Inductive types, are intuitvely viewed as types which are completely "determined" by their constructors, of which map into the inductive type being formed as *points*. They are positive types which contain no higher homotopical data. We would similarly like to state the same for sHoTT, that inductive types are objects without any higher *simplicial data*. We observe that this is trivially the case for the Unit type. Since all of its terms are definitionally $\star$, the morphism space of Unit is

$$\left\langle \Delta^1 \to \mathrm{Unit}\Big|^{\partial\Delta^1}_{[\star,\star]} \right\rangle \tag{1}$$

which clearly has contractible fibers. Since we've assumed relative functional extensionality, the entire type (**??**) will be contractible.

As a less degenerate task, we might want to do the same thing for the natural numbers. In service of that goal, we can define a type family

$$\mathrm{code} : \mathbb{N} \to \mathbb{N} \to \mathcal{U}_{\mathrm{cov}}$$

recursively, with defining equations

$$\begin{aligned}
&\mathrm{code}\, 0\, 0 :\equiv \mathrm{Unit}\\
&\mathrm{code}\, 0\, S(n) :\equiv \mathrm{Void}\\
&\mathrm{code}\, S(n)\, 0 :\equiv \mathrm{Void}\\
&\mathrm{code}\, S(n)\, S(m) :\equiv \mathrm{code}\, n\, m
\end{aligned}$$

We also define a map $r : \prod_{(n:\mathbb{N})} \mathrm{code}\, n\, n$ by induction as

$$\begin{aligned}
&r\, 0 :\equiv \star\\
&r\, S(n) :\equiv r(n)
\end{aligned}$$

Lemma 0.1. *For all $n, m : \mathbb{N}$,*

$$\hom_{\mathbb{N}}(n, m) \simeq \mathrm{code}\, n\, m \tag{2}$$

Proof. We define encode : $\prod_{(n:\mathbb{N})} \prod_{(m:\mathbb{N})} \hom_{\mathbb{N}}(n, m) \to \mathrm{code}\, n\, m$ as

$$\mathrm{encode}\, n :\equiv \mathrm{yon}_n^{\lambda m.\, \mathrm{code}\, n\, m}(r(n)) \tag{3}$$

and decode : $\prod_{(n:\mathbb{N})} \prod_{(m:\mathbb{N})} \mathrm{code}\, n\, m \to \hom_{\mathbb{N}}(n, m)$ as

$$\begin{aligned}
&\mathrm{decode}\, 0\, 0\, _ :\equiv \mathrm{idhom}_0\\
&\mathrm{decode}\, 0\, S(m)\, u :\equiv \mathrm{abort}(u)\\
&\mathrm{decode}\, S(n)\, 0\, u :\equiv \mathrm{abort}(u)\\
&\mathrm{decode}\, S(n)\, S(m)\, u :\equiv \lambda i.\, S((\mathrm{decode}\, n\, m\, u) i)
\end{aligned}$$

□

encode and decode, we define a map $\mathrm{pred} : \mathbb{N} \to \mathbb{N}$ as

$$\begin{aligned} \mathrm{pred}\, 0 &:\equiv 0 \\ \mathrm{pred}\, S(n) &:\equiv n \end{aligned}$$

Moreover, there is a map $\phi : \prod_{(n:\mathbb{N})} \prod_{(m:\mathbb{N})} \hom_{\mathbb{N}}(S(n), S(m)) \to \hom_{\mathbb{N}}(n, m)$ defined as

$$\begin{aligned} \phi\, 0\, 0\, \_ &:\equiv \mathrm{idhom}_0 \\ \phi\, S(n)\, S(m)\, f &:\equiv \lambda i.\, \mathrm{pred}(f(i)) \end{aligned}$$

and we We can also show that the fibers of code are propositional by defining a type $\psi : \prod_{(n:\mathbb{N})} \prod_{(m:\mathbb{N})} \mathrm{code}\, n\, m \to (\mathrm{code}\, n\, m = \mathrm{Unit})$ by

$$\begin{aligned} \psi\, 0\, 0\, \_ &:\equiv \mathrm{refl}_{\mathrm{Unit}} \\ \psi\, S(n)\, S(m)\, u &:\equiv \psi\, n\, m\, u \end{aligned}$$

Finally, we can show that these functions form a bi invertible equivalence by defining a map

$$\epsilon : \prod_{(n:\mathbb{N})} \prod_{(m:\mathbb{N})} \prod_{(u:\mathrm{code}\, n\, m)} \mathrm{encode}\, n\, m\, (\mathrm{decode}\, n\, m\, u) = u$$

as

$$\epsilon\, n\, m\, u :\equiv \mathrm{ap}_{((\psi\, n\, m\, u)^{-1})_*}(\mathrm{refl}_\star)$$

