1.1 #1, 11, 12, 17, 18, 21, 24

#1) Find $p_3(x)$ for $f(x) = \frac{1}{(x+1)}$ about $x_0 = 0$

$$\begin{cases} f(x) = (x+1)^{-1} & f'(x) = -(x+1)^{-2} & f''(x) = 2(x+1)^{-3} & f^3(x) = -6(x+1)^{-4} \\ f(0) = 1 & f'(0) = -1 & f''(0) = 2 & f^3(0) = -6 \end{cases}$$

Thus, $p_3(x) = f(0) + f'(0)x + f^2(0)\frac{x^2}{2!} + f^3(0)\frac{x^3}{3!}$

$= 1 - 1(x) + 2\frac{x^2}{2!} - 6\frac{x^3}{3!} = 1 - x + x^2 - x^3$ $\square$

#11 Find $p_3(x)$ about $x_0 = 0$ and then estimate an upper bound for $|R_3(x)|$.

a. $f(x) = e^{-x}$ $x \in [0,1]$

$p_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$ $|R_3(x)| = \left|\frac{x^4}{4!}(-e^{-\xi_x})\right|$ for some $\xi_x \in [0,1]$

Therefore, $|R_3(x)| \le \frac{|x|^4}{24}|e^{-\xi_x}| \le \frac{1}{24}$ on $[0,1]$

Solution manual is wrong.

b $f(x) = \ln(1+x)$ on $x = [-1,1]$

After work, $f(0) = 0$, $f'(0) = 1$, $f^2(0) = -1$, $f^3(0) = 2$, $f^4(x) = -6(1+x)^{-4}$

Therefore, $p_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. The remainder term $R_3(x) = \frac{x^4}{4!}\frac{-6}{(1+x)^4}$.

However, since $-1 \in [-1,1]$, the remainder term could blow up: $R_4(-1) = \frac{(-1)^4}{4!}\frac{6}{0}$

c. $f(x) = \sin x$, $x \in [0,\pi]$

$p_3(x) = x - \frac{x^3}{3!}$ $|R_3(x)| = \frac{|x|^5}{5!}|\cos(\xi)|$ for some $\xi \in [0,\pi]$

$|R_3(x)| \le \frac{\pi^5}{5!} \approx 2.55$ (huge!)

d. $f(x) = \ln(1+x)$, $x \in [{}^{1}/_{2}, {}^{1}/_{2}]$

$p_3(x)$ is given in b. $R_3(x) = \frac{x^4}{4!}\frac{(-6)}{(1+x)^4}$ and $|R_3(x)| = \frac{6}{24}\frac{|x|^4}{(1+x)^4} \le \frac{1}{4}\frac{(\frac{1}{2})^4}{(1-\frac{1}{2})^4} = \frac{1}{4}$

11e. $f(x) = (x+1)^{-1}$ $x \in [{}^{-1}\!/_2, {}^1\!/_2]$

From problem 1, $p_3(x) = 1 - x + x^2 - x^3$. Moreover, $f^4(x) = 24(x+1)^{-5}$

so that $R_3(x) = \frac{x^4}{4!} \cdot 24(\xi+1)^{-5} = \frac{x^4}{(1+\xi)^5}$ for some $\xi \in [-{}^1\!/_2, {}^1\!/_2]$

To bound this, $|R_3(x)| = \frac{|x|^4}{(1+\xi)^5} \le \frac{(\frac{1}{2})^4}{(1-{}^1\!/_2)^5} = 2.$ □

12. Find n so that $|R_n(x)| \le 10^{-3}$ for the following Taylor approximations.

a. $f(x) = \sin x$, $x \in [0,\pi]$. If $n = 2k+1$, then $R_n(x) = (-1)^k \frac{x^{2k+3}}{(2k+3)!} \cos\xi$

for some $\xi \in [0,\pi]$. Since $|\cos\xi| \le 1$ always,

$|R_n(x)| \le \frac{|x|^{2k+3}}{(2k+3)!}$. Now find k s.t. $\frac{\pi^{2k+3}}{(2k+3)!} \le 10^{-3}$, then

solve for n. Alternatively, find n odd, so that $\frac{\pi^{n+2}}{(n+2)!} \le 10^{-3}$.

