MATH 253
February 13, 2020

Name: SOLUTIONS

Instructions. (100 points) You have 60 minutes. Closed book, closed notes, and no calculators allowed. *Show all your work* in order to receive credit. Bald answers will receive little, if any, credit.

(15pts) **1.** Consider the three points $P(1,3,-1)$, $Q(3,2,4)$, $R(2,3,-3)$ in $\mathbb{R}^3$.

(a) (5 pts) Find the equation of the plane S containing the three points.

$\overrightarrow{PQ} = \langle 2,-1,5 \rangle$ $\quad \overrightarrow{PR} = \langle 1,0,-2 \rangle$ both lie in plane S.

Let $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 5 \\ 1 & 0 & -2 \end{vmatrix} = \langle 2, 9, 1 \rangle$. Then

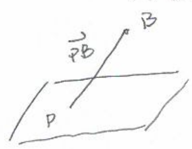
$\vec{x} \cdot \vec{n} = \vec{P} \cdot \vec{n} \Rightarrow \quad 2x + 9y + z = \langle 1,3,-1 \rangle \cdot \langle 2,9,1 \rangle = 2+27-1 = 28$

$$\boxed{2x + 9y + z = 28}$$

(b) (5 pts) By finding an appropriate point B on the line $\vec{\ell}(t) = \langle 1-3t,\ 2t,\ 2t+1 \rangle$, $t \in \mathbb{R}$, show that the line $\vec{\ell}(t)$ does **not** lie in the plane S.

Take $B = \vec{\ell}(0) = (1,0,1)$ and check if it satisfies the equation for S: $\quad 2(1) + 9(0) + 1 = 3 \neq 28$ $\quad B$ is not on S.

(c) (5 pts) Using your answer to part (b), find the distance between your point B and the plane S.



Let $\vec{v} = \overrightarrow{PB} = \langle 1,0,1 \rangle - \langle 1,3,-1 \rangle = \langle 0,-3,2 \rangle$

and find $\text{comp}_{\vec{n}} \vec{v}$

$$\text{Distance} = \left| \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|} \right| = \left| \frac{\langle 2,9,1 \rangle \cdot \langle 0,-3,2 \rangle}{\sqrt{2^2+9^2+1^2}} \right| = \left| \frac{0 - 27 + 2}{\sqrt{86}} \right| = \left| \frac{-25}{\sqrt{86}} \right|$$

$$= \boxed{\frac{25}{\sqrt{86}} = \frac{25\sqrt{86}}{86}}$$

Answers depend on B of course.

(12pts) **2.** Consider a particle moving in the plane with velocity:

$$\mathbf{v}(t) = \cos t\mathbf{i} + te^t\mathbf{j}.$$

(a) (8 pts) Find the position function at all times if $\mathbf{r}(0) = \mathbf{i}+\mathbf{j}$.

$\vec{v}(t) = \langle \cos t, te^t \rangle$ $\qquad \vec{r}(t) = \langle \int \cos t\, dt, \int te^t dt \rangle + \vec{c}$

integrate by parts

$= \langle \sin t, (t-1)e^t \rangle + \vec{c}$

$\vec{r}(0) = \langle 1,1 \rangle = \langle \sin(0), (-1)e^0 \rangle + \langle c_x, c_y \rangle$

$= \langle 0 + c_x, -1 + c_y \rangle \Rightarrow c_x = 1 \quad c_y = 2$

$\boxed{\vec{r}(t) = \langle 1 + \sin t, 2 + (t-1)e^t \rangle \quad t \in \mathbb{R}}$

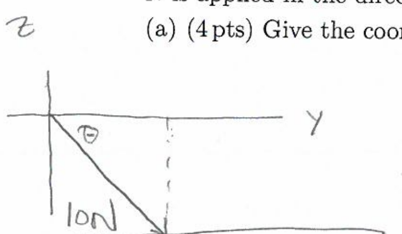
(b) (4 pts) Find the acceleration function at all times.

