Instructions: This exam is closed book and closed notes, and two hours in length. You may use **only** your brain and blank scratch paper in writing solutions.

   The problems in Part I are computational in nature, and full marks are awarded for correct answers. You need only justify your answers if you are explicitly asked to do so. Part II involves writing proofs and is theoretical in nature. You should prove results from first principles and not simply quote statements from the book. Your proofs will be graded not only on correctness, but points will be awarded/taken away for poor writing and exposition. Blank paper is supplied for scratch work, but final responses should be written in the space provided.

Part I.

   1. (10 pts.) Consider the cylic group $C_{4900} = \langle x \rangle$ of order $4900 = 2^2 \cdot 5^2 \cdot 7^2$.

      (a) (4 pts.) Give the number of generators of C_{4900}.

      (b) (6 pts.) List explicitly the elements x^a, with $0 \leq a \leq 4899$, of order 10.

         Answer: $|x^a| = 10$ if $a =$ _____.

         (If it helps, you can simply give the prime factorizations of a. I am not interested in your ability to multiply integers.)

   2. (10 pts.) Consider the cyclic groups $\mathbb{Z}/30\mathbb{Z}$ and $C_{18} = \langle x \rangle$ of orders 30 and 18 respectively, and suppose that

$$\varphi_a : \mathbb{Z}/30\mathbb{Z} \to C_{18}$$
$$1 \mapsto x^a$$

      extends to a well-defined group homomorphism from $\mathbb{Z}/30\mathbb{Z}$ to C_{18}.

      (a) (6 pts.) List the values of a with $0 \leq a \leq 17$ for which this is true. (I.e. The map defines a well-defined group homomorphism.)

      (b) (4 pts.) Give a brief explanation why such a well-defined group homomorphism can not be surjective.

3. (20 pts. – 4 pts. each) Consider the symmetric group $G = S_7$ and let $\sigma = (1\ 2\ 3\ 6\ 5\ 4\ 7)$ be an 7-cycle.

    (a) Express σ as the product of (not necessarily disjoint) transpositions.

    (b) Compute the number of conjugates of σ in S_7.

    (c) Let τ be the 7-cycle $(3\ 7\ 1\ 4\ 5\ 6\ 2)$. Give an element α that conjugates σ to τ, i.e. give α such that $\alpha\sigma\alpha^{-1} = \tau$.

    (d) Noting that S_7 acts on itself by conjugation, explicitly use the Orbit-Stabilizer theorem to find the size of the stabilizer of σ under this action and the elements of the Stabilizer subgroup of S_7.

    The stabilizer of σ in this context is better known as _____. (Using appropriate notation in place of words here is fine.)

    (e) Noting that $\sigma \in A_7$, what is the size of the conjugacy class of σ in A_7? Stated otherwise, how many conjugates in A_7 does σ have? Briefly, state a result that justifies your answer.

    Answer: The number of conjugates of σ in A_7 is _____

    because ....

4. (12 pts. – 6 pts. each) In the Table below, list a representative of each isomorphism class for groups G with $|G| = 6$ or p^2.

Group order $|G|$	Number of isomorphism types	Representatives
2	1	C_2
6		
p^2, for p prime		

2

Part II. Complete four of the following problems. You *MUST* choose exactly one problem from group A, and exactly three problems from group B and clearly indicate which problems you are submitting for grading. Each of the four problems is worth 12 pts.

A. Complete **exactly ONE** of the following two problems.

A.1 Suppose G is a group with H, K subgroups of G. Prove that if $H \leq N_G(K)$, then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G.

A.2 Suppose that a finite group G is of order 105, $|G| = 3 \cdot 5 \cdot 7$, and that G has normal subgroups of order 3, 5 and 7. (Do *NOT* assume results from the book here, but prove this result from first principles.)

Prove or disprove: G is cyclic.

B. Complete **exactly THREE** of the following problems.

B.1 Let P be a p-group, $|P| = p^a > 1$ for p a prime, and let A be a nonempty finite set. Suppose that P acts on A and define *the set of fixed points* of this action:

$$A_0 = \big\{ a \in A \mid g \cdot a = a \text{ for every } g \in P \big\}.$$

Prove that

$$|A| \equiv |A_0| \pmod{p}.$$

B.2 Let $\varphi(n)$ denote the Euler φ-function. Prove that if p is a prime and $n \in \mathbb{Z}^+$, then

$$n \,\Big|\, \varphi(p^n - 1).$$

(Hint: Compute the order of $\bar{p}$ in the appropriate group first.)

B.3 Let G be a finite Abelian group with $|G| = n$. Suppose that a is a positive integer relatively prime to n, $(a, n) = 1$. Prove that every element $g \in G$ can be written as $g = x^a$ for some $x \in G$.

(Hint: try thinking of an appropriate automorphism of G.)

B.4 Suppose G is a finite group of order $|G| = 14,553 = 3^3 \cdot 7^2 \cdot 11$ and that N is a normal subgroup of G of order $|N| = 11$. Prove that $N \leq Z(G)$.