Section 4.1

1. LET G BE A GROUP ACTING ON A, A NONEMPTY SET. PROVE THAT IF $a, b \in A$ AND $b = g \cdot a$ FOR SOME $g \in G$, THEN $G_b = gG_ag^{-1}$ WHERE G_a IS THE STABILIZER OF a. DEDUCE THAT IF G ACTS TRANSITIVELY ON A THEN THE KERNEL OF THE ACTION IS $\cap_{g\in G} gG_ag^{-1}$.

   Proof. (Bastille) Let $a, b \in A$, $g \in G$ such that $b = g \cdot a$. Let $c \in G_b$. Then note that

   $$c = (gg^{-1})c(gg^{-1}) = g(g^{-1}cg)g^{-1}.$$

   But

   $$\begin{aligned}(g^{-1}cg) \cdot a &= (g^{-1}c) \cdot (g \cdot a) = (g^{-1}c) \cdot b \quad \text{since } b = g \cdot a\\ &= g^{-1} \cdot (c \cdot b) = g^{-1} \cdot b \quad \text{since } c \in G_b\\ &= g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1 \cdot a = a.\end{aligned}$$

   Therefore $g^{-1}cg \in G_a$ and thus $c \in gG_ag^{-1}$ and so $G_b \subseteq gG_ag^{-1}$. (1)
   Now assume $d \in gG_ag^{-1}$. Then there exists $f \in G_a$ such that $d = gfg^{-1}$. Then

   $$\begin{aligned}d \cdot b &= (gfg^{-1}) \cdot (g \cdot a) = (gfg^{-1}g) \cdot a\\ &= (gf) \cdot a = g \cdot (f \cdot a) = g \cdot a \quad \text{since } f \in G_a\\ &= b.\end{aligned}$$

   Therefore, $d \in G_b$ and thus $gG_ag^{-1} \subseteq G_b$. (2) Combining (1) and (2) leads to $G_b = gG_ag^{-1}$.

   If G acts transitively on A, then the kernel of the action, K, is by definition:

   $$K = \cap_{b\in A} G_b,$$

   but if we fix $a \in A$, then because the action is transitive, for any $b \in A$, there exists $g \in G$ such that $g = b \cdot a$, so in particular $A = \{b \in A\} = \{g \cdot a | g \in G\}$. But by the result just proven, then we have: $G_b = gG_ag^{-1}$ when $b = g \cdot a$. Hence $K = \cap_{b\in A} G_b = \cap_{g\in G} gG_ag^{-1}$.

   □

2. LET G BE A PERMUTATION GROUP ON THE SET A, LET $\sigma \in G$ AND LET $a \in A$. PROVE THAT $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$. DEDUCE THAT IF G ACTS TRANSITIVELY ON A THEN

   $$\bigcap_{\sigma\in G} \sigma G_a \sigma^{-1} = 1.$$

   Proof. (Hazlett) Choose $\lambda \in \sigma G_a \sigma^{-1}$. Then $\lambda = \sigma\rho\sigma^{-1}$ for some $\rho \in G_a$. So $\sigma\rho\sigma^{-1} \cdot \sigma(a) = \sigma\rho \cdot a = \sigma \cdot a = \sigma(a)$. Thus $\lambda \in G_{\sigma(a)}$. Select $\tau \in G_{\sigma(a)}$. Notice that $\sigma^{-1}\tau\sigma \cdot a = \sigma^{-1}\tau \cdot \sigma(a) = \sigma^{-1} \cdot a = \sigma(a)$. Hence $\sigma^{-1}\tau\sigma \in G_a$. Consequently $\tau \in \sigma G_a \sigma^{-1}$. Therefore $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$.

   We can deduce that

   $$\bigcap_{\sigma\in G} \sigma G_a \sigma^{-1} = \bigcap_{\sigma\in G} G_{\sigma(a)} = \bigcap_{b\in A} G_b.$$

   If G is the trivial group then the only element in G will be 1 and $\bigcap_{b\in A} G_b = G = 1$. If G is not the trivial group there are no elements that fix each $b \in A$ except 1. Consequently $\bigcap_{b\in A} G_b = 1$. Therefore

   $$\bigcap_{\sigma\in G} \sigma G_a \sigma^{-1} = \bigcap_{b\in A} G_b = 1.$$

   □

Chapter 4.2

8. PROVE THAT IF H HAS FINITE INDEX n THEN THERE IS A NORMAL SUBGROUP K OF G WITH $K \leq H$ AND $|G : K| \leq n!$. (SCHAMEL)

