MATH 202
Midterm 1

Name : Solutions

October 7, 2009

Instructions: Show all work for full credit.

1. (23 pts.) Three points, with coordinates

$$A(1,1,1),\quad B(3,-1,0),\quad C(-2,1,-1),$$

lie in $\mathbb{R}^3$.

(a) Consider the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

i. (6 pts.) Determine the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
(You may leave your answer in a form involving inverse trigonometric functions, and you do not need to rationalize denominators.)

$\vec{AB} = (3-1, -1-1, 0-1) = (2,-2,-1)$ $\quad \vec{AC} = (-2-1, 1-1, -1-1) = (-3,0,-2)$

$$\theta = \arccos\left[\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right] \qquad \vec{a}\cdot\vec{b} = (2,-2,-1)\cdot(-3,0,-2) = -4 \qquad \therefore \boxed{\theta = \arccos\left(\frac{-4}{3\sqrt{13}}\right)}$$

$$\|\vec{a}\| = \sqrt{(2)^2+(-2)^2+(-1)^2} = 3$$

$$\|\vec{b}\| = \sqrt{(-3)^2+(0)^2+(-2)^2} = \sqrt{13}$$

ii. (2 pts.) Is the angle θ that you found in part (a) acute? Explain briefly.

Since $\vec{a}\cdot\vec{b} = -4 < 0$, $\pi/2 < \theta < \pi$. θ is obtuse, not acute.

(b) Now consider the plane in $\mathbb{R}^3$ containing the three points A, B, and C.

i. (5 pts.) Give a normal vector $\vec{n}$ to this plane.

$$\text{Let } \vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -1 \\ -3 & 0 & -2 \end{vmatrix} \Rightarrow \begin{aligned} &= 4\hat{i} - (-4-3)\hat{j} + -6\hat{k} \\ &= 4\hat{i} + 7\hat{j} - 6\hat{k} \\ &= \langle 4,7,-6\rangle \end{aligned}$$

ii. (5 pts.) Give the equation of the plane in $\mathbb{R}^3$ that contains these three points.

Use point A,

$$4x+7y-6z = (4,7,-6)\cdot(1,1,1)$$

$$\boxed{4x+7y-6z=5}$$

iii. (5 pts.) Is the point $D(-7,3,-2)$ on this plane? (YES or NO?) Why or why not.

Test: $4(-7) + 7(3) - 6(-2) \stackrel{?}{=} 5$

$-28 + 21 + 12 \stackrel{?}{=} 5$

$-28 + 33 \stackrel{?}{=} 5$

$5 = 5$ ✓

Yes. D is on the plane since it satisfies the equation.

2. (7 pts.) In $\mathbb{R}^3$, a constant force $\mathbf{F} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ N acts on a particle that is moved along the positive x-axis for a total distance of 5 m. Find the work done. (A complete answer includes units.)

$\vec{D} = (5,0,0)$

$W = \text{Work} = \vec{F} \cdot \vec{D} = (3,1,-1) \cdot (5,0,0) = \boxed{15 \text{ Nm}}$

3. (15 pts.) Two planes with equations

$$y - z = 1$$
$$-2x - y + z = 3$$

intersect in a line ℓ.

(a) (2 pts.) Fill in the blank to make a true statement:

These planes are not parallel since their normal vectors *are not scalar multiples of one another.*

(b) (5 pts.) Give the coordinates of a point P that lies on both planes.

Lots of possible answers: if $y=0$, then $z=-1$ by equation 1.

Thus, from equation (2): $-2x - y + z = 3$

$\Rightarrow -2x - (0) + -1 = 3$

$-2x = 4 \quad x = -2$

$\boxed{P(-2,0,-1)}$ for example.

Also: $(-2,1,0)$

(c) (8 pts.) Give the equation of the line ℓ of intersection. Give *both* the vector equation for this line and the parametric equations for this line.

$\vec{v} = \vec{n}_1 \times \vec{n}_2$ for $\vec{n}_1 = (0,1,-1)$ and $\vec{n}_2 = (-2,-1,1)$

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = 0\hat{i} + 2\hat{j} + 2\hat{k} = (0,2,2)$ ← direction vector

$(0,2,2)$

$\ell: \vec{p}_0 + t\vec{v}$ for $t \in \mathbb{R}$

$(-2,0,-1) + t(0,2,2) \quad t \in \mathbb{R}$

Vector Equation of ℓ: $(-2,0,-1) + t(0,2,2) \quad t \in \mathbb{R}$

Parametric Equation of ℓ: $x(t) = -2$, $y(t) = 2t$, $z(t) = -1 + 2t \quad t \in \mathbb{R}$

4. (25 pts.)

(a) (7 pts.) Determine if the vectors $\vec{a} = (1,-1,-1)$, $\vec{b} = (-1,1,1)$, and $\vec{c} = (2,0,1)$ are co-planar.

