File with Exam 1, Exam 2 and computer arithmetic worksheet on it for studying.

EXAM 1:

1. (15 pts.) Consider the function $f(x) = \ln(1 + x)$.

   (a) Find the Taylor polynomial $p_4(x)$, a polynomial of degree $n = 4$, that approximates $f(x)$ near the point $x_0 = 0$. Show all work for full credit.

   (b) Consider now the nth degree Taylor polynomial $p_n(x)$ approximating $f(x)$ at $x_0 = 0$. Give the value of the error term $R_n(x) = f(x) - p_n(x)$.

   (c) Determine the *smallest* number of terms n such that the absolute error $|R_n(x)| < 10^{-3}$ for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Justify your work by filling in the table below and showing your work.

k	$R_k(x)$
$n - 1 =$	$\lvert R_{n-1}(x)\rvert \leq$
$n =$	$\lvert R_n(x)\rvert \leq$

2. (6 pts.) Give the 200th Taylor polynomial $p_{200}(x)$ approximating $g(x) = 2001x^{79} - 1001x^{29} + x - 3$ near $x_0 = 0$ and the error $R_{200}(x)$. Briefly justify your answer.

   $p_{200}(x) =$ $\qquad\qquad$ $R_{200}(x) =$

   Justification:

3. (15 pts.) Pretend that a computer can only represent the floating point numbers 0, $\pm\infty$, and those which in base 2 have the form

$$\pm 1.a_1a_2a_3 \times 2^m,$$

where $a_1,\ a_2, 1_2, a_3$ are binary digits (i.e., 0 or 1) and m is an integer with $-5 \le m \le 5$. If x is a real number, let $fl(x)$ denote its floating point value and assume that this machine uses *truncation* for finding floating point equivalents. For parts (b)-(e), you must briefly justify your answer for full credit.

(a) Give the **floating point** representation and the **decimal** value of the smallest and largest *positive* floating point numbers on this computer. (Perform scratch work elsewhere.)

My answers:

smallest positive number:

in floating point ____________________ decimal equivalent: __________

largest positive number:

in floating point ____________________ decimal equivalent: __________

(b) Give an example of a real number x such that $fl(x) \neq x$.

(c) Give an example of two real numbers such that the sum $fl(x + y)$ gives an overflow error.

(d) Give an example of two non-zero real numbers such that the sum $fl(x + y) = x$.

(e) i. Give the definition of machine epsilon ϵ_M. (Give the general definition for any machine.)

ii. Give the value of machine epsilon ϵ_M for this computer. Justify your answer.

4. (6 pts.) Use Taylor's Theorem to give an expression for $f(x+h)$ at the point x and then show that the right difference approximation to $f'(x) = \dfrac{f(x+h) - f(x)}{h}$ is $\mathcal{O}(h)$.

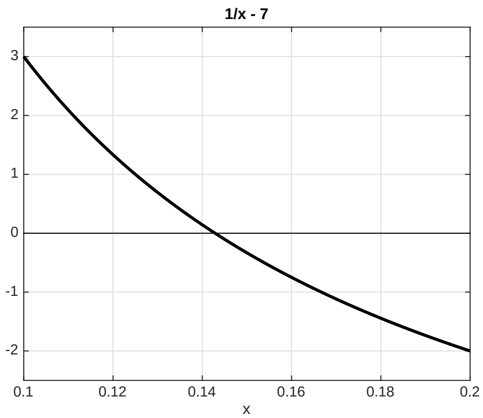
5. (5 pts.) Consider the system of three linear equations in three unknowns. Set up the augmented matrix and perform **one** step of Gaussian elimination on this system.

$$\begin{array}{llll} 3x & -\ 5y & +\ 2z & =\ 1 \\ 6x & +\ y & & =\ 1 \\ 2x & +\ 4y & +\ 2z & =\ 3 \end{array}$$

6. (5 pts.) Suppose the result of performing Gaussian elimination on a linear system of equations gives the following augmented matrix. Use backward substitution to solve for x, y, and z.

$$\left(\begin{array}{rrr|r} 2 & -1 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & 5 \end{array}\right)$$

7. (12 pts.) It is possible to check that $\frac{1}{7} \approx .14286$. Consider the graph of $f(x) = \frac{1}{x} - 7$ for $.1 \le x \le .2$ below. In this problem you should round your answer to five significant digits.



