16-811 HW3

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********** **Q1** **********

a)

$$f^{(0)}(x) = \frac{1}{3} + e^x - e^x, \quad f^{(0)}(0) = \frac{1}{3}$$
$$f^{(1)}(x) = e^x + e^x, \quad f^{(1)}(0) = 2$$
$$f^{(2)}(x) = e^x - e^x, \quad f^{(2)}(0) = 0$$

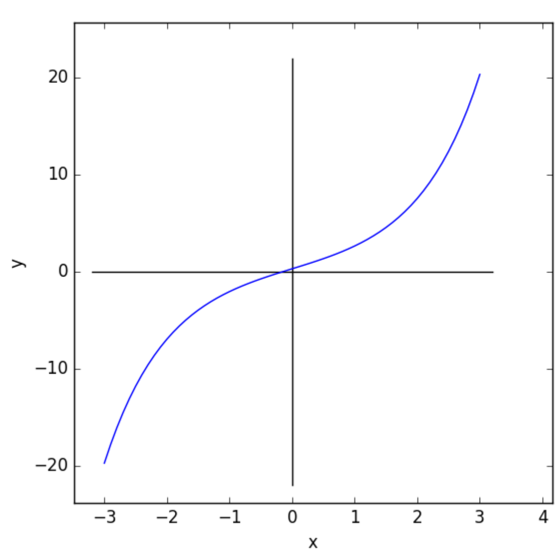
By **Methematical Induction**, it's easy to prove that:

$$f^{(2k-1)}(0) = 2, \quad f^{(2k)}(0) = 0 \quad (k \in \mathbb{N}+)$$

Thus,

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i \\ &= f^{(0)}(0) + \sum_{i=1}^{\infty} \frac{f^{(2i-1)}(0)}{(2i-1)!} x^{2i-1} + \sum_{i=1}^{\infty} \frac{f^{(2i)}(0)}{(2i)!} x^{2i} \\ &= \frac{1}{3} + 2\sum_{i=1}^{\infty} \frac{x^{(2i-1)}}{(2i-1)!} \end{aligned}$$

b) .



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c) Denote:

$$p(x) = a + bx + cx^2$$
$$err(x) = f(x) - p(x)$$
$$err(x_0) = -err(x_1) = err(x_2) = -err(x_3) = \|f(x) - p(x)\|_\infty$$

Since $f^{(n+1)}(x)$ doesn't change sign on [-3,3], we have $x_0 = -3$ and $x_3 = 3$.
From the taylor expansion in Q1(a), we learnt that the coefficient of x^2 is zero (x^2 is in the null space of $f(x)$), thus c is actually zero. Solve equation for a, b, x_1 and x_2 :

$$err(3) = -err(-3) \tag{1}$$
$$err'(x_1) = 0 \tag{2}$$
$$err'(x_2) = 0 \tag{3}$$
$$err(3) = err(x_1) \tag{4}$$
$$err(3) = -err(x_2) \tag{5}$$

Solve equation (1), we have

$$a = \frac{1}{2}$$

Solve equation (2)(3), with the constraints $x1 < x2$ we have

$$x1 = \ln\left(\frac{b - \sqrt{b^2-4}}{2}\right), \quad x2 = \ln\left(\frac{b + \sqrt{b^2-4}}{2}\right) \tag{6}$$

"Surprisingly"(, yet intuitively make sense especially if we shift $f(x)$ back down by $\frac{1}{3}$ to make $f(x)$ an odd function, but anyway), x_1 and x_2 are symmetric! Proof:

$$\begin{aligned} x_1 - x_2 &= \ln\left(\frac{b - \sqrt{b^2-4}}{2} \frac{b + \sqrt{b^2-4}}{2}\right) \\ &= \ln\left(\frac{b^2 - (b^2-4)}{4}\right) \\ &= \ln(1) = 0 \\ \therefore x_1 &= -x_2 \end{aligned}$$

