MATH 253 Name: Solutions

Quiz 7 March 27, 2020

Instructions: No books. No notes. No internet. No phone. No nothing except a writing implement and a printed version of this quiz or blank paper for responses. Only problems for which all work is shown will receive credit.

0. I started this exam at (fill in the time) ____________________.

0.5. This quiz took me ____________________ (include units) to complete.
(Include only time doing math.)

0.75. Use the extra space to write any comments you want to share with me.

∞. I attest that I did not cheat in any way on this quiz or violate the rules governing this assessment, and that this work is entirely my own.

Sign and date: ____

1. (3 pts.) Consider the function $f(x,y) = e^{-((x-1)^2+(y+2)^2)}$ on $\mathbb{R}^2$.

   (a) If you think about it, before doing *any* calculus you should know the this function has a global extremum at some point (a,b) in its domain. Quickly answer these three questions in the space at the right and below.

What are the coordinates of the global extremum (a,b)? $(1,-2)$

Is this a (circle one) global max/min/saddle point? Why?

$e^{-[\;\;]^2}$ has a max of 1 when $[\;\;]=0$.

What is the absolute (circle one and give value) min/max of $f(x,y)$ at this point?

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   (b) **Showing all your work**, find the critical points of $f(x,y)$. (Your answer should agree with your previous one, so check.)

$f_x(x,y) = e^{-(x-1)^2-(y+2)^2} \cdot -2(x-1) = -2(x-1)e^{-(x-1)^2-(y+2)^2}$

$f_y(x,y) = e^{-(x-1)^2-(y+2)^2} \cdot -2(y+2) = -2(y+2)e^{-(x-1)^2-(y+2)^2}$

Since $e^{-\alpha} \neq > 0$ for all α, requiring $f_x = 0$ and $f_y = 0 \Rightarrow (x-1)=0 \Rightarrow x=1$

$(y+2)=0 \Rightarrow y=-2$

Thus, $\boxed{(1,-2)}$ is the only critical point of $f(x,y)$.

From now on, let $\alpha = -(x-1)^2-(y+2)^2$

   (c) Now use the second derivative test, again **showing all work for credit**, to justify that the critical point is a (circle one) local maximum, local minimum or saddle point. (Note: it's also a *global* extremum for $f(x,y)$. For extra credit, explain why this is true.)

Find $D = f_{xx}f_{yy} - (f_{xy})^2$ first.

$f_{xx} = \frac{\partial}{\partial x} -2(x-1)e^{-\alpha} = -2[(x-1)e^{-\alpha}\cdot -2(x-1) + e^{-\alpha}] = -2e^{-\alpha}[-2(x-1)^2+1]$

$f_{yy} = \frac{\partial}{\partial y} -2(y+2)e^{-\alpha} = -2[(y+2)e^{-\alpha}\cdot -2(y+2) + e^{-\alpha}] = -2e^{-\alpha}[(y+2)^2+1]$

$f_{xy} = \frac{\partial}{\partial y}[-2(x-1)e^{-\alpha}] = -2(x-1)e^{-\alpha}[-2(y+2)] = 4e^{-\alpha}(x-1)(y+2)$

Evaluate at c.p. $(1,-2)$ noting that $e^{-\alpha} = 1$.

$f_{xx}(1,-2) = -2e^{-\alpha}[-2(x-1)^2+1]\Big|_{(1,-2)} = -2$

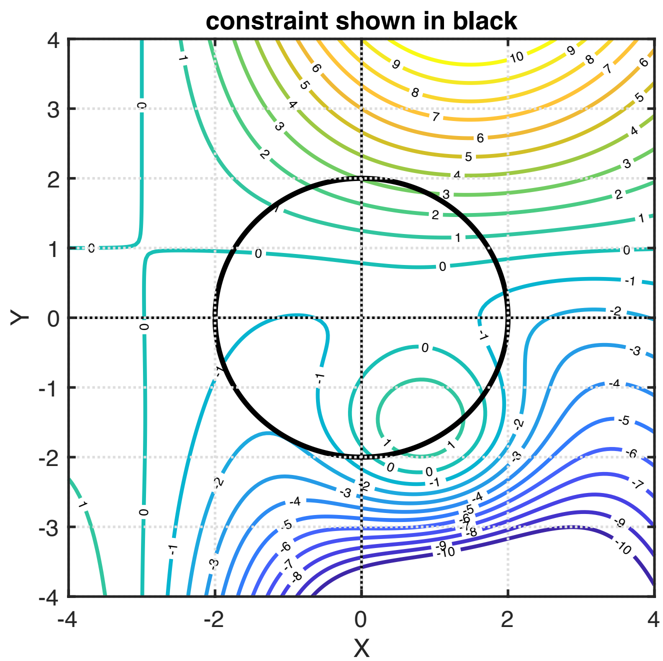
$f_{yy}(1,-2) = -2e^{-\alpha}[-2(y+2)^2+1]\Big|_{(1,-2)} = -2$

$f_{xy}(1,-2) = 4e^{-\alpha}(x-1)(y+2)\Big|_{(1,-2)} = 4\cdot 0 = 0$

Since $D = (-2)(-2) - 0^2 = 4 > 0$ and $f_{xx}(1,-2) < 0$, the point $(1,-2)$ is a local max!

2. (2 pts.) (FYI: there will be a more computationally oriented problem on Lagrange multipliers on Exam 2)

Consider the optimization problem: Find the minimum and maximum of $f(x,y)$ subject to the constraint $g(x,y) = x^2 + y^2 - 4 = 0$. The constraint equation is plotted in thick black, on top of a contour plot for $f(x,y))$ is given.



Give, as best you can, estimates for the maximum and minimum values of $f(x,y)$ subject to the constraint $g(x,y) = 0$.

Maximum value: ≈ 3.2

Minimum value: -2

Now, suppose in the last part, you answered that the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = 0$ occurred at the point (a,b).

Explain the relationship between the gradient vectors $\nabla g(a,b)$ and $\nabla f(a,b)$ and what they might look like at the point (a,b). Finally, as best you can, explain how this relates to the method of Lagrange multipliers.

$(a,b) \approx (-1, -1.5)$

Here $f(x,y) = -2$ and $g(a,b) = 0$ are tangent.

Since ∇f and ∇g are orthogonal to level curves,

they must be scalar multiples. I.e. $\nabla f(a,b) = \lambda \nabla g(a,b)$.

Part of Lagrange method.