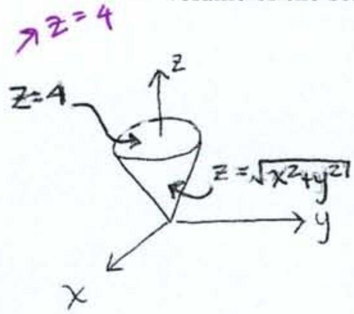
MATH 202
Midterm 3

Name : SOLUTIONS

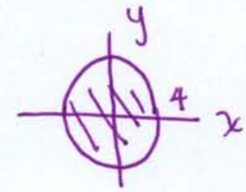
April 25, 2012

Instructions: Show all work for full credit. Poor notation or sloppy work will be penalized.

1. (12 pts.) Set up, but **do not integrate**, a triple integral **in cylindrical coordinates** that computes the volume of the solid that lies below $z = 4$ and above $z = \sqrt{x^2 + y^2}$. See figure.



$\to z = r$



$0 \leq r \leq 4$

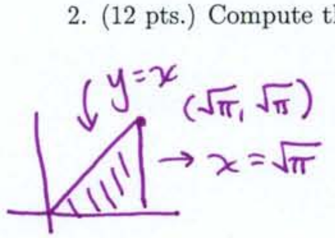
$0 \leq \theta \leq 2\pi$

$$\boxed{\int_0^{2\pi}\int_0^4\int_r^4 r\,dz\,dr\,d\theta}$$

Alternative: $\int_0^{2\pi}\int_0^4\int_0^z r\,dr\,dz\,d\theta$

2. (12 pts.) Compute the iterated integral:

$$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \sin(x^2)\,dx\,dy$$



Switch order:

$$= \int_0^{\sqrt{\pi}}\int_0^x \sin(x^2)\,dy\,dx = \int_0^{\sqrt{\pi}} \sin(x^2)\,y\Big|_0^x\,dx$$

$$= \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_0^{\sqrt{\pi}} = -\frac{1}{2}\left[\cos(\pi) - \cos(0)\right]$$

$$= \frac{-1}{2}\left[-1-1\right] = \boxed{1}$$

3. (14 pts.) A solid sphere B of radius 2 centered at the origin has charge density

$$\rho(x, y, z) = e^{(x^2+y^2+z^2)^{\frac{3}{2}}} \text{ coulombs/cm}^3$$

at any point (x, y, z) in the sphere in cm. Compute the total electrical charge of the solid B. Include units.

B

$r = 2$ $\quad \iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$ coulombs

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} e^{(\rho^2)^{3/2}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^2 e^{\rho^3} \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3} e^{\rho^3} \sin\varphi \Big|_0^2 \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}(e^8 - e^0) \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}(e^8 - 1) \sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}(e^8 - 1)(-\cos\varphi)\Big|_0^{\pi} \, d\theta = \int_0^{2\pi} \frac{1}{3}(e^8-1)\left[-\cos(\pi) - -\cos(0)\right] d\theta = \frac{2}{3}(e^8-1)\int_0^{2\pi} d\theta$$

$$= \boxed{\frac{4\pi}{3}(e^8 - 1) \text{ coulombs}}$$

4. (14 pts.) Use Green's theorem to evaluate the integral

$$\oint_C (e^{\cos(x)} - \frac{1}{3}y^3)\,dx + (\ln y + \frac{1}{3}x^3)\,dy$$

where C is the circle of radius 3 centered at the origin oriented in the counterclockwise direction.

$M = e^{\cos(x)} - \frac{1}{3}y^3 \qquad \frac{\partial M}{\partial y} = -y^2$

$N = (\ln(y) + \frac{1}{3}x^3) \qquad \frac{\partial N}{\partial x} = x^2$

Green's Theorem:

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Therefore, compute $\iint_R (x^2 - -y^2)\, dA = \iint_R x^2 + y^2 \, dA$ for R:

$$= \int_0^{2\pi}\int_0^3 r^2 (r\,dr\,d\theta) = \int_0^{2\pi}\int_0^3 r^3 \, dr\,d\theta = \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^3 \, d\theta = \int_0^{2\pi} \frac{81}{4} \, d\theta$$

$$= \frac{81}{4}(2\pi) = \boxed{\frac{81\pi}{2}}$$

5. (20 pts.) Consider the vector field with continuous partial derivatives defined on all of $\mathbb{R}^2$,

$$\mathbf{F}(x,y,z) = \left\langle\, ye^x + \sin\left(\frac{\pi}{2}y\right),\ e^x + \frac{\pi}{2}x\cos\left(\frac{\pi}{2}y\right) + 2y \,\right\rangle.$$

