



Taylor & Francis  
Taylor & Francis Group



AAG  
Association of American Geographers

---

Push-Pull Migration Laws

Author(s): Guido Dorigo and Waldo Tobler

Source: *Annals of the Association of American Geographers*, Mar., 1983, Vol. 73, No. 1 (Mar., 1983), pp. 1-17

Published by: Taylor & Francis, Ltd. on behalf of the Association of American Geographers

Stable URL: <https://www.jstor.org/stable/2569342>

---

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <https://about.jstor.org/terms>



JSTOR

Taylor & Francis, Ltd. and Association of American Geographers are collaborating with JSTOR to digitize, preserve and extend access to *Annals of the Association of American Geographers*

## Push-Pull Migration Laws

\*Guido Dorigo and †Waldo Tobler

\*Geographisches Institut, Universitaet Zürich, CH-8033, Zürich, Switzerland

†Department of Geography, University of California-Santa Barbara,  
Santa Barbara, CA 93106

---

**Abstract.** The mathematics of a push-pull model are shown to incorporate many of Ravenstein's laws of migration, to be equivalent to a quadratic transportation problem, and to be related to the mathematics of classical continuous-flow models. These results yield an improved class of linear spatial interaction models. Empirical results are presented for one country.

**Key Words:** geographical movement, Helmholtz equation, migration theory, quadratic programming, spatial interaction.

---

IT is now approximately one hundred years since the geographer Ernst Ravenstein reported his "Laws of Migration" to the statisticians of London (Ravenstein 1876, 1885, 1889). We commemorate this by outlining an elementary mathematical model of migration that incorporates several of his "laws" as direct and simple consequences. Having studied the literature,<sup>1</sup> grown large since Ravenstein's time, we believe that we can formulate the migration process as the resultant of a "push" factor and a "pull" factor, but must discount this combination by a distance deterrence between the places. The push factors are those life situations that give one reason to be dissatisfied with one's present locale; the pull factors are those attributes of distant places that make them appear appealing. We will specify this old idea as a very elementary equation system, and will make estimates using empirical data, but not in a regression format; rather, we will study the model from a structural point of view. By including distance discounting we place the model in the venerable class of "gravity" models, and show that it has properties similar to other models in this class (Stewart 1947; Carrothers 1956; Ewing 1981; Schneider 1959; Isard 1960; Batty 1976; Smith 1978; Tocalis 1979; Hua and Porell 1979; Er-

lander 1980). But we also show that it is simultaneously an optimizing model, with shadow prices and with a well-known and simple objective function, and thus belongs to the class of mathematical programming problems. We then present several computational algorithms, which give additional insight into the nature of the model. We will interpret the model in a discrete (network) form, and in a spatially continuous version, the latter as a system of linear partial differential equations. Our empirical example will use Census Bureau migration data for the United States.

### Mathematical Specification

Ravenstein based his "laws" on insightful, careful scrutiny of census tables. He did not formulate his ideas in algebraic form, which is what we now attempt to do. Our mathematical statement is extremely simple, consisting of one elementary equation for each directed exchange of migrants occurring during a specified interval of time between pairs of places. Yet this simple system has a deep structure with interesting properties. This is not unusual in scientific mathematics (May 1976). Specifically, we model migration as

$$M_{ij} = (R_i + E_j)/d_{ij}, i \neq j \quad (1)$$

*Annals of the Association of American Geographers*, 73(1), 1983, pp. 1-17  
© Copyright 1983 by Association of American Geographers

where  $M_{ij}$  is the magnitude (as a count of people) of the movement from place  $i$  to place  $j$  (of  $r$  places) in some specified time interval, and  $d_{ij}$  is the distance between these places, measured in appropriate units (kilometers, road lengths, dollar costs, travel time, social distance, employment opportunities, etc.). We have labeled our primary variables  $R$  and  $E$ , using  $R$  for "rejecting," "repelling," "repulsing" and  $E$  for "enticing."  $R_i$  is the "push" away from place  $i$ , and  $E_j$  is the "pull" toward place  $j$ . We combine these variables to say that migration is the resultant sum of the two, discounted for distance effects. This distance effect can be interpreted as an attenuation of information owing to the two-dimensional geometric nature of the surface of the earth, or as intervening obstacles to be overcome. Dimensional considerations lead to the conclusion that the pushes and pulls are in person-kilometers. Thus a push of two hundred will propel two hundred people one kilometer or one person two hundred kilometers, etc. In our model there are  $r^2$  simultaneous equations, one for each movement between the pairs of places. When the movements are between areal units, a distance often is not defined for the self-migrations  $M_{ii}$ . Then the system consists of only  $r(r-1)$  equations and the notational convention used here corresponds to this situation. Self-moves, however, can be incorporated by establishing a rule assigning nonzero distances for the intraregional moves.

Aggregating the basic Equations (1) over the  $r$  places yields

$$\sum_{\substack{j=1 \\ j \neq i}}^r M_{ij} = R_i \sum_{\substack{j=1 \\ j \neq i}}^r \frac{1}{d_{ij}} + \sum_{\substack{j=1 \\ j \neq i}}^r \frac{E_j}{d_{ij}} = O_i, \quad (2)$$

$$\sum_{\substack{i=1 \\ i \neq j}}^r M_{ij} = \sum_{\substack{i=1 \\ i \neq j}}^r \frac{R_i}{d_{ij}} + E_j \sum_{\substack{i=1 \\ i \neq j}}^r \frac{1}{d_{ij}} = I_j.$$

We call these the "outsums" and the "insums." The equations exhibit a certain symmetry, so that we can assert that

the process of dispersion is the inverse of that of absorption, and exhibits similar features (Ravenstein 1885, 199).

The important immediate consequence of the aggregation is that it shows that it is possible to solve for the numerical value of all of

the push factors ( $R$ ) and all of the pull factors ( $E$ ) if one knows only the outsums and insums for all of the places (see below for details). These marginal sums will of course be known if the full migration table is known. In other words, the equations allow the numerical calculation of the push and pull factors without our knowing in advance what is decisive for migration. We postulate that the estimated push and pull factors are combinations (not necessarily linear) of local traits or characteristics of the inhabitants, but we do not at present need to speculate as to the nature of these attributes of the places or people, and we do not deny that this is still an oversimplified view of reality. A "true" push factor might be a high unemployment rate, but this push must be reduced by the heavy inertial cost of leaving friends and a familiar environment. If we measure the combination of these two tendencies we may find that the total push at a place is negative. This is not the same as a positive pull toward that place, but algebraically it does reduce the importance of a pull from some other place. We will similarly find that an "attractive" place may have a large push value. These results have been anticipated in a theoretical paper by Lee (1966).

Our model, if valid, has consequences for regression studies. Consider the well known Lowry (1966) model (Rogers 1968):

$$M_{ij} = k \frac{U_i}{U_j} \frac{W_j}{W_i} L_i L_j / d_{ij},$$

where the  $U$ 's refer to unemployment rates,  $W$ 's are wage rates, and the  $L$ 's are the number of people in the respective labor markets. By substitution in (1), and canceling the distance term, we obtain

$$R_i + E_j = k \frac{U_i}{U_j} \frac{W_j}{W_i} L_i L_j,$$

which would normally be estimated (with exponents or elasticities) using the logarithmic form. But the push-pull factors cannot be unscrambled in this way, and the two models are clearly inconsistent; we will never get good estimates of the push-pull factors. But we can run a regression on either  $R$  or  $E$ , or both, after they have been computed by the methods outlined below. In this way a correspondence can be established between our model and the popular regression models of,

for example, Clark and Ballard (1980) or Greenwood (1981).

Once one has calculated the  $E$ 's and  $R$ 's, it is possible to estimate the gross migration table. That is, one can compute a full movement table from just the marginals and the distances. In this respect the push-pull model is similar to the origin-destination-constrained entropy model (Wilson 1967), which is its chief competitor.<sup>2</sup> Being able to compute the table from its marginals is important because this allows a test of the model. We see that the model predictions can be compared to observed values without any additional calibration of parameters. Empirical results will be given later. We first deduce more implications of the equation system.

For a place  $k$  with a given pull  $E_k$  and a hinterland containing places  $i$  and  $j$  of equal push  $R_i$  and  $R_j$ , and with  $d_{ik} < d_{jk}$ , it is immediately apparent that  $M_{ik} > M_{jk}$ . This shows that in these circumstances migration diminishes in strength with distance, or,

Migrants enumerated in a . . . center of absorption will . . . grow less with the distance proportionately (Ravenstein 1885, 199).

