MATH 202
Quiz 9

Name : Answers

April 25, 2008

Instructions: This quiz is worth ten points. You get one point for taking this quiz.

1. (2 pts.) Consider the curve C defined by an arc of a parabola $y = x^2$ from the point $(-1, 1)$ to $(2, 4)$. Assume distance is measured in meters along C.

   (a) Give a parameterization $\mathbf{r}(t)$ of this curve. A complete answer includes an interval $t_i \le t \le t_f$ giving the range of values for t.

   $\vec{r}(t) = \langle t, t^2 \rangle \qquad -1 \le t \le 2$

   (b) Find the work done by the force field $\mathbf{F}(x, y) = x \sin y\, \mathbf{i} + y\, \mathbf{j}$ Newtons in moving a particle along the parabola $y = x^2$ from the point $(-1, 1)$ to $(2, 4)$. Indicate units.

   $$W = \int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{2} \langle t \sin t^2, t^2 \rangle \langle 1, 2t \rangle \, dt$$

   $$= \int_{-1}^{2} t \sin t^2 + 2t^3 \, dt = -\frac{\cos t^2}{2}\Big|_{-1}^{2} + \frac{t^4}{2}\Big|_{-1}^{2}$$

   $$= -\frac{\cos 4}{2} + \frac{\cos 1}{2} + \frac{15}{2}$$

2. (2 pts.) Find the curl and divergence of $\mathbf{F} = xyz\,\mathbf{i} - x^2 y\,\mathbf{j}$.

   $P(x,y,z) = xyz \qquad Q(x,y,z) = -x^2 y \qquad R(x,y,z) = 0$

   $$\operatorname{curl} \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

   $$= \underline{\langle 0, xy, -2xy - xz \rangle}$$

   $$\operatorname{div} \vec{F} = \underline{yz - x^2}$$

3. (3 pts.) Consider the vector field $\mathbf{F}(x, y) = (e^x \sin y + 2y,\ e^x \cos y + 2x + 2y)$.

(a) Noticing that $\mathbf{F}$ is defined on all of $\mathbb{R}^2$, determine (with proof) if the vector field $\mathbf{F}$ is conservative.

$$\frac{\partial P}{\partial y} = e^x \cos y + 2 \qquad \frac{\partial Q}{\partial x} = e^x \cos y + 2$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so $\vec{F}$ is conservative

(b) If $\mathbf{F}$ is conservative, find a potential function $f(x, y)$ for $\mathbf{F}$.

$f_x = e^x \sin y + 2y$

$f_y = e^x \cos y + 2x + 2y$

Then $f(x,y) = e^x \sin y + 2yx + g(y)$

$f_y(x,y) = e^x \cos y + 2x + g_y(y)$, so $g_y(y) = 2y$

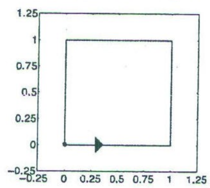
and $g(y) = \frac{y^2}{\cancel{2}}$. Thus $\boxed{f(x,y) = e^x \sin y + 2xy + y^2}$

(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for C the positively-oriented quarter of the unit circle $x^2 + y^2 = 1$ from $(0, 0)$ to $(0, 1)$.

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(0,1) - f(0,0)$$

$$= \boxed{\sin 1 + 1}$$

4. (2 pts.) Use Green's Theorem to evaluate the line integral $\int_C e^y\, dx + 2xe^y\, dy$ for C the positively-oriented curve determined by the square that joins $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$. See figure.



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_C e^y dx + 2x e^y dy = \iint_D 2e^y - e^y \, dA = \int_0^1 \int_0^1 e^y \, dy\, dx$$

$$= \boxed{e - 1}$$