(a) Perform three iterations of the bisection method to approximate $\frac{1}{7}$. Give your answer in the table. (Make sure at each step you list the left endpoint a, the right endpoint b, and the approximation to the root x_n.)

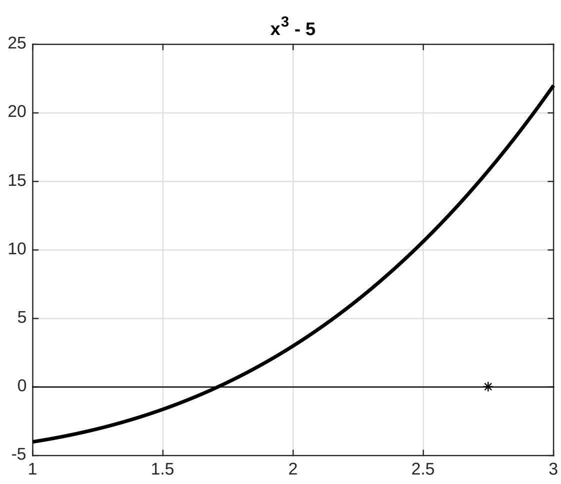
n	a	b	estimate $x_n = c$
1	.10000	.20000	
2			
3			

(b) Find the value of the absolute error and the relative error for the estimate x_3 from your table above. As part of your answers, give the formulas you use to compute these quantities.

(c) Give a formula involving a, b, and n for an upper bound on the absolute value of the error $|e_n|$ after the nth iteration. Then use this formula to determine how many iterations N you should perform if you want the estimate to satisfy $|e_N| \le 10^{-8}$.

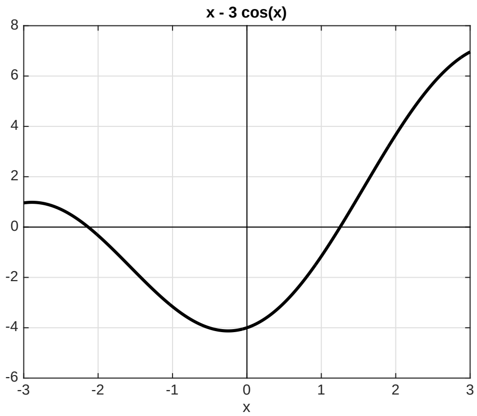
8. (7 pts.) Below is the graph of $g(x) = x^3 - 5$. This graph and Newton's method can be used to find an approximation to $\sqrt[3]{5}$.

(a) With an initial value of $x_0 = 2.75$, sketch three iterations of Newton's method. (Your picture should highlight the geometric interpretation of Newton's method. Clearly mark x_1, x_2, and x_3 on the graph.)



(b) Will Newton's method converge?

9. (15 pts.) In this problem you will perform Newton's Method to estimate a solution to the equation $x = 4\cos(x)$. For your convenience a graph of $f(x) = x - 4\cos(x)$ is shown.



(a) Give the formula for computing the $(n+1)$st estimate x_{n+1} from x_n:

$x_{n+1} =$

(b) Perform two iterations of Newton's Method to estimate a solution to the equation $x = 4\cos(x)$. Use $x_0 = 1$ for your initial guess. (Show all work in computing x_1 and x_2 and round your answers to four decimal places.)

Answer:

n	x_n
0	$x_0 = 1.0000$
1	$x_1 =$
2	$x_2 =$

(c) Let α be the root of the equation $x = 4\cos(x)$ that is near 1, and let e_k denote the error in the kth step $\alpha - x_k$ as usual. Note that $e_0 = \alpha - 1$.

It can be shown that the error in Newton's method satisfies $e_{n+1} = -\frac{1}{2}e_n^2 \frac{f''(\xi_n)}{f'(x_n)}$ for some ξ_n between α and x_n. Use this formula to give an explicit formula for e_1 in terms of e_0. (I.e. compute the derivatives and put them in the right places.)

Then give an upper bound for $|e_1|$ using this formula. (Your answer should be expressed as $|e_1| \le Ce_0^2$ for some C. Estimate this C.)

$e_1 =$

$|e_1| \le$

(d) There is a second root β to the equation $x = 4\cos(x)$ that is roughly $\beta \approx -2.1$. Is it possible to give a starting value x_0 in Newton's method that is close to β, yet the algorithm converges to α? If so, show this graphically in the plot above. If not, explain why this is impossible.

