MATH253X-UX1
Spring 2019

Midterm Exam 2

Name: Answer Key

Instructions. You have 120 minutes. Closed book, closed notes, no calculator. *Show all your work* in order to receive full credit.

1. Show that $\displaystyle\lim_{(x,y)\to(-2,1)} \frac{x+y+1}{xy+2}$ does not exist.

Solution: Setting $x = -2$ and letting $y \to 1$ to approach $(-2,1)$ along the line $(-2,y)$, we see $\displaystyle\lim_{y\to 1} \frac{y-1}{-2y+2} = -\frac{1}{2}$. Setting $y = 1$ and letting $x \to -2$ to approach $(-2,1)$ along the line $(x,1)$, we see $\displaystyle\lim_{x\to -2} \frac{x+2}{x+2} = 1$. Since these limits are different, $\boxed{\text{the original multivariable limit does not exist.}}$

2. Let $w = \dfrac{xy}{x-z}$.

(a) Verify that w satisfies the partial differential equation $xw_x + xw_z = yw_y$.

Solution: The first partial derivatives are:

$$w_x = \frac{y(x-z) - xy(1)}{(x-z)^2} = \frac{-yz}{(x-z)^2} \quad , \quad w_y = \frac{x}{x-z} \quad , \quad w_z = \frac{xy}{(x-z)^2}$$

And we have:

$$\boxed{xw_x + xw_z = \frac{-xyz}{(x-z)^2} + \frac{x^2y}{(x-z)^2} = \frac{xy(-z+x)}{(x-z^2)} = \frac{xy}{x-z} = y\frac{x}{x-z} = yw_y \quad \checkmark}$$

(b) Use the appropriate chain rule to find w_s for $(s,t) = (2,1)$ if $x = s^2t$, $y = t^2 - s$, $z = 3t$.

Solution: For $(s,t) = (2,1)$ we have $(x,y,z) = (2^2(1), 1^2 - 2, 3(1)) = (4,-1,3)$ and:

$$w_s = w_x x_s + w_y y_s + w_z z_s = \frac{-yz}{(x-z)^2}(2st) + \frac{x}{x-z}(-1) + \frac{xy}{(x-z)^2}(0)$$

$$\Rightarrow \quad w_s\Big|_{(s,t)=(2,1)} = \frac{-(-1)3}{(4-3)^2}(2(2)(1)) + \frac{4}{4-3}(-1) + 0 = 12 - 4 = \boxed{8}$$

3. Consider the surface $z = \frac{2}{3}x^{\frac{3}{2}} + 2y$ over the rectangular region $R = [1,4] \times [0,1]$.

(a) Compute the volume under the surface and over R.

Solution:

$$\begin{aligned} V &= \int_1^4 \int_0^1 \frac{2}{3}x^{\frac{3}{2}} + 2y \; dy \; dx = \int_1^4 \left[\frac{2}{3}x^{\frac{3}{2}}y + y^2\right]_0^1 dx \\ &= \int_1^4 \frac{2}{3}x^{\frac{3}{2}} + 1 \; dx = \left[\frac{2}{3}\left(\frac{2}{5}\right)x^{\frac{5}{2}} + x\right]_1^4 \\ &= \frac{4\left(2^5\right)}{15} + 4 - \frac{4}{15} - 1 = \frac{4(32-1)}{15} + 3 = \frac{124+45}{15} = \boxed{\frac{169}{15}} \end{aligned}$$

(b) Compute the surface area of $z = \frac{2}{3}x^{\frac{3}{2}} + 2y$ over the region R.

Solution: We have $z_x = \frac{2}{3}\left(\frac{3}{2}\right)x^{\frac{1}{2}} = \sqrt{x}$ and $z_y = 2$ so:

$$\begin{aligned} SA &= \int_1^4 \int_0^1 \sqrt{1 + z_x^2 + z_y^2}\, dy\, dx = \int_1^4 \int_0^1 \sqrt{1 + x + 4}\, dy\, dx \\ &= \int_1^4 \left[y\sqrt{x+5} \right]_0^1 dx = \int_1^4 \sqrt{x+5}\, dx \\ &= \left[\frac{2}{3}(x+5)^{\frac{3}{2}} \right]_1^4 = \frac{2}{3}(27 - 6\sqrt{6}) = \boxed{2(9 - 2\sqrt{3})} \end{aligned}$$

4. Find an equation of the tangent plane at $(2, 0, 1)$ to the surface

$$x^2 z - yz^2 + y^2 = 4.$$

Solution: Let $F(x, y, z) = x^2 z - yz^2 + y^2$. Then we find

$$\nabla F(x, y, z) = \langle 2xz, -z^2 + 2y, x^2 - 2yz \rangle,$$

so $\nabla F(2, 0, 1) = \langle 4, -1, 4 \rangle$. The tangent plane is thus given by

$$4(x - 2) - 1(y - 0) + 4(z - 1) = 0,$$

or

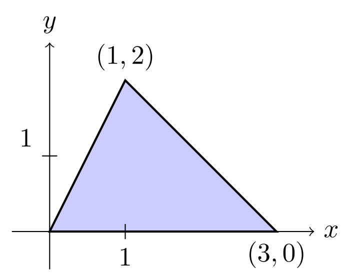
$$\boxed{4x - y + 4z = 12}.$$

5. Let $z = \ln(xy)$. Use the total differential to approximate Δz when moving from the point $(1, 2)$ to the point $(0.98, 2.1)$.

