MATH 310: Numerical Analysis
Details for Homework #7
due Wednesday, Oct 28 at 9 am

1. Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to zero.

2. Show that the sequence $p_n = 10^{-n^k}$ does *not* converge quadratically to zero, regardless of the size of the exponent $k > 1$.

3. (a) Show that the sequence $p_n = \dfrac{1}{n^2}$ for $n \geq 1$ converges linearly to 0.

   (b) Show that the sequence $p_n = \dfrac{1}{n^k}$ for $n \geq 1$ converges linearly to 0, for any integer $k \geq 1$.

4. Write a program or script that computes the errors in the Bisection method, Newton's Method, and the secant method for finding an estimate to $\alpha = \sqrt{7}$ by finding a zero of $f(x) = x^2 - 7$. Then use the table of errors you generate to see if you can detect the rates of convergence for each of these. Write a few paragraphs, explaining your results. You will be graded on your commentary.

5. The 'fixed slope' method. Another way of modifying Newton's method is to substitute a non-zero constant g for the $f'(x_n)$ in the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{g},$$

   where g is some reasonable value. This of course saves us the evaluation of a derivative and can be quite a reasonable algorithm in some situations.

   Answer the following questions for finding a root α to $f(x) = 2 - e^{-x}$.

   (a) Give the exact value of the root α, and use MATLAB or a calculator to give an estimate of α.

   (b) Give the iterative function $\varphi(x)$ for this algorithm. At this point this should be a function of g. Then compute symbolically the derivative $\varphi'(x)$.

   (c) Suppose you choose $g = 1$ and your initial guess is $x_0 = 2$. Perform a few iterations of this algorithm (by hand is fine) and determine if the algorithm converges to the root α.

   (d) By mimicking the error analysis we performed in class, give an expression for the error e_{k+1} as a function of $\varphi'(x)$ and e_k. Simplify.

   (e) Now suppose that you start with $x_0 = 0$ as your initial guess. Can you find a positive number a such that the fixed slope method converges for $g > a$ and diverges for $g < a$? Explain. (In class we saw that the fixed slope method converges for $g = 2$ and diverges for $g = .5$. This gives you a range of values .....) For you answer, either a theoretical one or empirical one is okay, but you should know you will be graded on your analysis.