Solutions to Take-home portion of midterm
March 21, 2008

1. Let $Z_{60} = \langle x \rangle$ be the cyclic group of order 60.

   (a) Compute $\phi(60)$, and list all generators of 60.

   Proof. Recall that the Euler-phi function $\phi(n)$ gives the number of positive integers a less than $n \in \mathbb{Z}^+$ that are relatively prime to n. Also, for prime numbers p we know that $\phi(p^a) = p^{a-1}(p-1)$. We also remember that $\phi(ab) = \phi(a)\phi(b)$ if a and b are relatively prime. Now $60 = 22 \cdot 3 \cdot 5$. Then $\phi(60) = \phi(2^2 \cdot 3 \cdot 5) = \phi(2^2)\phi(3)\phi(5) = 2^{2-1}(2-1) \cdot 3^{1-1}(3-1) \cdot 5^{1-1}(5-1) = 2 \cdot 2 \cdot 4 = 16$. These 16 integers are $k = 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59$, and thus, $\langle x^k \rangle = \langle x \rangle$ for any of these values of k. □

   (b) List all elements of Z_{60} of order 6.

   Proof. We begin by noticing that $|x^{10}| = 6$ in Z_{60}. We are looking for integers k such that $\frac{60}{\gcd(60,k)} = 6$. Therefore we need to have $\gcd(60, k) = 10$ for $1 \leq k \leq 60$. The $k's$ that satisfy this equation are $10, 50$. So the elements of order 6 in Z_{60} are x^{10} and x^{50}. □

   Comments: Generators are *elements* of the group Z_{60}, while $\langle x^{11} \rangle$ is a subgroup; indeed, it is the cyclic subgroup of Z_{60} generated by the element x^{11}.

2. Prove that the subset of elements of finite order in an Abelian group forms a subgroup. This group is known as the *torsion subgroup*. Is the same thing true for non-Abelian groups?

   Proof. Let G be an Abelian group. Let $H = \{g \in G \ : \ g^n = e, \ n < \infty\}$. We begin by noting $e \in H$, so $H \neq \varnothing$.

   Suppose $a, b \in H$. Then there exist $n, m < \infty$ such that $a^n = b^m = e$. Notice that $nm < \infty$ and $(ab)^{nm} = a^{nm}b^{nm}$ since G is Abelian. Since $a^n = b^m = e$ we have $(ab)^{nm} = e$. Hence $ab \in H$.

   Assume that $a \in H$ and $a^n = e$ for some $n < \infty$. Then $(a^{-1})^n = (a^n)^{-1} = e^{-1} = e$. Hence $a^{-1} \in H$.

   By the two-step subgroup test $H \leq G$.

   The same is not true for non-Abelian groups. Let $G = GL_2(\mathbb{R})$, the set of two-by-two matricies with non-zero determinant. Let H be defined as above. Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0.5 \\ 2 & 0 \end{bmatrix}$. Notice that $A^2 = B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore both A and B have finite order, so we have $A, B \in H$. However $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$. Now notice that $(AB)^n = \begin{bmatrix} 2^n & 0 \\ 0 & 0.5^n \end{bmatrix}$, which clearly does not have finite order. Therefore $AB \notin H$, hence H is not a subgroup. □

   Comments: To establish the second part of this problem, namely that the result is false if the hypothesis that G be Abelian is dropped, you need to provide a counter-example. Many students pointed out where in their proof they used that G was Abelian. However, this only shows that *your* proof fails if G is non-Abelian.

3. The *exponent* of a group is the smallest positive integer n such that $x^n = e$ for all x in the group. Prove that every finite group has exponent that divides the order of the group.

Proof. Let $|G| = k$, for $k \in \mathbb{Z}^+$, and let n denote the exponent of G. By Lagrange's theorem, we know that

$$x^k = e, \text{ for all } x \in G.$$

It follows that the exponent $n \leq k$.

