MATH 371 Name : SOLUTIONS
Quiz 6 December 8, 2008

Instructions: Round all answers to two significant digits (i.e. two decimal places, or if your answer is very small, give the first two non-zero digits following the decimal point). Also, you must show your work (integration, etc) and *use the method mentioned in the problem* to get full credit. There are 10 points on this quiz. Good luck.

1. (2 pts.) Suppose that for quality assurance at a large plant, a manager randomly samples 50 skis from a manufacturing line for testing each day. Let D denote the number of defectives in this random sample. An unknown proportion p of these skis are defective, and p varies from day to day. Suppose p has a uniform distribution on the interval $(0, .01)$.

   (a) Find the expected number of defectives observed among the 50 sample skis.

   $D \sim Binom(50, p)$ $\quad \mathbb{E}(D) = \mathbb{E}(\mathbb{E}(D|p)) = \mathbb{E}(50p) = 50\,\mathbb{E}(p) = 50\left(\frac{.01}{2}\right) = \boxed{.25}$

   (b) Find the variance of the number of defectives from among the 50 samples

   $Var(D) = Var(\mathbb{E}(D|p)) + \mathbb{E}(Var(D|p))$

   $= Var(50p) + \mathbb{E}(50p(1-p)) = 50^2 Var(p) + 50\,\mathbb{E}(p) - 50\,\mathbb{E}(p^2) = 50^2 \frac{(.01)^2}{12} + 50\frac{(.01)}{2} - \underbrace{50V(p) - 50\,\mathbb{E}}_{-50\,\mathbb{E}(p^2)}$

   $= 50^2\frac{(.01)^2}{12} + 50\frac{(.01)}{2} - 50\frac{(.01)^2}{12} - 50\left[\frac{(.01)}{2}\right]^2 \approx \boxed{.2692}$

2. (2 pts.) Let Y be a random variable with density function given by

$$f(y) = \begin{cases} \frac{3}{2}(1-y^2), & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

   Use the method of transformations to find the density function of $U = 1 - 3Y$. (A complete answer indicates the domain of the density function.)

   $u = h(Y) = 1 - 3Y$ decreasing

   $U = 1 - 3Y \Rightarrow h^{-1}(u) = \frac{1}{3}(1-u)$

   $\Rightarrow |h^{-1}(u)| = |-\frac{1}{3}| = \frac{1}{3}$

   $f_u(u) = f_y(h^{-1}(u))\,|\frac{dh^{-1}}{du}|$

   $0 \le y \le 1 \Rightarrow -2 \le u \le 1$

   $= \frac{3}{2}\left(1 - \left(\frac{1}{3}(1-u)\right)^2\right)\left[\frac{1}{3}\right]$

   $= \boxed{\frac{1}{2}\left(1 - \frac{1}{9}(1-u)^2\right) \quad -2 \le u \le 1}$

   $f_u(u)$

3. (3 pts.) Suppose that a gasoline supplier has daily sales X in thousands of gallons given by the density function

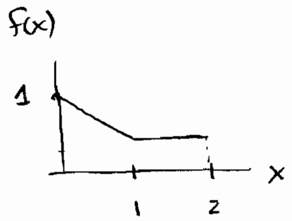
$$f(x) = \begin{cases} 1 - \frac{2}{3}x, & 0 \le x \le 1 \\ \frac{1}{3}, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

The daily profit in dollars is given by $U = 500X - 200$.

(a) Give a brief explanation of the formula for U in terms of profit.

The supplier makes \$500 per thousand of gals sold − \$200 overhead each day.

(b) Use the method of distribution functions to find the cumulative distribution function $F_U(u)$ for U, and then the probability density $f_u(u)$ for U. (A complete answer gives both the c.d.f. and the density function, and their domains.)



the density ↑ for X

If $U = 500X - 200$ and $0 \le X \le 2$, then $-200 \le U \le 800$

Note: $\left.\begin{array}{l} 0 \le X \le 1 \Rightarrow -200 \le U \le 300 \\ 1 \le X \le 2 \Rightarrow 300 \le U \le 800 \end{array}\right\}$ piecewise

Case I: $-200 \le u \le 300$

$F_u(u) = P(U \le u) = P(500X - 200 \le u) = P\left(X \le \frac{1}{500}(u+200)\right)$

$= \int_0^{\frac{1}{500}(u+200)} 1 - \frac{2}{3}x \, dx = x - \frac{x^2}{3}\Big|_0^{1/500(u+200)} = \frac{1}{500}(u+200) - \frac{1}{3}\left[\frac{1}{500}(u\ldots\right.$

Case II: $300 \le u \le 800$

$F_u(u) = F_u(300) + P(300 \le U \le u)$

$= \frac{2}{3} + P(300 \le 500X - 200 \le u)$

$= \frac{2}{3} + P\left(1 \le X \le \frac{1}{500}(u+200)\right)$

$= \frac{2}{3} + \int_1^{\frac{1}{500}(u+200)} \frac{1}{3} \, dx$

$= \frac{2}{3} + \frac{1}{3}\left(\frac{1}{500}(u+200) - 1\right)$

$= \frac{1}{3} + \frac{1}{1500}(u+200)$

Note: $F_u(300) = \frac{1}{500}(300+200) - \frac{1}{3}\left[\frac{1}{500}(300+200)\ldots\right.$

$= 1 - 1/3 = 2/3$

$$\therefore F_u(u) = \begin{cases} 0, & \text{if } u < -200 \\ \frac{1}{500}(u+200) - \frac{1}{3}\left[\frac{1}{500}(u+200)\right]^2, & \text{if } -200 \le u \le 3\ldots \\ \frac{1}{3} + \frac{1}{1500}(u+200), & \text{if } 300 < u \le 800 \\ 1, & \text{if } u > 800 \end{cases}$$

Phew!

$$\text{Finally, } f_u(u) = F_u'(u) = \begin{cases} \frac{1}{500} - \frac{1}{375000}(u+200) & \text{if } -200 \le u \le 3\ldots \\ \frac{1}{1500} & \text{if } 300 \le u \le 800 \\ 0 & \text{otherwise} \end{cases}$$

4. (3 pts.) Suppose in some molecular substance that a particle is emitted at a rate of $\lambda_1 = 20$ per hour, and a particle is absorbed at a rate of $\lambda_2 = 1$ per hour. Assume that the events *particle is emitted* and *particle is absorbed* are independent. If

$$Y_1 : \text{number of particles emitted in one hour,}$$
$$Y_2 : \text{number of particles absorbed in one hour,}$$

then define

$$U = Y_1 + Y_2.$$

(a) Give a brief explanation of the meaning of U.

# of particles emitted or absorbed in one hour

(b) Use the method of moment-generating functions to *prove* that U is Poisson distributed with mean $\lambda = 21$.

INDEPENDENCE!

$$m_U(t) = \mathbb{E}(e^{tU}) = \mathbb{E}(e^{t(Y_1+Y_2)}) = \mathbb{E}(e^{tY_1} \cdot e^{tY_2}) = \mathbb{E}(e^{tY_1})\mathbb{E}(e^{tY_2})$$

$$= m_{Y_1}(t)\, m_{Y_2}(t)$$

$$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

$$\Rightarrow \quad U \sim \text{Pois}(\lambda_1+\lambda_2) = \text{Pois}(21)$$

(c) Find the probability $P(U \geq 24)$.

$$= 1 - P(U \leq 23)$$

$$= 1 - .716 \approx \boxed{.284}$$

table p. 847 $\lambda = 21$, $a = 23$