MATH 371
Exam 1

Name : SOLUTIONS

October 25, 2019

Instructions: Give numerical answers unless instructed that a formula alone suffices. You may consult tables on the inside cover of your textbook and in the appendices, but indicate in your solution that you have done so. A 'dumb' calculator may be used for routine arithmetic, but nothing else. Bald answers will receive very limited, if any, credit; that is, show formulas before performing your computations. Good luck.

1. (16 pts. – 4 pts. each) An unfair coin has probability $p = p(H) = .4$ of coming up H = Heads.

   (a) i. Give the probability that the first H is flipped on the third coin toss. Give your answer to three decimal places.

   $(.6)^2(.4) = \boxed{.144}$ $\qquad G \sim \text{Geom}(.4)$

   ii. On which coin toss (first, second, etc.) do you expect to see the first H come up? That is, give the expected value for the random variable in the last problem. Round

   $E(G) = \frac{1}{p} = \frac{1}{.4} = \boxed{2.5}$

   (b) i. Give the probability that the second H is flipped on the fifth coin toss. Give your answer to three decimal places.

   $N \sim$ Negative Binomial $\quad r=2 \quad p = .4$

   $P(N=5) = \binom{4}{1}(.4)^2(.6)^3 \approx \boxed{.138}$

   ii. Give the variance for the random variable that computes the flip number on which the second H is observed. Give your answer to one decimal place.

   $Var(N) = \frac{r(1-p)}{p^2} = \frac{2(.6)}{.4^2} = \boxed{7.5}$

2. (5 pts.) A random variable X has the moment generating function $m(t) = \frac{.9e^t}{1 - .1e^t}$. What is the distribution of X?

   $X \sim \text{Geom}(.9)$

3. (~~32~~ 5 pts. – ~~4~~ 5 pts. each) Answer the following, showing work.

   (a) In a small class of 16 students, exactly 10 favor recalling Governor Dunleavy. Give a formula for the probability that exactly four students in a random sample of size 6 favor recalling Governor Dunleavy.

   $H \sim$ Hypergeometric $\quad N = 16, \; n = 6, \quad r = 10$

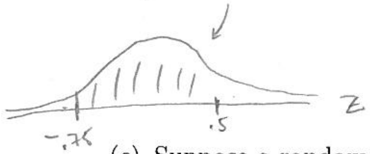
   $$P(H=4) = \frac{\binom{10}{4}\binom{6}{2}}{\binom{16}{6}}$$

(b) Under standard use, studded tires have an average lifetime of 5 seasons. Assuming that the length of life Y of a studded tire is approximately normally distributed with mean 5 seasons and standard deviation 2, find the probability that the studded tire will last between 3.5 and 6 seasons of use Give your answer to four decimal places.

$Y \sim Norm(5, 4)$ $\leftarrow \sigma^2$ $\qquad P(3.5 \le Y \le 6) = P\left(\frac{3.5-5}{2} \le Z \le \frac{6-5}{2}\right)$

$= P(-.75 \le Y \le .5) = 1 - P(Z \ge .5) + P(Z \ge .75) = 1 - .3085 - .2266 \approx .4649$

$\uparrow$ Table



(c) Suppose a random variable X is binomially distributed with $X \sim$ Binom(1000, .0068).

i. Give a formula for the probability that X is at least 12, $P(X \ge 12)$.

$P(X \ge 12) = \sum_{x=12}^{1000} \binom{1000}{x} (.0068)^x (.9932)^{1000-x} = 1 - P(X \le 11)$

ii. Use the Poisson approximation to the binomial distribution to estimate the probability $P(X \ge 12)$, rounding your answer to three decimal places.

$np = 1000(.0068) = 6.8 = \lambda$ ✓ Poisson! $\qquad Y \sim Pois(6.8)$

$P(X \ge 12) = 1 - P(X \le 11) \approx 1 - P(Y \le 11) = 1 - .955 \approx .045$

$\uparrow$ Table

(d) The proportion K of required material that a student knows for an exam is modeled with a Beta-distributed random variable with parameters $\alpha = 8$, $\beta = 2$, $K \sim$ Beta(8, 2).

i. What is the expected value of K? Explain informally what this means about anticipated student knowledge of material on an exam.