(a) (8 pts.) By finding a potential function f, prove that $\mathbf{F}$ is conservative.

if $\vec{F} = \nabla f$, then $\frac{\partial f}{\partial x} = ye^x + \sin(\frac{\pi}{2}y) \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x}\,dx = ye^x + x\sin(\frac{\pi}{2}y) + g(y)$

and $\frac{\partial f}{\partial y} = e^x + \frac{\pi}{2}x\cos(\frac{\pi}{2}y) + 2y \Rightarrow f(x,y) = \int \frac{\partial f}{\partial y}\,dy$

$= ye^x + x\sin(\frac{\pi}{2}y) + y^2 + K$

$$\boxed{f(x,y) = ye^x + x\sin(\tfrac{\pi}{2}y) + y^2 + K}$$

(b) (7 pts.) Supposing $\mathbf{F}$ represents a force field, compute the amount of work done by $\mathbf{F}$ on a particle moving along the path

$$\mathbf{r}(t) = \langle\, t,\ 2t+1 \,\rangle,\quad 0 \le t \le 1.$$

(Assume that $\mathbf{F}$ is measured in Newtons and distances are measured in meters.)

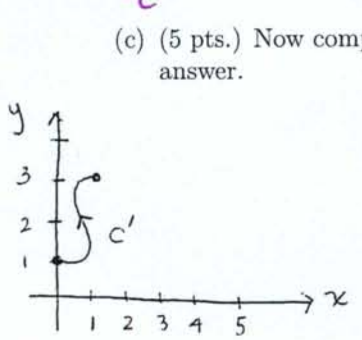
Work $= \int_C \vec{F}\cdot d\vec{r}$ Newtons

Since $\vec{F}$ is conservative, we need only evaluate the potential function $f(x,y)$ at the endpts.

beginning pt: $t=0 \to \vec{r}(0) = (0,1)$ endpoint: $t=1$, $\vec{r}(1) = (1,3)$

Thus, $\int_C \vec{F}\cdot d\vec{r} = f((1,3)) - f((0,1)) = (3e + 1\sin(\frac{3\pi}{2}) + 3^2) - (1(1) + 0 + 1^2) = (3e - 1 + 9) - (2)$ $= \boxed{3e + 6 \text{ Nm}}$ or joules

(c) (5 pts.) Now compute the work done by $\mathbf{F}$ on a particle moving along the path C' pictured. Explain your answer.



Again $3e+6$ Nm since $\int_C \vec{F}\cdot d\vec{r}$ is independent of path for this conservative vector field on a simply-connected domain.

6. (18 pts.) Consider the vector field

$$\mathbf{F}(x,y,z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - x^2\,\mathbf{k}$$

(a) (7 pts.) Compute curl $\mathbf{F}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -x^2 \end{vmatrix} = (0 - xy)\hat{i} - (-2x - x)\hat{j} + (yz - 0)\hat{k}$$

$$= \boxed{-xy\hat{i} + 3x\hat{j} + yz\hat{k} = \langle -xy, 3x, yz \rangle}$$

(b) (7 pts.) Compute div $\mathbf{F}$.

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle xz, xyz, -x^2 \rangle = \tfrac{\partial}{\partial x}(xz) + \tfrac{\partial}{\partial y}(xyz) + \tfrac{\partial}{\partial z}(-x^2)$$

$$= z + xz + 0 = \boxed{z + xz}$$

(c) (4 pts.) Suppose the vector field $\mathbf{F}$ represents the velocity field for some fluid. Compute the divergence of $\mathbf{F}$ at the point $(1,1,1)$ and indicate what $\text{div}\,\mathbf{F}(1,1,1)$ tells you about the net fluid flow at $(1,1,1)$.

$\text{div}\,\vec{F}(1,1,1) = 1 + (1)(1) = \boxed{2}$

net flow is out $\qquad\qquad$ $\boxed{\text{Source}}$

7. (10 pts.)

True or False? If the following statements are correct, mark them 'True.' If they are false, give corrected versions.

(a) If $-C$ denotes 'C backwards,' then $\int_C F \cdot d\mathbf{r} = \int_{-C} F \cdot d\mathbf{r}$.

False. $\displaystyle\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$

(b) If $-C$ denotes 'C backwards,' then $\int_C f\,ds = \int_{-C} f\,ds$.

True. $\qquad ds \geq 0$