Math 215 Fall 2011
Final Exam

Name: Key Lab section: ____________

Instructions:

- The exam consist of 8 problems for a total of 140 points. Please look through the exam booklet and make sure it has 16 pages. The next to last page is a list of formulas which may be useful. The last page is blank and is to be used as scratch paper. You may tear both of those pages apart from the rest of the exam.
- The exam duration is 120 minutes.
- No calculators or notes are allowed.
- For multiple choice problems there is no partial credit. For all other problems, show all your work to receive full credit.
- Make sure your answers are clearly marked (circled or boxed).

	Points	Possible
Problem 1		30
Problem 2		20
Problem 3		10
Problem 4		10
Problem 5		10
Problem 6		15
Problem 7		20
Problem 8		25
Total		140

Problem 1 (5 × 6 = 30 points) Indicate whether the statement is **true** or **false**. There is no need to explain your answers.

i) Let $f(x, y)$ be a smooth function, and suppose $\nabla f(a, b) = \mathbf{0}$, and $f_{xx}(a, b) < 0$. Then $f(a, b)$ is necessarily a local maximum.

(a) True
(b) False

ii) The area of a region in $\mathbb{R}^2$ can be written as an integral along its boundary if its boundary is a piecewise smooth simple closed contour.

(a) True
(b) False

iii) Let $f(x, y, z)$ be a smooth function, and (x_0, y_0, z_0) a point in the domain of f. There exists a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}} f(x_0, y_0, z_0) = 0$.

(a) True
(b) False

iv) Let $F(t)$ be a function of one variable such that $F'(t) = f(t)$. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle f(x), f(y), f(z) \rangle$. Let C be a circle in the xy-plane. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

(a) True
(b) False

$\vec{F}$ is gradient of $F(x) + F(y) + F(z)$

v) Suppose the point (x_0, y_0) is a local maximum of the function $f(x,y)$ subject to the constraint $g(x,y) = 4$, and that $f(x,y)$ and $g(x,y)$ are both smooth functions. Then $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel vectors.

(a) True

(b) False

vi)

$$\int_0^{2\pi} \int_0^{\pi} \int_a^b \rho^2 \sin\phi \;\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_a^b 1\, dx\, dy\, dz.$$

(a) True

(b) False

Problem 2 ($5 \times 4 = 20$ points) Multiple choice. Circle the best answer. There is no need to explain your choice.

i) The line $\mathbf{r}(t) = \langle t+1, 2t-1, -3t+16 \rangle$ is perpendicular to which of the following planes?

$\vec{r}\,'(t) = \langle 1, 2, -3 \rangle$

(a) $3z = x + 2(y-1)$ $\quad \vec{n} = \langle 1, 2, -3 \rangle$

(b) $-x - 2y + 3z = 11$ $\quad n = \langle -1, -2, 3 \rangle$

(c) $2x + 4y - 6z = 31$ $\quad \vec{n} = \langle 2, 4, -6 \rangle \sim \langle 1, 2, -3 \rangle$

(d) All of the them

(e) None of the them

ii) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u}$ is not the zero vector, then which of the following is necessarily true?

(a) $\mathbf{u} \cdot \mathbf{v} = 0$

(b) Either $\text{proj}_{\mathbf{u}}\mathbf{v} = \mathbf{v}$ or $\text{proj}_{\mathbf{u}}\mathbf{v} = -\mathbf{v}$

(c) $|\mathbf{u}| = |\mathbf{v}|$

(d) All of the above

(e) None of the above

iii) Let $f(x, y) = xye^{xy}$. Which of the following unit vectors gives the direction of the maximal rate of increase of f at the point $(1, 1)$?

(a) $\frac{1}{\sqrt{2}}\langle 1, 1\rangle$

(b) $\frac{1}{\sqrt{2}}\langle -1, 1\rangle$

(c) $\frac{1}{\sqrt{2}}\langle 1, -1\rangle$

(d) $\frac{1}{\sqrt{2}}\langle -1, -1\rangle$

(e) None of the above

$f_y = e^{xy}(x + x^2y)$; $f_x = e^{xy}(y + xy^2)$

$\nabla f(1,1) = e\langle 2, 2\rangle$

iv) Which of the following best describes the critical points of the function

$$f(x, y) = x + \frac{y^3}{3} + \frac{y^2}{2} - \frac{x^2}{2}?$$

(a) One critical point.

(b) 2 critical points. One local minimum and one local maximum.

(c) 2 critical points. One saddlepoint and one local maximum.

(d) 2 critical points. One saddlepoint and one local minimum.

(e) 3 critical points. One saddlepoint, one local maximum, and one local minimum.

