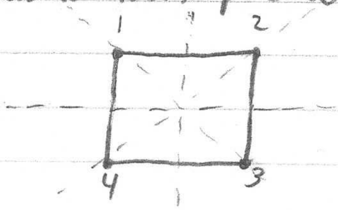
HW #4 Solutions

§8 #12. Find the orbit of 1 under the permutation defined prior to Exercise 1.

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

$$O_{1,\tau} = \{1\ 2\ 4\ 3\}$$

22. Write down eight matrices that form a group under matrix multiplication that is isomorphic to D_4



Consider the transformations that the elements of D_4 do to the column vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$R_{0°} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ "0° clockwise rotation" $R_{0°}$ corresponds with $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$R_{90°} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}$ "90° clockwise rotation" $R_{90°}$ corresponds with $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$R_{180°} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 2 \end{pmatrix}$ "180° clockwise rotation" $R_{180°}$ corresponds with $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$R_{270°} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ "270° clockwise rotation" $R_{270°}$ corresponds with $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$H \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}$ "horizontal reflection" H corresponds with $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$V\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}4\\3\\2\\1\end{pmatrix}$$ "vertical reflection" V corresponds with $\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$

$$D_{2,4}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}3\\2\\1\\4\end{pmatrix}$$ "reflection about diagonal 2,4" $D_{2,4}$ corresponds with $\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$

$$D_{1,3}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}1\\4\\3\\2\end{pmatrix}$$ "reflection about diagonal 1,3" $D_{1,3}$ corresponds with $\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}$

Let P be the group of permutation matrices corresponding to each of the above transformations on the square. Then $P \cong D_4$.

46. Show that S_n is a nonabelian group for $n \geq 3$.

Proof: Note that for $n \geq 3$, $(1\,2), (1\,2\,3) \in S_n$. Moreover, $(1\,2)(1\,2\,3) = (2\,3)$ but $(1\,2\,3)(1\,2) = (1\,3)$. Since $(1\,2)(1\,2\,3) \neq (1\,2\,3)(1\,2)$, S_n is nonabelian for $n \geq 3$. ■

48. Let $a, b \in A$ and $\sigma \in S_A$. Show that if $O_{a,\sigma}$ and $O_{b,\sigma}$ have an element in common, then $O_{a,\sigma} = O_{b,\sigma}$.

Proof: Define the relation $a \sim b$ if $\sigma^k(a) = b$ for some $k \in \mathbb{Z}$. We will show that $\sim$ is an equivalence relation.

1. Reflexivity: We have that $a \sim a$ since $a = \sigma^0(a) = \iota(a)$
2. Symmetry: Suppose that $a \sim b$. Then $b = \sigma^k(a)$ for some $k \in \mathbb{Z}$. Therefore, $\sigma^{-k}(b) = a$, so $b \sim a$.
3. Transitivity: Suppose that $a \sim b$ and $b \sim c$. Then $b = \sigma^m(a)$ and $c = \sigma^n(b)$ for some $m, n \in \mathbb{Z}$. Hence, $c = \sigma^n(\sigma^m(a)) = (\sigma^n \circ \sigma^m)(a) = \sigma^{n+m}(a)$. Thus, $a \sim c$.

We have established that $\sim$ is an equivalence relation. Moreover, the equivalence classes of A are precisely the orbits of A. Since equivalence classes form a partition of A, it follows that if $O_{a,\sigma} \cap O_{b,\sigma} \neq \emptyset$, then $O_{a,\sigma} = O_{b,\sigma}$. ■

49. If A is a set, then a subgroup H of S_A is transitive on A if for each $a, b \in A$ there exists $\sigma \in H$ such that $\sigma(a) = b$. Show that if A is a nonempty finite set, then there exists a finite cyclic subgroup H of S_A with $|H| = |A|$ that is transitive on A.

Proof: Let $A = \{a_1, a_2, \ldots, a_n\}$ and let $\sigma : A \to A$ be the permutation $\sigma(a_i) = a_{i +_n 1}$; that is, $\sigma = (a_1\ a_2 \ldots a_n)$. Let $H = \langle \sigma \rangle$. By definition, H is a cyclic subgroup of S_A. We will show that $|H| = |A|$ and that H is transitive on A. We have that $\sigma^n(a_i) = a_{i +_n n} = a_i$ for all $1 \le i \le n$. Therefore, $\sigma^n = \iota$, the identity permutation. Moreover, $k=n$ is the smallest positive integer such that $\sigma^k = \iota$. Thus, $|\sigma| = n$. However, $|\sigma| = |\langle \sigma \rangle| = |H|$ and $|A| = n$. Therefore, $|H| = |A|$. Now we will show that H is transitive on A. Let $a_i, a_j \in A$. We have that $\sigma^{j-i}(a_i) = a_{i +_n j-i} = a_j$. Thus, there exists $\sigma^{j-i} \in H$ such that $\sigma^{j-i}(a_i) = a_j$. Hence, H is transitive on A. Therefore, $H = \langle (a_1\ a_2 \ldots a_n) \rangle$ is a finite cyclic subgroup of S_A with $|H| = |A|$ that is transitive on A. ■

50. Show that for $\sigma \in S_A$, $\langle \sigma \rangle$ is transitive on A if and only if $O_{a,\sigma} = A$ for some $a \in A$.

