MATH 371 Name : SOLUTIONS

Exam 2 November 25, 2019

Instructions: Give numerical answers unless instructed that a formula alone suffices. You may consult tables on the inside cover of your textbook and in the appendices, but indicate in your solution that you have done so by writing 'Table.' A 'dumb' calculator may be used for routine arithmetic, but nothing else. Bald answers will receive very limited, if any, credit; that is, show formulas before performing your computations. Good luck. Happy Thanksgiving!

1. (10 pts.) A random variable W is known to have moment generating function $m_W(t) = e^{1.2t+10t^2}$. Find the distribution of W including all parameter values.

$W \sim$ Norm(1.2, 20)

2. (16 pts.)

In a certain populous state, political party affiliations for voters are given by the proportions listed in the table to the right. A random sample of size $n = 6$ is taken.

Party	Proportion
Republican	.50
Democrat	.10
Independent	.32
Unaffiliated	.08

(a) (5 pts.) Find the probability that the random sample of size $n = 6$ contains two Republicans, one Democrat, and one Independent party member. Give your answer to 4 decimal places

$(R, D, I, U) \sim \text{Multinomial}(.5, .1, .32, .08)$

$$P(R=2, D=1, I=1, U=2) = \binom{6}{2\ 1\ 1\ 2}(.5)^2(.1)^1(.32)^1(.08)^2 = \frac{6!}{2!2!}(.5)^2(.1)(.32)(.08)^2$$

$\approx .0092$

(b) (5 pts.) Give the distribution of the variables

R : number of Republicans in random sample of size 6

O : number of Independents and Unaffliated voters in random sample of size 6

pool I and U $.32 + .08 = .4$

Answer: $R \sim$ Binom(6, .5) $O \sim$ Binom(6, .4)

(c) (6 pts.) Find the variance of the function $U = 3R - 2O$. 2 decimal places

$$V(3R-2O) = 9\,\text{Var}(R) + 4\,\text{Var}(O) + 2(3)(-2)\,\text{Cov}(R, O)$$

$$= 9(6)(.5)(.5) + 4(6)(.4)(.6) + 2(3)(-2)[-6(.5)(.4)]$$

≈ 33.66

3. (14 pts. – 7 pts. each) To study the effects of a tuition increase on students, two people are to be selected at random for an interview from a group containing four current students, three alumni, and two parents of current students. Let

X_1: *number of current students on the committee*

X_2: *number of alumni on the committee*

(a) Give a formula and the exact value (as a fraction) for the joint probability function $p(x_1, x_2)$ for the pair (X_1, X_2) when $(x_1, x_2) = (1, 1)$ and $(0, 1)$.

denominator $= \binom{9}{2} = \frac{9 \cdot 8}{2!} = 36$

(x_1, x_2)	$p(x_1, x_2)$
$(1, 1)$	$\frac{\binom{4}{1}\binom{3}{1}\binom{2}{0}}{\binom{9}{2}} = \frac{4 \cdot 3}{36} = \boxed{\frac{1}{3}}$
$(0, 1)$	$\frac{\binom{4}{0}\binom{3}{1}\binom{2}{1}}{\binom{9}{2}} = \frac{3 \cdot 2}{36} = \boxed{\frac{1}{6}}$

(b) The marginal distribution $p_1(x_1)$ of X_1 is modeled by one of the 'well-known' discrete probability distributions we have studied, (i.e. it makes it onto the inside back cover of our textbook). What is the marginal distribution for X_1? Be sure to include the support of $f_1(x_1)$ and a listing of its parameter values.

Answer: $X_1 \sim$ Hypergeometric with parameters $N = 9$ $n = 2$ $r = 4$

and support $x_1 =$ 0, 1, 2.

4. (12 pts.) When a couple undertakes *in vitro fertilization* (IVF) the number of viable embryos might be modeled by a binomial random variable X.

(a) (4 pts.) Suppose that following an IVF intervention 10 embryos are saved. Why is it reasonable that

X: *the number of viable embryos*

might be modeled with a binomial random variable Binom$(10, p)$, $X \sim$ Binom$(10, p)$?

It is reasonable to assume the trials are independent and the eggs are either successfully fertilized (S) or not (F).

(b) (8 pts.) Suppose that the probability p of a viable embryo is unknown, but believed to be well modeled by $p \sim$ Beta(3, 2). Find the expected value and variance of X.

$p \sim \text{Beta}(3,2)$ $\mathbb{E}(p) = .6$

$V(p) = \frac{3 \cdot 2}{(3+2)^2(3+2+1)} = \frac{1}{25} = .04$

$\mathbb{E}(p^2) = V(p) + [\mathbb{E}(p)]^2$

$= .04 + (.6)^2 = .4$

$E(X) = E(E(X \mid p)) = E(10p) = 10 E(p) = 10 \cdot \frac{3}{3+2} = \boxed{6}$

$V(X) = V(E(X \mid p)) + E(V(X \mid p))$

$= V(10p) + E(10p(1-p))$

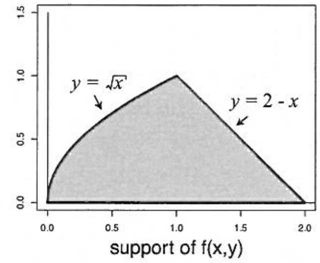
$= 10^2 V(p) + 10[\mathbb{E}(p) - \mathbb{E}(p^2)]$

