MATH 490 Name : SOLUTIONS

Midterm March 25, 2016

Instructions: Show all work for full credit. Poor notation or sloppy work will be penalized. For this exam, the symbol p represents a *prime number* in $\mathbb{Z}$, and $|\cdot|_p$ represents the p-adic norm on $\mathbb{Q}$ or $\mathbb{Q}_p$.

1. (15 pts.) Perform the following 5-adic arithmetic computations.

a)
$$\begin{array}{r} 134_\wedge \\ \times \quad 23_\wedge \\ \hline 1012 \\ 323 \\ \hline \boxed{4242_\wedge} \end{array}$$

b)
$$\begin{array}{r} {}^{2}\ \cancel{3}\overset{11}{\cancel{2}}1_\wedge \\ -\quad 24_\wedge \\ \hline \boxed{242_\wedge} \end{array}$$

c) Give the canonical 5-adic expansion of $\frac{7}{25}$.

$7 = 12_\wedge$

$\frac{1}{25}\cdot 7 = \frac{7}{25} = \boxed{{}_\wedge 12}$

2. (9 pts.) Give examples of the following, if they exist. Briefly justify your answer.

(a) A 7-adic integer $x \in \mathbb{Q} \setminus \mathbb{Z}$.

Eg $\boxed{x = \frac{1}{2}}$ $x \in \mathbb{Q}$, $|\frac{1}{2}|_7 = 1 \Rightarrow x \in \mathbb{Z}_7$ $x \notin \mathbb{Z}$

(b) An element $y \in \mathbb{Q}_7 \setminus \mathbb{Q}$.

$y = \ldots 1111011101101_\wedge$

Note the non-repeating canonical 7-adic expansion. By a HW problem, $y \notin \mathbb{Q}$.

(c) An element $z \in \mathbb{Z}_7 \setminus \mathbb{Q}$.

Note that $\omega = 2$ is a quadratic residue mod 7, since $3^2 \equiv 2 \pmod 7$. Since $p = 7$ is odd, it follows by the Corollary to Hensel's lemma that $\sqrt{2} \in \mathbb{Z}_7 \setminus \mathbb{Q}$.

3. (26 pts.) Give an example of a non-constant nor non-eventually constant sequence of integers $\{a_n\}$ such that $\{a_n\}$ converges to $a = 2$ with respect to the 3-adic norm $|\cdot|_3$, but $\{a_n\}$ does not converge with respect to the 5-adic norm $|\cdot|_5$.

Take

Eg. (a) *Answer:* $a_n = 2 + 3^n$ for $n \in \mathbb{N}$

(b) Now use the $\epsilon - N$ definition of 'a sequence converges to a limit a' to prove that $\{a_n\} \to 2$ with respect to the 3-adic norm $|\cdot|_3$.

Claim: The sequence $\{2+3^n\}$ converges to 2 with respect to $|\ |_3$.

Let $\varepsilon > 0$ be given. Take $N \in \mathbb{N}$ so that $\frac{1}{3^N} < \varepsilon$. Then if $n > N$, we have $|2+3^n-2|_3 = |3^n|_3 = \frac{1}{3^n} < \frac{1}{3^N} < \varepsilon$, as needed.

(c) Prove that $\{a_n\}$ does not converge with respect to the 5-adic norm $|\cdot|_5$.

Proof: Since $\mathbb{Q}_5$ is a complete metric space and all Cauchy sequences converge in complete metric spaces, it suffices to show that $\{2+3^n\}$ is <u>NOT</u> Cauchy in $\mathbb{Q}_5$. To this end, consider $|a_n - a_{n+1}|_5 = |2+3^n - (2+3^{n+1})|_5$ $= |3^n - 3^{n+1}|_5 = |3^n|_5\,|3-1|_5 = |3^n|_5\,|2|_5 = 1$. Thus, consequtive terms in the sequence are always a unit distance apart and $\{a_n\}$ is <u>not</u> Cauchy in $\mathbb{Q}_5$.

(d) Explain briefly, but rigorously, why this problem illustrates that $|\cdot|_3 \not\sim |\cdot|_5$ as norms on the rational numbers.

The sequence $\{2+3^n\}$ is Cauchy with respect to $|\ |_3$, but not $|\ |_5$

That is, by definition, these norms are inequivalent

4. (20 pts.)

(a) (5 pts.) State the Strong Triangle Inequality for a non-Archimedean norm $\|\cdot\|$ on a normed field F.

For all $x,y \in F$

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}$$

(b) (5 pts.) By Proposition 1.15, we have "*If the elements $x, a \in \mathbb{Q}_5$ satisfy the inequality $|x-a|_5 < |a|_5$, then $|x|_5 = |a|_5$*. Give an example x, a, both non-zero, illustrating this. (A complete answer includes computations of the relevant norms.)

Eg. $x = 1$, $a = 6$, then $|x-a|_5 = |5|_5 = \frac{1}{5} < |1|_5 = |6|_5 = 1$

(c) (10 pts.) Prove that if $B = B(a,r) = \{x \in \mathbb{Q}_5 \mid |x-a|_5 < r\}$ is the open ball centered at a of radius $r > 0$ and $b \in B$, then $B = B(b,r)$.