   Proof. Let G act by left multiplication on the left cosets of H in G. Let π_H denote the permutation represented by this action. By Theorem 4.2.3, then $\ker \pi_H \lhd G$ and $\ker \pi_H \leq H$. Furthermore, $G/\ker \pi_H \cong \pi_H(G)$ and $|\pi_H(G)| = n$, since this action is transitive. Thus, by Cayley's Theorem, $\pi_H(G)$ is isomorphic to a subgroup of S_n, and hence $|G : \ker \pi_H| = |G/\ker \pi_H| \leq |S_n| = n!$. □

9. PROVE THAT IF p IS A PRIME AND G IS A GROUP OF ORDER p^α FOR SOME $\alpha \in \mathbb{Z}^+$, THEN EVERY SUBGROUP OF INDEX p IS NORMAL IN G. DEDUCE THAT EVERY GROUP OF ORDER p^2 HAS A NORMAL SUBGROUP OF ORDER p.

   Proof. (Schamel) Since p is the smallest prime dividing p^α, we have by Corollary 4.2.5 that every subgroup of index p is normal in G. Let G be a group of order p^2. Then $|G| \geq 2^2$, so G has at least three elements of order greater than 1. If $x \in G$ and $|x| = p$ then $\langle x \rangle = p$ and we are done. By Lagrange's Theorem, the only possible order of an element $x \in G$ besides 1 and p is p^2. But then, if $|x| = p^2$ then G is cyclic of order p^2 and so contains an element of order p. Hence every group of order p^2 has a normal subgroup of order p. □

10. PROVE THAT EVERY NON-ABELIAN GROUP OF ORDER 6 HAS A NONNORMAL SUBGROUP OF ORDER 2. USE THIS TO CLASSIFY GROUPS OF ORDER 6.

    Proof. (Allman)

    Let G be a non-Abelian group where $|G| = 6$. Then by Lagrange's Theorem, G has an element x of order 2 and an element y of order 3. Since $[G : \langle y \rangle] = 2$, the subgroup $\langle y \rangle \lhd G$. Consider the element x and its conjugate $yxy^{-1} = z$. Then $z \neq e$, since $x \neq e$. In addition, $z \neq x$, since G is not Abelian and the elements x and y must generate G by counting. It follows that $\langle x \rangle$ is not a normal subgroup of G.

    Since $\langle y \rangle \lhd G$ and G is non-Abelian, it must be that $xyx^{-1} = y^2 = y^{-1}$. Thus, G has a presentation as $G = \langle x, y \mid x^2 = e,\ y^3 = e,\ xyx^{-1} = y^{-1} \rangle$ and G is isomorphic to D_6 (and therefore also to S_3). Finally, if G were Abelian, then it must be generated by the product of an element of order 2 with an element of order 3, and thus is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. □

Chapter 4.3

2. FIND ALL CONJUGACY CLASSES AND THEIR SIZES IN THE FOLLOWING GROUPS:
   (BAGGETT) *NOTE: ALL ORDERS FOR CENTRALIZERS OF ELEMENTS IN (A) AND (B) WERE COMPUTED IN EXERCISE 2.5.6

   A. D_8

CONJUGACY CLASS	SIZE
$O_1 = \{1\}$	$[D_8 : C_{D_8}(1)] = 8/8 = 1$
$O_r = \{r, r^3\}$	$[D_8 : C_{D_8}(r)] = 8/4 = 2$
$O_{r^2} = \{r^2\}$	$[D_8 : C_{D_8}(r^2)] = 8/8 = 1$
$O_s = \{s, sr^2\}$	$[D_8 : C_{D_8}(s)] = 8/4 = 2$
$O_{sr} = \{sr, sr^3\}$	$[D_8 : C_{D_8}(sr)] = 8/4 = 2$

   B. Q_8

Conjugacy Class	Size
$O_1 = \{1\}$	$[Q_8 : C_{Q_8}(1)] = 8/8 = 1$
$O_{-1} = \{-1\}$	$[Q_8 : C_{Q_8}(-1)] = 8/8 = 1$
$O_i = \{i, -i\}$	$[Q_8 : C_{Q_8}(i)] = 8/4 = 2$
$O_j = \{j, -j\}$	$[Q_8 : C_{Q_8}(j)] = 8/4 = 2$
$O_k = \{k, -k\}$	$[Q_8 : C_{Q_8}(k)] = 8/4 = 2$

c. A_4

Conjugacy Class	Size
$O_1 = \{1\}$	1
$O_{(12)(34)} = \{(12)(34), (13)(24), (14)(23)\}$	3
$O_{(123)} = \{(123), (134), (142), (243)\}$	4
$O_{(132)} = \{(132), (124), (143), (234)\}$	4

