Math 253 Final Exam
April 30, 2020

Name: Answer Key

Instructions. You have 140 minutes = 2 hours and 20 minutes to scan, complete, and upload this exam. In other words, you have up to a maximum of $2.\bar{3}$ hours for this exam. Closed book, closed notes, no internet, no calculators, and no help allowed. No cheating of any kind. **Show all your work** in order to receive credit. Incomplete answers with little work shown will be graded harshly.

1. Answer the following.

(a) Find the point of intersection of the line ℓ with parametric equations:

$$x(t) = 1 + 2t, \quad y(t) = -1 + 3t, \quad z(t) = 2 + t$$

and the plane $3x - 5y + z = 6$.

Solution:

Plug $x(t) = 1+2t$, $y(t) = -1+3t$, $z(t) = 2+t$ into the equation for the plane and find by solving that $t = \dfrac{1}{2}$. Thus, the point of intersection has coordinates $x(\frac{1}{2}) = 1+2\cdot\frac{1}{2}$, $y(\frac{1}{2}) = -1+3\cdot\frac{1}{2}$, $z(\frac{1}{2}) = 2+\frac{1}{2}$ or $(2, \frac{1}{2}, \frac{5}{2})$.

(b) Find the area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$, where

$$\mathbf{a} = \langle -3,\, 0,\, 1\rangle \text{ and } \mathbf{b} = \langle 2,\, -1, 1\rangle.$$

Solution:

The area is given by the magnitude of the cross product: $|\mathbf{a} \times \mathbf{b}| = \sqrt{35}$.

2. Let $f(x, y, z) = e^x z + 2y \ln\left(\dfrac{y}{4}\right)$

(a) What is the directional derivative of $f(x, y, z)$ at the point $P\,(0, 4, 3)$ in the direction toward the origin?

Solution: We want to take the derivative in the direction of $-\,(0, 4, 3) = (0, -4, -3)$. The unit vector $\mathbf{u}$ in that direction is $\mathbf{u} = \left(0, \dfrac{-4}{5}, \dfrac{-3}{5}\right)$. Since $D_{\mathbf{u}} f(0, 4, 3) = \nabla f(0, 4, 3) \cdot \mathbf{u}$, we first compute that gradient $\nabla f(x, y, z) = \left\langle e^x z,\, 2 + 2\ln\left(\dfrac{y}{4}\right),\, e^x\right\rangle$ and $\nabla f(0, 4, 3) = \langle 3, 2, 1\rangle$. Finally, $D_{\mathbf{u}} f(0, 4, 3) = \nabla f(0, 4, 3) \cdot \mathbf{u} = 0 - \dfrac{8}{5} - \dfrac{3}{5} = -\dfrac{11}{5}$.

(b) In what direction from $(0, 4, 3)$ should you move to increase $f(x, y, z)$ the most?

Solution: In the direction of the gradient, namely $\nabla f(0, 4, 3) = \langle 3, 2, 1\rangle$.

3. Consider the function $f(x, y) = x^2 y + 4xy + y^2$.

(a) Find all critical points of $f(x, y)$.

Solution: Set all partial derivatives equal to zero and find the simultaneous solutions:

$$\begin{aligned} f_x(x, y) &= 2xy + 4y = 0 \implies y(x + 2) = 0 \implies x = -2 \text{ or } y = 0, \\ f_y(x, y) &= x^2 + 4x + 2y = 0. \end{aligned}$$

Case 1: $x = -2$. Then $f_y(-2, y) = (-2)^2 + 4(-2) + 2y = 0 \implies y = 2$. Thus, $\boxed{(-2, 2)}$ is a critical point.

Case 2: $y = 0$. Then $f_y(x, 0) = x^2 + 4x = x(x + 4) = 0 \implies x = 0$ or $x = -4$. Thus, $\boxed{(0, 0) \text{ and } (-4, 0)}$ are critical points.

(b) Determine if the critical points from part (a) are local maxima, local minima, saddle points, or if there is not enough information to tell.

Solution: We need to compute $D = f_{xx}f_{yy} - f_{xy}^2$ at each of the critical points. To the end, $f_{xx}(x,y) = 2y$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = 2x+4$, and $D = 4y - 4(x+2)^2$.

Critical point	D	f_{yy}	conclusion.
$(-2,2)$	8	positive	local min
$(0,0)$	-16		saddle point
$(-4,0)$	-16		saddle point

4. Find the equation of the tangent plane to the implicitly defined surface

$$\sqrt{y^2+z^2+2} + x^2z - x = 8$$

at the point $(3,-1,1)$.