Proof: We show first that $B(b,r) \subseteq B(a,r)$. Let $x \in B(b,r)$ and note that $|x-b|_5 < r$ by definition. We have $|x-a|_5 = |(x-b)+(b-a)|_5 \leq \max\{|x-b|_5, |b-a|_5\}$ by the Strong Triangle Inequality. However, $|x-b|_5 < r$ since $b \in B(b,r)$ and $|b-a|_5 < r$ since $b \in B(a,r)$. and therefore $\max\{|x-b|_5, |b-a|_5\} < r$. Thus, $|x-a|_5 < r$ and $x \in B(a,r)$. By symmetry, $B(a,r) \subseteq B(b,r)$ too. Thus $B(a,r) = B(b,r)$. $\square$

5. (8 pts.) Give the value of $|3^6!|_3$, simplifying any exponents occurring in your answer. (No need to justify your answer, unless you hope for partial credit.)

First count the number of times $3 \mid 3^6!$. This number is

$$\underbrace{3^5}_{\text{\# of factors divisible by } 3} + \underbrace{3^4}_{\text{\# of factors divisible by } 3^2} + 3^3 + \cdots + 3 + \underbrace{1}_{\text{\# of factors divisible by } 3^6} = \frac{3^6-1}{3-1} = \frac{728}{2} = \underline{364}.$$

Thus, $|3^6!|_3 = \frac{1}{3^{364}} = 3^{-364}$

Very small

6. (22 pts.) Solving equations.

(a) (6 pts.) Let $F(x) \in \mathbb{Z}_p[x]$ be a polynomial, and $F'(x)$ denote its formal derivative. State Hensel's Lemma for finding a solution to $F(x) = 0$ in $\mathbb{Z}_p$.

Suppose $a_0 \in \mathbb{Z}_p$ such that $F(a_0) \equiv 0 \pmod p$ and $F'(a_0) \not\equiv 0 \bmod p$, Then there exists a unique $a \in \mathbb{Z}_p$ such that $F(a_0) \equiv F(a) \pmod p$ and $F(a) = 0$.

(b) (8 pts.) Let $p = 7$ and consider the quadratic equation $F(x) = x^2 + x + 2 = 0$.

i. Show that there exists some $a_0 \in \{0, 1, \dots 6\} \subset \mathbb{Z}_7$ such that $F(a_0) \equiv 0 \pmod 7$.

Let $a_0 = 3$, then $F(3) = 3^2 + 3 + 2 = 14 \equiv (0 \bmod 7)$

ii. Can a_0 be refined to find $a \in \mathbb{Z}_p$ with $F(a) = 0$? Explain. If so, find the first two terms in the 7-adic expansion of x, $x \equiv a_1 a_{0\wedge} \pmod{7^2}$.

Using Hensel's Lemma, test $F'(3)$. The formal derivative is $F'(x) = 2x + 1$ and $F'(3) = 2 \cdot 3 + 1 \equiv 0 \pmod 7$. Thus, by this version of Hensel's Lemma, $a_0 = 3$ can not be refined to a root of $F(x)$.

(c) (8 pts.) Let $p = 7$ and consider the equation $F(x) = x^2 - 2 = 0$ in $\mathbb{Z}_7$. Does there exist a root $a \in \mathbb{Z}_7$ to this equation? Explain. If so, find the first two digits in the 7-adic expansion of a, $a \equiv a_1 a_{0\wedge} \pmod{7^2}$.

Let $a_0 = 2$, then $3^2 \equiv 2 \pmod 7$ and by "Hensel's Lemma for Square Roots" or just Hensel's Lemma, $a_0 = 2$ can be refined to a p-adic integer root

Consider $b = 3 + 7a_1$, then $F(b) = F(3 + 7a) = (3 + 7a_1)^2 - 2 = 9 + 42a_1 + 7^2 a_1^2 - 2$

$= 7 + 42a_1 + 7^2 a_1^2 \equiv 7 + 42a \pmod{7^2}$. After division by 7, we must solve

(*) $1 + 6a_1 \equiv 0 \pmod 7$ Since we want to find a root mod 7^2. The digit $a_1 = 1$ works Thus, $a \equiv 13_\wedge \pmod{7^2}$.

Check:

$$\begin{array}{r} 13 \\ 13 \\ \hline 42 \\ 13 \\ \hline 02_\wedge \end{array}$$

The other square root is $(\bar{6}_\wedge)(13_\wedge) \pmod{7^2}$
$\equiv 54_\wedge \pmod{7^2}$

Extra credit: Consider the rational number x with canonical 3-adic expansion

$$x = \overline{121}_\wedge 2$$

Find $a, b \in \mathbb{Z}$ so that $x = \frac{a}{b}$.

Let $y = 3x = \overline{12}12_\wedge = \overline{12}_\wedge$.

Note that $9y = \overline{12}00_\wedge$ and thus $y - 9y = -8y = \overline{12}_\wedge - \overline{12}00_\wedge = 12_\wedge = 5$

It follows that $y = -5/8$ and $x = \frac{1}{3}y = \boxed{-5/24.}$