By the division algorithm for $\mathbb{Z}$, we have that there exists a quotient $q \in \mathbb{Z}$ and a remainder $r \in \mathbb{Z}$ so that

$$k = qn + r, \text{ for } 0 \leq r < n. \tag{1}$$

For each $x \in G$,

$$\begin{aligned} e = x^k &= x^{qn+r} \\ &= x^{qn}x^r \\ &= (x^n)^q x^r \\ &= x^r \text{ since } n \text{ is the exponent of } G. \end{aligned}$$

However, since n is the least positive integer so that $x^n = e$ for all elements in G, we find that $r = 0$ in Equation 1. Thus, $n \,\Big|\, k = |G|$, as needed. □

4. Prove that every group of order 77 is cyclic.

Proof. Let G be a group of order 77.

More on Wednesday..... For a model proof, you can look at Example 16 in Chapter 10.

□

5. Let N be a normal subgroup of G and let $H \leq G$. (H is not necessarily normal.) Prove that NH is a subgroup of G. Give an example to show that NH might not be a subgroup of G if neither N nor H is normal.

Proof. Let $NH = \{nh \mid n \in N, h \in H\}$. We begin by noting that both N and H are subgroups and therefore non-empty. Hence $e \in N, H$, so $e = e \cdot e \in NH$. Hence $NH \neq \varnothing$.

Suppose $nh \in NH$. Then $(nh)^{-1} = h^{-1}n^{-1} \in h^{-1}N$. Since N is normal, left and right cosets are equal: $h^{-1}N = Nh^{-1}$. In particular, $h^{-1}n^{-1} = n'h^{-1}$ for some $n' \in N$. Therefore, $(nh)^{-1} = n'h^{-1} \in NH$.

Next we assume $n_1h_1,\ n_2h_2 \in NH$. Since $N \lhd G$, we have $h_1N = Nh_1$. In particular, the element $h_1n_2 = n'h_1$ for some $n' \in N$.

Now we see that

$$(n_1h_1)(n_2h_2) = n_1(h_1n_2)h_2 = n_1(n'h_1)h_2 = (n_1n')(h_1h_2).$$

Notice that $n_1n' \in N$ and $h_1h_2 \in H$. Therefore $(n_1h_1)(n_2h_2) = (n_1n')(h_1h_2) \in NH$.

By the two-step subgroup test $NH \leq G$.

To show the statement does not hold when N and H are not normal consider $G = S_4$ and take $N = \{e, (12)\}$, $H = \{e, (13)\}$. Then, with a little computation, we see that

$$NH = \{e, (13), (12), (132)\},$$

and we notice that NH is not a subgroup. For example, the element (132) has no inverse in NH. Also, the product $(13)(12) = (123) \notin NH$. Therefore NH is not a subgroup of G. □

Comments: Look carefully at the third (or second) paragraph in the proof above. Several students incorrectly asserted that $h_1n_2 = n_2h_1$. This would mean that the elements h_1 and n_2 commute, which may not be true. Since $N \lhd G$, we know that right and left cosets are equal: $h_1N = Nh_1$. From this you obtain, that *there exists* some $n' \in N$ with $h_1n_2 = n'h_1$, but this n' may not be equal to n_2.

6. (a) Chapter 9, # 37: Let G be a finite group and let H be a normal subgroup of G. Prove that the order of an element gH in G/H must divide the order of g in G.

Proof. Let $g \in G$. Then $|g| = n < \infty$ and by Lagrange's theorem $n \,\Big|\, |G|$. Let k be the order of $(gH) \in G/H$. Notice that $(gH)^n = g^n H$ since H is normal. But $g^n = e$ so $(gH)^n = eH = H$. By Corollary 2 to Theorem 4.1 the order k of gH in G/H divides n. $\square$

(b) Chapter 10, # 4: Prove that the mapping given in Example 11 is a homomorphism. What is the kernel of this homomorphism?

Proof. Let $\phi : S_n \to \mathbb{Z}_2$ where

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Let E denote any even permutation and O any odd permutation. We have a few cases to consider to show ϕ is a homomorphism. Notice that $EE = E$, $OO = E$ and $OE = EO = O$. Now,

$$\begin{aligned} \phi(EE) &= \phi(E) = 0 = 0 + 0 = \phi(E) + \phi(E), \\ \phi(EO) &= \phi(O) = 1 = 0 + 1 = \phi(E) + \phi(O), \\ \phi(OO) &= \phi(E) = 0 = (1 \bmod 2) \ + \ (1 \bmod 2) = \phi(O) + \phi(O). \end{aligned}$$

Hence ϕ is a homomorphism.