Solution: Since we're looking at values of $x, y > 0$ we can rewrite $z = \ln x + \ln y$ so:

$$\Delta z \approx dz = z_x dx + z_y dy = \frac{dx}{x} + \frac{dy}{y} = \frac{(0.98 - 1)}{1} + \frac{2.1 - 2}{2} = -0.02 + 0.05 = \boxed{0.03}$$

6. Assume a planar lamina has density $\rho = x$ and occupies the following region:



(a) Give two equivalent expressions for the mass of the lamina first setting up bounds and integrand in $dx\, dy$ then in $dy\, dx$. DO NOT evaluate.

Solution: The first line is $y = 2x$ (or $x = \frac{y}{2}$) and the other is $y - 0 = \frac{0-2}{3-1}(x - 3)$ that is $y = 3 - x$ (or $x = 3 - y$):

$$\boxed{m = \int_0^2 \int_{\frac{y}{2}}^{3-y} x\, dx\, dy = \int_0^1 \int_0^{2x} x\, dy\, dx + \int_1^3 \int_0^{3-x} x\, dy\, dx}$$

(b) Compute M_x the moment of mass with respect to the x-axis for the lamina.

Solution:

$$\begin{aligned}
M_x &= \int_0^2 \int_{\frac{y}{2}}^{3-y} xy \, dx \, dy = \int_0^2 \left[\frac{x^2 y}{2} \right]_{x=\frac{y}{2}}^{x=3-y} dy \\
&= \int_0^2 \frac{y(3-y)^2}{2} - \frac{y^3}{8} \, dy = \left| \begin{matrix} u = y & du = dy \\ dv = (3-y)^2 dy & v = -\frac{(3-y)^3}{3} \end{matrix} \right| \\
&= \frac{1}{2}\left(\left[-\frac{y(3-y)^3}{3} \right]_0^2 - \int_0^2 -\frac{(3-y)^3}{3} \, dy \right) - \left[\frac{y^4}{32} \right]_0^2 \\
&= \frac{1}{2}\left(-\frac{2}{3} + 0 - \left[\frac{(3-y)^4}{12} \right]_0^2 \right) - \frac{1}{2} + 0 = -\frac{1}{3} - \frac{1}{2}\left(\frac{1}{12} - \frac{81}{12} \right) - \frac{1}{2} \\
&= \frac{80}{24} - \frac{5}{6} = \frac{20}{6} - \frac{5}{6} = \frac{15}{6} = \boxed{\frac{5}{2}}
\end{aligned}$$

7. Find and classify all critical points of

$$f(x, y) = x^3 + xy^2 - 4xy + x + 1.$$

Solution: The gradient is

$$\nabla f = \langle f_x, f_y \rangle = \left\langle 3x^2 + y^2 - 4y + 1, 2xy - 4x \right\rangle$$

is defined everywhere and when setting it to the zero vector, we get $f_y = 0 = 2x(y-2)$ for:

- either $x = 0$ then plugging into $f_x = 0$ that means $y^2 - 4y + 1 = 0$ so we get $y = 2 \pm \sqrt{3}$
- or $y = 2$ then plugging into $f_x = 0$ that means $3x^2 - 3 = 0$ so $x = \pm 1$

Hence we found four critical points: $\boxed{(0, 2 \pm \sqrt{3}) \ , \ (\pm 1, 2)}$.

To classify them, we use the Second Partials Test:

$$f_{xx} = 6x \quad , \quad f_{yy} = 2x \quad , \quad f_{xy} = 2y - 4 \quad \Rightarrow \quad d(x, y) = 12x^2 - 4(y-2)^2$$

- $d(0, 2 \pm \sqrt{3}) = -4(3) < 0$ so $\boxed{\text{saddle points at } (0, 2 \pm \sqrt{3}, 1)}$;
- $d(1, 2) = 12 - 0 > 0$ and $f_{xx} = 6 > 0$ so $\boxed{\text{relative minimum at } (1, 2)}$;
- $d(-1, 2) = 12 - 0 > 0$ and $f_{xx} = -6 < 0$ so $\boxed{\text{relative maximum at } (-1, 2)}$.

