Comments on HW 1 and preliminary exercises

1. Exercise 2 on p. 5 of the text: Prove that a number x is rational if, and only if, its representation by an infinite decimal fraction is eventually periodic.

   Proof. The result holds easily for $x = 0$, so we assume that $x \neq 0$. Indeed, we may reduce to the case that $x > 0$ since for a negative number y, the result would hold if, and only if, it held for $-y > 0$.

   Assume then that $x > 0$ and $x = \frac{a}{b} \in \mathbb{Q}$, and recall that by the division algorithm, when a positive integer c is divided by b there exist q, $r \in \mathbb{Z}$ with $0 < r < b$ so that $c = qb + r$. Significantly, there are only a *finite* number of remainders after division by b. Now, using long division to divide a by b, we note that this process involves repeated division by b with remainders. However, since there are only a finite number of possible remainders (b to be exact), it must be that after $b+1$ steps in long division (or fewer), the remainder is a repetition of one seen earlier. That is, the decimal expansion of x begins to repeat.

   For the converse, assume x is eventually periodic, and write $x = n.d_1 d_2 \cdots d_k \overline{a_1 \cdots a_j}$ for $n \in \mathbb{Z}$ and $d_i, a_l \in \{0, 1, \cdots 9\}$. Thus,

   $$x = n + \frac{d_1 \cdots d_k}{10^k} + 10^{-k}\big(.\overline{a_1 \cdots a_j}\big).$$

   Since n, $\frac{d_1 \cdots d_k}{10^k}$ are clearly rational, and $10^{-k}y$ is rational if y is rational, we can reduce to that case that x is periodic of the form $x = .\overline{a_1 \cdots a_j}$. Consider the difference

   $$\begin{aligned} 10^j x - x &= 10^j.\overline{a_1 \cdots a_j} - .\overline{a_1 \cdots a_j} \\ &= a_1 \cdots a_j.\overline{a_1 \cdots a_j} - .\overline{a_1 \cdots a_j} \\ &= a_1 \cdots a_j. \end{aligned}$$

   With a little algebra, we see that
   $$x = \frac{a_1 \cdots a_j}{10^j - 1}$$
   and $x \in \mathbb{Q}$, as desired. □

2. Exercise 5 on p. 5 of the text:

   (a) The closure of $\mathbb{R}$ with the 'arctan' distance is $\mathbb{R} \cup \{-\infty, \infty\}$.

   (b) The closure of $\mathbb{R}$ with the 'e^x' distance is $\mathbb{R} \cup \{-\infty\}$.

3. Exercise 7 on p. 5 of the text: Skyler next week.

4. Exercise 8 on p. 5 of the text: You need to check that the three axioms on page 2 for a metric hold. This is relatively straight forward.

Preliminary exercise. Suppose a, n are positive integers with $1 \leq a < n$. Prove that if the $\gcd(a, n) = 1$, then there exists an integer b with $1 \leq b \leq n-1$ such that

$$ab \equiv 1 \pmod{n}.$$

John Aarhus showed us that there exists an integer b' with $ab' \equiv 1 \pmod{n}$, but he did not show that b' was in the correct range. Here is the rest of the proof:

Assume that $ab' \equiv 1 \pmod{n}$, and note that for any integer k, if $b = b' + kn$, then

$$b \equiv b' \pmod{n} \text{ with } b' \not\equiv 0 \pmod{n}$$

and therefore

$$ab = ab' \equiv 1 \pmod{n}.$$

Thus, if b' fails to be in the range $1 \leq b' < n$, replace b' with $b \in \{0, 1, \ldots, n-1\}$. This is possible since the set $\{0, 1, \ldots, n-1\}$ contains a complete set of representatives of equivalence classes mod n and $b = b' + kn$ must be in this set for some k. Since $n \nmid b$, b satisfies $1 \leq b \leq n-1$.