Functors in a Web-Scalable Module System

Guided Research Presentation

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Outline

- Introduction
- MMT and Theory Representation
- Functor Representation
- Well-formed Expressions
- Conclusions and Future Work

Introduction

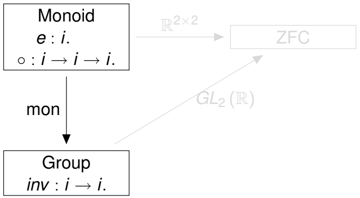
- mathematical knowledge is increasing in size - needs to be organized
- representation as networks of mathematical theories
- several systems can represent mathematical theories (MMT [1], PVS[2], Isabelle[3])
- **functors** cannot be represented in either of the systems $\longrightarrow$ we will present a **solution.**

Theory Representation in MMT

Representation:

- theory graph → theories → symbol declarations and axioms
- theory morphisms
- Representation of categories through theories

Monoid :={
symbols and axioms of FOL
e : i,
$\circ$: $i \to i \to i$,
e_l : true forall [x] e $\circ$ x eq x,
e_r : true forall [x] x $\circ$ e eq x,
assoc : true forall [x] forall [y] forall [z] (x $\circ$ y) $\circ$ z eq x $\circ$ (y $\circ$ z) }



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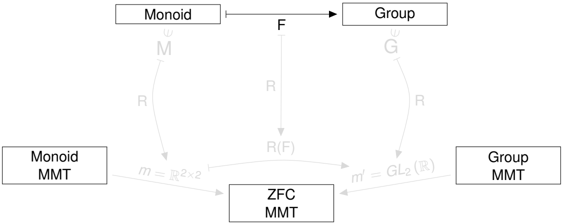
$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Monoid}\\ e : i.\\ \circ : i \to i \to i.\end{array}} \arrow[r, "\mathbb{R}^{2\times 2}"] \arrow[d, "\text{mon}"'] & \boxed{\text{ZFC}} \\ \boxed{\begin{array}{c}\text{Group}\\ inv : i \to i.\end{array}} \arrow[ur, "GL_2(\mathbb{R})"'] & \end{tikzcd}$$

Representing Functors



$R(F) := \lambda m : Monoid \longrightarrow ZFC($
$\quad universe \mapsto \{ x \in universe^m \},$
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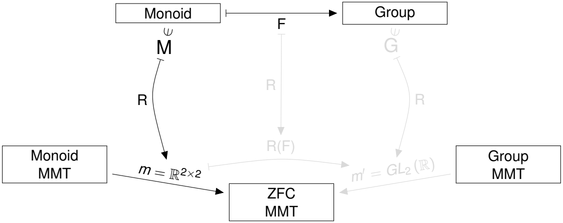

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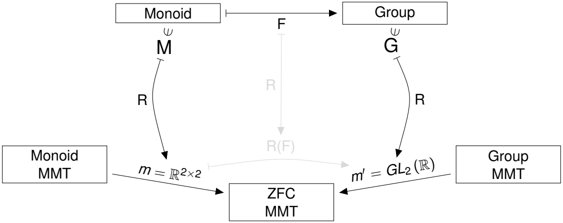

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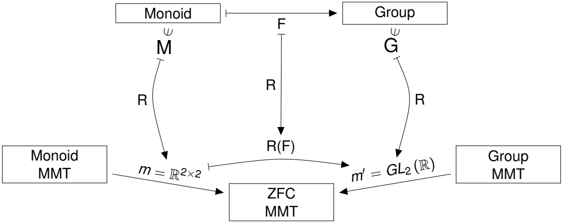

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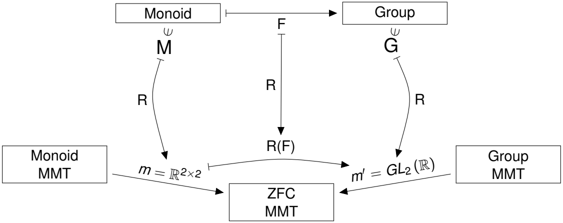

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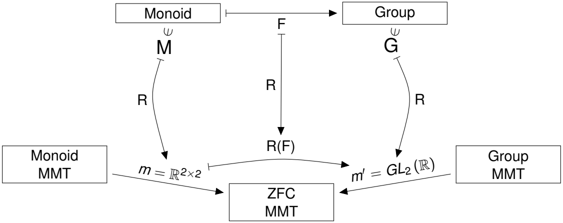

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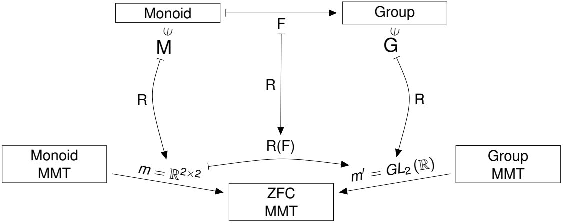

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Syntax

- We modify the flat grammar of MMT to represent functors.
- Changes are orthogonal to the ommited rules.
- Our solution can be extended to the general case without much difficulty.

View declarations	Viw	$::=$	$m := \{\sigma\} : \tau$
Type	τ	$::=$	$(S \to T) \mid \tau' \Rightarrow \tau$
Substitutions	σ	$::=$	$(c \mapsto \omega)^* \mid \lambda x : \tau.\sigma$
Morphisms	μ	$::=$	$m \mid \mu \bullet \mu' \mid m(\mu)$

Figure: Section from the grammar

Well-formed Expressions

- An expression is well-formed if it is semantically meaningful.
- We define a list of judgments that can appear in expressions.
- The context used by the judgements is γ or $\gamma;\Gamma$:

$$\gamma = (T := \{\vartheta\} \,|m := \{\sigma\} : \tau)^* \qquad \Gamma = (x : S \to T)^*$$

$$\frac{\gamma;\Gamma, x : \tau' \rhd \sigma : \tau}{\gamma;\Gamma \rhd \lambda x : \tau'.\sigma : \tau' \Rightarrow \tau}\,(Viw_\lambda)$$

$$\frac{\gamma;\Gamma \rhd_T \omega_i \quad dom(S) = \{c_1, ..., c_n\}}{\gamma;\Gamma \rhd c_1 \mapsto \omega_1, ..., c_n \mapsto \omega_n : S \to T}\,(Viw_{base})$$

Example: $Mon, Gr, ZFC; x : Mon \to ZFC, \sigma : Gr \to ZFC \rhd$

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Conclusions and Future Work

- We represented functors building upon the MMT language.
- We gave the rules for well-formed expressions.

- We will integrate functors in the complete MMT grammar.
- This is posible and requires the adjustment of the language to the new productions.
- We will try to integrate the morphism component of a functor in MMT.

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Thank you!

Questions?