$\boxed{\text{By trial and error, } n = 11 \text{ works.}}$

b. $f(x) = e^{-x}$, $x \in [0,1]$ Here $R_n(x) = \frac{x^{n+1}}{(n+1)!} e^{-\xi}$ for $\xi \in [0,1]$

Since the maximum value of $e^{-\xi}$ is 0 on $[0,1]$ and the maximum value of x^{n+1} is 1 on $[0,1]$, $|R_n(x)| \le \frac{1}{(n+1)!}$

Thus, again by trial and error, $|R_n(x)| \le \frac{1}{(n+1)!} \le 10^{-3}$ $\boxed{\text{if } n=6}$

c. $f(x) = \ln(1+x)$, $x \in [0, {}^3\!/_4]$.

$R_n(x) = \frac{x^{n+1}}{(n+1)!} \cdot \frac{(-1)^n n!}{(1+\xi)^{n+1}} = \frac{x^{n+1}(-1)^n}{(n+1)\;(1+\xi)^{n+1}}$ $\xi \in [0, {}^3\!/_4]$

Thus, $|R_n(x)| = \frac{|x|^{n+1}}{n+1} \cdot \frac{1}{(1+\xi)^{n+1}} \le \frac{({}^3\!/_4)^{n+1}}{(n+1)} \le 10^{-3}$ $\boxed{\text{if } n = 14}$

d. $f(x) = (1+x)^{-1}$, $x \in [0, {}^1\!/_2]$, $R_n(x) = \frac{(-1)^{n+1}(n+1)!}{(1+\xi)^{n+2}} \frac{x^{n+1}}{(n+1)!} = (-1)^{n+1} \frac{x^{n+1}}{(1+\xi)^{n+2}}$ $\xi \in [0, {}^1\!/_2]$

$|R_n(x)| = \frac{|x|^{n+1}}{(1+\xi)^{n+2}} \le |x|^{n+1} \le (\frac{1}{2})^{n+1} \le 10^{-3}$ $\boxed{\text{if } n = 9}$

12e. Just like 12c, require $|R_n(x)| = \dfrac{|x|^{n+1}}{(1+\xi)^{n+1}(n+1)} \leq 10^{-3}$ on $[0, 1/2]$ #3.

$|R_n(x)| \leq \dfrac{(\frac{1}{2})^{n+1}}{(1)(n+1)} \leq 10^{-3}$ $\boxed{\text{if } n=7}$

#17. $f(x) = x^4 + 1$ has $p_3(x) = 1$ at $x_0 = 0$

#18 $f(x) = x^4 + 1$ has $p_4(x) = x^4 + 1$ at $x_0 = 0$

#21. The Taylor polynomial $p_n(x)$ to a function $f(x)$ is

THE BEST POLYNOMIAL APPROXIMATION

to a function $f(x)$ near x_0.

If $f(x)$ is a polynomial of degree less than or equal to n, then $f^{(k)}(x_0) = 0$ for $k \geq n+1$, i.e. higher derivatives are zero!

Thus, $p_n(x) = f(x)$ exactly

#24. Use the MVT to find M such that $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$ for functions in problem 11.

Basic Idea: If f is continuous on $[a,b]$, differentiable on (a,b), then $\exists \xi \in [a,b]$ s.t. $|f(b) - f(a)| = |f'(\xi)|\,|b-a|$. If we can find M s.t. $|f'(\xi)| \leq M$ for all $x \in [x_1, x_2]$, then we have found M.

a) $f(x) = e^{-x}$ on $[0,1]$. $f'(x) = -e^{-x}$ and $|f'(x)| = |e^{-x}| \leq 1$ on $[0,1]$. $\boxed{\text{Take } M=1.}$

b) $f(x) = \ln(1+x)$ on $[-1,1]$ Ill defined since $-1 \notin$ domain of $f(x)$

c) $f(x) = \sin x$ on $[0,\pi]$ $f'(x) = \cos x \Rightarrow |f'(x)| = |\cos x| \leq 1$ on $[0,1]$. $\boxed{M=1}$

reversed e) $f(x) = (1+x)^{-1}$ on $[-1/2, 1/2]$ $f'(x) = -1(x+1)^{-2} \Rightarrow |f'(x)| = \dfrac{1}{|x+1|^2}$ which is maximized at $x = -1/2$: $|f'(x)| \leq \dfrac{1}{|(1-1/2)|^2} = 4$ $\boxed{M=4}$

d) $f(x) = \ln(x+1)$ on $[-1/2, 1/2]$: $f'(x) = \dfrac{1}{x+1}$ which is minimized when $x = -1/2$

$\therefore$ $\boxed{M=2}$