MATH 371
Midterm

Name : SOLUTIONS

October 22, 2008

Instructions: Round all answers to two significant digits (i.e. two decimal places, or if your answer is very small, give the first two non-zero digits following the decimal point). Good luck.

1. (10 pts. – 5 pts. each)

   (a) Give an example of two events, A and B, that are mutually exclusive, but not independent. Justify briefly that your example is correct.

Lots of egs possible.

Experiment: toss a fair coin

A: H comes up

B: T comes up

$A \cap B = \emptyset$

$P(A \mid B) = 0 \neq P(A)$

   (b) A manufacturing company makes washers for its livelihood. Suppose the mean diameter of a certain type of washer is .5 inch and the standard deviation is .01.
   Suppose a washer is selected at random. Use Tschebysheff's Theorem to give an interval $[x_{\text{lower}}, x_{\text{upper}}]$ so that the probability that the diameter of this washer lies within this interval is at least .75.

$$P(\underbrace{|Y-\mu| < 2\sigma}_{\text{Consider}}) \geq 1 - \frac{1}{(2)^2} = .75$$

Consider $[\mu - 2\sigma, \mu + 2\sigma]$ $= \boxed{[.48, .52]}$

2. (10 pts.) Suppose $0 < q < 1$.

   (a) Give a formula for $\displaystyle\sum_{y=1}^{\infty} y q^{y-1}$ by viewing the summand as a derivative and interchanging the order of summation and differentiation.

$$\sum_{y=1}^{\infty} y q^{y-1} = \sum_{y=1}^{\infty} \frac{d}{dq} q^y = \frac{d}{dq} \sum_{y=1}^{\infty} q^y = \frac{d}{dq}\left(\frac{q}{1-q}\right) = \frac{d}{dq} q(1-q)^{-1}$$

$$= q[(-1)(1-q)^{-2}(-1)] + (1)(1-q)^{-1} = (1-q)^{-2}[q + 1 - q] = \frac{1}{(1-q)^2}$$

   (b) Use your answer from part (a) to calculate that the expected value of a geometric random variable Y is $E(Y) = \frac{1}{p}$, where $p = P(\text{success})$.

$$E(Y) = \sum_{y=1}^{\infty} y p(y) = \sum_{y=1}^{\infty} y q^{y-1} p = p\left(\sum_{y=1}^{\infty} y q^{y-1}\right) \overset{\text{part (a)}}{=} p \frac{1}{(1-q)^2} = \frac{p}{p^2} = \boxed{\frac{1}{p}}$$

3. (10 pts.) Suppose two fair dice are thrown and consider the following events:

$P(A) = \frac{1}{6}$ $P(B) = \frac{5}{36}$

A: the roll of the first die is a 4
B: the sum of the two dice is 6
C: the sum of the two dice is 7

$P(C) = \frac{6}{36} = \frac{1}{6}$

(a) Are A and B independent? Prove your answer.

$P(A \cap B) = P(\text{Roll } 1 = 4 \text{ and Roll } 2 = 2) = \frac{1}{36} \neq P(A)P(B) = \frac{5}{216}$

OR

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/36}{1/6} = \frac{1}{6} \neq \frac{5}{36}$$

(b) Find $P(A \mid C)$.

$$\frac{P(A \cap C)}{P(C)} = \frac{P(\text{Roll } 1 = 4 \text{ and Roll } 2 = 3)}{P(C)} = \frac{1/36}{1/6} = \frac{1}{6}$$

4. (10 pts.) The average number of cars leaving a UAF parking lot between 9am and 10am on a school day is 2.2.

(a) If Y denotes the number of cars leaving the parking lot between 9am and 10am, what type of random variable is Y? Explain.

Poisson $\lambda = 2.2$ "rare" event in unit time

3 pts.

(b) Give the probability that exactly 3 cars leave the parking lot between 9am and 10am.

$P(Y=3) = e^{-2.2} \frac{(2.2)^3}{3!} \approx .1966 \approx .20$

3 pts.

(c) Give the probability that at most 5 cars leave the parking lot between 9am and 10am.

4 pts.

$P(Y \leq 5) = .975$ Table p. 844

5. (12 pts. – 3 pts. each) Consider the experiment of tossing a coin three times, where the probability of a head on an individual toss is 0.4. (This is an unfair coin.) Let X denote the number of heads that come up. Suppose that for each toss that comes up heads, you earn \$1 and for each toss that comes up tails, you lose \$1.

(a) Give the probability distribution $p(x)$ of X. (Your answer should include the domain of $p(x)$. If you use any parameters, be sure to indicate their value.)

$X \sim \text{Binom}(3, .4)$

$p(0) = (.6)^3 = .216$ $\qquad$ $p(2) = \binom{3}{2}(.4)^2(.6) = .288$

$p(1) = \binom{3}{1}(.4)(.6)^2 = .432$ $\qquad$ $p(3) = (.4)^3 = .064$

(b) Consider the random variable

Y: total earnings in playing this game.

i. Give a formula for Y in terms of X.

$$Y = \underbrace{1x}_{\text{win \$1}} - \underbrace{1(3-x)}_{\text{lose \$1}} = 2x - 3 \text{ dollars}$$

ii. Calculate the expected value $E(Y)$. Should you play this game?

$$E(Y) = E(2Y-3) = 2E(Y) - 3$$
$$= 2(3(.4)) - 3$$
$$= -.6 = -60¢$$

No. If you played a lot of times, on average you would lose 60¢ / game.

iii. Give the variance of your earnings, Var(Y).

$$\text{Var}(Y) = \text{Var}(2X-3) = 4\,\text{Var}(X)$$
$$= 4\underbrace{(3(.4)(.6))}_{npq} = 2.88$$

6. (18 pts.) Many people are reluctant to respond affirmatively to sensitive questions like "Have you ever used illegal drugs?" on surveys. Suppose that 80% of the population answers "no" to this question. Let X be the number of people you need to survey until you get a single affirmative reply.

