HW 5 SOLUTIONS

1. Gaussian Elimination with Back Substitution gives

$$\left(\begin{array}{ccc|c} 6 & 4 & 8 & -2 \\ 0 & 7/3 & -10/3 & 13/3 \\ 0 & 0 & -1/7 & 2/7 \end{array}\right)$$ with solution $\boxed{x=3,\ y=-1,\ z=-2}$

2. §3.1 #1.

Iteration	a	b	c = midpoint	f(c)
1	0	2	1	-1
2	1	2	1.5	1.375
3	1	1.5	$\boxed{1.25}$	-0.0469

$\alpha = \sqrt[3]{2} \approx 1.2599$

#3. $n \geq \dfrac{\log(b-a) - \log(\epsilon)}{\log 2}$ $\qquad \epsilon = 10^{-6}$

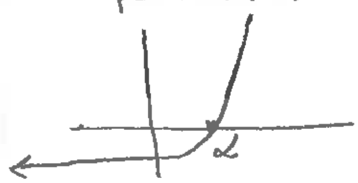
a) $n \geq 20.9 \Rightarrow n = 21$ b) $n \geq 19.9 \Rightarrow n = 20$ c) $n = 20$ d) $n = 21$ e) $n \geq 21.5 \Rightarrow n = 22$

f) $n = 21$ g) $n > 17.1 \Rightarrow n = 18$ h) $n \geq 21.5 \rightarrow n = 22$

#4. §3.2 #1. Newton's Method will estimate the value of $\sqrt{2}$.

$f'(x) = 2x$ Thus, $x_{n+1} = x_n - \dfrac{(x_n^2 - 2)}{2x_n} = x_n - \frac{1}{2}x_n + \frac{1}{x_n} = \boxed{\frac{1}{2}x_n + \frac{1}{x_n}}$

#5 §3.2 #7. Lots of possibilities:



If $x_0 < 0$, then Newton's Method doesn't work at all since $f'(x_0) = 0$.

#6. 3.2 #9. If $f(x) = \arctan(x)$, then $f'(x) = \dfrac{1}{1+x^2}$ and Newton's Method says

$x_{n+1} = x_n - \arctan(x_n)\,(1 + x_n^2)$. For the 2 cycle you need

$$-\beta = \beta - \arctan(\beta)\,(1+\beta^2) \quad \text{or} \quad 2\beta - \arctan(\beta)\,(1+\beta^2) = 0$$

This has 3 roots $\beta = 0$, $\beta \approx 1.39$

7. §3.5 #1. If $f(x)$ satisfies $|f''(x)| \le 3$, $|f'(x)| \ge 1$ for all x and $|e_0| < 1/2$, bound e_1, e_2, e_3.

By Theorem 3.2, $e_{n+1} = -\frac{1}{2} e_n^2 \frac{f^{(2)}(\xi_n)}{f'(x_n)}$ for $\xi_n \in [\alpha, x_n]$. By hypothesis,

make big

$$\left| \frac{f^{(2)}(\xi_n)}{f'(x)} \right| \le \frac{3}{1} = 3 \Rightarrow |e_1| \le \frac{1}{2} e_0^2 (3) \le \frac{3}{8}$$

make small

$$|e_2| \le \frac{1}{2}(e_1)^2(3) \le \frac{27}{128}$$

$$|e_3| \le \frac{1}{2}(e_2)^2(3) \le \frac{1}{2}\left(\frac{27}{128}\right)^2(3)$$

$$\boxed{\begin{aligned} |e_1| &\le \tfrac{3}{8} = .375 \\ |e_2| &\le \tfrac{27}{128} \approx .2109 \\ |e_3| &\le \approx .0667 \end{aligned}}$$

8. 3.5 #20. $f(x)$: $|f^{(2)}(x)| \le 4$ $|f'(x)| \ge 2$ for all x, $|e_0| \le 1/3$

Then $|e_{n+1}| = \left| -\frac{1}{2} \frac{f^{(2)}(\xi_n)}{f^{(1)}(x_n)} e_n^2 \right| \le \frac{1}{2} \cdot \frac{4}{2} e_n^2 = e_n^2$. Thus,

$|e_1| \le e_0^2 = \frac{1}{9}$, $|e_2| \le \frac{1}{81}$, $|e_3| \le \frac{1}{(81)^2} \approx .00015$

9. 3.6 #1. $f(x)$: unique root $\alpha \in [0,1]$, for all x $f'(x) \ge 2$ $0 \le f''(x) \le 3$

Let $x_0 = 1/2$.

