MATH253X-F01 Midterm Exam 2 Name: Solutions
Fall 2019

**Instructions.** (100 points) You have 60 minutes. Closed book, closed notes, no calculator. *Show all your work* in order to receive full credit.

(6pts) **1.** Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{3x^2+y^2}.$$

Suggests d.n.e.

Either show it does not exist, or give strong evidence for suspecting it does.

Approach along $x=0$: $\lim_{(0,y)\to(0,0)} \frac{x^2+3y^2}{3x^2+y^2} =$

$\lim_{(0,y)\to(0,0)} \frac{3y^2}{y^2} = \boxed{3}$.

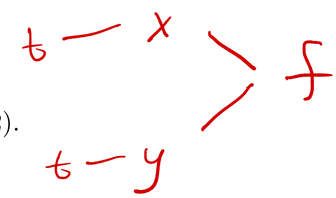
Along $y=0$, $\lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{3x^2+\cancel{y^2}}$

$= \boxed{1/3}$ $3 \neq 1/3 \Rightarrow$ d.n.e.

(10pts) **2.** The following table gives some information about a function $f(x,y)$:

| $(x,y)$ | $f$ | $f_x$ | $f_y$ |
|---|---|---|---|
| $(-1,3)$ | 3 | 2 | $-1$ |
| $(0,1)$ | $-5$ | $-1$ | 3 |
| $(3,4)$ | 1 | 4 | $-2$ |

(a) (5 pts) Use the chain rule to compute $\frac{dg}{dt}(0)$ where:




$$g(t) = f(t^2 - t + 3, 2e^{-3t} + 2).$$

x y

$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

at $t=0$, $x=3$ $y=4$ $\frac{dx}{dt} = 2t-1$ $\frac{dy}{dt} = -6e^{-3t}$

$\frac{dx}{dt}\Big|_{t=0} = -1$ $\frac{dy}{dt}\Big|_{t=0} = -6$

$\frac{dg}{dt} = 4\cdot(-1) + (-2)(-6)$

$= \boxed{+8}$

(b) (5 pts) Give an equation for the linear (tangent plane) approximation to $f$ at the point $(-1,3)$, and use it to estimate $f(-1.1, 3.2)$.

$f(x,y) \approx f(-1,3) + f_x(-1,3)\,dx + f_y(-1,3)\,dy$ $dx = -.1$ $dy = .2$

$f(-1.1, 3.2) \approx 3 + (2)(-.1) + (-1)(.2) = 3 - .2 - .2 = \boxed{2.6}$

Equation of tangent plane is

$L(x,y) = 3 + 2(x+1) - (y-3)$

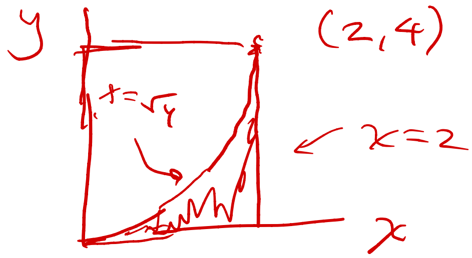
(12pts) **3.** Evaluate the integral

$$\int_0^4 \int_{\sqrt{y}}^2 e^{(x^3+1)}\,dx\,dy \quad = I$$

fully, by first drawing the region of integration, and then reversing the order of integration.

$\sqrt{y} \le x \le 2$

$0 \le y \le 4$




$x = \sqrt{y} \Rightarrow y = x^2$

$$I = \int_0^2 \int_0^{x^2} e^{x^3+1}\,dy\,dx = \int_0^2 x^2 e^{x^3+1}\,dx$$

$$= \frac{1}{3} e^{x^3+1} \Big|_0^2 = \boxed{\frac{1}{3}\left(e^9 - 1\right)}$$

u-Sub

(12pts) **4.** Find and classify (using the Second Derivatives Test) all critical points of

$$f(x, y) = x^2y - 2xy + y^2 - 3y + 1.$$

C.P.: $f_x = 2xy - 2y = 2y(x-1)$

$f_y = x^2 - 2x + 2y - 3$

Require $f_x = 0 \Rightarrow y = 0$ or $x = 1$

Case 1. $y = 0$. Then $f_y = 0 \Rightarrow x^2 - 2x + 3 = 0$

or $x = 3, -1$ $\quad \boxed{(3, 0)}$ $\boxed{(-1, 0)}$

Case 2: $x = 1$ Then $f_y = 0 \Rightarrow 1 - 2 + 2y - 3 = 0$ $\quad y = 2$

$\boxed{(1, 2)}$

2nd derivative test: $f_{xx} = 2y$ | $f_{xy} = f_{yx} = 2x - 2$ | $f_{yy} = 2$

$(3,0)$: $D = f_{xx}f_{yy} - f_{xy}f_{yx} = 0(2) - 4^2 = -16 < 0$ saddle pt.

