Final Exam
Math 202 F01

Name: Solutions
05/09/06

1. (10 pts.) A vector field F in $\mathbb{R}^3$ is given by

$$F(x,y,z) = \left(zye^{xz} + \ln(1+y),\ e^{xz} + \frac{x}{1+y},\ xye^{xz} + z \right)$$

and a curve C is parameterized by

$$\mathbf{r}(t) = (t, t^2 - t, 1 - t^2), \quad 0 \le t \le 1.$$

(a) Find *all* potential functions for F.

$\frac{\partial f}{\partial x} = zye^{xz} + \ln(1+y)$, so $f(x,y,z) = ye^{xz} + x\ln(1+y) + C(y,z)$

so $e^{xz} + \frac{x}{1+y} = \frac{\partial f}{\partial y} = e^{xz} + \frac{x}{1+y} + \frac{\partial C}{\partial y}(y,z)$

so $\frac{\partial C}{\partial y} = 0$, so $C(y,z) = D(z)$ and $f(x,y,z) = ye^{xz} + x\ln(1+y) + D(z)$

so $xye^{xz} + z = \frac{\partial f}{\partial z} = xye^{xz} + 0 + \frac{dD}{dz}(z)$

so $\frac{dD}{dz} = z$, so $D(z) = \frac{z^2}{2} + E$

and $f(x,y,z) = ye^{xz} + x\ln(1+y) + \frac{z^2}{2} + E$

(b) Use your answer in part (a) to compute $\int_C F \cdot ds$.

$\int_C F \cdot d\vec{s} = f(\text{end of } C) - f(\text{start of } C)$

end of $C = \vec{r}(1) = (1,0,0)$
start of $C = \vec{r}(0) = (0,0,1)$

$= f(1,0,0) - f(0,0,1)$ with $f(x,y,z) = ye^{xz} + x\ln(1+y) + z^2/2$

$= 0 - (\frac{1}{2}) = -\frac{1}{2}$

(c) What physical quantity might the integral in part (b) represent? With your interpretation of the integral, what do F and C represent physically? If F is a force field, and C the path an object follows, then $\int_C F \cdot d\vec{s}$ = work done on the object by F as it moves along C.

2. (10 pts.) Among the points (x, y) satisfying the constraint $3x^2 + y^2 = 3$, find all those maximizing the function $f(x, y) = x^2 - y$.

$g(x,y) = 3x^2 + y^2 = 3$ is the constraint

$f(x,y) = x^2 - y$ is the function to maximize

Using Lagrange multipliers,

$\nabla f = \lambda \nabla g \Rightarrow (2x, -1) = \lambda(6x, 2y)$ so $\left.\begin{array}{l} 2x = \lambda 6x \\ -1 = \lambda 2y \\ 3x^2 + y^2 = 3 \end{array}\right\}$ 3 eqs 3 unknowns

$2x = \lambda 6x \Rightarrow x = 0$ and so $3 \cdot 0^2 + y^2 = 3$, $y = \pm\sqrt{3}$ or $\lambda = \frac{1}{3}$ and so $-1 = \frac{2}{3}y$, $y = -3/2$, $3x^2 + (-3/2)^2 = 1$, so $x = \pm\frac{1}{2}$

Candidates: $(0, -\sqrt{3}), (0, +\sqrt{3})$, $(\frac{1}{2}, -\frac{3}{2}), (-\frac{1}{2}, -\frac{3}{2})$

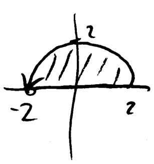
Evaluating f at all these pts gives the largest value at

$(\frac{1}{2}, -\frac{3}{2})$ and $(-\frac{1}{2}, -\frac{3}{2})$

3. (10 pts.) Use Green's Theorem to compute

$$\oint_C -yx^2\, dx + xy^2\, dy,$$

where C is the straight line from $(-2, 0)$ to $(2, 0)$, followed by the semicircular arc around the origin from $(2, 0)$ back to $(-2, 0)$.



$$\oint_{C=\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{D} (y^2 + x^2)\, dA = \int_0^{\pi}\int_0^2 r^2\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{\pi} \frac{r^4}{4}\Big|_0^2\, d\theta = \int_0^{\pi} 4\, d\theta = 4\pi$$

4. (10 pts.) Give an equation for the plane containing the three points $(2,1,1)$, $(1,-2,1)$ and $(1,2,2)$.

Let $\vec{a} = (2,1,1)-(1,-2,1) = (1,3,0)$

$\vec{b} = (2,1,1)-(1,2,2) = (1,-1,-1)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 0 \\ 1 & -1 & -1 \end{vmatrix} = (-3,+1,-4)$$

so $-3(x-2) + 1(y-1) + -4(z-0) = -4$, or $-3x+y-4z=-9$

5. (10 pts.) Consider the surface formed by the part of the graph of $z = 8 - 2x^2 - 2y^2$ where $z \geq 0$.

(a) Sketch a graph of the surface

$z = 8 - 2r^2$

(b) Give a parameterization of the surface.

$\Phi(u,v) = (u, v, 8-2u^2-2v^2)$ u,v such that $u^2+v^2 \leq 4$

or $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 8-2r^2)$ $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$

(c) Give an integral to compute the area of the surface. You do not need to fully evaluate the integral, provided that you leave it so that all that remains to be done is the evaluation of an iterated integral.

