MATH 253
Midterm 2

Name: SOLUTIONS

March 23, 2021

Instructions: 100 points total. Use only your brain and writing implement. You have 90 minutes to complete this exam. Good luck.

1. (8 pts.) Prove that the following limit does **NOT** exist.

Variable answers.

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2}$$

Approach along $x=0$: $\lim_{(0,y)\to(0,0)} \frac{4(0)^2y}{0^4+y^2} = 0$

Approach along $y=x^2$: $\lim_{(x,x^2)\to(0,0)} \frac{4x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{4x^4}{2x^4} = 2$

Since $0\neq 2$, this limit does not exist.

2. (8 pts.) Find the directional derivative of $f(x,y) = xy$ at the point $P(1,9)$ in the direction from P to $Q(4,5)$. Is $f(x,y)$ (circle one) increasing / decreasing / stationary at P?

Direction: $\vec{v} = \overrightarrow{PQ} = \langle 4-1, 5-9\rangle = \langle 3,-4\rangle$. A unit vector $\vec{u}$ in direction of $\vec{v}$ is $\vec{u} = \langle \frac{3}{5}, \frac{-4}{5}\rangle$. If $f(x,y)=xy$, Then $\nabla f = \langle y, x\rangle$ and $\nabla f(1,9) = \langle 9,1\rangle$. Finally, $D_{\vec{u}} f(1,9) = \nabla f(1,9)\cdot\vec{u} = \langle 9,1\rangle\cdot\langle\frac{3}{5},\frac{-4}{5}\rangle$

$= \frac{27}{5} - \frac{4}{3} = \boxed{\frac{23}{5}}$ Since $23/5 > 0$, $f(x,y)$ is increasing in the direction of $\vec{v}$.

3. (8 pts.) Suppose that

$$f(x,y) = x\,e^{xy} \quad \text{where } x=t^2,\ y=\ln(t).$$

Use the **Chain Rule** to find the derivative $\frac{df}{dt}$. Simplify your answer completely for full credit and make sure it is a function only of the variable t.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= e^{xy}(1+xy)(2t) + x^2e^{xy}\frac{1}{t}$$

$$= e^{t^2\ln(t)}(1+t^2\ln t)2t + (t^2)^2e^{t^2\ln t}\frac{1}{t}$$

substitute

$$= te^{t^2\ln t}\left(2+2t^2\ln t + t^2\right) = t^{1+t^2}(2+2t^2\ln t + t^2)$$

not necessary

$$\frac{\partial f}{\partial x} = xe^{xy}\cdot y + 1\cdot e^{xy} = e^{xy}(1+xy)$$

$$\frac{\partial f}{\partial y} = x^2e^{xy}$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = \frac{1}{t}$$

$$e^{xy} = e^{t^2\ln t} = (e^{\ln t})^{t^2} = t^{t^2}$$

4. (12 pts.) Consider the surface defined by $h(x, y) = 5x^2 + 3y^2$.

(a) Find the tangent plane to the surface $h(x, y) = 5x^2 + 3y^2$ at the point $(1, 1, h(1, 1))$.

$h(1,1) = 8 \quad h_x(x,y) = 10x \quad h_x(1,1) = 10 \quad h_y(x,y) = 6y \quad h_y(1,1) = 6$

$z - h(1,1) = h_x(1,1)(x-1) + h_y(1,1)(y-1)$

$z - 8 = 10(x-1) + 6(y-1)$

$z = 10x + 6y - 8$

$\boxed{10x + 6y - z = 8}$

(b) Estimate the value $h(.9, 1.01)$ using differentials. (Full credit only for using a linear approximation.)

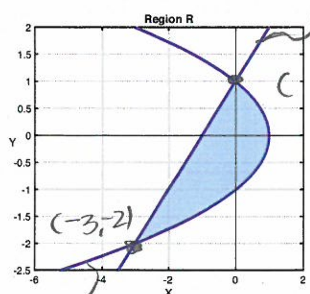
$h(.9, 1.01) \approx 10(.9) + 6(1.01) - 8$ ← Plug into tangent plane equation

$= 9 + 6.06 - 8$

$= \boxed{7.06}$

5. (12 pts.) The shaded lamina (plate or region) R below is bounded by the curves with equations $y^2 = 1-x$ and $y = x+1$. On this lamina, the charge density is given by $\sigma(x, y) = xy$ coulombs$/m^2$. Find the total charge of the lamina, including units in your final answer.

Key Computations boxed.