Let $K = \{\sigma \mid \sigma \text{ is even}\}$. Clearly if $\sigma \in K$ we know $\phi(\sigma) = 0$ and then we have $\sigma \in \ker \phi$. Therefore, $K \subseteq \ker \phi$. Suppose $\sigma \in \ker \phi$. Then $\phi(\sigma) = 0$. By definition of ϕ we know that σ is even. Hence $\sigma \in K$ and $\ker \phi \subseteq K$. Therefore $\ker \phi$ is the set of all even permutations; that is, $\ker(\phi) = A_n$.

Since kernels of homomorphisms are normal, we comment that this proves additionally that $\ker(\phi) = A_n \lhd S_n$. $\square$

7. Chapter 10

\# 6 Let G be the group of all polynomials with real coefficients under addition. For each f in G let $\int f$ denote the antiderivative of f that passes through the point $(0, 0)$. Show that the mapping $f \mapsto \int f$ from G to G is a homomorphism. What is the kernel of this mapping? Is this mapping a homomorphism if $\int f$ denotes the antiderivative that passes through $(0, 1)$?

Proof. Let G be the group of all polynomials with real coefficients under addition. Define $\phi : G \to G$ were $f \mapsto \int f$. (Where $\int f$ passes through the point $(0, 0)$.) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ be polynomials in G. Without loss of generality we assume $n \geq m$. Now,

$$\begin{aligned} \phi(f(x) + g(x)) &= \phi((a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \\ &\quad + (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)) \\ &= \phi(a_n x^n + \cdots + (a_{n-m} + b_{n-m})x^{n-m} + \cdots + (a_1 + b_1)x + (a_0 + b_0)) \\ &= \int (a_n x^n + \cdots + (a_{n-m} + b_{n-m})x^{n-m} + \cdots + (a_1 + b_1)x + (a_0 + b_0)) \\ &= \frac{a_n}{n+1}x^{n+1} + \cdots + \frac{(a_{n-m} + b_{n-m})}{n-m+1}x^{n-m+1} + \cdots + \frac{(a_1 + b_1)}{2}x^2 + (a_0 + b_0)x \\ &= \left(\frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + a_0 x\right) + \left(\frac{b_m}{m+1}x^{m+1} + \frac{b_{m-1}}{m}x^m + \cdots + b_0 x\right) \\ &= \phi(f(x)) + \phi(g(x)). \end{aligned}$$

Hence ϕ is a homomorphism.

Notice that the $\ker \phi = \{f \in G \mid \phi(f) = 0\}$. Notice that any non-zero polynomial will map to a non-zero element under ϕ. Hence $\ker \phi = \{0\}$.

Let ψ be the mapping above where $f \mapsto \int f$ and $\int f$ passes through $(0,1)$. Take $f(x) = x$. Then $\psi(f(x)) = \int f(x) = \frac{x^2}{2} + 1$. Notice that $\psi(f(x) + f(x)) = \psi(2x) = x^2 + 1$. However, $\psi(f(x)) + \psi(f(x)) = \frac{x^2}{2} + 1 + \frac{x^2}{2} + 1 = x^2 + 2$. Then $\psi(f(x) + f(x)) \neq \psi(f(x)) + \psi(f(x))$. Therefore ψ is not a homomorphism. □

\# 9 Prove that the mapping from $G \oplus H$ to G given by $(g,h) \to g$ is a homomorphism. What is the kernel? This mapping is called the *projection* of $G \oplus H$ onto G.

