HW 5 PROBLEMS

1-6. Hassett, Chapter 3, #1-6a. For 6a, only show that every finite subset $S \subset \mathbb{A}^n(k)$ is a variety.

7. The 'twisted cubic' in $\mathbb{A}^3(\mathbb{R})$ is the curve C defined by the vanishing of the polynomials $y - x^2$ and $z - x^3$. We have (or almost have) an alternative description of this variety given parametrically; that is, C is (the closure of) the image of the morphism

$$\begin{aligned} \phi : \mathbb{A}^1(\mathbb{R}) &\to \mathbb{A}^3(\mathbb{R}) \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

   Noting that $\mathrm{Im}(\phi) = \{(t, t^2, t^3) \mid t \in \mathbb{R}\}$, prove that $I = I(\mathrm{Im}(\phi)) = \langle y - x^2,\, z - x^3 \rangle$.

   Hint: One inclusion is easy. For the other, using some appropriate monomial ordering divide any $f \in I$ by $y - x^2$ and $z - x^3$ and consider the remainder term.

8. Generalize the last problem: Fix $d \in \mathbb{Z}^+$ and consider the curve in $\mathbb{A}^d(\mathbb{R})$ given by the image of the morphism:

$$\begin{aligned} \phi_d : \mathbb{A}^1(\mathbb{R}) &\to \mathbb{A}^d(\mathbb{R}) \\ t &\mapsto (t, t^2, \ldots, t^d). \end{aligned}$$

   In slightly modified form, this is usually considered a *projective* variety, but we will still informally call this affine version the 'rational normal curve.'

   Formulate and prove a generalization of problem 1 for the rational normal curve.

9. Let $V \subseteq \mathbb{A}^6(\mathbb{R})$ be the variety of 2×3 matrices of rank at most 1. That is, if $A \in V$, then $\mathrm{rank}(A) \leq 1$. Let $I \subset \mathbb{R}[a_{11}, \ldots, a_{23}]$ be the ideal generated by two polynomials f_1, f_2:

$$I = \langle a_{11}a_{22} - a_{12}a_{21}, a_{12}a_{23} - a_{13}a_{22} \rangle.$$

   Show explicitly that $V \subsetneq V(I)$.