Chapter 4.5

7. Exhibit all Sylow 2-subgroups of S_4 and find elements of S_4 which conjugate one of these into each of the others.

Proof. (Gillispie) Since $|S_4| = 4! = 2^3 \cdot 3$, if $P \in Syl_2(S_4)$, then $|P| = 8$.
Notice that

$$\begin{aligned}\langle(1\,2\,3\,4),\,(1\,2)(3\,4)\rangle &= \{(1),\,(1\,2)(3\,4),\,(1\,2\,3\,4),\,(1\,3)(2\,4),\\ &\quad (1\,4\,3\,2),\,(2\,4),\,(1\,3),\,(1\,4)(2\,3)\}\end{aligned}$$

is of order 8. Also

$$\begin{aligned}\langle(1\,3\,4\,2),\,(1\,3)(4\,2)\rangle &= \{(1),\,(1\,3)(2\,4),\,(1\,3\,4\,2),\,(1\,4)(2\,3),\\ &\quad (1\,2\,4\,3),\,(2\,3),\,(1\,4),\,(1\,2)(3\,4)\}\end{aligned}$$

and

$$\begin{aligned}\langle(1\,4\,2\,3),\,(1\,4)(2\,3)\rangle &= \{(1),\,(1\,4)(2\,3),\,(1\,4\,2\,3),\,(1\,2)(3\,4),\\ &\quad (1\,3\,2\,4),\,(3\,4),\,(1\,2),\,(1\,3)(2\,4)\}\end{aligned}$$

are distinct and of order 8.
Sylow's Theorem states that $n_2 \,\big|\, 3$. Either $n_2 = 1$, which we've shown to be untrue, or $n_2 = 3$, and so we have described every Sylow 2-subgroup.

Now, notice that

$$(2\,3\,4)(1\,2\,3\,4)(2\,4\,3) = (1\,3\,4\,2) \quad \text{and} \quad (2\,3\,4)(1\,2)(3\,4)(2\,4\,3) = (1\,3)(2\,4).$$

And so $(2\,3\,4)\langle(1\,2\,3\,4),\,(1\,2)(3\,4)\rangle(2\,3\,4)^{-1} = \langle(1\,3\,4\,2),\,(1\,3)(4\,2)\rangle$.

Also,

$$(2\,3\,4)(1\,3\,4\,2)(2\,4\,3) = (1\,4\,2\,3) \quad \text{and} \quad (2\,3\,4)(1\,3)(2\,4)(2\,4\,3) = (1\,4)(2\,3).$$

So $(2\,3\,4)\langle(1\,3\,4\,2),\,(1\,3)(2\,4)\rangle(2\,3\,4)^{-1} = \langle(1\,4\,2\,3),\,(1\,4)(2\,3)\rangle$.

Finally notice that

$$(2\,3\,4)(1\,4\,2\,3)(2\,4\,3) = (1\,2\,3\,4) \quad \text{and} \quad (2\,3\,4)(1\,4)(2\,3)(2\,4\,3) = (1\,2)(3\,4).$$

And so $(2\,3\,4)\langle(1\,4\,2\,3),\,(1\,4)(2\,3)\rangle(2\,3\,4)^{-1} = \langle(1\,2\,3\,4),\,(1\,2)(3\,4)\rangle$.

We've shown that powers of $(2\,3\,4)$ conjugate the elements of $Syl_2(S_4)$ into other elements of $Syl_2(S_4)$, and that every element of $Syl_2(S_4)$ can be taken into any other element of $Syl_2(S_4)$ by conjugation by $(2\,3\,4)$.

□

8. Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.

Proof. (Mobley) The order of S_5 is 120 or $2^3 \cdot 3 \cdot 5$. Since we are looking for Sylow 2-subgoups of S_5, we need the highest power of 2 that divides 120. The highest power of 2 is $2^3 = 8$. Recognizing that each of the subgroups from #7 is also contained in S_5, let our first Sylow 2-subgroup be

$$A = \{e, (12), (12)(34), (34), (13)(24), (14)(23), (1324), (1423)\}$$

from the previous problem. Another subgroup is

$$B = \{e, (15), (12)(35), (23), (13)(25), (15)(23), (1253), (1352)\}.$$

(Note that this is just one of the subgroups from #7 with fives replacing the fours.) The element of S_5 that conjugates one into the other is $\sigma = (123)(45)$ with $\sigma A \sigma^{-1} \subset B$.

