HW 4 Solutions

Book problems:

§2.2 # 8, 12, 15

#8. By Taylor's Theorem, $f(x+a) = f(x) + f'(x)a + f''(x)\frac{a^2}{2} + f^{(3)}(x)\frac{a^3}{3!} + f^{(4)}(x)\frac{a^4}{4!} + O(h^5)$

Thus, $f(x+h) = f(x) + f'(x)h + f^{(2)}(x)\frac{h^2}{2!} + f^{(3)}(x)\frac{h^3}{3!} + f^{(4)}(x)\frac{h^4}{4!} + O(h^5)$

$$f(x-h) = f(x) - f'(x)h + f^{(2)}(x)\frac{h^2}{2!} - f^{(3)}(x)\frac{h^3}{3!} + f^{(4)}(x)\frac{h^4}{4!} + O(h^5)$$

$$f(x+2h) = f(x) + f'(x)(2h) + f^{(2)}(x)\frac{(2h)^2}{2!} + f^{(3)}(x)\frac{(2h)^3}{3!} + f^{(4)}(x)\frac{(2h)^4}{4!} + O(h^5)$$

$$f(x-2h) = f(x) - f'(x)(2h) + f^{(2)}(x)\frac{(2h)^2}{2!} - f^{(3)}(x)\frac{(2h)^3}{3!} + f^{(4)}(x)\frac{(2R)^4}{4!} + O(h^5)$$

and $8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)$

$$= 0 + [8+8-2-2]f'(x)h + 0 + [8+8-8-8]f^{(3)}(x)\frac{h^3}{3!} + 0 + O(h^5)$$

$$= f'(x)12h + O(h^5).$$

Note decrement in power of h.

Dividing by $12h$ gives

$$f'(x) = \frac{1}{12h}\left(8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)\right) + O(h^4)$$

#12. By Taylor's Theorem, $f(x+a) = f(x) + f'(x)a + f^{(2)}(x)\frac{a^2}{2!} + f^{(3)}(x)\frac{a^3}{3!} + O(h^4)$

Thus,
$$f(x+h) = f(x) + f'(x)h + f^{(2)}(x)\frac{h^2}{2!} + f^{(3)}(x)\frac{h^3}{3!} + O(h^4)$$

$$f(x-h) = f(x) - f'(x)h + f^{(2)}(x)\frac{h^2}{2!} - f^{(3)}(x)\frac{h^3}{3!} + O(h^4)$$

and $f(x+h) - 2f(x) + f(x-h) =$

$$0 + 0 + f^{(2)}(x)h^2 + 0 + O(h^4).$$

Dividing by h^2 yields, $f^{(2)}(x) = \frac{1}{h^2}\left(f(x+h) - 2f(x) + f(x-h)\right) + O(h^2).$

#15. Let $f(x) = \ln\left(e^{\sqrt{x^2+1}} \sin(\pi x) + \tan(\pi x)\right)$

(b) Ugh. $f'(x) = \dfrac{e^{\sqrt{x^2+1}}(\cos \pi x)\pi + \sin(\pi x)\, e^{\sqrt{x^2+1}} \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x) + \sec^2(\pi x)\pi}{e^{\sqrt{x^2+1}} \sin(\pi x) + \tan(\pi x)}$

(a) If $x = 0$, the denominator is 0! Double ugh.

If $x = 1/2$, remarkably, it's not so bad.

(a) + (b) point: Even though you can find exactly the values of $f'(x)$ using the Chain Rule, sometimes a numerical estimate is easier.