$f_x = 1 - x$; $f_y = y^2 + y$ $\quad f_{xy} = 0$

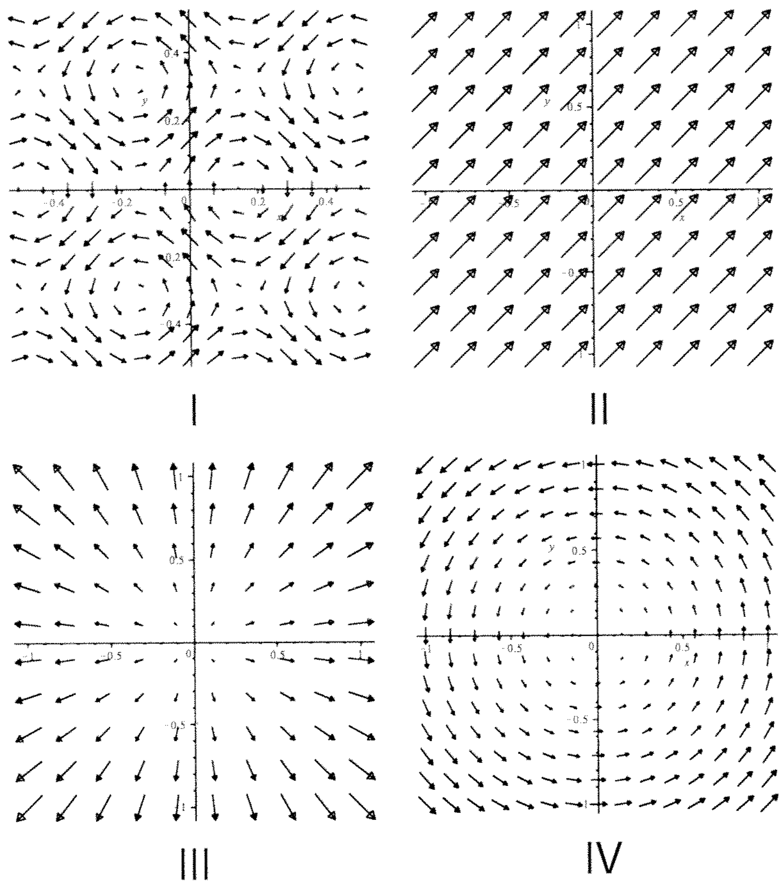
$f_{xx} = -1 \quad f_{yy} = 2y + 1$

Crit. pts: $(1, 0)$; $(1, -1)$

$D(1,0) = f_{xx}(1,0) f_{yy}(1,0) - f_{xy}(1,0)^2 = (-1)(1) - 0 < 0$

$D(1,-1) = (-1)(-1) - 0 = 1 > 0$

Problem 3 ($5 \times 2 = 10$ points) Circle the best answer. There is no need to explain your choice.



i) Which of the above vector fields could be be the gradient of some function?

(a) I and II

(b) II and III

(c) III and IV

(d) I and III

(e) I and IV

ii) Which of the above vector fields could be the gradient of some function f such that $f(0, -1/2) = f(1/2, 0)$?

(a) I

(b) II

(c) III

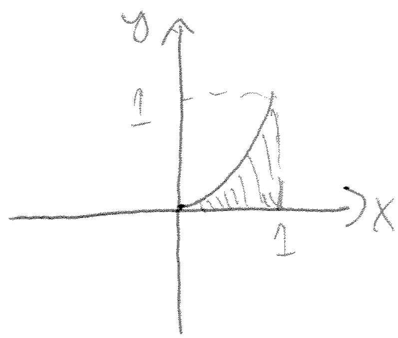
(d) IV

(e) More than one of the above

Problem 4 (5 + 5 = 10 points) Consider the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{5 - x^3}\, dx\, dy.$$

i) Draw the region in the xy-plane over which this integral is taken.



ii) Compute this integral.

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{5-x^3}\, dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{5-x^3}\, dy\, dx$$

$$= -\frac{1}{3}\int_0^1 (-3x^2)\sqrt{5-x^3}\, dx = \left(-\frac{1}{3}\right)(5-x^3)^{3/2}\left(\frac{2}{3}\right)\Bigg]_{x=0}^{1}$$

$$= -\frac{2}{9}\left[4^{3/2} - 5^{3/2}\right] = \boxed{\frac{2}{9}\left[5^{3/2} - 8\right]}$$

Problem 5 (5 + 5 = 10 points) Consider the vector field

$$\mathbf{F}(x, y) = \langle \sin y + 2xe^y, x\cos y + x^2e^y \rangle.$$

i) Is $\mathbf{F}$ conservative? If so, find a potential function $f(x, y)$ such that $\nabla f = \mathbf{F}$.

