Fun with p-adic arithmetic

Good morning. I was thinking last night about the 5-adic expansion of $\frac{1}{4}$ and .... realized I made a mistake in class. (The downside of changing one's presentation at random in class to try something new .....) The mistake is actually interesting and instructive ..... Thanks to Aven for asking the right question.

Give the 5-adic expansion for the number $\frac{1}{4}$.

Here is the first thing that made me realize something was afoul: The fraction $\frac{1}{4}$ is a 5-adic integer! This is because $\left|\frac{1}{4}\right|_5 = 1$. As a consequence, we should find a 5-adic expansion of the form

$$\frac{1}{4} = \ldots a_2 a_1 a_{0\wedge}\,.$$

The way to do this is to set up the appropriate arithmetic problem:

$$\begin{array}{cr} & \ldots a_2 a_1 a_{0\wedge} \\ \times & 4\ _\wedge \\ \hline = & 1\ _\wedge \end{array}$$

Now, *because we are doing arithmetic in the finite field $\mathbb{Z}/5\mathbb{Z}$*, we can find a unique solution to the equation

$$4\,a_0 = 1,$$

namely, $a_0 = 4$.

Moving to the next 'digit' we need to solve for a_1 in the following:

$$\begin{array}{cr} & \ldots a_2 a_1 4_\wedge \\ \times & 4_\wedge \\ \hline = & 1_\wedge \end{array}$$

By carrying the 3, $(4 \times 4 = 16 = 31_\wedge)$, we need to solve

$$4a_1 + 3 = 0,$$

which again has a unique solution in the finite field $\mathbb{Z}/5\mathbb{Z}$: $a_1 = 3$.

Now computing $34_\wedge \times 4$ we get $301_\wedge$ and must carry a 3 again. Now the pattern should be clear for solving for the rest of the digits.

Now why is it that ${}_\wedge\overline{1}$ is wrong? This is the good part. It because the sequence of partial sums of ${}_\wedge\overline{1}$ does **NOT** converge in the 5-adic metric. Using sigma notation,

$${}_\wedge\overline{1} = \sum_{i=1}^{\infty} \frac{1}{5^i},$$

and the point is that each subsequent term in the sum is **farther away** from zero than the previous one. Stated otherwise, as you move to the right of the $_\wedge$, the p-adic numbers get larger and larger:

$$5^i = \left|\frac{1}{5^i}\right|_5 < \left|\frac{1}{5^{i+1}}\right|_5 = 5^{i+1}.$$