This observation further makes (4)(5) equivalent, because $err(x)$ is an odd function:

$$\begin{aligned} err(x) &= \frac{1}{3} + 2sinh(x) - a - bx \qquad (a = 1 \text{ from equation(1)}) \\ &= 2sinh(x) - bx \qquad (sinh(x) \text{ and } bx \text{ are both odd}) \end{aligned}$$

Thus, we can any one of (4) or (5) with (6) (here I solved it numerically in my code *q1.py* : *q1c()* with Muller Root Finder I implemented for HW2):

$$b = 5.38724$$

Thus, the best uniform quadratic approximation is:

$$p(x) = \frac{1}{3} + 5.38724x$$

The errors are:

$$\begin{aligned}
&L_\infty = err(3) = 3.87403 \\
&L_2 = \sqrt{\int_{-3}^{3} err(x)^2 dx} \\
&= \sqrt{\int_{-3}^{3} (e^x - e^{-x} - bx)^2 dx} \\
&= \sqrt{\int_{-3}^{3} [(e^{2x} + e^{-2x} - 2) - 2(e^x bx - e^{-x} bx) + b^2x^2] dx} \\
&= \sqrt{\{\frac{e^{2x} + e^{-2x}}{2} - 2x - 2[e^x(bx - b)} \\
&\qquad + e^{-x}(bx + b)] + \frac{b^2x^3}{3}\}\Big|_{-3}^{3} \\
&= 6.62513 \text{ (Verified numerically in } \textit{q1.py : q1c()}\text{)}
\end{aligned}$$

d) Based on

$$\begin{aligned}
&p_{-1}(x) = 0 \\
&p_0(x) = 1 \\
&p_{i+1}x = xp_i - \frac{\langle xp_i, p_i \rangle}{\langle p_i, p_i \rangle} p_i - \frac{\langle p_i, p_i \rangle}{\langle p_{i-1}, p_{i-1} \rangle} p_{i-1}
\end{aligned}$$

, we calculate:

$$\begin{aligned}
&\langle p_0, p_0 \rangle = \int_{-3}^{3} (1 * 1) dx = 6 \\
&\langle xp_0, p_0 \rangle = \int_{-3}^{3} (x * 1) dx = 0 \\
&p_1 = x - 0 - 0 = x \\
&\langle p_1, p_1 \rangle = \int_{-3}^{3} (x * x) dx = 18 \\
&\langle xp_1, p_1 \rangle = \int_{-3}^{3} (x * x * x) dx = 0 \\
&p_2 = x * x - 0 * x - \frac{18}{6} * 1 = x^2 - 3
\end{aligned}$$

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Further we calculate:

$$\begin{aligned}
\langle f, p_0 \rangle &= \int_{-3}^{3} \frac{1}{3} + 2\sinh(x) dx = 2 \\
\langle f, p_1 \rangle &= \int_{-3}^{3} \frac{x}{3} + 2x\sinh(x) dx \\
&= [(x-1)e^x + (x+1)e^{-x}]\Big|_{-3}^{3} \\
&= 80.74044 \\
\langle f, p_2 \rangle &= \int_{-3}^{3} [\frac{1}{3} + 2\sinh(x)][x^2 - 3] dx \\
&= \int_{-3}^{3} [\frac{x^2}{3} - 1] dx \\
&= [\frac{x^3}{9} - x]\Big|_{-3}^{3} \\
&= 0
\end{aligned}$$

