MATH 202 Name : SOLUTIONS

Midterm 1 February 17, 2012

Instructions: Show all work for full credit. Poor notation or sloppy work will be penalized.

1. (6 pts.) Let $P(-1,2,0)$ and $Q(1,1,-3)$ be points in $\mathbb{R}^3$. Give parametric equations for the line containing P and Q.

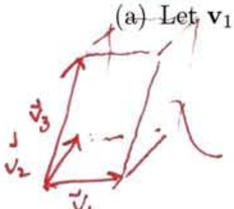
The direction vector $\vec{v} = \overrightarrow{PQ} = \langle 2,-1,-3 \rangle$. Using the point P,

$$\boxed{x(t) = -1 + 2t, \quad y(t) = 2 - t, \quad z(t) = -3t} \qquad t \in \mathbb{R}$$

Using the point Q, $\langle 1+2t, 1-t, -3-3t \rangle$

2. (20 pts. – 5 pts. each) Give brief clear explanations to the following questions.

(a) Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be three vectors in $\mathbb{R}^3$. What is the geometric meaning of $|\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)|$?



The volume of the parallelopiped spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

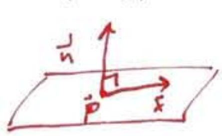
Volume

(b) Suppose $\mathbf{a}$ and $\mathbf{b}$ are unit vectors in $\mathbb{R}^3$ and $\mathbf{a} \cdot \mathbf{b} = -\dfrac{\sqrt{3}}{2}$, what can you say about the vectors?

$$\vec{a} \cdot \vec{b} = 1 \cdot 1 \cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \boxed{\theta = \frac{5\pi}{6}}$$

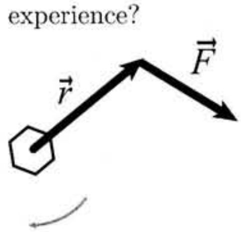
$\theta = 150^0$, if you prefer.

(c) Let $P(x_1, y_1, z_1)$ be a point in $\mathbb{R}^3$ at the tip of vector $\mathbf{p}$, and $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$, $\mathbf{x} = \langle x, y, z \rangle$. Explain why the equation $(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0$ gives the equation of a plane. (Drawing a picture might help.)



The vector $\vec{x} - \vec{p} = \overrightarrow{PX}$ and the plane is given by all points $\vec{x}$ with $\overrightarrow{PX}$ orthogonal to $\vec{n}$.

(d) The magnitude of the torque vector $\tau = \|\boldsymbol{\tau}\| = \|\mathbf{r} \times \mathbf{F}\|$ measures the tendency of an object (a bolt, for example) to rotate. (See figure.) Using the formula for the magnitude of the cross product of vectors, explain the effect of lengthening the lever arm on this tendency to rotate. Does this agree with your experience?



$$\tau = \|\vec{\tau}\| = \|\vec{r} \times \vec{F}\| = \|\vec{r}\| \, \|\vec{F}\| \sin\theta$$

If $\|\vec{r}\|$ increases, so does τ.

This is why, for example, longer wrenches make it easier to turn a bolt.

3. (17 pts.) Let $\mathbf{a} = < 1, -1, 3 >$ and $\mathbf{b} = < 2, 0, 1 >$ be vectors in $\mathbb{R}^3$.

(a) (8 pts.) Find the equation of the plane that contains the vectors **a** and **b**.

Find the normal $\vec{n}$ first: $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{vmatrix} = -\hat{i} + 5\hat{j} + 2\hat{k}$ $\quad (= \vec{a} \times \vec{b})$

Using $\vec{a}$, $\quad \vec{x} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$-x + 5y + 2z = \langle 1, -1, 3 \rangle \cdot \langle -1, 5, 2 \rangle$

$-x + 5y + 2z = -1 - 5 + 6$

$\boxed{-x + 5y + 2z = 0}$

OR use $(0,0,0)$ on plane.