$y = x+1 \Rightarrow x = y-1$

$y^2 = 1-x \Rightarrow x = 1-y^2$

Total Charge $= \iint_R \sigma(x,y)\,dA = \iint_R xy\,dx\,dy$ easier than $dy\,dx$

$-2 \le y \le 1$

$y-1 \le x \le 1-y^2$

$= \boxed{\int_{-2}^{1} \int_{y-1}^{1-y^2} xy\,dx\,dy}$

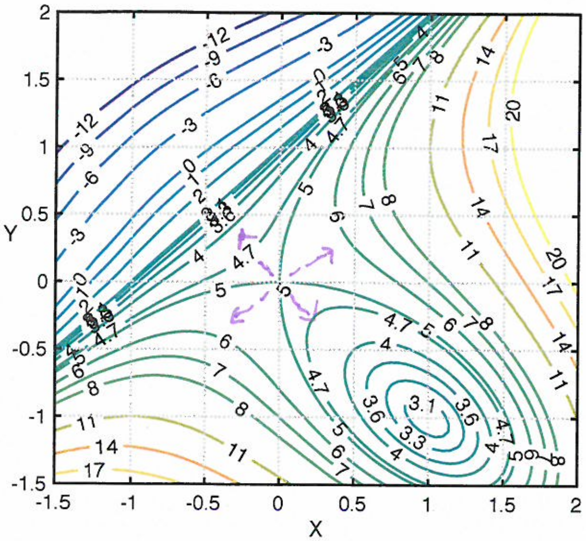
$= \int_{-2}^{1} \frac{1}{2}x^2 y \Big|_{y-1}^{1-y^2} dy = \int_{-2}^{1} \frac{1}{2}y\left[(1-y^2)^2 - (1-y)^2\right] dy$

$= \int_{-2}^{1} \frac{1}{2}y\left[1 - 2y^2 + y^4 - (1 - 2y + y^2)\right] dy = \int_{-2}^{1} \frac{1}{2}y\left[2y - 3y^2 + y^4\right] dy$

$= \int_{-2}^{1} \boxed{y^2 - \frac{3}{2}y^3 + \frac{1}{2}y^5}\,dy = \boxed{\frac{1}{3}y^3 - \frac{3}{8}y^4 + \frac{1}{12}y^6}\Big|_{-2}^{1} = \left(\frac{1}{3} - \frac{3}{8} + \frac{1}{12}\right) - \left(\frac{1}{3}(-8) - \frac{3}{8}(16) + \frac{1}{12}(64)\right)$

$= \frac{1}{3}(9) - \frac{3}{8}(-15) + \frac{1}{12}(-63) = 3 + \frac{45}{8} - \frac{21}{4} = \frac{24 + 45 - 42}{8} = \boxed{\frac{27}{8} \text{ coulombs}}$ Answer

6. (14 pts.) Pictured is a contour plot for the function $f(x, y) = 5 + 2x^3 - 2y^3 + 6xy$



(a) The function $f(x, y)$ has **two** local extrema at points (a, b), [i.e. a saddle point, a local maximum, or a local minimum at (a, b)]. In the table below, give the values of these extrema and the points at which they occur. Then briefly justify your answer.

	coordinates (a, b)	Value $f(a, b)$	min, max or saddle ?
1.	$(0,0)$	5	saddle point
2.	$(1,-1)$	≈ 3	local min

Justification: ← You could get 3 exactly as $f(1,-1)$.

(see purple lines) At $(0,0)$ the values of $f(x,y)$ decrease or increase depending which direction you move from $(0,0)$. Increase NE, SW

Decrease NW, SE.

From $(1,-1)$, the values of $f(x,y)$ increase.

(b) Use the second derivatives test to verify your answer. That is, find all critical points of $f(x, y)$ and classify them as local maxima, local minima, or saddle points.

$f(x,y) = 5 + 2x^3 - 2y^3 + 6xy$ $\qquad f_x = 6x^2 + 6y$ $\qquad f_y = -6y^2 + 6x$

$f_{xx} = 12x$ $\qquad f_{yy} = -12y$

$f_{xy} = 6$

Critical Points: Set $f_x = 0$, $f_y = 0$ and Solve:

$f_x = 0 \Rightarrow 6x^2 + 6y = 0 \Rightarrow 6(x^2 + y) = 0 \Rightarrow y = -x^2$

$f_y = 0 \Rightarrow -6y^2 + 6x = 0 \Rightarrow 6(-y^2 + x) = 0 \Rightarrow x = y^2$

Plugging $x = +y^2$ into top equation yields $y = -(+y^2)^2$ or $y = -y^4$ or $y^4 - y = 0$.

