MATH253
Fall 2019

Midterm Exam 1

Name: Answer Key

Instructions. You have 90 minutes. Closed book, closed notes, and no calculators allowed. *Show all your work* in order to receive full credit.

1. Consider points $A(4, -3, 2)$ and $B(2, 1, c)$ and vectors $\mathbf{u} = \langle 1, -2, 3\rangle$ and $\mathbf{v} = \langle -1, -1, 2\rangle$.

(a) Find the vector projection of $\mathbf{u}$ along $\mathbf{v}$.

Solution:

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\langle 1,-2,3\rangle\cdot\langle -1,-1,2\rangle}{1+1+4}\langle -1,-1,2\rangle = \frac{1(-1)-2(-1)+3(2)}{6}\langle -1,-1,2\rangle \\ &= \frac{-1+2+6}{6}\langle -1,-1,2\rangle = \frac{7}{6}\langle -1,-1,2\rangle = \boxed{\left\langle \frac{-7}{6}, \frac{-7}{6}, \frac{7}{3}\right\rangle}\end{aligned}$$

(b) Find the area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$.

Solution: The area of the parallelogram is $\|\mathbf{u}\times\mathbf{v}\|$. So we have:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -1 & -1 & 2 \end{vmatrix} = \langle -2(2)+1(3), -(1(2)+1(3)), 1(-1)+1(-2)\rangle = \langle -1, -5, -3\rangle$$

$$\Rightarrow \quad A = \|\mathbf{u}\times\mathbf{v}\| = \sqrt{1+25+9} = \boxed{\sqrt{35}}.$$

(c) Find all values of c such that the length of $\overrightarrow{AB}$ equals 5.

Solution: We have

$$\overrightarrow{AB} = \langle -2, 4, c-2\rangle.$$

So,

$$\begin{aligned}\left\|\overrightarrow{AB}\right\| = 5 \quad &\Longleftrightarrow \quad \sqrt{4+16+(c-2)^2} = 5 \quad \Longleftrightarrow \quad 20 + (c-2)^2 = 25 \\ &\Longleftrightarrow \quad (c-2)^2 = 5 \quad \Longleftrightarrow \quad \boxed{c = 2 \pm \sqrt{5}}.\end{aligned}$$

(d) Find all values of c such that $\overrightarrow{AB}$ is parallel to $\mathbf{u}$.

Solution: $\overrightarrow{AB}$ is parallel to $\mathbf{u}$ if there exists a real nonzero number k such that:

$$\vec{AB} = k\mathbf{u} \quad \Longleftrightarrow \quad \begin{cases} -2 = k(1) \\ 4 = k(-2) \\ c-2 = k(3) \end{cases}$$

From the first two equations, we get $k = -2$ and plugging it into the third, we have:

$$c - 2 = (-2)(3) \quad \Longleftrightarrow \quad \boxed{c = -4}.$$

(e) Find all values of c such that $\overrightarrow{AB}$ is orthogonal to $\mathbf{v}$.

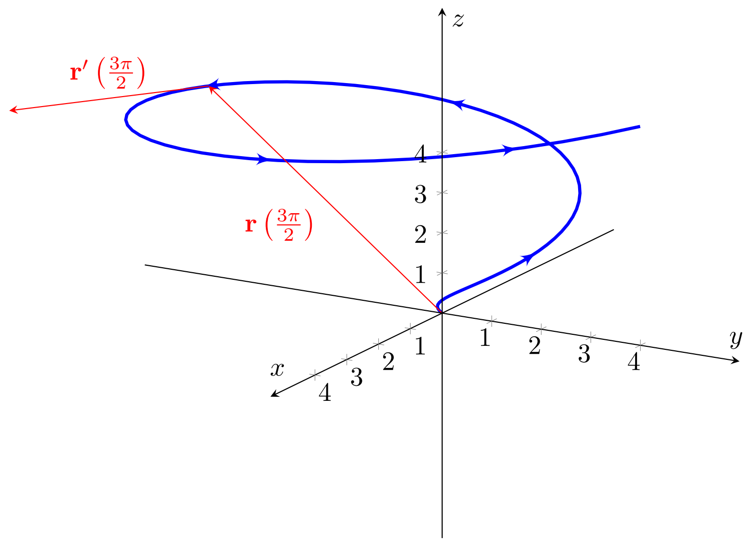
Solution: The vectors are orthogonal if their dot product is zero. So we have:

$$\begin{aligned}\overrightarrow{AB}\cdot\mathbf{v} = 0 \quad &\Longleftrightarrow \quad \langle -2, 4, c-2\rangle\cdot\langle -1,-1,2\rangle = 0 \quad \Longleftrightarrow \quad -2(-1)+4(-1)+2(c-2) = 0 \\ &\Longleftrightarrow \quad 2 - 4 + 2c - 4 = 0 \quad \Longleftrightarrow \quad 2c = 6 \quad \Longleftrightarrow \quad \boxed{c = 3}.\end{aligned}$$