(b) (4 pts.) Find the area of the parallelogram spanned by **a** and **b**. (Your answer will contain an angle θ.)

Solution 1: Area $= \|\vec{a}\| \|\vec{b}\| \sin\theta$ where θ is angle between $\vec{a}$ and $\vec{b}$

$\|\vec{a}\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$ $\quad \|\vec{b}\| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$ $\quad \therefore \boxed{\text{Area} = \sqrt{55} \sin\theta}$

Better answer: Find

$\|\vec{a} \times \vec{b}\| = \sqrt{(-1)^2 + (5)^2 + 2^2}$

$= \boxed{\sqrt{30}}$

(c) (5 pts.) Find the *vector projection* of **a** onto **b**; that is, find $\text{proj}_{\mathbf{b}}(\mathbf{a})$.

$\text{proj}_{\vec{b}} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b}$

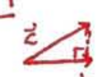
Computing: $\vec{a} \cdot \vec{b} = 2 + 0 + 3 = 5$

$\|\vec{b}\|^2 = 5$

Therefore,

$\text{proj}_{\vec{b}} \vec{a} = \frac{5}{5} \langle 2, 0, 1 \rangle$

$= \boxed{\langle 2, 0, 1 \rangle} = \vec{b}$!

Note: 

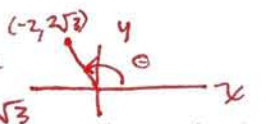
4. (12 pts. – 4 pts. each)

(a) Convert the point P with rectangular coordinates $P(-2, 2\sqrt{3}, 1)$ to cylindrical coordinates.

Find r, θ, z: $\quad z = 1$ easy

$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

θ: In the xy-plane



θ in QII with $\tan\theta = -\sqrt{3}$

$\tan\theta = -\sqrt{3}$ in QII if

$\theta = \frac{2\pi}{3}$ (or 120°)

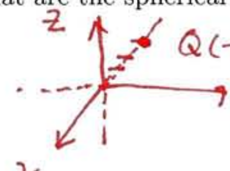
$\boxed{\text{Answer: } \left(4, \frac{2\pi}{3}, 1\right)}$

(b) If Q is the point with rectangular coordinates $Q(-3, 0, 0)$, what are the spherical coordinates of Q?

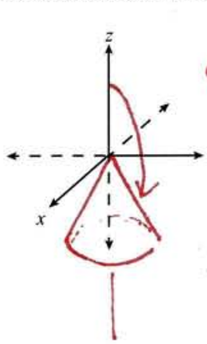
Find ρ, θ, φ:

$\boxed{\rho = 3, \ \theta = \pi, \ \varphi = \pi/2}$

See figure.



(c) Describe and sketch the surface in $\mathbb{R}^3$ whose equation in spherical coordinates is given by $\phi = \dfrac{3\pi}{4}$.

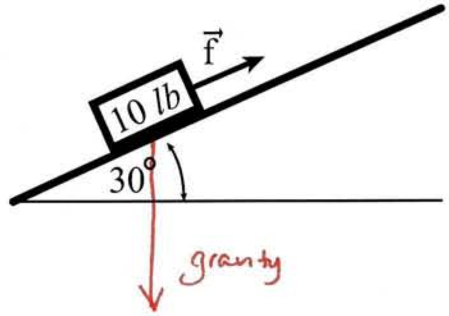


$\varphi = 3\pi/4$

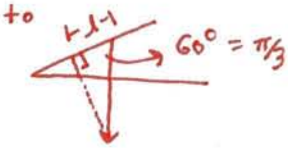
Cone

$\frac{3\pi}{4}$ is 135° from positive z-axis

5. (5 pts.) A 10 lb weight is placed on an incline forming an angle of 30° with the horizontal as shown. Find the magnitude of the force **f** needed to keep the weight from sliding.