Factoring $y(y^3+1) = 0$ $\quad y(y+1)(y^2 - y + 1) = 0$ $\quad$ or $\boxed{y = 0, -1}$

If $y = 0$, then $x = +(0)^2 = 0$

If $y = -1$, then $x = (-1)^2 = 1$

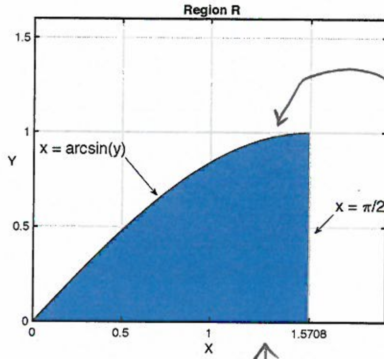
Thus, the critical points are $(0,0)$, $(1,-1)$ ✓

Now $D = \begin{vmatrix} 12x & 6 \\ 6 & -12y \end{vmatrix} = -144xy - 36$. $\qquad D(0,0) = -36 < 0 \Rightarrow (0,0)$ is a Saddle point ✓

$D(1,-1) = -144(1)(-1) - 36 = 108 > 0$. Moreover, $f_{xx}(1,-1) = 12(1) = 12 > 0 \Rightarrow (1,-1)$ is a local min. ✓

7. (12 pts.) Compute the double integral over the region R of integration by **reversing the order of integration.**

$$\int_0^1 \int_{\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{3+\cos^2(x)}\,dx\,dy$$



$x = \arcsin(y) \Rightarrow$

$y = \sin x$

$y=0$

$$= \int_0^{\pi/2}\int_0^{\sin(x)} \cos x\sqrt{3+\cos^2 x}\;dy\,dx$$

$$= \int_0^{\pi/2} \cos x\sqrt{3+\cos^2 x}\; y\Big|_0^{\sin x}\,dx$$

$$= \int_0^{\pi/2} \sin x\cos x\sqrt{3+\cos^2 x}\;dx$$

$R:\quad 0 \le x \le \pi/2$

$\quad\;\; 0 \le y \le \sin x$

Let $u = 3+\cos^2 x$, $du = -2\cos x\sin x\,dx \rightarrow$

$-\frac{1}{2}du = \sin x\cos x\,dx.$

$x=0 \Rightarrow u = 3+\cos^2(0) = 4$

$x = \pi/2 \Rightarrow$

$u = 3+\cos^2(\pi/2) = 3$

Substituting,

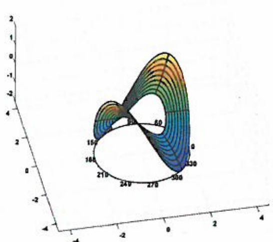
$$\int_4^3 -\tfrac{1}{2}u^{\frac{1}{2}}\,du$$

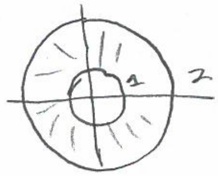
$$= -\tfrac{1}{3}u^{3/2}\Big|_4^3$$

$$= -\tfrac{1}{3}\left(3^{3/2} - 4^{3/2}\right)$$

$$= \boxed{\tfrac{8}{3} - \sqrt{3}}$$

8. (12 pts.) Find the surface area of the part of the saddle $z = x^2 - y^2$ that lies between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 4$. (A picture is included for help with visualization, but is unnecessary.)





R

annulus.

Polar coordinates:

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$$S(A) = \iint_R \sqrt{1+f_x^2+f_y^2}\,dA$$