$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \langle -\sin t, te^t + e^t \rangle$

$= \boxed{\langle -\sin t, e^t(t+1) \rangle \quad t \in \mathbb{R}}$

(10pts) **3.** A wrench 15 cm long lies along the positive y-axis and grips a bolt at the origin (0,0,0). A force $\mathbf{F}$ of 10 N is applied in the direction of $\langle 0, 1, -\sqrt{3} \rangle$.

(a) (4 pts) Give the coordinates of the force vector $\mathbf{F}$.



$\theta = \arctan(-\sqrt{3}/1) = -\pi/3$

$\vec{F} = \langle f_y, f_z \rangle$ where

Making $0 < \theta < \pi/2 \Rightarrow \theta = \pi/3$

$f_y = 10\cos\theta \quad f_z = 10\sin\theta$

if you want to take care of the sign yourself

$= 10\left(\frac{1}{2}\right) = 5 \quad = 10\left(-\frac{\sqrt{3}}{2}\right) = -5\sqrt{3}$

$\boxed{\vec{F} = \langle 0, 5, -5\sqrt{3} \rangle}$

(b) (4 pts) What is the magnitude of the torque $\boldsymbol{\tau}$? Give units with your answer.

$|\tau| = |\vec{F}||\vec{r}|\sin\theta \qquad 0 < \theta < \pi/2$

$= 10(.15)\frac{\sqrt{3}}{2} = \frac{1.5}{2}\sqrt{3} = \boxed{.75\sqrt{3} \text{ Nm}}$

(c) (2 pts) In what direction does the torque vector $\boldsymbol{\tau}$ point? Place a ✓ by the correct answer. You need not justify your answer.

~~(A) $\mathbf{i}$~~	(B) $\mathbf{j}$	(C) $\mathbf{k}$
(D) $-\mathbf{i}$	(E) $-\mathbf{j}$	(F) $-\mathbf{k}$

Tough: Positive x-axis comes out of page

Negative x-axis goes into page

The right hand rule makes $\vec{\tau}$ point into page $\equiv -\hat{\imath}$.

(22[pts]) **4.** Consider the equations of two planes:

$$\begin{array}{llcl} \text{Plane 1:} & 6x \phantom{{}+y} + 6z & = & 5 \\ \text{Plane 2:} & x + y & = & 9 \end{array}$$

(a) (3 pts) Show that the two planes are not parallel.

$\vec{n}_1 = \langle 6, 0, 6\rangle \quad \vec{n}_2 = \langle 1, 1, 0\rangle \qquad \vec{n}_1 \neq k\,\vec{n}_2$ for any $k \in \mathbb{R}$

The normals, and therefore the planes, are not parallel.

(b) (5 pts) Find the angle θ between the two planes.

Θ = angle between $\vec{n}_1$ and $\vec{n}_2$ $\qquad \cos\Theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{\langle 6,0,6\rangle \cdot \langle 1,1,0\rangle}{\sqrt{6^2+6^2}\,\sqrt{1^2+1^2}}$

$= \dfrac{6}{\sqrt{72}\sqrt{2}} = \dfrac{6}{\sqrt{144}} = \dfrac{1}{2} \qquad \therefore \boxed{\Theta = \pi/3}$

(c) (7 pts) Give the **vector** equation of the line $\vec{\ell}(t)$ passing through $P(4, -2, 1)$ and parallel to the line of intersection of Plane 1 and Plane 2.

A direction vector $\vec{v}$ for the line of intersection can be taken as

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \leftarrow$ Scaled $\vec{n}_1$! $= \langle -1, 1, 1\rangle$

$\vec{\ell}(t) = \vec{P} + t\vec{v} = \langle 4, -2, 1\rangle + t\langle -1, 1, 1\rangle = \boxed{\langle 4 - t, -2 + t, 1 + t\rangle \quad t \in \mathbb{R}}$

(d) (7 pts) Give the vector projection $\text{proj}_{\vec{n_2}}\,\vec{v}$ of $\vec{v} = \langle -2, 5, 1\rangle$ onto the normal vector $\vec{n_2}$ of Plane 2.

