Review problems

1. Consider the jointly continuous uniformly distributed random variables (X, Y) on the domain bounded by $x = 0$, $y = 2$, $xy = 4$, $x = 4$, and $y = 0$. (It is **easy** to check your answers without integrating.)

   (a) Draw the *support* of the joint density function $f(x, y)$; that is, the region S where $f(x, y) > 0$.

   (b) Find the value of c so that $f(x, y)$ is a valid density function on S.

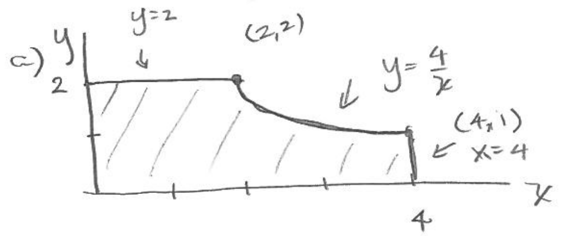
   (c) Set up an integral to find the marginal density $f_X(x)$ and include the domain of this function.

   (d) Set up an integral to find the marginal density $f_Y(y)$ and include the domain of this function.

   (e) Set up an integral that computes the conditional probability $P(X \geq 1 \mid Y = \frac{3}{2})$.

   (f) Set up a computation that computes the conditional probability that $P(X \geq 1 \mid Y \geq \frac{1}{2})$.

SOLUTION:

a) 

$xy = 4 \Rightarrow y = \frac{4}{x}$

b) Find the area of S. $\text{Area}(S) = 4 + \int_2^4 \int_0^{\frac{4}{x}} dy\,dx = 4 + \int_2^4 \frac{4}{x}\,dx = 4 + 4\ln x \Big|_2^4$

$= 4 + 4(\ln 4 - \ln 2) = 4 + 4\ln 2$

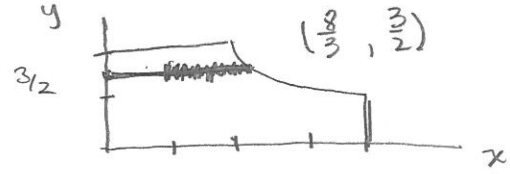
≈ 6.77

Thus, $c = \frac{1}{4+4\ln 2} \approx \boxed{.1477}$

c) $f_X(x) = \underset{y\text{-values}}{\int} \frac{1}{4+4\ln 2}\,dy = \begin{cases} \int_0^2 \frac{1}{4+4\ln 2}\,dy & 0 \leq x \leq 2 \quad \frac{2}{4+4\ln 2} = \frac{1}{2+2\ln 2} \\ \int_0^{\frac{4}{x}} \frac{1}{4+4\ln 2}\,dy & 2 \leq x \leq 4 \quad \frac{4}{x} \cdot \frac{1}{4+4\ln 2} = \frac{1}{x(1+\ln 2)} \\ 0 & \text{otherwise} \end{cases}$

d) $f_Y(y) = \underset{x\text{-values}}{\int} \frac{1}{4+4\ln 2}\,dx = \begin{cases} \int_0^4 \frac{1}{4+4\ln 2}\,dx = \frac{1}{1+\ln 2} & 0 \leq y \leq 1 \\ \int_0^{\frac{4}{y}} \frac{1}{4+4\ln 2}\,dx = \frac{1}{y(1+\ln 2)} & 1 < y \leq 2 \\ 0 & \text{otherwise} \end{cases}$

e) $P(X \geq 1 \mid Y = \frac{3}{2})$



$y = \frac{3}{2} = \frac{4}{x}$

$\Rightarrow x = \frac{8}{3}$

The conditional density is

$$f(x \mid Y = 3/2) = \frac{f(x,y)}{f_Y(3/2)} = \frac{\frac{1}{4+4\ln 2}}{\frac{2}{3}(1+\ln 2)} \qquad f_Y(3/2) = \frac{2}{3(1+\ln 2)}$$

$$= \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8}$$

Clearly correct since $0 \leq x \leq \frac{8}{3}$ and $f_X(x)$ should be uniform on for $y = 3/2$.

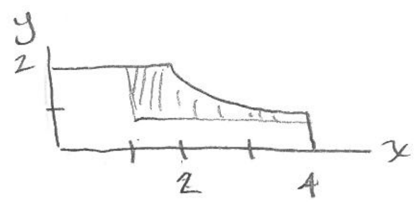
The answer as an integral is

$$\int_1^{8/3} f(x \mid 3/2)\,dx = \int_1^{8/3} \frac{3}{8}\,dx = \frac{5}{8}$$

f) $P(X \geq 1 \mid Y \geq 1/2) = \dfrac{P(X \geq 1, Y \geq 1/2)}{P(Y \geq 1/2)}$

Numerator:

$P(X \geq 1, Y \geq 1/2)$



Must be written as a sum of two integrals

$$= \int_1^2 \int_{1/2}^2 \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\,dx + \int_2^4 \int_{1/2}^{\frac{4}{x}} \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\,dx$$

Denominator:

$$P(Y \geq 1/2) = \int_{\frac{1}{2}}^2 f_Y(y)\,dy = \int_{1/2}^1 \frac{1}{1+\ln 2}\,dy + \int_1^2 \frac{1}{y(1+\ln 2)}\,dy$$

Thus, $$P(X \geq 1 \mid Y \geq 1/2) = \frac{\int_1^2 \int_{1/2}^2 \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\,dx + \int_2^4 \int_{1/2}^{4/x} \frac{1}{4}\left(\frac{1}{1+\ln 2}\right) dy\,dx}{\int_{1/2}^1 \frac{1}{1+\ln 2}\,dy + \int_1^2 \frac{1}{y(1+\ln 2)}\,dy}$$