$$\frac{\partial Q}{\partial x} = \cos y + 2x e^y \quad ; \quad \frac{\partial P}{\partial y} = \cos y + 2xe^y$$

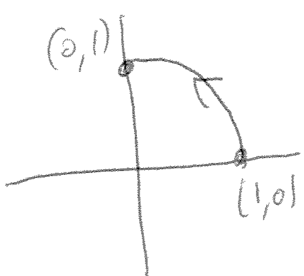
$\boxed{\text{Yes conservative!}}$

$$f_x = \sin y + 2xe^y \quad ; \quad f_y = x\cos y + x^2e^y \Rightarrow f = x\sin y + x^2e^y + g_2(x)$$

$$f = x\sin y + x^2e^y + g_1(y) \quad ; \quad \boxed{f(x,y) = x\sin y + x^2e^y}$$

ii) Let C_1 be the quarter of the unit circle which lies in the first quadrant oriented in the counterclockwise direction. Calculate the contour integral

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}.$$



$$\int_{C_1} \vec{F} \cdot d\vec{r} = f(0,1) - f(1,0)$$

$$= 0 - 1 = \boxed{-1}$$

Problem 6 (5 + 10 = 15 points) Consider the vector field

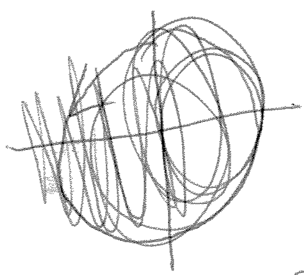
$$\mathbf{F}(x, y) = \langle -y + \sin(\cos(x)), y^{2y^{3y+2011}} + 2x \rangle.$$

i) Is $\mathbf{F}$ conservative? If so, find a potential function $f(x, y)$ such that $\nabla f = \mathbf{F}$.

$$\frac{\partial Q}{\partial x} = 2; \quad \frac{\partial P}{\partial y} = -1$$

$\boxed{\text{Not conservative}}$

ii) What is the amount of work performed by the field $\mathbf{F}$ in moving an object once around the unit circle centered at the point $(0, 2)$ in the counterclockwise direction?



Green's theorem:
$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

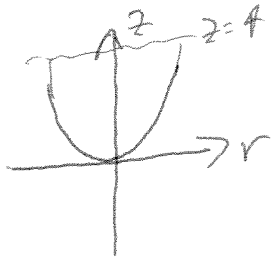
$$= \iint_D 3 \, dA = \boxed{3\pi}$$

Problem 7 (5 + 5 + 10 = 20 points) Let S be the surface described by the equation in cylindrical coordinates

$$z = r^2,$$

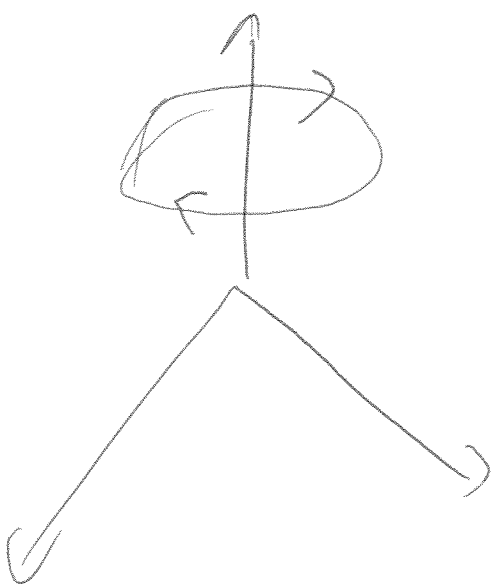
and the inequality $0 \le z \le 4$, oriented such that the unit normal vector points downwards.

i) Draw the surface S, along with a unit normal vector to show the orientation.



ii) Describe ∂S, the boundary of S. Draw this contour in $\mathbb{R}^3$ with the orientation inherited from S.

It is a circle of radius 2 in the plane $z=4$, centered around the z-axis



iii) Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = \langle xz, yz^2, (z+2)^{(z+1)^{xy}} \rangle .$$

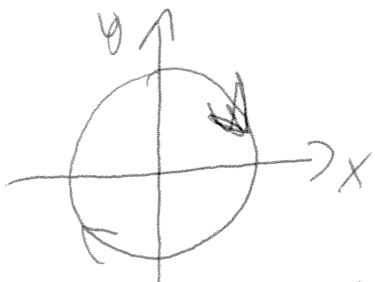
Evaluate $\displaystyle\iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}$.