If we let  $A_k = E_k - R_k$ , then it is easily shown that the net migration flows are

$$M_{ij} - M_{ji} = (A_i - A_j)/d_{ij}.$$

In words, the attractivity of a place is the difference between the pull factor and the push factor at that place. And the net movement between two places is equal to the difference of their attractivities, discounted by distance, as a gradient. This is the same net migration model used with some success by Somer-meyer (1971) and recently studied in depth by Tobler (1981). It is seen to be a derivative of the push-pull model.

If we call  $T_k = E_k + R_k$ , then the total, two-way, movement between two places is

$$M_{ij} + M_{ji} = (T_i + T_j)/d_{ij}.$$

This "T-factor" measures the total exchange, or turnover, at a place; both the in and out movements are involved. The empirical observations,

each main current of migration produces a compensating counter current (Ravenstein 1885, 199),

and

inspection of data indicates . . . outward and in-

ward flows being almost in balance in most cases (Gleave and Cordey-Hayes 1977, 17),

are both described by this turnover factor. High values of  $T$  imply an active migration market, with lots of movement in both directions. Low values of  $T$  imply a quiescent place, and intermediate values may mean high in-migration with low out-migration, or the reverse. Some numerical estimates are given later, as well as a direct computational method by which to estimate the  $T$  values. It would appear that this combination, the sum of the push and pull factors at a single place, should be an interesting candidate for correlation and regression studies. But it is not identical with the gross migration sum  $M_{ij} + M_{ji}$  or with the total  $I_k + O_k$ .

The Equations (2) can be rewritten as

$$R_k = \left\{ O_k - \sum_{\substack{i=1 \\ i \neq k}}^r \frac{E_i}{d_{ik}} \right\} / \sum_{\substack{i=1 \\ i \neq k}}^r \frac{1}{d_{ik}},$$

$$E_k = \left\{ I_k - \sum_{\substack{i=1 \\ i \neq k}}^r \frac{R_i}{d_{ik}} \right\} / \sum_{\substack{i=1 \\ i \neq k}}^r \frac{1}{d_{ik}}. \quad (3)$$

This form of the equations shows that the push factor at a place depends on the pull factor of all of the other places and the number of people leaving the place; similarly, the pull factor at a place depends on the push factor at all of the other places and the number of people entering the place. The dependence in both cases is that of a distance decay in a normalized linear combination, i.e., it is a spatially discounted weighted average. Every place is related to every other place, but near places are more related, and the push and pull factors are structurally intertwined. A substantive implication is that if one makes a place less appealing, thus increasing its "rejection" or pushing factor, then this will change the enticing pull at all of the other places. The change in the  $E$ 's will then propagate to the alternative places, changing their  $R$  values, etc. Similar ramifications occur if one makes a single place more appealing.

Further, and perhaps less obvious, raising the number of out-migrants (in-migrants) at a place, for whatever reason or by whatever means, but without changing any other attributes of the place, increases the push (pull) factors and these changes also propagate through the system. Notice that these effects

are geographically uneven, being more pronounced for outlying places, and include geographical competition and shadowing. This suggests the following type of problem (MacKinnon 1975): where should one make a unit change for it to have the greatest (least) impact on the migration pattern? To the extent that the model describes important properties of actual behavior, this calculation of impacts of observed or proposed changes at places on the total migration system can be an extremely useful tool. And, because of the interrelatedness of push, migration, and pull in the model, there is some prospect that it can be used for short-range prediction. This is seen by the fact that a change at one place is dampened spatially relatively quickly (in a short distance; there is no time in the model) and the effect remains local. The numerical values of most of the  $R$  and  $E$  factors can therefore be expected to change only rather slowly with time at most locations. This sketch of model dynamics resembles relations outlined by Hollingsworth (1970) for migration in Scotland.

It has often been observed that substantial differences in migration rates occur for different age, occupational, social, and educational classes; Ravenstein was able to distinguish between male and female migrants. In order to investigate the behavior of different groups let  $M^a$  and  $M^b$  be the migration tables for groups  $a$  and  $b$ . Then the combined table is

$$M^{a+b} = M^a + M^b,$$

and the components are

$$M_{ij}^{a+b} = (R_i^{a+b} + E_j^{a+b})/d_{ij},$$

on the assumption that the distances are perceived similarly for each group. We notice that "exponent additivity" holds for  $M$ ,  $R$ ,  $E$ ,  $I$ ,  $O$ ,  $T$  and  $A$ , and that the model has this property for any number  $G$  of groups,

$$\begin{aligned} \sum_{g=1}^G M_{ij}^g &= \left( \sum_{g=1}^G R_i^g + \sum_{g=1}^G E_j^g \right) / d_{ij} \\ &= \frac{1}{d_{ij}} \sum_{g=1}^G (R_i^g + E_j^g). \end{aligned}$$

The push and pull terms are different for each place and each group, as is clear from the notation used, but we can add these terms to

get the same results as would be obtained from an analysis of the sum of the groups.

Now let each group consist of a single individual. It may be helpful here to imagine the complete migration table as the sum of a large number of individual migration tables, each containing only a single entry, with value one, the rest of the entries in that table being zero. Our empirical example (Table 5) is thus thought of as the sum of some 12 million tables, each of which describes the movement of one individual. The first  $O_1$  of these tables describe the out-migrations from region one, and so on. For the  $k^{\text{th}}$  individual we must have

$$1 = (R_i^k + E_j^k)/d_{ij},$$

or

$$R_i^k + E_j^k = d_{ij},$$

which seems to say that the dissatisfaction at the origin and the expected satisfaction at a destination for each individual must, at the margin, balance the cost of moving before the move will occur. Here we have shown that if the push factor is zero then the pull toward  $j$  for the moving individual is proportional to distance. Conversely, if there is no pull then the push is equal to the cost or distance of moving. In this system each individual is allowed different satisfactions. Summing over all the individuals who are moving from  $i$  to  $j$ , we have

$$\begin{aligned} \sum_k M_{ij}^k &= M_{ij} = \left( \sum_k R_i^k + \sum_k E_j^k \right) / d_{ij} \\ &= (R_i + E_j)/d_{ij}, \end{aligned}$$

which shows that a consistent grouping from individuals to aggregates is possible for this model. The possibility of such aggregation is generally regarded as being a desirable quality in a migration model (Speare, Goldstein, and Frey 1974, 163–205), and the push-pull model has this property.

Another interesting computational possibility is to estimate the  $R$  and  $E$  values for a single row (or column) from the full migration table. Set all entries to zero except the row (column) of interest. The computational result is analogous to the estimation of a mean migration field (Hägerstrand 1957; Tobler 1979). Or divide the row entries by the outsum for that row, setting all other entries equal to zero

(i.e.,  $M_{ij} = O_i(R_i + E_j)/d_{ij}$ ). The migration spread function that is estimated for the row then bears a resemblance to the impulse response for a unit migrant (Tobler 1969, 1970).

### Variations on the Theme

The extreme simplicity of the model gives it a desirable tinkability and robustness. For example, most of the following variations pose no particular problem of calculation or interpretation. Several of these are of substantive importance.

As noted by Ravenstein (1885, 198),

In forming an estimate of [the] displacement [of people] we must take into account the number of natives of each county which furnishes the migrants, as also the population of the . . . districts which absorb them.

Thus some prefer to work with rates and should use

$$M_{ij} = P_i P_j (R_i + E_j) / d_{ij},$$

where the  $P$ 's refer to populations. Adding two such equations, we get

$$M_{ij} + M_{ji} = P_i P_j (T_i + T_j) / d_{ij},$$

which is our version of Zipf's (1946) formulation. Subtraction yields

$$M_{ij} - M_{ji} = P_i P_j (A_j - A_i) / d_{ij},$$

an attractivity gradient model for net movement rates. The equations for  $R$  and  $E$  in the above case are slightly different from those given in (3), but are easily derived by analogous deductions.

Those who prefer different distance-decay functions may use

$$M_{ij} = (R_i + E_j) e^{-\beta d_{ij}},$$

$$M_{ij} = (R_i + E_j) d_{ij}^{-\alpha}$$

or

$$M_{ij} = B_{ij} (R_i + E_j),$$

where  $B_{ij}$  is the length of the boundary connecting places  $i$  and  $j$ . This latter satisfies the intuitive notion that

Counties having an extended boundary in proportion to their area, naturally offer greater facilities for an inflow . . . than others with a restricted boundary (Ravenstein 1885, 175).