(a) What type of discrete probability distribution does X have? Give a brief explanation for your choice including parameter values.

4pts.

X is geometric with $p = P(S) = P(\text{answers "Yes"}) = .2$

since X counts the trial on which the 1st affirmative occurs and all trials are independent and identical.

(b) Compute the probability that $X < 11$.

4pts.

$$P(X<11) = 1 - P(X \geq 11) = 1 - \sum_{x=11}^{\infty} q^{x-1} p = 1 - \sum_{x=11}^{\infty} (.8)^{x-1} (.2)$$

$$= 1 - .2 \left(\sum_{x=11}^{\infty} (.8)^{x-1} \right) = 1 - (.2) \frac{.8^{10}}{1-.8} = 1 - .8^{10} \approx .8926 \approx \boxed{.89}$$

geometric

(c) Suppose you survey individuals until you receive 5 affirmative responses.

i. How many individuals should you expect to survey?

3 pts.

$$\mathbb{E}(Y) = \frac{r}{p} = \frac{5}{.2} = 25$$

$Y \sim$ Negative Binomial with $p = .2$ $r = 5$

ii. What type of discrete random variable did you use to answer (i)? (Do not give an explanation here.)

3pts.

Negative Binomial with $p = .2$ $r = 5$

iii. Give the probability that you would need to survey exactly 20 individuals in order to get 5 affirmative responses.

4pts.

Let Y: trial in which you get the fifth affirmative response

Then $P(Y=20) = \binom{19}{4} (.2)^5 (.8)^{14} \approx \boxed{.0436}$

7. (12 pts. – 4 pts. each) In Denali National Park, ecologists try to estimate the number N of grizzly bears that live within the park boundaries. In summer 2005 rangers caught $m = 45$ bears, marked them, and released them. During the summer of 2006, rangers caught a random sample of $n = 20$ grizzlies. Let Y denote the number of marked grizzly bears in the random sample of size 20.

(a) What type of discrete probability distribution is a good model for Y? Explain.

Hypergeometric

Y counts the number of "successes" (= marked grizzly bears) in a random sample of size n=20 from a population of size N

(b) Suppose $N = 300$. What is the probability that exactly four of the twenty grizzlies caught in summer 2006 were marked?

$$P(Y=4) = \frac{\binom{45}{4}\binom{255}{16}}{\binom{300}{20}} \approx \boxed{.188}$$

(c) Suppose N is unknown. (This means ignore part (b) of this problem.) Suppose further that the park ecologists estimate that the probability of finding four marked grizzlies in their sample of size twenty is approximately $P(Y = 4) \approx .07633$ when N is unknown. What is the best estimate of N you can give?

$$\frac{\binom{45}{4}\binom{N-45}{16}}{\binom{300}{20}} \approx .07633$$

Test different values of N gives

$N \approx 480$

8. (8 pts.) Suppose Y is a random variable with moment generating function $m(t) = \frac{.2e^t}{1 - .8e^t}$.

(a) Give the probability $P(Y = 3)$.

$Y \sim$ geometric with $p = .2$

$p(3) = (.8)^2(.2) = .128$

(b) Use the moment generating function to determine the expected value of Y. (Note: there is an easy way to check your answer here. However, the question asks you to *use* the m.g.f. to find $E(Y)$.)

$$m(t) = \frac{.2e^t}{(1-.8e^t)} \Rightarrow m'(t) = \frac{(1-.8e^t)(.2e^t) - (.2e^t)(-.8e^t)}{(1-.8e^t)^2}$$

$$= \frac{.2e^t\left[1 - .8e^t + .8e^t\right]}{(1-.8e^t)^2} = \frac{.2e^t}{(1-.8e^t)^2} \quad \therefore \quad E(Y) = m'(0) = \frac{.2e^0}{(1-.8e^0)^2} = \frac{.2}{.2^2} = \boxed{5}$$

9. (10 pts.) The length of time required by students to complete a one hour exam is a random variable T with density given by

$$f(t) = \begin{cases} e(1 - e^{-t}), & 0 \le t \le 1 \text{ hours} \\ 0, & \text{otherwise.} \end{cases}$$

(a) After observing that $f(t) \ge 0$ for all values of t, evaluate an appropriately chosen definite integral $\int_a^b f(t)dt$, to show that $f(t)$ is indeed a density function for a random variable T.

4pts

$$\int_0^1 e(1-e^{-t})\,dt = e(t + e^{-t})\Big|_0^1 = e\left[(1+e^{-1}) - (0+1)\right]$$

$$= e(e^{-1}) = \boxed{1}$$

(b) Now give the cumulative distribution function $F(t)$ for T. (A complete answer includes the domain.)

3pts

$$F(t) = \begin{cases} 0, & t < 0 \\ e(t+e^{-t}) - e & 0 \le t \le 1 \\ 1, & t > 1 \end{cases} \qquad F(t) = \int_0^t f(x)\,dx$$

(c) Give the probability that a randomly selected student finishes the exam with less than 15 minutes to spare.

$$P(.75 \le T \le 1) = F(1) - F(.75) = 1 - \left[e(.75 + e^{-.75}) - e\right] \approx .3955$$

3 pts.

OR $\quad P(.75 \le T \le 1) = \int_{.75}^1 e(1-e^{-t})\,dt \approx .3955$