16-811 HW1

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9 Sep 2019

Q1

Execute python code *q1.py* to see the performance of my algorithm on 5 examples.

Q2

SVD decomposition:

The result and my work can be found in python code *q2.py*.

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LDU decomposition:

$$\begin{array}{cccc}
\textbf{A1} & P & A' & L \\
 & \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & \begin{pmatrix}10&9&2\\5&3&1\\2&2&2\end{pmatrix} & \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \\
\xrightarrow{Permutation} & \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} & \begin{pmatrix}2&2&2\\5&3&1\\10&9&2\end{pmatrix} & \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \\
\xrightarrow[Row3-5Row1]{Row2-\frac{1}{2}Row1} & \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} & \begin{pmatrix}2&2&2\\0&-2&-4\\0&-1&-8\end{pmatrix} & \begin{pmatrix}1&0&0\\\frac{5}{2}&1&0\\5&0&1\end{pmatrix} \\
\xrightarrow{Row3-\frac{1}{2}Row2} & \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} & \begin{pmatrix}2&2&2\\0&-2&-4\\0&0&-6\end{pmatrix} & \begin{pmatrix}1&0&0\\\frac{5}{2}&1&0\\5&\frac{1}{2}&1\end{pmatrix}
\end{array}$$

$$\begin{array}{ccc} I & & D \\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & \xrightarrow{Copy\ Diag(A')\ to\ form\ D} & \begin{pmatrix}2&0&0\\0&-2&0\\0&0&-6\end{pmatrix} \end{array}$$

$$\begin{array}{ccc} A' & & U \\ \begin{pmatrix}2&2&2\\0&-2&-4\\0&0&-6\end{pmatrix} & \xrightarrow{Normalize\ A'\ based\ on\ D} & \begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix} \end{array}$$

$Check\ \ the\ \ result:$

$$\begin{array}{ccccccccc} L & D & U & & & & P & A \\ \begin{pmatrix}1&0&0\\\frac{5}{2}&1&0\\5&\frac{1}{2}&1\end{pmatrix} & \begin{pmatrix}2&0&0\\0&-2&0\\0&0&-6\end{pmatrix} & \begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix} & = & \begin{pmatrix}2&2&2\\5&3&1\\10&9&2\end{pmatrix} & = & \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} & \begin{pmatrix}10&9&2\\5&3&1\\2&2&2\end{pmatrix} \end{array}$$

$$\begin{array}{cccc} \mathbf{A2} & P & A' & L \\ & \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} & \begin{pmatrix}16&16&0&0\\4&0&-2&0\\0&1&-1&0\\0&0&0&1\\0&0&1&1\end{pmatrix} & \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \\ \xrightarrow{Permutation} & \begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&1&0&0&0\end{pmatrix} & \begin{pmatrix}16&16&0&0\\0&1&-1&0\\0&0&1&1\\0&0&0&1\\4&0&-2&0\end{pmatrix} & \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \\ \xrightarrow[\substack{Row5+6Row3\\Row5-6Row4}]{\substack{Row5-\frac{1}{4}Row1\\Row5+4Row2}} & \begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&1&0&0&0\end{pmatrix} & \begin{pmatrix}16&16&0&0\\0&1&-1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\\frac{1}{4}&-4&-6&6&1\end{pmatrix} \end{array}$$

$$\overset{I}{\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}} \xrightarrow{Copy\ Diag(A')\ to\ form\ D} \overset{D}{\begin{pmatrix}16&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}}$$

$$\overset{A'}{\begin{pmatrix}16&16&0&0\\0&1&-1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}} \xrightarrow{Normalize\ A'\ based\ on\ D} \overset{U}{\begin{pmatrix}1&1&0&0\\0&1&-1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}}$$

$Check\ \ the\ \ result:$

$$\overset{L}{\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\\frac{1}{4}&-4&-6&6&1\end{pmatrix}} \overset{D}{\begin{pmatrix}16&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}} \overset{U}{\begin{pmatrix}1&1&0&0\\0&1&-1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}} = \begin{pmatrix}16&16&0&0\\0&1&-1&0\\0&0&1&1\\0&0&0&1\\4&0&-2&0\end{pmatrix} = \overset{P}{\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&1&0&0&0\end{pmatrix}} \overset{A}{\begin{pmatrix}16&16&0&0\\4&0&-2&0\\0&1&-1&0\\0&0&0&1\\0&0&1&1\end{pmatrix}}$$