$\square$

14. Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

Proof. (Hazlett) Note, $n_{13} \equiv 1 \mod 13$ and $n_{13}|24$. Hence $n_{13} = 1$. Then there exists a unique Sylow 13-subgroup P. Therefore $P \trianglelefteq G$ and G is not simple. $\square$

17. Prove that if $|G| = 105$, then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

Proof. (Baggett) We have that $|G| = 3 \cdot 5 \cdot 7$. From Sylow's Theorem, we have that $n_5 \equiv 1(mod\ 5)$ and $n_5 \,\big|\, 21$. Thus, $n_5 = 1$ or 21. Similarly, we have that $n_7 \equiv 1(mod\ 7)$ and $n_7 \,\big|\, 15$. Thus, $n_7 = 1$ or 15. We will show that $n_5 = 1$ or $n_7 = 1$. Suppose to the contrary that $n_5 = 21$ and $n_7 = 15$. Then there are $21(4) = 84$ elements of order 5 in G and $15(6) = 90$ elements of order 7 in G. This is a contradiction since $84 + 90 > 105$. Thus, either $n_5 = 1$ or $n_7 = 1$.
Suppose that $n_5 = 1$. Then there is a unique Sylow 5-subgroup $P \triangleleft G$. Let $Q \in Syl_7(G)$. Then $Q \leq N_G(P) = G$, so $PQ \leq G$. Furthermore, $|PQ| = \frac{|P||Q|}{|P \cap Q|} = 35$ and $[G : PQ] = 3$, the smallest prime dividing $|G|$. Thus, $PQ \triangleleft G$. Moreover, $Q \leq PQ$, $n_7(PQ) \equiv 1(mod 7)$, and $n_7(PQ) \,\big|\, 5$. Thus, $n_7(PQ) = 1$ and Q is the unique Sylow 7-subgroup of PQ. Therefore, Q is characteristic in PQ. Since Q char PQ and $PQ \triangleleft G$, it follows that $Q \triangleleft G$. Thus, there is a normal Sylow 5-subgroup P in G and a normal Sylow 7-subgroup Q in G. The proof for the case when $n_7 = 1$ is similar. $\square$

20. Prove that if $|G| = 1365$ then G is not simple.

Proof. (Buchholz) Let $|G| = 1365 = 3 \cdot 5 \cdot 7 \cdot 13$.
Then $n_{13} \equiv 1 \mod 13$ and $n_{13} \,\big|\, 105$, implies that $n_{13} = 1$ or 105.
Then $n_7 \equiv 1 \mod 7$ and $n_7 \,\big|\, 195$, implies that $n_7 = 1$ or 15.
Then $n_5 \equiv 1 \mod 5$ and $n_5 \,\big|\, 273$, implies that $n_5 = 1, 21$ or 91.
First consider $n_{13} = 105$, $n_7 = 15$, and $n_5 = 21$. Then there are $12(105) = 1260$ elements of order 13, $6(15) = 90$ elements of order 7, and $4(21) = 84$ elements of order 5. But $1260 + 90 + 84 = 1434$, which is larger then the order of G. Hence either $n_{13} = 1$,$n_7 = 1$, or $n_5 = 1$ and so there is a normal subgroup. (We have either $P_{13} \in \mathrm{Syl}_{13}(G)$, $P_7 \in \mathrm{Syl}_7(G)$, or $P_5 \in \mathrm{Syl}_5(G)$ is a non-trivial, proper normal subgroup of G.)

$\square$

21. Prove that if $|G| = 2907$, then G is not simple.

Proof (Granade). Let G be a group such that $|G| = 2907$. Then, note that $2907 = 3^2 \cdot 17 \cdot 19$. Using the Sylow theorems, we list the possible candidates for n_3, n_{17} and n_{19}:

$$\begin{array}{ccl|ccl|ccl} n_3 & \equiv & 1 \bmod 3 & n_{17} & \equiv & 1 \bmod 17 & n_{19} & \equiv & 1 \bmod 19 \\ & | & 323 & & | & 171 & & | & 153 \\ & \in & \{1, 19\} & & \in & \{1, 171\} & & \in & \{1, 153\} \end{array}$$

Next, note that if any of n_3, n_{17}, n_{19} is 1, then by Corollary 20 in the text, we have that there exists a normal subgroup of order 9, 17 or 19, respectively. Thus, in that case, we are done.