10. (8 pts.) In analyzing an (unspecified) algorithm, you discover that the error terms are related by $e_{k+1} = .79e_k$.

    Prove or disprove: This algorithm converges.

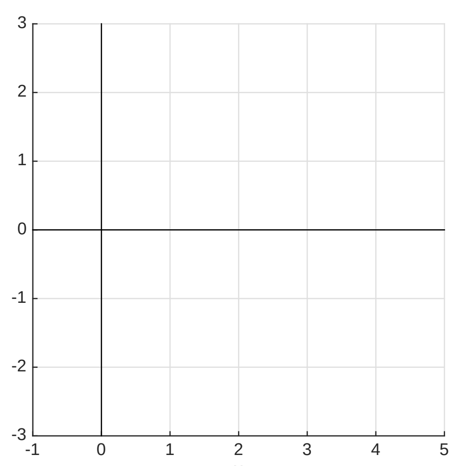
    If the algorithm converges, give the order of convergence for the estimates to the true value.

    Answer: (Circle one.) The algorithm DOES / DOES NOT converge.

11. (8 pts.) Suppose $f(x)$ is a differentiable function, and that one of two methods discussed in class (right and/or central difference approximations) was used to approximate $f'(a)$ at some value $x = a$. For this algorithm, the values of h were halved with each successive iteration, and the values of the error $e_n = f'(a) - x_n$ are displayed in the table below. Determine whether the right difference approximation or the central difference approximation was used to estimate $f'(a)$. Justify your answer.

n	e_n
1	0.06445292
2	0.01552106
3	0.00384407
4	0.00095877
5	0.00023955
6	0.00005988
7	0.00001497
8	0.00000374
9	0.00000094
10	0.00000023

12. (8 pts.) Sketch a function $f(x)$ on the axes below for which the bisection method for finding a root of $f(x)$ is **better** than Newton's method. Briefly explain your answer.



Instructions: Show all work for full credit. You may use a calculator for simple 'adding machine'-like computations.

1. (16 pts.) Consider the function $f(x) = \dfrac{1}{x}$ for $x > 0$ and equally spaced nodes $x_0 = \frac{1}{2}$, $x_1 = 1$, $x_2 = \frac{3}{2}$.

   (a) (2 pts.) Give the degree n of the interpolating polynomial $p_n(x)$ that passes through the points $(x_i, f(x_i))$ for each i?

   (b) (6 pts.) Find the Lagrange form of the interpolant $p(x)$ that passes through these points. You may leave your answer as the sum of polynomials, but simplify the coefficients.

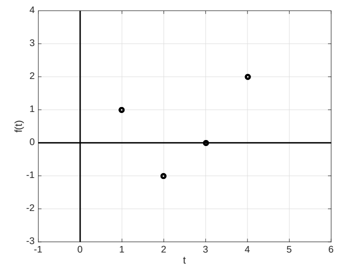
   (c) (4 pts.) Compute the value of the interpolating polynomial you found in (b) at the argument $x = \dfrac{3}{4}$. Then compute the absolute error for this approximation to $f(x)$ at $x = \dfrac{3}{4}$.

   Answer: $p(\frac{3}{4}) =$ ________________ and the absolute error is ____________________________.

   (d) (4 pts.) Suppose now that $x \in [.5,\ 1.5]$; that is, x is any point in the interval. Give an upper bound for the absolute error between $f(x)$ and $p(x)$ that works for every $x \in [.5,\ 1.5]$. (Round your answer to four decimal places, if necessary.)

2. (18 pts.) Consider the data tabulated in the following table and depicted in the plot below:

t	1	2	3	4
$f(t)$	1	-1	0	2



(a) (3 pts.) On the axes provided above, sketch a rough graph of the interpolating polynomial $p_3(x)$.

(b) (5 pts.) With the nodes in increasing order, $t_0 = 1 < \cdots < t_3 = 4$, create a divided differences table to find the coefficients c_i in the Newton form of the interpolant. Show all work for credit, and give your answers as rational numbers, not decimals, in the table provided.

k	x_k	$f_0(x_k)$	$f_1(x_k)$	$f_2(x_k)$	$f_3(x_k)$

(c) (3 pts.) Using your divided differences table from part (a), give the Newton form of the interpolating polynomial $p_3(t)$ for these data.

(d) (3 pts.) Now give the Newton form of the interpolating polynomial $p_3^{reverse}(x)$ for the nodes in the *reverse order*: $x_0 = 4$, $x_1 = 3$, $x_2 = 2$, $x_3 = 1$.