The crossing density, in persons per kilometer, is  $M_{ij}/B_{ij} = R_i + E_j$ . This version of the model allows for easy combination of adjacent regions. Let areas  $j$  and  $k$  be contiguous to area  $i$ . Then

$$B_{ij+k} = B_{ij} + B_{ik},$$

The migration magnitudes are

$$M_{ij} = B_{ij} (R_i + E_j) \text{ and}$$

$$M_{ik} = B_{ik} (R_i + E_k),$$

and, for the combined region,

$$M_{ij+k} = B_{ij+k} (R_i + E_{j+k}),$$

which must equal the sum of the first two. A little algebra now demonstrates that the pull factor of the combined region is the weighted average of the pull factors of the regions that are combined,

$$E_{j+k} = (B_{ij}E_j + B_{ik}E_k)/B_{ij+k},$$

and thus that

$$M_{ij+k} = B_{ij+k}R_i + B_{ij}E_j + B_{ik}E_k,$$

and this extends to combinations of more than two regions. Aggregation of regions is thus easily and consistently achieved, and the model does not require re-estimation. This suggests that the analysis always be performed on the finest level of geographic detail available.

### An Equivalent Model

The model given by (1) is the solution to the variational problem

$$\text{Minimize: } \sum_{i=1}^r \sum_{j=1}^r M_{ij}^2 d_{ij}, i \neq j$$

$$\text{Subject to: } \sum_{j=1}^r M_{ij} = O_i \quad (4)$$

$$\sum_{i=1}^r M_{ij} = I_j.$$

The equivalence of these two models, (1) and (4), is easily established by writing out the equations in full (see below). We find that the place-specific push-pull factors in (1) are directly proportional to the Lagrangians of the constrained optimization problem (4). The



quadratic functional in (4) is the same as the classical definition of the "work" in a resistive electrical network, and it minimizes this quantity while satisfying the constraints. Zipf's (1949) postulate regarding movements is thus equivalent to the postulates of the push-pull model of migration. In a transportation context it can be shown to be equivalent to minimizing transport costs, proportional to distance, when congestion causes the costs on each link to increase in direct proportion to the flow magnitude on that link. T. Smith (in correspondence) has pointed out that this model also has a simple interpretation as a network equilibrium flow pattern, in the sense of Wardrop (1952).

An obvious difficulty of the model (1), or of its equivalent (4), is that it is possible to obtain negative values for the number of people migrating. The condition  $M_{ij} \geq 0$  must be added. This can be done in a number of ways, as discussed below, but slightly complicates the mathematics. Empirically we have discovered that this constraint is not active when the population-weighted form is used.

## Computational Considerations

Equations (2) can be written in matrix form as

$${}_{2r} \begin{bmatrix} Q & H \\ H & Q \end{bmatrix} {}_{2r} \begin{bmatrix} R \\ E \end{bmatrix}_1 = \begin{bmatrix} O \\ I \end{bmatrix},$$

where  $Q$  is an  $r$ -by- $r$  diagonal matrix with

$$q_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^r \frac{1}{d_{ij}};$$

$H$  is also  $r$ -by- $r$  with  $h_{ij} = 1/d_{ij}$ ,  $i \neq j$  and with a null diagonal. To solve (4) we multiply the constraints by Lagrangians  $\alpha$  and  $\beta$  and add them to the objective function, then set the derivatives equal to zero, in the usual fashion. Thus the minimum of (4) occurs where the partial derivatives of the constrained objective function

$$\begin{aligned} \epsilon = & \sum_i \sum_j M_{ij}^2 d_{ij} + \sum_i \alpha_i \left( O_i - \sum_j M_{ij} \right) \\ & + \sum_j \beta_j \left( I_j - \sum_i M_{ij} \right) \end{aligned}$$

are equal to zero. The first  $r^2$  of these derivatives are

$$\partial \epsilon / \partial M_{ij} = 2M_{ij}d_{ij} - \alpha_i - \beta_j.$$

When these values are zero we have

$$M_{ij} = \frac{1}{2}(\alpha_i + \beta_j)/d_{ij},$$

which completes the necessary part of the proof showing the proportional equivalence of (1) and (4). The particular values of  $\alpha_i$  and  $\beta_j$  are then determined by the partials  $\partial \epsilon / \partial \alpha$  and  $\partial \epsilon / \partial \beta$ . This system of  $2r$  equations is of rank  $2r-1$ , because

$$\sum_{i=1}^r O_i = \sum_{j=1}^r I_j,$$

and the  $R$ 's and  $E$ 's (or  $\alpha$ 's and  $\beta$ 's) are determined only up to an additive constant  $C$ . Thus the model can always be written as

$$M_{ij} = [(R_i + C) + (E_j - C)]/d_{ij}.$$

It is convenient to fix any one  $R$  or  $E$  and then to solve the reduced set of  $2r-1$  equations by taking

$${}_{2r-1} \begin{bmatrix} R \\ E \end{bmatrix}_1 = \begin{bmatrix} Q & H \\ H & Q \end{bmatrix}^{-1} \begin{bmatrix} O \\ I \end{bmatrix}. \quad (5)$$

The inversion should be calculated using a program for sparse matrices (Jacoby and Kowalik 1980). An alternative approach is to solve Equations (3) directly by a coupled iterative relaxation technique (Southwell 1956), or to compute a generalized inverse (Bjerrhammer 1972). Observe that the inverse matrix depends only on the geometry of the region; it completely characterizes the area for this purpose and normally would not differ drastically for sufficiently close time periods. But this matrix also specifies the interrelationships of all places to each other; any single change effects everything else in the equations.

One way of approximating the nonnegativity condition is to set all negative computed  $M_{ij}$  values to zero and then to use biproportional adjustment (Leontief 1941; Bacharach 1970; Fienberg 1970) to satisfy the constraints on the marginals. The result is a model of the form

$$M_{ij} = a_i b_j (R_i + E_j)/d_{ij},$$

which, of course, differs from what had been intended. In practice the negative computed migrations are small, and zeros are forced only for widely separated places, i.e., where small  $M_{ij}$  are expected anyway and the  $a_i b_j$  do

not have a large impact on the numerical estimates. The  $a_i$  and  $b_j$  are close to one. The big advantage of this approach is its simplicity, and only a small amount of computational space is needed. As already noted, the nonnegativity constraint is generally not required when the population-weighted version of the model is used.

The system (4) with the additional requirement  $M_{ij} \geq 0$  can be solved directly as a quadratic programming problem with linear constraints (Fletcher 1971). We have used the computer program given in Fletcher (1970). Kunzi, Tzschach, and Zehnder (1971) also give such a program. The main disadvantage of this approach is the size of the computer memory needed (about  $r^4$ ), but an exact solution is obtained. Because migration tables on the order of 3,000 by 3,000 are now becoming available, the program cited would need to be run on a computer with nearly  $10^{14}$  memory locations, and this is impractical. The sparse structure of the problem suggests several possible improvements, in addition to the alternatives already cited.

The Equations (3) can be uncoupled as follows. Let  $T = E + R$  and  $A = E - R$ , and thus, when  $d_{ij} = d_{ji}$ ,

$$T_k = \left\{ F_k - \sum_{\substack{i=1 \\ i \neq k}}^r \frac{T_i}{d_{ik}} \right\} / \sum_{\substack{i=1 \\ i \neq k}}^r \frac{1}{d_{ik}},$$

$$A_k = \left\{ \Delta_k + \sum_{\substack{i=1 \\ i \neq k}}^r \frac{A_i}{d_{ik}} \right\} / \sum_{\substack{i=1 \\ i \neq k}}^r \frac{1}{d_{ik}},$$

where  $F_k = I_k + O_k$  and  $\Delta_k = I_k - O_k$ .  $T$  and  $A$  are seen to be independent of each other, and each equation can be separately solved by iteration, with  $\pm$  switches in a single computer program. Notice the dependence of the turnover  $T$  on the sum of the insums and outsums at a place, and the dependence of the attractivity  $A$  on the difference of these values. It is direct that  $E = (T + A)/2$  and  $R = (T - A)/2$ . One value is still arbitrary, and care must be taken that the values satisfy the insum, outsum, and nonnegativity constraints.