Proof. Let $\phi : G \oplus H$ by $(g,h) \mapsto g$. Suppose $(g,h), (a,b) \in G \oplus H$. Then

$$\begin{aligned}\phi((g,h) + (a,b)) &= \phi((g+a, h+b)) \\ &= g + a \\ &= \phi((g,h)) + \phi((a,b)).\end{aligned}$$

Hence ϕ is a homomorphism.
Let $K = \{(0,h) \mid h \in H\}$. We claim that $K = \ker\phi$. Indeed, if $(0,h) \in K$ then $\phi((0,h)) = 0$. Hence $K \subseteq \ker\phi$. Suppose $(g,h) \in \ker\phi$. Then $\phi(g,h) = g = 0$. Hence $(g,h) = (0,h) \in K$. Therefore $\ker\phi \subseteq K$. Together we have $\ker\phi = \{(0,h) \mid h \in H\}$. □

8. Chapter 10

\# 14 Explain why the correspondence $x \to 3x$ from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ is not a homomorphism.

Proof. Notice that $\phi(6+7) = \phi(1) = 3 \bmod 10$, since $13 \bmod 12 \equiv 1$.
However,

$$\begin{aligned}\phi(6) + \phi(7) &= (3 \cdot 6 \bmod 10) + (3 \cdot 7 \bmod 10) \\ &= (18 \bmod 10) + (21 \bmod 10) \\ &= (8 \bmod 10) + (1 \bmod 10) \\ &= 9 \bmod 10.\end{aligned}$$

Since $\phi(6+7) \neq \phi(6) + \phi(7)$, we see that ϕ is not a homomorphism. □

\# 15 Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and $\ker\phi = \{0, 10, 20\}$. If $\phi(23) = 9$ determine all elements that map to 9.

Proof. Recall $\phi^{-1}(9) = \{x \in \mathbb{Z}_{30} \mid \phi(x) = 9\}$. Using property 6 of Theorem 10.1 we know that $\phi^{-1}(9) = 23 + \ker\phi$. Hence the set of all elements that map to 9 are given by $\phi^{-1}(9) = \{23, 3, 13\}$. □

\# 16 Prove that there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Proof. Note that the element $(1,0) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$ has order 8. However, given any $(a,b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_4$ we see that $4(a,b) = (4a, 4b) \equiv (0,0)$. Therefore the order of $(a,b) \leq 4$. Since $(a,b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_4$ was arbitrary we can conclude that the maximum order of any element in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ is 4. Hence $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ has no element of order 8. Since homomorphisms preserve orders of elements and there is not an element of order 8 in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ we conclude that there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. □

QUESTION: A number of students indicated that if ϕ in the last problem were a homomorphism, then $\ker(\phi)$ must be $\langle(4,1)\rangle$, $\langle(0,1)\rangle$, or $\langle(4,0)\rangle$, (or some variation thereof). I did not follow your reasoning here.

9. Chapter 10 # 23 Suppose ϕ is a homomorphism from $\mathbb{Z}_{36}$ to a group of order 24.

(a) Determine all the possible homomorphic images.

Proof. By property 2 of Theorem 10.1, a homomorphism is completely specified by the image of 1. So, if $1 \mapsto a$, then $x \mapsto xa$. By Lagrange's Theorem and property 7 of Theorem 10.1, we need $|a|$ to divide 36 and 24. Thus $|a| = 1, 2, 3, 4, 6, 12$. Then possible homomorphic images are $\text{Im}(\phi_1) \cong \mathbb{Z}_1$, $\text{Im}(\phi_2) \cong \mathbb{Z}_2$, $\text{Im}(\phi_3) \cong \mathbb{Z}_3$, $\text{Im}(\phi_4) \cong \mathbb{Z}_4$, $\text{Im}(\phi_6) \cong \mathbb{Z}_6$ and $\text{Im}(\phi_{12}) \cong \mathbb{Z}_{12}$. □

(b) For each image in part a, determine the corresponding kernel of ϕ.

Proof. By the First Isomorphism Theorem we know $G/\ker\phi \cong \text{Im}(\phi)$. Then for each ϕ_i we have $\mathbb{Z}_{36}/\ker(\phi_i) \cong \text{Im}(\phi_i)$. We then see that $\ker(\phi_1) = \mathbb{Z}_{36}$, $\ker(\phi_2) = \langle 2\rangle \cong \mathbb{Z}_{18}$, $\ker(\phi_3) = \langle 3\rangle \cong \mathbb{Z}_{12}$, $\ker(\phi_4) = \langle 4\rangle \cong \mathbb{Z}_9$, $\ker(\phi_6) = \langle 6\rangle \cong \mathbb{Z}_6$, and $\ker(\phi_{12}) = \langle 12\rangle \cong \mathbb{Z}_3$. □

10. Chapter 10

35 Prove that the mapping $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ given by $(a, b) \to a - b$ is a homomorphism. What is the kernel of ϕ? Describe the set $\phi^{-1}(3)$, that is the set of all elements that map to 3.