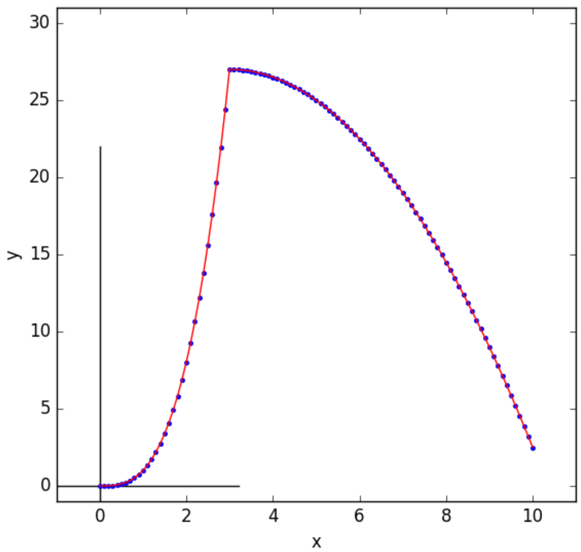
Thus,

$$\begin{aligned}
p(x) &= \sum_{i=0}^{2} \frac{\langle f, p_i \rangle}{\langle p_i, p_i \rangle} p_i(x) \\
&= \frac{2}{6} * 1 + \frac{80.74044}{18} * x + 0 \\
&= \frac{1}{3} + 4.48558 * x
\end{aligned}$$

The errors are:

$$\begin{aligned}
&err'(\hat{x}) = e^{\hat{x}} + e^{-\hat{x}} - b = 0 \quad (b = 4.48558) \\
&\therefore \hat{x} = \ln(\frac{b \pm \sqrt{b^2 - 4}}{2}) = \pm 1.44699 \\
&err(\hat{x}) = 2.47556 \\
&L_\infty = \max[err(\hat{x}), err(3), err(-3)] = 6.57901 \\
&L_2 = 5.40912 \quad \text{(Same derivation and verification with Q1c.)}
\end{aligned}$$

********** **Q2** **********



$$f(x) = \begin{cases} x^3 & x \leq 3 \\ 22.5 + 3x - 0.5x^2 & x > 3 \end{cases}$$

The function basis I picked are [1,x,x^2,x^3,x^4,x^5]:

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$$

Data are group into two ($x \leq 3$ and $x > 3$). For each group, matrix A has $A_{ij} = (x_i)^j$ and vector b has $bi = f(x_i)$, and linear equation $Ac = b$ is solved using SVD.
Code is in *q2.py* .

********** **Q3** **********

a)

$$\begin{aligned}
&T_1(\cos\theta) = \cos\theta \\
\therefore &T_1(x) = x \\
&T_2(\cos\theta) = \cos 2\theta = 2\cos x^2 - 1 \\
\therefore &T_2(x) = 2x^2 - 1
\end{aligned}$$

Thus,

$$\begin{aligned}
&T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x \\
&T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1
\end{aligned}$$

b)

$$\begin{aligned}
T_3(x) * T_4(x) &= (4x^3 - 3x)(8x^4 - 8x^2 + 1) \\
&= 32x^7 - 56x^5 + 28x^3 - 3x
\end{aligned}$$

It is an odd function. $\frac{1}{\sqrt{1-x^2}}$ is even, thus the function being integrated is odd. Integrating an odd function over [-1,1] results in zero. Thus $T_3(x)$ and $T_4(x)$ are orthogonal.

c)

$$x = \cos\theta$$

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Thus,

$$x \in [-1, 1] \to \theta \in [(2k-1)\pi, 2k\pi],\ k \in \mathbb{N}$$

$$\begin{aligned}
\|T_n(x)\|^2 &= \langle T_n(x), T_n(x) \rangle \\
&= \int_{-1}^{1} \frac{T_n(x)^2}{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} \frac{T_n(\cos\theta)^2}{\sqrt{1-\cos\theta^2}} d\cos\theta \\
&= \int_{\pi}^{2\pi} \frac{T_n(\cos\theta)^2}{-\sin\theta} (-\sin\theta) d\theta \\
&= \frac{1}{2} \int_{\pi}^{2\pi} 2\cos n\theta^2 - 1 + 1 d\theta \\
&= \frac{1}{2} \int_{\pi}^{2\pi} \cos 2n\theta + 1 d\theta \\
&= \frac{1}{2} [(\frac{\sin 2n\theta}{2n} + \theta)\Big|_{\pi}^{2\pi}] \\
&= \frac{1}{2} [(0 + 2\pi) - (0 + \pi)] \\
&= \frac{\pi}{2} \\
\|T_n(x)\| &= \sqrt{\frac{\pi}{2}}
\end{aligned}$$

