HW 7 Solutions

§14 #11. Give the order of the element in the factor group.

$(2,1) + \langle(1,1)\rangle$ in $\mathbb{Z}_3 \times \mathbb{Z}_6 / \langle(1,1)\rangle$

Here is one approach: $\left| \frac{\mathbb{Z}_3 \times \mathbb{Z}_6}{\langle(1,1)\rangle} \right| = \frac{|\mathbb{Z}_3| \cdot |\mathbb{Z}_6|}{|\langle(1,1)\rangle|} = \frac{3 \cdot 6}{6} = 3$

From Lagrange's Theorem $|(2,1) + \langle(1,1)\rangle|$ must divide 3. Hence, $(2,1) + \langle(1,1)\rangle$ must have either order 1 or 3. If $(2,1) + \langle(1,1)\rangle$ has order 1, then $(2,1) + \langle(1,1)\rangle$ must be the identity element in $\mathbb{Z}_3 \times \mathbb{Z}_6 / \langle(1,1)\rangle$, which means that $(2,1) + \langle(1,1)\rangle = \langle(1,1)\rangle$. This could only happen if $(2,1) \in \langle(1,1)\rangle$, which it is not. Thus, $(2,1) + \langle(1,1)\rangle$ has order 3.

Alternatively, one can check that $1 \cdot (2,1) \notin \langle(1,1)\rangle$, $2 \cdot (2,1) \notin \langle(1,1)\rangle$, but $3 \cdot (2,1) = (0,3) \in \langle(1,1)\rangle$. Hence, $(2,1) + \langle(1,1)\rangle$ has order 3.

#34. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

Proof: Let $\phi_g : G \to G$ be defined by the rule $\phi_g(x) = gxg^{-1}$ where $g \in G$. First, we will show that ϕ_g is an isomorphism for all $g \in G$. Take any $y \in G$. We have that $\phi_g(g^{-1}yg) = g(g^{-1}yg)g^{-1} = y$. Thus, ϕ_g is surjective. Next, consider $\phi_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \phi_g(x)\phi_g(y)$. Thus, ϕ_g is a homomorphism. Lastly, consider

$$\begin{aligned} \operatorname{Ker}\phi_g &= \{x \in G \mid gxg^{-1} = e\} = \{x \in G \mid x = g^{-1}eg\} \\ &= \{x \in G \mid x = e\} \\ &= \{e\} \end{aligned}$$

Therefore, ϕ_g is injective. Thus, ϕ_g is an isomorphism. Since ϕ_g is an isomorphism, $\phi_g(H) \leq G$. Moreover, ϕ_g is one-to-one, so $|H| = |\phi_g(H)|$. But H is the only subgroup of G with order $|H|$. Thus, $\phi_g(H) = H$. This means that $gHg^{-1} = H$. But this is true for all $g \in G$, so H is normal. ∎

#37. Show that if G is nonabelian, then the factor group $G/Z(G)$ is not cyclic.

<u>Proof</u>: We will prove instead the contrapositive: If $G/Z(G)$ is cyclic, then G is abelian. Suppose that $G/Z(G)$ is cyclic. Then $G/Z(G)$ has a generator, $gZ(G)$ for some $g \in G$. Take any $x, y \in G$. Then $xZ(G) = (gZ(G))^m = g^m Z(G)$ and $yZ(G) = (gZ(G))^n = g^n Z(G)$ for some $m, n \in \mathbb{Z}$. This means that $x = g^m z_1$ and $y = g^n z_2$ for some $z_1, z_2 \in Z(G)$. Since z_1 and z_2 are elements of $Z(G)$, they commute with all elements of G. Thus,

$$xy = g^m z_1 g^n z_2 = g^m g^n z_1 z_2 = g^{m+n} z_2 z_1$$
$$= g^n g^m z_2 z_1 = g^n z_2 g^m z_1 = yx.$$

Therefore, G is abelian. Since we have proven the contrapositive, we have proven the original statement. ∎