MATH 371
Review problems

1. Consider the jointly continuous uniformly distributed random variables (X, Y) on the domain bounded by $x = 0$, $y = 2$, $xy = 4$, $x = 4$, and $y = 0$. (It is **easy** to check your answers without integrating.)

   (a) Draw the *support* of the joint density function $f(x, y)$; that is, the region S where $f(x, y) > 0$.

   (b) Find the value of c so that $f(x, y)$ is a valid density function on S.

   (c) Set up an integral to find the marginal density $f_X(x)$ and include the domain of this function.

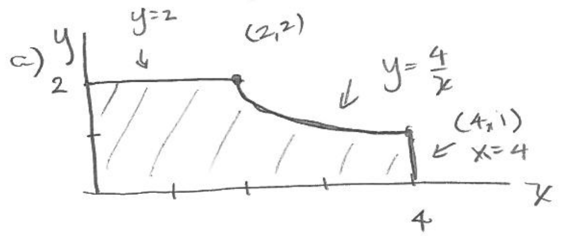
   (d) Set up an integral to find the marginal density $f_Y(y)$ and include the domain of this function.

   (e) Set up an integral that computes the conditional probability $P(X \geq 1 \mid Y = \frac{3}{2})$.

   (f) Set up a computation that computes the conditional probability that $P(X \geq 1 \mid Y \geq \frac{1}{2})$.

SOLUTION:

a)



$xy = 4 \Rightarrow y = \frac{4}{x}$

b) Find the area of S. $\text{Area}(S) = 4 + \int_2^4 \int_0^{\frac{4}{x}} dy\,dx = 4 + \int_2^4 \frac{4}{x}\,dx = 4 + 4\ln x \Big|_2^4$

$= 4 + 4(\ln 4 - \ln 2) = 4 + 4\ln 2$

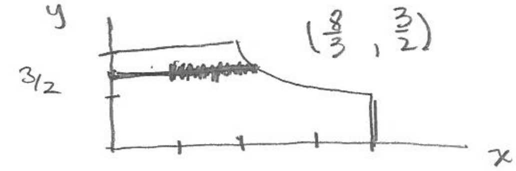
≈ 6.77

Thus, $c = \frac{1}{4 + 4\ln 2} \approx \boxed{.1477}$

c) $f_X(x) = \int_{y\text{-values}} \frac{1}{4+4\ln 2}\,dy = \begin{cases} \int_0^2 \frac{1}{4+4\ln 2}\,dy & 0 \leq x \leq 2 & \frac{2}{4+4\ln 2} = \frac{1}{2+2\ln 2} \\ \int_0^{\frac{4}{x}} \frac{1}{4+4\ln 2}\,dy & 2 \leq x \leq 4 & \frac{4}{x} \cdot \frac{1}{4+4\ln 2} = \frac{1}{x(1+\ln 2)} \\ 0 & \text{otherwise} \end{cases}$

d) $f_Y(y) = \int_{x\text{-values}} \frac{1}{4+4\ln 2}\,dx = \begin{cases} \int_0^4 \frac{1}{4+4\ln 2}\,dx = \frac{1}{1+\ln 2} & 0 \leq y \leq 1 \\ \int_0^{\frac{4}{y}} \frac{1}{4+4\ln 2}\,dx = \frac{1}{y(1+\ln 2)} & 1 < y \leq 2 \\ 0 & \text{otherwise} \end{cases}$

e) $P(X \geq 1 \mid Y = \frac{3}{2})$



$y = \frac{3}{2} = \frac{4}{x}$

$\Rightarrow x = \frac{8}{3}$

The conditional density is

$$f(x \mid Y = 3/2) = \frac{f(x,y)}{f_Y(3/2)} = \frac{\frac{1}{4+4\ln 2}}{\frac{2}{3}(1+\ln 2)} \qquad f_Y(3/2) = \frac{2}{3(1+\ln 2)}$$

$$= \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$$

Clearly correct since $0 \leq x \leq \frac{8}{3}$ and $f_X(x)$ should be uniform on for $y = 3/2$.

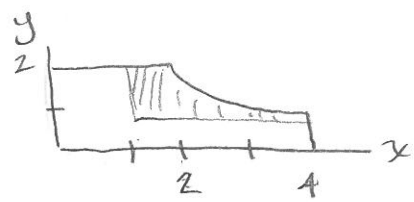
The answer as an integral is

$$\int_1^{8/3} f(x \mid 3/2)\, dx = \int_1^{8/3} \frac{3}{8}\, dx = \frac{5}{8}$$

f) $$P(X \geq 1 \mid Y \geq 1/2) = \frac{P(X \geq 1, Y \geq 1/2)}{P(Y \geq 1/2)}$$

Numerator:

$P(X \geq 1, Y \geq 1/2)$



Must be written as a sum of two integrals

$$= \int_1^2 \int_{1/2}^2 \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\, dx + \int_2^4 \int_{1/2}^{\frac{4}{x}} \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\, dx$$

Denominator:

$$P(Y \geq 1/2) = \int_{\frac{1}{2}}^2 f_Y(y)\, dy = \int_{1/2}^1 \frac{1}{1+\ln 2}\, dy + \int_1^2 \frac{1}{y(1+\ln 2)}\, dy$$

Thus, $$P(X \geq 1 \mid Y \geq 1/2) = \frac{\int_1^2 \int_{1/2}^2 \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\, dx + \int_2^4 \int_{1/2}^{4/x} \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\, dx}{\int_{1/2}^1 \frac{1}{1+\ln 2}\, dy + \int_1^2 \frac{1}{y(1+\ln 2)}\, dy}$$

2. In a large calculus class of 200 students, 40 earn an A on a text, 60 earn a B, and the remaining students earn a C, D, or F. Suppose a random sample of size 25 is taken.

