Computational Imaging and Spectroscopy: Sparse and redundant representations

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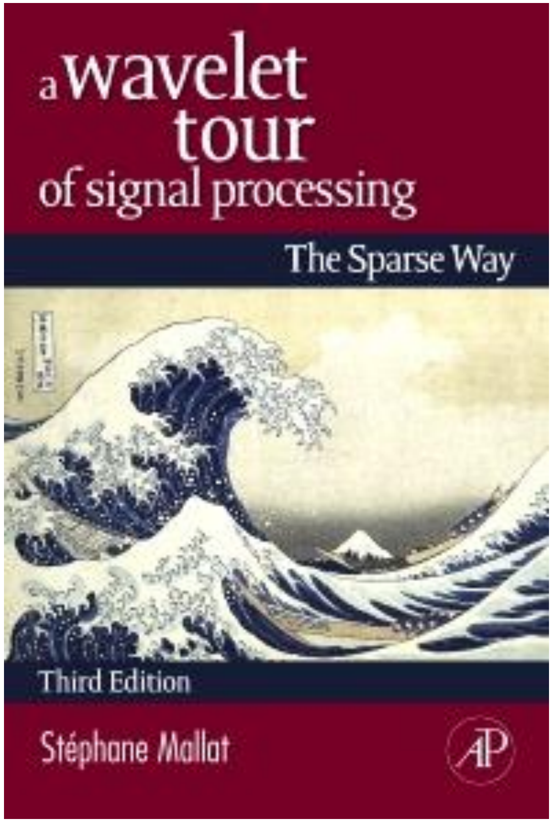
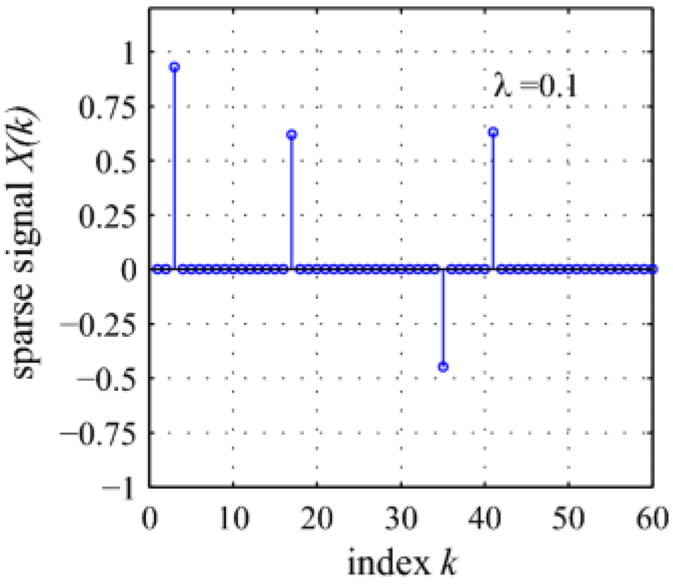
Reference

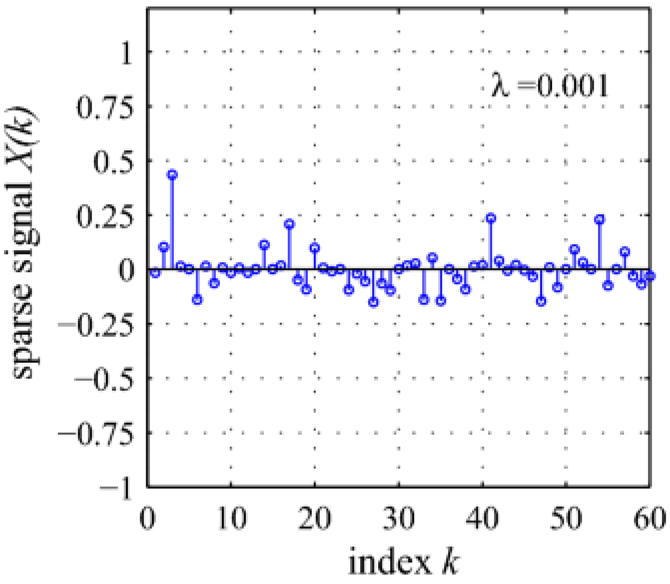


Stephane Mallat

A Wavelet Tour of Signal Processing 3rd Edition

What is sparsity?





What is sparsity?

Strictly sparse signals

A signal $x \in \mathbb{R}^N$ is said to be k-sparse if its support is of cardinality $k \ll N$

Therefore, a k-sparse has k non zeros entries

If a signal is not sparse it can be sparsified in appropriate transform domain. We can model a signal x as a linear combination of T elementary waveforms called atoms

$$x = \Phi \alpha = \sum_{i=1}^{T} \alpha[i] \varphi_i$$

The $N \times T$ matrix $\Phi = [\varphi_1, \cdots, \varphi_T]$ is called a dictionary, in general $\|\varphi_i\|^2 = \sum_{n=1}^{N} |\varphi_i[n]|^2 = 1$

What is sparsity?

Compressible signal

In practice signal are in general not strictly sparse but they may be compressible or weakly sparse. In this case the sorted magnitudes of the representation coefficients $\alpha = \Phi^T x$ decay quickly:

$$\left|\alpha_{(i)}\right| \le C_i^{-1/s}, \qquad i = 1,\ldots,T$$

The nonlinear approximation error decays of its k largest coefficients decays as

$$\left\|x - x_k\right\| \le C\left(2/s - 1\right)^{1/2} K^{1/2-1/s}, \qquad s < 2$$

What is sparsity?

Atoms

An atom is an elementary signal representing template. Examples include Wavelets, sinusoids, gaussians, etc.

Dictionary

A dictionary $\boldsymbol{\Phi}$ is an indexed collection of atoms $\left(\varphi_\gamma\right)_{\gamma \in \Gamma}$, where Γ is a countable set of cardinality T

The index γ depends on the dictionary: frequency for Fourier, scale and position for Wavelets, etc.

In discrete time finite length signals a dictionary is viewed as a $N \times T$ matrix whose columns are the atoms, considered as column vectors. When $N < T$ the dictionary is said to be overcomplete or redundant

What is sparsity?

Analysis and synthesis

Analysis is the operation which associates with each signal x a vector of coefficients α attached to an atom

$$\boldsymbol{\alpha} = \boldsymbol{\Phi}^T \boldsymbol{x}$$

Synthesis is the operation of reconstructing the signal by superposing atoms:

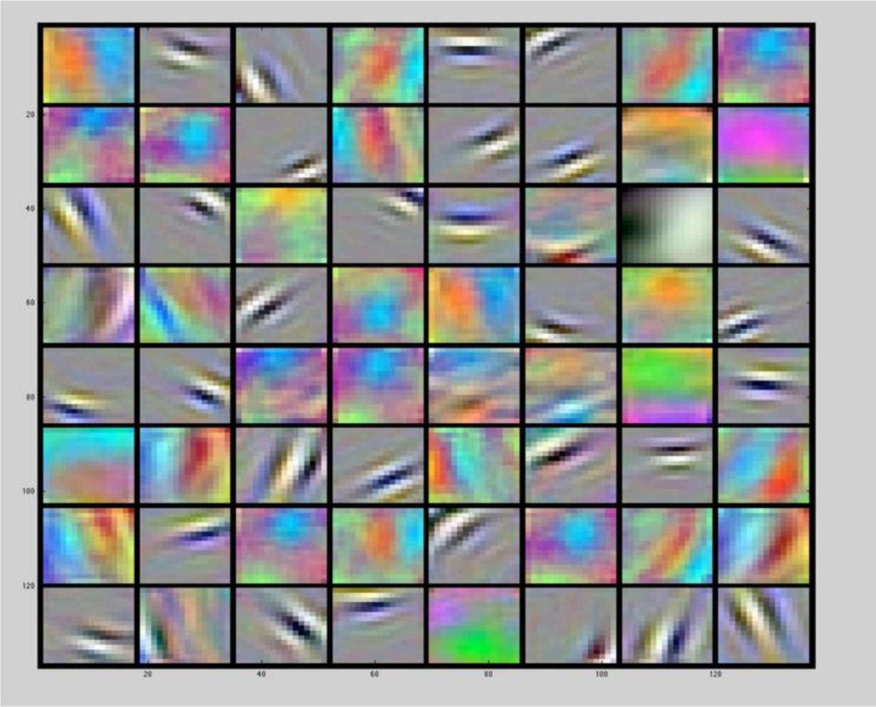
$$\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{\alpha}$$

What is sparsity?

