MATH 310: Numerical Analysis

Homework # 4 **due Wednesday, October 7 at 9 am**

Textbook problems

Section 2.2:

8.

12. modified as follows:

    Prove that an estimate for the second derivative of a well-behaved function $f(x)$ is given by

$$f''(x) \approx \frac{1}{h^2}\Big[f(x+h) - 2f(x) + f(x-h)\Big]. \tag{1}$$

Then using Taylor's theorem with remainder term prove that

$$f''(x) = \frac{1}{h^2}\Big[f(x+h) - 2f(x) + f(x-h)\Big] + \mathcal{O}(h^2).$$

15. (Feel free to use MATLAB for the values in (a).)

Programming Assignment

1. Examine the function that your instructor provided that computes estimates for $f'(x)$. In particular, make sure you understand how the MATLAB function `inline` is working. This function computes the left, right, and central difference approximations to the derivative and takes four arguments. The arguments are explained below.

```
%
% fname   a string with the name of the function to differentiate
% x       the point at which to compute f'
% h       a small number used initially in the difference quotient
% M       number of times to halve h in computing derivatives
%
% derivate(fname, x, h, M)
```

In addition, the program formats output nicely so that you can compare the various results. For example, the following code

```
disp(' ')
disp(sprintf('      Your estimates for the derivative of %s at x = %g are', fname, x))
disp(' ')
disp(sprintf('         h                Left                Right               Center'))
disp(' ')
```

produces output as follows:

```
       Your estimates for the derivative of x^2 at x = 2 are

          h                Left                Right               Center

   0.10000000000000   4.10000000000000   3.90000000000000   4.00000000000000
   0.05000000000000   4.04999999999999   3.95000000000000   4.00000000000000
```

which is quite nice for comparing methods for computing derivatives.

Now use this program to answer the following questions.

(a) Calculate $f'(3)$ for $f(x) = \ln x$ exactly. Then run your program to estimate the derivative $f'(3)$. Which of the three methods is the most accurate? What value of h gives you the best estimate?

(b) Calculate $g'(\arcsin(.8))$ for $g(x) = \tan x$ exactly. Then run your program to estimate the derivative $g'(\arcsin(.8))$. Which of the three methods is the most accurate? What value of h gives you the best estimate?

(c) Calculate $h'(0)$ for $h(x) = \sin(x^2 + \frac{1}{3}x)$ exactly. Then run your program to estimate the derivative $h'(0)$. Which of the three methods is the most accurate? What value of h gives you the best estimate?

(d) Calculate $j'(0)$ for $j(x) = |x|$ exactly. Then run your program to estimate the derivative. Which of the three methods is the most accurate? What value of h gives you the best estimate? Explain what happened.

2. In this problem you'll want to investigate some inherent difficulties in numerical differentiation. Turn to p. 50 in your text and examine Table 2.2 shown there. You want to produce similar results. Write a program that will produce a table of errors from using the left, right, and central difference approximations. Your function will need all the arguments that the differentiation program needed, plus the true value of the derivative. For example, the following would work well:

```
%  deriverr(fname, x, h, M, truevalue)
%
%  fname        a string with the name of the function to differentiate
%  x            the point at which to compute f'
%  h            a small number used initially in the difference quotient
%  M            number of times to halve h in computing derivatives
%  truevalue    the value of f'(x) for error calculations
%
```

Moreover, the output of your program could look something like this:

```
>> deriverr('log(x)',1,.5,3,1);

 k      h           f(x+h)              f(x-h)           f(x+h)-f(x-h)   (f(x+h)-f(x-h))/2h
-------------------------------------------------------------------------------------------
 1  5.00e-01  0.4054651081081644  -0.6931471805599453  1.0986122886681096 1.0986122886681096
 2  2.50e-01  0.2231435513142098  -0.2876820724517809  0.5108256237659907 1.0216512475319814
 3  1.25e-01  0.1177830356563835  -0.1335313926245226  0.2513144282809061 1.0052577131236244


Press any key to continue....

  The errors in your calculations for the derivative of log(x) at x = 1 are

        h                Left                Right               Center

   0.500000000000000   0.189069783783671  -0.386294361119891  -0.098612288668110
   0.250000000000000   0.107425794743161  -0.150728289807124  -0.021651247531981
   0.125000000000000   0.057735714748932  -0.068251140996181  -0.005257713123624


Press any key to continue to see the ratios E(h)/E(h/2) ....

  The ratio of subsequent errors in your calculations for the derivative of log(x) at x = 1 are

        k                Left                Right               Center

   1.000000000000000   1.760003584201624   2.562852412206125   4.554577676064513
   2.000000000000000   1.860647178445946   2.208436190327684   4.117997125916982
```

The following code displays output to the screen quite nicely.

```
disp(' ')
disp(' k      h           f(x+h)              f(x-h)           f(x+h)-f(x+h)       f''(x)  ')
disp('--------------------------------------------------------------------------------------------')

                          (lines of code deleted)

disp(sprintf('%2d  %0.2e  %5.16f  %5.16f  %5.16f %5.16f', k, h, right, left, dcenter, centerestimate))
```

Now answer the following questions.

(a) Test your program by duplicating the results from the output above. (Do not hand this in.) Now use your program on the function $f(x) = \arctan(x)$, $x = \sqrt{2}$, $h = 1$, $M = 26$. Now run your program again with $M = 40$. Fiddle with the number of iterations M to get a feel for the error.

   i. Compute $f'(\sqrt{2})$ exactly.

   ii. Find the number of iterations M when cancelation error begins to set in. (Don't change the values of the other arguments, i.e. use $f(x) = \arctan(x)$, $x = \sqrt{2}$, $h = 1$.)

   iii. What value of h gives you the best estimate in the central derivative?

(b) Run your program to calculate the derivative of $g(x) = \cos(x)$ at $x = \frac{\pi}{2}$.

   i. Which of the three methods for computing derivatives is the most accurate? Explain.

   ii. What value of h gives you the best estimate in the central derivative?

   iii. Suppose you run the program to compute $g'(\frac{\pi}{2})$ and start with $h = .01$. Find the number of iterations M when cancelation error begins to set in.

3. Write a program to compute $f''(x)$ using Equation (1).

   (a) Calculate $f''(3)$ for $f(x) = \ln x$ by hand. Then run your program to estimate the derivative. What value of h gives you the best estimate?

   (b) Calculate $g''(\arcsin(.8))$ for $g(x) = \tan x$. Then run your program to estimate the derivative. What value of h gives you the best estimate?