DS 3000: Foundations of Data Science **(Practice Solutions)**

Midterm Exam (In-Class Math Practice Solutions)

Professor: Dr. Gerber *Name:* ______________________

Instructions: Please read the below carefully before continuing:

1. Write your name **legibly** in the box above.
2. You have **100 minutes** to complete the exam. There are **??? parts**. You should move on if a part is taking you more than ??? minutes.
3. Write **only on the front side of the exam**; the exams will be scanned into Gradescope one-sided. Anything written on the back will not be considered when grading.
4. You may use a **calculator** (**NOT** a mobile phone) and **one note sheet** as resources on this exam. You may not use any other resources (aside from your brain). You will sign an academic integrity pledge before beginning the exam.
5. Show as much work as possible to obtain partial credit. Correct answers **without supporting work will NOT receive any credit**. Also **clearly indicate your answers by circling/putting boxes around them**.
6. Round all numeric answers with decimals to **3 decimal places**.
7. When you finish the exam, please bring the exam to the front of the room. You can keep your note sheet. You may leave as soon as you hand in your exam.

Failure to sign the below academic integrity pledge will result in a score of 0 on this exam!

Academic Integrity Pledge

I pledge on my personal honor and integrity that the work on this exam is entirely my own, and that no outside sources were used in helping me answer the questions.

Signature: ____

Problem 1: Vectors (12 points)

Find the resultant vector and its magnitude (Euclidean length/L2-norm) given the vectors in each part:

1. Find $p \odot q + r \cdot s$ and it's length.

$$p = \begin{bmatrix} -7 \\ -6 \end{bmatrix} \quad q = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad r = \begin{bmatrix} -3 \\ -9 \end{bmatrix} \quad s = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

2. Find $-3a - 7(b \odot c)$ and it's length.

$$a = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ \frac{1}{7} \end{bmatrix} \quad c = \begin{bmatrix} \frac{1}{4} \\ 2 \end{bmatrix}$$

SOLUTIONS

These operations should be very straightforward, but if you really want to check your work, please see the TAs or Professors if you want to see these solutions.

Problem 2: Span (12 points)

Is the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the span of the following set in R^3?

$$T = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

SOLUTION

The question is whether there are scalars α, β such that:

$$\alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Can show:

$$-\alpha + \beta = 1$$
$$\beta = 2$$
$$\alpha + \beta = 3$$

Suggests that

$$\alpha = 1$$

and

$$\beta = 2$$

. This means that the given vector **is in** the span of T.

Problem 3: Matrix Operations I (16 points)

Given:

$$w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, Y = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 5 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

Perform the following operations. If the operation is not possible, indicate the reason. You must show all work/derivations to receive full credit.

1. $Y^T w$
2. $Y \odot Z$
3. wZ
4. YZ^T

SOLUTIONS

These operations should be very straightforward, but if you really want to check your work, please see the TAs or Professors if you want to see these solutions.

Problem 4: Matrix Operations II (12 points)

Mr. Hwan writes the following matrix on the board and asks his students to write down a matrix that they could multiply to that matrix (i.e., that would work as a right matrix in a matrix multiplication):

$$H = \begin{bmatrix} -3 & 1 \\ 2 & 0 \\ 4 & -3 \end{bmatrix}$$

Three students, Wanda (W), Xavier (X) and Zach (Z) write:

$$W = \begin{bmatrix} -1 & 2 \\ 5 & 3 \\ 0 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & -3 \\ 7 & 1 \end{bmatrix}$$

$$Z = \begin{bmatrix} -4 & 1 & 0 \\ 8 & 5 & -3 \end{bmatrix}$$

"One of you has made a mistake," Mr. Hwan says. Is Mr. Hwan correct? Who made the mistake? Calculate the final matrices for those who did not make a mistake.

SOLUTION

If we call Mr. Hwang's matrix H, we know that it is 3 by 2 (3 rows, 2 columns). So is W, while X is 2 by 2, and Z is 2 by 3. Of these, W cannot be used as a right matrix; if you try to do HW, the inner dimensions don't match. We can, however, do the other two operations:

$$HX = \begin{bmatrix} 1 & 10 \\ 4 & -6 \\ -13 & -15 \end{bmatrix}$$

$$HZ = \begin{bmatrix} 20 & 2 & -3 \\ -8 & 2 & 0 \\ -40 & -11 & 9 \end{bmatrix}$$

Problem 5: Projections I (12 points)

For each of the below, find the point in the span of the $\vec{a}$ vectors closest to the $\vec{b}$ vector.

