Section 4.3

34. Prove that if p is a prime and P is a subgroup of S_p of order p, then $|N_{S_p}(P)| = p(p-1)$. [Argue that every conjugate of P contains exactly $p-1$ p-cycles and use the formula for the number of p-cycles to compute the index of $N_{S_p}(P)$ in S_p.]

Proof. (Hazlett) Note, by Cauchy's Theorem we can find an element $\sigma \in P$ where $|\sigma| = p$. However, the only elements of order p in S_p are p-cycles. Hence P contains a p-cycle σ. Additionally we know $P = \langle \sigma \rangle$ since $|P| = p$. We also note that powers of p-cycles are also p-cycles. Hence P contains the identity and $p-1$ p-cycles. Since conjugation preserves cycle structure we know that every conjugate of P will also have $p-1$ p-cycles. The number of p-cycles in S_p is $(p-1)!$. We deduce that there are $\frac{(p-1)!}{p-1} = (p-2)!$ conjugates of P. If we let G act on P by conjugation we can count the number of elements in any orbit, ie. the number of conjugates of P, by the equation $|O_P| = [S_p : N_{S_p}(P)]$. This gives us $(p-2)! = \frac{|S_p|}{|N_{S_p}(P)|} = \frac{p!}{|N_{S_p}(P)|}$. Therefore $|N_{S_p}(P)| = \frac{p!}{(p-2)!} = p(p-1)$. $\qquad\square$

Chapter 4.4

1. If $\sigma \in \mathrm{Aut}(G)$ and φ_g is a conjugation by g prove $\sigma \varphi_g \sigma^{-1} = \varphi_{\sigma(g)}$. Deduce that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

Proof. (Schamel) Consider $a \in G$. Then

$$
\begin{aligned}
(\sigma \varphi_g \sigma^{-1})(a) &= \sigma(\varphi_g(\sigma^{-1}(a))) \\
&= \sigma(g\sigma^{-1}(a)g^{-1}) \\
&= \sigma(g)\sigma(\sigma^{-1}(a))\sigma(g^{-1}) \\
&= \sigma(g)a(\sigma(g))^{-1} \\
&= \varphi_{\sigma(g)}(a).
\end{aligned}
$$

Thus $\sigma \varphi_g \sigma^{-1} = \varphi_{\sigma(g)}$. Hence, for all $\sigma \in \mathrm{Aut}(G)$, $\sigma \mathrm{Inn}(G)\sigma^{-1} = \mathrm{Inn}(G)$ and so $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. $\quad\square$

6. Prove that characteristic subgroups are normal. Give an example of a normal subgroup that is not characteristic.

Proof. (Bastille) Let $H\,\mathrm{char}\,G$, and let $g \in G$. Define

$$
\sigma_g : G \to gGg^{-1} = G
$$
$$
\sigma_g(x) = gxg^{-1}.
$$

Note that by Proposition 13, since $G \trianglelefteq G$, $\sigma_g \in \mathrm{Aut}(G)$. Furthermore since $H\,\mathrm{char}\,G$, we must have $\sigma_g(H) = H$, i.e. $gHg^{-1} = H$. Since g was chosen arbitrarily, it follows that H is normal in G. $\quad\square$

<u>Remark</u> Not all normal subgroups are characteristic. Let $G = (\mathbb{Q}, +)$, and consider the subset $S = \mathbb{Z} \subseteq \mathbb{Q}$. Note that $0 \in S$ so S is not empty and if $a, b \in \mathbb{Z}$ so is $a - b$ so $S \le G$. Furthermore, $S \trianglelefteq G$ since G is Abelian. Define the following map:

$$
\sigma_{1/2} : \mathbb{Q} \to \mathbb{Q}
$$
$$
\sigma_{1/2}(q) = \frac{1}{2}q.
$$

Note that $\sigma_{1/2}$ is a homomorphism (1): if $q_1, q_2 \in \mathbb{Q}$,

$$\sigma_{1/2}(q_1 + q_2) = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}q_1 + \frac{1}{2}q_2 = \sigma_{1/2}(q_1) + \sigma_{1/2}(q_2).$$

Furthermore $\sigma_{1/2}$ is bijective (2):

- if $\sigma_{1/2}(q_1) = \sigma_{1/2}(q_2)$ then $\frac{1}{2}q_1 = \frac{1}{2}q_2 \Rightarrow q_1 = q_2$ so $\sigma_{1/2}$ is injective;
- if $r \in \mathbb{Q}$ then $2r \in \mathbb{Q}$ and $\sigma_{1/2}(2r) = \frac{1}{2}(2r) = r$ so $\sigma_{1/2}$ is surjective.