8. Find the absolute minimum and maximum of

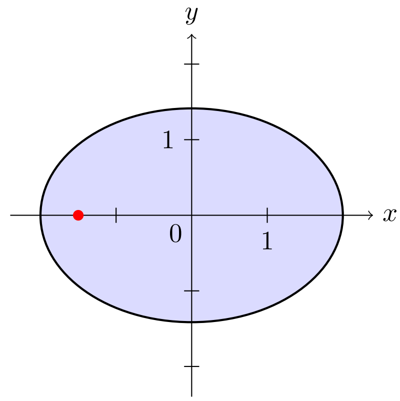
$$f(x, y) = x^2 - y^2 + 3x$$

in the region $x^2 + 2y^2 \leq 4$.

Solution: The absolute min/max will happen either at the critical point(s) if in the region or on the boundary. We have:

$$\nabla f = \langle 2x + 3, -2y \rangle = \overrightarrow{0} \quad \Longleftrightarrow \quad (x, y) = \left(-\frac{3}{2}, 0 \right)$$

Plug in the point into the inequality of the region to see if it satisfies it: $\frac{9}{4} + 2(0) = \frac{9}{4} \leq 4$ indeed. So the critical point is within the region. We can also sketch the region and the critical point:



Now for the boundary, we use Lagrange multipliers by defining the constraint as $g(x, y) = x^2 + 2y^2 = 4$:

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad \langle 2x + 3, -2y \rangle = \lambda \langle 2x, 4y \rangle \quad \Longrightarrow \quad \begin{cases} 2x + 3 = 2\lambda x \\ -2y = 4\lambda y \end{cases}$$

The second equation has two solutions:

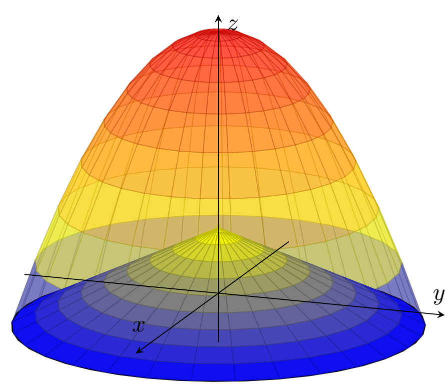
- either $y = 0$ then from the constraint $x^2 = 4$ so $x = \pm 2$;
- or $\lambda = -\frac{1}{2}$ then from the first equation $2x + 3 = -x$ so $x = -1$ which in turns when putting it into the constraint gives $1 + 2y^2 = 4$ so $y = \pm\sqrt{\frac{3}{2}}$

We now put all these points into a table and evaluate the function value for each:

x	y	$f(x, y)$	
$-\frac{3}{2}$	0	$-\frac{9}{4}$	
2	0	10	absolute maximum
-2	0	-2	
-1	$\pm\sqrt{\frac{3}{2}}$	$-\frac{7}{2}$	absolute minimum

9. Fully SET UP bounds and integrand in polar coordinates to represent the volume of the solid bounded by the cone $z = 2 - \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 8 - x^2 - y^2$. DO NOT evaluate.

Solution: Let's start with a picture:



The inverted cone $z = 2 - r$ (with $r \geq 0$) is below and the inverted paraboloid $z = 8 - r^2$ is above. The base or shadow R in the xy-plane is a disk with radius satisfying

$$2 - r = 8 - r^2 \quad \Longleftrightarrow r^2 - r - 6 = 0 \quad \Longleftrightarrow r = -2, 3$$

So here $r = 3$ and so the volume is:

$$\begin{aligned} V &= \iint_R (8 - x^2 - y^2) - (2 - \sqrt{x^2 + y^2})\, dA \\ &= \int_0^{2\pi} \int_0^3 \left[(8 - r^2) - (2 - r)\right]\, r\, dr\, d\theta \\ \Rightarrow &\boxed{V = \int_0^{2\pi} \int_0^2 6r + r^2 - r^3\, dr\, d\theta} \end{aligned}$$

10. Let

$$f(x, y) = x^2 y + \sin(\pi y).$$

(a) Find the directional derivative of f at $(1, -1/2)$ in the direction of $\langle -3, 4 \rangle$.

Solution: First compute the gradient:

$$\nabla f(x, y) = \left\langle 2xy, x^2 + \pi \cos(\pi y) \right\rangle.$$

Now the direction we consider is

$$\mathbf{u} = \frac{\langle -3, 4 \rangle}{||\langle -3, 4 \rangle||} = \frac{\langle -3, 4 \rangle}{\sqrt{9 + 16}} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle.$$

Therefore,

$$D_{\mathbf{u}} f(1, -1/2) = \nabla f(1, -1/2) \cdot \mathbf{u} = \langle -1, 1 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = \frac{3}{5} + \frac{4}{5} = \boxed{\frac{7}{5}}.$$

(b) What is the maximum rate of change of f at the point $(1, -1/2)$?

Solution:

$$\|\nabla f(1, -1/2)\| = \|\langle -1, 1 \rangle\| = \sqrt{1 + 1} = \boxed{\sqrt{2}}$$