MATH 253
Midterm 1

Name: SOLUTIONS

February 9, 2021

Instructions: 100 points total. Use only your brain and writing implement. You have 90 minutes to complete this exam. Good luck.

1. (24 pts. – 6 pts. each) A particle at point $P(15, -2, 3)$ in $\mathbb{R}^3$ is constrained so that it can only move along the line segment joining the point P to the point $Q(25, -2, 3)$.

   (a) Find the displacement vector $\overrightarrow{PQ}$ along which the particle may move, and the length $|\overrightarrow{PQ}|$ in meters of $\overrightarrow{PQ}$.

   $\overrightarrow{PQ} = \langle 25-15, -2-(-2), 3-3 \rangle = \langle 10, 0, 0 \rangle$

   Answer: $\overrightarrow{PQ} =$ $\langle 10, 0, 0 \rangle$ $\qquad |\overrightarrow{PQ}| =$ 10.

   (b) A constant force vector $\mathbf{F} = 2\sqrt{3}\,\mathbf{i} + \mathbf{j} + \sqrt{3}\,\mathbf{k}$ Newtons acts on this particle and moves it from the point P to Q. Find the work done. Include units in your final answer.

   $\text{Work} = \vec{F} \cdot \overrightarrow{PQ} = \langle 2\sqrt{3}, 1, \sqrt{3} \rangle \cdot \langle 10, 0, 0 \rangle$

   $= 20\sqrt{3}$ Nm or joules

   (c) Find the angle θ between the force vector $\mathbf{F}$ and $\overrightarrow{PQ}$.

   $$\theta = \arccos\left(\frac{\vec{F} \cdot \overrightarrow{PQ}}{|\vec{F}|\,|\overrightarrow{PQ}|}\right) = \arccos\left(\frac{20\sqrt{3}}{10 \cdot \sqrt{(2\sqrt{3})^2 + 1^2 + \sqrt{3}^2}}\right)$$

   $$= \arccos\left(\frac{20\sqrt{3}}{10\sqrt{12+1+3}}\right) = \arccos\left(\frac{2\sqrt{3}}{\sqrt{16}}\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$$

   $$= \pi/6$$

   Answer: $\theta =$ $\boxed{\pi/6}$.

   (d) Suppose you wish to **maximize** the work done in moving the particle from P to Q. Find a force vector $\mathbf{G}$ that has the same magnitude as $\mathbf{F}$, but would maximize the workwork done. Briefly justify your answer.

   $\cos\theta = 1$ maximizes Work, i.e., $\theta = 0$ and $\vec{F}$ and $\overrightarrow{PQ}$ are parallel. We need $|\vec{G}| = 4$ $(= |\vec{F}|)$ and a unit vector $\vec{u}$ in the direction of $\overrightarrow{PQ}$, $\vec{u} = \langle 1, 0, 0 \rangle$. Thus, $\vec{G} = \langle 4, 0, 0 \rangle$

   Answer: $\mathbf{G} =$ $\langle 4, 0, 0 \rangle$.

2. (14 pts.) Compute the definite integral

$$\int_0^2 \left\langle 4te^{2t}, 0, \frac{1}{1+4t^2} \right\rangle dt.$$

We need $\left\langle \int_0^2 4te^{2t}\,dt, \int_0^2 0\,dt, \int_0^2 \frac{1}{1+4t^2}\,dt \right\rangle$

$\int_0^2 4te^{2t}\,dt$: $u = 4t$ $du = 4\,dt$; $dv = e^{2t}dt$ $v = \frac{1}{2}e^{2t}$ $\therefore$ $uv - \int v\,du$

$= 2te^{2t} - \int 2e^{2t}\,du = \boxed{2te^{2t} - e^{2t}}\Big|_0^2 = (2t-1)e^{2t}\Big|_0^2 = (3e^4) - (-1)$

antiderivative.

$= 1+3e^4$

$\int_0^2 \frac{1}{1+4t^2}\,dt = \boxed{\frac{1}{2}\arctan(2t)}\Big|_0^2 = \frac{1}{2}\left[\arctan(4) - \arctan(0)\right] = \frac{1}{2}\arctan(4)$

antiderivative

Answer: $\langle 1+3e^4, 0, \frac{1}{2}\arctan(4) \rangle$.

3. (12 pts.)

(a) (9 pts.) Find the equation of the plane containing the points

$$P(-1,2,1), \qquad Q(-1,5,2), \qquad R(-2,1,4)$$

The vectors $\vec{PQ} = \langle 0,3,1\rangle$ and $\vec{PR} = \langle -1,-1,3\rangle$ lie in the plane.

Let $\vec{n} = \vec{PQ} \times (\vec{RP}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = -10\hat{i} + \hat{j} - 3\hat{k} = \langle -10, 1, -3\rangle$

For algebraic ease, use $\vec{n} = -\langle -10, 1, -3\rangle = \langle 10, -1, 3\rangle$

$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{p} \Rightarrow$ $10x - y + 3z = \langle 10,-1,3\rangle \cdot \langle -1,2,1\rangle = -10-2+3 = -9$

Answer: $\boxed{10x - y + 3z = -9}$.