Combining (1) and (2) leads to $\sigma_{1/2} \in \text{Aut}(\mathbb{Q})$. Yet $\sigma_{1/2}(\mathbb{Z}) \neq \mathbb{Z}$ since for example $3 \in \mathbb{Z}$ but $\sigma_{1/2}(3) = \frac{3}{2} \notin \mathbb{Z}$. So $\mathbb{Z}$ is not characteristic in $\mathbb{Q}$.

7. If H is the unique subgroup of a given order in a group G, prove H is characteristic in G.

   Proof. (Baggett) Take any $\varphi \in Aut(G)$. Since φ is an isomorphism, $H \cong \varphi(H)$. Thus, $\varphi(H) \leq G$ and $|\varphi(H)| = |H|$. Since H is the unique subgroup of a given order in G, it follows that $\varphi(H) = H$. Thus, H is characteristic in G. ∎

8. Let G be a group with subgroups H and K with $H \leq K$.

   (a) Prove that if H is characteristic in K and K is normal in G, then H is normal in G.

      Proof. (Lawless)
      ∎

   (b) Prove that if H is characteristic in K and K is characteristic in G, then H is characteristic in G. Use this to prove that the Klein 4-group is characteristic in S_4.

      Proof. (Lawless) Let H char K char G. Let $\sigma \in Aut(G)$. Then since K char G, $\sigma(K) = K$. Thus, $\sigma|_K$ is an automorphism of K. Since H char K, then $\sigma|_K(H) = H$. Therefore, $\sigma(H) = H$, and so H char G.

      We will show V_4 char A_4 char S_4. First, we know A_4 is a subgroup of order 12 in S_4. We will show A_4 is the only subgroup of S_4 of order 12. Assume $H \leq S_4$ and $|H| = 12$. Thus, there exists some $\sigma \in H$ with $|\sigma| = 3$. Since $[S_4 : H] = 2$, we know that $H \trianglelefteq S_4$, and as such all conjugates of σ must be in H. Thus, all 3-cycles are in H. Since A_4 is generated by the 3-cycles of S_4, we know $G = A_4$. Thus, A_4 is the only subgroup of S_4 of order 12, so A_4 char S_4.

      Since V_4 is the only subgroup of order 4 in A_4, then we know V_4 char A_4. Thus, V_4 char S_4. ∎

   (c) Give an example to show that if H is normal in K and K is characteristic in G, then H need not be normal in G.

      Proof. (Lawless) We have shown V_4 char A_4. Consider the subgroup $\langle (1\,2)(3\,4) \rangle \leq V_4$. Notice, however,
      $$(1\,2\,3)(1\,2)(3\,4)(1\,3\,2) = (1\,4)(2\,3) \notin \langle (1\,2)(3\,4) \rangle.$$
      Thus, $\langle (1\,2)(3\,4) \rangle$ is not a normal subgroup of A_4.
      ∎

Chapter 5.2

2. In each of parts (a) to (e) give the lists of invariant factors for all abelian groups of the specified order:
   (a) order 270, **(b)** order 9801, **(c)** order 320, **(d)** order 105, **(e)** order 44100.

   Proof. (Mobley)

   (a)

n_k values	Invariant Factor
$n_1 = 2 \cdot 3^3 \cdot 5$	$A \cong \mathbb{Z}/270\mathbb{Z}$
$n_1 = 2 \cdot 3^2 \cdot 5$	
$n_2 = 3$	$A \cong \mathbb{Z}/90\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$n_1 = 2 \cdot 3 \cdot 5$	
$n_2 = 3$	
$n_3 = 3$	$A \cong \mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

   (b)

n_k values	Invariant Factor
$n_1 = 3^4 \cdot 11^2$	$A \cong \mathbb{Z}/9801\mathbb{Z}$
$n_1 = 3^3 \cdot 11^2$	
$n_2 = 3$	$A \cong \mathbb{Z}/3267\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$n_1 = 3^2 \cdot 11^2$	
$n_2 = 3$	
$n_3 = 3$	$A \cong \mathbb{Z}/1089\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$n_1 = 3^2 \cdot 11^2$	
$n_2 = 9$	$A \cong \mathbb{Z}/1089\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$
$n_1 = 3 \cdot 11^2$	
$n_2 = 3$	
$n_3 = 3$	
$n_4 = 3$	$A \cong \mathbb{Z}/363\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$n_1 = 3^4 \cdot 11$	
$n_2 = 11$	$A \cong \mathbb{Z}/891\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$
$n_1 = 3^3 \cdot 11$	
$n_2 = 3 \cdot 11$	$A \cong \mathbb{Z}/297\mathbb{Z} \times \mathbb{Z}/33\mathbb{Z}$
$n_1 = 3^2 \cdot 11$	
$n_2 = 3^2 \cdot 11$	$A \cong \mathbb{Z}/99\mathbb{Z} \times \mathbb{Z}/99\mathbb{Z}$
$n_1 = 3^2 \cdot 11$	
$n_2 = 3 \cdot 11$	
$n_3 = 3$	$A \cong \mathbb{Z}/99\mathbb{Z} \times \mathbb{Z}/33\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$n_1 = 3 \cdot 11$	
$n_2 = 3 \cdot 11$	
$n_3 = 3$	
$n_4 = 3$	$A \cong \mathbb{Z}/33\mathbb{Z} \times \mathbb{Z}/33\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

