MATH 371 Name :______________________

Quiz 1 September 27, 2019

**Instructions:** Give numerical answers unless instructed that a formula suffices. A 'dumb' calculator may be used for routine arithmetic, but nothing else. Good luck.

1. (5 pts.) Answer the following:

   (a) (2 pts.) In order to fulfill general education requirements, a student must take one Math class, one Science class, two English classes, and one History class. The college offers 4 Math general education classes, 5 Science general education classes, 5 English general education classes and 4 History general education classes. How many different ways could a student fulfill this college's general education requirement?

   (b) (3 pts.) There are ten quantitative skills which an employer wishes a job applicant to possess. Five of the ten skills are selected at random for a skills test, the applicant is asked to perform them, and passes if they get at least 4 out of five correct. Assuming that the student knows 8 of the ten skills, give a formula that computes the probability that the student passes the skills test (gets *at least 4* skills correct). You **only** need to give the formula here and not its decimal value.

2. (10 pts.) Consider the random variable

$$Y\text{: } \textit{Winnings in dollars in a game of chance}$$

and its probability distribution given in the table below:

| $Y$ | -2 | -1 | 0 | 3 | 5 |
|---|---|---|---|---|---|
| $p(y)$ | .1 | .2 | .6 | | .05 |

   (a) (1 pt.) Assuming the only possible winnings (where a negative value means a loss) in one round of the game are -2, -1, 0, 3, or 5 dollars, find the probability $P(Y = 3)$.

   (b) (4 pts.) Is this game favorable? I.e. Do you expect to win money if you were to play this game repeatedly? To justify your answer, perform the relevant computation and give a brief justification.

   (c) (5 pts.) Give a general formula for the variance $V = V(Y)$ and the standard deviation $\sigma = \sigma(Y)$ of a random variable. Then compute $V(Y)$ for the $Y$ above.

3. (10 pts.) Consider the data below collected on placement exams and pass rates of a population of Calculus I students. Five events are labelled including **A**: *Student passes with grade of C or better* and **B**: *Student earns grade less than C.*, etc.

Proportions of Calculus I students

| Placement Test Range | (**A**) Passes with C or better | (**B**) Earns grade less than C |
|---|---|---|
| (**C**) 78-100 | .28 | .10 |
| (**D**) 60-78 | .20 | .25 |
| (**E**) 0-60 | .01 | .16 |

(a) (6 pts.) Give the following probabilities, rounding answers to two decimal places.

i. (1 pt.) $P(B \text{ and } D) = P($ Student earns grade less than C and Student scores in range 60-78 on placement test $)$

ii. (2 pts.) $P(A) = P($ Student passes with grade of C or better $)$

iii. (3 pts.) $P(A \mid D) = P($ Student passes with grade C or better $\mid$ Student scores in range 60-78 on placement test $)$

(b) (4 pts.) Are the events $A$ and $C$ independent? Prove your answer.

4. (6 pts.) A statistician works setting rates for health insurance premiums. This statistician has to determine the annual cost $C$ of a premium. On average, many insured people do not use their health insurance at all, though 35% of the insured people do make claims for an average annual amount of $1200. Suppose that it costs the insurance company $15 per person for annual enrollment. How much should the statistician charge for the annual cost $C$ of the premium if the insurance company wants to make $50 per insured person?

5. (6 pts.) A population of students contains 55% from UAA and 45% from UAF. It is known that 10% of UAA students and 20% of UAF students favor joining the two universities into a single university. A student selected at random from this population is found to favor joining the two universities. Find the conditional probability that this student is from UAF. Round your answer to two decimal places.

6. (5 pts.) The College Board finds that student scores on the Mathematics portion of the SAT are approximately normally distributed with a mean $\mu = 510$ and a standard deviation $\sigma = 90$. Use the empirical rule to estimate the percentage of students who score in the range $[420, 690]$ on the Mathematics portion of the SAT, rounding your answer to two decimal places.