$E(K) = \frac{\alpha}{\alpha+\beta} = \frac{8}{8+2} = \boxed{.8}$

ii. Explicitly give the density function $f(k)$ for K. Your answer should contain only numerical constants and the variable k (i.e. no symbolic parameters), and carefully indicate the support of $f(k)$ (the subset of real numbers where $f(k) > 0$).

$f(k) = \frac{\Gamma(10)}{\Gamma(8)\Gamma(2)} k^7 (1-k) = \frac{9!}{7!\,1!} k^7 (1-k) = 72\, k^7 (1-k)$ for $0 \le k \le 1$

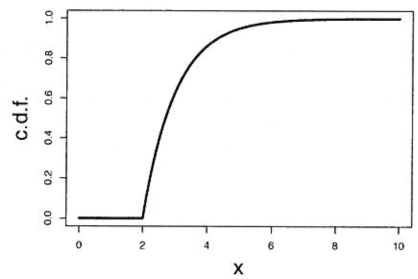
iii. Give the probability that a student knows at least 90% of the required material. Give full details of your computation, only using your calculator for the final evaluation. Round to three decimal places.

$P(K \ge .9) = \int_{.9}^{1} 72\, y^7 (1-y)\, dy = 72\left[\int_{.9}^{1} y^7 - y^8\, dy\right] = 72\left[\frac{1}{8}y^8 - \frac{1}{9}y^9\right]\Big|_{.9}^{1}$

$= 1 - 72\left(\frac{(.9)^8}{8} - \frac{(.9)^9}{9}\right) \approx .225$

4. (15 pts.) Below the graph of the c.d.f (the distribution function) $F(x)$ of a random variable X and its equation are given.

$$F(x) = \begin{cases} 0, & \text{if } x < 2 \\ 1 - e^{-(x-2)}, & \text{if } x \geq 2. \end{cases}$$



(a) (2 pts.) Is X a discrete or continuous random variable? Why?

Continuous since its c.d.f is continuous

(b) (4 pts.) Find the probability that X is greater than or equal to 4, $P(X \geq 4)$. Give a formula for the value of the probability, then round your answer to three decimal places.

$P(X \geq 4) = 1 - P(X \leq 4) = 1 - F(4) = 1 - (1 - e^{-(4-2)}) = e^{-2} \approx \boxed{.135}$

(c) (4 pts.) Give a formula for the density function $f(x)$ for X, making sure that the domain of $f(x)$ is defined on the entire real number line.

$f(x) = F'(x) = \begin{cases} e^{-(x-2)} & x \geq 2 \\ 0 & \text{otherwise} \end{cases}$

(d) (5 pts.) Find the expected value $\mathbb{E}(X)$. *Hint:* There is an easy (and fast) way to do this, thinking about linear transformations or shifts of random variables.

If $Y \sim \text{Exp}(1)$, then $X = Y + 2$. Thus, $\mathbb{E}(X) = \mathbb{E}(Y+2) = \mathbb{E}(Y) + 2$

shift 2 units

$= 1 + 2 = \boxed{3}$

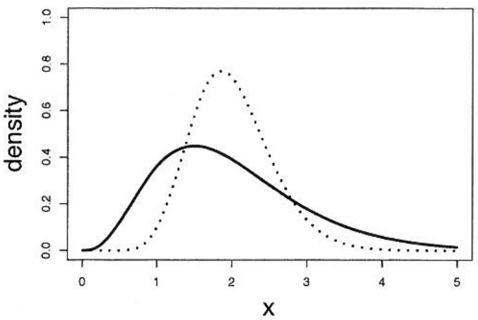
Otherwise, integrate by parts.