Solution: If the surface is given by $g(x,y,z) = 8$, then a normal vector for the tangent plane is $\mathbf{n} = \nabla g(3,-1,1)$. Computing $\nabla g(x,y,z) = \left\langle 2xz-1,\ \frac{y}{\sqrt{y^2+z^2+2}},\ \frac{z}{\sqrt{y^2+z^2+2}} + x^2 \right\rangle$, and $\mathbf{n} = \nabla g(3,-1,1) = \left\langle 5,\ \frac{-1}{2},\ \frac{19}{2} \right\rangle$. For ease, we will take $\mathbf{n}$ to be $\mathbf{n} = \langle 10,\ -1,\ 19\rangle$. Thus, the equation of the plane if $\mathbf{n}\cdot\overline{x} = \mathbf{n}\cdot\langle 3,-1,1\rangle$ or $\boxed{10x - y + 19z = 50}$.

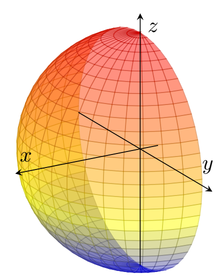
5. Draw the region of integration D for the iterated integral below. Then set up the double integral obtained by reversing the order of integration

$$\int_0^1 \int_0^{x^3} f(x,y)\,dy\,dx + \int_1^e \int_0^{1-\ln(x)} f(x,y)\,dy\,dx$$

Solution:

$$\int_0^1 \int_{\sqrt[3]{y}}^{e^{1-y}} f(x,y)\,dx\,dy$$

6. An object fills the solid hemisphere region B of a sphere of radius 3 shown below. The charge density at any point is given by $\sigma(x,y,z) = x\sqrt{x^2+y^2+z^2}$. Here x, y, z are measured in cm and $\sigma(x,y,z)$ in coulombs/cm^3. Use spherical coordinates to compute the total electric charge of the solid. Include units in your final answer.



Solution: Using $x = \rho\sin(\varphi)\cos(\theta)$, $\rho = \sqrt{x^2+y^2+z^2}$, and $dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$, we compute that

$$\int_0^{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^3 \rho\sin(\varphi)\cos(\theta)\rho\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^3 \rho^4\sin^2(\varphi)\cos(\theta)\,d\rho\,d\theta\,d\varphi$$
$$= \frac{243\pi}{5} \text{ coulombs.}$$

One needs the trigonometric identity $\sin^2(\varphi) = \frac{1}{2}(1 - \cos(2\varphi))$ for the "$d\varphi$" integral.

7. The position with $\langle x, y\rangle$ in kilometers of an object in the plane is given by

$$\mathbf{r}(t) = \left\langle 2t-2,\ \frac{1}{3}(4t-1)^{\frac{3}{2}}\right\rangle, \quad \text{for } t \geq \frac{1}{2} \text{ minutes.}$$

(a) Find the distance traveled by the object (i.e. arc length) between $t = 1$ min and $t = 4$ min.

Solution: Arc length $= s = \displaystyle\int_1^4 |\mathbf{r}'(t)|\,dt$, where $\mathbf{r}'(t) = \langle 2, 2\sqrt{4t-1}\rangle$. Thus, $s = \displaystyle\int_1^4 \sqrt{4 + 4(4t-1)}\,dt = \int_1^4 \sqrt{16t}\,dt = 4\int_1^4 \sqrt{t}\,dt = 4\left(\frac{2}{3}\right)t^{\frac{3}{2}}\Big|_1^4 = \frac{8}{3}\,(8-1) = \frac{56}{3}$ km.

(b) Consider the function $f(x, y) = xy + \ln(y^2)$ that gives the temperature in degrees C at a point $(x, y) \in \mathbb{R}^2$:

Use the chain rule (no direct substitution) to find the rate of change $\dfrac{df}{dt}$ of temperature along the trajectory $\mathbf{r}(t)$ at $t = \frac{5}{4}$ min. What does the sign of the derivative tell you about the temperature?

