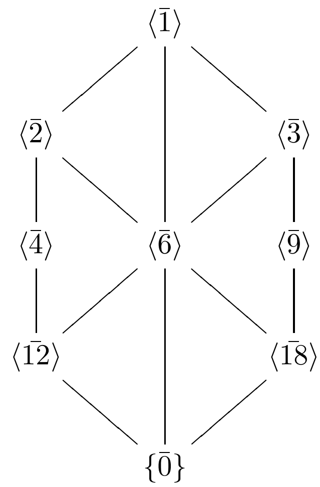
Selected Midterm Solutions
and other miscellany....

Part I. 1. Draw a lattice diagram for the abelian group $\mathbb{Z}/36\mathbb{Z}$.



Part II. 1. Let $g_1, g_2, \cdots, g_r$ be representatives of conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.

Proof. If G acts on itself by conjugation, then by the class equation,

$$|G| = |Z(G)| + \sum_{i=1}^{k} \big[G : C_G(g_i)\big],$$

where the sum is over distinct conjugacy classes of G of size greater than 1. (We may assume that the representatives g_i are ordered so that $g_i \notin Z(G)$ for $i = 1, \ldots, k$.) Letting $j = |Z(G)|$, we note that G has $r = j + k$ distinct conjugacy classes with $j \geq 1$ and $r > k$.

For each of the elements g_i, $i = 1, \ldots, k$, notice that $|C_G(g_i)| \geq r$, since by hypothesis g_i commutes with each g_j, $j = 1, \ldots, r$. Then for each of these k conjugacy classes $\mathcal{O}_{g_i}$ of size greater than 1,

$$|\mathcal{O}_{g_i}| = \big[G : C_G(g_i)\big] = \frac{|G|}{|C_G(g_i)|} \leq \frac{|G|}{r}.$$

Combining this with the class equation, we have

$$\begin{aligned}
&|G| = |Z(G)| + \sum_{i=1}^{k} \big[G : C_G(g_i)\big] \\
\implies &|G| \leq (r-k) + \sum_{i=1}^{k} \frac{|G|}{r} \\
\implies &|G| \leq (r-k) + k\frac{|G|}{r} \\
\implies &|G|\big(1 - \frac{k}{r}\big) \leq (r-k).
\end{aligned}$$

Now since $(r-k) \neq 0$, we find

$$|G| \leq r.$$

Clearly, $|G| \geq r$. Thus we conclude

$$|G| = r.$$

This means that G has $|G|$ distinct conjugacy classes and necessarily then, each is of size 1. Thus, for all $g \in G$, $g \in Z(G)$. $\square$

2. See class notes.

3. (a) While it would not suffice to quote the theorem "Any group of order 77 with $7 \nmid 10 = 11 - 1$ is cyclic, mimicking the proof of this statement for groups of order pq, p, q prime and $p < q$, $p \nmid q - 1$ will lead to a proof.
   (b) Hint: A counting argument suffices.

4. (a) Hint: Try a counting argument.
   (b) forthcoming ....

5. See class notes. This follows from the Class Equation,

A good exercise:

Problem: In the symmetric group S_9, count the number of conjugates of
(1) (123)(456)(789)
(2) (12)(34)(56)
(3) (123)(456)
(4) a k-cycle σ, for $k = 2, \ldots, 9$.
(5) (1234)(56)