   (c)

n_k values	Invariant Factor
$n_1 = 2^6 \cdot 5$	$A \cong \mathbb{Z}/320\mathbb{Z}$
$n_1 = 2^5 \cdot 5$	
$n_2 = 2$	$A \cong \mathbb{Z}/160\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$n_1 = 2^4 \cdot 5$	
$n_2 = 2^2$	$A \cong \mathbb{Z}/80\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
$n_1 = 2^4 \cdot 5$	
$n_2 = 2$	
$n_3 = 2$	$A \cong \mathbb{Z}/80\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$n_1 = 2^3 \cdot 5$	
$n_2 = 2^3$	$A \cong \mathbb{Z}/40\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$
$n_1 = 2^3 \cdot 5$	
$n_2 = 2^2$	
$n_3 = 2$	$A \cong \mathbb{Z}/40\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$n_1 = 2^3 \cdot 5$	
$n_2 = 2$	
$n_3 = 2$	
$n_4 = 2$	$A \cong \mathbb{Z}/40\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$n_1 = 2^2 \cdot 5$	
$n_2 = 2^2$	
$n_3 = 2^2$	$A \cong \mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
$n_1 = 2^2 \cdot 5$	
$n_2 = 2^2$	
$n_3 = 2$	
$n_4 = 2$	$A \cong \mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$n_1 = 2^2 \cdot 5$	
$n_2 = 2$	
$n_3 = 2$	
$n_4 = 2$	
$n_5 = 2$	$A \cong \mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$n_1 = 2 \cdot 5$	
$n_2 = 2$	
$n_3 = 2$	
$n_4 = 2$	
$n_5 = 2$	
$n_6 = 2$	$A \cong \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(d)

n_k values	Invariant Factor
$n_1 = 3 \cdot 5 \cdot 7$	$A \cong \mathbb{Z}/105\mathbb{Z}$

(e)

n_k values	Invariant Factor
$n_1 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$	$A \cong \mathbb{Z}/44100\mathbb{Z}$
$n_1 = 2 \cdot 3^2 \cdot 5^2 \cdot 7^2$	
$n_2 = 2$	$A \cong \mathbb{Z}/22050\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$n_1 = 2 \cdot 3 \cdot 5^2 \cdot 7^2$	
$n_2 = 2 \cdot 3$	$A \cong \mathbb{Z}/7350\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$
$n_1 = 2 \cdot 3 \cdot 5 \cdot 7^2$	
$n_2 = 2 \cdot 3 \cdot 5$	$A \cong \mathbb{Z}/1470\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$
$n_1 = 2 \cdot 3 \cdot 5 \cdot 7$	
$n_2 = 2 \cdot 3 \cdot 5 \cdot 7$	$A \cong \mathbb{Z}/210\mathbb{Z} \times \mathbb{Z}/210\mathbb{Z}$
$n_1 = 2^2 \cdot 3 \cdot 5^2 \cdot 7^2$	
$n_2 = 3$	$A \cong \mathbb{Z}/14700\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$n_1 = 2^2 \cdot 3 \cdot 5 \cdot 7^2$	
$n_2 = 3 \cdot 5$	$A \cong \mathbb{Z}/2940\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$
$n_1 = 2^2 \cdot 3 \cdot 5 \cdot 7$	
$n_2 = 3 \cdot 5 \cdot 7$	$A \cong \mathbb{Z}/420\mathbb{Z} \times \mathbb{Z}/105\mathbb{Z}$
$n_1 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^2$	
$n_2 = 5$	$A \cong \mathbb{Z}/8820\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$
$n_1 = 2 \cdot 3^2 \cdot 5 \cdot 7^2$	
$n_2 = 2 \cdot 5$	$A \cong \mathbb{Z}/4410\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$
$n_1 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7$	
$n_2 = 7$	$A \cong \mathbb{Z}/6300\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$
$n_1 = 2 \cdot 3^2 \cdot 5^2 \cdot 7$	
$n_2 = 2 \cdot 7$	$A \cong \mathbb{Z}/3150\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$
$n_1 = 2^2 \cdot 3 \cdot 5^2 \cdot 7$	
$n_2 = 3 \cdot 7$	$A \cong \mathbb{Z}/2100\mathbb{Z} \times \mathbb{Z}/21\mathbb{Z}$
$n_1 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$	
$n_2 = 5 \cdot 7$	$A \cong \mathbb{Z}/1260\mathbb{Z} \times \mathbb{Z}/35\mathbb{Z}$
$n_1 = 2 \cdot 3 \cdot 5^2 \cdot 7$	
$n_2 = 2 \cdot 3 \cdot 7$	$A \cong \mathbb{Z}/1050\mathbb{Z} \times \mathbb{Z}/42\mathbb{Z}$
$n_1 = 2 \cdot 3^2 \cdot 5 \cdot 7$	
$n_2 = 2 \cdot 5 \cdot 7$	$A \cong \mathbb{Z}/630\mathbb{Z} \times \mathbb{Z}/70\mathbb{Z}$