5. If the center of G is of index n, prove that every conjugacy class has at most n elements.

Proof. (Baggett) Let $g \in G$ and let O_g be the conjugacy class containing g. Note that $Z(G) \leq C_G(g)$. We have that $[G : Z(G)] = [G : C_G(g)][C_G(g) : Z(G)]$. Since $[C_G(g) : Z(G)] \geq 1$, it follows that $[G : Z(G)] \geq [G : C_G(g)]$. Therefore, $|O_g| = [G : C_G(g)] \leq [G : Z(G)] = n$. Thus, every conjugacy class has at most n elements. □

6. Assume G is a non-abelian group of order 15. Prove that $Z(G) = 1$. Use the fact that $\langle g \rangle \leq C_G(g)$ for all $g \in G$ to show that there is at most one possible class equation for G. [Use Exercise 36, Section 3.1.]

Proof. (Mobley) We know from Exercise 4 in Section 3.2 that if $|G| = pq$ where p and q are prime that either G is abelian or $Z(G) = 1$. Since we know G is non-abelian and $15 = 3 \cdot 5$, it must be the case that $Z(G) = 1$.

By the class equation, $15 = 1 + \sum_{i=1}^{r} |G : C_G(g_i)|$. Thus, $\sum_{i=1}^{r} |G : C_G(g_i)| = 14$. Since the size of each orbit must divide G, we need to consider $1, 3$ and 5 such that $14 = a \cdot 1 + b \cdot 3 + c \cdot 5$ where a, b and c are the number of orbits of the respective size and $0 \leq a \leq 1, 0 \leq b \leq 3$ and $0 \leq c \leq 2$. However, orbits of size one are contained in the center. Therefore, $14 = b \cdot 3 + c \cdot 5$ and $b = 3$ and $c = 1$. There is no other combination of b and c that will produce 14.

□

7. For $n = 3, 4, 6$ and 7 make lists of the partitions of n and give representatives for the corresponding conjugacy classes of S_n.

Proof. (Mobley) For S_3, the table would be as follows.

Element Representative	Number of each type
e	1
(1 2)	3
(1 2 3)	2
Total	6

For S_4, the table would be as follows.

Element Representative	Number of each type
e	1
(1 2)	6
(1 2)(3 4)	3
(1 2 3)	8
(1 2 3 4)	6
Total	24

For S_6, the table would be as follows.

Element Representative	Number of each type
e	1
(1 2)	15
(1 2)(3 4)	45
(1 2)(3 4)(5 6)	15
(1 2 3)	40
(1 2 3)(4 5 6)	40
(1 2 3 4)	90
(1 2 3 4)(5 6)	90
(1 2 3 4 5)	144
(1 2 3 4 5 6)	120
(1 2 3)(4 5)	120
Total	720

For S_7, the table would be as follows.

Element Representative	Number of each type
e	1
(1 2)	21
(1 2)(3 4)	105
(1 2)(3 4)(5 6)	105
(1 2 3)	70
(1 2 3)(4 5 6)	280
(1 2 3 4)	210
(1 2 3 4)(5 6)	630
(1 2 3 4 5)	504
(1 2 3 4 5 6)	840
(1 2 3)(4 5)	420
(1 2 3)(4 5)(6 7)	210
(1 2 3 4 5 6 7)	720
(1 2 3 4 5)(6 7)	504
(1 2 3 4)(5 6 7)	420
Total	5040

□

9. SHOW $|C_{S_n}((1\,2)(3\,4))| = 8 \cdot (n-4)!$. FOR ALL $n \geq 4$. DETERMINE THE ELEMENTS IN THIS CENTRALIZER EXPLICITLY.

Proof. We know the order of $C_{S_n}(1\,2)(3\,4)$ is the order of S_n divided by the number of conjugates of $(1\,2)(3\,4)$. Notice

$$\mathcal{O}_{(1\,2)(3\,4)} = \binom{n}{4} \cdot 3 = \frac{3 \cdot n!}{4!(n-4)!} = \frac{n!}{8(n-4)!}.$$

Therefore:

$$|C_{S_n}((1\,2)(3\,4))| = n! \cdot \left(\frac{8(n-4)!}{n!}\right) = 8(n-4)!.$$