The variational form of the model (4) with the nonnegativity constraint  $M_{ij} \geq 0$  suggests a simpler solution. We can consider this a "quadratic transportation problem," a generalization of the well known linear case (Koopmans 1949). The usual tableau is estab-

lished, but now the source places are also the destination places, and the tableau is square. The only restriction is that a place cannot send migrants to itself. Then each place seems to desire people of one kind, and exports another type, a reasonable enough interpretation. If we can find a simple efficient algorithm for this quadratic transportation problem, we will not need to use the space-consuming general quadratic programming procedures. An initial feasible solution is easily obtained, and this is then improved by appropriate changes in the tableau. The constraints are thus always satisfied. By restricting the changes to integers, and being careful never to subtract more from a cell than is already there, we move toward a nonnegative integral solution. But the true solution is in general not integral valued (in contrast to the linear transportation problem), so that we must allow fractional movements in order to achieve the optimum. Philosophically this is slightly disappointing. Once we have found the optimum, the push, pull factors (Lagrangians) emerge as shadow prices, that is, as supply and demand prices, like those in the spatial price equilibrium model of Samuelson (1952), a difference being that in his model linear transportation costs are minimized. In our spatial equilibrium a quadratic functional is minimized and "location costs" are assigned to people. The push-pull factors can now be called "prices"; economists often assign prices to persons, which are then called "labor." We further observe that the quadratic optimum contains more than the small number of values that would be obtained from the equivalent (two-way) linear transportation problem. In fact, the extreme sparsity of entries in the solution to the conventional linear transportation problem suggests that attempts to model actual, as opposed to normative, behavior with that model are unrealistic, as has been noted several times (Polenska 1966; Morrill 1967; Mera 1971; Nijkamp 1975). Recently proposals have been made to overcome this shortcoming, e.g., Hodgson (1978) and Bröcker (1980), to which we add our present suggestion. The advantage of the quadratic functional is that it forces a larger number of smaller flows, and this is a better descriptor of real tables than is provided by the linear functional. We are optimistic that the method of Beale (1959) can be improved to obtain a workable primal



solution to this nonlinear transportation problem.

After discussing so many computational approaches, we describe a simple algorithm that appears to work with only modest storage and computation time requirements, and that satisfies the condition for nonnegative movements.<sup>3</sup>

Step (1): Solve the problem as given by Equation (5) using a matrix inversion, or by operating on Equations (3) using a coupled convergent iterative scheme.

Step (2): Compute the resulting migrations  $M_{ij}$ .

Step (3): Whenever any  $M_{ij}$  is negative replace the corresponding  $d_{ij}$  by a large number ( $= \infty$ ,  $1/d_{ij} = 0$ ) setting the values in  $Q$  and  $H$  appropriately, and then go to Step (1). Several of the  $M_{ij}$  may be negative at one time, and all of the appropriate  $d_{ij}$  are set to large values. Once a  $d_{ij}$  has been modified in this manner it is left large on subsequent passes. If there are no negative  $M_{ij}$  in the entire array on this step, then stop because you are done.

The numerical answers obtained by this simple procedure, which seems to require only a few iterations, agree with those obtained using a full quadratic programming procedure, e.g., that of Fletcher (1970, 1971).

## The Geographically Continuous Version

Ravenstein (1885, 198) asserts that he has "proved that the great body of our migrants only proceed a short distance," and many others have verified this observation (for example, Hägerstrand 1957). Consequently, it does not do great violence to the data to assume movement only between neighboring places, especially if observations are taken over very short time intervals. Many types of geographical movement display this characteristically local process. This is also not as severe an assumption as might appear at first glance because computationally it simply means that a migrant from, say, New York to California is routed to pass through all intermediate places.

For expository simplicity allow every place to have only four neighbors among whom exchanges are possible. Index these by subscripts from one to four, and take them to be

equidistant, at distance  $d \equiv 1$ , and spaced as on a square mesh, for which  $B_{ij} \equiv 1$  also. Then Equations (3) become

$$4E = I - (R_1 + R_2 + R_3 + R_4),$$

$$4R = O - (E_1 + E_2 + E_3 + E_4).$$

Now add  $-4R$  to both sides of the first equation, and  $-4E$  to both sides of the second, to obtain

$$\begin{aligned} R_1 + R_2 + R_3 + R_4 - 4R &= I - 4(E + R), \\ E_1 + E_2 + E_3 + E_4 - 4E &= O - 4(E + R). \end{aligned}$$

The left hand sides are recognized as finite difference versions of the Laplacian. Thus we can write, approximately and for a limiting fine mesh,

$$\begin{aligned} \nabla^2 E(x, y) &= I(x, y) - 4[R(x, y) + E(x, y)], \\ \nabla^2 R(x, y) &= O(x, y) - 4[R(x, y) + E(x, y)], \end{aligned}$$

assuming that  $R$  and  $E$  are differentiable spatial functions.<sup>4</sup> This is a coupled system of two simultaneous partial differential (Helmholz) equations, each of which separately is the Euler equation of, or makes stationary, a functional of the form

$$\int \int_{\mathcal{R}} \left\{ \frac{\partial z^2}{\partial x} + \frac{\partial z^2}{\partial y} + \lambda^2 z^2 + 2\beta z \right\} dx dy,$$

with  $\partial z / \partial n = 0$  on the boundary. The equations can be uncoupled as before. By addition we obtain

$$\nabla^2 T + 8T = F,$$

where  $F(x, y) = I(x, y) + O(x, y)$  is the forcing function. This is a single Helmholtz equation in the single unknown turnover function  $T(x, y)$ , minimizing an integral of the given type. Alternatively, by subtraction we have

$$\nabla^2 A = \Delta,$$

using  $\Delta(x, y) = I(x, y) - O(x, y)$ ; this result gives Poisson's equation for the unknown  $A(x, y)$ . Efficient computer algorithms exist for solving both Helmholtz's equation and Poisson's equation (Proskurowski and Widlund 1976; Buzbee, Golub, and Nielson 1970). The arbitrary additive constant of integration must still be supplied.  $A(x, y)$  is interpreted as the attractivity at each location. Assuming that movement is proportional to the gradient of this attractivity, i.e.,

$$\vec{V} = \text{grad } A,$$

allows one to make interesting maps of the net geographical movement pattern, as is shown in a later figure. The movement across the border surrounding the region being studied gives the Neumann condition for these partial differential equations. In Table 5 (below) this flux is not recorded and is treated as implying no international migration into or out of the United States.

The occurrence of Helmholtz's equation, normally used to describe dynamic wave motion, in a static equilibrium paper on migration, was quite unanticipated. Hydrologic terminology (streams, currents, flows, eddies, waves) is common in the migration literature (e.g., Redford 1964), but the equations are not.

### Empirical Example

We have performed a few computations contrasting the push, pull model with US Bureau of the Census (1973) migration estimates for 1965/1970. These estimates are based on a 15 percent sample enumeration of the migrations between the nine census regions defined by the government for data publication purposes. Nine regions covering the contiguous United States yield an average geographic resolution of 944 kilometers, and patterns of circa 2,000 kilometers across or greater should be observable. Tables 1 and 2 give input data for the model, including high-way driving distances between cities located near the center of gravity of the regions and the length of the boundaries between the regions, the latter computed from a magnetic tape containing the boundaries of all of the counties of the United States in coordinate (latitude, longitude) form. Table 3 presents

our estimates for the basic push-pull model and has been computed by the constrained quadratic programming algorithm. Whenever a negative flow is computed from the push-pull factors of this table, it must be set to zero to obtain the correct minimum of the functional. The fit of the model and its variants to the data can be measured in several ways; all indicate about the same results as are usually obtained for gravity models ( $R^2 > 80$  percent). More interesting are the maps (Figure 1), which show the resulting push, pull, turnover, and attractivity values in their geographical context. The Pacific Region is seen to be the most "repulsive." High housing costs or metropolitan air pollution might be the reason. Conversely, this same area is also the most "enticing," perhaps for its beautiful scenery, coastal climate, or life styles. Our model does not consider these contrasts contradictory but instead incorporates, by summation, these two effects into a large turnover, and also assigns a high net attractivity to the region. One may compare regions known to be losing population, the Mid-Atlantic for example, in this same way. This area still has a large drawing power, but this is overcome by the push effect. Each region on the maps can be examined in this manner.

Figure 2 shows the comparable results for the model

$$M_{ij} = P_i P_j (R_i + E_j) / d_{ij};$$

the effect of removing the population sizes is seen. Here is a case in which the matrix formulation (Equation 5, as modified to include population) was used directly and the  $M_{ij} \geq 0$  constraint was not needed; all values turned out to be positive even without the use of this constraint. Table 4 shows the estimated

**Table 1.** Model Input

Region	Insum	Outsum	I + O	I - O	Population (1970)
New England	675,408	679,180	1,354,588	-3,772	11,848,000
Mid Atlantic	1,155,811	1,874,320	3,030,131	-718,509	37,056,000
East North Central	1,789,112	2,134,267	3,923,379	-345,155	40,266,000
West North Central	942,162	1,212,105	2,154,267	-269,943	16,327,000
South Atlantic	2,484,387	1,765,650	4,250,007	718,737	29,920,000
East South Central	819,222	986,050	1,805,272	-166,828	13,096,000
West South Central	1,237,079	1,146,498	2,383,577	90,581	19,025,000
Mountain	1,067,069	987,331	2,054,400	79,738	8,289,000
Pacific	2,143,172	1,528,021	3,671,193	615,151	25,476,000

Source: U.S. Bureau of the Census (1973).