7. (8 pts. – 4 pts. each) In a round of the Stanley Cup, two teams compete in a best-of-seven series. That is, they play until one team wins four hockey games. Let $S$ be the sample space containing all ways a round in the series can end.

   Suppose Team A and B are playing each other. Using notation like $AAAA$ to denote the event that Team A wins the series in exactly four games, or $AABBBB$ to denote the event that Team B wins the series in six games after losing the first two, answer the following.

   Let $Y$ be the event

   *Y: Team A wins the series in exactly* 5 *games.*

   (a) Write down the atomic events in the sample space $S$ that correspond to event $Y$.

   (b) Suppose that Team A is better than Team B and in any game the probability that Team A wins is .6. Compute the probability of event $Y$. Round your answer to two decimal places.

   (c) **Extra Credit:** Using notation as above, define the probability distribution on the random variable

   $X$: *the game number on which the round in the series ends*

   and prove it is a probability distribution (i.e. it sums to one on its support.)

MATH 371 Name :______________________________________
Exam 1 October 25, 2019

**Instructions:** Give numerical answers unless instructed that a formula alone suffices. You may consult tables on the inside cover of your textbook and in the appendices, but indicate in your solution that you have done so by writing 'Table.' A 'dumb' calculator may be used for routine arithmetic, but nothing else. Bald answers will receive very limited, if any, credit; that is, show formulas before performing your computations. Good luck. TOO LONG!

1. (16 pts. – 4 pts. each) An unfair coin has probability $p = p(H) = .4$ of coming up $H$ = Heads.

   (a) i. Give the probability that the first $H$ is flipped on the third coin toss. Give your answer to three decimal places.

   ii. On which coin toss (first, second, etc.) do you expect to see the first $H$ come up? That is, give the expected value for the random variable in the last problem. Round your answer to one decimal place.

   (b) i. Give the probability that the second $H$ is flipped on the fifth coin toss. Give your answer to three decimal places.

   ii. Give the variance for the random variable that computes the flip number on which the second $H$ is observed. Give your answer to one decimal place.

2. (5 pts.) A random variable $X$ has the moment generating function $m(t) = \dfrac{.9e^t}{1 - .1e^t}$. What is the distribution of $X$?

3. (32 pts.) Answer the following, showing work.

   (a) In a small class of 16 students, exactly 10 favor recalling Governor Dunleavy. Give a formula for the probability that exactly four students in a random sample of size 6 favor recalling Governor Dunleavy.

(b) Under standard use, studded tires have an average lifetime of 5 seasons. Assuming that the length of life $Y$ of a studded tire is approximately normally distributed with mean 5 seasons and standard deviation 2, find the probability that the studded tire will last between 3.5 and 6 seasons of use Give your answer to four decimal places.

(c) Suppose a random variable $X$ is binomially distributed with $X \sim \text{Binom}(1000, .0068)$.

i. Give a formula for the probability that $X$ is at least 12, $P(X \geq 12)$.

ii. Use the Poisson approximation to the binomial distribution to estimate the probability $P(X \geq 12)$, rounding your answer to three decimal places.

(d) The proportion $K$ of required material that a student knows for an exam is modeled with a Beta-distributed random variable with parameters $\alpha = 8$, $\beta = 2$, $K \sim \text{Beta}(8, 2)$.

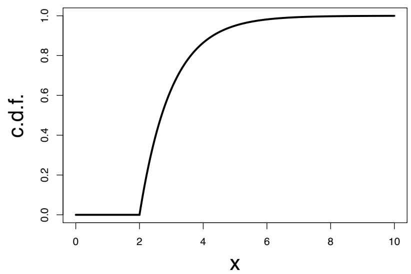
i. What is the expected value of $K$? Explain informally what this means about anticipated student knowledge of material on an exam.

ii. Explicitly give the density function $f(k)$ for $K$. Your answer should contain only numerical constants and the variable $k$ (i.e. no symbolic parameters), and carefully indicate the support of $f(k)$ (the subset of real numbers where $f(k) > 0$).

iii. Give the probability that a student knows at least 90% of the required material. Give full details of your computation, only using your calculator for the final evaluation rounding your answer to two decimal places.

