HW #2 Solutions

§1.2 #2, 4, 6, 7 § 1.3 #2

§1.2 #2. Show $\frac{1-\cos x}{x} = \frac{1}{2}x + \mathcal{O}(x^2)$ for x sufficiently small

$$\frac{1-\cos x}{x} = \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right)}{x} = \frac{\frac{x^2}{2}-\frac{x^4}{24}+\dots}{x} = \frac{x}{2}-\frac{x^3}{4!}+\dots$$

i.e. $\frac{1-\cos x}{x} = \frac{x}{2} + \mathcal{O}(x^3)$ since the remainder term $-\frac{x^3}{4!}+\frac{x^5}{6!}-\dots$ is on the order of x^3 near zero.

#4. Show $(1+x)^{-1} = 1-x+x^2+\mathcal{O}(x^3)$ for x sufficiently small.

$f(x) = (1+x)^{-1}$ $\quad f'(x) = -(1+x)^{-2}$ $\quad f^2(x) = 2(1+x)^{-3}$ $\Rightarrow f^n(x) = (-1)^n (1+x)^{-n+1}$

$f(0) = 1$ $\quad f'(0) = -1$ $\quad f^2(0) = 2$

Taylor's theorem $\Rightarrow$ $(1+x)^{-1} \approx 1-x+x^2$ with $R_2(x) = f^3(\xi)\frac{x^3}{3!}$. Since $|f^3(c)|$ is bounded near $x_0 = 0$, $|R_3(x)| \le c\,x^3$ and the appoximation is $\mathcal{O}(x^3)$.

#6. Show $\sum_{k=0}^{n} r^n = \frac{1}{1-r} + \mathcal{O}(r^{n+1})$ using the formula for a finite geometric sum

Ans. $$\frac{1}{1-r} = 1+r+r^2+\dots+r^n+r^{n+1}+\dots \quad \text{for } r\approx 0$$

$$= \underbrace{\sum_{k=0}^{n} r^k}\; \underbrace{+r^{n+1}+r^{n+2}+\dots}_{\mathcal{O}(r^{n+1}) \text{ for } r\approx 0}$$

$$= \sum_{k=0}^{n} r^k + \mathcal{O}(r^{n+1})$$

#7. Show that $k=9$ is all that is needed so that $S = \sum_{k=0}^{\infty} e^{-k}$ is approximated to within 10^{-4} absolute accuracy

The error R_n is exactly $R_n = \sum_{k=n+1}^{\infty} r^k = \frac{r^{n+1}}{1-r} = \frac{e^{-(n+1)}}{1-e^{-1}}$

for $r = \frac{1}{e}$. Plugging in for n, $R_7 \approx .0005$, $R_8 \approx .00019$, $R_9 \approx .00007$ so $n=9$ suffices.

§1.3 #2. (i) exactly (ii) 3 digit decimal w/ chopping (iii) 3 digit decimal w/ rounding

a. $\frac{1}{6}+\frac{1}{10}$ (i) Exactly: $16/60 = \boxed{4/15 = .2\overline{6}}$

(ii) Chopping: $.166 + .100 = \boxed{.266}$

(iii) Rounding: $.167 + .100 = \boxed{.267}$

Error: Exactly will always have zero error

Chopping: $|16/60 - 266/1000| = |\frac{1}{1500}| = .000\overline{6}$ Absolute Error

$$\frac{|16/60 - 266/1000|}{|16/60|} = \frac{1}{400} = .0025$$ Relative Error

Rounding: $|16/60 - 267/1000| = 1/3000 = .000\overline{3}$ Absolute Error

$$\frac{|16/60 - 267/1000|}{|16/60|} = 1/800 = .00125$$ Rel. Error

b. $\frac{1}{6}\cdot\frac{1}{10}$ $\boxed{\text{(i)} = 1/60 = .01\overline{6} \quad \text{(ii)}\ .016 \quad \text{(iii)}\ .017}$

Error with chopping: $|1/60 - .016| = |.01\overline{6} - .016| = .000\overline{6}$ Abs. error

$$\frac{|1/60 - .016|}{1/60} = \frac{|.01\overline{6} - .016|}{|.01\overline{6}|} = .04$$ Rel. error

Error with Rounding:

$|.01\overline{6} - .017| = .000\overline{3}$ and $|.01\overline{6} - .017| / |.01\overline{6}| = 1/50 = .02$ Rel. Err.

c. $\frac{1}{9}+\left(\frac{1}{7}+\frac{1}{6}\right)$ Exactly, the sum is $\frac{1}{9}+\frac{13}{42} = \frac{42+9\cdot13}{9\cdot42} = \frac{159}{378} = \boxed{\frac{53}{126}} \approx .4206$

Chopping: $\frac{1}{7}+\frac{1}{6} \approx .142 + .166 = .308 = 308/100$

Absolute Error = $|53/126 - 419/1000| = \frac{1}{9000}$

Then $\frac{1}{9}+\left(\frac{1}{7}+\frac{1}{6}\right) \approx .111 + .308 = \boxed{.419}$

Absolute Error = $|53/126 - 419/1000| = 1/9000 = .000\overline{1}$

Rel. Error = $$\frac{|53/126 - 419/1000|}{53/126} = 1/3772 \approx .000265$$

Rounding: $\frac{1}{7} + \frac{1}{6} \approx .143 + .167 = .310$

$\frac{1}{9} + \left(\frac{1}{7} + \frac{1}{6}\right) \approx .111 + .310 = \boxed{.421}$

Abs. Error: $\left| 53/126 - .421 \right| \approx .000365$

Rel. Error $= \dfrac{\left| 53/126 - .421 \right|}{\left| 53/126 \right|} \approx .000868$

(d) Looks identical to (c) to me.