Section 12.3

4. Prove that the Jordan canonical form for the matrix $\begin{pmatrix} 9 & 4 & 5 \\ -4 & 0 & -3 \\ -6 & -4 & -2 \end{pmatrix}$ is that given at the beginning of the chapter. Explicitly determine why this matrix cannot be diagonalized.

Proof. Let A be the matrix given above, then the Jordan canonical form of A is

$$J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

and $P = \begin{pmatrix} 3 & 4 & -2 \\ -2 & -2 & 2 \\ -2 & -4 & 2 \end{pmatrix}$ puts A in its Jordan form.

This matrix can not be diagonalized because the geometric multiplicity of the eigenvalue $\lambda = 2$ is only one, while its algebraic multiplicity is 2. This can be determined from the Jordan block for $\lambda = 2$. □

6. Determine which of the four matrices are similar:

$$A = \begin{pmatrix} -1 & 4 & -4 \\ 2 & -1 & 3 \\ 0 & -4 & 3 \end{pmatrix}, \; B = \begin{pmatrix} -3 & -4 & 0 \\ 2 & 3 & 0 \\ 8 & 8 & 1 \end{pmatrix}, \; C = \begin{pmatrix} -3 & 2 & -4 \\ 2 & 1 & 0 \\ 3 & -1 & 3 \end{pmatrix}, \; D = \begin{pmatrix} -1 & 4 & -4 \\ 0 & -3 & 2 \\ 0 & -4 & 3 \end{pmatrix}.$$

Proof. Put each of these matrices in Jordan canonical form and then use Theorem 23 on page 493. The Jordan canonical forms are, respectively,

$$J_A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; J_B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; J_C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; J_D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

From this canonical form, we see that only A and C are similar. We also see that A and C are not diagonalizable, but both B and D are. However, B and D have different (multi)-sets of eigenvalues.