4. (15 pts.) Below the graph of the c.d.f (the distribution function) $F(x)$ of a random variable $X$ and its equation are given.

$$F(x) = \begin{cases} 0, & \text{if } x < 2 \\ 1 - e^{-(x-2)}, & \text{if } x \geq 2. \end{cases}$$




(a) (2 pts.) Is $X$ a discrete or continuous random variable? Why?

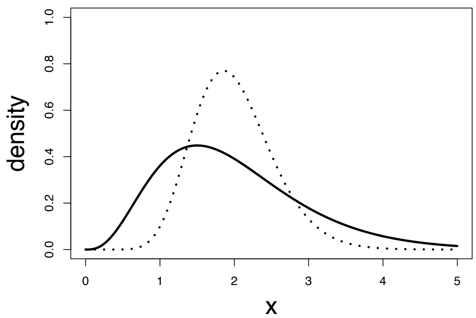
(b) (4 pts.) Find the probability that $X$ is greater than or equal to 4, $P(X \geq 4)$. Give a formula for the value of the probability, then round your answer to three decimal places.

(c) (4 pts.) Give a formula for the density function $f(x)$ for $X$, making sure that the domain of $f(x)$ is defined on the entire real number line.

(d) (5 pts.) Find the expected value $\mathbb{E}(X)$. *Hint:* There is an easy (and fast) way to do this, thinking about linear transformations or shifts of random variables.

5. (8 pts.) Use the moment generating function $m(t)$ for a binomial random variable $B \sim \text{Binom}(10, .8)$ to show that the expected value is $\mathbb{E}(B) = 8$.

6. (10 pts.) Below are two graphs (solid, dotted) of density functions for distinct Gamma-distributed random variables $X_{solid}$ and $X_{dotted}$, both with mean equal to 2.




$\Gamma(.2, 10)$ $\Gamma(.4, 5)$

$\Gamma(1, 1)$ $\Gamma(1, 2)$

$\Gamma(3, 1/6)$ $\Gamma(4, 1/2)$ $\Gamma(14, 1/7)$

Listed on the right are the values of possible parameter choices for $X_{solid}$ and $X_{dotted}$, two of which are correct. Identify the correct parameter values for the random variables, and explain how you determined this.

$X_{solid} \sim$ ___________ because .... $X_{dotted} \sim$ ___________ because ....

7. (14 pts.) Consider a uniform random variable $U \sim \text{Unif}(10, 30)$.

(a) (3 pts.) Give the mean, variance, and standard deviation of $U$. Give the exact values for the mean and variance, and round the standard deviation to two decimal places.

$\mu =$ ___________ $\sigma^2 =$ ___________ $\sigma =$ ___________

(b) (5 pts.) Use Tschebysheff's Theorem to give an interval that contains the mean $\mu$ with probability at least .75. Round answers to one decimal place.

(c) (5 pts.) Use your knowledge of the uniform distribution to find a (symmetric) interval about the mean $\mu$ that has probability exactly .75.

(d) (1 pt.) How good is Tschebysheff's approximation for $U$? (Give a one word answer.)

Page for scratch work, or extra credit.

**EXTRA CREDIT:**

If $X \sim \text{Geom}(p)$ is a geometric random variable, then it can be shown that $X$ is *memoryless*.

State the *memoryless* property for a geometric random variable, and then prove that $X \sim \text{Geom}(p)$ is memoryless.