$\square$

3. In each of parts (a) to (e) give the lists of elementary divisors for all abelian groups of the specified order and then match each list with the corresponding list of invariant factors found in the preceding exercise: **(a)** order 270, **(b)** order 9801, **(c)** order 320, **(d)** order 105, **(e)** order 44100.

Proof. (Buchholz)

(a)

Orders p^β	Partitions of β	Abelian Groups
2^1	1	Z_2
3^3	3; 2, 1; 1, 1, 1	$Z_{27}, Z_9 \times Z_3, Z_3 \times Z_3 \times Z_3$
5^1	1	Z_5

Taking all the possible ways gives all the isomorphism types:

- $Z_{27} \times Z_5 \times Z_2$

– $Z_9 \times Z_3 \times Z_5 \times Z_2$

– $Z_5 \times Z_3 \times Z_3 \times Z_3 \times Z_2$

Matching with the list of invariant factors we get

– $Z_{27} \times Z_5 \times Z_2 \cong \mathbb{Z}/270\mathbb{Z}$

– $Z_9 \times Z_3 \times Z_5 \times Z_2 \cong \mathbb{Z}/90\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

– $Z_5 \times Z_3 \times Z_3 \times Z_3 \times Z_2 \cong \mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

(b)

Orders p^β	Partitions of β	Abelian Groups
3^4	$4; 3,1; 2,2; 2,1,1;$ $1,1,1,1$	$Z_{81}, Z_{27} \times Z_3, Z_9 \times Z_9, Z_9 \times Z_3 \times Z_3,$ $Z_3 \times Z_3 \times Z_3 \times Z_3$
11^2	$2; 1,1$	$Z_{121}, Z_{11} \times Z_{11}$

Taking all the possible ways gives all the isomorphism types:

– $Z_{121} \times Z_{81}$

– $Z_{121} \times Z_{27} \times Z_3$

– $Z_{121} \times Z_9 \times Z_9$

– $Z_{121} \times Z_9 \times Z_3 \times Z_3$

– $Z_{121} \times Z_3 \times Z_3 \times Z_3 \times Z_3$

– $Z_{81} \times Z_{11} \times Z_{11}$

– $Z_{27} \times Z_{11} \times Z_{11} \times Z_3$

– $Z_{11} \times Z_{11} \times Z_9 \times Z_9$

– $Z_{11} \times Z_{11} \times Z_9 \times Z_3 \times Z_3$

– $Z_{11} \times Z_{11} \times Z_3 \times Z_3 \times Z_3 \times Z_3$

Matching with the list of invariant factors we get

– $Z_{121} \times Z_{81} \cong \mathbb{Z}/9801\mathbb{Z}$

– $Z_{121} \times Z_{27} \times Z_3 \cong \mathbb{Z}/3267\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

– $Z_{121} \times Z_9 \times Z_9 \cong \mathbb{Z}/1089\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$

– $Z_{121} \times Z_9 \times Z_3 \times Z_3 \cong \mathbb{Z}/1089\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

– $Z_{121} \times Z_3 \times Z_3 \times Z_3 \times Z_3 \cong \mathbb{Z}/363\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

– $Z_{81} \times Z_{11} \times Z_{11} \cong \mathbb{Z}/891\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$

– $Z_{27} \times Z_{11} \times Z_{11} \times Z_3 \cong \mathbb{Z}/297\mathbb{Z} \times \mathbb{Z}/33\mathbb{Z}$

– $Z_{11} \times Z_{11} \times Z_9 \times Z_9 \cong \mathbb{Z}/99\mathbb{Z} \times \mathbb{Z}/99\mathbb{Z}$

– $Z_{11} \times Z_{11} \times Z_9 \times Z_3 \times Z_3 \cong \mathbb{Z}/99\mathbb{Z} \times \mathbb{Z}/33\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

