Exam 1
Math 202 F01

Name: Solutions
02/15/06

1. A particle is initially at the point $\mathbf{r}(0) = (-1, -2, 0)$. It moves so that its velocity is given by $\mathbf{v}(t) = (3\cos(5t), 4, 3\sin(5t))$.

   (a) (6 pts.) Find the acceleration of the object at $t = \pi/2$. (You should simplify your answer so that no trigonometric functions appear.)

   $$\vec{a}(t) = \vec{v}'(t) = (-15\sin(5t), 0, 15\cos(5t))$$
   $$\vec{a}\left(\tfrac{\pi}{2}\right) = \left(-15\sin\left(\tfrac{5\pi}{2}\right), 0, 15\cos\left(\tfrac{5\pi}{2}\right)\right) \quad \text{but } \tfrac{5\pi}{2} = 2\pi + \tfrac{\pi}{2} \text{ so}$$
   $$= \left(-15\sin\left(\tfrac{\pi}{2}\right), 0, 15\cos\left(\tfrac{\pi}{2}\right)\right)$$
   $$= (-15, 0, 0)$$

   (b) (6 pts.) Find the position of the object at all times.

   $$\vec{r}'(t) = \vec{v}(t) = (3\cos(5t), 4, 3\sin(5t))$$
   $$\text{so } \vec{r}(t) = \left(\tfrac{3}{5}\sin(5t) + C, 4t + D, \tfrac{-3}{5}\cos(5t) + E\right)$$
   But $(-1, -2, 0) = \vec{r}(0) = (C, D, -3/5 + E)$, so $C = -1$, $D = -2$, $E = 3/5$
   $$\vec{r}(t) = \left(\tfrac{3}{5}\sin(5t) - 1, 4t - 2, -3/5\cos(5t) + 3/5\right)$$

   (c) (8 pts.) Find the length of the path the object has followed between the times $t = \pi$ and $t = 3\pi$.

   $$\text{Length} = \int_{\pi}^{3\pi} \|\vec{r}'(t)\| \, dt = \int_{\pi}^{3\pi} \|\vec{v}(t)\| \, dt = \int_{\pi}^{3\pi} \sqrt{9\cos^2(5t) + 16 + 9\sin^2(5t)} \, dt$$
   $$= \int_{\pi}^{3\pi} \sqrt{9+16} \, dt = \int_{\pi}^{3\pi} 5 \, dt = 10\pi$$

   (d) (3 pts.) At what time (if ever) does the object cross the xz-plane?

   The xz-plane is $y = 0$. Setting $y(t) = 0$ from (b) gives $4t - 2 = 0$, so $t = \frac{1}{2}$.

2. Consider the three points in space: $A = (1,2,1)$, $B = (0,2,-1)$, and $C = (2,1,1)$.

(a) (5 pts.) Which of B and C is the closest to A?

$\vec{AB} = (0,2,-1)-(1,2,1) = (-1,0,-2)$, $\|\vec{AB}\| = \sqrt{1+0+4} = \sqrt{5}$

$\vec{AC} = (2,1,1)-(1,2,1) = (1,-1,0)$, $\|\vec{AC}\| = \sqrt{1+1+0} = \sqrt{2}$ $\quad \sqrt{2} < \sqrt{5}$

so C is closest to A

(b) (7 pts.) Give a parameterization of a line through B that is parallel to the line between A and C.

$$\vec{\ell}(t) = (0,2,-1) + t\underbrace{(1,-1,0)}_{\vec{AC}} = (t, 2-t, -1)$$

(c) (9 pts.) Give an equation for the plane in which the points A, B, and C lie.

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -2 \\ 1 & -1 & 0 \end{vmatrix} = (-2,-2,1)$$

$$(-2,-2,1)\cdot(x,y,z) = (-2,-2,1)\cdot(1,2,1)$$

$$-2x - 2y + z = -5$$

or

$$2x + 2y - z = 5$$

3. (6 pts.) An object is acted on by a force of $\mathbf{F} = (2,1,1)N$. However, other constraints on the object allow it to move only in the direction given by $\mathbf{d} = (1,1,-1)$. Calculate a vector representing the part of the force $\mathbf{F}$ that can actually affect the motion of the object.

$$\text{proj}_{\vec{d}}\vec{F} = \frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{(2,1,1)\cdot(1,1,-1)}{(1,1,-1)\cdot(1,1,-1)}(1,1,-1)$$

$$= \frac{2}{3}(1,1,-1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

4. Consider the two parameterized paths $\mathbf{r}(t) = (t^2, 2, t)$ and $\mathbf{s}(t) = (1 + \ln t, 2t, 3t - 2)$.

(a) (5 pts.) Show that particles following these paths would collide.

