Section 2.5

6. Use the given lattices to help find the centralizers of every element in the following groups:

- a. D_8
- b. Q_8
- c. S_3
- d. D_{16}

(Schamel)

a.

b.

c.

d.

- 8. In each of the following groups find the normalizer of each subgroup:
 - a. S_3
 - b. Q_8

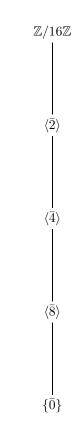
(Schamel)

a.

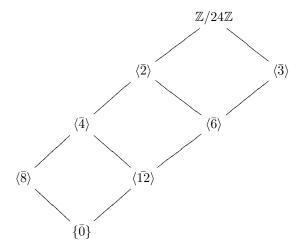
$$\begin{array}{c|ccccc} A & \{e\} & \langle (1\ 2) \rangle & \langle (1\ 3) \rangle & \langle (2\ 3) \rangle & \langle (1\ 2\ 3) \rangle & S_3 \\ \hline \\ N_{S_3}(A) & S_3 & \langle (1\ 2) \rangle & \langle (1\ 3) \rangle & \langle (2\ 3) \rangle & S_3 & S_3 \end{array}$$

b.

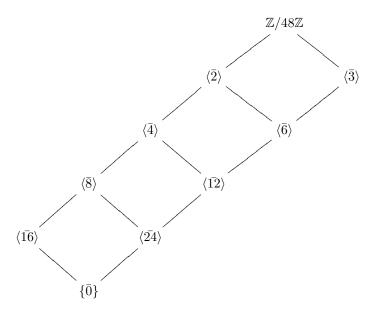
- 9. Draw the lattices of subgroups of the following groups: (Baggett) $\,$
 - a. $\mathbb{Z}/16\mathbb{Z}$



b. $\mathbb{Z}/24\mathbb{Z}$



c. $\mathbb{Z}/48\mathbb{Z}$



Section 3.1

3. Let A be an Abelian group and let B be a subgroup of A. Prove that A/B is Abelian. Give an example of a non-Abelian group G containing a proper normal subgroup N such that G/N is Abelian.

Proof. (Bastille) First we show that B is normal. Let $b \in B \le A$ and let $a \in A$. Then $aba^{-1} \in aBa^{-1}$ and

$$aba^{-1} = aa^{-1}b$$
 since A is Abelian (and $a^{-1}, b \in A$)
= $b \in B$.

Since a, b were chosen arbitrarily, it follows that $aBa^{-1} \subseteq B$ for all $a \in A$; hence B is normal in A. Now consider $A/B = \{aB | a \in A\}$. Let $a_1B, a_2B \in A/B$. Then we have

$$(a_1B)(a_2B) = (a_1a_2)B$$
 since B is normal so the operation is well-defined $= (a_2a_1)B$ since $a_1, a_2 \in A$, A Abelian $= (a_2B)(a_1B)$ since B is normal.

Therefore, A/B is Abelian.

Remark: There exist non-Abelian groups G containing a proper normal subgroup N such that G/N is Abelian. For example, take $G = Q_8$ and $N = \{1, -1\}$. Then G is non-Abelian since for example $ij = k \neq -k = ji$, and $N \triangleleft G$ since for all $a \in Q_8$:

$$a1a^{-1} = 1 \in N$$
 and $a(-1)a^{-1} = -1 \in N$.

Now we have: $G/N = \{N, iN, jN, kN\}$. Note that if $a, b \in Q_8$, then

- if WLOG $b = \pm 1$ then ab = ba so

$$(aN)(\pm 1N) = (a(\pm 1))N = ((\pm 1)a)N = (\pm 1N)(aN).$$

- if $a, b \neq \pm 1$, note that if ab = c then ba = -c but for any coset cN we have $cN = \{c, -c\}$ so

$$(aN)(bN) = (ab) N = cN = (-c)N = (ba)N = (bN)(aN).$$

Therefore G/N is Abelian.

27. Let N be a finite subgroup of a group G. Show that $gNg^{-1} \subseteq N$ if and only if $gNg^{-1} = N$. Deduce that $N_G(N) = \{g \in G \mid gNg^{-1} \subseteq N\}$.

Proof. (Lawless) Clearly, if $gNg^{-1}=N$, then $gNg^{-1}\subseteq N$. Assume $gNg^{-1}\subseteq N$. Consider the map $\varphi:N\to gNg^{-1}$ defined by $\varphi(n)=gng^{-1}$. We can see this map is injective, since if $\varphi(n)=\varphi(m)$, then $gng^{-1}=gmg^{-1}$, and thus n=m. Since $|N|<\infty$, and the map is injective, then we know this is a bijection from $N\to gHg^{-1}$. This, combined with our assumption that $gNg^{-1}\subseteq N$ gives us that $gNg^{-1}=N$.