On the other hand, suppose that $n_3, n_{17}, n_{19} \neq 1$. Then, by the list of candidates above, we have that $n_3 = 19$, $n_{17} = 171$ and $n_{19} = 153$. Since 17 and 19 are both prime, we have that the intersection between two arbitrary elements of $\mathrm{Syl}_{17}(G) \cup \mathrm{Syl}_{19}(G)$ is $\{e\}$. Thus, G contains $16 \cdot 171 = 2736$ elements of order 17 and $18 \cdot 153 = 2754$ elements of order 19. Thus, $|G| \geq 2736 + 2754 + 1 = 5491$. This contradicts that $|G| = 2907$, and so we conclude that at least one of n_3, n_{17}, n_{19} is 1. $\square$

Proof (Granade). Let G be a group such that $|G| = 132$. Then, note that $132 = 2^2 \cdot 3 \cdot 11$. Using the Sylow theorems, we list the possible candidates for n_2, n_3 and n_{11}:

$$\begin{array}{rcl|rcl|rcl} n_2 & \equiv & 1 \bmod 2 & n_3 & \equiv & 1 \bmod 3 & n_{11} & \equiv & 1 \bmod 11 \\ & \mid & 33 & & \mid & 44 & & \mid & 12 \\ & \in & \{1,3,11,33\} & & \in & \{1,4,22\} & & \in & \{1,12\} \end{array}$$

Next, note that if any of n_2, n_3, n_{11} is 1, then by Corollary 20 in the text, we have that there exists a normal subgroup of order 4, 3 or 11, respectively. Thus, in that case, we are done.

On the other hand, suppose that $n_2, n_3, n_{11} \neq 1$. Then, by the list of candidates above, we have that $n_{11} = 12$. By the same counting argument as in Problem 21, we thus have that there are $10 \cdot 12 = 120$ elements of order 11 in G. We can therefore exclude exclude that $n_3 = 22$, since that would imply that $|G| \geq 2 \cdot 22 + 120 = 142$. Thus, $n_3 = 4$, and so we have 8 elements of order 3, leaving 3 non-identity elements to choose from for elements in our Sylow 2-subgroups. Since each Sylow 2-subgroup has order 4, that implies that we have a unique subgroup of order 4, contradicting that $n_2 \neq 1$. We can therefore conclude that at least one of n_2, n_3, n_{11} is 1, and so we are done. $\square$

25. Prove that if G is a group of order 385 then $Z(G)$ contains a Sylow 7-subgroup of G and a Sylow 11-subgroup is normal in G.

Proof. (Bastille) We note that $|G| = 385 = 5 \cdot 7 \cdot 11$ and for $p = 5, 7, 11$, since $|\mathrm{Syl}_p(G)| = n_p$ must satisfy $n_p \equiv 1 \bmod p$ and $n_p \mid \frac{385}{p}$ (because each prime divisor divides exactly $|G|$), we have the following table of possibilities:

p	1 mod p	divisors of $\frac{385}{p}$
5	1,6,11,16,21,26,31,36,41,46,51,56,61,66,71,76	1,7,11,77
7	1,8,15,22,29,36,43,50,57	1,5,11,55
11	1,12,23,34	1,5,7,35