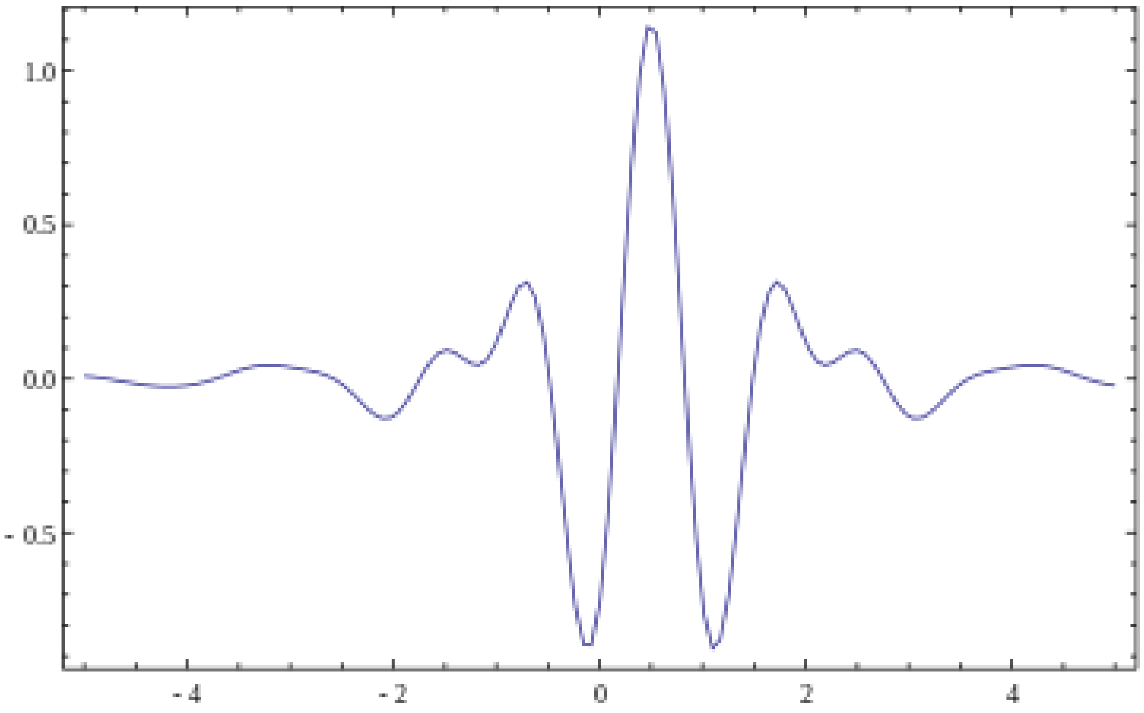
Dictionaries

Several dictionaries have been proposed depending on the underlying nature of the signals:

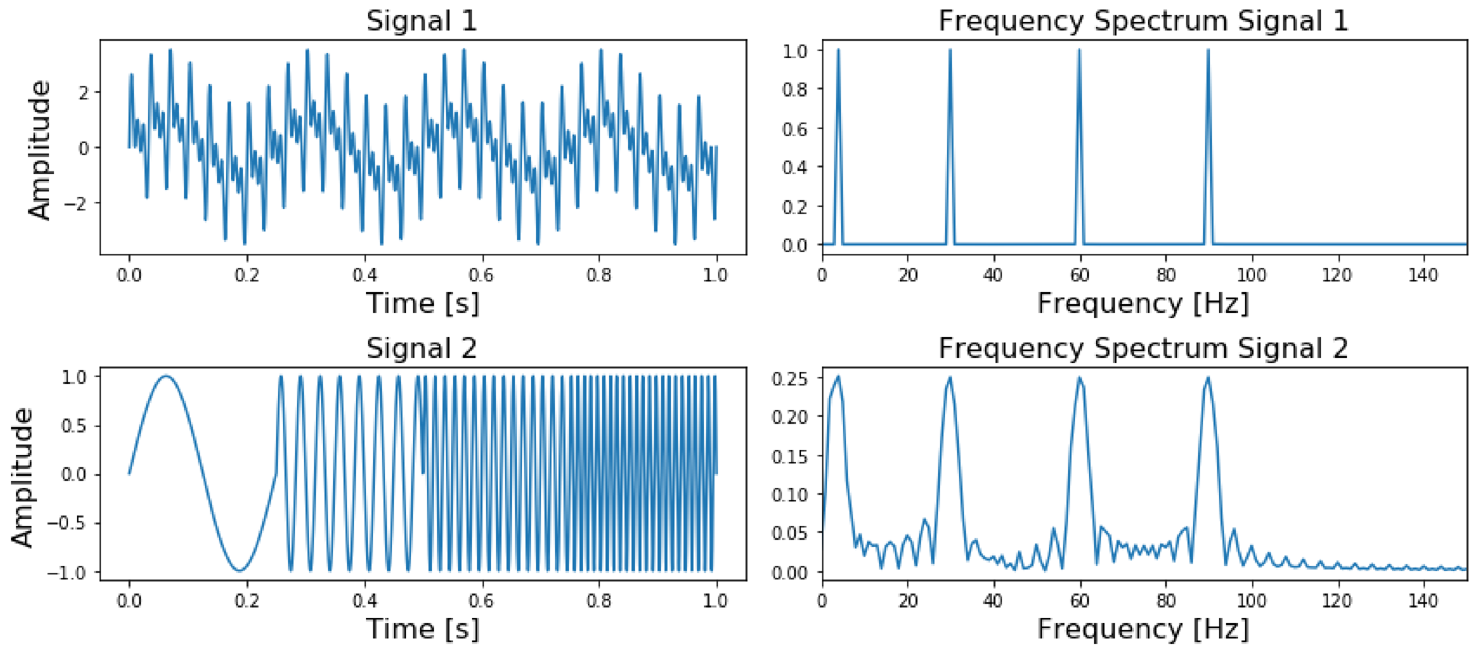
- ❑ Fourier,
- ❑ Wavelets,
- ❑ Curvelets
- ❑ Bandelets
- ❑ Ridgelets
- ❑ Noiselets
- ❑ etc



From Fourier to Wavelets



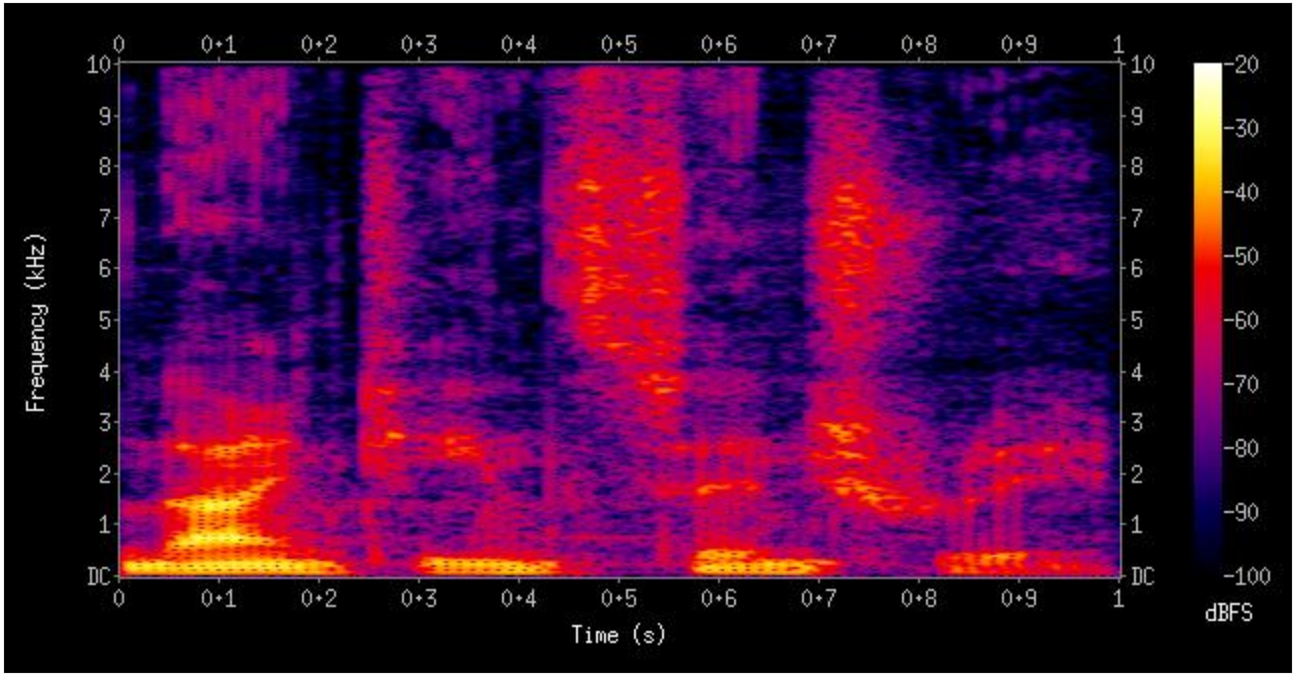
From Fourier to Wavelets



FFT *cannot* tell *when* the frequency peaks occur!