A3

$$\begin{array}{cccc} & P & A' & L \\ & \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & \begin{pmatrix}10&6&4\\5&3&2\\1&1&0\end{pmatrix} & \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \\ \xrightarrow{Permutation} & \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} & \begin{pmatrix}10&6&4\\1&1&0\\5&3&2\end{pmatrix} & \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \\ \xrightarrow[Row3-\frac{1}{2}Row1]{Row2-\frac{1}{10}Row1} & \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} & \begin{pmatrix}10&6&4\\0&\frac{4}{10}&-\frac{4}{10}\\0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0\\\frac{1}{10}&1&0\\\frac{1}{2}&0&1\end{pmatrix} \end{array}$$

$$\begin{array}{c}I\\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\end{array} \xrightarrow{Copy\ Diag(A')\ to\ form\ D} \begin{array}{c}D\\ \begin{pmatrix}10&0&0\\0&\frac{4}{10}&0\\0&0&1\end{pmatrix}\end{array}$$

$$\begin{array}{c}A'\\ \begin{pmatrix}10&6&4\\0&\frac{4}{10}&-\frac{4}{10}\\0&0&0\end{pmatrix}\end{array} \xrightarrow{Normalize\ A'\ based\ on\ D} \begin{array}{c}U\\ \begin{pmatrix}1&\frac{6}{10}&\frac{4}{10}\\0&1&-1\\0&0&0\end{pmatrix}\end{array}$$

$Check\ \ the\ \ result:$

$$\begin{array}{c}L\\ \begin{pmatrix}1&0&0\\\frac{1}{10}&1&0\\\frac{1}{2}&0&1\end{pmatrix}\end{array} \quad \begin{array}{c}D\\ \begin{pmatrix}10&0&0\\0&\frac{4}{10}&0\\0&0&1\end{pmatrix}\end{array} \begin{array}{c}U\\ \begin{pmatrix}1&\frac{6}{10}&\frac{4}{10}\\0&1&-1\\0&0&0\end{pmatrix}\end{array} = \begin{pmatrix}10&6&4\\1&1&0\\5&3&2\end{pmatrix} = \begin{array}{c}P\\ \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\end{array} \begin{array}{c}A\\ \begin{pmatrix}10&6&4\\5&3&2\\1&1&0\end{pmatrix}\end{array}$$

Q3

(a)

One unique solution.

$$x = \begin{pmatrix}1\\-2\\3\end{pmatrix}$$

To verify: $Ax = \begin{pmatrix}-2\\2\\4\end{pmatrix} = b$

(b)

Many solutions.

The SVD solution is, $\bar{x} = \frac{1}{3}\begin{pmatrix}1\\-4\\5\end{pmatrix}$.

All solutions can be represented as, $x = \frac{1}{3}\begin{pmatrix}1\\-4\\5\end{pmatrix} + p\begin{pmatrix}1\\-1\\-1\end{pmatrix}$, where $p \in R$.

To verify: $Ax = \begin{pmatrix}2\\1\\-1\end{pmatrix} + p\vec{0} = b$

(c)

No solutions.

The SVD solution below is the least-square solution of the system. It is also the exact solution of $\mathbf{A}x = \bar{b}$, where $\bar{b}$ is the projection of b in the column space of A.

$$\bar{x} = \frac{1}{3}\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

To verify: $A(A\bar{x} - b) = A(\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - b) = A\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \vec{0}$

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Similarity:

- The SVD solutions of (b) and (c) are the same, because the two systems have the same matrix A and the projections of their b on the column space of A are the same which is also the b of (b).
- The SVD solutions of (b) and (c) are both least-square solutions.

Difference:

- Vector b of (b) is in the column space of A, vector b of (c) is the sum of a vector that is in the column space and a vector that is perpendicular to the column space (null space of A^T). Therefore, the SVD solution of (b) is an exact solution that actually satisfies the system, while (c)'s is not.

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Q4

(a)

The matrix A makes x disappear on the dimension of u and in the other (n-1) dimensions keeps x unchanged.
(Also, the matrix A makes a vector x to be the normal line for its own Householder reflection. The new vector Ax falls on the 2-dimensional plane spanned by u and x, and Ax is perpendicular to u.)

Proof:

$$\begin{aligned} Ax &= (I - uu^T)x \\ &= x - uu^Tx \\ &= x - < u, x > u \end{aligned}$$

, where $< u, x > u$ is the projection of x onto u, the difference between x and this projection must be perpendicular to u.

(b)

The eigenvalues consist of 1 zero and (n-1) ones.
Because after transformation A, on the direction of u, Ax becomes nothing, while Ax on any other directions that are perpendicular to u remain the same.

(c)

The null space of A is a 1-dimensional space spanned by vector u.