$\vec{r}(t) = \vec{s}(t)$

$(t^2, 2, t) = (1 + \ln t, 2t, 3t-2)$

$\left.\begin{array}{l} t^2 = 1 + \ln t \\ 2t = 2 \\ t = 3t-2 \end{array}\right\}$ all have solution $t=1$

So at $t=1$

$\vec{r}(1) = (1,2,1) = \vec{s}(1)$

(b) (5 pts.) At what angle would the particles hit one another? Your answer may involve an inverse trigonometric function.

tangent vectors to paths at $t=1$ are

$\vec{r}\,'(1) = (2t, 0, 1)\Big|_{t=1} = (2,0,1)$

$\vec{s}\,'(1) = (\tfrac{1}{t}, 2, 3)\Big|_{t=1} = (1,2,3)$

and $\theta = \cos^{-1}\left(\dfrac{(2,0,1)\cdot(1,2,3)}{\|(2,0,1)\|\,\|(1,2,3)\|}\right) = \cos^{-1}\left(\dfrac{5}{\sqrt{5}\sqrt{14}}\right) = \cos^{-1}\left(\dfrac{\sqrt{5}}{\sqrt{14}}\right)$

(c) (3 pts.) Is the angle in part (b) acute (less than $\pi/2$), right, or obtuse (greater than $\pi/2$)? Explain how you know. (No points will be given for an answer without explanation.)

θ is acute since $\dfrac{\sqrt{5}}{\sqrt{14}} > 0$ so $\cos^{-1}\left(\dfrac{\sqrt{5}}{\sqrt{14}}\right) < \dfrac{\pi}{2}$

5. Suppose you are given three vectors **a**, **b**, and **c**. What simple formulas could ~~use~~ you use to calculate each of the following?

(a) (4 pts.) The volume of the parallelepiped with edges **a**, **b**, and **c**.

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$

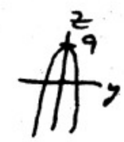
(b) (4 pts.) The area of the parallelogram with edges **a** and **b**.

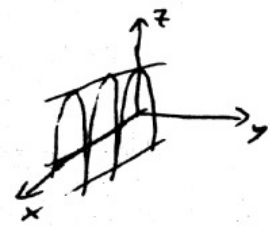
$\|\vec{a}\times\vec{b}\|$

6. The following equations can all be graphed relatively easily in $\mathbf{R}^3$. For each, with *one phrase or sentence* indicate what about the equation makes it possible to graph it without much effort, and then give the graph.

   (a) (7 pts.) $z = 9 - y^2$

   No x appears, so we graph in a yz-plane and then draw this for <u>every</u> x in $\mathbb{R}^3$

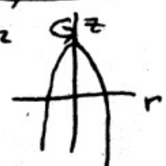
   

   (b) (7 pts.) $z = 9 - x^2 - y^2$

   x^2+y^2 indicates <u>cylindrical</u> symmetry

   $z = 9 - r^2$

   now draw for <u>every</u> θ

   (c) (7 pts.) $x + 2y + z = 1$

   All linear (first-degree) terms indicates a plane.

   Normal is $\vec{n} = (1,2,1)$

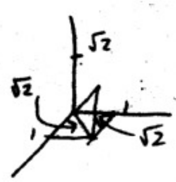
   Points along axes are easy to locate

   plane contains this triangle

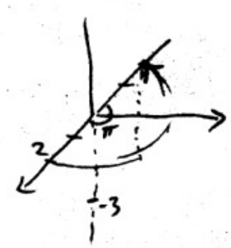
7. Convert between coordinate systems, as indicated.

   (a) (4 pts.) $(1, 1, \sqrt{2})$ in rectangular coordinates = ? in spherical coordinates

   $(\rho, \theta, \phi) = (2, \frac{\pi}{4}, \frac{\pi}{4})$

   (b) (4 pts.) $(2, \pi, -3)$ in cylindrical coordinates = ? in rectangular coordinates

   $(x, y, z) = (-2, 0, -3)$