From Fourier to Wavelets

Short time Fourier Transform

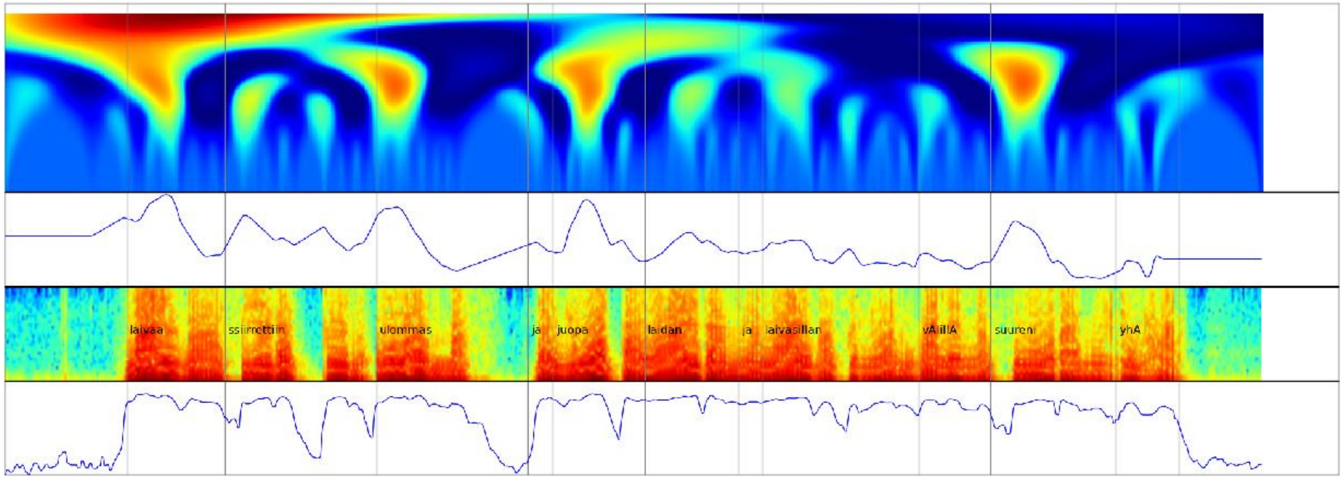


$$\mathbf{STFT}\{x(t)\}(\tau,\omega) \equiv X(\tau,\omega) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-i\omega t}\,dt$$

Continuous Wavelet transform

Continuous Wavelet transform $(f \in \boldsymbol{L_2}(\mathbb{R}))$

$$W(a,b)=\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} f(t)\psi^{*}\left(\frac{t-b}{a}\right)dt$$

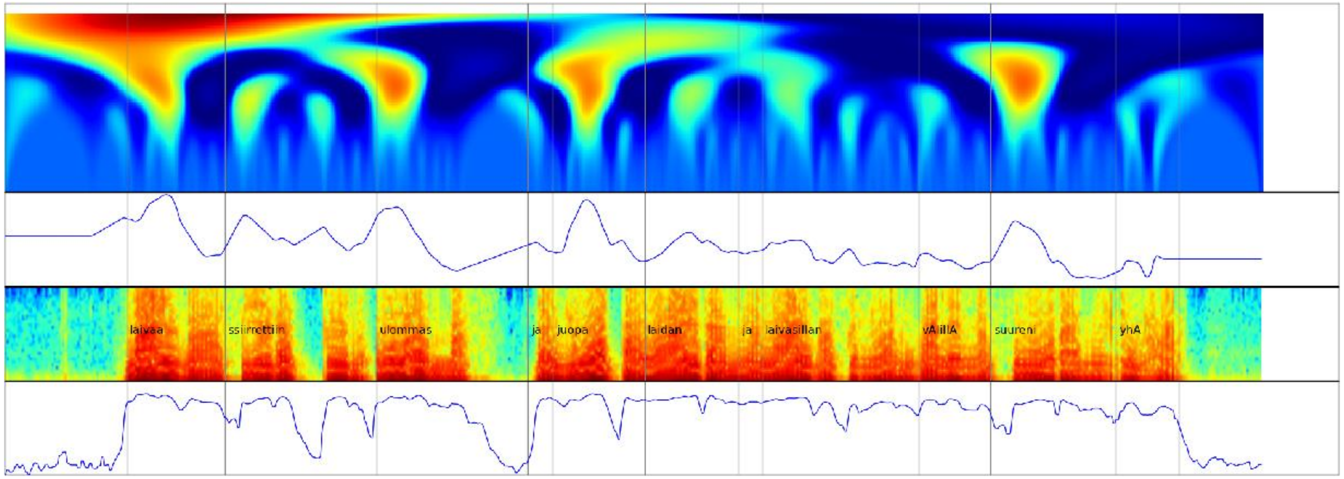


Continuous Wavelet transform

Continuous Wavelet transform $(f \in \boldsymbol{L_2}(\mathbb{R}))$

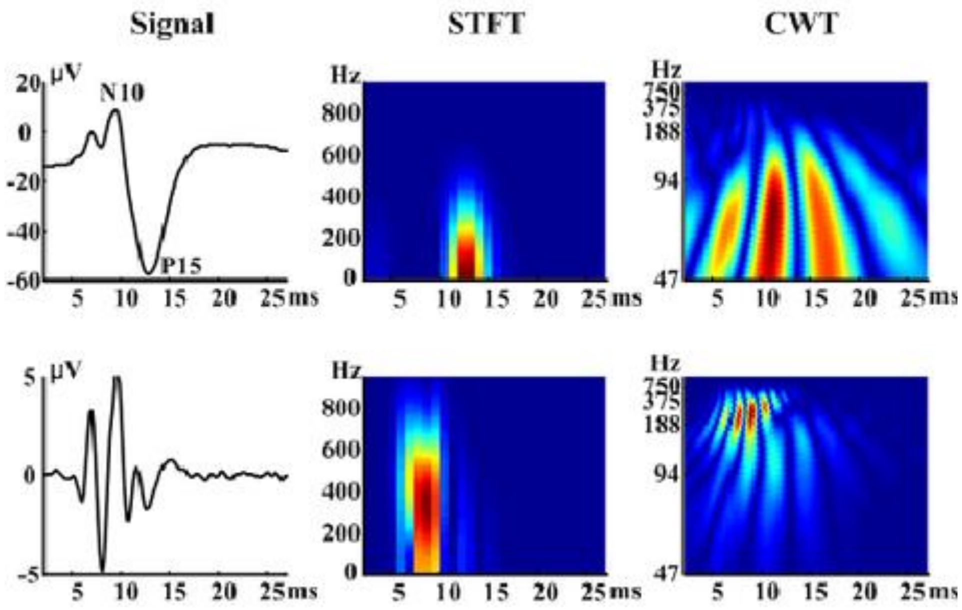
In the Fourier domain we have

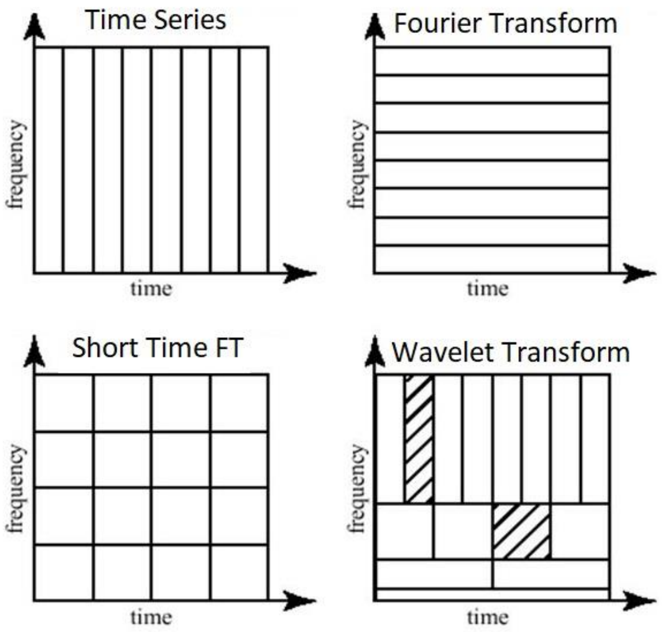
$$\hat{W}(a,v) = \sqrt{a}\,\hat{f}(v)\hat{\psi}^*(av)$$



