Comments on HW 3

Overall the homeworks were good, but I want to make sure I clarified a few things. See comments below and look at some of the proofs.

p. 14, Ex 10: For a normed field $(F, ||\cdot||)$, prove that

$$\Big| ||x|| - ||y|| \Big| \le ||x \pm y|| \text{ for all } x, y \in F.$$

Recall that the *definition of absolute valu*e means you must prove the two inequalities

$$-||x \pm y|| \le ||x|| - ||y|| \le ||x \pm y|| \text{ for all } x, y \in F.$$

To do this, note that by the triangle inequality

$$||x|| = ||(x - y) + y|| \le ||x - y|| + ||y||, \tag{1}$$

and

$$||y|| = ||(y - x) + x|| \le ||x - y|| + ||x||. \tag{2}$$

Here we used that $||z|| = || - z||$ in Equation 2.

Performing some algebra, from (1) we get

$$||y|| - ||x|| \ge -||x - y||,$$

and from (2)

$$||y|| - ||x|| \le ||x - y||.$$

These are the two inequalities we need to prove the result for the norm of $x - y$. For the analogous result for $||x + y||$, simply replace y with $-y$ □

11 (3), (4): Good proofs.

12: The hint in the back of the book essentially gave it all away.

13: The hint here also essentially completes the problem, but I include a proof since handling the three cases was a tad delicate.

Prove that if two norms are equivalent $||\cdot||_1 \sim ||\cdot||_2$ and $x \in F$, then $||x||_1 < 1$ if, and only if, $||x||_2 < 1$ if, and only if, $||x||_1 > 1$ if, and only if, $||x||_2 > 1$ if, and only if, $||x||_1 = 1$ if, and only if, $||x||_2 = 1$.

Proof. To begin, note that the result holds for $x = 0$ since $||0|| = 0$ for any norm, so we may assume that $x \neq 0$. We first show that $||x||_1 < 1$ if, and only if, $||x||_2 < 1$. For the sake of contradiction, assume that $||x||_1 < 1$ and $||x||_2 \ge 1$. Note that $x \neq 1$, since for any norm $||1|| = 1$ by Proposition 1.6, yet by assumption $||x||_1 < 1$. That is, $x \neq 0, 1$.

Consider $||\cdot||_1$ and note that the sequence $\{x^n\}$ is Cauchy with respect to this norm. This follows from Proposition 1.8 since that the proof of that Proposition shows that $\{x^n\}$ converges to 0 and any convergent sequence in F is Cauchy. (Alternatively, prove this directly: for any $\epsilon > 0$, choose $N \in \mathbb{N}$ such that for all $m > N, ||x^m||_1 < \frac{\epsilon}{2}$. Then by the triangle inequality, if $m, k > N$, it follows that $||x^m - x^k||_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.)

Now consider the sequence $\{x^n\}$ with respect to $||\cdot||_2$. Since $x \neq 1$, $||x-1||_2 > 0$. Let $\delta = ||x-1||_2$ be this positive number. Because norms are multiplicative, for all indices n,

$$||x^{n+1} - x^n||_2 = ||x^n(x-1)||_2 = ||x^n||_2||x-1||_2 \geq \delta > 0,$$

and $\{x^n\}$ is not Cauchy with respect to $||\cdot||_2$. This contradicts that the two norms are equivalent and establishes that $||x||_1 < 1$ implies $||x||_2 < 1$. The converse holds by the identical argument after interchanging the role of the two norms, and the first equivalence is established.

Now assume that $||x||_1 > 1$. Since norms are multiplicative and $x \neq 0$, $||\frac{1}{x}||_1 < 1$. By the previous equivalence, $||\frac{1}{x}||_2 < 1$. By the multiplicative property of norms, $||x||_2 > 1$, establishing the second equivalence.

Finally, if $||x||_1 = 1$, then $||x||_1 \not< 1$ and $||x||_1 \not> 1$. By the equivalences already established, the only possibility is that $||x||_2 = 1$. $\square$