Solution: By the multivariate chain rule,

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= 2y + \left(x + \frac{2}{y}\right)2\sqrt{4t-1},\end{aligned}$$

since $\frac{\partial f}{\partial x} = y$, $\frac{dx}{dt} = 2$, $\frac{\partial f}{\partial y} = x + \frac{2}{y}$ (do not forget the chain rule!), and $\frac{dy}{dt} = 2\sqrt{4t-1}$. Before substituting, we need the values of $x = x(\frac{5}{4})$ and $y = y(\frac{5}{4})$ which can be computed as $x = \frac{1}{2}$ and $y = \frac{1}{3}\left(4(\frac{5}{4}) - 1\right)^{\frac{3}{2}} = \frac{1}{3}\cdot 4^{\frac{3}{2}} = \frac{8}{3}$. Substituting for x, y, and t, we find that $\dfrac{df}{dt}\Big|_{t=\frac{5}{4}} = 2\cdot\dfrac{8}{3} + \left(\dfrac{1}{2} + \dfrac{3}{4}\right)2(2) = \dfrac{31}{3}$. Since the derivative is positive, f is increasing when $t = \frac{5}{4}$.

(c) Set up, but **do NOT evaluate**, the line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field $\langle 9y^2, \frac{3}{2}y\rangle$ and C is the path paramaterized by $\mathbf{r}(t)$ for $\frac{5}{4} \leq t \leq \frac{5}{2}$. A complete answer has a simplified integrand and includes limits of integration.

Solution: The vector $\mathbf{F}(\mathbf{r}(t)) = \langle 9y^2, \frac{3}{2}y\rangle\Big|_{\mathbf{r}(t)} = \left\langle (4t-1)^3, \frac{1}{2}(4t-1)^{\frac{3}{2}}\right\rangle$, and $d\mathbf{r} = \langle 2, 2\sqrt{4t-1}\rangle\ dt$ so

$$\int_c \mathbf{F}\cdot d\mathbf{4} = \int_{\frac{5}{4}}^{\frac{5}{2}} \left\langle (4t-1)^3,\ \frac{1}{2}(4t-1)^{\frac{3}{2}}\right\rangle\cdot\left\langle 2,\ 2\sqrt{4t-1|}\right\rangle\ dt = \boxed{\int_{\frac{5}{4}}^{\frac{5}{2}} 2(4t-1)^3 + 2(4t-1)^2\,dt}$$

(d) Let C' be the line segment joining the end points of C from part (c). Give a smooth parametrization $\mathbf{r_2}(t)$ of C'.

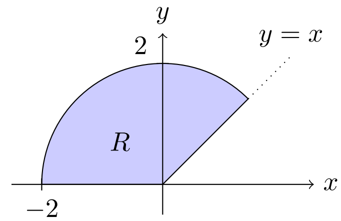
Solution: The endpoints of $\mathbf{r}(t)$ are: beginning point: $\mathbf{r}(\frac{5}{4}) = \langle\frac{1}{2}, \frac{8}{3}\rangle$ and end point: $\mathbf{r}(\frac{5}{2}) = \langle 3, 9\rangle$. Thus the oriented line segment joining these points is

$$(1-t)\left\langle\frac{1}{2}, \frac{8}{3}\right\rangle + t\,\langle 3, 9\rangle, \quad 0 \leq t \leq 1.$$

(e) It is possible to compute that $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = K$ (for some number K) and that $\displaystyle\int_{C'} \mathbf{F}\cdot d\mathbf{r_2} \neq K$. Why does this show that $\mathbf{F}$ is not a conservative vector field? Explain.

Solution: A conservative vector field defined on $\mathbb{R}^2$ would be independent of path, which is false for this particular $\mathbf{F}$.

8. Consider the region R shown below:



(a) Find the mass of the planar lamina R, if its density is given by $\rho(x, y) = x^2 y$.

Solution: The mass is given by

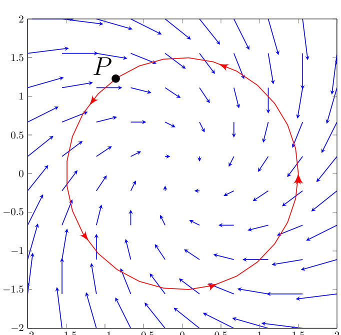
$$\int_{\frac{\pi}{4}}^{\pi} \int_0^2 (r\cos(\theta))^2\, r\sin(\theta)\, r dr d\theta = \int_{\frac{\pi}{4}}^{\pi} \int_0^2 r^4 \cos^2(\theta)\sin(\theta)\, dr d\theta = \boxed{\frac{32}{15} + \frac{8\sqrt{2}}{15}}$$

(b) Consider the vector field $\mathbf{F}(x, y) = \left\langle e^{xy},\, x^2 e^{xy} \right\rangle$ in $\mathbb{R}^2$. Use Green's Theorem to set up, **but do not evaluate**, a double integral to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $C = \partial R$ is the boundary of the region R oriented in the *counter-clockwise* direction. Your answer should look like $\iint_R \underline{\qquad\qquad} \, dA$; that is, you need to find the integrand and display your answer nicely.