Continuous Wavelet transform

Continuous Wavelet transform $(f \in L_2(\mathbb{R}))$





Continuous Wavelet transform

Properties of the wavelet transform

i. CWT is a linear transform, for any scalar $\rho_1$and ρ_2

If $f(t)$=$\rho_1 f_1(t) + \rho_2 f_2(t)$ then $W_f(a,b) = \rho_1 W_{f_1}(a,b) \quad + \rho_2 W_{f_2}(a,b)$

ii. CWT is covariant under translation

If $f_0(t) = f(t - t_0)$ then $W_{f_0}(a,b)$=$W_f(a, b - t_0)$

iii. CWT is covariant under dilation

If $f_s(t) = f(st)$ then $W_{f_s}(a,b)$=$\frac{1}{\sqrt{s}} W_f(sa, sb)$

Continuous Wavelet transform

Inverse transform

$$f\left(t\right)=\frac{1}{C_{\chi}}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{a}}W\left(a,b\right)\chi\left(\frac{t-b}{a}\right)\frac{dadb}{a^{2}}$$

$$C_{\chi}=\int_{0}^{+\infty}\frac{\hat{\psi}^{*}\left(v\right)\hat{\chi}\left(v\right)}{v}dv=\int_{-\infty}^{0}\frac{\hat{\psi}^{*}\left(v\right)\hat{\chi}\left(v\right)}{v}dv$$

With the admissibility condition $\mathrm{C}_{\chi}<+\infty \Rightarrow \hat{\psi}(0)=0$

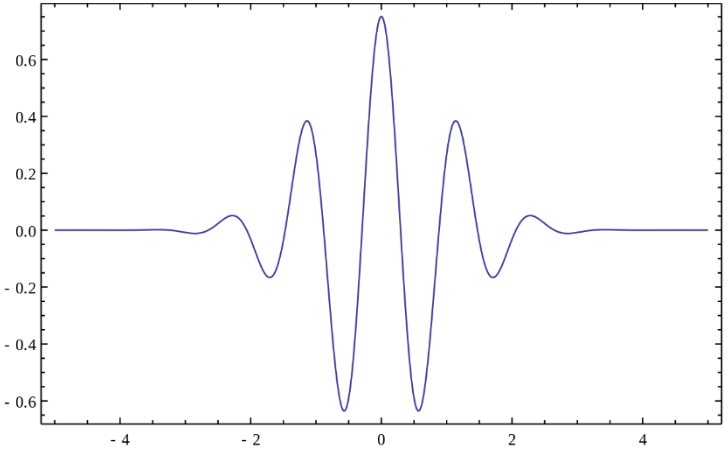
Continuous Wavelet transform

Morlet's wavelet

$$\hat{\psi}(v) = e^{-2\pi^2(v-v_0)^2}$$

$$\Re(\psi(t)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \cos(2\pi v_0 t)$$

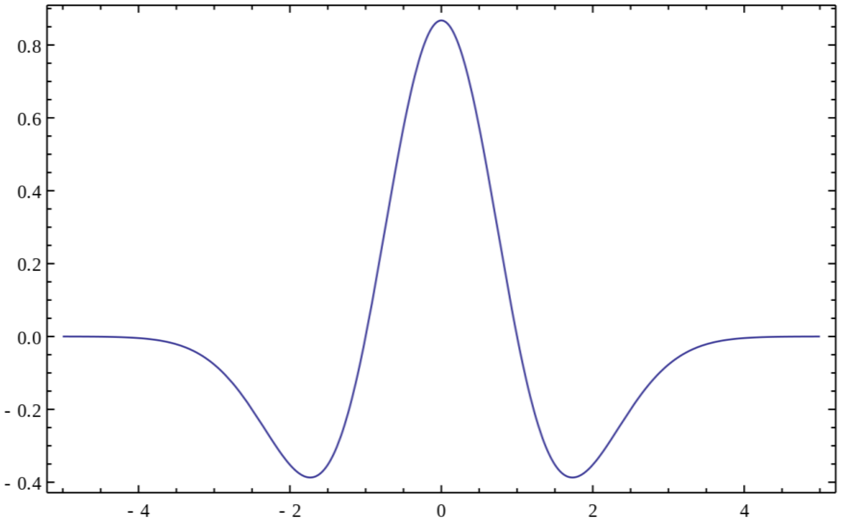
$$\Im(\psi(t)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \sin(2\pi v_0 t)$$



Continuous Wavelet transform

Mexican Hat

$$\psi(t) = \left(1 - t^2\right) e^{-\frac{t^2}{2}}$$



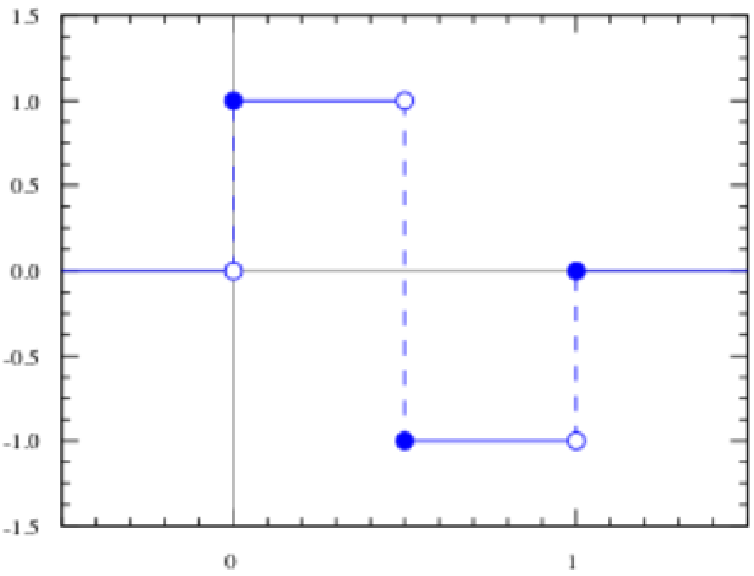
Continuous Wavelet transform

Haar wavelet

$$\psi_{m,n}(t) = a_0^{-m/2}\psi\left(a_0^{-m}\left(t - nb_0 a_0^m\right)\right)$$

With $a_0 = 2$ and $b = 1$

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Multiresolution analysis and DWT

Discrete wavelet transform (DWT)

Multiresolution analysis

$$W(a,b) = \frac{1}{\sqrt{2^j}} \int_{-\infty}^{+\infty} f(t)\psi^*\left(\frac{t-b}{2^j}\right)dt$$

Approximation subspaces

$$\ldots \subset V_3 \subset V_2 \subset V_1 \subset V_0 \ldots$$

If $f(t) \in V_j \rightarrow f(2t) \in \boldsymbol{V_{j+1}}$. The function $f(t)$ is projected at each level j into the subspace $\boldsymbol{V_j}$. This projection is defined by the approximation coefficient $c_j[l]$

Multiresolution analysis and DWT

Discrete wavelet transform

Multiresolution analysis (scaling function)

$$c_j[l] = \left\langle f, \phi_{j,l} \right\rangle = \left\langle f, 2^{-j}\phi\left(2^{-j}\cdot - l\right)\right\rangle$$

The scaling function ϕ having the following properties

$$\frac{1}{2}\phi\left(\frac{t}{2}\right) = \sum_k h[k]\phi(t-k)$$

The coefficients c_{j+1} can be computed directly from $c_j[l]$

$$c_{j+1}[l] = \sum_k h[k-2l]c_j[k]$$

Multiresolution analysis and DWT

Discrete wavelet transform

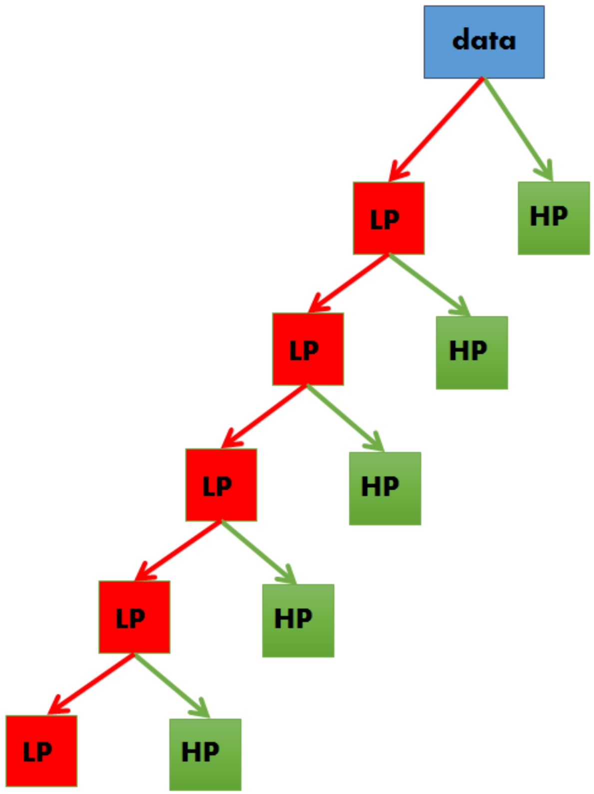
Multiresolution analysis (Wavelet function)

