MATH 631: Midterm Exam

Instructions: Complete all problems in Part I. Submit solutions to three out of the five problems from Part II. Good luck.

Part I. Complete all of the following.

1. Draw a lattice diagram for the abelian group $\mathbb{Z}/36\mathbb{Z}$.
2. Consider the symmetric group S_5.
   (a) List all conjugates of $\sigma = (123)(45)$.
   (b) Find $\big[S_5 : N_{S_5}(<\sigma>)\big]$. Give a one sentence explanation of your answer.
   A better question would have been Find $\big[S_5 : C_{S_5}(<\sigma>)\big]$.
3. Let $Z_{450} = <x>$ denote the cyclic group of order $450 = 2 \cdot 3^2 \cdot 5^2$.
   (a) Compute the number of generators of Z_{450}.
   (b) List all the elements of Z_{450} of order 9.
4. List, up to isomorphism, all abelian groups of order 72.
5. (a) Carefully state the Sylow Theorems.
   (b) Prove that a group of order 55 is not simple.

Part II. Choose any three of the following.

1. Let $g_1, g_2, \cdots, g_r$ be representatives of conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.
2. Suppose G is a finite group with $|G| = n$ and that p is the smallest prime so that $p \Big| n$. Prove that if $H \leq G$ and $\big[G : H\big] = p$, then $H \lhd G$.
3. (a) Let G be a finite group with $|G| = 77$. Prove that G is cyclic.
   (b) Prove that a group G with order $|G| = 12$ is not simple.
4. (a) Prove that a group G with order $|G| = 380 = 2^2 \cdot 5 \cdot 19$ is not simple.
   (b) Let G be a group with order $|G| = 315 = 3^2 \cdot 5 \cdot 7$. Suppose that a Sylow 3-subgroup P of G is normal. Prove that $P \leq Z(G)$.
5. Let p be a prime number and suppose that $|P| = p^\alpha$ for some $\alpha \in \mathbb{Z}^+$. Prove that the center of P is not trivial, $Z(P) \neq \{e\}$.