi kT}\right]^{3/2} 4\pi v^2 \ \exp\left(-\frac{mv^2}{2kT}\right)$$

Setting $b = \frac{m_H}{2kT}$

$$\langle v_H \rangle = 4\pi \left[\frac{b}{\pi}\right]^{3/2} \int_0^\infty v^3 e^{-bv^2} dv$$

$$\langle v_H \rangle = 4\pi \left[\frac{b}{\pi}\right]^{3/2} \frac{1}{2b^2}$$

$$\langle v_H \rangle = \sqrt{\frac{4}{\pi b}} = \sqrt{\frac{8k_B T}{\pi m_H}}$$

(b) Calculate the time τ_M for the grain to be hit by its own mass M in gas atoms. Express τ_M in terms of M, a, n_H, and $\langle v_H \rangle$.

SOLUTION:

To solve this, we need to know the number of collisions per time (the collision rate R), and the mass gained per each collision (this is just m_H). Together, this gives the mass gained per time, and we can then determine the time to accumulate a total mass M:

$$\frac{M}{\tau_M} = R\, m_H = \sigma \langle v_H \rangle n_H m_H$$

$$\tau_M = \frac{M}{m_H} \left(\frac{1}{\langle v_H \rangle \pi (2a)^2 n_H} \right)$$

(c) Evaluate $\langle v_H \rangle$ and τ_M for a grain of radius $a = 10^{-5}$ cm and density $\rho = 3$ g cm^{-3}, in a gas with $n_H = 30$ cm^{-3} and $T = 10^2$ K.

SOLUTION:

First, we have to calculate M from the given information:

$$M = \rho \frac{4}{3} \pi a^3$$

$$M = (3 \text{ g cm}^{-3}) \, \frac{4}{3} \pi (10^{-5} \text{ cm})^3$$

$$M = 1.3 \times 10^{-14} \text{ g}$$

Then, plugging in values to the equations we previously derived:

$$\langle v_H \rangle = \sqrt{\frac{8(1.38 \times 10^{-16} \text{ cm}^2 \text{ g s}^{-2} \text{ K}^{-1})(100 \text{ K})}{\pi \; (1.67 \times 10^{-24} \text{ g})}}$$

$$\langle v_H \rangle \; = \; 1.45 \times 10^5 \text{ cm s}^{-1} \; = \; 1.45 \text{ km s}^{-1}$$

$$\tau_M = \frac{M}{m_H} \left(\frac{1}{\langle v_H \rangle \pi (2a)^2 n_H} \right)$$

$$\tau_M = \frac{1.3 \times 10^{-14} \text{ g}}{1.67 \times 10^{-24} \text{ g}} \left(\frac{1}{(1.45 \times 10^5 \text{ cm s}^{-1})(\pi)(10^{-5} \text{ cm})^2 (30 \text{ cm}^{-3})} \right)$$

$$\tau_M = 5.7 \times 10^{12} \text{ s} \; = \; 1.8 \times 10^5 \text{ yrs}$$