Gravity acts in $-\hat{j}$ direction with magnitude 10 lbs. $\vec{a} = -10\hat{j}$. We want to know the length ℓ



Thus, $\ell = \|10 \text{ lbs}\| \cos 60^\circ$
$= 10\frac{1}{2} = \boxed{5 \text{ lbs}}$

6. (15 pts.) Consider the planes in $\mathbb{R}^3$ given by the equations below:

$$\begin{array}{ll} \text{Plane 1:} & 2x - 3y + z = 1 \\ \text{Plane 2:} & -4x + 6y - 2z = 2 \end{array}$$

(a) (5 pts.) Are the planes parallel or do they intersect? Explain your answer briefly.

The normals are $\vec{n}_1 = \langle 2, -3, 1 \rangle$ and $\vec{n}_2 = \langle -4, 6, -2 \rangle$; these vectors are parallel $\vec{n}_2 = -2\vec{n}_1$. Thus, the planes are parallel.

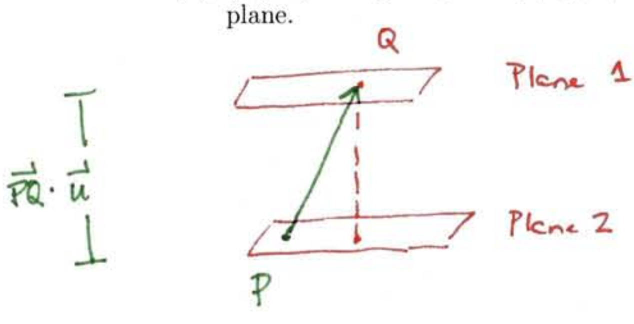
(b) (3 pts.) Give the equation of a plane parallel to Plane 1 that passed through the origin.

$2x - 3y + z = 0$

(c) (2 pts.) Find the point Q on the first plane with x-coordinate 1 and y-coordinate 1.

$(1, 1, 2)$ lies on plane 1

(d) (5 pts.) Using the point $Q(x, y, z)$ found in the previous problem, find the distance from Q to the second plane.



This is actually the distance between the two planes. If P is a point on plane 2, then $\vec{PQ} \cdot \vec{u}$ where $\vec{u}$ is a unit normal vector is the distance.

P is arbitrary; I'm choosing $(0, 0, -1)$ for a point on Plane 2.

$\vec{u}$ can be computed either with $\vec{n}_1$ or $\vec{n}_2$ (they are parallel). $\vec{u} = \frac{1}{\sqrt{14}} \langle 2, -3, 1 \rangle$

$\vec{PQ} = \langle 1, 1, 2 \rangle - \langle 0, 0, -1 \rangle = \langle 1, 1, 3 \rangle$. Finally, d = distance = $\vec{PQ} \cdot \vec{u}$

$= \langle 1, 1, 3 \rangle \cdot \frac{1}{\sqrt{14}} \langle 2, -3, 1 \rangle = \frac{1}{\sqrt{14}}(2 - 3 + 3) = \boxed{\frac{2}{\sqrt{14}}}$

7. (8 pts. – 4 pts. each) The trajectory in the plane of two particles are given by the equations

$$\text{Particle 1:} \quad \mathbf{r}(u) = < e^{2u},\, 2\cos(u) - 1 >$$
$$\text{Particle 2:} \quad \mathbf{s}(v) = < \tan(v),\, 1 + \ln(\cancel{\tfrac{\pi}{4}}v) >$$

Should read $\ln\left(\frac{4}{\pi}v\right)$!

where u and v are both measured in seconds.

(a) Show that the graphs of $\mathbf{r}(u)$ and $\mathbf{s}(v)$ intersect at the point $(1,1)$. (Justify your answer for full credit.)

If $u=0$, $\vec{r}(0) = \langle 1,1 \rangle$.

If $v = \pi/4$, then $\vec{s}(v) = \langle 1, ? \rangle$

This problem was graded generously. Everyone got 8/8.