Proof: Suppose that $\langle \sigma \rangle$ is transitive on A. Let $a \in A$. Since $\langle \sigma \rangle$ is transitive, for all $b \in A$, $\sigma^k(a) = b$ for some $k \in \mathbb{Z}$. Hence, for all $b \in A$, $b \in O_{a,\sigma}$. Thus, $O_{a,\sigma} = A$.

Conversely, suppose that $O_{a,\sigma} = A$ for some $a \in A$. We must show that for all $b, c \in A$ that $\sigma^k(b) = c$ for some $k \in \mathbb{Z}$. Let $b, c \in A$. Then $b, c \in O_{a,\sigma}$. Therefore, $\sigma^m(a) = b$ and $\sigma^n(a) = c$ for some $m, n \in \mathbb{Z}$. Hence, $a = \sigma^{-m}(b)$. Thus, $c = \sigma^n(\sigma^{-m}(b)) = (\sigma^n \circ \sigma^{-m})(b) = \sigma^{n-m}(b)$. Thus, there exists $\sigma^{n-m} \in \langle \sigma \rangle$ such that $\sigma^{n-m}(b) = c$. Hence, $\langle \sigma \rangle$ is transitive on A. ■

52. Let G be a group. Prove that the permutations $\rho_a : G \to G$, where $\rho_a(x) = xa$ for $a \in G$ and $x \in G$, do form a group isomorphic to G.

Proof: One can easily check that ρ_a is indeed a permutation. Let $P = \{\rho_a : G \to G \mid a \in G \text{ and } \rho_a(x) = xa\}$. Let $\phi : G \to S_G$ such that $\phi(x) = \rho_{x^{-1}}$. We will show that ϕ is one-to-one and a homomorphism. Suppose that $\phi(x) = \phi(y)$. Then $\rho_{x^{-1}} = \rho_{y^{-1}}$. In particular, this means that $\rho_{x^{-1}}(e) = \rho_{y^{-1}}(e)$. However, $\rho_{x^{-1}}(e) = ex^{-1} = x^{-1}$ and similarly $\rho_{y^{-1}}(e) = ey^{-1} = y^{-1}$. Thus, $x^{-1} = y^{-1}$. However, inverses are unique, so $x = y$. Therefore, ϕ is one-to-one. Next, consider $\phi(xy) = \rho_{(xy)^{-1}} = \rho_{y^{-1}x^{-1}}$ and $\phi(x)\phi(y) = \rho_{x^{-1}}\rho_{y^{-1}}$. We have that

$$\rho_{y^{-1}x^{-1}}(g) = gy^{-1}x^{-1} = (gy^{-1})x^{-1} = \rho_{y^{-1}}(g)x^{-1} = \rho_{x^{-1}}(\rho_{y^{-1}}(g))$$
$$= (\rho_{x^{-1}} \circ \rho_{y^{-1}})(g) = (\rho_{x^{-1}}\rho_{y^{-1}})(g). \text{ for all } g \in G.$$

Therefore, $\phi(xy) = \rho_{y^{-1}x^{-1}} = \rho_{x^{-1}}\rho_{y^{-1}} = \phi(x)\phi(y)$ and ϕ is a homomorphism. From Lemma 8.15, $G \cong \phi(G) = P$. ■

§9 #32. Let A be an infinite set. Let K be the set of all $\sigma \in S_A$ that move at most 50 elements of A. Is K a subgroup of S_A? Why?

K is not a subgroup of S_A. Let $\sigma_1 = (a_1\ a_2\ \dots\ a_{50})$ and $\sigma_2 = (a_{51}\ a_{52}\ \dots\ a_{100})$. Then $\sigma_1\sigma_2$ moves 100 elements of A, so $\sigma_1\sigma_2 \notin K$. Thus, K is not closed under permutation multiplication.

34. Show that if σ is a cycle of odd length, then σ^2 is a cycle.

Proof: Without loss of generality, let $\sigma = (1\ 2\ \dots\ 2n+1)$ for some $n \in \mathbb{Z}^+$. Then $\sigma(x) = x +_{2n+1} 1$ for $1 \le x \le 2n+1$ and $\sigma(x) = x$ for $x > 2n+1$. Now consider σ^2. We have that $\sigma^2(x) = \sigma(\sigma(x)) = \sigma(x +_{2n+1} 1) = (x +_{2n+1} 1) + 1 = x +_{2n+1} 2$ for $1 \le x \le 2n+1$ and $\sigma^2(x) = x$ for $x > 2n+1$. This means that $\sigma^2 = (1\ 3\ 5\ \dots\ 2n+1\ 2\ 4\ \dots\ 2n)$. Thus, σ^2 is a cycle. ■