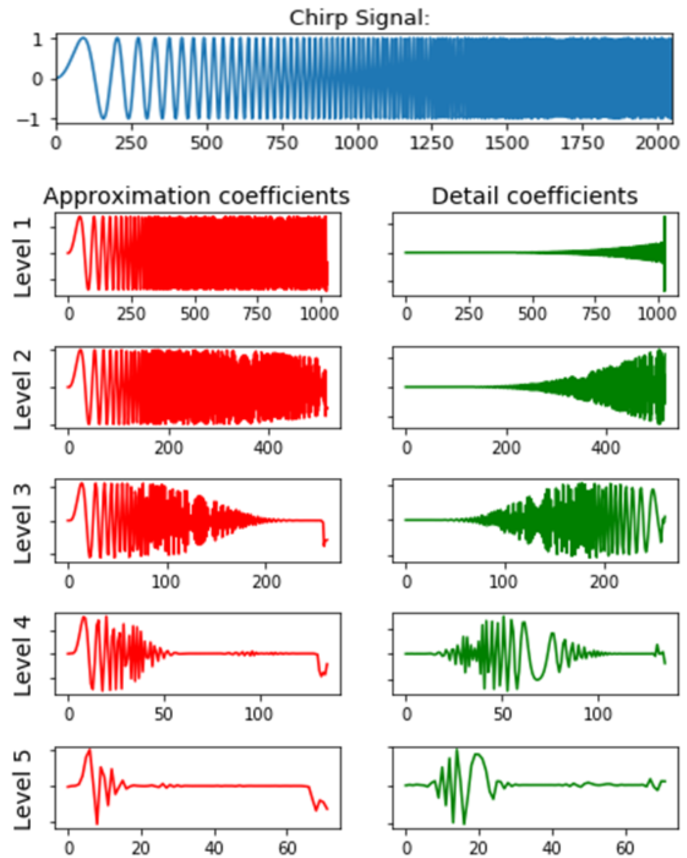
$$\frac{1}{2}\psi\left(\frac{t}{2}\right)=\sum_{k}h[k]\phi(t-k)$$

The coefficients are computated by the following inner product:

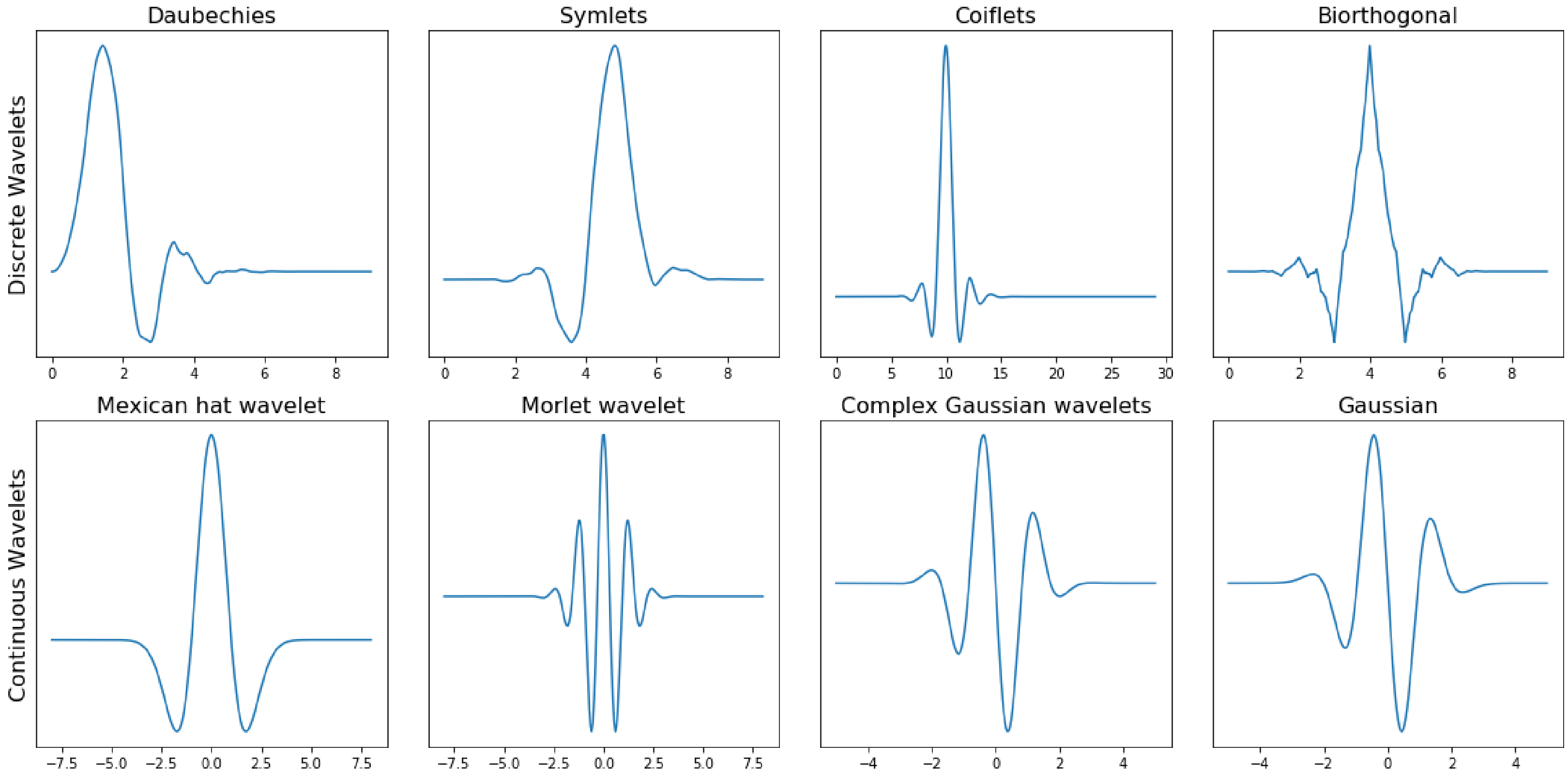
$$w_{j+1}[l]=\sum_{k}g[k-2l]c_{j}[k]$$

Multiresolution analysis and DWT





Multiresolution analysis and DWT

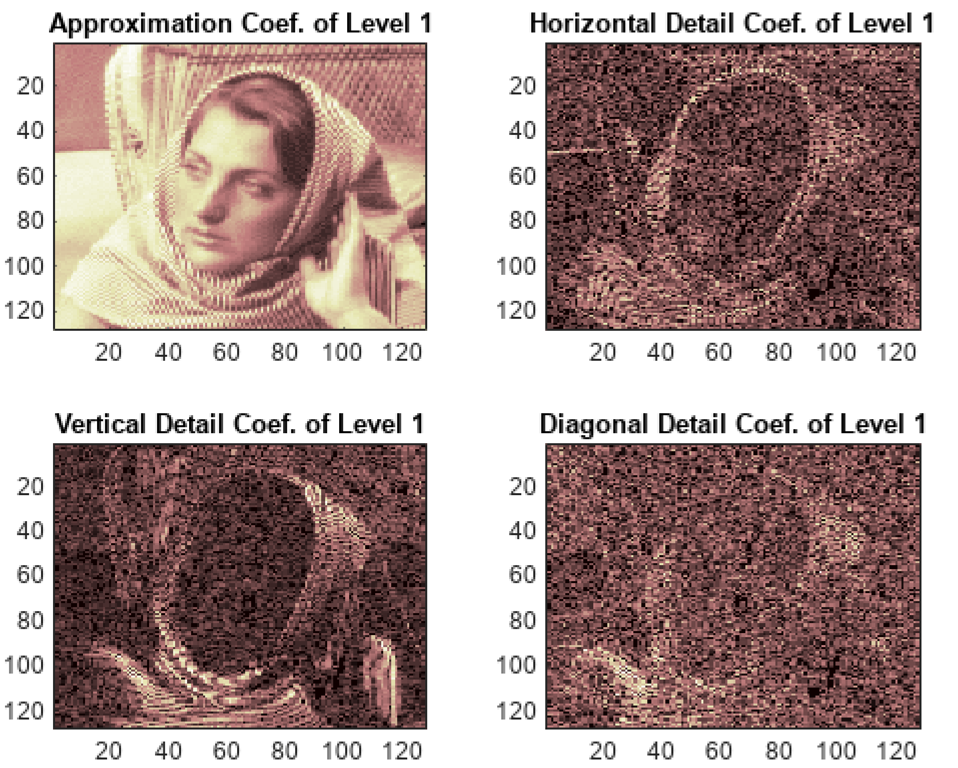


Multiresolution analysis and DWT

Discrete wavelet transform

Two dimensional decimated wavelet transform

$$c_{j+1}[l] = \sum_k h[m-2k]h[n-2l]c_j[m,n]$$
$$= \left[\overline{hh} * c_j\right]_{\downarrow 2,2}[k,l]$$



