MATH 202
Midterm 2

Name : ____________________

April 8, 2008

Instructions: Show all work for full credit. Poor or sloppy mathematical notation will be penalized.

1. (10 pts.) Explain why $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Approaching (0,0) along $y=x \Rightarrow \frac{xy}{x^2+y^2} = \frac{x^2}{2x^2} = \frac{1}{2}$

Since $\frac{1}{2} \neq \frac{-1}{2}$, the limit d.n.e.

Approaching (0,0) along $y=-x \Rightarrow \frac{xy}{x^2+y^2} = \frac{-x^2}{2x^2} = \frac{-1}{2}$

2. (15 pts.)

(a) (8 pts.) Find the equation of the tangent plane to the elliptic paraboloid $f(x,y) = x^2 + 2y^2$ at the point $(1,1,3)$.

$$\left.\begin{array}{l} f_x(x,y) = 2x \Rightarrow f_x(1,1) = 2 \\ f_y(x,y) = 4y \Rightarrow f_y(1,1) = 4 \end{array}\right\}$$

Thus, $z = 3 + 2(x-1) + 4(y-1)$

or $\boxed{z = 2x + 4y - 3}$ is the equation of the tangent plane

(b) (7 pts.) Give the value of the best linear approximation for $f(.9, 1.01)$.

$f(.9, 1.01)$ is best approximated linearly by the tangent plane

$f(.9, 1.01) \approx$ 2.84

$$\therefore f(.9, 1.01) \approx 2(.9) + 4(1.01) - 3$$

$$= 1.8 + 4.04 - 3$$

$$= 1.8 + 1.04$$

$$= 2.84$$

3. (10 pts.) Suppose the height in tens of meters of a kite is given by the function

$$h(x, y) = \frac{1}{5-x}y^2 \text{ tens of } m,$$

and the kite flyer is located at the position $(x, y) = (4, 1)$. (Assume the kite flies directly overhead.)

(a) (4 pts.) In what direction from $(4, 1)$ should the kite flyer move to increase the height of the kite the most?

In the direction of $\nabla h(4,1)$

$\therefore \nabla h(x,y) = \left(\frac{-y^2}{(5-x)^2}, \frac{2y}{5-x} \right) \Rightarrow \nabla h(4,1) = \left(\frac{-1^2}{(5-4)^2}, \frac{2(1)}{5-4} \right) = \boxed{(-1,2)}$

(b) (4 pts.) If the kite flyer moves in the direction indicated by the vector $\mathbf{v} = (-1, 1)$, what is the rate of change of the kite's height?

Let $\vec{u} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ be the unit vector in the direction of $\vec{v}$

Compute $D_{\vec{u}} f(4,1) = \nabla f(4,1) \cdot \vec{u} = (1,2) \cdot \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$

(units = 10m/m)

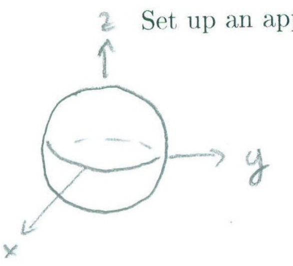
(c) (2 pts.) Using your answer to part (b), do you expect the kite to rise or fall as the kite flyer moves in the direction of $\mathbf{v}$? Justify briefly.

Since $\frac{\sqrt{2}}{2} > 0 \Rightarrow$ the kite should rise.

4. (10 pts.) It is well known that the volume of the solid ball of radius R is given by

$$V = \frac{4}{3}\pi R^3.$$

Set up an appropriate iterated triple integral in *spherical coordinates* that computes this volume.



$0 \le \rho \le R$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

Sphere

$\Rightarrow \text{Vol} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

5. (15 pts.) Consider the function $f(x, y) = xy^2 - y^2 - \frac{1}{2}x^2$.

(a) (8 pts.) Find all critical points of $f(x, y)$.

$f_x(x,y) = y^2 - x = 0 \Rightarrow y^2 = x \quad (1)$

$f_y(x,y) = 2xy - 2y = 0 \Rightarrow 2y(x-1) = 0 \quad (2)$

From (2): $y=0$ or $x=1$

If $y=0$, then from (1) we have $0^2 = x$ or $x=0$ $\Rightarrow$ $\boxed{(0,0) \text{ is a c.p.}}$

If $x=1$, then from (1) we have $y^2 = 1$ or $y = \pm 1$

$\Rightarrow$ $\boxed{(1,1) \text{ and } (1,-1) \text{ are c.p.'s.}}$

(b) (7 pts.) Use the second derivative test to determine if the critical points are local maxima, local minima, saddle points or if there is not enough information to tell.

Let D be the determinant of $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -1 & 2y \\ 2y & 2x-2 \end{pmatrix}$

Since $f_x(x,y) = y^2 - x \Rightarrow f_{xx}(x,y) = -1$ and $f_{yx}(x,y) = 2y$

agree(!)