28. Let N be a *finite* subgroup of a group G and assume $N = \langle S \rangle$ for some subset S of G. Prove that an element $g \in G$ normalizes N if and only if $gSg^{-1} \subseteq N$.

Proof. (Bastille) If g normalizes N then for all $n \in N$, $gng^{-1} \in N$. In particular, for all $s \in S \subseteq N$, $gsg^{-1} \in N$. Hence $gSg^{-1} \subseteq N$.

Now if $gSg^{-1} \subseteq N$, note that since N is finite, so must be S and all its elements have finite order. So we can write $S = \{a_1, a_2, \dots, a_k\}$ for some fixed k and any $n \in N$ can be expressed in the form:

$$n = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}$$
 where $\alpha_i \ge 0$ since $|a_i|$ is finite.

We also note that for any $a, b \in G$,

$$gabg^{-1} = ga1bg^{-1} = ga(g^{-1}g)bg^{-1} = (gag^{-1})(gbg^{-1}).$$

So inductively we find that $ga^{\ell}g^{-1}=(gag^{-1})^{\ell}$ for any $\ell\geq 0$. Therefore,

$$gng^{-1} = ga_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} g^{-1}$$

$$= (ga_1^{\alpha_1} g^{-1}) (ga_2^{\alpha_2} g^{-1}) \cdots (ga_k^{\alpha_k} g^{-1})$$

$$= (ga_1 g^{-1})^{\alpha_1} (ga_2 g^{-1})^{\alpha_2} \cdots (ga_k g^{-1})^{\alpha_k}.$$

But for all $1 \le i \le k$, ga_ig^{-1} is an element of gSg^{-1} so $ga_ig^{-1} \in N$ since we assume gSg^{-1} is contained in N, and hence by closure under the operation in N, $(ga_ig^{-1})^{\alpha_i} \in N$. Therefore,

$$gng^{-1} = \prod_{i=1}^{k} (ga_ig^{-1})^{\alpha_i} \in N,$$

and hence since n was chosen arbitrarily, we have $gNg^{-1} \subseteq N$, and so g normalizes N.

31. Prove that if $H \leq G$ and N is a normal subgroup of H then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of G in which N is normal (i.e., is the join of all subgroups H for which $N \leq H$).

Proof. (Mobley) Since H and $N_G(N)$ are both groups, it is sufficient to show that $H \subseteq N_G(N)$. To this end, pick $h \in H$. Since $N \subseteq H$, for all $h \in H$, $hNh^{-1} = N$. Thus, $h \in N_H(N)$. Since H is a subgroup of G, it must be the case that $H \subseteq N_G(N)$.

Suppose that $K \leq G$ and $N \subseteq K$. Using arguments similar to those above, we can show that $K \leq N_G(N)$. Thus, any arbitrary normal subgoup of G is contained in $N_G(N)$ and $N_G(N)$ is the largest subgroup of G in which N is normal.

33. Find all normal subgroups of D_8 and for each of these find the isomorphism types of its corresponding quotient.

Proof. (Buchholz) The normal subgroups of D_8 are $\langle s, r^2 \rangle$, $\langle r \rangle$, $\langle r \rangle$, $\langle r^2 \rangle$, 1 and D_8 . The first three of these have index 2 in D_8 and therefore are normal. For the subgroup $\langle r^2 \rangle$, we note that $rr^2r^{-1} = r^2$ and $sr^2r^{-1} = r^{-2} = r^2$. Since r and s generate D_8 , it follows that $g \langle r^2 \rangle g^{-1} = \langle r^2 \rangle$ for any $g \in G$. Thus, $\langle r^2 \rangle \triangleleft D_8$.

Now we must find the isomorphism types of each corresponding quotients. First note that $|D_8/< s, r^2>|=|D_8/< rs, r^2>|=|D_8/< r> >|=2$ so all are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Now $|D_8/< r^2>|=4$ and $|D_8/< r^2>$ is not cyclic so $|D_8/< r^2>\cong \mathbb{Z}/2\mathbb{Z}\oplus \mathbb{Z}/2\mathbb{Z}$.

35. Prove that $SL_n(F) \subseteq GL_n(F)$ and describe the isomorphism type of the quotient group.

Proof. (Hazlett) Let $\varphi: GL_n(F) \to F$ such that $\varphi(A) = \det(A)$. Note, $\varphi(A)\varphi(B) = \det(A)\det(B) = \det(AB) = \varphi(AB)$. Then φ is a homomorphism. Note, the kernel of φ is $SL_n(F)$. Consequently $SL_n(F)$ is normal in $GL_n(F)$.