   (a) Find the probability that five students in the sample earned an A on the exam.

   (b) Find the marginal probability function for the variable
   A: number of students who earned an A on the exam.

   (c) Write down a formula that computes the probability of the event
   E: Between 2 and 5 students in the sample earn a B on the exam, given that 10 students in the sample earned an A.

   (d) Give the bivariate probability function for (A, B).

Use hypergeometric random variable ...

a) $P(A=5) = \dfrac{\binom{40}{5}\binom{160}{20}}{\binom{200}{25}}$

b) $p_A(a) = \dfrac{\binom{40}{a}\binom{160}{25-a}}{\binom{200}{25}}$ $\quad a = 0, 1, \ldots, 25$

c) $P(E) = P(2 \le B \le 5 \mid A = 10) = \dfrac{P(A=10,\ 2 \le B \le 5,\ C = 15 - B)}{p_A(10)}$

$$= \frac{\displaystyle\sum_{b=2}^{5} \frac{\binom{40}{10}\binom{60}{b}\binom{100}{15-b}}{\binom{200}{25}}}{\dfrac{\binom{40}{10}\binom{160}{15}}{\binom{200}{25}}}$$

$$= \frac{\displaystyle\sum_{b=2}^{5} \binom{60}{b}\binom{100}{15-b}}{\binom{160}{15}}$$

d) $p(a,b) = \dfrac{\binom{40}{a}\binom{60}{b}\binom{100}{25-a-b}}{\binom{200}{25}}$, $\quad 0 \le a+b \le 25$ $\quad$ Not independent ...

3. Consider the jointly distributed random variables (X, Y) with joint density function

$$f(x,y) = \begin{cases} ce^{-y}, & \text{for } 0 \le x \le e^2 - 1,\ 0 \le y \le \ln(x+1) \\ 0, & \text{otherwise.} \end{cases}$$

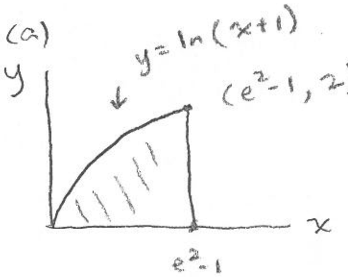
(a) Draw the *support* of the joint density function $f(x,y)$; that is, the region S where $f(x,y) > 0$. Then find the value of c so that $f(x,y)$ is a valid density function on S.

(b) Set up an integral to find the marginal density $f_Y(y)$ and include the domain of this function.

(c) Verify that your marginal density $f_Y(y)$ is correct by integrating it on the support of Y.

(d) Find the value of the conditional probability $P(X \ge 4 \mid Y = \ln(3))$. *Answer:* $\dfrac{e^2-5}{(e^2-3)} \approx .54$.

(a)



$$\int_0^{e^2-1}\int_0^{\ln(x+1)} e^{-y}\,dy\,dx = \int_0^{e^2-1} -e^{-y}\Big|_0^{\ln(x+1)} dx$$

$$= \int_0^{e^2-1} -e^{-\ln(x+1)} - -e^{0}\,dx = \int_0^{e^2-1} 1 - \frac{1}{x+1}\,dx$$

$$= x - \ln(x+1)\Big|_0^{e^2-1} = (e^2-1) - \ln(e^2-1+1) = e^2 - 3. \text{ Thus, } c = \frac{1}{e^2-3}.$$

(b.) With $c = \frac{1}{e^2-3}$, $f_Y(y) = \int_{\text{all } x} ce^{-y}dx = \int_{e^y-1}^{e^2-1} ce^{-y}\,dx$ $\qquad y = \ln(x+1) \Rightarrow x = e^y - 1$

$$= ce^{-y}\left([e^2-1] - [e^y-1]\right) = ce^{-y}(e^2 - e^y)$$

$$f_Y(y) = \begin{cases} \dfrac{e^{-y}(e^2-e^y)}{e^2-3} = \dfrac{1}{e^2-3}(e^{2-y}-1) & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) $$1 \overset{?}{=} \int_0^2 \frac{e^2}{e^2-3}e^{-y} - \frac{1}{e^2-3}\,dy$$

$$= -\frac{e^2}{e^2-3}e^{-y}\Big|_0^2 - \frac{1}{e^2-3}\,y\Big|_0^2 = \frac{1}{e^2-3}\left[\left[-e^2e^{-2} + e^2\right] - 2\right] = \frac{1}{e^2-3}\left[-1+e^2-2\right]$$

$$= \frac{e^2-3}{e^2-3} = 1$$

Review Problem # 3

(d) $P(X \geq 4 \mid Y = \ln(3))$

The "correct" density is $f(x \mid y = \ln 3) = \dfrac{f(x, y = \ln 3)}{f_Y(\ln 3)} = \dfrac{\dfrac{e^{-\ln 3}}{e^2 - 3}}{\dfrac{e^{-\ln 3}(e^2 - e^{\ln 3})}{e^2 - 3}}$

$$= \frac{e^{-\ln 3}}{e^{-\ln 3}(e^2 - e^{\ln 3})} = \frac{1}{e^2 - 3}$$

which is easy to check is correct for $2 \leq x \leq e^2 - 1$

Since $f(x,y) = f(y)$ (i.e. no dependency on x) implies $f(x \mid y = \ln 3)$ should be constant constant on an interval of length $(e^2 - 1) - (2) = e^2 - 3$.

Thus, $P(X \geq 4 \mid Y = \ln 3) =$

$$\int_4^{e^2 - 1} f(x \mid y = \ln 3)\, dx$$

$$= \int_4^{e^2 - 1} \frac{1}{e^2 - 3}\, dx$$

$$= \frac{e^2 - 5}{e^2 - 3} \approx .54$$