Recall that if $\sigma \in C_{S_a}(\tau)$ for some $\sigma, \tau \in S_a$, then $\sigma \in C_{S_n}(\tau)$ for all $n \geq a$. Since $|C_{S_4}((1\,2)(3\,4))| = 8$, Then $\sigma\tau \in C_{S_n}((1\,2)(3\,4))$ where $\sigma \in C_{S_4}((1\,2)(3\,4))$, and τ is an element in S_n which does not contain 1, 2, 3, or 4 in its cycle decomposition. □

12. FIND A REPRESENTATIVE FOR EACH CONJUGACY CLASS OF ORDER 4 IN S_8 AND S_{12}.

Proof. In S_8:

Representative	class size
$(1\,2\,3\,4)$	420
$(1\,2\,3\,4)(5\,6)$	2520
$(1\,2\,3\,4)(5\,6)(7\,8)$	1260
$(1\,2\,3\,4)(5\,6\,7\,8)$	1260

In S_{12}:

Representative	class size
$(1\,2\,3\,4)$	2970
$(1\,2\,3\,4)(5\,6)$	83160
$(1\,2\,3\,4)(5\,6)(7\,8)$	623700
$(1\,2\,3\,4)(5\,6\,7\,8)$	623700
$(1\,2\,3\,4)(5\,6)(7\,8)(9\,10)$	1247400
$(1\,2\,3\,4)(5\,6\,7\,8)(9\,10)$	374220
$(1\,2\,3\,4)(5\,6)(7\,8)(9\,10)(11\,12)$	311850
$(1\,2\,3\,4)(5\,6\,7\,8)(9\,10)(11\,12)$	187110
$(1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12)$	1247400

□

26. LET G BE A TRANSITIVE PERMUTATION GROUP ON THE FINITE SET A WITH $|A| > 1$. SHOW THAT THERE IS SOME $\sigma \in G$ SUCH THAT $\sigma(a) \neq a$ FOR ALL $a \in A$.

Proof. (Gillispie) Because A is finite, we know that $G \leq S_A$, which has order $|S_A| = |A|!$, and so $|G| \leq |A|!$ and is thus finite.
Consider $\cup_{a\in A} G_a$.
Because G is a transitive group, and by proposition 4.1.2, $|A| = |\mathcal{O}_a| = |G : G_a|$.
Since G is finite $|G : G_a| = \frac{|G|}{|G_a|}$, and so $|G_a| = \frac{|G|}{|G:G_a|} = \frac{|G|}{|A|}$.
I claim that $|\cup_{a\in A} G_a| < |G|$, which implies the existance of $\sigma \in G$ such that $\sigma \notin \cup_{a\in A} G_a$, that is, $\sigma(a) \neq a$ for all A.
But, notice that $e(a) = a$ for all $a \in A$, and so $e \in G_a$ for all $a \in A$, so all stabilizers contain e.
So

$$\begin{aligned}
|\cup_{a\in A} G_a| &\leq 1 + \sum_{a\in A} |G_a - \{e\}| \\
&= 1 + \sum_{a\in A} |G_a| - 1 \\
&= 1 + \sum_{a\in A} \frac{|G|}{|A|} - 1 \\
&= 1 + |A|\frac{|G|}{|A|} - |A| \\
&= 1 + |G| - A \\
&\leq |G| - 1 \\
&< |G|.
\end{aligned}$$

As needed. $\square$

30. If G is a group of odd order, prove for any non-identity element $x \in G$ that x and x^{-1} are not conjugate in G.

Proof (Granade). Choose some $x \in G \setminus \{e\}$. Then, $|\mathcal{O}_x|$ divides $|G|$. But then, since $|G|$ is odd, we have that $2 \nmid |G|$, and so $2 \nmid |\mathcal{O}_x|$. Suppose for the sake of contradiction that there exists $x \in G$ such that x is conjugate to x^{-1}. Then $\mathcal{O}_x = \mathcal{O}_{x^{-1}}$.

Next, choose $y \in \mathcal{O}_x$. By definition, there exists $g \in G$ such that $y = gxg^{-1}$, and so $y^{-1} = gx^{-1}g^{-1} \in \mathcal{O}_{x^{-1}} = \mathcal{O}_x$. Since $|y| \mid |G|$, and since $2 \nmid |G|$, we have that $|y| \neq 2$ and thus that $y \neq y^{-1}$. We may therefore partition $\mathcal{O}_x$ into pairs of the form $\left\{y, y^{-1}\right\}$, showing that $\mathcal{O}_x$ is of even order. This is a contradiction, and so we conclude that there does not exist any $x \in G$ such that $x^{-1} \in \mathcal{O}_x$. $\square$