HW 8 PROBLEMS

1. Chapter 3, # 21

2. Prove that $V \subseteq \mathbb{A}^n(k)$ is irreducible if, and only if, $I(V)$ is a prime ideal.

3. Consider the $V \subseteq \mathbb{A}^3(\mathbb{C})$ defined by the three quadric equations:

$$\begin{array}{llll} f_1 & = 2xz + 2y^2 + 3y + z^2 & = & 0 \\ f_2 & = x + yz + 2z & = & 0 \\ f_3 & = xz + y^2 + 2y & = & 0 \end{array}$$

   Prove that V is isomorphic to $\mathbb{A}^1(\mathbb{C})$. Show explicitly that the coordinate rings $k[V]$ and $k[\mathbb{A}^1(\mathbb{C})]$ are isomorphic.

   Hint: Use Singular and an 'appropriate' term order to find a Groebner basis for $I(V)$.

4. Let C denote the twisted cubic in $\mathbb{A}^3(\mathbb{R})$. Prove that the maps $\phi_1, \phi_2 : \mathbb{A}^3(\mathbb{R}) \to \mathbb{A}^2(\mathbb{R})$ given below define the same morphism from $C \to A^2(\mathbb{R})$.

$$\begin{aligned} \phi_1(x, y, z) &= (2x^2 + y^2,\, z^2 - y^3 + 3xz), \\ \phi_2(x, y, z) &= (2y + xz,\, 3y^2). \end{aligned}$$

5. Consider the morphism $\phi : \mathbb{A}^2(\mathbb{R}) \to \mathbb{A}^5(\mathbb{R})$ defined by

$$(u,\, v) \mapsto (u,\, v,\, u^2,\, uv,\, v^2).$$

   The image of this map is closed and called the *Veronese surface*, $V = \mathrm{Im}(\phi) = \overline{\mathrm{Im}(\phi)}$.

   (a) Find an implicit description of V.
   (b) Prove or disprove: $V \cong \mathbb{A}^2(\mathbb{R})$. Is V rational?

6. In this problem, we will show that the affine line $\mathbb{A}^1(\mathbb{R})$ is *not* isomorphic to the 'non-singular' cubic $V = V(y^2 - x^3 + x)$. Note that V is an example of an *elliptic curve* (or at least gives the $\mathbb{R}$-points on an elliptic curve).

   (a) Sketch a graph of V. (Be quick with this; feel free to use software.)
   (b) Suppose $\phi : \mathbb{A}^1(\mathbb{R}) \to V$ is given by $t \mapsto (a(t), b(t))$. Explain why it must be true that $b(t)^2 = a(t)\big(a(t)^2 - 1\big)$.
   (c) Viewing $a(t)\big(a(t)^2 - 1\big) \in \mathbb{R}[t]$, explain why the two factors must be relatively prime.
   (d) Using the unique factorizations of $a(t)$ and $b(t)$ into products of irreducible polynomials, show that $b^2 = ac^2$ for some polynomial $c(t) \in \mathbb{R}[t]$.
   (e) From the last part, it follows that $c^2 = a^2 - 1$. Deduce from this equation that c, a, and, hence, b must be constant polynomials, and that V is not isomorphic to the affine line.

7. Let $V \subseteq A^n(k)$ be a hypersurface defined by the single equation $x_n - f(x_1, \ldots, x_{n-1}) = 0$. Show that V is isomorphic to $A^{n-1}(k)$.

8. Consider the variety $V = V(y^3 - x^2) \subseteq \mathbb{A}^2(\mathbb{R})$.

(a) Show that $y^3 - x^2$ is irreducible in $\mathbb{R}[x, y]$. Then conclude that V is irreducible, and $\mathbb{R}[V]$ is an integral domain, and $\mathbb{R}(V)$ is a field.

(b) In *one sentence*, explain why problem 3 from the take-home part of your midterm shows that V is not isomorphic to $\mathbb{A}^1(\mathbb{R})$.

(c) Using the term order $>_{\text{lex}}$ with $x > y$ for polynomials in $\mathbb{R}[x, y]$, then the coordinate ring of V is

$$\mathbb{R}[V] = \{a(y) + x\,b(y) \mid a(y),\, b(y) \in \mathbb{R}[y]\}.$$

   i. Justify that $\mathbb{R}[V]$ has the form claimed above.

   ii. Define multiplication for elements in $\mathbb{R}[V]$.

(d) Give an explicit description of the elements of the field of rational functions $\mathbb{R}(V)$ as follows.

   i. Suppose $0 \neq c + x\,d \in \mathbb{R}[V]$, compute $\dfrac{a + x\,b}{c + x\,d} = \left(\dfrac{a + x\,b}{c + x\,d}\right)\left(\dfrac{c - x\,d}{c - x\,d}\right)$.

   ii. From i, conclude that $\mathbb{R}(V) = \mathbb{R}(y) + x\,\mathbb{R}(y)$.

(e) Now show that V is rational by

   i. explicitly giving an isomorphism of their rational function fields (*i.e.* show $k(V) \cong k(\mathbb{A}^1(\mathbb{R}))$).

   ii. if you have not done so in the last part, then explicitly give rational maps $\rho : \mathbb{A}^1(\mathbb{R}) \dashrightarrow V$ and $\psi : V \dashrightarrow \mathbb{A}^1(\mathbb{R})$ that correspond to the maps of function fields from part i. On what open sets U are these maps defined? Show informally that $\phi \circ \rho : \mathbb{A}^1(\mathbb{R}) \dashrightarrow \mathbb{A}^1(\mathbb{R})$ and $\rho \circ \psi : V \dashrightarrow V$ are defined at some points of their domain and are the identity on these points.

(f) Finally, make sure that you understand the point of this problem: Give a summary of the main conclusion.

9. Consider the rational maps $\rho : \mathbb{A}^1(\mathbb{R}) \dashrightarrow \mathbb{A}^3(\mathbb{R})$ and $\psi : \mathbb{A}^3(\mathbb{R}) \dashrightarrow \mathbb{A}^1(\mathbb{R})$ given by

$$\rho(t) = \left(t, \frac{1}{t}, t^2\right) \text{ and } \psi(x, y, z) = \frac{x + yz}{x - yz}.$$

Show that $\psi \circ \rho$ is not defined at any points of $\mathbb{A}^1(\mathbb{R})$. (*Moral*: Compositions of rational maps may not be defined.)

10. Chapter 3, # 25, modified as follows:

(a) Sketch (or somehow get an image in your mind of) the surface $xyz - 1 = 0$. We will call this surface S.

(b) Outline the steps you would follow to prove that $S = V(xyz - 1)$ is rational. Use Proposition 3.57 for your outline. (There is one VERY HARD step which you should point out.)

(c) For an slightly alternative proof that S is rational, try to show that $k(S) \cong k(u, v)$ directly.

   i. Give an isomorphism of $k[S]$ with $k[x, y, \frac{1}{xy}]$. (Note that $k[x, y, \frac{1}{xy}]$ is the localization of $k[x, y]$ with respect to the multiplicatively closed set $\{1, \frac{1}{xy}, \frac{1}{(xy)^2}, \dots\}$.)

ii. Convince yourself by proving (or at the very least justifying without proof) that $k[S]$ is an integral domain.

iii. Compute the field $k(S)$ by computing the quotient field of $k[x, y, \frac{1}{xy}]$.

iv. Now show that $k(S) \cong k(u, v) = k(\mathbb{A}^2)$.