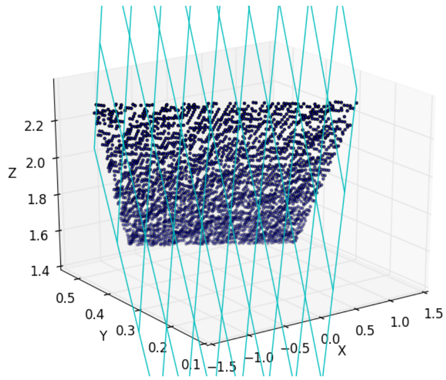
The length of $T_n(x)$ is a constant, thus all $T_n(x)$ have the same length.

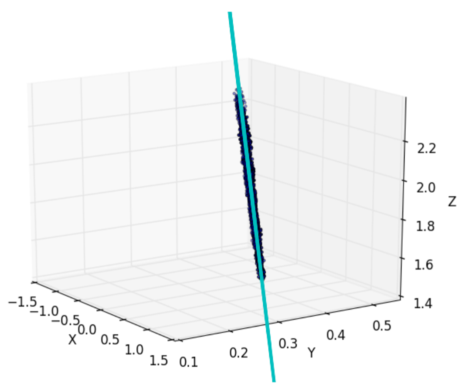
d)

$$\begin{aligned}
\langle T_i(x), T_j(x) \rangle &= \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} \frac{T_i(\cos\theta)T_j(\cos\theta)}{\sqrt{1-\cos\theta^2}} d\cos\theta \\
&= \int_{\pi}^{2\pi} \frac{\cos i\theta \cos j\theta}{-\sin\theta} (-\sin\theta) d\theta \\
&= \frac{1}{2} \int_{\pi}^{2\pi} \cos[(i+j)\theta] + \cos[(i-j)\theta] d\theta \\
&= \frac{1}{2} [(\frac{\sin[(i+j)\theta]}{(i+j)} + \frac{\sin[(i-j)\theta]}{(i-j)})\Big|_{\pi}^{2\pi}] \\
&= \frac{1}{2} [(0+0) - (0+0)] = 0
\end{aligned}$$

********** **Q4** **********

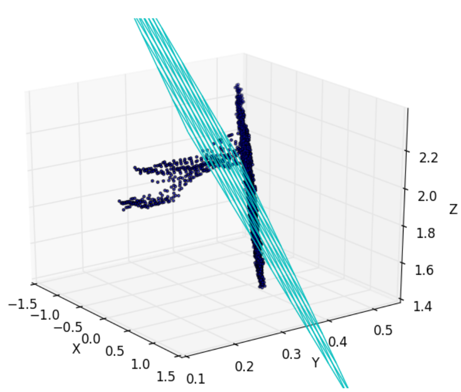
a) .

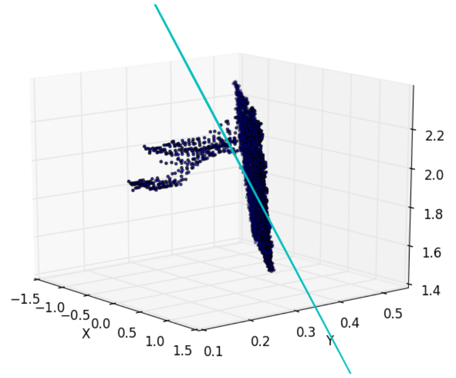




The average distance is: 0.00274

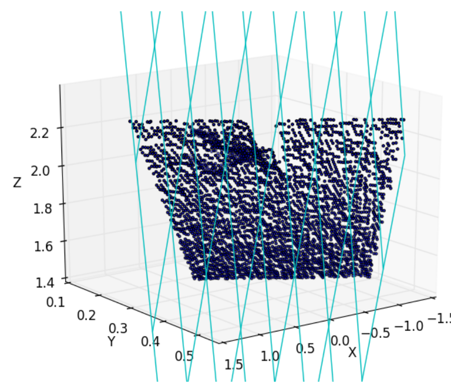
b) .