Table 2. Model Inputs\*

	1	2	3	4	5	6	7	8	9
1 NE: Boston	—	301	0	0	0	0	0	0	0
2 MA: New York	219	—	91	0	350	0	0	0	0
3 ENC: Chicago	1,009	831	—	972	264	703	0	0	0
4 WNC: Omaha	1,514	1,336	505	—	0	166	755	937	0
5 SA: Charleston	974	755	1,019	1,370	—	1,295	0	0	0
6 ESC: Birmingham	1,268	1,049	662	888	482	—	1,140	0	0
7 WSC: Dallas	1,795	1,576	933	654	1,144	662	—	637	0
8 MTN: Salt Lake City	2,420	2,242	1,451	946	2,278	1,795	1,287	—	1,542
9 PAC: San Francisco	3,174	2,996	2,205	1,700	2,862	2,380	1,779	754	—

\* Distance between cities below the diagonal; length of border between regions above the diagonal. All in miles.

flows, and Table 5 repeats the census figures for easy direct comparison.

Tables 6 and 7 show the solution using

$$M_{ij} = B_{ij}(R_i + E_j) \tag{6}$$

with the lengths of boundaries between regions instead of distances. This result is particularly interesting because we have had to introduce pass-through transients in order that all of the flows come out correctly. It is not possible to get all of the necessary people to the Pacific Region from the Mountain Region alone. Nor can this latter region absorb all of the out-migrants from the Pacific Region. We must increase the insum to the Mountain Region, and increase its outsum also, to get enough people to move from the East to the West, and in the opposite direction. The following algorithm seems to work well. Compute a solution using Equations (5), appropriately modified to use boundary lengths instead of distances. Negative migration flows will occur when this solution is inserted into the basic model (6). Now increment the insums and outsums by the absolute value of the sum of the negative computed

migrations in each row or column. Use these new marginals to get a new estimate via (5)—the inverse depends only on the geometric structure of the region and need not be recomputed. Continue this iterative procedure until there are no negative flows. This appears to be a new way of solving the “traffic assignment problem” (Florian 1976; Boyce 1980), routing all flows through adjacent places, while minimizing congestion via the quadratic functional.<sup>5</sup> Table 7 gives the number of estimated border-crossing transients by region. These transshipment migrations

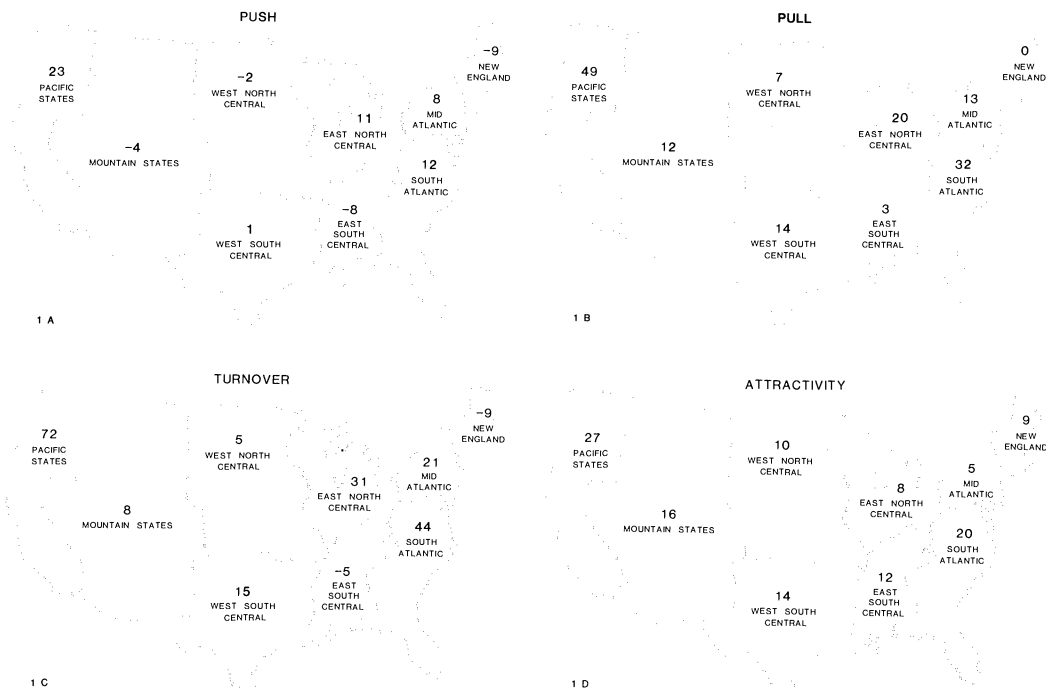
... sweep along with them many of the natives of the counties through which they pass ... [and] ... deposit, in their progress, many of the migrants which have joined them at their origin (Ravenstein 1885, 191).

Unfortunately we know of no actual published counts on the part of statistical agencies of crossings of internal boundaries by migrants. International estimates may be easier to obtain. Until such data are found there is no method of testing the realism of our procedure. But it is easily seen that

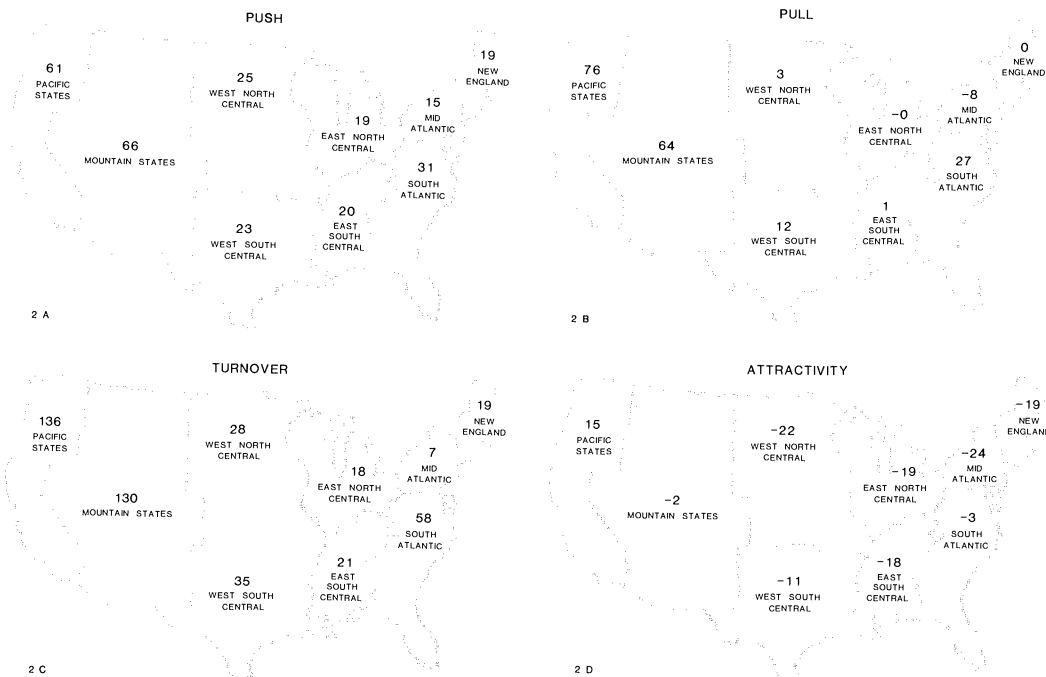
Table 3. Model Estimates.\* Estimated by Minimization of the Functional (4) with  $M_{ij} \geq 0$

Region	Push (R)	Pull (E)	Turnover (T = E + R)	Attractivity (A = E - R)
New England	-9,150	0	-9,150	9,150
Mid Atlantic	8,010	12,885	20,895	4,875
East North Central	11,410	19,825	31,235	8,415
West North Central	-2,420	7,085	4,665	9,505
South Atlantic	11,905	31,875	43,780	19,970
East South Central	-8,415	3,255	-5,160	11,670
West South Central	555	14,355	14,910	13,800
Mountain	-4,000	12,385	8,385	16,385
Pacific	22,615	49,475	72,090	26,860

\* All values to be multiplied by 10<sup>4</sup>.