MATH 371 Name : ______________________________

Exam 2 November 25, 2019

**Instructions:** Give numerical answers unless instructed that a formula alone suffices. You may consult tables on the inside cover of your textbook and in the appendices, but indicate in your solution that you have done so by writing 'Table.' A list of some useful formulas is included as the last page of this exam. A 'dumb' calculator may be used for routine arithmetic, but nothing else. Bald answers will receive very limited, if any, credit; that is, show formulas before performing your computations. Good luck. Happy Thanksgiving!

1. (10 pts.) A random variable $W$ is known to have moment generating function $m_W(t) = e^{1.2t+10t^2}$. Find the distribution of $W$ including all parameter values.

   $W \sim$ ______________________________

2. (16 pts.)

   In a certain populous state, political party affiliations for voters are given by the proportions listed in the table to the right. A random sample of size $n = 6$ is taken.

| Party | Proportion |
|---|---|
| Republican | .50 |
| Democrat | .10 |
| Independent | .32 |
| Unaffiliated | .08 |

   (a) (5 pts.) Find the probability that the random sample of size $n = 6$ contains two Republicans, one Democrat, and one Independent party member. Give your answer to four decimal places.

   (b) (5 pts.) Give the distribution of the variables

   $R$ : number of Republicans in random sample of size 6

   $O$ : number of Independents and Unafflliated voters in random sample of size 6

   *Answer:* $R \sim$ ________________ $O \sim$ ________________

   (c) (6 pts.) Find the variance of the function $U = 3R - 2O$. Give your answer to two decimal places.

3. (14 pts. – 7 pts. each) To study the effects of a tuition increase on students, two people are to be selected at random for an interview from a group containing four current students, three alumni, and two parents of current students. Let

$X_1$: *number of current students interviewed*

$X_2$: *number of alumni inteviewed*

(a) Give a formula and the exact value (as a fraction) for the joint probability function $p(x_1, x_2)$ for the pair $(X_1, X_2)$ when $(x_1, x_2) = (1, 1)$ and $(0, 1)$.

| $(x_1, x_2)$ | $p(x_1, x_2)$ |
|---|---|
| $(1, 1)$ | |
| $(0, 1)$ | |

(b) The marginal distribution $p_1(x_1)$ of $X_1$ is modeled by one of the 'well-known' discrete probability distributions we have studied, (i.e. it makes it onto the inside back cover of our textbook). What is the marginal distribution for $X_1$? Be sure to include the support of $f_1(x_1)$ and a listing of its parameter values.

*Answer:* $X_1 \sim$ ______________________________ with parameters ______________________________

and support $x_1 =$ ____________________________________

4. (12 pts.) When a couple undertakes *in vitro fertilization* (IVF) the number of viable embryos might be modeled by a binomial random variable $X$.

(a) (4 pts.) Suppose that following an IVF intervention 10 embryos are saved. Why is it reasonable that

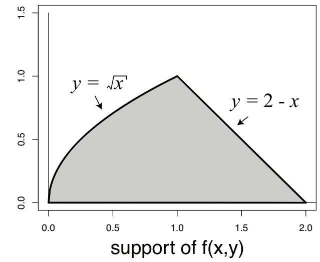
$X$: *the number of viable embryos*

might be modeled with a binomial random variable Binom$(10, p)$, $X \sim$ Binom$(10, p)$?

(b) (8 pts.) Suppose that the probability $p$ of a viable embryo is unknown, but believed to be well modeled by $p \sim$ Beta$(3, 2)$. Find the expected value and variance of $X$.

5. (25 pts.) Consider the jointly distributed random variables $(X, Y)$ with joint density $f(x, y) = \frac{6}{7}$ on the region shown below. A graph of the support of $f(x, y)$ has been provided for you.

$$f(x,y) = \begin{cases} \frac{6}{7}, & \text{on the region shown} \\ 0, & \text{otherwise.} \end{cases}$$




(a) (7 pts.) Find the marginal distribution $f_y(y)$ for $Y$. Be sure to include the support of $f_y(y)$ in your final answer.