$\vec{n}_2 = \langle 1, 1, 0\rangle \qquad \vec{v} = \langle -2, 5, 1\rangle$

$\text{proj}_{\vec{n}_2}\,\vec{v} = \dfrac{\vec{v} \cdot \vec{n}_2}{|\vec{n}_2|^2}\,\vec{n}_2 = \dfrac{\langle -2, 5, 1\rangle \cdot \langle 1, 1, 0\rangle}{1^2 + 1^2 + 0^2}\,\langle 1, 1, 0\rangle$

$= \dfrac{-2 + 5 + 0}{2}\,\langle 1, 1, 0\rangle = \boxed{\dfrac{3}{2}\langle 1, 1, 0\rangle = \left\langle \dfrac{3}{2}, \dfrac{3}{2}, 0\right\rangle}$

(21 pts) **7.** Consider a space curve parameterized by:

$$\mathbf{r}(t) = \left\langle t^2 - 3, t^3 + 1, \frac{5t^2\sqrt{2}}{2} \right\rangle \quad , \quad t \in \mathbb{R}.$$

(a) (7 pts) Find **parametric** equations for the tangent line to the curve at the point $(1, 9, 10\sqrt{2})$.

By trial and error, notice $\vec{r}(2) = \langle 4-3, 9, \frac{5\cdot 4\sqrt{2}}{2} \rangle = \langle 1, 9, 10\sqrt{2} \rangle \Rightarrow \boxed{t=2}$

The direction vector for the tangent line is $\vec{v} = \vec{r}\,'(2)$. Thus

$\vec{r}\,'(t) = \langle 2t, 3t^2, 5\sqrt{2}\,t \rangle$ at $t=2$, $\vec{v} = \vec{r}\,'(2) = \langle 4, 12, 10\sqrt{2} \rangle$

We can also take $\vec{v} = \frac{1}{2}\langle 4, 12, 10\sqrt{2} \rangle = \langle 2, 6, 5\sqrt{2} \rangle$ (points the same direction)

$\vec{\ell}(t) = P + t\vec{v} \Rightarrow$

$$\boxed{\begin{aligned} x(t) &= 1 + 2t \\ y(t) &= 9 + 6t \qquad t \in \mathbb{R} \\ z(t) &= 10\sqrt{2} + 5\sqrt{2}\,t \end{aligned}}$$

(b) (5 pts) Compute the unit tangent vector $\mathbf{T}(t)$ for all times $t > 0$. Simplify your answer.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 2t, 3t^2, 5\sqrt{2}\,t \rangle}{\sqrt{(2t)^2 + (3t^2)^2 + (5\sqrt{2}t)^2}} = \frac{1}{\sqrt{4t^2 + 9t^4 + 50t^2}} \langle 2t, 3t^2, 5\sqrt{2}t \rangle$$

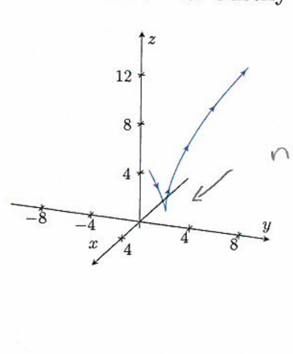
$$= \frac{1}{\sqrt{t^2}\sqrt{54 + 9t^2}} \langle 2t, 3t^2, 5\sqrt{2}t \rangle = \frac{1}{3|t|}\frac{1}{\sqrt{6+t^2}} \langle 2t, 3t^2, 5\sqrt{2}t \rangle = \boxed{\frac{1}{3\sqrt{6+t^2}} \langle 2, 3t, 5\sqrt{2} \rangle}$$

(c) (6 pts) Briefly explain why the unit tangent vector $\mathbf{T}(t)$ is undefined at $t = 0$. Then find the limit as $t \to 0^+$ of $\mathbf{T}(t)$.