**Figure 1.** Push, pull, turnover, and attractivity values from the basic model (see Table 3).



**Figure 2.** Push, pull, turnover, and attractivity values from the  $M_{ij} = P_i P_j (R_i + E_j) / d_{ij}$  model.

**Table 4.** Interregional Migration Estimated by the Model  $M_{ij} = P_i P_j (R_i + E_j) / d_{ij}$

	1	2	3	4	5	6	7	8	9
1 New England	—	209,765	87,194	27,776	167,447	24,659	38,952	33,558	89,828
2 Mid-Atlantic	306,575	—	265,342	82,397	625,135	76,888	122,621	108,421	286,941
3 East North Central	88,727	184,682	—	280,552	540,023	158,734	252,950	189,748	438,853
4 West North Central	31,454	73,525	313,749	—	185,031	62,427	174,250	126,534	245,134
5 South Atlantic	110,965	325,530	353,479	119,273	—	258,490	212,207	102,649	283,058
6 East South Central	23,889	51,566	151,174	53,973	379,659	—	141,955	50,385	133,449
7 West South Central	29,338	66,847	187,570	124,332	251,411	110,458	—	106,828	269,743
8 Mountain	26,709	78,673	150,209	98,271	101,187	40,562	95,525	—	396,196
9 Pacific	57,752	165,222	280,396	155,589	234,494	87,004	198,619	348,946	—

... even in the case of 'counties of dispersion,' which have a population to spare for other counties, there takes place an inflow of migrants across that border which lies furthest away from the great centers of absorption (Ravenstein 1885, 191).

In the present instance the West North Central states form one such interior-lying dispersing region; the reader will be able to find others and can easily identify the direction of movement across each of their borders. A striking contemporary international case is Mexico, which has an illegal immigrant problem—with people coming in from the south of the country!

The final maps (Figure 3) display the results of a computation based on the continuous attractivity model using Poisson's equation

$$\nabla^2 A(x,y) = I(x,y) - O(x,y).$$

The contiguous United States is approximated, somewhat crudely, by a 61-by-95 lat-

tice of 5,795 nodes, with one finite difference equation for each node. The changes in population resulting from migration are distributed over each of the nine census regions to yield the source/sink field  $\Delta(x,y)$  on this mesh. Computation of the potential  $A(x,y)$  and the flow vectors then uses standard procedures (Wachspress 1966) with a Neumann condition on the boundary.<sup>6</sup> The continuous case closely resembles the case in which transients cross over internal borders, and it is in fact possible to count the number of transients in this model; the method is described in detail in Tobler (1981). The resulting maps indicate that

... migratory currents flow along certain well defined geographical channels (Ravenstein 1889: 284),

although this shows up much better when higher-resolution data are used as input or when time series are available. The variation

**Table 5.** Interregional Migration Estimated by the US Bureau of the Census (1973)

	1	2	3	4	5	6	7	8	9
1 New England	—	180,048	79,223	26,887	198,144	17,995	35,563	30,528	110,792
2 Mid-Atlantic	283,049	—	300,345	67,280	718,673	55,094	93,434	87,987	268,458
3 East North Central	87,267	237,229	—	281,791	551,483	230,788	178,517	172,711	394,481
4 West North Central	28,977	60,681	286,580	—	143,860	49,892	185,618	181,868	274,629
5 South Atlantic	130,830	382,565	346,407	92,308	—	252,189	192,223	89,389	279,739
6 East South Central	21,434	53,772	287,340	49,828	316,650	—	141,679	27,409	87,938
7 West South Central	30,287	64,645	161,645	144,980	199,466	121,366	—	134,229	289,880
8 Mountain	21,450	43,749	97,808	113,683	89,806	25,574	158,006	—	437,255
9 Pacific	72,114	133,122	229,764	165,405	266,305	66,324	252,039	342,948	—

Source: U.S. Bureau of the Census (1973).

**Table 6.** Interregional Migration Estimated by the Model  $M_{ij} = B_{ij}(R_i + E_j)$ 

	1	2	3	4	5	6	7	8	9
1 New England	—	679,180	0	0	0	0	0	0	0
2 Mid-Atlantic	675,408	—	196,658	0	1,002,254	0	0	0	0
3 East North Central	0	65,877	—	1,375,996	543,813	162,193	0	0	0
4 West North Central	0	0	1,104,506	—	0	0	994,245	481,429	0
5 South Atlantic	0	410,754	478,785	0	—	876,111	0	0	0
6 East South Central	0	0	22,775	13,465	938,320	—	242,499	0	0
7 West South Central	0	0	0	902,833	0	11,928	—	327,124	0
8 Mountain	0	0	0	17,942	0	0	95,722	—	2,143,172
9 Pacific	0	0	0	0	0	0	0	1,528,021	—

in the density of the streaklines, and the spacing of the contoured potentials, also allows one to see that

The more distance from the fountainhead which feeds them, the less swiftly do these currents flow . . . (Ravenstein 1885: 191).

The streaklines should be interpreted as ensemble averages rather than as paths of individual migrants. It is also apparent that no account has been taken of transportation facilities in the computation for Figure 3. The figure does not show fine detail, such as movement to the suburbs, because of the low resolution of the migration observations.

## Conclusion

We believe that we have adequately demonstrated that several of Ravenstein's "laws" hold for our equations. There remain many challenging problems, as should be obvious from all of those aspects of migration that we

have not modeled, and we are aware of these inadequacies. Still, as the Nobel laureate Paul Samuelson so aptly remarks in his *Economics* (1976):

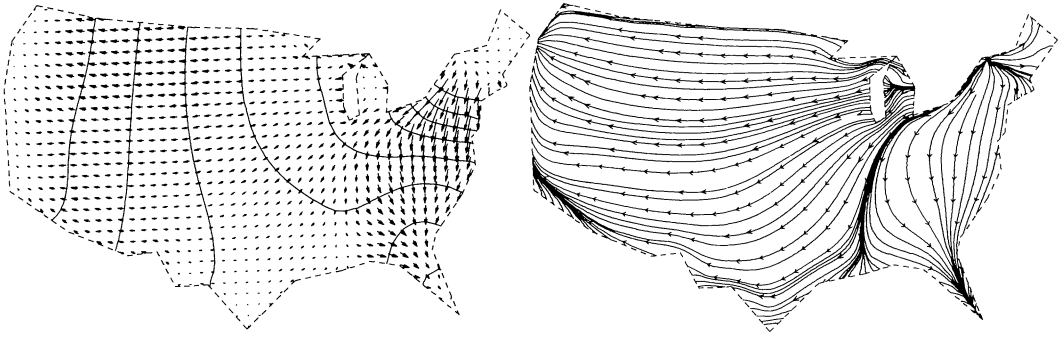
Every theory, whether in the physical or biological or social sciences, distorts reality in that it oversimplifies. But if it is a good theory, what is omitted is outweighed by the beam of illumination and understanding thrown over the diverse empirical data.

Our model obviously does not fit the observations perfectly; perhaps the residuals will suggest alternative research directions. But we believe that some issues have been "illuminated." We have given a specific mathematical form, previously lacking, to the push-pull idea and have shown that many of Ravenstein's "laws" can be deduced from these formulae. We have established relations to Zipf's "principle" and to conventional spatial interaction models, and have shown how these can be contrasted with the now classical transportation/transshipment problem

**Table 7.** Model Results. Estimated from  $M_{ij} = B_{ij}(R_i + E_j)$ 

Region	Push (R)	Pull (E)	Turnover (T = E + R)	Attractivity (A = E - R)	Transients
New England	2,969	0	2,969	-2,969	0
Mid Atlantic	2,240	-715	1,525	-2,955	0
East North Central	1,440	-74	1,366	-1,514	13,611
West North Central	1,210	-24	1,186	-1,234	1,368,074
South Atlantic	1,886	618	2,504	-1,268	0
East South Central	106	-1,210	-1,104	-1,316	231,004
West South Central	1,221	106	1,327	-1,115	95,387
Mountain	43	-707	-664	-750	1,269,506
Pacific	1,698	1,347	3,045	-351	0





**Figure 3.** Potential field, gradients, and streamlines for 1965/1970 from the model  $\nabla^2 A = \Delta$ , with  $\partial A / \partial n = 0$  on the boundary, solved as a system of 5,795 simultaneous equations.

and the spatial price equilibrium concept. A spatially continuous version of our model is articulated; this should become increasingly important as migration tables increase in size to provide greater spatial resolution. Our model yields two-way migrations within the context of pushes and pulls and is consistent at individual and aggregate levels; both attributes have been difficult of specification in previous work. The astute reader will also have observed that we have described an abstract geographical movement system and not only, or even specifically, migration. The model equations can therefore be applied to information flows, to commodity movements, to commuting patterns, or to shopping behavior, and so on. The model may also have relevance to sociometric matrices (Holland and Leinhardt 1981).