Multiresolution analysis and DWT

Discrete wavelet transform

Two dimensional decimated wavelet transform (detail coefficients)

- **Vertical wavelet:** $\psi^1(t_1, t_2) = \phi(t_1)\psi(t_2)$
- **Horizontal wavelet:** $\psi^2(t_1, t_2) = \psi(t_1)\phi(t_2)$
- **Diagonal wavelet:** $\psi^3(t_1, t_2) = \psi(t_1)\psi(t_2)$

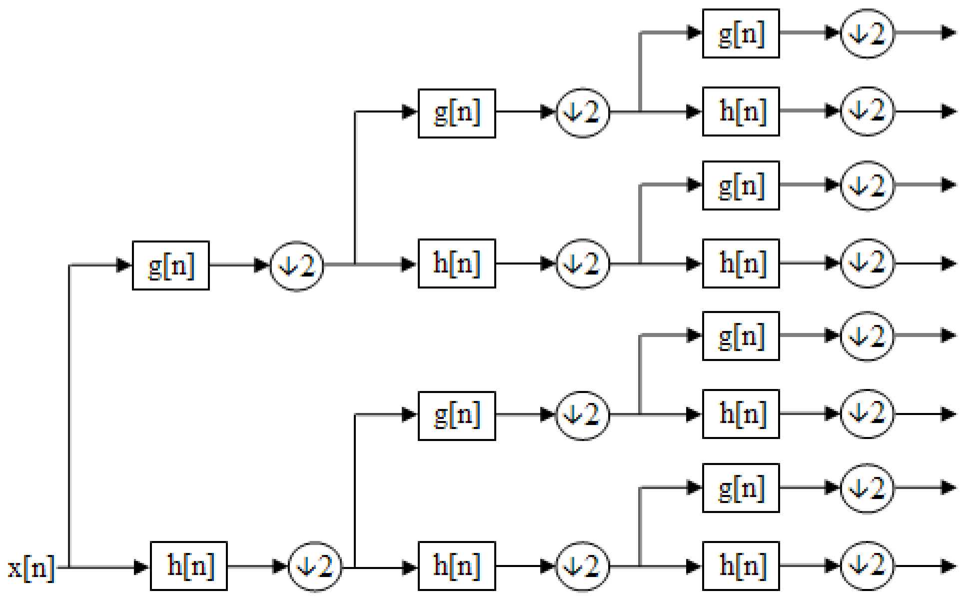
$$w_{j+1}^{1}[k,l] = \sum_{m,n} g[m-2k]h[n-2l]c_j[m,n]$$

$$w_{j+1}^{2}[k,l] = \sum_{m,n} h[m-2k]g[n-2l]c_j[m,n]$$

$$w_{j+1}^{3}[k,l] = \sum_{m,n} g[m-2k]g[n-2l]c_j[m,n]$$

Wavelet packets

Wavelets packet



Wavelet packets

Wavelet packets

Wavelets packets are a generalization of the wavelets, in the sense that the details spaces are also divided.

Let the sequence of functions be defined recursively as follow

$$\psi^{2p}\left(2^{-(j+1)}t\right) = 2\sum_{l\in\mathbb{Z}} h[l]\psi^{p}\left(2^{-j}t - l\right)$$
$$\psi^{2p+1}(2^{-(j+1)}t) = 2\sum_{l\in\mathbb{Z}} g[l]\psi^{p}(2^{-j}t - l)$$

In the Fourier domain

$$\hat{\psi}^{2p}\left(2^{-(j+1)}v\right) = \hat{h}\left(2^{j}v\right)\hat{\psi}^{p}\left(2^{j}v\right)$$
$$\hat{\psi}^{2p+1}(2^{-(j+1)}v) = \hat{g}(2^{j}v)\hat{\psi}^{p}(2^{j}v)$$

Wavelet packets

Wavelet packets

So we have the following recursive expressions

$$w_{j+1}^{2p}[l] = \sum_k h[k-2l]w_j^p[k] = \left[\bar{h} \star w_j^p\right]_{\downarrow 2}[l]$$

$$w_{j+1}^{2p+1}[l] = \sum_k g[k-2l]w_j^p[k] = \left[\bar{g} \star w_j^p\right]_{\downarrow 2}[l]$$

Reconstruction:

$$w_j^p[l] = \sum_k \left(h[k+2l]w_{j+1}^{2p}[k] + g[k+2l]w_{j+1}^{2p+1}[k]\right)$$

Redundant representations

Undecimated Wavelet Transform (Translation Invariant or Stationary)

$$c_{j+1}[l] = \left(\bar{h}^{(j)} * c_j\right)[l] = \sum_k h[k]\, c_j\left[l + 2^j k\right]$$

$$w_{j+1}[l] = \left(\bar{g}^{(j)} * c_j\right)[l] = \sum_k g[k]\, c_j\left[l + 2^j k\right]$$

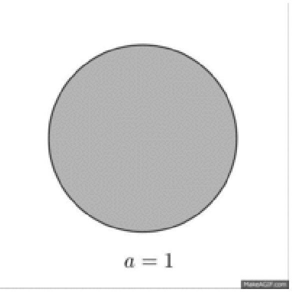
$$\begin{cases} h^{(j)}[l] = h^{(j)}[l] & \text{if } \; l/2^j \; \text{is integer} \\ 0 & \text{otherwise} \end{cases}$$

Reconstruction

$$c_j[l] = \left(\tilde{h}^{(j)} * c_{j+1}\right)[l] + \left(\tilde{g}^{(j)} * w_{j+1}\right)[l]$$

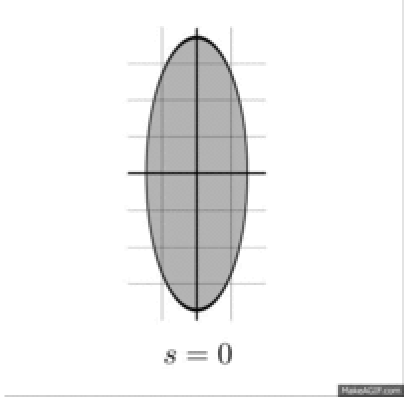
Shearlets Transform

Continous shearlets system



Parabolic scaling matrix (resolution)

$$A_a = \begin{bmatrix} a & 0 \\ 0 & a^{1/2} \end{bmatrix}$$ with a strictly positive



Shear matrix (orientation)

$$S_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \quad s \in \mathbb{R}$$

Shearlets Transform

Continous shearlets system

For $\psi \in \boldsymbol{L}^2(\mathbb{R}^2)$ we have

$$\mathrm{SH}_{cont}(\psi) = \left\{\psi_{a,s,t} = a^{3/4}\left(\boldsymbol{S}_s \boldsymbol{A}_a(\cdot - t)\right) \middle|\, a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2\right\}$$

Then the transform is defined as

$$\left\langle f, \psi_{a,s,t} \right\rangle$$

Shearlets Transform

Discrete shearlet systems

A discrete version of the shearlet system can be obtained from the continous system by discretizing the parameters set

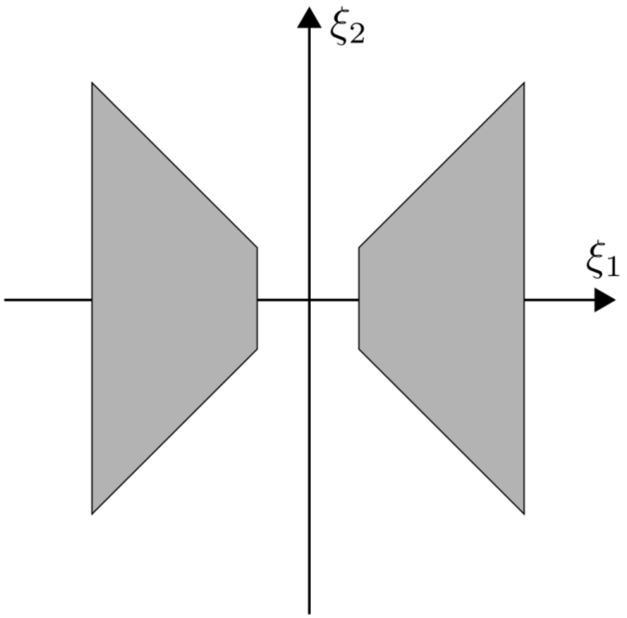
$$\mathrm{SH}_{discrete}(\psi) = \left\{\psi_{j,k,m} = 2^{3j/4}\left(\boldsymbol{S}_k\boldsymbol{A}_{2^j}\cdot - m\right) \middle|\, j \in \mathbb{Z}, k \in \mathbb{Z}, m \in \mathbb{Z}^2\right\}$$