Noting that

$$\mathrm{encode}\, n\, m\, (\mathrm{decode}\, n\, m\, u) =_{\mathrm{code}\, n\, m} u \simeq \star =_{\mathrm{Unit}} \star \tag{4}$$

since $\mathrm{ap}_{(\psi\, n\, m\, u)_*} : \mathrm{encode}\, n\, m\, (\mathrm{decode}\, n\, m\, u) =_{\mathrm{code}\, n\, m} u \to \star =_{\mathrm{Unit}} \star$ is an equivalence with inverse as $\mathrm{ap}_{((\psi\, n\, m\, u)^{-1})_*}$ as transporting is an equivalence.

On the other hand, if we try to define

$$\eta : \prod_{(n:\mathbb{N})} \prod_{(m:\mathbb{N})} \prod_{(f:\hom_{\mathbb{N}}(n,m))} \mathrm{decode}\, n\, m\, (\mathrm{encode}\, n\, m f) = f,$$

we run into issues as we have no way of showing that, in the base case, that all $f : \hom_{\mathbb{N}}(0, 0)$ have the property that

$$f = \mathrm{refl}_0.$$

In fact, we will ultimately have to inhabit the type

$$\prod_{(n:\mathbb{N})} \prod_{(m:\mathbb{N})} \hom_{\mathbb{N}}(n, m) \to \hom_{\mathbb{N}}(n, m) = \mathrm{Unit}.$$

That is, we want to show that the two-sided representable for the naturals also has propositional fibers. From there, we can easily complete η. To do so, though, is equivalent to a proof that $\mathbb{N}$ is Segal, which ultimately requires us to have a characterization of the morphism types in the naturals.

Lemma 0.2. $\prod_{(n:\mathbb{N})}\prod_{(m:\mathbb{N})}\hom_{\mathbb{N}}(n,m) \to \hom_{\mathbb{N}}(n,m) = \mathrm{Unit}$ *if and only if* $\mathbb{N}$ *is Segal.*

Proof. In the forward direction, we assume there is a term

$$\zeta : \prod_{(n:\mathbb{N})}\prod_{(m:\mathbb{N})}\hom_{\mathbb{N}}(n,m) \to \hom_{\mathbb{N}}(n,m) = \mathrm{Unit}.$$

Then, we can complete the definition of η very quickly:

$$\epsilon\, n\, m\, f :\equiv \mathrm{ap}_{((\zeta\, n\, m\, f)^{-1})_*}(\mathrm{refl}_\star) \tag{5}$$

So, we conclude that for all $n, m : \mathbb{N}$,

$$\hom_{\mathbb{N}}(n,m) \simeq \mathrm{code}\, n\, m.$$

The astute reader will notice that code defined above is exactly code defined in theorem 2.13.1. Hence,

$$(n = m) \simeq \mathrm{code}(n,m) \simeq \hom_{\mathbb{N}}(n,m).$$

That is, $\mathbb{N}$ is discrete, so it is Segal. On the other hand, given a proof that $\mathbb{N}$ is Segal, we can characterize the morphism space in a way that looks much more similar to book HoTT. Define a map

$$\mathrm{code}' : \mathbb{N} \to \mathbb{N} \to \mathcal{U}_{\mathrm{cov}}$$

as

$$\begin{aligned}
&\mathrm{code}'\, 0\, 0 :\equiv \mathrm{Unit}\\
&\mathrm{code}'\, 0\, S(n) :\equiv \mathrm{Void}\\
&\mathrm{code}'\, S(n)\, 0 :\equiv \mathrm{Void}\\
&\mathrm{code}'\, S(n)\, S(m) :\equiv \mathrm{code}'\, n\, m
\end{aligned}$$

□

We start with Unit.

define Unit in section 2

Lemma 0.3. *Unit is discrete*

Proof. It suffices to show that the map induced by idtoarr between total spaces

$$\sum_{v:\mathrm{Unit}} u = v \to \sum_{v:\mathrm{Unit}} \hom_{\mathrm{Unit}}(u,v) \tag{6}$$

is an equivalence. The domain is contractible as it is a based path space. In the codomain, both □

We also expect the type of natural numbers to be discrete too.

define natural numbers

Lemma 0.4. $\mathbb{N}$ *is discrete.*

Proof. To show this □

The unit type corresponds to the category with one object and one morphism (the identity).