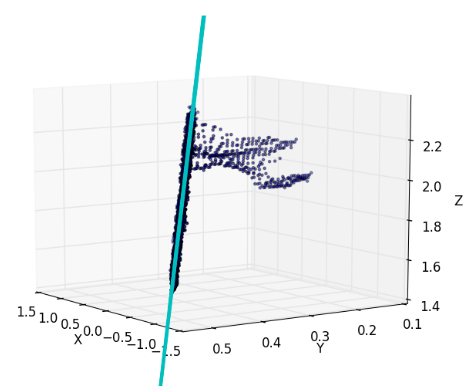




It doesn't look like the plane fitted the table surface, because least square approximation is affected by outliers.

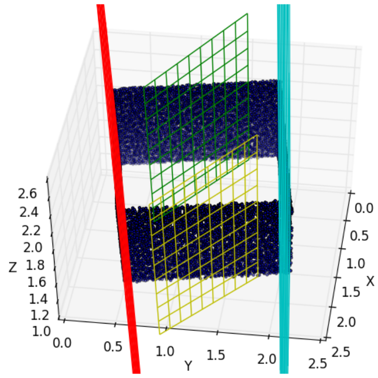
c) .

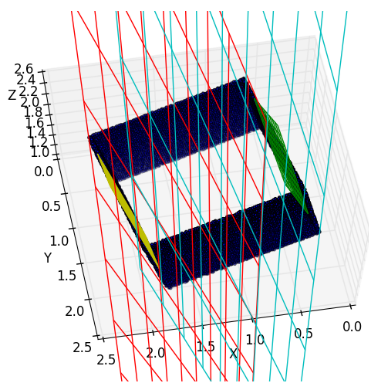




I applied RANSAC to reject outliers and fit a plane to the rest of the good data.

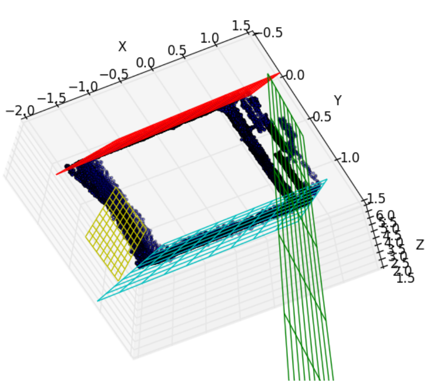
d) .

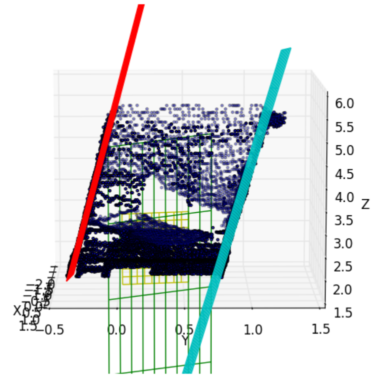


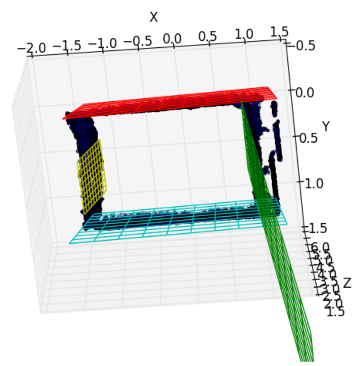


The planes of four walls are in color cyan, red, yellow, green. I used solution from Q4c, to find one plane at a time, and then remove the points on this plane from data set, and look for the next plane, until I have all four planes.

e) .







The planes of four walls are in color cyan, red, yellow, green.

I took the concept of "average distance" from Q4a, and my smoothness scoring system for a particular plane works like: (1) from dataset, find all points on this plane (close enough to this plane judged by a constant distance threshold), (2) calculate the average distance of these points to this plane, call it "smoothness score". The smaller the score is, the smoother the plane is.

plane0(cyan) score: 0.048

plane1(red) score: 0.052

plane2(yellow) score: 0.098

plane3(green) score: 0.119

The most smooth plane is plane0(cyan).