2. Below is a sketch of the space curve:

$$\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle \quad , \quad 0 \le t \le \frac{7\pi}{3}.$$

Solution:



(a) Draw on the above the position and velocity vectors for $t = \dfrac{3\pi}{2}$.

Solution: We need to compute both $\mathbf{r}\left(\dfrac{3\pi}{2}\right)$ and $\mathbf{r}'\left(\dfrac{3\pi}{2}\right)$ then draw the first in standard position and the second starting at the tip of the first.

$$\begin{aligned} \mathbf{r}\left(\frac{3\pi}{2}\right) &= \left\langle 0, -\frac{3\pi}{2}, \frac{3\pi}{2}\right\rangle \\ \mathbf{r}'(t) &= \langle \cos t - t\sin t, \sin t + t\cos t, 1\rangle \\ \Rightarrow \quad \mathbf{r}'\left(\frac{3\pi}{2}\right) &= \left\langle \frac{3\pi}{2}, -1, 1\right\rangle \end{aligned}$$

(b) Find the speed at time t and simplify your result.

Solution:

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1} \\ &= \sqrt{\cos^2 t - \cancel{2\cos t\sin t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\cos t\sin t} + t^2\cos^2 t + 1} \\ &= \sqrt{1 + t^2(\sin^2 t + \cos^2 t) + 1} = \boxed{\sqrt{t^2+2}} \end{aligned}$$

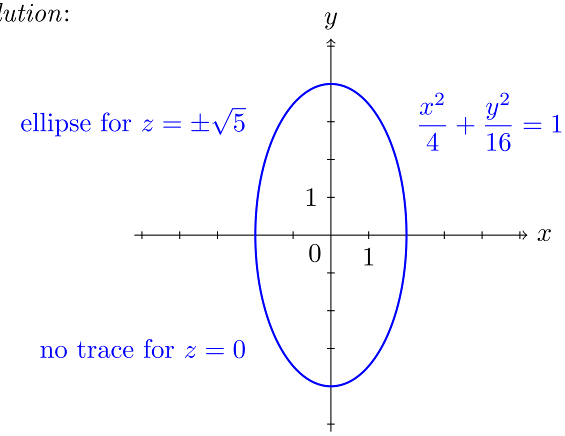
(c) At what time(s) is the acceleration horizontal (i.e. normal to $\mathbf{k}$)?

Solution: We need to solve for t in $\mathbf{r}''(t)\cdot\mathbf{k} = 0$ but any dot product with $\mathbf{k} = \langle 0,0,1\rangle$ only leaves you with the z-component of the vector. Here, since the z-component in $\mathbf{r}'(t)$ is constant, then the z-component of $\mathbf{r}''(t)$ is zero for all t. So $\boxed{\text{the acceleration is horizontal for all times } t}$.

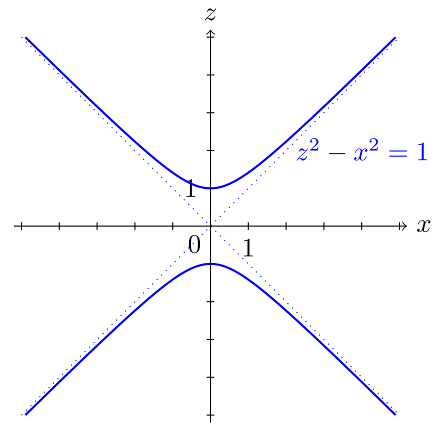
3. Time to sketch some surfaces!

(a) For $x^2 + \dfrac{y^2}{4} - z^2 = -1$, sketch the given traces, then the surface in 3D.

