MATH 310: Numerical Analysis
Homework # 4

Selected solutions to the Programming Assignment

1. Examine the function that your instructor provided that computes estimates for $f'(x)$.

   Now use this program to answer the following questions.

   (a) Calculate $f'(3)$ for $f(x) = \ln x$ exactly. Then run your program to estimate the derivative $f'(3)$. Which of the three methods is the most accurate? What value of h gives you the best estimate?

   $f'(3) = \frac{1}{3}$ and MATLAB estimates can be easily calculated.
   Beginning with $h = 1$ and halving h with each iterate, you can make the following table:

Your estimates for the derivative of log(x) at x = 3 are

	h	Right	Left	Center
1	0.100000000000000	0.327898228229910	0.339015516756813	0.333456872493362
2	0.050000000000000	0.330586039024210	0.336142366327623	0.333364202675916
3	0.025000000000000	0.331952112587803	0.334729986820657	0.333341049704230
4	0.012500000000000	0.332640811893121	0.334029712838451	0.333335262365786
5	0.006250000000000	0.332986592611952	0.333681038563398	0.333333815587675
6	0.003125000000000	0.333159842691373	0.333507065101912	0.333333453896643
7	0.001562500000000	0.333246557906932	0.333420169041432	0.333333363474182
8	0.000781250000000	0.333289938089365	0.333376743647591	0.333333340868478
9	0.000390625000000	0.333311633828544	0.333355036606235	0.333333335217390
10	0.000195312500000	0.333322483110123	0.333344184498401	0.333333333804262
11	0.000097656250000	0.333327908106185	0.333338758796344	0.333333333451264
12	0.000048828125000	0.333330620692323	0.333336046032855	0.333333333362589
13	0.000024414062500	0.333331977008129	0.333334689676121	0.333333333342125
14	0.000012207031250	0.333332655172853	0.333334011484112	0.333333333328483
15	0.000006103515625	0.333332994268858	0.333333672388108	0.333333333328483
16	0.000003051757813	0.333333163871430	0.333333502858295	0.333333333364862
17	0.000001525878906	0.333333248709096	0.333333417947870	0.333333333328483
18	0.000000762939453	0.333333291055169	0.333333375456277	0.333333333255723
19	0.000000381469727	0.333333312300965	0.333333354210481	0.333333333255723
20	0.000000190734863	0.333333323942497	0.333333343733102	0.333333333837800

and compute the error estimates too ....

The error estimates for the derivative of log(x) at x = 3 are

	h	Right	Left	Center
1	0.100000000000000	0.005435105103423	-0.005682183423480	-0.000123539160028
2	0.050000000000000	0.002747294309123	-0.002809032994290	-0.000030869342583
3	0.025000000000000	0.001381220745530	-0.001396653487323	-0.000007716370897
4	0.012500000000000	0.000692521440212	-0.000696379505117	-0.000001929032453
5	0.006250000000000	0.000346740721381	-0.000347705230065	-0.000000482254342
6	0.003125000000000	0.000173490641960	-0.000173731768579	-0.000000120563309
7	0.001562500000000	0.000086775426401	-0.000086835708098	-0.000000030140849
8	0.000781250000000	0.000043395243968	-0.000043410314258	-0.000000007535145
9	0.000390625000000	0.000021699504790	-0.000021703272902	-0.000000001884056
10	0.000195312500000	0.000010850223210	-0.000010851165068	-0.000000000470929
11	0.000097656250000	0.000005425227149	-0.000005425463011	-0.000000000117931
12	0.000048828125000	0.000002712641011	-0.000002712699522	-0.000000000029255
13	0.000024414062500	0.000001356325204	-0.000001356342788	-0.000000000008792
14	0.000012207031250	0.000000678160480	-0.000000678150779	0.000000000004851
15	0.000006103515625	0.000000339064475	-0.000000339054774	0.000000000004851
16	0.000003051757813	0.000000169461903	-0.000000169524962	-0.000000000031529
17	0.000001525878906	0.000000084624238	-0.000000084614536	0.000000000004851
18	0.000000762939453	0.000000042278164	-0.000000042122944	0.000000000077610
19	0.000000381469727	0.000000021032368	-0.000000020877148	0.000000000077610
20	0.000000190734863	0.000000009390836	-0.000000010399769	-0.000000000504466

Examining these tables, it is clear that the best estimates are given by the central difference approximation for the 14, 15, and 17 iterates by the central approximation:

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  14  0.000012207031250    0.000000678160480    -0.000000678150779    0.000000000004851
  15  0.000006103515625    0.000000339064475    -0.000000339054774    0.000000000004851
...
  17  0.000001525878906    0.000000084624238    -0.000000084614536    0.000000000004851
```

Then the error in the central estimate starts to *grow*.

(b) Calculate $g'(\arcsin(.8))$ for $g(x) = \tan x$ exactly. Then run your program to estimate the derivative $g'(\arcsin(.8))$. Which of the three methods is the most accurate? What value of h gives you the best estimate?

The exact value of the derivative is $2.\overline{7}$. The central estimate gives you the best answer, and I found $h \approx 0.000000762939453$ (or half this) to give me the best estimate before cancellation error set in.

(c) Calculate $h'(0)$ for $h(x) = \sin(x^2 + \frac{1}{3}x)$ exactly. Then run your program to estimate the derivative $h'(0)$. Which of the three methods is the most accurate? What value of h gives you the best estimate?

Again you need the chain rule:

$$h'(x) = \cos(x^2 + \frac{1}{3}x) \cdot \left(2x + \frac{1}{3}\right),$$

and evaluating at $x = 0$, we find $h'(0) = \cos(0) \cdot \frac{1}{3} = \frac{1}{3}$.
All three methods eventually get the answer of 0.333333333333333. The happens for the central difference approximation for $h = \frac{.1}{2^{19}} \approx 1.9073e-07$, but for the right and left difference approximations this occurs for much smaller values of h. Here are the values for $h = \frac{.1}{2^{47}}$

h	Right	Left
0.000000000000001	0.333333333333334	0.333333333333333

(d) Calculate $j'(0)$ for $j(x) = |x|$ exactly. Then run your program to estimate the derivative. Which of the three methods is the most accurate? What value of h gives you the best estimate? Explain what happened.

The derivate does not exist at $x = 0$!
However, the program happily computes -1, 1, or 0 for the left, right and central difference approximations. These correctly express the slope of the tangent line as you approach $x = 0$ from the left, right, or take an average of these.

2. Write a program that will produce a table of errors from using the left, right, and central difference approximations. Now answer the following questions.

(a) Now use your program on the function $f(x) = \arctan(x)$, $x = \sqrt{2}$, $h = 1$, $M = 26$. Now run your program again with $M = 40$. Fiddle with the number of iterations M to get a feel for the error.

i. Compute $f'(\sqrt{2})$ exactly.

$f'(x) = \dfrac{1}{1+x^2}$ and $f'(\sqrt{2}) = \frac{1}{3}$.

ii. Find the number of iterations M when cancelation error begins to set in. (Don't change the values of the other arguments, i.e. use $f(x) = \arctan(x)$, $x = \sqrt{2}$, $h = 1$.)

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Your estimates for the derivative of atan(x) at x = 1.41421 are

           h                  Right               Left                Center

17  0.000015258789062   0.333330935660342   0.333335731033003   0.333333333346673
18  0.000007629394531   0.333332134498050   0.333334532173467   0.333333333335759
19  0.000003814697266   0.333332733920543   0.333333932765527   0.333333333343035
20  0.000001907348633   0.333333033602685   0.333333633025177   0.333333333313931
```

For the central difference approximation, you can see cancellation error starting after the 18th iterate. For the right and left difference approximations, you see cancellation error starting around the 25th and 26th iterate respectively.

iii. What value of h gives you the best estimate in the central derivative?
The value of h given in the 18th iterate, $h = 0.000007629394531$.

(b) Run your program to calculate the derivative of $g(x) = \cos(x)$ at $x = \frac{\pi}{2}$.

i. Which of the three methods for computing derivatives is the most accurate? Explain.

Because of the symmetry of $\cos(x)$ at $x = \frac{\pi}{3}$, you get exactly the same result. Indeed, $\cos(\frac{\pi}{2}) = 0$ and repeated use of the formulas for cosine of sums (and differences) we see that

$$\frac{\cos(\frac{\pi}{2}+h) - \cos(\frac{\pi}{2})}{h} = \frac{\cos(\frac{\pi}{2}) - \cos(\frac{\pi}{2}-h)}{h} = -\frac{\sin(h)}{h}.$$

Using formulas for the cosine of the sum, these quantities are also equal to the central approximation:

$$\frac{\cos(\frac{\pi}{2}+h) - \cos(\frac{\pi}{2}-h)}{2h}.$$

If you don't remember these formulas, here is one of them:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

ii. What value of h gives you the best estimate in the central derivative?

My investigations led me to say h as displayed in the 26th iterate.

```
Your estimates for the derivative of cos(x) at x = 1.5708 are

           h                  Right               Left                Center

25  0.000000059604645   -0.999999999999999   -0.999999999999999   -0.999999999999999
26  0.000000029802322   -1.000000000000000   -1.000000000000000   -1.000000000000000
```

iii. Suppose you run the program to compute $g'(\frac{\pi}{2})$ and start with $h = .01$. Find the number of iterations M when cancelation error begins to set in.

3. Write a program to compute $f''(x)$ using Equation (1).

$$f''(x) \approx \frac{1}{h^2}\Big[f(x+h) - 2f(x) + f(x-h)\Big]. \tag{1}$$

(a) Calculate $f''(3)$ for $f(x) = \ln x$ by hand. Then run your program to estimate the derivative. What value of h gives you the best estimate?

Differentiating twice yields $f''(x) = -x^{-2} = \dfrac{-1}{x^2}$ so $f''(3) = -\frac{1}{9}$.
The closest value I computed came from $h = 0.000488281250000$.

iterate	h	f''(x)	error
12.000000000000000	0.000488281250000	-0.111111111007631	-0.000000000103480

(b) Calculate $g''(\arcsin(.8))$ for $g(x) = \tan x$. Then run your program to estimate the derivative. What value of h gives you the best estimate?

Differentiating twice yields $g''(x) = 2\sec^2(x)\tan(x)$. You can compute $g''(x)$ exactly at $\arcsin(.8)$ to get the true value of $\frac{200}{27} = 7.\overline{407}$. I found $h \approx 0.000012207031250$ to give the best estimate. Then cancellation error sets in.