MATH 253 Name: Solutions

Quiz 8 April 10, 2020

Instructions: You may upload this quiz any time before 8 am on Monday, April 13.

This quiz is untimed and, if you do it correctly, you will both get a perfect score (!) and get some feedback on what you should study more. Please answer all questions on this first page.

There are two copies of this quiz in this pdf file. (There is way more space provided for your solutions than needed, but since you are scanning answers, I am not trying to save trees.)

- When you find time, sit down and take this quiz with no notes or aid. Time yourself, then answer question 1 below.
- Take a break if you like. No need to do this right away.
- Now using your book or any other resources, but making sure you think, redo any problems on the quiz that you did not get correct.
- Finally, answer question 2 below.

1. Completing the quiz for the first time without any aids took me (fill in the time) ____________________.

2. Use the extra space to explain what was difficult for you, or generally to provide feedback on your solutions. After writing about your learning experience, please add comments that might help for class review for the final exam.

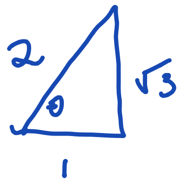
1. Consider a point P whose rectangular coordinates are $(x, y, z) = (\sqrt{3}, 3, 2)$.

   Convert these coordinates to cylindrical and spherical coordinates.

Cylindrical: $z = 2$

$r = \sqrt{(\sqrt{3})^2 + (3)^2} = \sqrt{12} = 2\sqrt{3}$

$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{3}{\sqrt{3}}\right) = \arctan(\sqrt{3})$



$\rightarrow \theta = \pi/3$

Spherical coordinates:

(ρ, θ, φ). $\underline{\theta = \pi/3}$

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(\sqrt{3})^2 + (3)^2 + (2)^2} = \sqrt{16} = \underline{4}$

$z = 2 = \rho \cos\varphi = 4\cos\varphi \Rightarrow \cos\varphi = \frac{1}{2} \Rightarrow \varphi = \underline{\pi/3}$

My answers:

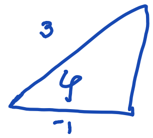
Cylindrical coordinates are: $(2\sqrt{3}, \pi/3, 2)$ Spherical coordinates are: $(4, \pi/3, \pi/3)$

2. Consider a point P whose spherical coordinates are $(\rho, \theta, \phi) = \left(3, \frac{\pi}{4}, \arccos(\frac{-1}{3})\right)$.

   Convert these coordinates to rectangular and cylindrical coordinates.

Rectangular: $z = \rho\cos\varphi = 3\cos(\arccos(-1/3)) = 3(-1/3) = -1$,

Compute $r = \rho\sin\varphi = 3\sin(\arccos(-1/3))$



$\sqrt{9 - (-1)^2} = \sqrt{8} = 2\sqrt{2}$ $\Rightarrow \sin\varphi = \frac{2\sqrt{2}}{3}$

$\hookrightarrow r = 3\,\frac{2\sqrt{2}}{3} = 2\sqrt{2}$

$x = r\sin\varphi\cos\theta = 2\sqrt{2}\cos(\pi/2) = 2\sqrt{2}\,\frac{\sqrt{2}}{2} = 2$,

$y = r\sin\varphi\sin\theta = 2$,

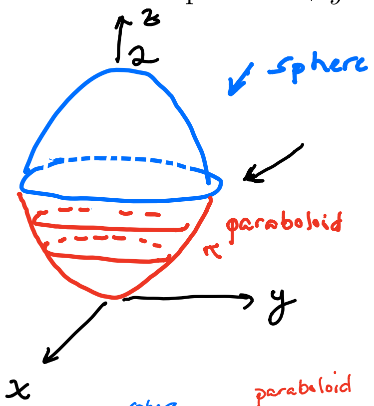
Cylindrical Coords: $r = 2\sqrt{2}$, $\theta = \pi/4$, $z = -1$

all done above

My answers:

Rectangular coordinates are: $(2, 2, -1)$ Cylindrical coordinates are: $(2\sqrt{2}, \frac{\pi}{4}, -1)$

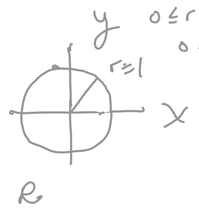
3. Use the cylindrical coordinate system to find the volume **above** the paraboloid $z = x^2 + y^2$ and **below** the sphere $x^2 + y^2 + z^2 = 2$. (Hint: begin by drawing a sketch of the volume.)



Need: Equation of intersection

Solve $x^2+y^2 = \sqrt{2-x^2-y^2}$ or guess $\boxed{z=1}$: $\underbrace{x^2+y^2=1}_{\text{paraboloid}}$ ✓ $\quad \underbrace{x^2+y^2+(1)^2=2}_{\text{sphere}} \Rightarrow x^2+y^2=1$ ✓

came from my drawing



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

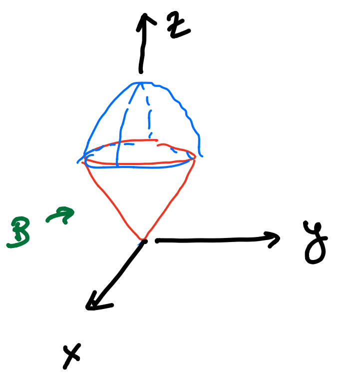
$$\text{Vol} = V = \int_R \underbrace{\sqrt{2-x^2-y^2}}_{\text{sphere}} - \underbrace{(x^2+y^2)}_{\text{paraboloid}}\,dA = \int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2}-r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\sqrt{2-r^2} - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{4}r^4 \Big|_0^1 \,d\theta = \int_0^{2\pi}\left[\left(-\frac{1}{3}-\frac{1}{4}\right) - \left(-\frac{1}{3}(2)^{3/2}\right)\right] d\theta$$

$$= 2\pi\left[-\frac{7}{12} + \frac{2}{3}\sqrt{2}\right]$$

$$= \boxed{\pi\left[\frac{4}{3}\sqrt{2} - \frac{7}{6}\right]}$$

4. A solid B lies **above** the cone $\phi = \frac{\pi}{3}$ and **below** the sphere $\rho = 4\cos\phi$. The charge density in *coulombs*$/m^3$ at any point (x, y, z) within the solid B is proportional to its distance from the origin. Find the total charge of B. Include units in your answer.



$\sigma(x,y,z) = k\rho$ coulombs$/m^3$

$$\text{Total Charge} = \iiint_B k\rho\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\varphi} k\rho\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= 2\pi k\int_0^{\pi/3}\int_0^{4\cos\varphi} \rho^3\sin\varphi\,d\rho\,d\varphi = 2\pi k\int_0^{\pi/3}\sin\varphi\,\frac{1}{4}\rho^4\Big|_0^{4\cos\varphi} d\varphi$$

$$= 2\pi k\int_0^{\pi/3}\sin\varphi\,\frac{1}{4}\,256\cos^4\varphi\,d\varphi = 128\pi k\int_0^{\pi/3}\sin\varphi\cos^4\varphi\,d\varphi$$

$$= 128\pi k\left[-\frac{1}{5}\cos^5\varphi\right]\Big|_0^{\pi/3} = 128\pi k\left(-\frac{1}{5}\right)\left[\cos^5\pi/3 - \cos^5(0)\right]$$

$$= -\frac{128\pi k}{5}\left[\frac{1}{2^5}-1\right] = \frac{128\pi k}{5}\left(\frac{31}{32}\right) = \boxed{\frac{124\pi k}{5}\text{ coulombs}}$$