We have specified several model formulations and must now discriminate between these variants and between other models in the literature, especially those incorporating the entropy approach introduced by A. Wilson (1967). Some recent suggestions have been made for this purpose (Bishop, Fienberg, and Holland 1975, Chap. 9; Kau and Sirmans 1979; Hubert and Golledge 1981) but need refinement because several of our proposed variants fit so well that they cannot be distinguished statistically on the basis of the data used here. The linear property of our model gives it mathematical advantages over the multiplicative entropy class, especially with respect to aggregation. No analogy to physics has been used in the development of our model, but the resulting equations clearly bear a resemblance to those encountered in continuum mechanics. The entropy equa-

tions are derived from ideas also used in statistical mechanics. Thus the two classes of models of spatial interaction may bear the same relation to each other as the two approaches to mechanics. Each approach attempts to describe the same events as the other and uses comparable empirical observations. As outlined here the push-pull model is deterministic, but a stochastic version may also be obtainable (Hersh and Griego 1969). Investigation of the relation between these classes of models may lead to deeper insights, and this offers a challenge for students.

Computationally, we have assumed either the availability of the migrant insums and outsums (or their combination  $I \pm O$ ), or the equivalent availability of place-specific person prices (push-pull factors). Much of the migration literature is concerned with explaining these factors. We have shown how to compute them *a posteriori*. This has advantages and disadvantages. Instead of attempting to include complex human behavior in our equations, we estimate the effects of this behavior, allowing other researchers to estimate the relation of the computed pushes and pulls to life situations. As a consequence we cannot claim to have explained why people migrate; we only assert a tolerably good description of the spatial pattern when one is given table marginals and a measure of the strength of driving impulses, whatever these might be. Because of this, our model is equally applicable to refugees, who constitute a large fraction of all migrants, as well as to the more benign internal movements. A remaining difficulty is in the dynamic forecasting of the insums and outsums, the feedback these have on subsequent events, and in

the details of the temporal and spatial lag structures.

## Acknowledgments

Dr. Dorigo's postdoctoral studies in the United States were supported by the Swiss National Fund for Scientific Research. We thank Teresa Everett, Susan Pond, and Joanne Mustacchia for their assistance in completing the manuscript and Ralph Milliff for computation of the flow maps.

## Notes

1. The reader is directed to the several bibliographic and critical surveys that have recently been published, e.g., Adams (1968), Bennett and Gade (1979), Clark (1982), Cordey-Hayes (1972), Courgeau (1980), DaVanzo (1981), Davis (1974), Golledge (1980), Greenwood (1975), Hagerstrand (1957), Hoffmann-Novotny (1970), Hotelling (1978), Kuznets (1957–1964), Ledent (1981), Mangalam (1968), Morrison (1973), Mueller (1982), Olsson (1965), Petersen (1978), Petersen and Thomas (1968), Pierson (1973), Price and Sikes (1975), Pryor (1981), Quigley and Weinberg (1977), Ritchey (1976), Shaw (1975), Thomas (1938), Thorntwaite (1934), Weaver and Downing (1976), Welch (1971), Willis (1974), Wolpert (1965), the *International Migration Review*, a periodical published by the Center for Migration Studies of New York, and the *Population Index* of Princeton University.
2. Wilson's model is multiplicative

$$M_{ij} = a_i b_j O_i I_j \exp(\beta d_{ij}),$$

and maximizes a constrained entropy function. An additive analog with the same constraints is

$$\text{Minimize: } \sum_i \sum_j M_{ij}^2 \quad i, j = 1 \dots r.$$

Subject to:

$$\sum_j M_{ij} = O_i, \quad \sum_i M_{ij} = I_j,$$

$$\sum_i \sum_j M_{ij} d_{ij} = D, \quad M_{ij} \geq 0.$$

Here  $D$  is a constraint on the total movement, and this leads to the slightly different push-pull, distance-deterrence model:

$$M_{ij} = R_i + E_j + \gamma d_{ij}.$$

Empirically, using the nine-region census data (Tables 2 and 5), we find that  $\gamma$  is negative, as expected. The computational procedures are similar to those described for the model (1); also see Wansbeek (1977). We deduce for this model that

$$M_{ij} + M_{ji} = T_i + T_j + 2\gamma d_{ij}, \text{ and} \\ M_{ij} - M_{ji} = A_j - A_i,$$

when  $d_{ij} = d_{ji}$ . ( $T$  and  $A$  are as defined later in the body of the text.)  $M_{ij}$  is also estimated if included with the in- and outflows.

3. The computer program is available from the authors.
4. We use the standard notation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

for the Laplacian operator.

5. On a network one might also wish to add capacity constraints.
6. The boundary condition is only approximately satisfied by our numerical technique; see Milliff (1980) for a discussion of this problem.

## References

- Adams, R. B. 1968. Migration geography: a methodological inquiry and United States case study. Ph.D. dissertation, Department of Geography, University of Minnesota, Minneapolis.
- Bacharach, M. 1970. *Biproportional matrices and input-output change*. London: Cambridge University Press.
- Batty, M. 1976. *Urban modelling*. London: Cambridge University Press.
- Beale, E. 1959. An algorithm for solving the transportation problem when the shipping cost over each route is convex. *Naval Research Logistics Quarterly* 6:43–56.
- Bennett, D., and Gade, O. 1979. *Geographic perspectives in migration research*. Studies in Geography No. 12, University of North Carolina, Chapel Hill.
- Bishop, Y., Fienberg, S., and Holland, P. 1975. *Discrete multivariate analysis: theory and practice*. Cambridge, Mass.: MIT Press.
- Bjerhammer, A. 1972. *Theory of errors and generalized inverses*. Amsterdam: Elsevier.
- Boyce, D. 1980. A framework for constructing network equilibrium models of urban location. *Transportation Science* 14:77–96.
- Bröcker, J. 1980. An application of economic interaction models to the analysis of spatial effects of economic integration. *Environment and Planning A* 12:321–38.
- Buzbee, G., Golub, G., and Nielson, C. 1970. On direct methods for solving Poisson's equation. *SIAM Journal of Numerical Analysis* 7:627–56.
- Carrothers, G. 1956. An historical review of the gravity and potential concepts of human interaction. *Journal of the American Institute of Planners* 22:94–102.
- Clark, G., and Ballard, K. 1980. Modelling out-migration from depressed regions: the significance of origin and destination characteristics. *Environment and Planning A* 12:799–812.
- Clark, W. A. V. 1982. Recent research on migration and mobility: a review and interpretation. *Progress in Planning* 18:56.