First compute $M = \frac{\max |f^{(2)}(x)|}{2 \min |f'(x)|} \le \frac{3}{2(2)} = \frac{3}{4}$ and $|\alpha - x_0| < 1/2$

Thus, $M|\alpha - x_0| \le \frac{3}{4}$ and by Theorem 3.3 Newton's Method converges

Also, by Theorem 3.3., $|e_n| = |\alpha - x_n| \le \frac{1}{M}\left(M|\alpha - x_0|\right)^{2^n}$

$$\le \frac{1}{(3/4)} \left(\frac{3}{4} \cdot \frac{1}{2}\right)^{2^n} = \frac{4}{3} \cdot \left(\frac{3}{8}\right)^{2^n}$$

If $n = 4$, $|e_4| \le .0000002 < 10^{-6}$!

For Bisection, $b - a = 1$ and $\left(\frac{1}{2}\right)^n (b-a) \le 10^{-6}$ if $n = 20$

Programming Assignment:

1. 3.2.5 # 11 a, b, c

Trap Rule Error Uniform Grid

$$|e_n| = \frac{|b-a|}{12} h^2 f^{(2)}(\xi_n)$$

11a. $f(x) = x^2 e^{-x}$ $[0,2]$ $I(f) = 2 - \frac{10}{e^2}$

Note the number of trapezoids varies from $n = 2, 4, 8, \ldots, 128 = 2^7$ and $h_{n+1} = \frac{1}{2} h_n$

Actually, the improvement is much better than expected. $e_{n+1} \approx \frac{1}{16} e_n$, though one would expected $\frac{1}{4}$ as $h \to h/2$

b. $f(x) = \frac{1}{1+25x^2}$ on $[0,1]$ $I(f) = \frac{1}{5}\arctan(5)$

Here, indeed, the ratios of e_{n+1}/e_n are .0699, -.0183, .3487, .2501, .25, .25, .25

So the error behaves as expected.

c. $f(x) = \sqrt{1-x^2}$ on $[-1,1]$ $I(f) = \pi/2$

For this function, the error ratios $e_{n+1}/e_n \approx .35$ and not the expected .25.

#2. 3.3 #3c This is straight forward, if you wrote your programs $x \approx .56714$

#3 Let $f(x) = x - \tan x$ $x \approx 4.493407$ $x \approx 7.72525$ Convergence is fast.

#4. If $f(x) = x^{-2}\tan x$, then $f'(x) = x^{-2}(\sec^2 x) + (-2)x^{-3}\tan x = \frac{\sec^2 x}{x^2} - \frac{2\tan x}{x^3}$

and differentiating $f'(x) = x^{-2}\sec^2 x - 2x^{-3}\tan x$ yields

$$f''(x) = x^{-2}(2\sec x)(\sec x \tan x) - 2x^{-3}\sec^2 x - 2[x^{-3}\sec^2 x - 3x^{-4}\tan x]$$

$$= \frac{2\sec^2 x \tan x}{x^2} - \frac{4\sec^2 x}{x^3} + \frac{6\tan x}{x^4}$$

This should make you glad about the secant method and Bisection.

Using my Newton's Method program, the root $\alpha \approx .9477471335 16990$

5. $x \approx .12132$ and $x \approx .1231056$

This required x_0 very close to the root.

#6. $f(x) = x^3 + 94x^2 - 389x + 294$

Newton's Method homes in on $x = -98$ since with $x_0 = 2$, $x_1 = -100$! and $x_2 = -98$.

#7. This is related to a problem of the written assignment. One solution is to note that we need

$$\underline{x_{n+1} = -x_n} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{and so} \quad 0 = 2x_n - \frac{f(x_n)}{f'(x_n)}.$$

Using Newton's Method on $g(x) = 2x - \arctan(x)\,(1+x^2)$ gives

$x_n \simeq 1.3917452\ldots$ and $x_{n+1} = -x_n$.

#8. You should see the error in the bisection method converges LINEARLY while Newton's method is <u>much</u> faster. [Recall quadratic convergence.]