$(-1,0)$: $D = 0(2) - (-4)^2 < 0$ saddle point

$(1,2)$: $D = 4(2) - (0) = 8 > 0$ $\quad f_{xx}(1,2) = 4 > 0 \Rightarrow$ local min

(8pts) **5.** Give an equation for the tangent plane to the surface

$$\frac{xy}{y+z} + e^{-z}\ln(x+2y) = 3 \qquad g(x,y,z) = 3$$

at the point $(3,-1,0)$. $\vec{p}$

Ideas: • Implicitly defined surface so $\vec{n} = \nabla g(x,y,z)$.

• then use $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p}$

$\nabla g = \langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \rangle$ where $\frac{\partial g}{\partial x} = \frac{y}{y+z} + \frac{e^{-z}}{x+2y}$, $\frac{\partial g}{\partial y} = x\left[y(-1)(y+z)^{-2} + \frac{1}{(y+z)}\right] + \frac{2e^{-z}}{x+2y}$,

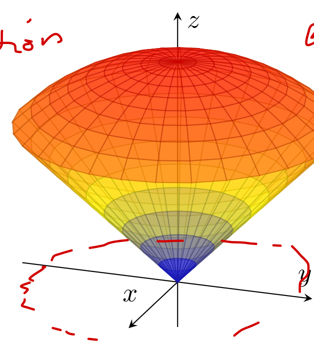
$= \frac{-xy}{(y+z)^2} + \frac{x}{(y+z)} + \frac{2e^{-z}}{x+2y}$

and $\frac{\partial g}{\partial z} = xy(-1)(y+z)^{-2} - e^{-z}\ln(x+2y)$.

Evaluating: $\nabla g(3,-1,0) = \langle \frac{-1}{-1+0} + \frac{e^{-0}}{3-2}, \frac{-3(-1)}{(-1)^2} + \frac{3}{-1+0} + \frac{2e^{-0}}{3+2(-1)}, \frac{-3(-1)}{(-1)^2} - e^{-0}\ln(3-2) \rangle$

$= \boxed{\langle 2,2,3\rangle = \vec{n}}$ $\quad \vec{n}\cdot\vec{p} = \vec{n}\cdot\vec{x} \Rightarrow \boxed{2x + 2y + 3z = 4}$

(10pts) **6.** Use polar coordinates to find the volume of the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the top half of the sphere $x^2+y^2+z^2 = 6$.




$z_\ell = z_u$ intersection $\quad z_u = \sqrt{6-x^2-y^2}$

$\sqrt{6-x^2-y^2}$

$= \sqrt{x^2+y^2}$ $\quad z_\ell = \sqrt{x^2+y^2}$

$\Rightarrow 6-x^2-y^2 = x^2+y^2$ $\quad R \to x^2+y^2 = 3$

$3 = x^2+y^2$

$r^2 = x^2+y^2$

$$\text{Vol} = \iint_R \sqrt{6-x^2-y^2} - \sqrt{x^2+y^2}\, dA \qquad dA = r\,dr\,d\theta$$

$R: \ 0 \le r \le \sqrt{3}$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{6-r^2} - \sqrt{r^2}\ r\,dr\,d\theta$$

u-sub

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{6-r^2} - r^2\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(6-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3\right]_0^{\sqrt{3}} d\theta$$

$$= 2\pi\left[\left[-\frac{1}{3}(6-3)^{3/2} - \sqrt{3}\right] - \left[-\frac{1}{3}6^{3/2}\right]\right] = 2\pi\left[-2\sqrt{3} + 2\sqrt{6}\right]$$

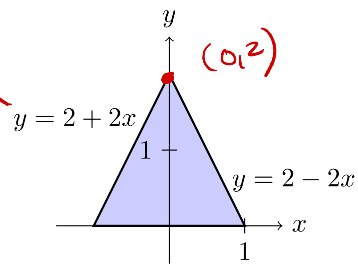
$$= \boxed{4\pi(\sqrt{6}-\sqrt{3})}$$

(16pts) **7.** A flat triangular plate is bounded by the lines $y = 2 - 2x$, $y = 2 + 2x$ and the $x$-axis, where $x, y$ are in $m$. The mass density is given by

$$\rho(x, y) = y^2 \text{ kg/m}^2.$$

$y = 2 + 2x \Rightarrow x = \frac{1}{2}(y-2) = \frac{1}{2}y - 1$

$y = 2 - 2x \Rightarrow x = -\frac{1}{2}y + 1$




From the symmetry of the plate and the density, you can see that the center of mass of the plate must be on the $y$-axis, so $\bar{x} = 0$. ← Why?

(a) (8 pts) Give an expression involving integrals for $\bar{y}$, including appropriate limits of integration.