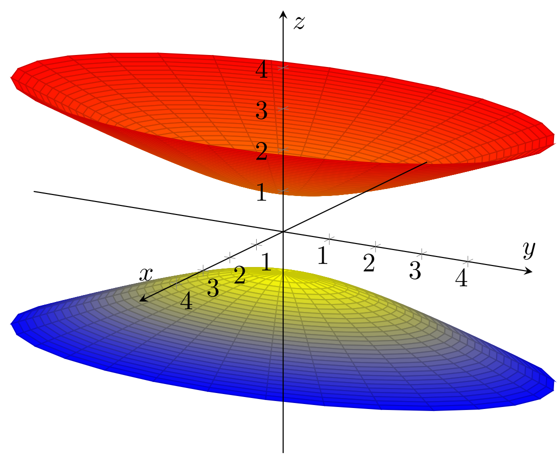
Solution:



traces: $z = 0, \pm\sqrt{5}$

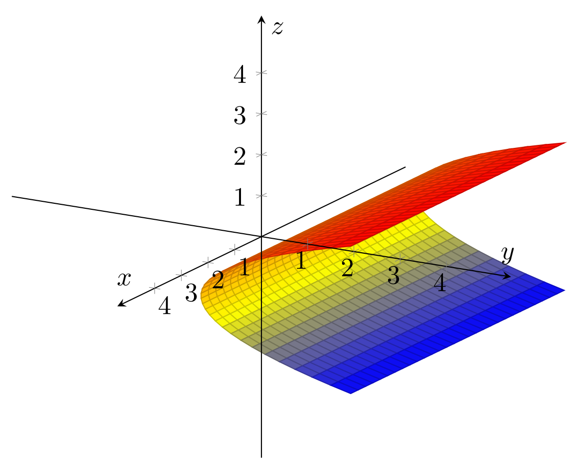


trace: $y = 0$



(b) Sketch the surface $y = z^2 + 1$.

Solution:



4. Consider the following point, line, and plane:

$$A = (3, -2, 5),$$

$$\vec{\ell}(t) = \langle 1 - 2t, t, 3 + 4t \rangle,$$

$$P: \quad 2x - 3y + z = -4,$$

(a) Give the equation of a plane parallel to the plane P that passes through A.

Solution: The plane will have the same normal as P, i.e. $\mathbf{n} = \langle 2, -3, 1 \rangle$ and so using point A, we have:

$$2(x-3) - 3(y+2) + (z-5) = 0 \quad \Longleftrightarrow \quad \boxed{2x - 3y + z = 17}.$$

(b) Find the point of intersection of the line $\vec{\ell}(t)$ and the plane P.

Solution: Plug in the coordinates of the line $\vec{\ell}(t)$ into the plane and solve for t:

$$\begin{aligned} 2(1-t) - 3(t) + (3+4t) = -4 \quad &\Longleftrightarrow \quad 2 - \cancel{4t} - 3t + 3 + \cancel{4t} = -4 \\ &\Longleftrightarrow \quad -3t = -9 \quad \Longleftrightarrow \quad t = 3 \end{aligned}$$

So the position vector for the point is $\vec{\ell}(3)$ and thus in coordinate notation the point is $\boxed{(-5, 3, 15)}$.

(c) Find the angle the line $\vec{\ell}(t)$ makes with the normal to the plane P. (Your answer may involve an inverse trigonometric function.)

Solution: We need the consider the angle between the direction vector of the line: $\mathbf{u} = \langle -2, 1, 4 \rangle$ and the normal $\mathbf{n} = \langle 2, -3, 1 \rangle$ to the plane. We have:

$$\begin{aligned} \cos\theta &= \frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{u}\| \, \|\mathbf{n}\|} = \frac{\langle -2, 1, 4 \rangle \cdot \langle 2, -3, 1 \rangle}{\sqrt{4+1+16}\sqrt{4+9+1}} = \frac{-2(2) + 1(-3) + 4(1)}{\sqrt{21}\sqrt{14}} \\ &= \frac{-\cancel{4} - 3 + \cancel{4}}{7\sqrt{3}\sqrt{2}} = \frac{-3}{7\sqrt{6}} = -\frac{3\sqrt{6}}{7(6)} = -\frac{\sqrt{6}}{14} \quad \Longrightarrow \quad \boxed{\theta = \arccos\left(-\frac{\sqrt{6}}{14}\right)} \end{aligned}$$

(d) Find an equation for the plane containing the point A and the line $\vec{\ell}(t)$.

