MATH 202 Name : SOLUTIONS

Midterm 2 November 18, 2009

Instructions: Show all work for full credit.

WARNING: Poor or incorrect use of notation will be penalized severely. Carefully communicate your answer.

1. (30 pts. – 10 pts. each)

(a) Prove that the following limit does not exist: $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

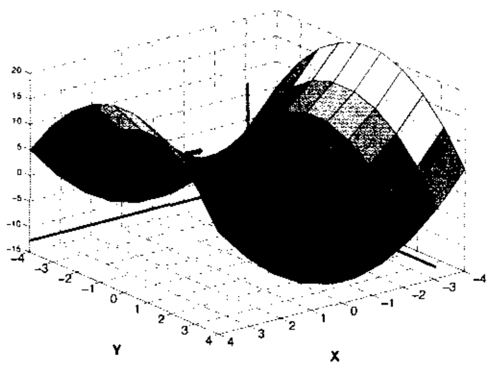
First, approach $(0,0)$ along $y=x$:

$$\frac{xy^3}{x^2+y^6} = \frac{y\cdot y^3}{y^2+y^6} = \frac{y^4}{y^2(1+y^4)} = \frac{y^2}{1+y^4}$$ which approaches 0 as $y\to 0$

Along the curve $x=y^3$, $\frac{xy^3}{x^2+y^6} = \frac{y^3\cdot y^3}{(y^3)^2+y^6} = \frac{y^6}{2y^6} = \frac{1}{2}$. Since $0 \neq \frac{1}{2}$, the limit d.n.e.

(b) Carefully examine the graph of the surface defined by $z = f(x, y)$ shown in the following figure, paying particular attention to the labeling of the axes. The point P with coordinates $(1, 1, f(1, 1))$ is indicated by a large black dot in the figure.

Note: This figure is drawn with the normal orientation for the axis.



i. What is the sign of the partial derivative $f_x(1, 1)$? Briefly explain your answer.

Positive. $f(x,y)$ increases as you move from $f(1,1)$ to $f(1+\Delta x, 1)$ for $\Delta x > 0$. See figure.

ii. Now consider the directional derivative of $f(x, y)$ in the direction of $\mathbf{v} = (1, 2)$ at the point given by $x = 1$, $y = 1$. What is the sign of this directional derivative? Briefly explain your answer.

Negative.

As you move from $(1,1,f(1,1))$ in the direction of $\vec{v} = (1,2)$, $f(x,y)$ decreases.

(c) Give the linear approximation of the function $g(x,y) = \sin(4x+3y)$ at the point $P(-3,4)$.

$g(-3,4) = \sin(4(-3)+3(4)) = 0$

$g_x(x,y) = 4\cos(4x+3y)$ and $g_x(-3,4) = 4\cos(0) = 4$

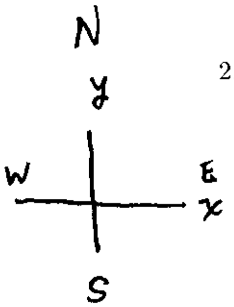
$g_y(x,y) = 3\cos(4x+3y)$ and $g_y(-3,4) = 3\cos(0) = 3$

$\therefore$ $L(x,y)$ = Equation of tangent plane to $g(x,y)$ at $P(-3,4)$

$= g(-3,4) + g_x(-3,4)(x-(-3)) + g_y(-3,4)(y-4)$

$= 0 + 4(x+3) + 3(y-4) = 4x+3y$

$\boxed{L(x,y) = 4x+3y}$



2. (15 pts.) Suppose you are climbing up out of a canyon whose shape is given by

$$z = -100 + .05x^2 + .01x + .02y^2 \quad \text{meters},$$

and your current position in this canyon is given by the point with coordinates $(10, 20, -86.9)$. Assume that the positive x-axis points due east and the positive y-axis points due north.

(a) (8 pts.) If you walk due south, will you start to ascend or descend? At what rate? (A complete answer gives a brief justification.)