$z = f = x^2-y^2 \Rightarrow f_x = 2x \quad f_y = -2y$

Thus,

$$S(A) = \iint_R \sqrt{1+(2x)^2+(-2y)^2}\,dA$$

$$= \iint_R \sqrt{1+4x^2+4y^2}\,dA$$

Now use polar coordinates

$$= \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\;r\,dr\,d\theta$$

$$= 2\pi\int_1^2 \sqrt{1+4r^2}\;r\,dr$$

Let $u = 1+4r^2$

$du = 8r\,dr$

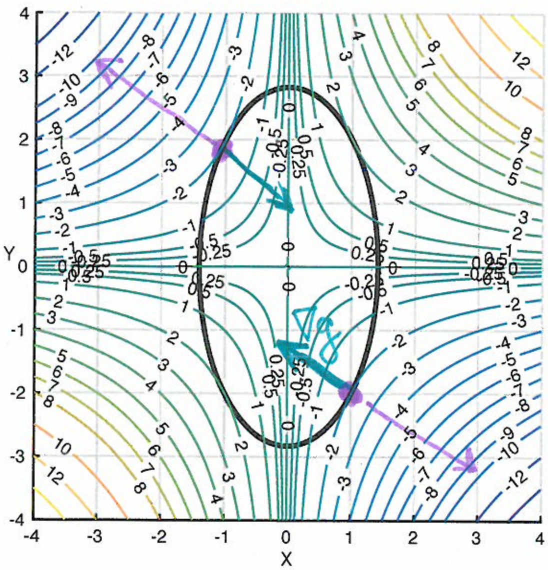
$\frac{1}{8}du = r\,dr$

$$= 2\pi\int \tfrac{1}{8}u^{\frac{1}{2}}\,du = \tfrac{\pi}{4}\cdot\tfrac{2}{3}u^{3/2} = \tfrac{\pi}{6}\left(1+4r^2\right)^{3/2}\Big|_1^2$$

$$= \boxed{\tfrac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]}$$

9. (14 pts.) Consider the function $f(x, y) = xy$ and its contour plot shown below.

Contour plot of f(x,y) = xy
Constraint g(x,y) = 8 in black.



∇g purple.

∇f blue

(up to scaling)

(a) The function $f(x, y)$ has two local minima subject to the constraint $g(x, y) = 4x^2 + y^2 = 8$. (The constraint $g(x, y) = 8$ is plotted in black in the figure.) By examining the contour plot give the coordinates of the two local minima (a, b) and the value $f(a, b)$ at those points.

	(a, b)	Minimum value $f(a, b)$
1.	$(a_1, b_1) = (1, -2)$	-2
2.	$(-1, +2)$	-2

(3pts)

Shown in purple.

(b) Give the equations you must solve simultaneously in order to use the method of Lagrange multipliers to find the minimum values of $f(x, y)$ subject to the constraint $4x^2 + y^2 = 8$. (Be careful; it might be easy to leave out one equation.) (4pts)

$\nabla f = \lambda \nabla g \Rightarrow \langle y, x \rangle = \lambda \langle 8x, 2y \rangle$. Thus:

(1.) $\boxed{y = \lambda 8x}$ and (2.) $\boxed{x = \lambda 2y}$ and (3) $\boxed{4x^2 + y^2 = 8}$ ↙ (x,y) must be on constraint.

(c) Now verify that the first point, call its coordinates (a_1, b_1), in your list from part (a) satisfies these equations.

I chose $(1, -2)$: If $\lambda = -\frac{1}{4}$, then (1.) $y \overset{?}{=} (-\frac{1}{4})8x \Rightarrow y \overset{?}{=} -2x \Rightarrow -2 = -2(1)$ ✓

(2.) $x \overset{?}{=} (-\frac{1}{4})(2y) \Rightarrow x \overset{?}{=} -\frac{1}{2}y \Rightarrow 1 \overset{?}{=} -\frac{1}{2}(-2) \Rightarrow 1 = 1$ ✓ (3pts)

(3.) $4x^2 + y^2 = 4(1)^2 + (-2)^2 = 8$ ✓

If you chose $(-1, 2)$, then $\lambda = -1/4$ too.

(d) One of the equations you gave in (b) should involve the gradient vector ∇f. Compute the gradient vectors $\nabla f(a_1, b_1)$ and $\nabla g(a_1, b_1)$, then plot them (up to a positive scaling factor) in the contour plot above. Then in the space to the right, explain briefly why the method of Lagrange multipliers works. (4pts)

∇f in blue

∇g in purple

$\nabla f(a_1, b_1) =$

for $(1,-2)$: $\nabla g(a_1, b_1) =$

* $\nabla f(1,-2) = \langle -2, 1 \rangle$

* $\nabla g(1,-2) = \langle 8, 2(-2) \rangle = \langle 8, -4 \rangle$

Note: $\nabla f(1,-2) = \langle -2, 1 \rangle = \underbrace{-\frac{1}{4}}_{\text{multiple } \lambda} \langle 8, -4 \rangle$

Explanation: The curves $g(x,y) = 0$ and $f(x,y) = -2$ have parallel normal vectors since they are tangent at $(1,-2)$ (and $(-1,2)$) where the minimum value occurs.

for $(-1,2)$ $\nabla f(-1,2) = \langle 2, -1 \rangle = \underbrace{-\frac{1}{4}}_{\lambda} \langle -8, 4 \rangle$