Then the transform is defined as

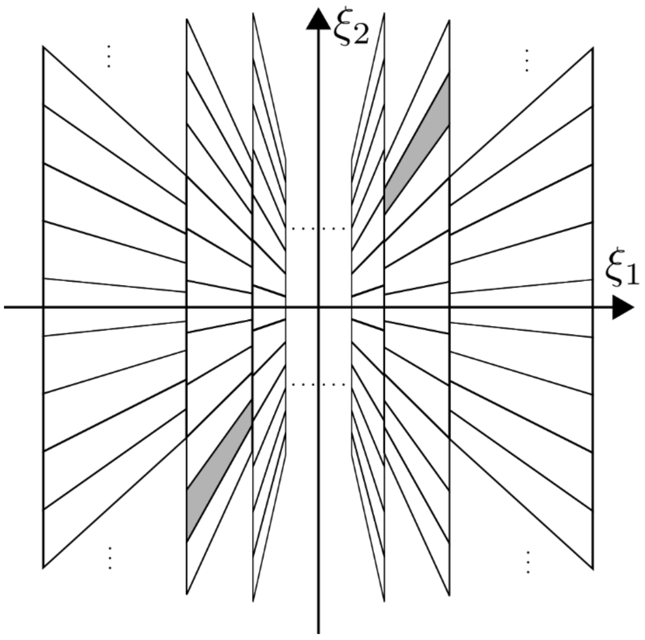
$$\langle f, \psi_{j,k,m} \rangle$$

Shearlets Transform

Discrete shearlet systems



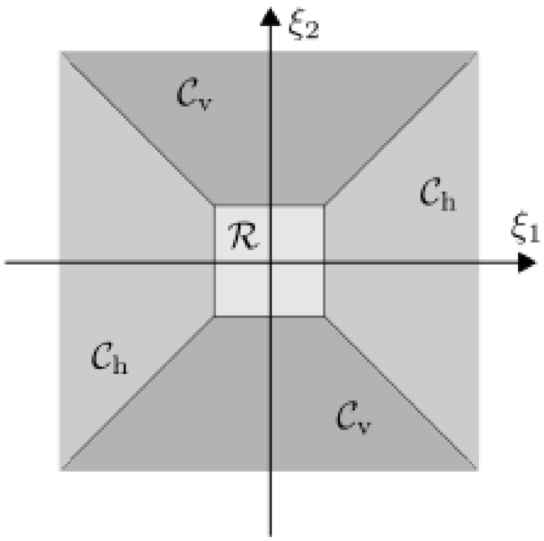
Trapezoidal frequency support of the classical shearlet



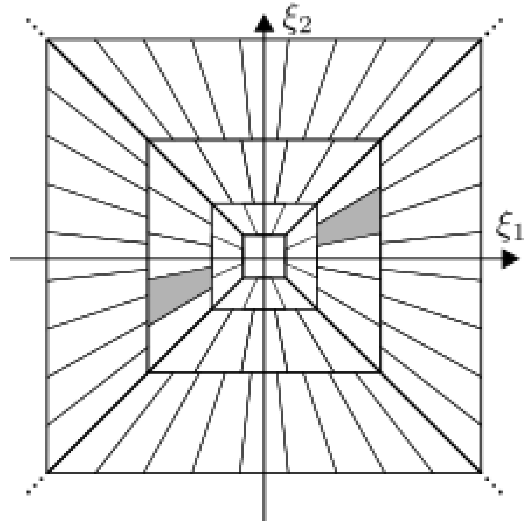
Frequency tiling of the (discrete) classical shearlet system.

Shearlets Transform

Discrete shearlets system (cone-adapated)



Decomposition of the frequency domain into cones.



Frequency tiling of the cone-adapted shearlet system generated by the classical shearlet

Nonlinear approximation

Application 1

Nonlinear approximation by DWT and UWT

Load the cameraman image

Apply a DWT, sort and keep only the 10% larger coefficients
Reconstruct

Apply a UWT, sort and keep only the 10% larger coefficients
Reconstruct

Plot
Compare results (visually and with SSIM)
https://en.wikipedia.org/wiki/Structural_similarity_index_measure

Nonlinear approximation

PyWavelets

Nonlinear approximation by DWT and SWT

Discrete Wavelet Transform
wavedec2
waverec2

Stationary (Undecimated, Shift-invariant) Wavelet Transform
swt2
iswt2

Continuous Wavelet Transform
cwt

SSIM
Ex. https://scikit-image.org/docs/stable/auto_examples/transform/plot_ssim.html

Nonlinear approximation

Application 2 (OPTIONAL!)
(PyShearLab and Pillow (PIL))

Comparison of sparse representations of Shearlets and SWT

Load the Barbara image

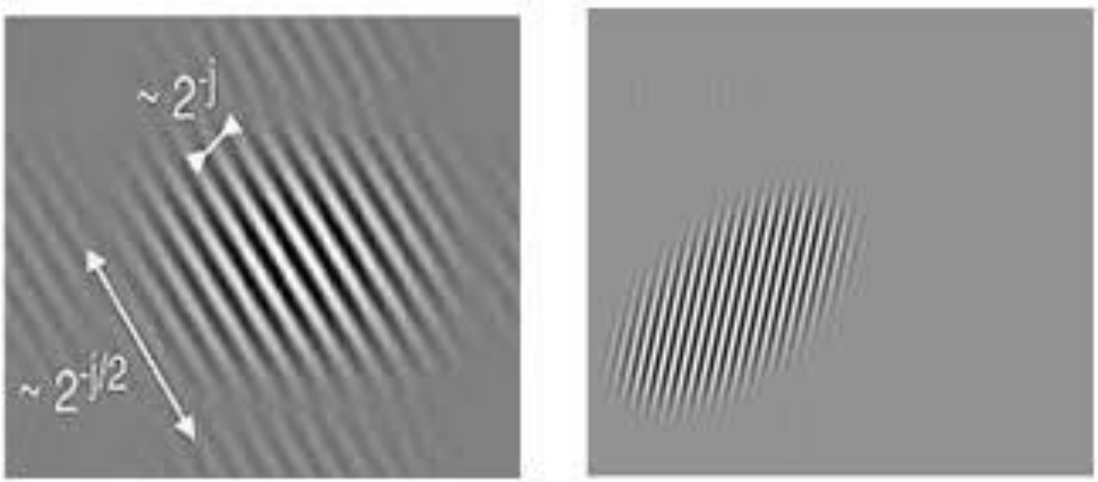
Apply a shearlet transform, sort and keep only the 10% larger coefficients
Reconstruct

Apply a SWT, sort and keep only the 10% larger coefficients
Reconstruct

Plot, and compare the perfomance of Shearlet and SWT

pyshearlab.SLgetShearletSystem2D
pyshearlab.SLsheardec2D
pyshearlab.SLshearrec2D

Annexe: Curvelet Transform



The second-Generation Curvelet Transform

Construction of the DCTG2

The second-generation curvelets are defined at scale 2^{-j}, orientation θ_ℓ and position $\mathbf{t}_{\boldsymbol{k}}^{j,\ell} = R_{\theta_\ell}^{-1}(2^{-j}k, 2^{-j/2}l)$, by translation and rotation of a mother wavelet φ_j as

$$\varphi_{j,\ell,\mathbf{k}}(\mathbf{t}) = \varphi_{j,\ell,\mathbf{k}}(t_1, t_2) = \varphi_{j,}\left(R_{\theta_\ell}\left(\mathbf{t} - \mathbf{t}_{\mathbf{k}}^{j,\ell}\right)\right)$$

Where R_{θ_ℓ} is the rotation by θ_ℓ radians. θ_ℓ is the equispaced sequence of rotation angles $\theta_\ell = 2\pi 2^{-\lfloor j/2 \rfloor}$, $0 \leq \theta_\ell \leq 2\pi$. $k = (k, l) \in \mathbb{Z}^2$ is the subspace of translation parameters. φ_j is defined by its Fourier transform in polar coordinates:

$$\hat{\varphi}_j = 2^{-3j/4}\widehat{\varpi}\left(2^{-j}r\right)\hat{v}\left(\frac{2^{\lfloor j/2 \rfloor}\theta}{2\pi}\right)$$

The second-Generation Curvelet Transform

Construction of the DCTG2

The support of $\hat{\varphi}_j$ is the polar parabolic wedge defined by the support of $\widehat{\varpi}$ and $\hat{\upsilon}$, respectively the radial and angular windows.