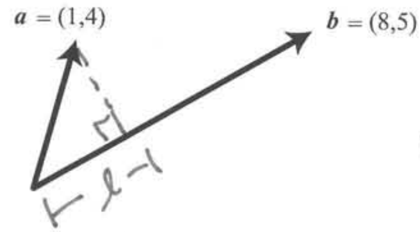
$\vec{a}, \vec{b}, \vec{c}$ are coplanar $\iff$ the scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Easiest solution: notice $\vec{a}$ is parallel to $\vec{b}$. Thus, $\vec{a} \times \vec{b} = \vec{0}$ and $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

Other solution: $\vec{b} \times \vec{c} = (1, 3, -2)$ and $\vec{a} \cdot (\vec{b} \times \vec{c}) = 1(1) + (-1)(3) + (-1)(-2) = 0$

$\boxed{\text{CO-PLANAR}}$

(b) In the figure below, two vectors $\vec{a}$ and $\vec{b}$ are shown.



length l = $\text{Comp}_{\vec{b}}\, \vec{a}$, a scalar

i. (3 pts.) By making appropriate markings in the figure above, indicate $\text{comp}_{\vec{b}}\, \vec{a}$. (You need to clearly indicate whether your answer is a scalar or a vector.)

ii. (3 pts.) Give $\text{comp}_{\vec{b}}\, \vec{a}$.

$$\text{comp}_{\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} = \frac{(1,4)\cdot(8,5)}{\sqrt{8^2+5^2}} = \frac{8+20}{\sqrt{64+25}} = \boxed{\frac{28}{\sqrt{89}} = \frac{28\sqrt{89}}{89}}$$

iii. (5 pts.) Give $\text{proj}_{\vec{b}}\, \vec{a}$.

$$\text{proj}_{\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\, \vec{b} = \boxed{\frac{28}{89}(8,5)}$$

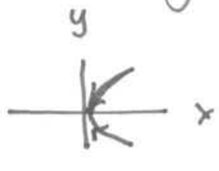
(c) (7 pts.) Show that

$$\lim_{(x,y)\to(0.0)} \frac{xy^2}{x^2+y^4}$$

does not exist.

Approach along $x = 0$ (the y-axis): $\dfrac{xy^2}{x^2+y^4} = \dfrac{0}{y^4} \to 0$

Approach along $x = y^2$: $\dfrac{xy^2}{x^2+y^4} = \dfrac{(y^2)y^2}{(y^2)^2+y^4} = \dfrac{y^4}{2y^4} = \dfrac{1}{2}$

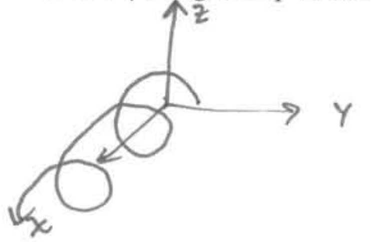


Since $0 \neq \frac{1}{2}$, the limit d.n.e.

5. (20 pts.) An object moves along a trajectory so that its position in the space $\mathbb{R}^3$, as a function of time $t \geq 0$ in seconds, is given by

$$\mathbf{r}(t) = (t,\ \cos(2t),\ \sin(2t)) \text{ cm.}$$

(a) (6 pts.) By thinking about the parametric equations for $\mathbf{r}(t)$, describe the motion of this object in $\mathbb{R}^3$. (It might help to draw the coordinate axes to explain your answer.)



It's a circular helix, coming out along the positive x-axis and rotating in the positive direction with respect to the yz-plane

(b) (8 pts.) What is the length of its trajectory between times $t = 0$ and $t = 2$?

$$\|\vec{r}\,'(t)\| = \sqrt{1^2 + (-2\sin 2t)^2 + (2\cos(2t))^2} = \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{5}$$

$$\text{length } L = \int_0^2 \|\vec{r}\,'(t)\|\,dt = \int_0^2 \sqrt{5}\,dt = \boxed{2\sqrt{5} \text{ cm}}$$

(c) (6 pts.) Give the unit tangent vector $\mathbf{T} = \mathbf{T}(\pi)$ at time $t = \pi$ seconds.

$$\vec{r}\,'(t) = (1, -2\sin(2t), 2\cos(2t))$$

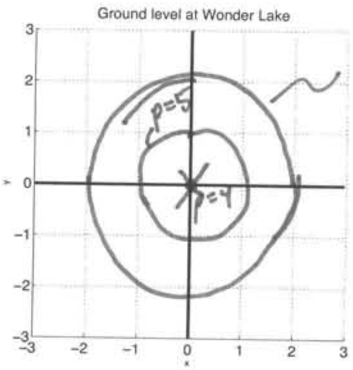
$$\vec{r}\,'(\pi) = (1, -2\sin(2\pi), 2\cos(2\pi)) = (1, 0, 2). \quad \therefore \vec{T} = \frac{1}{\|\vec{r}\,'(\pi)\|}\vec{r}\,'(\pi) = \boxed{\frac{1}{\sqrt{5}}(1, 0, 2) = \vec{T}}$$

6. (10 pts.) The density of mosquitoes on the (flat) ground at Wonder Lake is given by

$$\rho(x, y) = 4 + x^2 + y^2$$

thousands of mosquitoes/sq mile for $-3 \leq x \leq 3$ miles and $3 \leq y \leq 3$ miles.

(a) (6 pts.) Draw a contour plot of ρ that shows the level curves where $\rho = 4,\ 5$, and 8.



$\rho = 8$

$\rho = 4: \ 0 = x^2 + y^2$

$\rho = 5: \ 1 = x^2 + y^2$

$\rho = 8: \ 4 = x^2 + y^2$

circles of $r = 0, 1, 2$ respectively

(b) (4 pts.) Assuming you find mosquitoes annoying, where would you like to camp? Indicate this with an 'X' on the grid.

at the origin!