§6 An isomorphism of a group with itself is an automorphism of the group. Find the number of automorphisms of the given group.

14. $\mathbb{Z}_8$

The automorphism of a group is determined by the image of one of its generators. Moreover, the image of a generator is also a generator. There are 4 generators of $\mathbb{Z}_8$, namely 1, 3, 5, and 7 — the integers in $\mathbb{Z}_8$ that are relatively prime to 8. Therefore, there are 4 automorphisms of $\mathbb{Z}_8$, namely the automorphisms that map $1 \mapsto 1$, $1 \mapsto 3$, $1 \mapsto 5$, and $1 \mapsto 7$.

16. $\mathbb{Z}_{12}$

We need only count the number of integers in $\mathbb{Z}_{12}$ that are relatively prime to 12 (See above). This can be found by
$\varphi(12) = \varphi(2^2 \cdot 3) = \varphi(2^2)\,\varphi(3) = 2^1(2-1)\cdot(3-1) = 4$. Thus, there are 4 automorphisms of $\mathbb{Z}_{12}$, namely the automorphisms that map $1 \mapsto 1$, $1 \mapsto 5$, $1 \mapsto 7$, and $1 \mapsto 11$.

20. Find the number of elements in the indicated cyclic group.

The cyclic subgroup of the group $\mathbb{C}^*$ generated by $(1+i)/\sqrt{2}$

We have that $\frac{1+i}{\sqrt{2}} = e^{i\pi/4}$, an eighth root of 1. Therefore, $\langle \frac{(1+i)}{\sqrt{2}} \rangle = U_8$ and $|\langle \frac{(1+i)}{\sqrt{2}} \rangle| = 8$. Alternatively, $\langle \frac{(1+i)}{\sqrt{2}} \rangle = \{ \frac{(1+i)}{\sqrt{2}}, i, \frac{(-1+i)}{\sqrt{2}}, -1, \frac{(-1-i)}{\sqrt{2}}, -i, \frac{(1-i)}{\sqrt{2}}, 1 \}$, so $|\langle \frac{(1+i)}{\sqrt{2}} \rangle| = 8$.

34. Give an example of a group with the property described, or explain why no example exists.

An infinite group that is not cyclic

$\langle \mathbb{Q}, + \rangle$, $\langle \mathbb{R}, + \rangle$, $\langle \mathbb{C}, + \rangle$, $\langle M_n(\mathbb{R}), + \rangle$
$\langle \mathbb{C}^*, \cdot \rangle$, $\langle GL(2,\mathbb{R}), \cdot \rangle$ are all infinite, non-cyclic groups, just to name a few

44. Let G be a cyclic group with generator a, and let G' be a group isomorphic to G. If $\phi: G \to G'$ is an isomorphism, show that, for every $x \in G$, $\phi(x)$ is completely determined by the value $\phi(a)$. That is, if $\phi: G \to G'$ and $\psi: G \to G'$ are two isomorphisms such that $\phi(a) = \psi(a)$, then $\phi(x) = \psi(x)$ for all $x \in G$.

Proof: Let $x \in G$. Since G is cyclic with generator a, $x = a^n$ for some $n \in \mathbb{Z}$. If $n = 0$, then $x = e$ and $\phi(x) = \psi(x) = e'$.

If $n > 0$, then $\phi(x) = \phi(a^n) = \underbrace{\phi(a)\cdot\phi(a)\cdot\ldots\cdot\phi(a)}_{n \text{ times}} = (\phi(a))^n = (\psi(a))^n = \psi(a^n) = \psi(x)$.

Similarly, if $n < 0$, then

$$\phi(x) = \phi(a^n) = \phi((a^{-1})^{|n|}) = \underbrace{\phi(a^{-1})\phi(a^{-1})\ldots\phi(a^{-1})}_{|n| \text{ times}} = \phi(a^{-1})^{|n|} = [\phi(a)^{-1}]^{|n|}$$

$$= [\psi(a)^{-1}]^{|n|} = \psi(a^{-1})^{|n|} = \psi((a^{-1})^{|n|}) = \psi(a^n) = \psi(x)$$

In all cases, we have that $\phi(x) = \psi(x)$ for all $x \in G$. Thus, $\phi(x)$ is completely determined by the value $\phi(a)$. ■

50. Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the unique such element. Show that $ax = xa$ for all $x \in G$.

Proof: Let $x \in G$. We have that

$(xax^{-1})^2 = (xax^{-1})(xax^{-1}) = xa^2x^{-1} = xx^{-1} = e$ since $a^2 = e$.

Therefore, either xax^{-1} has order 1 or 2; that is, $xax^{-1} = e$, or since a is the unique element of G of order 2, $xax^{-1} = a$. We will show that the first case cannot be true. Suppose that $xax^{-1} = e$. Then $xa = (xax^{-1})x = ex = xe$. From the left cancellation law, we have that $a = e$. However, $a \neq e$ since a has order 2 (and e has order 1). Thus, $xax^{-1} \neq e$ and hence, $xax^{-1} = a$. Multiplying on the right by x gives us then that $xa = ax$. ■

§7#2. List the elements of the subgroup generated by the given subset

The subset $\{4,6\}$ of $\mathbb{Z}_{12}$

$\langle 4, 6 \rangle = \{0, 2, 4, 6, 8, 10\}$