$T_u = (1, 0, -4u)$

$T_v = (0, 1, -4v)$

$T_u \times T_v = (4u, 4v, 1)$

$dS = \sqrt{16u^2+16v^2+1}\, du\, dv$

$$SA = \int_{-2}^{2}\int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} \sqrt{1+16u^2+16v^2}\, du\, dv$$

or

$T_r = (\cos\theta, \sin\theta, -4r)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

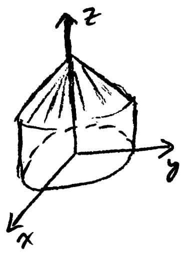
$T_r \times T_\theta = (4r^2\cos\theta, 4r^2\sin\theta, \underbrace{r\cos^2\theta + r\sin^2\theta}_{= r})$

$dS = \sqrt{\underbrace{16r^4\cos^2\theta + 16r^4\sin^2\theta}_{16r^4} + r^2}\, dr\, d\theta$

$= r\sqrt{16r^2+1}\, dr\, d\theta$

$$SA = \int_0^{2\pi}\int_0^{2} r\sqrt{16r^2+1}\, dr\, d\theta$$

6. (10 pts.) The inside of a yurt is shaped like the region between $z = 0$ and $z = 10 - \sqrt{x^2 + y^2}$, with $x^2 + y^2 \leq 25$, where x, y, z are measured in meters. If the density of mosquitoes inside the yurt is given by $\rho(x, y, z) = 12z$ mosquitoes/m^3, what is the total number of mosquitoes inside the yurt? $z = 10 - \sqrt{x^2+y^2} = 10 - r$



$$\iiint_V \rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^5\int_0^{10-r} (12z)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^5 6z^2 r\Big|_{z=0}^{10-r} dr\,d\theta = \int_0^{2\pi}\int_0^5 6(10-r)^2 r\,dr\,d\theta$$

$$= 6\int_0^{2\pi}\int_0^5 100r - 20r^2 + r^3\,dr\,d\theta = 6\int_0^{2\pi}\left(50r^2 - \frac{20}{3}r^3 + \frac{r^4}{4}\right)\Big|_0^5 d\theta$$

$$= 6\int_0^{2\pi} 50(5^2) - \frac{20}{3}(5^3) + \frac{(5^4)}{4}\,d\theta = 12\pi\left(50(5^2) - \frac{20}{3}(5^3) + \left(\frac{5^4}{4}\right)\right)$$

$$= \pi(5^2)\left(600 - 80(5) + 3(25)\right) = \pi(25)(275) = 6875\pi$$

7. (10 pts.) Give a linear approximation, valid for points near $(-1, 1, 2)$, to the function $f(x, y, z) = 3x^2 + 2yz^2 - 5y - 2$.

$f(-1,1,2) = 3 + 8 - 5 - 2 = 4$

$\frac{\partial f}{\partial x}(-1,1,2) = 6x\Big|_{(-1,1,2)} = -6$

$\frac{\partial f}{\partial y}(-1,1,2) = 2z^2 - 5\Big|_{(-1,1,2)} = 8 - 5 = 3$

$\frac{\partial f}{\partial z}(-1,1,2) = 4yz\Big|_{(-1,1,2)} = 8$

$L(x,y,z) = 8 - 6(x+1) + 3(y-1) + 8(z-2)$

8. (8 pts.) Let $F(x, y, z) = (2xy, z - y^2, x^2 + y^2 + z)$, and let S be a $2 \times 2 \times 2$ cube, centered at the origin, with edges parallel to the coordinate axes. Compute the flux of F through S. (Hint: Use the Divergence Theorem.)

$$\nabla \cdot F = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(z - y^2) + \frac{\partial}{\partial z}(x^2+y^2+z) = 2y - 2y + 1 = 1$$

$$\iint_{\substack{S = \partial V \\ \text{surface of cube}}} F \cdot d\vec{S} = \iiint_{\substack{V \\ \text{cube}}} (\nabla \cdot F)\, dV = \iiint_V 1\, dV = \text{volume of } V = 8$$

9. (10 pts.) The electric potential in a certain region of space is given by the function $V(x, y, z) = 5x^2 - yz$.

(a) What is the rate of change of V at the point $(1, 2, 2)$ in the direction given by the vector $(-1, -1, 1)$?

$$\nabla V\big|_{(1,2,2)} = (10x, -z, -y)\big|_{(1,2,2)} = (10, -2, -2) \qquad \vec{u} = \frac{(-1,-1,1)}{\sqrt{3}}$$

$$\nabla V\big|_{(1,2,2)} \cdot \vec{u} = (10, -2, -2) \cdot \frac{(-1,-1,1)}{\sqrt{3}} = \frac{-10+2-2}{\sqrt{3}} = \frac{-10}{\sqrt{3}}$$

(b) At the point $(1, 2, 2)$, what is the direction in which V increases most rapidly?

$\nabla V_{(1,2,2)} = (10, -2, -2)$ gives the direction.

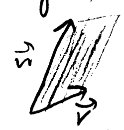
10. (12 pts. — 3 pts. each) Give short answers.

(a) What does it mean for a vector field F to be irrotational? (Indicate both the mathematical condition F must satisfy, and the physical meaning.) $\nabla \times \vec{F} = \vec{0}$

If $\vec{F}$ represents the velocity field of a fluid, "irrotational" means a paddle ball placed in the fluid would not rotate.

(b) What is the geometric meaning of the length of the vector $\mathbf{v} \times \mathbf{w}$?

$\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram with sides $\vec{v}$ & $\vec{w}$



(c) Stokes' Theorem states that $\iint_S \nabla \times F \cdot d\mathbf{S} = \int_C F \cdot d\mathbf{s}$. What is the relationship between S and C here? (Be sure you explain the relationship of their orientations as part of your answer.)

S is a surface and $C = \partial S$ is its boundary. Orientations are chosen so that if you walk around C with your head in the direction given by the normal to S and S on your left, you will go in the direction C should be traversed.

(d) Give formulas expressing the rectangular coordinates x, y, z in terms of spherical coordinates.

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$