> To show that u is one of the vectors spanning the null space of A: $Au = u - uu^T u = u - u = \vec{0}$.
> To show that u is the only vector spanning the null space:
> One way to look at it: Because any other direction perpendicular to u will just remain the same, as shown(/proven) in (a), the null space of A does not have any other dimension.
> Another way to look at it: From (b) we have known that A only has 1 zero eigenvalue and (n-1) non-zero eigenvalues, which means it has rank as (n-1), which leaves null space at most 1 dimension.

(d)

$$
\begin{aligned}
A^2 &= (I - uu^T)(I - uu^T) \\
&= I^2 - uu^T - uu^T + uu^T uu^T \\
&= I - 2uu^T + u(u^T u)u^T \\
&= I - 2uu^T + uIu^T \\
&= I - uu^T \\
&= A
\end{aligned}
$$

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Q5

The objective is to minimize f:

$$f = \sum (Rp_i + t - q_i)^T (Rp_i + t - q_i)$$

To find the optimal t for the objective, let $\frac{\partial f}{\partial t} = 0$:

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= 0 \\
\sum 2(Rp_i + t - q_i) &= 0 \\
\sum t &= \sum q_i - R \sum p_i \\
t &= \bar{q} - R\bar{p}
\end{aligned}
$$

Now f becomes:

$$\begin{aligned}
f &= \sum [Rp_i + (\bar{q} - R\bar{p}) - q_i]^T [Rp_i + (\bar{q} - R\bar{p}) - q_i] \\
&= \sum [R(p_i - \bar{p}) - (q_i - \bar{q})]^T [R(p_i - \bar{p}) - (q_i - \bar{q})] \\
&= \sum (p_i - \bar{p})^T R^T R (p_i - \bar{p}) - (q_i - \bar{q})^T R (p_i - \bar{p}) - (p_i - \bar{p})^T R^T (q_i - \bar{q}) + (q_i - \bar{q})^T (q_i - \bar{q}) \\
&= \sum (p_i - \bar{p})^T (p_i - \bar{p}) - 2 \sum (q_i - \bar{q})^T R (p_i - \bar{p}) + \sum (q_i - \bar{q})^T (q_i - \bar{q})
\end{aligned}$$

$\sum (p_i - \bar{p})^T (p_i - \bar{p})$ and $\sum (q_i - \bar{q})^T (q_i - \bar{q})$ are both constant, so to minimize f is to maximize g:

$$g = \sum (q_i - \bar{q})^T R (p_i - \bar{p})$$

Denote:

$$\begin{aligned}
Q &= \begin{bmatrix} (q_1 - \bar{q}) & \dots & (q_n - \bar{q}) \end{bmatrix} \\
P &= \begin{bmatrix} (p_1 - \bar{p}) & \dots & (p_n - \bar{p}) \end{bmatrix}
\end{aligned}$$

Therefore:

$$\begin{aligned}
g &= \sum (q_i - \bar{q})^T R (p_i - \bar{p}) \\
&= Tr(Q^T R P) \\
&= Tr(R P Q^T)
\end{aligned}$$

SVD factorize (PQ^T) as $U \Sigma V^T$:

$$\begin{aligned}
g &= Tr(R U \Sigma V^T) \\
&= Tr(\Sigma V^T R U)
\end{aligned}$$

Denote $M = V^T R U$, then:

$$g = Tr(\Sigma M) = \sum \sigma_i M_{ii}$$

All σ_i are constant derived from set $\mathbf{p}$ and $\mathbf{q}$, therefore, in order to maximize g, each M_{ii} should be its possible maximum value.
M is orthogonal, because $M = V^T R U$ and V^T, R and U are all orthogonal. Therefore, each column of M must be a unit vector, and elements of a unit vector must be ≤ 1.
We will then let all M_{ii} to be the maximum value 1, which means $M = I$.
Therefore,

$$\begin{aligned}
I &= M = V^T R U \\
R &= V U^T
\end{aligned}$$

To eliminate the possible reflection encoded in R, we should force $det(R) = 1$. Therefore we should let

$$R = \begin{cases} V \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} U^T & if \quad det(VU^T) = 1 \\ V \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} U^T & if \quad det(VU^T) = -1 \end{cases}$$

$$R = V \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & det(VU^T) \end{pmatrix} U^T$$

Execute my python code *q5.py* to see the performance of this method on randomly generated examples.

I read this paper for this problem:
Least-Squares Rigid Motion Using SVD, by *Olga Sorkine-Hornung* and *Michael Rabinovich* from *Department of Computer Science, ETH Zurich.*
https://igl.ethz.ch/projects/ARAP/svd_rot.pdf