HW #1 Solutions to Even Problems

§2 #30 Let F be the set of all real-valued functions having as domain the set $\mathbb{R}$ of all real numbers. Either prove the given statement or give a counterexample.

Function subtraction $-$ on F is commutative.

This statement is in general false. As a counterexample, let $f(x) = 2x$ and $g(x) = x$. Both $f, g \in F$ and moreover,

$$(f-g)(x) = f(x) - g(x) = 2x - x = x$$

but $(g-f)(x) = g(x) - f(x) = x - 2x = -x$.

Since $f - g \neq g - f$, $-$ on F is not commutative.

§3 Determine whether the given map ϕ is an isomorphism of the first binary structure with the second. If it is not an isomorphism, why not?

#2 $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = -n$ for $n \in \mathbb{Z}$

ϕ is an isomorphism. First, we will show that ϕ is injective (or one-to-one). Suppose $\phi(n) = \phi(m)$ for some $n, m \in \mathbb{Z}$. Then $-n = -m$, so $n = m$. Therefore, ϕ is injective. Second, we will show that ϕ is surjective (or onto). Take any element $m \in \mathbb{Z}$. We have that $\phi(-m) = -(-m) = m$. Thus, ϕ is surjective. Thirdly, we will show that ϕ is a homomorphism. We have that for any $n, m \in \mathbb{Z}$

$$\phi(n+m) = -(n+m) = (-n) + (-m) = \phi(n) + \phi(m).$$

Thus, ϕ is a homomorphism. Since ϕ is injective, surjective, and a homomorphism, ϕ is an isomorphism.

#4 $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = n+1$ for $n \in \mathbb{Z}$.

ϕ is not an isomorphism. We will show that ϕ is not a homomorphism. As a counterexample,

$$\phi(1+1) = \phi(2) = 3$$

but $\phi(1) + \phi(1) = 2 + 2 = 4$

Since $\phi(1+1) \neq \phi(1) + \phi(1)$, ϕ is not a homomorphism. Thus, ϕ is not an isomorphism either.

#8. $\langle M_2(\mathbb{R}), \cdot \rangle$ with $\langle \mathbb{R}, \cdot \rangle$ where $\phi(A)$ is the determinant of matrix A.
ϕ is not an isomorphism. We will show that ϕ is not injective (one-to-one).
Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
We have that $\phi(A) = \det(A) = 1$ and $\phi(B) = \det(B) = 1$. However, $A \neq B$. Thus, ϕ is not injective, and ϕ is not an isomorphism.

§4 #10 Let n be a positive integer and let $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$.
a. Show that $\langle n\mathbb{Z}, + \rangle$ is a group.
Proof: First, we must show that $n\mathbb{Z}$ is closed under the operation $+$. Take any elements $a, b \in \mathbb{Z}$. We have that $na + nb = n(a+b) \in n\mathbb{Z}$. Thus, $n\mathbb{Z}$ is closed under the operation $+$. Second, we must show that $+$ is associative. Let $a, b, c \in \mathbb{Z}$. We have that $na + (nb + nc) = (na + nb) + nc$. Thus, $+$ is associative. Thirdly, we must show that $n\mathbb{Z}$ has an identity element. We have that $0 = n(0) \in \mathbb{Z}$. Furthermore, for any $a \in \mathbb{Z}$, $na + 0 = 0 + na = na$. Thus, 0 is an identity element in $n\mathbb{Z}$. Fourthly, we must show the existence of inverses. Let $a \in \mathbb{Z}$. We have that $n(-a) \in n\mathbb{Z}$ and moreover, $na + n(-a) = n(-a) + na = 0$. Thus, every element $na \in n\mathbb{Z}$ has an additive inverse $n(-a) \in n\mathbb{Z}$. We have shown that $\langle n\mathbb{Z}, + \rangle$ is a group. ∎

b. Show that $\langle n\mathbb{Z}, + \rangle \simeq \langle \mathbb{Z}, + \rangle$.
Proof: We must find an isomorphism ϕ between $\langle n\mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}, + \rangle$. Let $\phi : \mathbb{Z} \to n\mathbb{Z}$ be defined by the rule $\phi(a) = na$. We will show that ϕ is an isomorphism. First, ϕ is injective since for any $a, b \in \mathbb{Z}$, if $\phi(a) = \phi(b)$, then $na = nb$ so $a = b$. Second, ϕ is surjective since for any element $na \in n\mathbb{Z}$, there exists an element $a \in \mathbb{Z}$ such that $\phi(a) = na$. Thirdly, ϕ is a homomorphism since for all $a, b \in \mathbb{Z}$
$\phi(a+b) = n(a+b) = na + nb = \phi(a) + \phi(b)$. Since ϕ is a bijective (ie. injective and surjective) homomorphism, ϕ is an isomorphism. Therefore, since there exists an isomorphism between $\langle n\mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}, + \rangle$, $\langle n\mathbb{Z}, + \rangle \simeq \langle \mathbb{Z}, + \rangle$. ∎