Hence our choices for n_p are:

$$n_5 = 1, 11; \qquad n_7 = 1; \qquad n_{11} = 1.$$

Since $n_7 = 1$, $\mathrm{Syl}_7(G) = \{P_7\}$ and P_7 must be cyclic since $|P_7| = 7$, a prime. Let $a \in P_7$ such that $P_7 = \langle a \rangle$. Let G act on P_7 by conjugation. Then the associated homomorphism:

$$\varphi : \quad G \to S_{P_7} \cong S_7$$

is well-defined (since $gP_7g^{-1} = P_7$ for all $g \in G$ by Corollary 20). Furthermore, S_7 can be viewed as the group of automorphisms of P_7, and since P_7 is cyclic, $\mathrm{Aut}(P_7) \cong \mathbb{Z}/(7-1)\mathbb{Z}$ by Proposition 17 (1) p. 136. Therefore, if K denotes the kernel of φ, we have

$$G/K \cong \varphi(G) \leq \mathbb{Z}/6\mathbb{Z}.$$

So

$$\frac{|G|}{|K|} \,\Big|\, 6 \quad \Rightarrow \quad K = G \quad \text{since } 2, 3 \nmid |G|.$$

But by definition, $K = \ker \varphi = \{g \in G \mid ga^kg^{-1} = a^k \ \forall k \in \mathbb{Z}\}$, so

$$\forall\ a^k \in P_7 : \qquad ga^k = a^kg \quad \text{for all } g \in G.$$

Hence $P_7 \leq Z(G)$.

We also have that $n_{11} = 1$, so the unique Sylow 11-subgroup of G is normal in G (by Corollary 20). $\square$

26. Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal then G is abelian. (Schamel) Note that $|G| = 3 \cdot 5 \cdot 7$. Let P_3 be our unique Sylow 3-subgroup of G. Since 3 is the smallest prime dividing the order of G and $|P_3| = 3$ so P_3 is cyclic, by problem 4.5.44 we have that $N_G(P_3) = C_G(P_3)$. Since P_3 is normal in G, we conclude $C_G(P_3) = G$ and hence $P_3 \in Z(G)$. But then $|G/Z(G)|$ divides 35. If $|G/Z(G)|$ is one of 1,5, or 7, then $G/Z(G)$ is cyclic. If $|G/Z(G)| = 35$, we also have the $G\mathbb{Z}(G)$ is cyclic, since this quotient group is of order pq for $p \nmid q - 1$. Thus $G/Z(G)$ is cyclic. Hence, by 3.1.36, G is abelian.

44. Let p be the smallest prime dividing the order of a group G. If $P \in Syl_p(G)$ and P is cyclic, prove that $N_G(P) = C_G(P)$.

    Proof. (Lawless) Let $|G| = p^\alpha m$ where p is the smallest prime dividing $|G|$, and $p \nmid m$. Let P be a Sylow p-subgroup of G, and let P be cyclic. Notice $|P| = p^\alpha$.

    Consider the action of $N_P(G)$ on P by conjugation. Notice this gives rise to a homomorphism $\psi : N_G(P) \to S_P$, with $ker(\psi) = C_G(P)$. By the first isomorphism theorem, $N_G(P)/C_G(P) \cong K = \mathrm{Im}(\psi)$. Since conjugation by elements of $N_G(P)$ induce a subgroup of the automorphism group of P, then we know $K \leq Aut(P)$, and so $|K| \,\big|\, |Aut(P)|$. Since P is cyclic, we know $|Aut(P)| = \varphi(|P|) = p^\alpha - p^{\alpha-1}$. So $|K| \,\big|\, p^{\alpha-1}(p-1)$.

    Since P is cyclic, we know P is abelian, and so $P \leq C_G(P)$. Moreover, we know $C_G(P) \leq N_G(P) \leq G$. Thus, there exist some integers m_1, m_2 such that $p^\alpha m_2 = C_G(P)$ and $p^\alpha m_1 = N_G(P)$, and so

    $$|N_G(P)/C_G(P)| = \frac{m_1}{m_2} \,\big|\, p^{\alpha-1}(p-1).$$

    If $m_1/m_2 = 1$, then we know $C_G(P) = N_G(P)$. Assume that $m_1/m_2 \neq 1$. However, since p^α was the highest power of p dividing the order of G, then we know $m_1/m_2 \nmid p^{\alpha-1}$. And since p was the smallest prime dividing the order of G, then we know $m_1/m_2 \nmid (p-1)$. So $m_1/m_2 \nmid p^{\alpha-1}(p-1)$, a contradiction.

    Therefore, $m_1/m_2 = 1$, and thus $|C_G(P)| = |N_G(P)|$. Thus, $C_G(P) = N_G(P)$, as desired.

    □