Stokes' theorem: $\displaystyle\iint_S (\text{curl } \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$

∂S parameterized by $\vec{r}(\theta) = \langle 2\cos\theta, -2\sin\theta, 4 \rangle$

$\vec{r}\,'(\theta) = \langle -2\sin\theta, -2\cos\theta, 0 \rangle$

$0 \leq \theta \leq 2\pi$



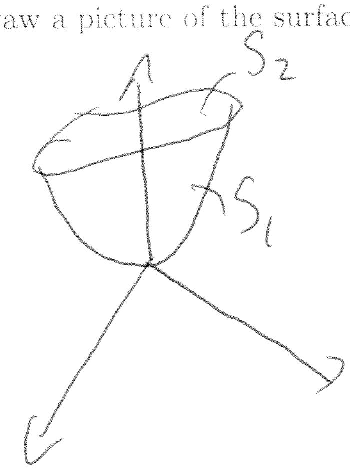
$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 8\cos\theta, -32\sin\theta, * \rangle \cdot \langle -2\sin\theta, -2\cos\theta, 0 \rangle d\theta$$

$$= \int_0^{2\pi} \left(-16\cos\theta\sin\theta + 64\sin\theta\cos\theta \right) d\theta$$

$$= \int_0^{2\pi} 48 \sin\theta\cos\theta \, d\theta = 24 \sin^2\theta \Big]_{\theta=0}^{2\pi} = \boxed{0}$$

Problem 8 ($5+5+5+5+5=25$ points) Suppose that the electric field in a charged plasma at some instant of time is given by $\mathbf{E}(x, y, z) = < 1-2x, 2+3y, 1+z >$. According to Gauss's Law, the total electric charge contained within a closed surface S is proportional to the outward flux of the electric field across S. Let S be the surface whose sides S_1 are given by the piece of the paraboloid $x^2+y^2=z$ for $0 \leq z \leq 4$, and whose top S_2 is the disc of radius 2 which lies in the plane $z=4$, centered at $(0,0,4)$.

i) Draw a picture of the surface S, and label S_1 and S_2 clearly.

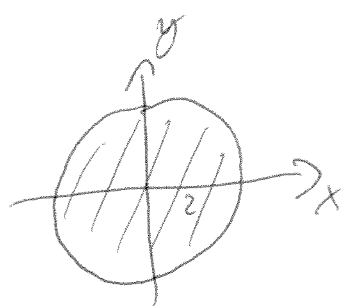


ii) Both S_1 and S_2 can be parametrized such that the domain in the parameter space is a disc of radius 2. Give these parametrizations.

$$S_1: \vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$$

$$S_2: \vec{s}(x,y) = \langle x, y, 4 \rangle$$

D :



iii) Use the parametrizations from part ii) to write a **single** area integral over the disc of radius 2, which gives the outward flux of **E** across S. Do not evaluate this integral.

S_1: $\vec{r}_x \times \vec{r}_y = \langle 1, 0, 2x \rangle \times \langle 0, 1, 2y \rangle = \langle -2x, -2y, 1 \rangle$

wrong orientation.

Flux through S_1: $\iint_D \langle 1-2x, 2+3y, 1+x^2+y^2 \rangle \cdot \langle 2x, 2y, -1 \rangle \, dA$

S_2: $\vec{S}_x \times \vec{S}_y = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$ right orientation

Flux through S_2: $\iint_D \langle 1-2x, 2+3y, 5 \rangle \cdot \langle 0, 0, 1 \rangle \, dA$

$\boxed{\text{Total flux } \iint_D \left(2x - 4x^2 + 4y + 6y^2 - 1 - x^2 - y^2 + 5 \right) dA}$

iv) Now write a **single** volume integral in cylindrical coordinates which gives the outward flux of **E** across S. Do not evaluate this integral.

By divergence theorem, total flux is

$$\iiint_E (\operatorname{div} \vec{E}) \, dV = \iiint_E (-2+3+1) \, dV$$

$$= \iiint_E 2 \, dV$$

where E is region enclosed by S_1, S_2.

v) Evaluate the outward flux of $\mathbf{E}$ across S using either your answer to part iii) or your answer to part iv).

$$\iiint_E 2\,dV = 2\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta.$$

$$= 2\int_0^{2\pi}\int_0^2 r(4-r^2)\,dr\,d\theta$$

$$= 2\int_0^{2\pi} 2r^2 - \frac{r^4}{4}\Big]_{r=0}^{2}\,d\theta$$

$$= 4\pi\left[8 - \frac{16}{4}\right] = \boxed{16\pi}$$