HW 4 PROBLEMS

1. Hassett, Chapter 2, #12, modified as follows:

   (a) Before beginning, read the entire problem carefully and then consider a particular instance of the algebra homomorphism ψ in problem 12:

   Consider the ring $k[x, y_1, y_2]$ and $\phi_1(x) = x$, $\phi_2(x) = x^2$.

   i. Carefully define the map ψ in this example.
   ii. Find the image of $f = x + y_1 - 2y_2$ and $g = xy_1 - y_2$ under ψ.
   iii. Is ψ surjective? Prove your answer for the general case.
   iv. By direct computation, show that $y_i - \phi_i \in \ker(\psi)$.
   v. By direct computation, show that $g \in I$ as defined in the problem.

   (b) Now do problem 12.

2. (a) Hassett, Chapter 2, #17a. You may (and are strongly encouraged to) use `Singular` to answer this question. Commands you will need are `matrix, minor, print(A)`.

   (b) View any 2×3 matrix $A = (a_{ij})$ as an element of $\mathbb{A}^6(\mathbb{R})$. Then consider the set of points $V \subseteq \mathbb{A}^6(\mathbb{R})$ satisfying the three equations g_1, g_2, g_3. That is, V is the *zero set* of g_1, g_2, g_3.

   Using ideas from linear algebra, give a concrete description of this set V.

3. (a) By hand, compute a Groebner basis for the ideal $I = \langle f_1, f_2 \rangle = \langle x^2 - y^2, xy - 1 \rangle$ with respect to $>_{\text{lex}}$ for $x > y$. For your write up, please include all S-polynomials.

   (b) You should be able to find a smaller (in terms of number) Groebner basis by taking only a subset of your answer to (a). More formally, a Groebner basis $f_1, \ldots, f_r$ is called *minimal* if for all i,

   $$< \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_{i-1}), \mathrm{LT}(f_{i+1}), \ldots, \mathrm{LT}(f_r)\rangle \neq < \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_r)\rangle.$$

   Give a minimal Grobner basis for I constructed from your answer to (a).

   (c) Give a concrete description of the zero set of the polynomials f_1 and f_2.

4. `Singular` exercises.

   (a) Determine whether $f = xy^3 - z^2 + y^5 - z^3$ and $g = -z^4yx - z^4y + z^4 + z^2y^2 + z^2yx + zy^2x + zy^2 - zy - yx - x^2$ are in the ideal $I = \langle -x^3 + y, x^2y - z \rangle$. If not, give the normal form $(\operatorname{mod} I)$ with respect to $>_{\text{grevlex}}$.

   (b) (Calculus III) Consider the function of two variables

   $$f(x, y) = x^3y^2 + x^2y^3 + x^2y + xy^2.$$

   Use `Singular` to compute the critical points of f. You will need the command `diff`, and once you have computed a Groebner basis G for the appropriate ideal, you will need the commands

   ```
   LIB "solve.lib";
   solve G;
   ```

   (c) (Lagrange multiplier problem – easy) Find the maximum and minimum values of $f(x, y) = 2x^2 + y^2$ subject to the constraint $g(x, y) = x^2 + y^2 - 9 = 0$. Do this by hand and by using `Singular`.