(b) (3 pts.) Is the origin on this plane? Why, or why not?

No. $\underset{x}{10(0)} - \underset{y}{0} + \underset{z}{3(0)} = 0$ and not -9.

4. (24 pts. – 6 pts. each) The formulas for curvature $\kappa(t)$ for a space curve are:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \qquad \kappa = \frac{d\mathbf{T}}{ds}$$

Consider the space curve given parametrically by

$$\mathbf{r}(t) = \left\langle 4t,\, t^2,\, \frac{1}{6}t^3 \right\rangle \text{ for } t \in \mathbb{R}.$$

(a) Find the length of the curve $\mathbf{r}(t)$ for $0 \leq t \leq 1$.

$L = s = \int_0^1 |\vec{r}\,'(t)|\,dt$

$= \int_0^1 \frac{1}{2}t^2 + 4\,dt$

$= \frac{1}{6}t^3 + 4t \Big|_0^1$

$= (\frac{1}{6} + 4) - 0 = \frac{25}{6}$

$\vec{r}\,'(t) = \langle 4, 2t, \frac{1}{2}t^2 \rangle$

$|\vec{r}\,'(t)| = \sqrt{16 + (2t)^2 + (\frac{1}{2}t^2)^2}$

$= \sqrt{16 + 4t^2 + \frac{1}{4}t^4}$

$= \sqrt{\frac{t^4 + 16t^2 + 64}{4}}$

$= \frac{\sqrt{(t^2+8)^2}}{2} = \frac{t^2+8}{2}$

$= \frac{1}{2}t^2 + 4$

Answer: $\boxed{\frac{25}{6}}$.

(b) Find the curvature of $\mathbf{r}(t)$ at the time $t = 1$.

$\vec{r}\,'(t) = \langle 4, 2t, \frac{1}{2}t^2 \rangle$, $|\vec{r}\,'(t)| = \frac{1}{2}t^2 + 4$ from above

$\vec{r}\,'(1) = \langle 4, 2, \frac{1}{2} \rangle$ $|\vec{r}\,'(1)| = \frac{1}{2}(1)^2 = \frac{9}{2}$

$\vec{r}\,''(t) = \langle 0, 2, t \rangle$ $\vec{r}\,''(1) = \langle 0, 2, 1 \rangle$

$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & 1/2 \\ 0 & 2 & 1 \end{vmatrix} = (2-1)\hat{i} - 4\hat{j} + 8\hat{k} = \langle 1, -4, 8 \rangle$

$|\vec{r}\,'(1) \times \vec{r}\,''(1)| = \sqrt{1^2 + (-4)^2 + 8^2}$

$= \sqrt{1 + 16 + 64}$

$= \sqrt{81} = 9$

$\kappa(1) = \frac{|\vec{r}\,'(1) \times \vec{r}\,''(1)|}{|\vec{r}\,'(1)|^3} = \frac{9}{(\frac{9}{2})^3} = \frac{8}{81}$

Answer: $\boxed{8/81}$.

(continued ...)

$$\mathbf{r}(t) = \left\langle 4t,\, t^2,\, \frac{1}{6}t^3 \right\rangle \text{ for } t \in \mathbb{R}.$$

(c) Suppose that a particle's trajectory is given by $\mathbf{r}(t)$ at time t. Give a unit vector $\mathbf{u}$ that points in the direction of travel at time $t = 2$.

$\vec{r}\,'(t) = \langle 4, 2t, \frac{1}{2}t^2 \rangle$ $\quad$ $\vec{r}\,'(2) = \langle 4, 4, 2 \rangle$ $\quad$ $|\vec{r}\,'(2)| = \sqrt{4^2 + 4^2 + 2^2}$

$= \sqrt{36}$

$= 6$

$\vec{u} = \dfrac{1}{|\vec{r}\,'(2)|}\vec{r}\,'(2) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$

Answer: The unit vector is $\mathbf{u} =$ $\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$.

(d) Give the parametric equations of the tangent line to $\mathbf{r}(t)$ at the point $\mathbf{r}(2)$.