Before simplification, there was a $|t|$ in the denominator, i.e. $|\vec{r}\,'(0)| = 0$

$$\lim_{t \to 0^+} \frac{1}{3\sqrt{6+t^2}} \langle 2, 3t, 5\sqrt{2} \rangle = \boxed{\frac{1}{3\sqrt{6}} \langle 2, 0, 5\sqrt{2} \rangle = \left\langle \frac{\sqrt{6}}{9}, 0, \frac{5\sqrt{3}}{9} \right\rangle}$$

(d) (3 pts) The graph of the space curve is shown below for $-1 \le t \le 2$. Indicate the position on the curve when $t = 0$. Justify your answer.

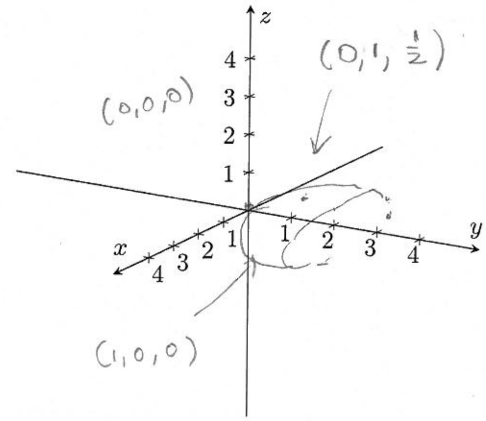


non-differentiable at cusp i.e. $\vec{r}\,'(0)$ d.n.e.

(10[pts]) **5.** Draw the requested sketches.

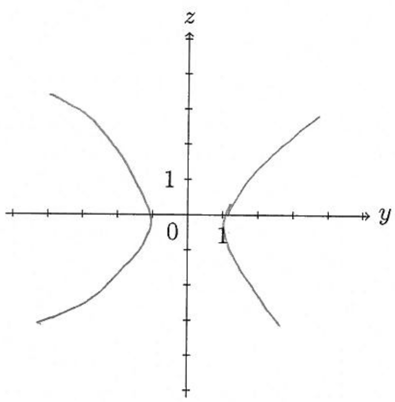
$y = x^2 + 4z^2$

(a) (5 pts) Sketch $x^2 - y + 4z^2 = 0$ in 3D.



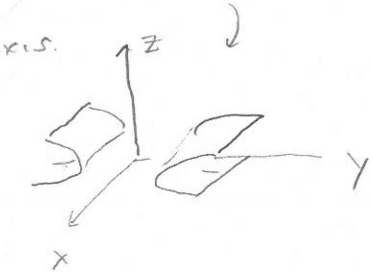
Elliptic Paraboloid

(b) (5 pts) Consider the surface $y^2 - z^2 = 1$.
Sketch the trace $x = 2$, then describe in words what the surface looks like.



trace: $x = 2$

This is a cylinder (hyperbolic cylinder) where all cross-sections look like for planes parallel to the x-axis.



(10[pts]) **6.** Find the arc length of the curve described by $\mathbf{r}(t) = \left\langle \sin t, 2t^{\frac{3}{2}}, \cos t \right\rangle$ for $1 \le t \le 2$.

$L = S = \int_1^2 |\vec{r}\,'(t)|\, dt$

$\vec{r}\,'(t) = \langle \cos t, 3t^{1/2}, -\sin t \rangle$

$|\vec{r}\,'(t)| = \sqrt{(\cos t)^2 + (3t^{1/2})^2 + (-\sin t)^2}$

$= \sqrt{\cos^2 t + \sin^2 t + 9t} = \sqrt{9t+1}$

✓ u-substitution

$= \int_1^2 \sqrt{9t+1}\, dt = \int_1^2 (9t+1)^{1/2}\, dt$

$= \frac{1}{9} \cdot \frac{2}{3}(9t+1)^{3/2} \Big|_1^2 = \frac{2}{27}(9t+1)\sqrt{9t+1}\Big|_1^2$

$= \frac{2}{27}\left[(18+1)\sqrt{19} - (10)\sqrt{10}\right] = \boxed{\frac{2}{27}\left(19\sqrt{19} - 10\sqrt{10}\right)}$