Proof. Let $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $(a, b) \mapsto a - b$. Suppose $(a, b), (c, d) \in \mathbb{Z} \oplus \mathbb{Z}$. Then

$$\begin{aligned}
\phi((a, b) + (c, d)) &= \phi((a + c, b + d)) \\
&= a + c - (b + d) \\
&= a + c - b - d \\
&= (a - b) + (c - d) \\
&= \phi(a, b) + \phi(c, d).
\end{aligned}$$

Therefore ϕ is a homomorphism.

Let $K = \{(a, a) \mid (a, a) \in \mathbb{Z} \oplus \mathbb{Z}\}$. We show that $K = \ker\phi$. Suppose $(a, a) \in K$. Then $\phi(a, a) = a - a = 0$. Hence $(a, a) \in \ker\phi$ and $K \subseteq \ker\phi$. Suppose $(a, b) \in \ker\phi$. Then $\phi(a, b) = a - b = 0$. Hence $a - b = 0$ or $a = b$. Then $(a, b) = (a, a) \in K$. Now $\ker\phi \subseteq K$. Therefore $\ker\phi = K$.

Let $T = \phi^{-1}(3) = \{(a + 3, a) | a \in \mathbb{Z}\}$. Given $(a + 3, a) \in T$, we see $\phi(a + 3, a) = a + 3 - a = 3$. Hence $(a + 3, a) \in \ker\phi$. Next we assume $(a, b) \in \phi^{-1}(3)$. Hence $\phi(a, b) = a - b = 3$ or $a = b + 3$. Hence $(a, b) = (b + 3, b)$ and $(a, b) \in T$. Therefore $T = \phi^{-1}(3)$. □

38 For each pair of positive integers m and n, we can define a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_m \oplus \mathbb{Z}_n$ by $x \to (x \bmod m, x \bmod n)$. What is the kernel when $(m, n) = (3, 4)$ What is the kernel when $(m, n) = (6, 4)$?

Proof. Let $\phi : \mathbb{Z} \to \mathbb{Z}_3 \oplus \mathbb{Z}_4$ be given by $x \mapsto (x \bmod 3,\ x \bmod 4)$.

Claim. $\ker\phi = \langle 12\rangle$.

If $a \in \langle 12\rangle$ we know $a = 12k$ for some $k \in \mathbb{Z}$. Then $\phi(a) = \phi(12k) = (12k \bmod 3, 12k \bmod 4) = (0, 0)$. Hence $a \in \ker\phi$. Therefore $\langle 12\rangle \subseteq \ker\phi$.

Suppose $x \in \ker\phi$. Then $\phi(x) = (x \bmod 3, x \bmod 4) = (0, 0)$, and we have both $0 \equiv x \bmod 3$ and $0 \equiv x \bmod 4$. Therefore, $x = 3 \cdot 4 \cdot k$ for some $k \in \mathbb{Z}$. Hence, $x \in \langle 12\rangle$ and $\ker\phi \subseteq \langle 12\rangle$. It follows that $\ker\phi = \langle 12\rangle$, and the claim is established.

Now let $\psi : \mathbb{Z} \to \mathbb{Z}_6 \oplus \mathbb{Z}_4$ where $x \mapsto (x \bmod 6, x \bmod 4)$. Again, we claim that $\ker\phi = \langle 12\rangle$. Suppose $a \in \langle 12\rangle$. Then $a = 12k$ for some $k \in \mathbb{Z}$. Then $\phi(a) = \phi(12k) = (12k \bmod 6, 12k \bmod 4) = (0, 0)$. Assume $x \in \ker\phi$. Then $\phi(x) = (x \bmod 6, x \bmod 4) = (0, 0)$. Hence $0 \equiv x \bmod 6$ and $0 \equiv x \bmod 4$. So x must be a multiple of both 6 and 4. Stated otherwise, $x = k\text{lcm}(6, 4) = k12$. Therefore $\ker\phi \subseteq \langle 12\rangle$. We conclude $\ker\phi = \langle 12\rangle$.