(e) (4 pts.) Suppose you used $p_3(t)$ and then $p_3^{reverse}(t)$ to interpolate the value of $f(t)$ at the value $t = \pi$. Which gives the better approximation to $f(\pi)$. Explain.

3. (12 pts.) Give two reasons/situations why using the Newton form of the interpolating polyonomial $p_n(x)$ might be preferable to the Lagrange form. (You will be graded on your explanation as well as your answer.)

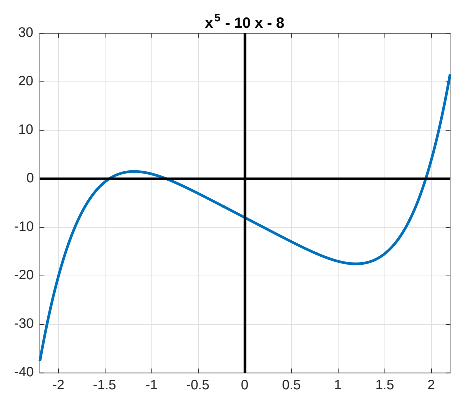
   i. The Newton form of the interpolating polynomial is preferable when ....

   ii. The Newton form of the interpolating polynomial is better when ....

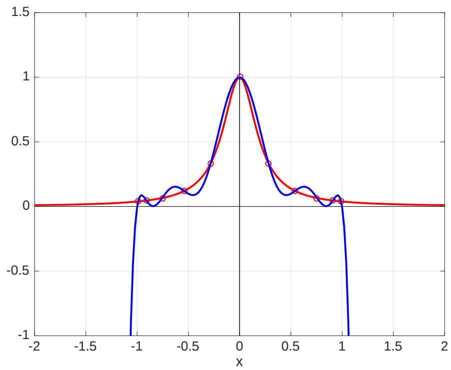
4. (7 pts.) Suppose you have $n+1$ data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ and the polynomial interpolant $p_n(x)$ has degree strictly less than n. What must be true about these data points? Explain.

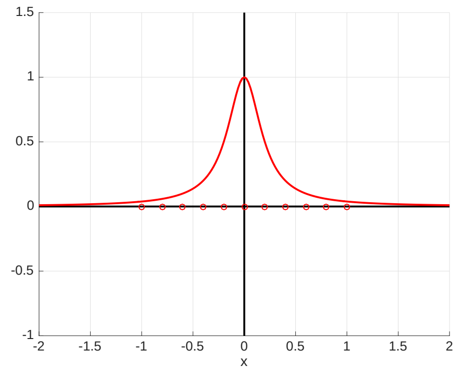
5. (10 pts.) Consider the graph of the quintic function $f(x) = x^5 - 10x - 8$ shown below. This function has only two local extrema, both of which are shown.

   Give a possible initial 'guess' that you might use if you were to use the Durand-Kerner method to find the roots of $f(x)$. Justify why you chose such a guess for starting values as a way to ensure convergence to the roots. (You will be graded both on correctness and completeness of your explanation.)



6. (15 pts.) Consider the graph of the 'Runge example' given below, where $f(x) = \dfrac{1}{1+25x^2}$ is plotted on the interval $[-1,\ 1]$. On the left, you can see the graph of the polynomial interpolant when the Chebyshev nodes are used, and on the right you can see a plot of $f(x)$ and equally spaced nodes.





(a) (4 pts.) Explain by looking at the graph on the left how you can tell that the Chebyshev nodes, *and not equally spaced nodes*, were used to construct the polynomial interpolant.

(b) (2 pts.) What is the degree n of the polynomial interpolant pictured on the left?

(c) (5 pts.) To the best of your ability, sketch on the axes on the right a polynomial interpolant to $f(x)$ at the equally spaced nodes displayed there. In the space here, briefly explain what features you are trying to highlight.

(d) (4 pts.) Using the Theorem on error in polynomial interpolation, explain carefully why the Chebyshev nodes might/might not be a better choice than equally spaced nodes for interpolation.

7. (10 pts.)

   (a) Give the definition that a sequence p_n converges quadratically to 0.

   (b) Prove or disprove: The sequence $p_n = \dfrac{1}{3^{2^n}}$ converges quadratically to 0.

