MATH 490 Name : Solutions

Midterm March 21, 2013

Instructions: Show all work for full credit. Poor notation or sloppy work will be penalized. Point values as indicated.

1. (30 pts. – 6 pts. each) p-adic norms

   (a) Give examples of the following:

   i. A 7-adic integer $x \in \mathbb{Q} \setminus \mathbb{Z}$. Compute the p-adic norm $|x|_7$ for justification.

   $x = \frac{1}{2} \in \mathbb{Q} \setminus \mathbb{Z}$ with $\left|\frac{1}{2}\right|_7 = 7^{-0} = 1$ Since $\left|\frac{1}{2}\right|_7 \le 1$, $\frac{1}{2} \in \mathbb{Z}_7$

   ii. An element $y \in \mathbb{Q}_7 \setminus \mathbb{Q}$. Justify briefly.

   $y = \ldots 01111101111011101101_\wedge$

   is not eventually periodic to the left.

   (b) i. Two sequences (a_n) and (b_n) in $(\mathbb{Q}_7, |\cdot|_7)$ with $a_n \neq b_n$ that both converge to 5.

   $(a_n) = (5)$

   $(b_n) = (5 + 7^n)$

   ii. Prove that $(a_n) \sim (b_n)$ in $\mathbb{Q}_7$.

   $|a_n - b_n|_7 = |5 - (5 + 7^n)|_7 = |-7^n|_7 = |7^n|_7 = \frac{1}{7^n}$.

   Since as $n \to \infty$, $d(a_n, b_n) \to 0$, so $(a_n) \sim (b_n)$

   Since they differ by a null sequence.

   (c) By Proposition 1.15, we have "*If the elements* $x, a \in \mathbb{Q}_5$ *satisfy the inequality* $|x - a|_5 < |a|_5$, *then* $|x|_5 = |a|_5$. Give an example x, a illustrating this. (A complete answer shows you computed all necessary norms.)

   Eq. $x = \frac{8}{5}$ $a = \frac{3}{5}$ Then $\left|\frac{8}{5} - \frac{3}{5}\right|_5 = |1|_5 = 1$

   and $\left|\frac{3}{5}\right|_5 = 5$, so $|x - a|_5 \le 1 < 5 = |a|_5$.

   Finally, $\left|\frac{8}{5}\right|_5 = \left|\frac{3}{5}\right|_5 = 5$.

(b) (15 pts.) Let $p = 7$ and consider the quadratic equation $F(x) = x^2 + x + 2 = 0$.

i. Show that there exists some $a_0 \in \{0, 1, \dots 6\} \subset \mathbb{Z}_7$ such that $F(a_0) \equiv 0 \mod 7$.

Testing for a_0, we see $\boxed{a_0 = 3}$ is a root mod 7

Since $3^2 + 3 + 2 = 14 \equiv 0 \mod 7$

ii. Can a_0 be refined to find $a \in \mathbb{Z}_p$ with $F(a) = 0$? Explain. If so, find the first three terms in the 7-adic expansion of x, $x \equiv a_2 a_1 a_{0\wedge} \mod 7^3$.

Computing $F'(x) = 2x + 1$ and noting $F'(a_0) = F'(3) = 6 + 1 \equiv 0 \mod 7$

then Hensel's Lemma does not apply. (You can not solve for b_1.)

(c) (15 pts.) Let $p = 7$ and consider the equation $F(x) = x^2 - 2 = 0$ in $\mathbb{Z}_7$. Does there exist a root $x \in \mathbb{Z}_7$ to this equation? Explain. If so, find the first three terms in the 7-adic expansion of x, $x \equiv a_2 a_1 a_{0\wedge} \mod 7^3$.

Yes. Both 3 and 4 ($= -3 \mod 7$) satisfy $F(\bar{a}_0) \equiv 0 \mod 7$.

Also, $F'(x) = 2x$ and $F'(3) = 6 \not\equiv 0 \mod 7$. [Similarly, $F'(4) = 1 \not\equiv 0 \mod 7$.]

By Hensel's Lemma, a root $a \in \mathbb{Z}_7$ exists with $a \equiv 3$ (or 4) mod 7.

After some algebra, $a \equiv 213_\wedge \mod 7^3$ works.

[Similarly, $a \equiv 454_\wedge \mod 7^3$ works.]

5. (Extra Credit) Show that if $B = B(a, r) = \{x \in \mathbb{Q}_5 \mid |x - a|_5 < r\}$ is the open ball centered at a of radius $r > 0$ and $b \in B$, then $B = B(b, r)$.