(b) (**Extra Credit.** Do on scrap paper.) Since the joint density $f_{x.y}(x, y)$ is constant, it seems that $f_y(y)$ might be uniform on $(0, 1)$, but it is not. Why? Explain.

(c) (5 pts.) Are the variables $X$ and $Y$ independent? Why? (Credit only for a correct justification.)

(d) (7 pts.) Find the conditional probability $P(X \leq 1 \mid Y = \frac{1}{2})$. Show all work for credit.

(e) (6 pts.) Set up an expression, but do not integrate , to evaluate $P(X \leq 4/9 \mid Y \leq 1/2)$. For credit your answer should not have any 'formal' expressions, but integrals with explicit integrands and explicit limits of integration. (Extra credit: On a piece of scrap paper, get the exact value of this integral.)

6. (13 pts.) In thousands of dollars, the profit $M$ made by an apartment owner per month is $M = 2 - Y$ where $Y \sim \text{Unif}(0, 3)$ and $Y$ is also measured in thousands of dollars. That is, if a renter causes \$500 of damage to the apartment in a month, then $Y = .5$ and $M = 2 - .5 = 1.5$, or the profit $M$ is \$1,500 that month. Too easy: Better problem $M = 2 - Y^2$ and $Y \sim \text{Unif}(0, 2)$. Middling difficulty: $M = 3 - 2Y$, $Y \sim \text{Unif}(0, 1)$.

(a) (3 pts.) Give the support of the function $M = h(Y)$ for the apartment owner's monthly profit. That is, find the values of $m$ such that the density $f_m(m)$ is non-zero.

(b) (10 pts.) Either using the Method of Distributions functions or the Method of Tranformations, find the density $f_m(m)$ for $M$.

7. (10 pts.) Suppose $X_1$ and $X_2$ are independent Poisson-distributed random variables where $X_1 \sim \text{Pois}(\lambda_1)$ and $X_2 \sim \text{Pois}(\lambda_2)$. Use the method of moment-generating functions to find the distribution of $U = X_1 + X_2$.

$U \sim$ ______________________

- Tschebysheff's Theorem: Let $Y$ be a random variable with mean $\mu$ and finite variance $\sigma^2$. Then for any constant $k > 0$,

$$P(\,|Y-\mu| < k\sigma\,) \geq 1 - \frac{1}{k^2},$$

or, equivalently,

$$P(\,|Y-\mu| \geq k\sigma\,) \leq \frac{1}{k^2}.$$

- $\mathbb{E}(Y_1) = \mathbb{E}\big(\mathbb{E}(Y_1 \mid Y_2)\big) \qquad V(Y_1) = \mathbb{E}\big(V(Y_1 \mid Y_2)\big) + V\big(\mathbb{E}(Y_1 \mid Y_2)\big)$

- Let $Y_1, \ldots Y_n$ be random variables with mean $\mu_i$, $i = 1, \ldots n$, and $X_1, \ldots X_m$ be random variables with mean $\eta_j$, $j = 1, \ldots m$. Let $U = \sum_{i-1}^{n} a_i Y_i$ and $W = \sum_{j=1}^{m} b_j X_j$. Then

$$V(U) = \sum_{i-1}^{n} a_i^2 V(Y_i) + 2 \sum\sum_{i<j} a_i a_j \operatorname{Cov}(Y_i, Y_j)$$

$$\operatorname{Cov}(U, W) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \operatorname{Cov}(Y_i, X_j)$$

- Variance and covariances of $(X_1, \ldots, X_k) \sim \text{Multinomial}(n; p_1, \ldots, p_k)$:

$$\mathbb{E}(Y_i) = np_i,\ V(Y_i) = np_i(1-p_i) \qquad \operatorname{Cov}(Y_i, Y_y) = -np_ip_j,\ i \neq j$$

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Scrap work