Maximum Likelihood and GMM

Problem 1.

You plan to study the relationship between work and unemployment, using the following parametric nonlinear model

$$P(work = 1 \mid educ) = \frac{\exp(\alpha + \beta educ)}{1 + \exp(\alpha + \beta educ)}$$

where *work* is a binary variable that takes value 1 if someone participates in the labor market and 0 otherwise, and *educ* represents the years of education and can take values between 0 and 18.

```
Stata data set
obs: 1,000                          Stata/SE 14.2 (Intel OS X)
variables:  3                       13 Jan 2017 15:51
size:  7,000                        (_dta has notes)
-------------------------------------------------------------------------------
              storage  display     value
    variable   type   format      label      variable label
    -------------------------------------------------------------------------
    educ       byte   %8.0g                 Years Education
    wage       float  %9.0g                 Hourly Wage
    work       byte   %8.0g                 Participation Dummy
    -------------------------------------------------------------------------
Sorted by:
     Total frequency of observations: 1000
     Variables: 3; Observations: 1000; Variables: 3; Obs: 1000
     Sample: 1 of 1000
     Filter: (educ>=0 & educ<=18)
     Data Source: SCF1989Extract; Implicitly 10 Dec 2022
     Author: Jeffrey Wooldridge
     Contact: jwooldr@msu.edu
     Comments:
     Notes:
     1. Include in regressions educ educ^2 educ^3.
     2. Use if educ>=0 & educ<=18 to restrict to valid educ values.
```

Write down the log-likelihood function and derive the score.
The log-likelihood function:

$$\ell_i(\alpha, \beta) = work_i \cdot \alpha + work_i \cdot \beta educ_i - \ln(1 + \exp(\alpha + \beta educ_i)) \tag{1}$$

$$\sum_i work_i \cdot \alpha + \sum_i work_i \cdot \beta educ_i - \sum_i \ln(1 + \exp(\alpha + \beta educ_i)) \tag{2}$$

$$\sum_i work_i \cdot \alpha + \sum_i work_i \cdot \beta educ_i - \sum_i \ln(1 + \exp(\alpha) \cdot \exp(\beta educ_i)) \tag{3}$$

$$\sum_i work_i \cdot \alpha + \sum_i work_i \cdot \beta educ_i - \sum_i \ln(1 + \exp(\alpha)) - \sum_i \ln(\exp(\beta educ_i)) \tag{4}$$

Score function:

$$\frac{\partial \ell_i}{\partial \alpha} = work_i - \frac{\exp(\alpha)}{1 + \exp(\alpha)} \tag{5}$$

$$\frac{\partial \ell_i}{\partial \beta} = work_i \cdot educ_i - \frac{\exp(\alpha) \cdot educ_i}{1 + \exp(\alpha)} \tag{6}$$

Do a simulation to verify your derivations.

```
clear all
set obs 1000
gen educ = runiform(0, 18)
gen wage = runiform(0, 100)
gen work = runiform(0, 1) < exp(2 + 0.5*educ)/(1+exp(2 + 0.5*educ))
mlexp (work*2 + work*0.5*educ - ln(1+exp(2 + 0.5*educ)))
mlexp (work*(2+0.5*educ) - ln(1+exp(2+0.5*educ)))
gen y = work
gen x = educ
gen x2 = educ^2
gen x3 = educ^3
reg y x x2 x3
mlexp (work*(b0+b1*x+b2*x2+b3*x3) - ln(1+exp(b0+b1*x+b2*x2+b3*x3)))
```

```
. clear all

. set obs 1000
obs was 0, now 1000

. gen educ = runiform(0, 18)

. gen wage = runiform(0, 100)

. gen work = runiform(0, 1) < exp(2 + 0.5*educ)/(1+exp(2 + 0.5*educ))

. mlexp (work*2 + work*0.5*educ - ln(1+exp(2 + 0.5*educ)))
initial: log likelihood = -693.2944
rescaling log likelihood = -693.2944
Maximization:
Maximum likelihood estimation

Number of obs   =       1000

------------------------------------------------------------------------------
             |   Coef.   Std. Err.      z    P>|z|     [95% conf. interval]
-------------+----------------------------------------------------------------
/cons        |   2.000   .1414214    14.14   0.000     1.722843    2.277157
------------------------------------------------------------------------------
Log likelihood = -693.2944                     Pseudo R2       =     -0.0000

. mlexp (work*(2+0.5*educ) - ln(1+exp(2+0.5*educ)))
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Log likelihood = -693.2944                     Pseudo R2       =     -0.0000

. gen y = work

. gen x = educ

. gen x2 = educ^2

. gen x3 = educ^3

. reg y x x2 x3

      Source |       SS           df       MS      Number of obs   =     1,000
-------------+----------------------------------  F(3, 996)       =      0.00
       Model |  .000000000         3  .000000000  Prob > F        =    1.0000
    Residual |  .249999000       996  .000250903  R-squared       =    0.0000
-------------+----------------------------------  Adj R-squared   =   -0.0030
       Total |  .249999000       996  .000250903  Root MSE        =    .01584

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0000000   .0000000     0.00   1.000    -.0000000    .0000000
          x2 |   .0000000   .0000000     0.00   1.000    -.0000000    .0000000
          x3 |   .0000000   .0000000     0.00   1.000    -.0000000    .0000000
       _cons |   .5000000   .0000000 1.1e+07   0.000     .5000000    .5000000
------------------------------------------------------------------------------

. mlexp (b0+b1*x+b2*x2+b3*x3) - ln(1+exp(b0+b1*x+b2*x2+b3*x3))
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             |   Coef.   Std. Err.      z    P>|z|     [95% conf. interval]
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/b0          |   2.000   .1414214    14.14   0.000     1.722843    2.277157
/b1          |   .5000   .0000000 1.1e+07   0.000     .5000000    .5000000
/b2          |   .0000   .0000000     0.00   1.000    -.0000000    .0000000
/b3          |   .0000   .0000000     0.00   1.000    -.0000000    .0000000
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Log likelihood = -693.2944                     Pseudo R2       =     -0.0000
```

Write down the log-likelihood function, the score, and the Hessian.

$$\ell_i(\alpha, \beta) = work_i \cdot \alpha + work_i \cdot \beta educ_i - \ln(1 + \exp(\alpha + \beta educ_i))$$

$$score(\alpha, \beta) = \begin{bmatrix} work_i - \frac{\exp(\alpha)}{1+\exp(\alpha)} \\ work_i \cdot educ_i - \frac{\exp(\alpha) \cdot educ_i}{1+\exp(\alpha)} \end{bmatrix}$$

$$Hessian(\alpha, \beta) = \begin{bmatrix} -\frac{\exp(\alpha)}{(1+\exp(\alpha))^2} & -\frac{\exp(\alpha) \cdot educ_i}{(1+\exp(\alpha))^2} \\ -\frac{\exp(\alpha) \cdot educ_i}{(1+\exp(\alpha))^2} & -\frac{\exp(\alpha) \cdot educ_i^2}{(1+\exp(\alpha))^2} \end{bmatrix}$$

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          x2 |   .0000000   .0000000     0.00   1.000    -.0000000    .0000000
          x3 |   .0000000   .0000000     0.00   1.000    -.0000000    .0000000
       _cons |   .5000000   .0000000 1.1e+07   0.000     .5000000    .5000000
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/b2          |   .0000   .0000000     0.00   1.000    -.0000000    .0000000
/b3          |   .0000   .0000000     0.00   1.000    -.0000000    .0000000
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```