- Courageau, D.** 1980. *Analyse quantitative de migrations humaines*. Paris: Masson.
- Cordey-Hayes, M.** 1972. Dynamic frameworks for spatial models. *Socio-Economic Planning Sciences* 6:365–85.
- DaVanzo, J.** 1981. Microeconomic approaches to studying migration decisions. In *Migration decision making*, ed. G. DeJong and R. Gardner, pp. 90–129. New York: Pergamon.
- Davis, K.** 1974. The migrations of human populations. *Scientific American* 231:92–109.
- Ewing, G.** 1981. *Papers in spatial modelling*. Department of Geography Reprints, McGill University, Montreal.
- Erlander, S.** 1980. *Optimal spatial interaction and the gravity model*. Berlin: Springer.
- Fienberg, S.** 1970. An iterative procedure for estimation in contingency tables. *The Annals of Mathematical Statistics* 41:907–17.
- Fletcher, R.** 1970. A Fortran subroutine for general quadratic programming. AERE-R 6370, Atomic Energy Research Establishment, Harwell, England.
- . 1971. A general quadratic programming algorithm. *Journal of the Institute for Mathematics and its Applications* 7:76–91.
- Florian, M., ed.** 1976. *Traffic equilibrium methods*. Heidelberg: Springer.
- Gleave, D., and Cordey-Hayes, M.** 1977. Migration dynamics and labour market turnover. *Progress in Planning* 8:1–95.
- Golledge, R.** 1980. A behavioral view of mobility and migration research. *The Professional Geographer* 32:14–21.
- Greenwood, M.** 1975. Research on internal migration in the United States. *Journal of Economic Literature* 13:397–433.
- . 1981. *Migration and economic growth in the United States*. New York: Academic Press.
- Hägerstrand, T.** 1957. Migration and area. In *Migration in Sweden*, ed. D. Hannerberg et al., pp. 17–128. Lund Studies in Geography No. 13, University of Lund, Sweden.
- Hersh, R., and Griego, R.** 1969. Brownian motion and potential theory. *Scientific American* 220:67–74.
- Hodgson, M.** 1978. Toward more realistic allocation in location-allocation models: an interaction approach. *Environment and Planning A* 10:1273–85.
- Hoffmann-Novotny, H.** 1970. *Migration—ein Beitrag zu einer soziologischen Erklärung*. Stuttgart: Enke.
- Holland, P., and Leinhardt, S.** 1981. An exponential family of probability distributions for directed graphs. *Journal of the American Statistical Association* 76:33–65.
- Hollingsworth, T.** 1970. *Migration: a study based on Scottish experience between 1939 and 1964*. Edinburgh: Oliver and Boyd.
- Hotelling, H.** 1978. A mathematical theory of migration. *Environment and Planning A* 10:1233–39.
- Hua, C., and Porell, F.** 1979. A critical review of the development of the gravity model. *International Regional Science Review* 4:97–126.
- Hubert, L., and Golledge, R.** 1981. A heuristic method for the comparison of related structures. *Journal of Mathematical Psychology* 23:214–26.
- Isard, W.** 1960. *Methods of regional science*. Cambridge, Mass.: MIT Press.
- Jacoby, S., and Kowalik, J.** 1980. *Mathematical modeling with computers*. Englewood Cliffs, N.J.: Prentice-Hall.
- Kau, J., and Sirmans, C.** 1979. The functional form of the gravity model. *International Regional Science Review* 4:127–36.
- Koopmans, T.** 1949. Optimum utilization of the transportation system. *Econometrica*, Supplement:136–49.
- Kunzi, H., Tzschach, H., and Zehnder, C.** 1971. *Numerical methods of mathematical optimization*. New York: Academic Press.
- Kuznets, S., ed.** 1957–1964. *Population redistribution and economic growth: United States 1870–1950*. Memoirs 45, 51, and 61, American Philosophical Society, Philadelphia.
- Ledent, J.** 1981. On the relationship between Alonso's theory of movement and Wilson's family of spatial interaction models. *Environment and Planning A* 13:217–24.
- Lee, E.** 1966. A theory of migration. *Demography* 3:47–57.
- Leontief, W.** 1941. *The structure of the American economy 1919–1929*. New York: Oxford University Press.
- Lowry, I.** 1966. *Migration and metropolitan growth: two analytical models*. San Francisco: Chandler.
- MacKinnon, R.** 1975. Controlling inter-regional migration processes of a Markovian type. *Environment and Planning A* 7:781–92.
- Mangalam, J.** 1968. *Human migration: a guide to migration literature in English 1955–1962*. Lexington, Ky.: University of Kentucky Press.
- May, R.** 1976. Simple mathematical models with very complicated dynamics. *Nature* 261:459–68.
- Mera, K.** 1971. An evaluation of gravity and linear programming transportation models for predicting interregional commodity flows. In *Techniques of Transportation Planning*, Vol. 1, ed. J. Meyer, pp. 297–309. Washington: Brookings.
- Milliff, R.** 1980. Boundary condition specification for a model of geographical movement. M.A. thesis, Department of Geography, University of California-Santa Barbara.
- Morrill, R.** 1967. The movement of persons and the transportation problem. In *Quantitative Geography*, ed. W. Garrison and D. Marble, pp. 84–94. Northwestern University Studies in Geography No. 14, Evanston, Illinois.
- Morrison, P.** 1973. Theoretical issues in the design of population mobility models. *Environment and Planning A* 5:125–34.
- Mueller, C.** 1982. *The economics of labor migration*. New York: Academic Press.
- Nijkamp, P.** 1975. Reflections on gravity and entropy models. *Regional Science and Urban Economics* 5:203–26.
- Olsson, G.** 1965. *Distance and human interaction*. Philadelphia: Regional Science Research Institute.
- Petersen, W.** 1978. International migration. *Inter-*

- national Encyclopedia of the Social Sciences* 10:286–300.
- Pierson, G.** 1973. *The moving American*. New York: A. Knopf.
- Polenska, K.** 1966. A case study of transportation models used in multiregional analysis. Ph.D. dissertation, Department of Economics, Harvard University, Cambridge, Mass.
- Price, D., and Sikes, M.** 1975. *Rural-urban migration research in the United States*. Washington: Department of Health, Education and Welfare.
- Proskurowski, W., and Widlund, O.** 1976. On the numerical solution of Helmholtz's equation by the capacitance matrix method. *Mathematics of Computation* 30:433–68.
- Pryor, R.** 1981. Integrating international and internal migration theories. In *Global trends in migration*, ed. M. Kritz, C. Keely, and S. Tomasi, pp. 110–29. New York: Center for Migration Studies.
- Quigley, J., and Weinberg, D.** 1977. Intra-urban residential mobility: a review and synthesis. *International Regional Science Review* 2:41–66.
- Ravenstein, E.** 1876. The birthplaces of the people and the laws of migration. *The Geographical Magazine* III:173–77, 201–6, 229–33.
- . 1885. The laws of migration. *Journal of the Statistical Society* 48:167–235.
- . 1889. The laws of migration: second paper. *Journal of the Royal Statistical Society* 52:241–305.
- Redford, A.** 1964. *Labour migration in England 1800–1850*, 2nd ed. Manchester: The University Press.
- Ritchey, P.** 1976. Explanations of migration. *Annual Review of Sociology* 2:363–404.
- Rogers, A.** 1968. *Matrix analysis of interregional population growth and distribution*. Berkeley: University of California Press.
- Samuelson, P.** 1952. Spatial price equilibrium and linear programming. *American Economic Review* 42:283–303.
- . 1976. *Economics*. New York: McGraw-Hill.
- Schneider, M.** 1959. Gravity models and trip distribution theory. *Papers and Proceedings of the Regional Science Association* 5:51–56.
- Shaw, R.** 1975. *Migration: theory and fact*. Philadelphia: Regional Science Research Institute.
- Smith, T.** 1978. A general efficiency principle of spatial interaction. In *Spatial interaction theory and planning models*, ed. A. Karlqvist, L. Lundqvist, F. Snickars, and J. Weibull, pp. 97–118. Amsterdam: North Holland.
- Sommeyer, W.** 1971. Multi-polar human flow models. *Papers and Proceedings, the Regional Science Association* 26:131–44.
- Southwell, R.** 1956. *Relaxation methods in theoretical physics*, Vol. II. Oxford: Clarendon Press.
- Speare, A., Goldstein, S., and Frey, W.** 1974. *Residential mobility, migration and metropolitan change*. Cambridge, Mass.: Ballinger.
- Stewart, J.** 1947. Empirical mathematical rules concerning the distribution and equilibrium of population. *Geographical Review* 37:461–85.
- Thomas, D.** 1938. *Research memorandum on migration differentials*. Bulletin No. 43, Social Science Research Council, New York.
- Thorntwaite, W.** 1934. *Internal migration in the United States*. Philadelphia: University of Pennsylvania Press.
- Tobler, W.** 1969. Geographical filters and their inverses. *Geographical Analysis* 1:234–53.
- . 1970. A computer movie simulating urban growth in the Detroit region. *Economic Geography* 42:234–40.
- . 1979. A geographical migration probability density function. *The Ontario Geographer* 13:41–46.
- . 1981. A model of geographical movement. *Geographical Analysis* 13:1–20.
- Tocalis, T.** 1979. Changing theoretical foundations of the gravity concept of human interaction. In *The nature of change in geographical ideas*, ed. B. Berry, pp. 65–124. Dekalb, Ill.: Northern Illinois University Press.
- U.S. Bureau of the Census.** 1973. *Lifetime and recent migration*. Subject Report PC (2)-2D. Washington: Department of Commerce.
- Wachspress, E.** 1966. *Iterative solution of elliptic systems*. Englewood Cliffs, N.J.: Prentice-Hall.
- Wansbeek, T.** 1977. Least-squares estimation of trip distribution parameters. *Transportation Research* 11:429–31.
- Wardrop, J.** 1952. Some theoretical aspects of road traffic research. *Proceedings of the Institute of Civil Engineers*, Part II, 1:325–78.
- Weaver, T., and Downing, T., eds.** 1976. *Mexican migration*. Tucson: University of Arizona, Bureau of Ethnic Research.
- Welch, R.** 1971. *Migration research and migration in Britain*. Occasional Paper No. 14, Centre for Urban and Regional Studies, Birmingham, England.
- Willis, K.** 1974. *Problems in migration analysis*. Westmead, England: Saxon House.
- Wilson, A.** 1967. A statistical theory of spatial distribution models. *Transportation Research* 1:253–69.
- Wolpert, J.** 1965. Behavioral aspects of the decision to migrate. *Papers and Proceedings of the Regional Science Association* 15:159–69.
- Zipf, G.** 1946. The  $P_1P_2/D$  hypothesis: on the intercity movement of persons. *American Sociological Review* 11:677–86.
- . 1949. *Human behavior and the principle of least effort*. Cambridge, Mass.: Addison-Wesley.