Let $\psi: GL_n(F)/SL_n(F) \to F \setminus \{0\}$ such that $\psi(ASL_n(F)) = \det(A)$. We claim that $ASL_n(F)$ is the set of all things with determinant equal to $\det(A)$. Suppose we have a matrix B such that $\det(B) = \det(A)$. Then $\det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{1}{\det(A)}\det(A) = 1$. So $A^{-1}B \in SL_n(F)$ and $ASL_n(F) = BSL_n(F)$. Choose $C \in ASL_n(F)$. Then C = AS where $S \in SL_n(F)$. Hence $\det(C) = \det(AS) = \det(A)\det(S) = \det(A)$. Consequently $ASL_n(F)$ is the set of all matrixes in $GL_n(F)$ with the same determinant as A. We can conclude then that ψ is not only well defined but also an injection. Also, given $f \in F \setminus \{0\}$ the $n \times n$ matrix H with $h_{1,1} = f$, $h_{i,i} = 1$ for $1 \le i \le n$ and $1 \le n$ bijection. Finally, note that $1 \le n$ and $1 \le n$ but $1 \le n$ and $1 \le n$ but $1 \le n$ and $1 \le n$ but $1 \le n$ but 1

36. Prove that if G/Z(G) is cyclic then G is abelian.

Proof. (Gillispie) Suppose G/Z(G) is cyclic with generator xZ(G).

Proposition 3.2.4 shows us that the sets aZ(G) partition G and if we pick some $g \in G$, we know that $g \in gZ(G)$

- . We also know that there exists some $n \in \mathbb{N}$ s.t. $gZ(G) = x^n Z(G)$. Thus there exists some $z \in Z(G)$ so that $g = x^n z$
- . Now, pick $g, h \in G$. We showed that there are $m, n \in \mathbb{Z}$ and $z_1, z_2 \in Z(G)$ so that $g = x^n z_1$ and $h = x^m z_2$. Notice that since z_1 and z_2 commute with anything in G we have

$$gh = x^{n}z_{1}x^{m}z_{2}$$

$$= x^{n}x^{m}z_{1}z_{2}$$

$$= x^{m}x^{n}z_{2}z_{1}$$

$$= x^{m}z_{2}x^{n}z_{1}$$

$$= hg$$

And so G is abelian. \square

41. Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$ is a normal subgroup of G and that G/N is Abelian.

Proof (Granade). We claim that for all $g \in G$, gN = Ng. To see this, note that it suffices to show that for all $g \in G$ and $n \in N$, there exists n' such that gn = n'g. Thus, pick $x, y, g \in G$. By the definition

of N, $x^{-1}y^{-1}xy \in N$. Next, let $x' = gxg^{-1}$ and $y' = gyg^{-1}$, so that $x = g^{-1}x'g$ and $y = g^{-1}y'g$. Substituting, we get that:

$$gx^{-1}y^{-1}xy = g(g^{-1}x'^{-1}g)(g^{-1}y'^{-1}g)(g^{-1}x'g)(g^{-1}y'g)$$

$$= (gg^{-1})x'^{-1}(gg^{-1})y'^{-1}(gg^{-1})x'(gg^{-1})y'g$$

$$= x'^{-1}y'^{-1}x'y'g$$

But then, $x'^{-1}y'^{-1}x'y' \in N$ and so $x'^{-1}y'^{-1}x'y'g \in Ng$. Therefore, $gN \subseteq Ng$. Reversing the argument above gives that gN = Ng, as required.

Corollary. G/N is Abelian.

Proof (Granade). Let $aN, bN \in G/N$. Then, we claim that abN = baN. It is thus sufficient to show that $ab(ba)^{-1} \in N$. But then, $ab(ba)^{-1} = aba^{-1}b^{-1} \in N$.

42. Assume both H and K are normal subgroups of G with $H \cap K = \{1\}$. Prove that xy = yx for all $x \in H$ and $y \in K$.

Proof. (Baggett) Take any elements $x \in H$ and $y \in K$. Since H is normal, we have that $y^{-1}xy \in y^{-1}Hy = H$; since $x^{-1} \in H$ and H is closed under multiplication, $x^{-1}y^{-1}xy \in H$. Similarly, we have that $x^{-1}y^{-1}x \in x^{-1}Kx = K$; since $y \in K$ and K is closed under multiplication, $x^{-1}y^{-1}xy \in K$. Thus, $x^{-1}y^{-1}xy \in H \cap K$. However, $H \cap K = \{1\}$, so $x^{-1}y^{-1}xy = 1$. Equivalently, we have that xy = yx.