Walking due south means compute the directional derivative in the direction of $\vec{v} = \vec{u} = (0,-1)$.

We are at $P(10,20)$ and the partial derivatives are $\frac{\partial z}{\partial x} = .1x + .01$, $\frac{\partial z}{\partial y} = .04y$.

Thus, $\nabla z = (.1x+.01, .04y)$ and $\nabla z(10,20) = (1.01, .8)$.

Finally, $D_{\vec{u}} z = \nabla z(10,20) \cdot (0,-1) = (1.01, .8) \cdot (0,-1) = -.8$.

Since $D_{\vec{u}} z(10,20) < 0$, we descend. The rate is $-.8$.

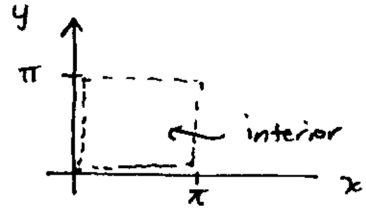
(b) (7 pts.) In what direction should you walk from your current coordinates of $(x,y) = (10,20)$ to climb in the direction of steepest ascent? Briefly justify your answer.

The gradient vector $\nabla z(10,20)$ points in the direction of steepest ascent.

$\nabla z(10,20) = (1.01, .8)$ (from above)

You should head in the direction of $(1.01, .8)$ from the point $(10,20)$.

R:



3. (15 pts.) Consider the function

$$f(x, y) = (\cos x)(\cos y)$$

on the square region R defined by $0 < x < \pi, \quad 0 < y < \pi$.

(a) (7 pts.) Find all critical points of $f(x, y)$ within R.

If $f(x,y) = (\cos x)(\cos y)$, then $f_x(x,y) = (-\sin x)\cos y$

$f_y(x,y) = \cos x\,(-\sin y)$

Setting these partial derivatives equal to zero:

$-\sin x(\cos y) = 0$ and $-\cos x \sin(y) = 0$

In R, the only solution to this system is $x = \frac{\pi}{2}$, $y = \pi/2$ [which makes $\cos\frac{\pi}{2} = 0$!]

The critical points are $\boxed{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)}$

(b) (8 pts.) Use the Second Derivative Test to determine if the critical points are local maxima, local minima, or saddle points.

Let $D = \begin{vmatrix} f_{xx}(\pi/2, \pi/2) & f_{xy}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \\ f_{yx}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) & f_{yy}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \end{vmatrix}$.

$f_{xx}(x,y) = -\cos x \cos y \Rightarrow f_{xx}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -\cos\frac{\pi}{2}\cos\frac{\pi}{2} = 0$

$f_{xy}(x,y) = \sin x \sin y \Rightarrow f_{xy}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 1 \quad = f_{yx}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f_{yy}(x,y) = -\cos x \cos y \Rightarrow f_{yy}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) = 0$

Therefore, $D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0^2 - 1 = -1 < 0$

By the second derivative test, $f(x,y)$ has a $\boxed{\text{SADDLE POINT}}$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

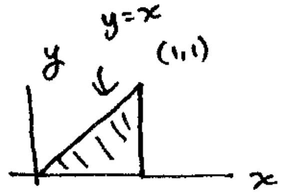
4. (15 pts.) Consider the iterated integral

$$\int_0^1 \int_y^1 e^{x^2}\, dx dy$$

(a) (5 pts.) Sketch the region of integration D.

$y \le x \le 1$

$0 \le y \le 1$



(b) (10 pts.) Evaluate the integral.

You must interchange the order of integration. otherwise, the computation is impossible.

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 y e^{x^2} \Big|_0^x \, dx$$

$$= \int_0^1 [x e^{x^2} - 0]\, dx = \int_0^1 x e^{x^2}\, dx = \frac{1}{2} e^{x^2} \Big|_0^1 = \boxed{\frac{1}{2}(e-1)}$$

5. (10 pts.) Set up, **but do not evaluate**, a triple integral in spherical coordinates that evaluates

$$I = \iiint_E x\, e^{x^2+y^2+z^2}\, dV$$

over the solid E between the spheres of radius 2 and radius 3, both centered at the origin.

