MATH 631 Name : SOLUTIONS

Midterm October 9, 2020

Instructions: This exam is closed book and closed notes, and two hours in length. You may use **only** your brain and blank scratch paper in writing solutions. Do not linger on any one problem for too long; there is not time for that. There are 106 points available on the exam and you earn your score out of 100.

The problems in Part I are computational in nature, and full marks are awarded for correct answers. You need only justify your answers if you are explicitly asked to do so. Part II involves writing proofs and is theoretical in nature. You should prove results from first principles and not simply quote statements from the book. Your proofs will be graded not only on correctness, but points will be awarded/taken away for poor writing and exposition. Blank paper is supplied for scratch work, but final responses should be written in the space provided.

Part I. (70 pts.) Short answer. Briefly justify your responses.

1. (7 pts.)

   (a) (2 pts.) Carefully state the First Isomorphism Theorem for groups.

   Look up.

   I am proving more!

   (b) (5 pts.) Letting $\mathbf{F}_{37}$ denote the finite field with 37 elements, now use the First Isomorphism Theorem to prove that $\mathrm{SL}_7(\mathbf{F}_{37}) \trianglelefteq \mathrm{GL}_7(\mathbf{F}_{37})$. (You must use the First Isomorphism Theorem for credit.)

Note that $\det : GL_7(\mathbb{F}_{37}) \longrightarrow (\mathbb{F}_{37}^{\times}, \cdot)$ is a group homomorphism since $\det(AB) = \det(A)\det(B)$ for all $A, B \in GL_7(\mathbb{F}_{37})$. Moreover, $\det$ is surjective with kernel $K = SL_7(\mathbb{F}_{37}) = \{A \in GL_7(\mathbb{F}_{37}) \mid \det(A) = 1\}$. By the First Isomorphism Theorem, $SL_7(\mathbb{F}_{37}) \trianglelefteq GL_7(\mathbb{F}_{37})$ and $GL_7(\mathbb{F}_{37}) / SL_7(\mathbb{F}_{37}) \cong \mathbb{F}_{37}^{\times}$

2. (10 pts.) Consider the cylic group $C_{10648} = \langle x \rangle$ of order $10648 = 2^3 \cdot 11^{\cancel{2}}.3$

   (a) (4 pts.) Give, with short justification, the number of elements $x \in C_{10648}$ with $|x| = 484 = 2^2 \cdot 11^2$.

C_{10648} has a unique cyclic subgroup of order 484 which is generated by $\varphi(484) = \varphi(2^2)\varphi(11^2) = 2 \cdot 11(10) = 220$ elements. This all follows from the Fundamental Theorem of Cyclic groups.

   (b) (6 pts.) List explicitly the elements x^a, with $0 \leq a \leq 10647$, of order 22.

$22 = 2 \cdot 11$ so we must find a such that $\dfrac{10{,}648}{(10{,}648,\ a)} = 22$

or that $\gcd(2^3 \cdot 11^3, a) = 2^2 \cdot 11^2 = 484$. Those a with $0 \leq a \leq 10{,}647$ satisfying this are $a = 484K$ with $(2 \cdot 11, K) = 1$ and $1 \leq K \leq 22$

i.e.

Answer: $|x^a| = 22$ if $a =$ $484K$ for $K = 1, 3, 5, 7, 9, 13, 15, 17, 19, 21$.

3. (25 pts. – 5 pts. each) Consider the symmetric group S_9 and let $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9) \in S_9$.

(a) Give the order of σ in S_9.

$|\sigma| = \text{lcm}(|(123)|, |(456789)|) = \boxed{6}$

(b) Is $\sigma \in A_9$? Why or why not?

$\sigma \notin A_9$ (123) is even, but $(4\ 5\ 6\ 7\ 8\ 9)$ is not.

(c) Let τ be the element $(6\ 7\ 8)(1\ 2\ 3\ 4\ 5\ 9)$. Give an element α that conjugates σ to τ, i.e. give α such that $\alpha\sigma\alpha^{-1} = \tau$.

$\alpha = (1\,6\,3\,8\,5\,2\,7\,4)$ is one choice.

(d) Now give a second element β, $\beta \neq \alpha$, that conjugates σ into τ. Lots of options here

One example: σ: $(2\ 3\ 1)(4\ 5\ 6\ 7\ 8\ 9)$

τ: $(6\,7\,8)(1\ 2\ 3\ 4\ 5\ 9)$

$\Rightarrow \beta = (2\ 6\ 3\ 7\ 4\ 1\ 8\ 5)$ should work

(e) What is the order of the centralizer subgroup $C_{S_9}(\sigma)$ in S_9. Why?

The number of conjugates of σ in S_9 equals $[S_9 : C_{S_9}(\sigma)]$.

Thus, $\binom{9}{3}2!\,5! = \dfrac{9!}{|C_{S_9}(\sigma)|} \Rightarrow |C_{S_9}(\sigma)| = \dfrac{9!}{\binom{9}{3}2!\,5!} = \dfrac{9\cdot 8\cdot 7\cdot 6}{\frac{9\cdot 8\cdot 7}{6}\cdot 2} = \boxed{18}$

4. (8 pts.) Explicitly list the conjugacy classes of elements in D_8 and then (explicitly) write the class equation for D_8.

Using the book's notation. $D_8 = \langle r, s \mid r^4 = 1,\ s^2 = 1,\ srs^{-1} = r^3 \rangle$

$\{e\}, \{r^2\}$ since $Z(D_8) = \{e, r^2\}$

$\{r, r^3\}$ Lots of ways to see this: $|C_{D_8}(r)| = |\langle r \rangle| = 4$

$\{s, sr^2\}$ $r^{-i} s r^{i} = sr^{2i}$ $i = 0, 1$

$\{sr, sr^3\}$ left-overs.

5. (20 pts. – 5 pts. each) Consider the alternating group A_4.

(a) List all elements of A_4.

Answer: $A_4 = \{e, (12)(34), (13)(24), (14)(23), (123), (321), (124), (421), (134), (431), (234), (432)\}$

(b) List all the left cosets of the Klein 4-group K_4 in A_4.

$K_4 = \{e, (12)(34), (13)(24), (14)(23)\}$

Compute directly.

$(123)\, K_4 = \{(123), (134), (243), (142)\}$

$(321)\, K_4 = \{(321), (234), (124), (143)\}$

(c) Prove or disprove: The Klein 4-group (circle one) IS / IS NOT a normal subgroup of A_4.

$K_4 \trianglelefteq A_4$: Conjugation by $g \in A_4$ preserves cycle structure

so for all $g \in A_4$, $x \in K_4$ $\quad gxg^{-1} \in K_4$.

Alternative: K_4 is the unique subgroup of order 4 is A_4.

(d) Let K denote the Klein 4-group from parts (b,c). Viewing K as a subgroup of A_5, prove or disprove: (circle one) K IS / IS NOT a normal subgroup of A_5

$K \ntrianglelefteq A_5$

Pf 1: A_5 is not simple.

Pf 2: $\underbrace{(152)}_{g}\underbrace{(12)(34)}_{x}\underbrace{(251)}_{g^{-1}} = (15)(34) \notin K$