8. (12 pts.) Recall the 'fixed slope' method for finding a root of a polynomial $f(x)$, a variation on Newton's method. This problem will consider convergence of this method to a root of the function $f(x) = \ln(1 - x)$.

   (a) (1 pt.) Give the value of the single root α of $f(x) = \ln(1 - x)$.
   Answer. $\alpha =$ ________________

   (b) (1 pt.) In the 'fixed slope' method, we will use the value of the $g = -2$ in place of the derivative: $x_{n+1} = x_n - \dfrac{f(x_n)}{g}$. Substitute and simplify this expression.

   (c) (2 pts.) Give the iterative function $\varphi(x)$ for this algorithm.

   (d) (3 pts.) Letting $e_k = \alpha - x_k$ denote the error in the kth iteration as usual, give an expression for e_{k+1} as a function e_k that involves iterative function. Simplify.

   (e) (5 pts.) Suppose the intial guess to a root of $f(x)$ is a negative number x_0. Will this fixed slope method converge to the root α? Justify your answer.

Theorem: (Polynomial Interpolation Error) Let $f \in \mathcal{C}^{n+1}([a,b])$ and let $x_0 < x_1 < \cdots < x_n$ be nodes in the interval $[a,b]$. Then, for each $x \in [a,b]$, there is a $\xi_x \in [a,b]$ such that

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{n+1}(\xi_x)(x - x_0)(x - x_1)\cdots(x - x_n).$$

Theorem: Let $p_n(x)$ be the polynomial interpolant for points $\big(x_i, f(x_i)\big)$ where $f \in \mathcal{C}^{n+1}(I)$ for some appropriately chosen interval I and equally spaced nodes, then

- $\big|f(x) - p_1(x)\big| \leq \frac{1}{8}(x - x_0)^2 ||f^{''}||_{\infty,I}$
- $\big|f(x) - p_2(x)\big| \leq \frac{1}{9\sqrt{3}}\, h^3\, ||f^{'''}||_{\infty,I}$
- $\big|f(x) - p_3(x)\big| \leq \frac{1}{24}\, h^4\, ||f^{(4)}||_{\infty,I}$

Supplementary Homework Problems
due Thursday, September 17

Pretend that a computer can only represent the floating point numbers 0, $\pm\infty$, and those which in base 2 have the form

$$\pm 1.a_1a_2 \times 2^m,$$

where $a_1,\ a_2$ are binary digits (i.e., 0 or 1) and m is an integer with $-2 \le m \le 2$. When a real number x is provided as input, it is converted to the machine number $fl(x)$, which is the closest number to x of the above form. When an operation such as addition is performed on inputs x and y, they are first converted to machine numbers, then added exactly, and finally the sum is converted to a machine number, so that the output is $fl(fl(x) + fl(y))$. Assume that this computer uses *rounding* to compute floating point numbers.

1. Give decimal or rational expressions for all 20 of the finite positive machine numbers. Then illustrate them all on a number line.
2. Express 3.5 and 6.5 in base 2. Is either of these a machine number? Find $fl(3.5)$ and $fl(6.5)$.
3. Express $\frac{7}{3}$ in base 2. Find $fl\left(\frac{7}{3}\right)$.
4. Give examples of machine numbers x and y (other than 0 or $\pm\infty$) such that:
   (a) $fl(x + y) = x$
   (b) $fl(x \cdot y)$ produces an overflow
   (c) $fl(x + y)$ produces an overflow
   (d) $fl(x/y)$ produces an underflow
5. Give examples of real numbers x and y such that:
   (a) $fl(x + y) \ne fl(fl(x) + fl(y))$
   (b) $fl(x \cdot y) \ne fl(fl(x) \cdot fl(y))$

For the next problems, suppose that (a different) computer can only represent the floating point numbers 0, $\pm\infty$, and those which in base 2 have the form

$$\pm 1.a_1a_2 \times 2^m,$$

where $a_1,\ a_2$ are binary digits (i.e., 0 or 1) and m is an integer with $-4 \le m \le 4$. Moreover, this computer uses *truncation* rather than rounding. To be clear, this computer only differs from the first in that it can store a larger range of exponents and it truncates numbers with more than three binary significant digits.

6. What is machine epsilon ϵ_M for this computer? Recall that ϵ_M it is the largest floating point number such that $fl(1 + \epsilon_M) = 1$.
7. Find $fl(2 + \epsilon_M)$ and $fl\left(\frac{1}{2} + \epsilon_M\right)$.