In continuous frequency ν the CurveletG2 coefficients of a 2D function$f(\mathbf{t})$ are given by:

$$\alpha_{j,\ell,\mathbf{k}} \equiv \left\langle f, \varphi_{j,\ell,\mathbf{k}} \right\rangle = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\nu})\hat{\varphi}_j\left(R_{\theta_\ell}\nu\right)e^{i\mathbf{t}_k^{j,\ell}\cdot\boldsymbol{\nu}}d\boldsymbol{\nu}$$

The second-Generation Curvelet Transform

Construction of the DCTG2

i. The curveletG2 defines a tight frame of $L_2(\mathbb{R}^2)$

ii. The effective length and width of these curvelets obey the parabolic scaling relation $\left(2^{-j} = \mathrm{width}\right) = \left(\mathrm{length} = 2^{-j/2}\right)^2$

iii. The curvelets exhibit and oscillation behavior in the direction perpendicular to their orientation

NB: This construction implies complex valued output

The second-Generation Curvelet Transform

Discrete coronization

The discrete transform takes as input data defined on a Cartesian grid. The continuous-space definition of the CurveletG2 uses coronae and rotation adapted to Cartesian arrays. These concepts are replaced by the Cartesian counterparts.

The Cartesian equivalent of the radial window $\hat{\varpi}_j(\nu) = \hat{\varpi}\left(2^{-j}\nu\right)$ is a band pass frequency localized window:

$$\hat{\varpi}_j(\boldsymbol{\nu}) = \sqrt{\hat{h}_{j+1}^2(\nu) \; - \hat{h}_j^2(\nu)} \;\; \forall j \geq 0, \qquad \hat{\varpi}_0(\boldsymbol{\nu}) = \hat{h}(\nu_1)\hat{h}(\nu_2)$$

Where $\hat{h}_j$ is separable

$$\hat{h}_j(\boldsymbol{\nu}) = \hat{h}_{1-\mathrm{D}}\left(2^{-j}\nu_1\right)\hat{h}_{1-\mathrm{D}}\left(2^{-j}\nu_2\right)$$

The second-Generation Curvelet Transform

Discrete coronization

Each corona has four quadrants: East, North, West and South. Separated into $2^{\lfloor j/2 \rfloor}$ orientations with the same areas.

We define the angular window for the $\ell\mathrm{th}$ direction as

$$\hat{v}_{j,\ell}(\boldsymbol{\nu}) = \hat{v}\left(2^{\lfloor j/2 \rfloor}\frac{\nu_2 - \nu_1 \tan\theta_\ell}{\nu_1}\right)$$

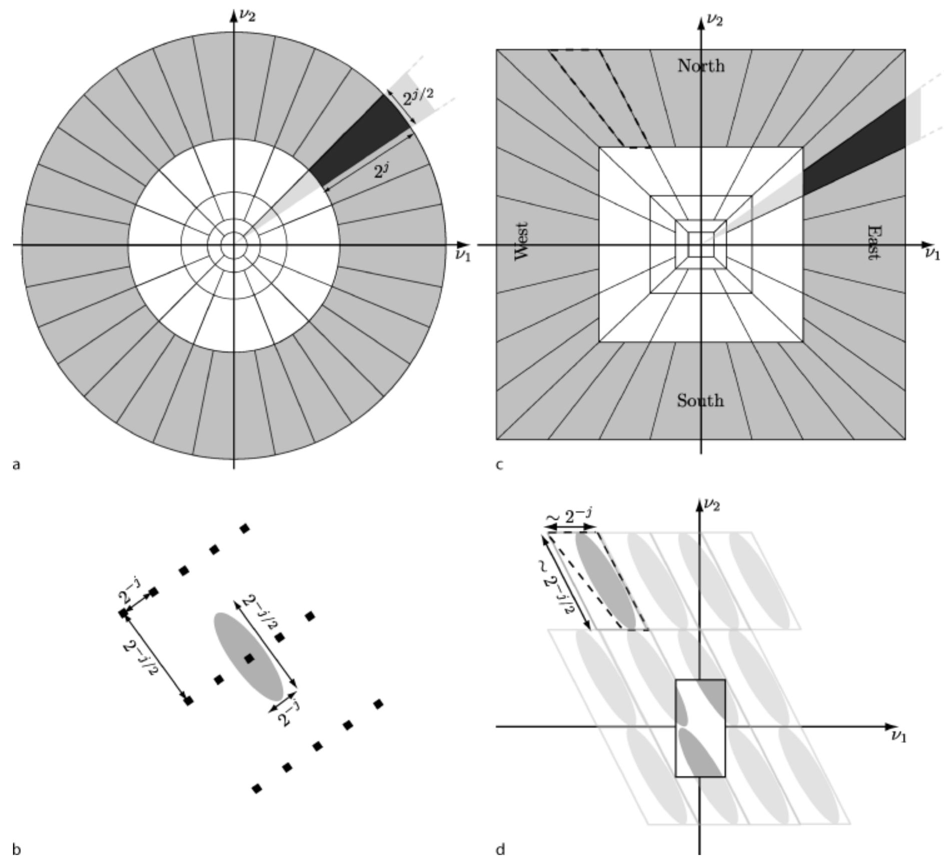
With the sequences of equispaced slopes $\tan\theta_\ell = 2^{-\lfloor j/2 \rfloor}\ell$, with $\ell = -2^{\lfloor j/2 \rfloor}, \dots, 2^{\lfloor j/2 \rfloor} - 1$. The Cartesian analog window of $\hat{\varphi}_j$ is defined as:

$$\hat{u}_{j,\ell}(\boldsymbol{\nu}) = \hat{\varpi}_j(\boldsymbol{\nu})\hat{v}_{j,\ell}(\boldsymbol{\nu}) = \hat{\varpi}_j(\boldsymbol{\nu})\hat{v}_{j,0}\left(S_{\theta_\ell}\boldsymbol{\nu}\right)$$

Where S_{θ_ℓ} is the shear matrix

The second-Generation Curvelet Transform

Discrete coronization



The second-Generation Curvelet Transform

Discrete coronization

From $\hat{u}_{j,\ell}(\boldsymbol{\nu})$ the DCTG2 construction suggests Cartesian curvelets that are translated and sheared versions of a mother Cartesian curvelet $\hat{\varphi}_j{}^D(\boldsymbol{\nu}) = \hat{u}_{j,0}(\boldsymbol{\nu})$, where

$$\varphi^D_{j,\ell,\mathbf{k}}(\mathbf{t}) = 2^{3j/4}\varphi^D_{j,}\left(S_{\theta_\ell}\mathbf{t} - \mathbf{m}\right)$$

With $\mathbf{m} = \left(2^{-j}k, 2^{-j/2}l\right)$

Digital implementation of DCTG2

$$\alpha_{j,\ell,\mathbf{k}} \equiv \left\langle f, \varphi^D_{j,\ell,\mathbf{k}}\right\rangle = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\nu})\varphi^D_j\left(S^{-1}_{\theta_\ell}\boldsymbol{\nu}\right)e^{iS^{-T}_{\theta_\ell}\mathbf{m}\cdot\boldsymbol{\nu}}d\boldsymbol{\nu}$$

The second-Generation Curvelet Transform

Digital implementation of DCTG2

i. Compute the 2-D FFT of f to obtain $\hat{f}$
ii. Form the windowed data $\hat{f}\hat{u}_{j,\ell}$
iii. Apply the inverse Fourier transform

ℓ_2 norm of curvelets is given by $1/\sqrt{\text{redundancy of the frame}}$

Annexe: Frames

Redundant transforms

Frames

An operator $\mathbf{F}$ from a Hilbert space $\mathcal{H}$ to $\mathcal{K}$ is the frame synthesis operator associated with a frame of $\mathcal{K}$. Its adjoint, i.e. the analysis operator $\mathbf{F}^*$, satisfies the generalized Parseval relation with lower and upper bounds a_1 and a_2:

$$a_1\|u\|^2 \leq \|\mathbf{F}^* u\|^2 \leq a_2 \; \|u\|^2 \;, \qquad 0 < a_1 \leq a_2 < +\infty$$

The frame is said to be tight when $a_1 = a_2$, in which case $\mathbf{F}\mathbf{F}^* = a\mathbf{I}$. When $a_1 = a_2 = a$ we have a Parseval relation.