Solution:

$$\boxed{\iint_R x^2 y e^{xy} + x e^{xy}\, dA}$$

9. Consider the continuous vector field $\mathbf{F}(x, y)$ pictured below, the curve C with counter-clockwise orientation (red), and the point P (black).



(a) Is div($\mathbf{F}$) positive, zero, or negative? Why?

Solution: Negative. There is a net inflow. Notice that C encloses a sink.

(b) What is the sign of $\mathbf{F} \cdot d\mathbf{r}$ at the point P? Discuss the dot product when answering.

Solution: Negative, since the angle θ between $\mathbf{F}(P)$ and $\mathbf{r}'(t)$ is obtuse, $\frac{\pi}{2} < \theta < \pi$ and $\mathbf{F} \cdot d\mathbf{r} = |\mathbf{F}|\,|\mathbf{r}'(t)|\, \cos(\theta)\, dt$.

(c) Is the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ zero, positive or negative? Why?

Solution: Negative. Viewing the line integral as computing the work done by $\mathbf{F}$, the work done is always opposite the orientation of C.

10. In $\mathbb{R}^3$, consider the space curve C given parametrically by

$$\mathbf{r}(t) = \langle\, 2\cos(t),\, \sin(t),\, t\,\rangle \qquad \text{for } 0 \le t \le \pi$$

Many parts of this problem are independent, so try them all even if you miss a previous one.

(a) Sketch the path C in $\mathbb{R}^3$, and describe the path simply in words (particularly if your sketch is poor). A complete answer has the coordinates of the beginning point a and the point b of C labelled, and its orientation is indicated by drawing an arrow. *Hint:* If you have trouble, start by ignoring the z-coordinate.

Solution: This is half an arc of an elliptical helix. The beginning point is $(2, 0, 0)$ and the end point is $(-2, 0, \pi)$. Another point on the helix is $(0, 1, \pi/2)$.

(b) If $\mathbf{r}(t)$ represents the position at (x, y, z) in meters of a particle at time t in seconds, then consider the point $P = \left(0, 1, \frac{\pi}{2}\right)$ on C.

1. How fast is the particle travelling when it is at P? Include units.

   Solution: The particle is at P when $t = \frac{\pi}{2}$ and its speed is $|\mathbf{r}'(\dfrac{\pi}{2})|$ or $|\langle -2, 0, 1\rangle| = \sqrt{5}$ m/s.

2. Find the equation of the tangent line to $\mathbf{r}(t)$ at the point P.

   Solution: The direction vector is $\mathbf{r}'(\frac{\pi}{2}) = \langle -2, 0, 1\rangle$, so the equation of the tangent line is $\left\langle 0, 1, \frac{\pi}{2}\right\rangle + t\,\langle -2, 0, 1\rangle$ or, in parametric form $x(t) = -2t$, $y(t) = 1$, $z(t) = t + \frac{\pi}{2}$, $t \in \mathbb{R}$.

(c) Now consider the continuous vector field defined on all of $\mathbb{R}^3$

$$\mathbf{F}(x, y, z) = \left\langle\, 3x^2 + 2x\cos(z),\; z^2,\; -x^2\sin(z) + 2yz \,\right\rangle$$

on the curve C parameterized by $\mathbf{r}(t)$ given above. That is, $\mathbf{r}(t) = \langle\, 2\cos(t), \sin(t), t\,\rangle$ for $0 \le t \le \pi$ as in the previous parts.

1. By finding a potential function $f(x, y, z)$, prove that $\mathbf{F}$ is conservative.

   Solution: By integrating with respect to x, y, and z, (that is, $\displaystyle\int 3x^2 + 2x\cos(z)\,dx$, $\displaystyle\int z^2\,dy$, and $\displaystyle\int -x^2\sin(z) + 2yz\,dz$, we find

$$\boxed{f(x, y, z) = x^3 + x^2\cos(z) + yz^2 + C}$$

2. Supposing that $\mathbf{F}$ represents a force field, compute the amount of work done by $\mathbf{F}$ in moving a particle along the path parameterized by $\mathbf{r}(t)$.

   Solution: Use the potential function $f(x, y, z)$ or else spend a lot of (wasted) time .....

   The work done is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(-2, 0, \pi) - f(2, 0, 0) = \big(-8 + 4(-1)\big) - \big(8 + 4(1)\big) = \boxed{-24}.$$