5. (8 pts.) Use the moment generating function $m(t)$ for a binomial random variable $B \sim \text{Binom}(10, .8)$ to show that the expected value is $\mathbb{E}(B) = 8$.

$m(t) = (.8e^t + .2)^{10}$ Thus, $m'(t) = 10(.8e^t + .2)^9 (.8) = 8(.8e^t + .2)^9$

and $m'(t)\big|_{t=0} = 8(.8e^0 + .2)^9 = \boxed{8}$ $\square$

6. (10 pts.) Below are two graphs (solid, dotted) of density functions for distinct Gamma-distributed random variables X_{solid} and X_{dotted}, both with mean equal to 2.



$\Gamma(.2, 10)$ $\Gamma(.4, 5)$

$\Gamma(1, 1)$ $\Gamma(1, 2)$

$\Gamma(3, 1/6)$ $\Gamma(4, 1/2)$ $\Gamma(14, 1/7)$

$\mu = 1/2$ No. $\mu = 2$ $\sigma^2 = 1$ $\mu = 2$, $\sigma^2 = 2/7 \approx .29$

Listed on the right are the values of possible parameter choices for X_{solid} and X_{dotted}, two of which are correct. Identify the correct parameter values for the random variables, and explain how you determined this.

$X_{solid} \sim$ $\Gamma(4, 1/2)$ because .... $X_{dotted} \sim$ $\Gamma(14, 1/7)$ because ....

Because of the shape of the plots, necessarily $\alpha > 1$. This restricts attention to those in row 3 above. However, $\Gamma(3, 1/6)$ does not have mean $\mu = 2$. Thus, the plots must be for $\Gamma(4, 1/2)$ and $\Gamma(14, 1/7)$. Of these, the variances are $\sigma^2 = 1, 2/7$ respectively. Since $f(x_{dotted})$ shows less spread (i.e. σ^2 smaller), $X_{dotted} \sim \Gamma(14, 1/7)$ and $X_{solid} \sim \Gamma(4, 1/2)$

7. (14 pts.) Consider a uniform random variable $U \sim \text{Unif}(10, 30)$.

(a) (3 pts.) Give the mean, variance, and standard deviation of U. Give the exact values for the mean and variance, and round the standard deviation to two decimal places.

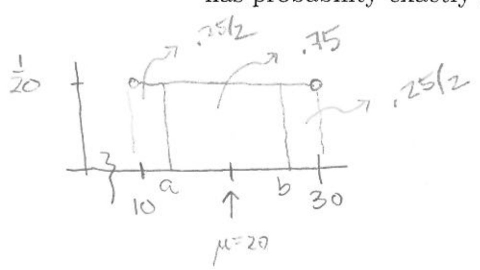
$\mu =$ 20 $\sigma^2 = \frac{20^2}{12} = \frac{100}{3}$ $\sigma = \sqrt{\frac{100}{3}} \approx 5.77$

(b) (5 pts.) Use Tschebysheff's Theorem to give an interval that contains the mean μ with probability at least .75. Round answers to one decimal place.

$P(|U - 20| < k(5.77)) \geq 1 - 1/k^2 = .75 \Rightarrow k = 2$. Thus, the interval is

$|Y - 20| < 2(5.77) \approx 11.5$ $20 \pm 11.5 = (8.5, 31.5)$ or in truth $(10, 30)$

(c) (5 pts.) Use your knowledge of the uniform distribution to find a (symmetric) interval about the mean μ that has probability exactly .75.



We need a, b so that $P(10 \leq u \leq a) = .125 = P(b \leq u \leq 30)$

If $\Delta u = 1$ unit, then $f(x)\Delta u = (1/20)(1) = .05$. Thus, for any interval of width $\Delta u = 2.5$ we have $f(x)\Delta x = (\frac{1}{20})(2.5) = .125$

Take $a = 10 + 2.5 = 12.5$ and $b = 30 - 2.5 = 27.5$

$\boxed{(12.5, 27.5)}$

(d) (1 pt.) How good is Tschebysheff's approximation for U? (Give a one word answer.) Horrible.