1. $\vec{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

   SOLUTION

   $\vec{p} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\vec{a} = \frac{1}{2}\vec{a} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$

2. $\vec{a_0} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{a_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

   SOLUTION

   $\vec{p} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \left(\begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3/7 \\ -5/7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ Note that this is $\vec{b}$, because the span of the two $\vec{a}$ is the entire plan, $\vec{b}$ is already in the span of the $\vec{a}$ vectors.

3. $\vec{a_0} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\vec{a_1} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

   SOLUTION

   $$\vec{p} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1/54 \\ -43/54 \end{bmatrix} = \begin{bmatrix} -85/54 \\ 46/54 \\ -1/54 \end{bmatrix}$$

Problem 6: Projections II (12 points)

Let $W = \text{span}\{v_1, v_2\}$ where

$$v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Find the closest point $\vec{w}$ in W to:

$$\vec{x} = \begin{bmatrix} 0 \\ 14 \\ -4 \end{bmatrix}$$

SOLUTION

$$\vec{w} = V(V^T V)^{-1} V^T \vec{x}$$

where

$$V = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 6 & 14 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 14 \\ -4 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \\ 2 \end{bmatrix}$$

Problem 7: Dependence/Independence (12 points)

Decide whether each set of vectors is linearly dependent or linearly independent.

1. $\left\{ \vec{a}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix} \right\}$

   SOLUTION

   The given set is linearly dependent, as you can easily see that $2\vec{a}_1 = -\vec{a}_2$

2. $\left\{ \vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

   SOLUTION

   One way: Solving RREF for the set of vectors gives:

   $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

   which implies the set is linearly dependent. Equivalently, you should be able to show that $\vec{a}_3 = -\vec{a}_1 + 2\vec{a}_2$.

3. $\left\{ \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$

   SOLUTION

   One way: Solving RREF for the set of vectors gives:

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   which implies the set is linearly independent. Equivalently, you can simply state that there is no way for you to calculate any of the vectors as a linear combination of the others.

Problem 8: Eigenvectors/Eigenvalues (12 points)

Find the eigenvalues and eigenvectors for the following matrices

1. $$A = \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}$$

2. $$B = \begin{bmatrix} 2 & 0 \\ 4 & -4 \end{bmatrix}$$

3. $$C = \begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix}$$

SOLUTION FOR A (it is the same process, but perhaps even simpler, for B and C)

1. First, find the eigenvalues:
$$(A - \lambda I) = \begin{bmatrix} -1-\lambda & -2 \\ -3 & 1-\lambda \end{bmatrix}$$
$$det(A - \lambda I) = (-1-\lambda)(1-\lambda) - (-2)(-3) \rightarrow \lambda^2 = 7$$
The eigenvalues are $\lambda = \{-\sqrt{7}, \sqrt{7}\}$.

2. Then find the corresponding eigenvectors for each eigenvalue:
   - $\begin{bmatrix} -1+\sqrt{7} & -2 \\ -3 & 1+\sqrt{7} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (\sqrt{7}-1)x - 2y = 0 \rightarrow y = \frac{\sqrt{7}-1}{2}x$ Any vector of the form $\vec{v} = \begin{bmatrix} 1 \\ \frac{\sqrt{7}-1}{2} \end{bmatrix}$ (i.e. any multiple of this vector) is an eigenvector when $\lambda = -\sqrt{7}$
   - $\begin{bmatrix} -1-\sqrt{7} & -2 \\ -3 & 1-\sqrt{7} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -3x-(1-\sqrt{7})y = 0 \rightarrow y = \frac{3}{1-\sqrt{7}}x$ Any vector of the form $\vec{v} = \begin{bmatrix} 1 \\ \frac{3}{1-\sqrt{7}} \end{bmatrix}$ (i.e. any multiple of this vector) is an eigenvector when $\lambda = \sqrt{7}$