$\vec{r}(2) = \langle 8, 4, \frac{8}{6} \rangle = \langle 8, 4, \frac{4}{3} \rangle$ $\quad \uparrow$ point

direction vector $\vec{v} = \vec{r}\,'(2) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$

$\vec{v} = 3\vec{r}\,'(2) = \langle 2, 2, 1 \rangle$

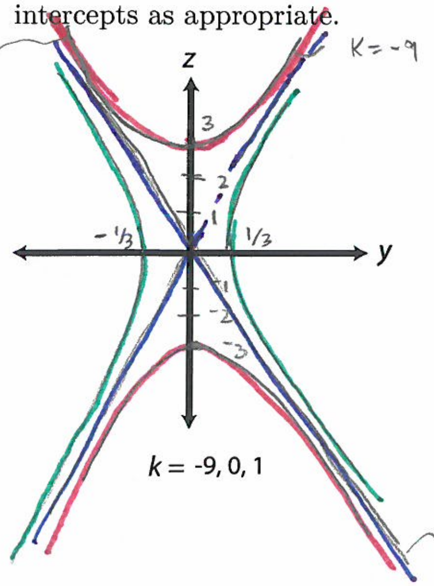
works too.

$\langle 8, 4, \frac{4}{3} \rangle + t\langle 2, 2, 1 \rangle$

Answer: $x(t) = 8 + 2t \quad y(t) = 4 + 2t \quad z(t) = \frac{4}{3} + t$.

5. (9 pts. – 3 pts. each) On the axes below, sketch the x-traces for the values of $k = -9, 0, 1$ for the 'saddle' given by the equation $x = 9y^2 - z^2$. Label the traces with their equations and indicate intercepts as appropriate.

$1 = -y^2 + \frac{z^2}{9}$



$\boxed{K = -9:}$ $\quad -9 = 9y^2 - z^2$ $\quad \boxed{9 = -9y^2 + z^2}$

$1 = -y^2 + \frac{z^2}{(3^2)}$

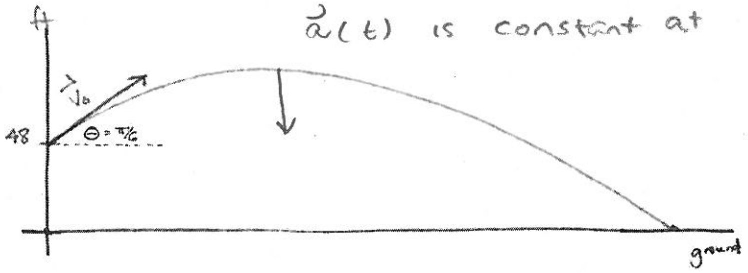
$\boxed{K = 0:}$ $\quad 0 = 9y^2 - z^2$ $\quad z^2 = 9y^2$

$z = \pm 3y$

$\boxed{K = 1:}$ $\quad 1 = 9y^2 - z^2$

6. (17 pts.) A projectile is fired from a height of 48 ft with an initial speed of 64 ft/s, and an angle $\theta = \frac{\pi}{6}$ of elevation. See figure.

$\vec{a}(t)$ is constant at $\langle 0, -32 \rangle$ ft/s^2



(a) (3 pts.) It is not difficult to show that the velecity of the projectile at time t is given by the vector equation:

$$\mathbf{v}(t) = \langle v_x, \ -32t + v_y \rangle \text{ ft/s}$$

where $\mathbf{v_0} = \langle v_x, \ v_y \rangle$ is the *initial velocity* of the projectile. Find $\mathbf{v_0}$.

$|\vec{v}_0| = 64$

$\cos \pi/6 = \frac{\sqrt{3}}{2}$ $\quad \sin \frac{\pi}{6} = \frac{1}{2}$

$V_x = 64 \cos \pi/6$

$V_y = 64 \sin \pi/6$

Answer: $\mathbf{v_0} =$ $\langle 32\sqrt{3}, \ 32 \rangle$ ft/s

(b) (6 pts.) Find the position $\mathbf{r}(t)$ of the projectile at any time t. Include units in your answer.

$\vec{r}(t) = \int_0^t \vec{v}(u)\,du = \langle 32\sqrt{3}\,t, -16t^2 + 32t \rangle + \vec{r}_0$ $\qquad \vec{r}_0 = \langle 0, 48 \rangle$ ft

$= \langle 32\sqrt{3}\,t + 0, -16t^2 + 32t + 48 \rangle$ ft

$\vec{v}(u) = \langle 32\sqrt{3}, -32t + 32 \rangle$

Answer: $\mathbf{r}(t) =$ $\langle 32\sqrt{3}\,t, \ -16t^2 + 32t + 48 \rangle$ ft

(c) (6 pts.) Find the time that the projectile hits the ground, and the horizontal distance it traveled.

If $\vec{r}(t) = \langle x(t), y(t) \rangle$, then the projectile hits the ground when $y(t) = 0$.

Solve $-16t^2 + 32t + 48 = 0$ $\qquad t = 3$ s is the relevant time.

$-16(t^2 - 2t + 3) = 0$ $\qquad x(3) = 96\sqrt{3}$

$-16(t-3)(t+1) = 0$

$t = +3, -1$

Answer: $t = 3$ and $x(3) = 96\sqrt{3}$ ft

(d) (2 pts.) On the drawing above, sketch an acceleration vector $\vec{a}(t)$ at some time t (you choose) before the projectile hits the ground.

See above.