– $Z_{11} \times Z_{11} \times Z_3 \times Z_3 \times Z_3 \times Z_3 \cong \mathbb{Z}/33\mathbb{Z} \times \mathbb{Z}/33\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

(c)

Orders p^β	Partitions of β	Abelian Groups
2^6	$6; 5,1; 4,2; 4,1,1;$ $3,3; 3,2,1; 3,1,1,1;$ $2,2,2; 2,2,1,1;$ $2,1,1,1,1; 1,1,1,1,1,1$	$Z_{64}, Z_{32} \times Z_2, Z_{16} \times Z_4, Z_{16} \times Z_2 \times Z_2,$ $Z_8 \times Z_8, Z_8 \times Z_4 \times Z_2, Z_8 \times Z_2 \times Z_2 \times Z_2,$ $Z_4 \times Z_4 \times Z_4, Z_4 \times Z_4 \times Z_2 \times Z_2,$ $Z_4 \times Z_2 \times Z_2 \times Z_2 \times Z_2, Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$
5^1	1	Z_5

Taking all the possible ways gives all the isomorphism types:

- $Z_{64} \times Z_5$
- $Z_{32} \times Z_5 \times Z_2$
- $Z_{16} \times Z_5 \times Z_4$
- $Z_{16} \times Z_5 \times Z_2 \times Z_2$
- $Z_8 \times Z_8 \times Z_5$
- $Z_8 \times Z_5 \times Z_4 \times Z_2$
- $Z_8 \times Z_5 \times Z_2 \times Z_2 \times Z_2$
- $Z_5 \times Z_4 \times Z_4 \times Z_4$
- $Z_5 \times Z_4 \times Z_4 \times Z_2 \times Z_2$
- $Z_5 \times Z_4 \times Z_2 \times Z_2 \times Z_2 \times Z_2$
- $Z_5 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$

Matching with the list of invariant factors we get

- $Z_{64} \times Z_5 \cong \mathbb{Z}/320\mathbb{Z}$
- $Z_{32} \times Z_5 \times Z_2 \cong \mathbb{Z}/160\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Z_{16} \times Z_5 \times Z_4 \cong \mathbb{Z}/80\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
- $Z_{16} \times Z_5 \times Z_2 \times Z_2 \cong \mathbb{Z}/80\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Z_8 \times Z_8 \times Z_5 \cong \mathbb{Z}/40\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$
- $Z_8 \times Z_5 \times Z_4 \times Z_2 \cong \mathbb{Z}/40\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Z_8 \times Z_5 \times Z_2 \times Z_2 \times Z_2 \cong \mathbb{Z}/40\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Z_5 \times Z_4 \times Z_4 \times Z_4 \cong \mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
- $Z_5 \times Z_4 \times Z_4 \times Z_2 \times Z_2 \cong \mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Z_5 \times Z_4 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \cong \mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Z_5 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \cong \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(d)

Orders p^β	Partitions of β	Abelian Groups
3^1	1	Z_3
5^1	1	Z_5
7^1	1	Z_7

Taking all the possible ways gives all the isomorphism types:

- $Z_7 \times Z_5 \times Z_3$

Matching with the list of invariant factors we get

- $Z_7 \times Z_5 \times Z_3 \cong \mathbb{Z}/105\mathbb{Z}$

(e)

Orders p^β	Partitions of β	Abelian Groups
2^2	2; 1, 1	Z_4, $Z_2 \times Z_2$
3^2	2; 1, 1	Z_9, $Z_3 \times Z_3$
5^2	2; 1, 1	Z_{25}, $Z_5 \times Z_5$
7^2	2; 1, 1	Z_{49}, $Z_7 \times Z_7$

Taking all the possible ways gives all the isomorphism types:

- $Z_{49} \times Z_{25} \times Z_9 \times Z_4$
- $Z_{49} \times Z_{25} \times Z_9 \times Z_2 \times Z_2$
- $Z_{49} \times Z_{25} \times Z_3 \times Z_3 \times Z_2 \times Z_2$
- $Z_{49} \times Z_{25} \times Z_4 \times Z_3 \times Z_3$
- $Z_{49} \times Z_9 \times Z_5 \times Z_5 \times Z_2 \times Z_2$
- $Z_{49} \times Z_9 \times Z_5 \times Z_5 \times Z_4$
- $Z_{49} \times Z_5 \times Z_5 \times Z_3 \times Z_3 \times Z_2 \times Z_2$
- $Z_{49} \times Z_5 \times Z_5 \times Z_4 \times Z_3 \times Z_3$
- $Z_{25} \times Z_9 \times Z_7 \times Z_7 \times Z_2 \times Z_2$
- $Z_{25} \times Z_9 \times Z_7 \times Z_7 \times Z_4$
- $Z_{25} \times Z_7 \times Z_7 \times Z_4 \times Z_3 \times Z_3$
- $Z_{25} \times Z_7 \times Z_7 \times Z_3 \times Z_3 \times Z_2 \times Z_2$
- $Z_9 \times Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_4$
- $Z_9 \times Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_2 \times Z_2$
- $Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_4 \times Z_3 \times Z_3$
- $Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_3 \times Z_3 \times Z_2 \times Z_2$