Solution: We need two nonparallel vectors in the plane to cross. We already have $\mathbf{u} = \langle -2, 1, 4 \rangle$ from the line, so now we pick a point $B(1, 0, 3)$ from the line to form $\overrightarrow{AB} = \langle -2, 2, -2 \rangle = -2 \langle 1, -1, 1 \rangle$. So a normal vector to the plane is:

$$\overrightarrow{AB} \times \mathbf{u} = -2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 1 & 4 \end{vmatrix} = -2 \langle -4 - 1, -(4+2), 1 - 2 \rangle = -2 \langle -5, -6, -1 \rangle$$

and taking the scalar multiple $\langle 5, 6, 1 \rangle$, we have the equation of the plane as:

$$\boxed{5(x-3) + 6(y+2) + z - 5 = 0} \quad \text{or} \quad \boxed{5x + 6y + z = 8}.$$

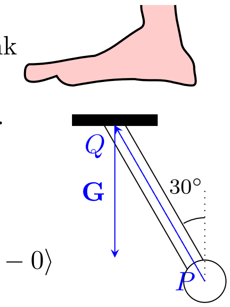
(e) Find the distance from the point A to the plane P.

Solution: Rewriting P as $2x - 3y + z + 4 = 0$, we have:

$$d = \frac{|2(3) - 3(-2) + 5 + 4|}{\sqrt{4+9+1}} = \frac{|6 + 6 + 5 + 4|}{\sqrt{14}} = \frac{21}{\sqrt{14}} = \frac{21\sqrt{14}}{14} = \boxed{\frac{3\sqrt{14}}{2}}$$

5. A bicycle pedal is attached to a 17 cm crank. When the crank is at an angle of 30° with the vertical (as shown) a foot applies a downward force of 200 N.

(a) What is the resulting torque? Give your answer as a vector.



Solution: Set up the force as $\mathbf{G} = \langle 0, -200\rangle = -200\langle 0, 1\rangle$ and then along the crank

$$\overrightarrow{PQ} = \langle 0.17\cos(120°), 0.17\sin(120°)\rangle = 0.17\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \frac{0.17}{2}\left\langle -1, \sqrt{3}\right\rangle.$$

Then add a zero $\mathbf{k}$-component to both to take the cross product for the torque:

$$\begin{aligned}\overrightarrow{\tau} = \overrightarrow{PQ} \times \mathbf{G} &= \frac{-200(0.17)}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & \sqrt{3} & 0 \\ 0 & 1 & 0 \end{vmatrix} = -100(0.17)\langle 0, 0, -1-0\rangle \\ &= -17\langle 0, 0, -1\rangle = \boxed{17\mathbf{k} = \langle 0, 0, 17\rangle}.\end{aligned}$$

(b) What is the magnitude of the torque? Indicate units.
Solution:

$$\|\overrightarrow{\tau}\| = \sqrt{0 + 0 + 17^2} = \boxed{17 \text{ Nm}}$$

(c) What is the direction of the torque vector? (Into the page $\otimes$, or out of the page $\odot$, in the figure).
Solution: By the right hand rule, the torque is coming $\boxed{\text{out of the page}}$, i.e. $\odot$.

6. An object moves in the plane with acceleration

$$\mathbf{a}(t) = \left\langle \frac{1}{t^2}, \frac{t}{(1+t^2)^2}\right\rangle.$$

At time $t = 1$ it is located at the point $(1, 0)$ and has velocity $\langle 2, 1\rangle$. Find a function $\mathbf{r}(t)$ giving its position at all times $t > 0$.
Solution: We start integrating, first the acceleration to get the velocity for all times $t > 0$:

$$\begin{aligned}
& & \mathbf{a}(t) &= \left\langle \frac{1}{t^2}, \frac{t}{(1+t^2)^2}\right\rangle \\
&\implies & \mathbf{v}(t) - \mathbf{v}(1) &= \int_1^t \left\langle \frac{1}{u^2}, \frac{u}{(1+u^2)^2}\right\rangle du = \left\langle -\frac{1}{u}, -\frac{1}{2(1+u^2)}\right\rangle\Big|_{u=1}^{u=t} \\
& & &= \left\langle -\frac{1}{t}, -\frac{1}{2(1+t^2)}\right\rangle - \left\langle -1, -\frac{1}{4}\right\rangle = \left\langle -\frac{1}{t}, -\frac{1}{2(1+t^2)}\right\rangle + \left\langle 1, \frac{1}{4}\right\rangle \\
&\iff & \mathbf{v}(t) &= \left\langle -\frac{1}{t}, -\frac{1}{2(1+t^2)}\right\rangle + \left\langle 1, \frac{1}{4}\right\rangle + \langle 2, 1\rangle = \left\langle 3 - \frac{1}{t}, \frac{5}{4} - \frac{1}{2(1+t^2)}\right\rangle \\
&\implies & \mathbf{r}(t) - \mathbf{r}(1) &= \int_1^t \left\langle 3 - \frac{1}{u}, \frac{5}{4} - \frac{1}{2(1+u^2)}\right\rangle du = \left\langle 3u - \ln|u|, \frac{5u}{4} - \frac{1}{2}\arctan u\right\rangle\Big|_{u=1}^{u=t} \\
& & &= \left\langle 3t - \ln t, \frac{5t}{4} - \frac{1}{2}\arctan t\right\rangle - \left\langle 3 - 0, \frac{5}{4} - \frac{1}{2}\left(\frac{\pi}{4}\right)\right\rangle \\
&\iff & \mathbf{r}(t) &= \left\langle 3t - \ln t, \frac{5t}{4} - \frac{1}{2}\arctan t\right\rangle - \left\langle 3, \frac{5}{4} - \frac{\pi}{8}\right\rangle + \langle 1, 0\rangle \\
&\implies & & \boxed{\mathbf{r}(t) = \left\langle 3t - \ln t - 2, \frac{5}{4}(t-1) - \frac{1}{2}\arctan t + \frac{\pi}{8}\right\rangle}
\end{aligned}$$

7. A particle moves with *velocity* $\mathbf{v}(t) = \langle t^2, 2t, 2\rangle$.

(a) Find the distance the particle travels between times $t = 1$ and 2.

Solution: The distance is the integral of the speed over time and since the speed is:

$$\|\mathbf{v}(t)\| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = \left|t^2 + 2\right| = t^2 + 2$$

then the distance traveled is:

$$\begin{aligned} L &= \int_1^2 \|\mathbf{v}(t)\| \; dt = \int_1^2 t^2 + 2 \; dt \\ &= \left[\frac{t^3}{3} + 2t\right]_1^2 = \frac{8}{3} + 4 - \left(\frac{1}{3} + 2\right) = \frac{7}{3} + 2 = \boxed{\frac{13}{3}} \end{aligned}$$

(b) Calculate the curvature of the trajectory at time $t = 1$.

Solution: The acceleration at time t is $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2t, 2, 0\rangle$ and so plugging in at $t = 1$, we have:

$$\mathbf{v}(1) = \langle 1, 2, 2\rangle \quad , \quad \mathbf{a}(1) = \langle 2, 2, 0\rangle = 2\langle 1, 1, 0\rangle \quad , \quad \|\mathbf{v}(1)\| = t^2 + 1\Big|_{t=1} = 3$$

and the cross product is:

$$\mathbf{v}(1) \times \mathbf{a}(1) = 2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 2\langle 0 - 2, -(0-2), 1 - 2\rangle = 2\langle -2, 2, -1\rangle .$$

Therefore, the curvature at $t = 1$ is

$$\kappa(1) = \frac{\|\mathbf{v}(1) \times \mathbf{a}(1)\|}{\|\mathbf{v}(1)\|^3} = \frac{2\sqrt{4+4+1}}{3^3} = \frac{2(3)}{27} = \boxed{\frac{2}{9}}.$$

(c) Find the unit tangent vector $\mathbf{T}(t)$ and the tangential component of acceleration $a_{\mathbf{T}}$ at $t = 1$.

Solution: The unit tangent vector is the normalized velocity vector:

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle t^2, 2t, 2\rangle}{t^2+2} = \left\langle \frac{t^2}{t^2+2}, \frac{2t}{t^2+2}, \frac{2}{t^2+2}\right\rangle \quad \Rightarrow \quad \boxed{\mathbf{T}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle}$$

and the tangential component of acceleration is:

$$a_{\mathbf{T}}(t) = \|\mathbf{v}(t)\|' = (t^2+2)' = 2t \quad \Rightarrow \quad \boxed{a_{\mathbf{T}}(1) = 2}.$$

We could also have used the formula:

$$a_{\mathbf{T}}(1) = \mathbf{a} \cdot \mathbf{T}\Big|_{t=1} = 2\langle 1, 1, 0\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle = 2\left(\frac{1}{3} + \frac{2}{3} + 0\right) = 2. \qquad \checkmark$$