$f_y(x,y) = 2xy - 2y \Rightarrow f_{yy}(x,y) = 2x-2$ and $f_{xy}(x,y) = 2y$

For $(x,y) = (0,0)$, we have $D = \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 > 0$. Since $f_{xx}(0,0) = -1 < 0$

$\Rightarrow$ $\boxed{(0,0) \text{ is a Local max}}$

For $(x,y) = (1,1)$, we have $D = \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0 \Rightarrow$ $\boxed{(1,1) \text{ is a saddle point}}$

For $(x,y) = (1,-1)$, we have $D = \begin{vmatrix} -1 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0 \Rightarrow$ $\boxed{(1,-1) \text{ is a saddle point}}$

6. (12 pts.) Find the maximum value of $f(x, y) = xy^2$ subject to the constraint $x^2 + y^2 = 1$.

Use Lagrange multipliers:

Notice first that a maximum value will occur for $x > 0$ and $y \neq 0$ since $f(x,y) = xy^2$

This will save time.

3 Eq's: (1) $y^2 = \lambda(2x)$ (2) $2xy = \lambda(2y)$ (3) $x^2 + y^2 = 1$

From (2), $2y(x - \lambda) = 0 \Rightarrow y = 0$ or $x = \lambda$. However, only $x = \lambda$ could yield a max.

For $x = \lambda$, (1) $\Rightarrow$ $y^2 = 2\lambda^2$ and plugging into (3), we get $\lambda^2 + 2\lambda^2 = 1$

or $3\lambda^2 = 1$ $\lambda = \pm\sqrt{1/3}$

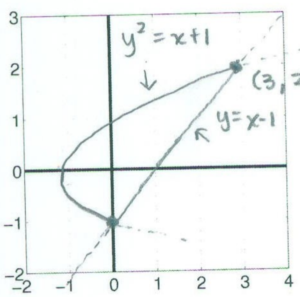
Since $x = \lambda$, only $x = +\sqrt{1/3}$ can yield a maximum value.

Since $y^2 = 2\lambda^2 = 2(\frac{1}{3}) \Rightarrow y = \pm\sqrt{2/3}$

Both points $(\sqrt{1/3}, \sqrt{2/3})$ and $(\sqrt{1/3}, -\sqrt{2/3})$ yield a maximum value

of $f(\sqrt{1/3}, \sqrt{2/3}) = \sqrt{1/3} \cdot 2/3 = \frac{2}{3\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{9}}$

7. (10 pts.) Set up, but do not compute, an iterated integral that evaluates $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = x + 1$.



Easiest "$dx\,dy$"

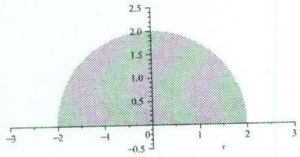
$$\iint_D xy\,dA = \int_{-1}^{2}\int_{y^2-1}^{1+y} xy\,dx\,dy$$

$y^2 = x + 1 \Rightarrow x = y^2 - 1$ ← smallest value of x for fixed y

$y = x - 1 \Rightarrow x = 1 + y$ ← largest value of x for fixed y

8. (15 pts.) Electric charge is distributed over the semi-circular plate pictured below so that the charge density is $\rho(x, y) = 1 - \sqrt{x^2 + y^2}$ coulombs/cm^2.

(a) (10 pts.) Find the total electric charge of the semi-circular plate. A complete answer includes units.



Use polar coordinates.

$0 \leq r \leq 2$

$0 \leq \theta \leq \pi$

Total Charge $= \int_0^\pi \int_0^2 (1-r)\, r\, dr\, d\theta = \int_0^\pi \int_0^2 r - r^2 \, dr\, d\theta$

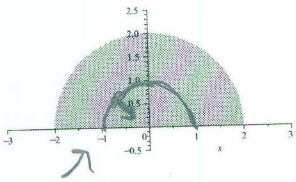
$= \int_0^\pi \left[\frac{1}{2}r^2 - \frac{1}{3}r^3 \right|_0^2 d\theta = \int_0^\pi \left(\frac{1}{2}(2)^2 - \frac{1}{3}(2)^3\right) - 0 \, d\theta$

$= \int_0^\pi 2 - \frac{8}{3} \, d\theta = \int_0^\pi -\frac{2}{3} \, d\theta = \boxed{-\frac{2\pi}{3} \text{ Coulombs}}$

(b) (5 pts.)

i. (5 pts.) Indicate on the plate the direction of the gradient vector $\nabla \rho(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$\nabla \rho(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ points to origin



$x^2 + y^2 = 1$ or $\boxed{\rho(x,y) = 0}$

ii. (Extra credit – 3 pts.) Give the formula for the level curve $\rho(x, y) = k$ that passes through the point $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and sketch this level curve on the plate.

$1 - \sqrt{(\frac{-1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} = 1 - \sqrt{\frac{1}{2} + \frac{1}{2}} = \underset{k}{0} \Rightarrow$ level curve is $\rho(x,y) = 0$

This is the same as

$x^2 + y^2 = 1$