Matching with the list of invariant factors we get

- $Z_{49} \times Z_{25} \times Z_9 \times Z_4 \cong \mathbb{Z}/44100\mathbb{Z}$
- $Z_{49} \times Z_{25} \times Z_9 \times Z_2 \times Z_2 \cong \mathbb{Z}/22050\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Z_{49} \times Z_{25} \times Z_3 \times Z_3 \times Z_2 \times Z_2 \cong \mathbb{Z}/7350\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$
- $Z_{49} \times Z_{25} \times Z_4 \times Z_3 \times Z_3 \cong \mathbb{Z}/14700\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
- $Z_{49} \times Z_9 \times Z_5 \times Z_5 \times Z_2 \times Z_2 \cong \mathbb{Z}/4410\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$
- $Z_{49} \times Z_9 \times Z_5 \times Z_5 \times Z_4 \cong \mathbb{Z}/8820\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$
- $Z_{49} \times Z_5 \times Z_5 \times Z_3 \times Z_3 \times Z_2 \times Z_2 \cong \mathbb{Z}/1470\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$
- $Z_{49} \times Z_5 \times Z_5 \times Z_4 \times Z_3 \times Z_3 \cong \mathbb{Z}/2940\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$
- $Z_{25} \times Z_9 \times Z_7 \times Z_7 \times Z_2 \times Z_2 \cong \mathbb{Z}/3150\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$
- $Z_{25} \times Z_9 \times Z_7 \times Z_7 \times Z_4 \cong \mathbb{Z}/6300\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$
- $Z_{25} \times Z_7 \times Z_7 \times Z_4 \times Z_3 \times Z_3 \cong \mathbb{Z}/2100\mathbb{Z} \times \mathbb{Z}/21\mathbb{Z}$
- $Z_{25} \times Z_7 \times Z_7 \times Z_3 \times Z_3 \times Z_2 \times Z_2 \cong \mathbb{Z}/1050\mathbb{Z} \times \mathbb{Z}/42\mathbb{Z}$
- $Z_9 \times Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_4 \cong \mathbb{Z}/1260\mathbb{Z} \times \mathbb{Z}/35\mathbb{Z}$
- $Z_9 \times Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_2 \times Z_2 \cong \mathbb{Z}/630\mathbb{Z} \times \mathbb{Z}/70\mathbb{Z}$
- $Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_4 \times Z_3 \times Z_3 \cong \mathbb{Z}/420\mathbb{Z} \times \mathbb{Z}/105\mathbb{Z}$
- $Z_7 \times Z_7 \times Z_5 \times Z_5 \times Z_3 \times Z_3 \times Z_2 \times Z_2 \cong \mathbb{Z}/210\mathbb{Z} \times \mathbb{Z}/210\mathbb{Z}$

$\square$

4. In each of parts (a) to (d) determine which pair of abelian groups listed are isomorphic. (Gillispie)

(a) $\{4, 9\}$, $\{6, 6\}$, $\{8, 3\}$, $\{9, 4\}$, $\{6, 4\}$, $\{64\}$.

The first and fourth are isomorphic.

Proof. Since this is simply a rearrangement of the cyclic groups generating the groups we have $\{4, 9\} \cong \{9, 4\}$.

Also the invariant factor representation of $\{8, 3\}$ is $\{2^3 \cdot 3\}$ and $\{6, 4\}$ is it's own invariant factor representation. Since the invariant factor representation is unique by the fundamental theorem of finitely generated abelian groups, we know these are not isomorphic. Also the order of each of these groups is 24 whereas the order of $\{4, 9\}$ is 36, and so neither $\{8, 3\}$ nor $\{6, 4\}$ is isomorphic to $\{4, 9\}$.

Also the invariant factor representation of $\{4, 9\}$ is $\{36\}$ and the invariant factor representation of $\{6, 6\}$ is itself, by the FToFGAG we have that these are not isomorphic groups.

None of the above groups is of order 64, and as such none are isomorphic to $\{64\}$. $\qquad\square$

(b) $\{2^2, \, 2 \cdot 3^2\}$, $\{2^2 \cdot 3, \, 2 \cdot 3\}$, $\{2^3 \cdot 3^2\}$, $\{2^2 \cdot 3^2, \, 2\}$.