The solid E in spherical coordinates is given by:

$2 \le \rho \le 3 \qquad 0 \le \theta \le 2\pi \qquad 0 \le \varphi \le \pi$

Using $x = \rho \sin\varphi \cos\theta$, $x^2+y^2+z^2 = \rho^2$, and $dV = \rho \sin\varphi \, d\varphi\, d\theta\, d\rho$

we obtain

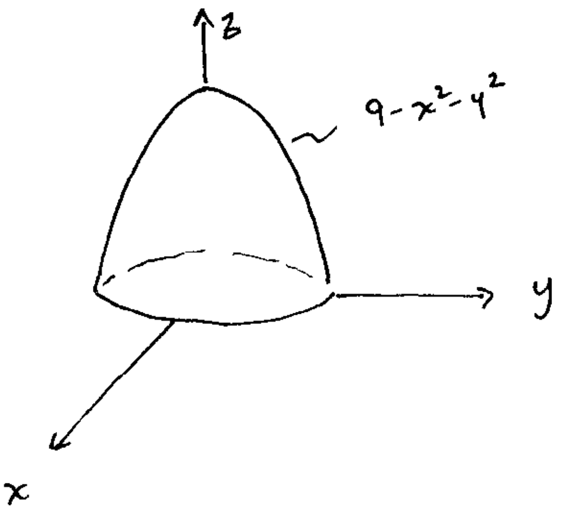
$$\int_0^{\pi} \int_0^{2\pi} \int_2^3 \rho \sin\varphi \cos\theta\, e^{\rho^2} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \boxed{\int_0^{\pi} \int_0^{2\pi} \int_2^3 \rho^3 \sin^2\varphi \, d\rho\, d\theta\, d\varphi}$$

6. (15 pts.) Suppose S is the solid bounded by $9 - y^2 - x^2$ and the plane $z = 0$. Electrical charge is distributed inside this solid with charge density given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$ coulombs/mm^3. Find the total charge of the solid. (Include units.)

S:

S is interior, a solid.



$$\text{Total charge} = \iiint_S \rho(x,y,z)\, dV$$

$0 \le z \le 9 - x^2 - y^2$

$0 \le z \le 9 - r^2$

Easiest solution: Use cylindrical coordinates.

$0 \le r \le 3$
$0 \le \theta \le 2\pi$
} projection in xy-plane

$$\text{Total Charge} = \int_0^3 \int_0^{2\pi} \int_0^{9-r^2} \rho(r\cos\theta, r\sin\theta, z)\, r\, dz\, d\theta\, dr$$

$\rho(x,y,z) = \sqrt{x^2+y^2}$
$= r$ in polar coordinates

$$= \int_0^3 \int_0^{2\pi} \int_0^{9-r^2} r\, r\, dz\, d\theta\, dr$$

$$= \int_0^3 \int_0^{2\pi} \int_0^{9-r^2} r^2\, dz\, d\theta\, dr$$

$$= \int_0^3 \int_0^{2\pi} \left[z r^2 \Big|_0^{9-r^2} \right] d\theta\, dr$$

$$= \int_0^3 \int_0^{2\pi} (9 - r^2) r^2\, d\theta\, dr$$

$$= \int_0^3 \int_0^{2\pi} 9r^2 - r^4\, d\theta\, dr$$

$$= 2\pi \int_0^3 9r^2 - r^4\, dr$$

$$= 2\pi \left[3r^3 - \frac{1}{5} r^5 \right]_0^3$$

$$= 2\pi \left[3(3^3) - \frac{3^5}{5} \right]$$

$$= 2\pi \left[81 - \frac{243}{5} \right]$$

$$= 2\pi (81) \left(\frac{2}{5} \right)$$

$$= \boxed{\frac{324\pi}{5} \text{ Coulombs}}$$