MATH 631 Name :________________________________
Midterm Part I November 14, 2018

Instructions: This exam is closed book and closed notes, and one hour in length. Part II is on Friday. You may use **only** your brain and blank scratch paper in writing solutions.

Questions involving quick computational answers will be clearly indicated. You do not need to show work for these. For more theoretical questions, you should prove results from first principles and not simply quote statements from the book. Your proofs will be graded not only on correctness, but points will be awarded/taken away for poor writing and exposition. Blank paper is supplied for scratch work, but final responses should be written in the space provided. Do your best!

x

1. (6 pts.) Answer briefly.

   (a) Give an example of a finite ring of characteristic zero, if possible. Otherwise explain why no such ring exists.

   (b) Give the definition of a nilpotent element in a ring R. Then prove that the set of nilpotent elements in $M_2(\mathbb{Q})$ is **not** an ideal.

2. (6 pts.) Suppose G is a non-cyclic group of order $205 = 5 \cdot 41$. Give, with proof, the number of elements of order 5 in G.

3. (6 pts.) Find **ALL** solutions x in the integers to the simultaneous congruences.

$$\begin{aligned} x &\equiv 7 \mod 11 \\ x &\equiv 2 \mod 5 \end{aligned}$$

4. (12 pts.) Suppose G is a group, $H \leq G$, and $\mathrm{Aut}(H)$ the group of automorphisms of H.

   (a) Using the First Isomorphism theorem, give a **full** proof of the following statement.
   The quotient group $N_G(H)/C_G(H) \cong A \leq \mathrm{Aut}(H)$.

   (b) Suppose now that P is a Sylow p-subgroup of S_p for a prime p. Prove that

$$N_{S_p}(P)/C_{S_p}(P) \cong \mathrm{Aut}(P).$$

5. (10 pts.) Prove **one** of the following statements. Circle the statement you want graded.

   (a) Every nonzero prime ideal in a PID is a maximal ideal.

   OR

   (b) In a PID every nonzero element is a prime if, and only if, it is irreducible.

6. (10 pts.) Suppose R is a commutative ring with 1 and for each $x \in R$, there is a positive integer $n > 1$ so that $x^n = x$. Prove that every nonzero prime ideal is maximal.

MATH 631 Name :____

Midterm Part II November 16, 2018

Instructions: This exam is closed book and closed notes, and one hour in length. You may use **only** your brain and blank scratch paper in writing solutions.

Questions involving quick computational answers will be clearly indicated. You do not need to show work for these. For more theoretical questions, you should prove results from first principles and not simply quote statements from the book. Your proofs will be graded not only on correctness, but points will be awarded/taken away for poor writing and exposition. Blank paper is supplied for scratch work, but final responses should be written in the space provided. Do your best!

7. (5 pts.) Recall that an *integral domain* or a *domain* is a commutative ring with 1 that has no zero divisors.

   Suppose that R is a commutative ring with 1. Give, with proof, a necessary and sufficient condition on ideals of R so that R is an integral domain.

   Answer: A ring R that is commutative with multiplicative identity 1 is an integral domain if, and only if

   ____.

   Proof:

8. (5 pts.) Find, with brief justification, all ring homomorphisms from $\mathbb{Z} \to \mathbb{Z}/12Z$.

9. (10 pts.) Suppose that A is an Abelian group of order $1323 = 3^3 \cdot 7^2$. Give the isomorphism classes for A in the table below. In the left hand column, give the elementary divisor decomposition and in the right hand column, give the invariant factor decomposition. **Groups on the same row should be isomorphic.** You do not need to show your work.

10. (10 pts.) Consider the ring of Gaussian integers $\mathbb{Z}[i]$.

    (a) (6 pts.) Prove that if $\alpha = a + bi$ for $a, b \in \mathbb{Z}$ is a Gaussian integer with $N(\alpha) = p$ for p a prime of $\mathbb{Z}$, then α is irreducible.

    (b) (4 pts.) Give an example of a prime number $p \in \mathbb{Z}$ such that p is irreducible in $\mathbb{Z}[i]$. Justify your answer by stating an appropriate result.

11. (10 pts.) Let G be a finite group of order 22, $|G| = 22$. Prove that G is either cyclic or isomorphic to the dihedral group D_{22}.

12. (10 pts.) Let D be a square-free integer, and consider the quadratic number field $\mathbb{Q}(\sqrt{D})$ and its subring of integers $\mathcal{O}$. Let $N : \mathbb{Q}(\sqrt{D}) \to \mathbb{Z}$ denote the field norm map which is multiplicative. The restriction of N to the ring of integers $\mathcal{O}$ will also denoted by N.

    (a) (3 pts.) Prove that an element $\alpha \in \mathcal{O}$ is a unit if, and only if, $N(\alpha) = \pm 1$.

    (b) (3 pts.) When $D = -3$, the ring of integers is $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\left(\dfrac{1+\sqrt{-3}}{2}\right)$. Find a unit in $\mathcal{O} \setminus \mathbb{Z}$.

    (c) (4 pts.) Let $D = -5$. Give, with proof, an example of an element $x = a + b\sqrt{-5}$ for $a, b \in Z$ such that x is irreducible, but x is not prime in $\mathbb{Z}[\sqrt{-5}]$.