The first and fourth are isomorphic.

Proof. Note that $\{2^2, \, 2 \cdot 3^2\} \cong \{2^2, \, 2, \, 3^2\} \cong \{2^2 \cdot 3^2, \, 2\}$.

The highest element order of $\{2^2, \, 2 \cdot 3^2\}$ is $lcm(2^2, 2 \cdot 3^2) = 2^2 3^2$.

But the highest element order of $\{2^3 \cdot 3^2\}$ is $2^3 \cdot 3^2$ and so $\{2^3 \cdot 3^2\} \not\cong \{2^2, \, 2 \cdot 3^2\}$.
And the highest order element of $\{2^2 \cdot 3, \, 2 \cdot 3\}$ is $lcm(2^2 \cdot 3, 2 \cdot 3) = 2^2 \cdot 3$, and so $\{2^2, \, 2 \cdot 3^2\} \not\cong \{2^2 \cdot 3, \, 2 \cdot 3\} \not\cong \{2^3 \cdot 3^2\}$. $\qquad\square$

(c) $\{5^2 \cdot 7^2, \, 3^2 \cdot 5 \cdot 7\}$, $\{3^2 \cdot 5^2 \cdot 7, \, 5 \cdot 7^2\}$, $\{3 \cdot 5^2, \, 7^2, \, 3 \cdot 5 \cdot 7\}$, $\{5^2 \cdot 7, \, 3^2 \cdot 5, \, 7^2\}$.

The first, second, and fourth are isomorphic.

Proof. Note that $\{5^2 \cdot 7^2, \, 3^2 \cdot 5 \cdot 7\} \cong \{5^2, 7^2, 3^2, 5, 7 \cong \{7^2 \cdot 5^2 \cdot 3^2, 7 \cdot 5\}$.
And $\{3^2 \cdot 5^2 \cdot 7, 5 \cdot 7^2\} \cong \{3^2, 5^2, 7, 5, 7^2\} \cong \{7^2 \cdot 5^2 \cdot 3^2, 7 \cdot 5\}$.
Finally $\{5^2 \cdot 7, 3^2 \cdot 5, 7^2\} \cong \{5^2, , 3^2, 5, 7^2\} \cong \{7^2 \cdot 5^2 \cdot 3^2, 7 \cdot 5\}$.
But notice that the highest element order in $\{7^2 \cdot 5^2 \cdot 3^2, 7 \cdot 5\}$ is $7^2 \cdot 5^2 \cdot 3^2$, and the highest element order in $\{3 \cdot 5^2, 7^2, 3 \cdot 5 \cdot 7\}$ is $lcm(3 \cdot 5^2 \cdot 7, 3 \cdot 5 \cdot 7^2) = 3 \cdot 5^2 \cdot 7^2$ and hence $\{3 \cdot 5^2, \, 7^2, \, 3 \cdot 5 \cdot 7\} \not\cong \{7^2 \cdot 5^2 \cdot 3^2, 7 \cdot 5\}$. $\qquad\square$

(d) $\{2^2 \cdot 5 \cdot 7, \, 2^3 \cdot 5^3, \, 2 \cdot 5^2\}$, $\{2^3 \cdot 5^3 \cdot 7, \, 2^3 \cdot 5^3\}$, $\{2, \, 2^2 \cdot 7, \, 2^3, \, 5^3, \, 5^3\}$, $\{2 \cdot 5^3, \, 2^2 \cdot 5^3, \, 2^3, \, 7\}$.

The third and fourth are isomorphic.

Proof. Note that, $\{2, \, 2^2 \cdot 7, \, 2^3, \, 5^3, \, 5^3\} \cong \{2, \, 2^2, \, 7, \, 2^3, \, 5^3, \, 5^3\} \cong \{7 \cdot 5^3 \cdot 2^3, \, 5^3 \cdot 2^2, \, 2\}$.
And $\{2 \cdot 5^3, \, 2^2 \cdot 5^3, \, 2^3, \, 7\} \cong \{2, \, 5^3, \, 2^2, \, 5^3, \, 2^3, \, 7\} \cong \{7 \cdot 5^3 \cdot 2^3, \, 5^3 \cdot 2^2, \, 2\}$.
And so $\{2, \, 2^2 \cdot 7, \, 2^3, \, 5^3, \, 5^3\} \cong \{7 \cdot 5^3 \cdot 2^3, \, 5^3 \cdot 2^2, \, 2\}$.

