Comments on HW 11.

2: Consider the metric space $(\mathbb{R}, |\cdot|)$ with the Euclidean topology. Give, with brief proof, an example to show a countably infinite intersection of open sets can be a closed set.

One solution is to look at

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

The intervals are open, but the singleton set, as the complement of an open set, is closed.

3: Consider $(\mathbb{R}, |\cdot|)$ with the Euclidean topology, and $X = (0, 1]$ the topological space with the topology induced by $(\mathbb{R}, |\cdot|)$. Give an example of a set $A \subsetneq X \subseteq \mathbb{R}$ that is open in the induced topology on X, but not in $\mathbb{R}$. Give an example of a set $B \subsetneq X \subseteq \mathbb{R}$ that is closed in the induced topology on X, but not in $\mathbb{R}$. Justify briefly.

Let $A = (\frac{1}{2}, 1]$. Then $A = X \cap (\frac{1}{2}, 2)$ and since $(\frac{1}{2}, 2)$ is an open set in $\mathbb{R}$, A is open in X. However, A is not open as a subset of $\mathbb{R}$, since $1 \in A$, but no open ball $B(1, r) \subset \mathbb{R}$ is contained in A.

Let $B = (0, \frac{1}{2}]$. Then $B = X \cap [-\frac{1}{2}, \frac{1}{2}]$ and since the closed interval is a closed subset of $\mathbb{R}$, B is a closed subset of X. However, the half-closed interval is not closed as a subset of $\mathbb{R}$.

4: Problem 58 (2) and (3).

My one comment is to note that under the given map, the pre-image of

$$1 = .(p-1)\ldots(p-1)\cdots = \sum_{i=1}^{\infty}\left(\frac{p-1}{p}\right)^{i}$$

is ± 1. Thus, the map is not 1-1 on $\mathbb{Q}_p$.