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{\int_0^2 \int_{\frac{1}{2}y-1}^{-\frac{1}{2}y+1} y^3\,dA}{\int_0^2 \int_{\frac{1}{2}y-1}^{-\frac{1}{2}y+1} y^2\,dA}$$

(b) (8 pts) The total mass of the plate is $m = \dfrac{4}{3}$ kg. Use this to calculate $\bar{y}$.

We only need the moment $M_x$.

$$\int_0^2 \int_{\frac{1}{2}y-1}^{-\frac{1}{2}y+1} y^3\,dx\,dy = \int_0^2 y^3 x \Big|_{\frac{1}{2}y-1}^{-\frac{1}{2}y+1} dy$$

$$= \int_0^2 y^3\left[-\tfrac{1}{2}y + 1 - \left(\tfrac{1}{2}y - 1\right)\right] dy = \int_0^2 y^3(2-y)\,dy =$$

$$\int_0^2 2y^3 - y^4\,dy = \frac{1}{2}y^4 - \frac{1}{5}y^5 \Big|_0^2 = 8 - \frac{32}{5} = \frac{8}{5}. \quad \bar{y} = \frac{\frac{8}{5}}{\frac{4}{3}} = \boxed{\frac{6}{5} \text{ meters}}$$

(10pts) **8.** Use Lagrange multipliers to find the maximum product of two positive numbers satisfying $x^2 + y = 6$.

need more space!

(16pts) **9.** Let $f(x, y) = x^2y - x + y^2$.

($10^{\text{pts}}$) **8.** Use Lagrange multipliers to find the $\boxed{\text{maximum product}}$ of two positive numbers satisfying $x^2 + y = 6$.

Maximize $f(x,y) = xy$ subject to $x^2+y=6$ and $x, y > 0$.

$\nabla f = \langle y, x \rangle$ $\quad \nabla g = \langle 2x, 1 \rangle$ $\quad g(x,y) = x^2+y-6 = 0$

Solve Simultaneously

$\nabla f = \lambda \nabla g$ $\quad x^2+y^2-6=0$ (2) $\quad x, y > 0$

$f_x$: $y = \lambda 2x$ (1)

$f_y$: $x = \lambda$

Plug $x=\lambda$ into (1): $y = x(2x) = \boxed{y = 2x^2}$

Plug $y = 2x^2$ into (2): $x^2 + 2x^2 = 6 \Rightarrow 3x^2 = 6$

$\Rightarrow x = \pm\sqrt{2}$

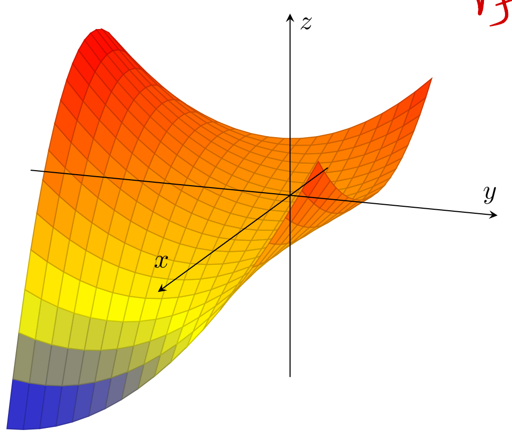
However, $x > 0 \Rightarrow x = +\sqrt{2}$

and so $y = 2(\sqrt{2})^2 = 4$

Final answer is $xy = \sqrt{2}\,4 = \boxed{4\sqrt{2}}$

(16pts) **9.** Let $f(x, y) = x^2y - x + y^2$.

$\nabla f = \langle 2xy - 1, x^2 + 2y \rangle$




(a) (5 pts) Compute the directional derivative of $f$ when moving in the direction of $-\mathbf{j}$ when you are at the point $(1, -1)$. Interpret your result in terms of change in values of $f$.

Note that $-\hat{j} = \langle 0, -1 \rangle$ is a unit vector.

$\nabla f(1,-1) = \langle 2(1)(-1) - 1, (1)^2 + 2(-1) \rangle = \langle -3, -1 \rangle$

$D_u f(1,-1) = \nabla f(1,-1) \cdot \langle 0, -1 \rangle = 1$. $f(x,y)$ is increasing at a rate of 1 here.

(b) (5 pts) Give the direction and magnitude of maximum decrease of $f$ when at the point $(1, -1)$.

$f(x,y)$ decreases the fastest in the direction of $-\nabla f(1,-1) = \langle 3, 1 \rangle$. The rate of decrease is

$|-\nabla f(1,-1)| = \sqrt{10}$.

(c) (6 pts) Fully set up bounds and integrand for computing the surface area of $f$ over the region $[-1, 2] \times [-2, 1]$. DO NOT EVALUATE.

$A(S) = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA = \int_{-1}^{2} \int_{-2}^{1} \sqrt{1 + (2xy-1)^2 + (x^2+2y)^2}\, dy\, dx$