But we have that $\{2^2 \cdot 5 \cdot 7, \, 2^3 \cdot 5^3, \, 2 \cdot 5^2\} \cong \{2^2, \, 5, \, 7, \, 2^3, \, 5^3, \, 2, \, 5^2\} \cong \{7 \cdot 5^3 \cdot 2^3, \, 5^2 \cdot 2^2, \, 2 \cdot 5\}$.
And $\{2^3 \cdot 5^3 \cdot 7, \, 2^3 \cdot 5^3\} \cong \{2^3, \, 5^3, \, 7, \, 2^3, \, 5^3\} \cong \{7 \cdot 5^3 \cdot 2^3, \, 2^3 \cdot 5^3\}$.

These are the invariant factor decomposition representations of these abelian groups, and by the fundamental theorem of finitely generated Abelian groups we know that these are unique and hence the second two groups are not isomorphic to each other or the prior groups. $\qquad\square$

9. Let $A = Z_{60} \times Z_{45} \times Z_{12} \times Z_{36}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A.

(Granade). Let $Z_{60} = \langle a \rangle$, $Z_{45} = \langle b \rangle$, $Z_{12} = \langle c \rangle$ and $Z_{36} = \langle d \rangle$. Then, let $x \in A$ such that $x = \left(a^i, b^j, c^k, d^l\right)$ for i, j, k, l being reduced to the smallest non-negative power (that is, reduce each modulo the order of the respective groups). We shall adopt the abbreviated notation $\overline{(i, j, k, l)}$. The order of x is then given by $|x| = \operatorname{lcm}(i, j, k, l)$. Thus, if $|x| = 2$, we must have that $|i|, |j|, |k|, |l| \in \{1, 2\}$ and that at least one of $|i|, |j|, |k|, |l|$ is 2. But then, Z_{45} contains no elements of order 2, and so the elements of order 2 in A are given by:

$$\overline{(0, 0, 0, 18)}, \, \overline{(0, 0, 6, 0)}, \, \overline{(0, 0, 6, 18)}, \, \overline{(30, 0, 0, 0)}, \, \overline{(30, 0, 0, 18)}, \, \overline{(30, 0, 6, 0)}, \, \overline{(30, 0, 6, 18)}$$

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That is, there are 7 elements of order 2 in A.

As for subgroups H of index 2, we have that $[A : H] = 2$ implies that $H \triangleleft A$ and so we may consider A/H. But then, this means that H is the kernel of a surjective homomorphism $\varphi : G \to Z_2$. Thus, the question *really* asks us to find distinct homomorphisms $Z_{60} \times Z_{45} \times Z_{12} \times Z_{36} \twoheadrightarrow Z_2$. Consider:

$$\varphi_{0001}\left(\overline{(i, j, k, l)}\right) = l \bmod 2$$

Then, since modular arithmetic is well-behaved, we have that φ_{0001} is a homomorphism. To see that φ_{0001} is surjective, note that $(0, 0, 0, 0) \mapsto 0$ and that $(0, 0, 0, 1) \mapsto 1$. The odd name given to this homomorphism reflects that we consider the element l, but that we do not consider i, j, k. Thus, name similar functions:

$$\varphi_{s_1 s_2 s_3 s_4}\left(\overline{(i, j, k, l)}\right) = (s_1 i + s_2 j + s_3 k + s_l l) \bmod 2$$

Then, we claim that $\varphi_{s_1 s_2 s_3 s_4}$ is a surjective homomorphism unless $s_1 s_2 s_3 s_4 = 0000$ or unless $s_2 = 1$. To see this, note that $\varphi_{s_1 1 s_3 s_4}$ fails to be a homomorphism as it fails to even be a function:

$$\begin{aligned}
\varphi_{s_1 1 s_3 s_4}\left(\overline{(0, 0, 0, 0)}\right) &= 0 \\
\varphi_{s_1 1 s_3 s_4}\left(\overline{(0, 45, 0, 0)}\right) &= 1
\end{aligned}$$

Similarly, note that φ_{0000} fails to be surjective, since $\varphi_{0000}(x) = 0$ for all $x \in A$. From there, it remains to be shown that all other functions of the form $\varphi_{s_1 s_2 s_3 s_4}$ are surjective homomorphisms. To see this, note that since if $s_2 = 0$, then φ is well-defined, as each of $60, 12, 36$ is divisible by 2. Moreover, since each of the functions of this form has at least one of $s_1, s_3, s_4 \neq 0$, we may choose exactly one selected coordinate to be one, and thus each function is surjective.

This gives us that each subgroup corresponds to an element of $(Z_2 \times Z_2 \times Z_2) \setminus (0, 0, 0)$ for a total of seven subgroups:

$$\ker \varphi_{0001}, \ker \varphi_{0010}, \ker \varphi_{0011}, \ker \varphi_{1000}, \ker \varphi_{1001}, \ker \varphi_{1010}, \ker \varphi_{1011}$$