(b) Do the two particles collide at $(1,1)$. Explain briefly.

No. The particles were present at different times.

8. (17 pts.) An particle moves in $\mathbb{R}^3$ with position given by

$$\mathbf{r}(t) = < \cos(t), t^2, 4\sin(t) > \text{ meters for } t \geq 0,$$

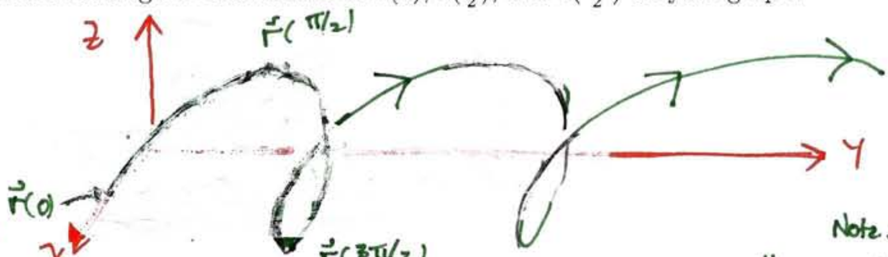
where t is measured in seconds.

(a) (5 pts.) On the axes below, sketch the trajectory of the particle. Indicate the orientation of the trajectory with an arrow. Include the rectangular coordinates of $\mathbf{r}(0)$, $\mathbf{r}(\frac{\pi}{2})$, and $\mathbf{r}(\frac{3\pi}{2})$ on your graph.

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}(\pi/2) = \langle 0, \frac{\pi^2}{4}, 4 \rangle$

$\vec{r}(\frac{3\pi}{2}) = \langle 0, \frac{9\pi^2}{4}, -4 \rangle$



Note: this "slinky" gets stretched along the y-axis as t increases.

(b) (5 pts.) Find the velocity $\mathbf{v}(t)$ of the particle at time t. Include units in your answer.

$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin(t), 2t, 4\cos t \rangle$ m/s.

(c) (4 pts.) Find the speed of the particle at $t = \frac{\pi}{6}$.

$\vec{v}(\pi/6) = \langle -\sin(\pi/6), 2\frac{\pi}{6}, 4\cos(\pi/6) \rangle = \langle -\frac{1}{2}, \frac{\pi}{3}, \frac{4\sqrt{3}}{2} \rangle = \langle -\frac{1}{2}, \frac{\pi}{3}, 2\sqrt{3} \rangle$

Speed $= \|\vec{v}(\pi/6)\| = \sqrt{(-\frac{1}{2})^2 + (\frac{\pi}{3})^2 + (2\sqrt{3})^2} = \sqrt{\frac{1}{4} + \frac{\pi^2}{9} + 12} = \sqrt{\frac{49}{4} + \frac{\pi^2}{9}}$ m/s

(d) (3 pts.) At time $t = e^{\pi}$, are the velocity and acceleration vectors perpendicular? Explain briefly.

A priori: HIGHLY unlikely since $\|\vec{v}(t)\|$ is not constant.

$\vec{v}(e^{\pi}) = \langle -\sin(e^{\pi}), 2e^{\pi}, 4\cos(e^{\pi}) \rangle$

$\vec{a}(t) = \vec{v}\,'(t) = \langle -\cos(t), 2, -4\sin(e^{\pi}) \rangle$

$\vec{a}(e^{\pi}) = \langle -\cos(e^{\pi}), 2, -4\sin(e^{\pi}) \rangle$

$\vec{v}(e^{\pi}) \cdot \vec{a}(e^{\pi}) = \sin(e^{\pi})\cos(e^{\pi}) + 4e^{\pi} - 16\sin(e^{\pi})\cos(e^{\pi}) = 4e^{\pi} - 15\sin(e^{\pi}) \cdot \cos(e^{\pi})$

$\neq 0$. Not orthogonal.