Notice that the general statement for $m,\ n \in Z^+$ is that $\ker(\phi) = \langle \text{lcm}(m, n)\rangle$. □

Other comments: If $\phi : \mathbb{Z}_m \to \mathbb{Z}_n$, it is customary to write $\phi(a) + \phi(b)$ using a " + ", since the operation in the group is addition.

Comments on the in-class exam:

1. (a) A non-cyclic Abelian group A.
   $\mathbb{Z} \oplus \mathbb{Z}$, but not $\mathbb{Z}$.
   (b) A group G and two elements $a, b \in G$ with $|a| < \infty$ and $|b| = \infty$.
   $a = 0$ and $b = 1$ in $\mathbb{Z}$.
   (c) A normal subgroup N of D_4.
   Several students cleverly wrote $\{e\}$ or D_4. I should have asked for a proper normal subgroup of D_4 that is not trivial. The subgroup $\langle R_{90} \rangle \lhd D_4$ is one such subgroup.
   (d) A group G whose only subgroups are $\{e\}$ and G.
   $\mathbb{Z}_{11}$ or $\mathbb{Z}_p$ for p prime.
   (e) Three non-isomorphic groups of order 34.
   This is impossible, since any group of order 34 must be isomorphic to $\mathbb{Z}_{34}$ (in which case it is cyclic) or D_{17} (in which case it is not cyclic).
   (f) An element of order 6 in S_5.
   $(123)(45)$ or any product of a disjoint 3-cycle and transposition.
   (g) An element of order 6 in A_5.
   There are not any. Why?

2. (3 pts.) Consider the permutation group S_6, and let $\sigma = (123)(45)(56)(13)$.

   Give in disjoint cycle notation the element $\sigma^{100} = [(123)(45)(56)(13)]^{100}$

   First, compute σ as a product of *disjoint* cycles, $\sigma = (23)(456)$. The order of σ then is 6. Now noticing that $100 \equiv 4 \bmod 6$, then

   $$\sigma^{100} = \sigma^4 = (23)^4 (456)^4, \text{ since disjoint cycles commute.}$$

   Thus, $\sigma^{100} = (456)$.

3. Consider the quotient group $G = 4\mathbb{Z}/24\mathbb{Z}$.

   (a) What is the order of G? List all elements of G.
   The order is six and the elements are cosets:

   $$4\mathbb{Z}/24\mathbb{Z} = \{24\mathbb{Z}, 4 + 24\mathbb{Z}, 8 + 24\mathbb{Z}, 12 + 24\mathbb{Z}, 16 + 24\mathbb{Z}, 20 + 24Z\}.$$

   (b) Is G cyclic? Justify your answer by computing the order of elements in G.
   Yes, G is cyclic because you can compute that $|4 + 24\mathbb{Z}| = |20 + 24\mathbb{Z}| = 6 = |G|$.

4. Consider the cyclic group C_{24} of order 24 generated by x, $C_{24} = \langle x \rangle$.

   Students did very well on this problem.

5. Fix $n \in \mathbb{Z}^+$ with $n \geq 2$. Prove that $\mathrm{SL}(n, \mathbb{R}) \lhd \mathrm{GL}(n, \mathbb{R})$.

   Quick outline of proof: Define the determinant map from $\mathrm{GL}(n, \mathbb{R})$ to the multiplicative group of non-zero real numbers $\mathbb{R}^*$,

   $$\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*.$$

   Show that det is a homomorphism with $\ker(\det) = \mathrm{SL}(n, \mathbb{R})$. Therefore, $\mathrm{SL}(n, \mathbb{R}) \lhd \mathrm{GL}(n, \mathbb{R})$.

6. Prove that any group A of order 4 is Abelian. Then classify (describe up to isomorphism) all groups of order 4.

   Write a solution to this problem and hand it in with your HW on Wednesday.