$= 100(.04) + 10[.6 - .4]$

$= 4 + 2 = \boxed{6}$

5. (25 pts.) Consider the jointly distributed random variables (X, Y) with joint density $f(x,y) = \frac{6}{7}$ on the region shown below. A graph of the support of $f(x,y)$ has been provided for you.

$$f(x,y) = \begin{cases} \frac{6}{7}, & \text{on the region shown} \\ 0, & \text{otherwise.} \end{cases}$$



(a) (7 pts.) Find the marginal distribution $f_y(y)$ for Y. Be sure to include the support of $f_y(y)$ in your final answer.

$$f_y(y) = \int_{y^2}^{2-y} f(x,y)\,dy = \frac{6}{7}(2-y-y^2) \quad \text{for } 0 \le y \le 1$$

$$f_y(y) = \begin{cases} 6/7\,(2-y-y^2) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (**Extra Credit.** Do on scrap paper.) Since the joint density $f_{x,y}(x,y)$ is constant, it seems that $f_y(y)$ might be uniform on $(0,1)$, but it is not. Why? Explain. While f_{xy} is constant, the length of (y_{min}, y_{max}) is not.

(c) (5 pts.) Are the variables X and Y independent? Why? (Credit only for a correct justification.)

No. $f(x,y) \neq f_x(x) f_y(y)$ Since $f_y(y)$ depends on the value of y, it can not be canceled by $f_x(x)$

(d) (7 pts.) Find the conditional probability $P(X \le 1 \mid Y = \frac{1}{2})$. Show all work for credit.

$$P(X \le 1 \mid Y = 1/2) = \int_0^1 f(x \mid y = 1/2)\,dx \qquad f(x \mid y = 1/2) = \frac{f(x, \frac{1}{2})}{f_y(1/2)}$$

$$= \frac{\frac{6}{7}}{\frac{6}{7}(2 - \frac{1}{2} - (\frac{1}{2})^2)} = \frac{1}{5/4} = 4/5$$

Thus, $\int_0^1 f(x \mid y = 1/2)\,dx = \int_{\frac{1}{4}}^1 \frac{4}{5}\,dx = \frac{3}{5} = \boxed{.6}$

The minimum value of $f(x \mid y = 1/2) > 0$ is $x = \frac{1}{4}$.

(e) (6 pts.) Set up an expression, but do not integrate , to evaluate $P(X \le 4/9 \mid Y \le 1/2)$. For credit your answer should not have any 'formal' expressions, but integrals with explicit integrands and explicit limits of integration. (Extra credit: On a piece of scrap paper, get the exact value of this integral.)

$$P(X \le \tfrac{4}{9} \mid Y \le \tfrac{1}{2}) = \frac{\int_0^{1/2} \int_{y^2}^{4/9} \frac{6}{7}\,dx\,dy}{\int_0^{1/2} \frac{6}{7}(2 - y - y^2)\,dy}$$

EC

$$= \frac{13/72}{5/6} = \frac{13}{60}$$

6. (13 pts.) In thousands of dollars, the profit M made by an apartment owner per month is $M = 2 - Y$ where $Y \sim \text{Unif}(0, 3)$ and Y is also measured in thousands of dollars. That is, if a renter causes \$500 of damage to the apartment in a month, then $Y = .5$ and $M = 2 - .5 = 1.5$, or the profit M is \$1,500 that month.

(a) (3 pts.) Give the support of the function $M = h(Y)$ for the apartment owner's monthly profit. That is, find the values of m such that the density $f_m(m)$ is non-zero.

$0 \leq y \leq 3 \Rightarrow \quad -1 \leq m \leq 2$

(b) (10 pts.) Either using the Method of Distributions functions or the Method of Tranformations, find the density $f_m(m)$ for M.

$h(Y) = 2 - Y$ is strictly decreasing $\quad h^{-1}(m) = 2 - m \quad \frac{d\,h^{-1}(m)}{dm} = -1$

$f_m(m) = f_y(h^{-1}(m)) \left| \frac{d\,h^{-1}(m)}{dm} \right|$

$= \frac{1}{3} |-1| = \frac{1}{3} \qquad M \sim \text{Unif}(-1, 2)$

7. (10 pts.) Suppose X_1 and X_2 are independent Poisson-distributed random variables where $X_1 \sim \text{Pois}(\lambda_1)$ and $X_2 \sim \text{Pois}(\lambda_2)$. Use the method of moment-generating functions to find the distribution of $U = X_1 + X_2$.

$m_{X_1}(t) = m_1(t) = e^{\lambda_1(e^t - 1)} \qquad m_{X_2}(t) = m_2(t) = e^{\lambda_2(e^t-1)}$

independence

$m_{X_1+X_2}(t) = \mathbb{E}\left(e^{t(X_1+X_2)}\right) = \mathbb{E}\left(e^{tX_1} \cdot e^{tX_2}\right) \overset{\downarrow}{=} \mathbb{E}(e^{tX_1})\,\mathbb{E}(e^{tX_2})$

$= m_1(t)\, m_2(t) = e^{\lambda(e^t - 1)} \cdot e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$

$U \sim$ Pois $(\lambda_1 + \lambda_2)$

Scrap work