2. (7 pts.) Consider the permutation group S_7, and let $\sigma = (123)(13)(25)(146)(57)(752)$. Give in disjoint cycle notation the element $\sigma^{100} = [(123)(13)(25)(146)(57)(752)]^{100}$

$\sigma = (146)(23)$

$|\sigma| = \operatorname{lcm}\{2,3\} = 6$

$\sigma^{100} = \sigma^{96}\sigma^{4} = \sigma^{4}$ since $6 \mid 96$

$= (146)^4(23)^4 = \boxed{(146)}$

3. (7 pts.) Suppose G is a cyclic group of order $141,582,168 = 2^3 \cdot 3^4 \cdot 7^5 \cdot 13$. How many elements of order 49 does G contain? Briefly justify your answer.

G contains a unique subgroup H with $|H| = 49$

H has $\varphi(49) = 7 \cdot 6 = 42$ generators.

4. (10 pts.) Use the Fundamental Theorem of Finite Abelian Groups to list, up to isomorphism, all Abelian groups of order $756 = 2^2 \cdot 3^3 \cdot 7$. You do not need to justify your answer here; simply give a complete list without repetitions.

$\mathbb{Z}_4 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_7$ $\qquad$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_7$

$\mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$ $\qquad$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$

$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$ $\qquad$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$

Six total.

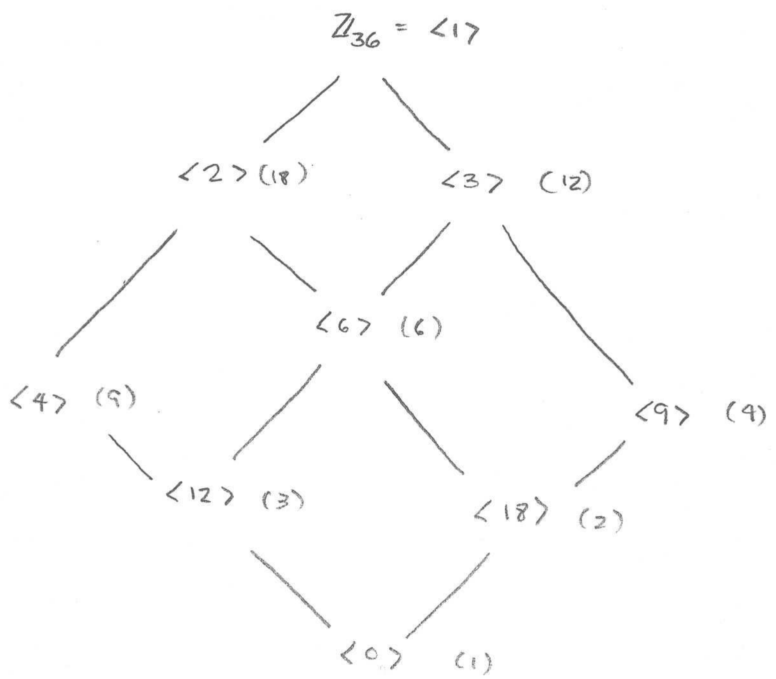
5. (16 pts.) Suppose G is a cyclic group of order n, $G \cong \mathbb{Z}_n$.

(a) Part of the Fundamental Theorem for Cyclic Groups characterizes all subgroups of a *finite* cyclic group G. (There are both existence and uniqueness statements.). State this part of the theorem.

5pts

(b) Now assume that $n = 36$ and so $G \cong \mathbb{Z}_{36}$. Draw a subgroup lattice for $\mathbb{Z}_{36}$. Clearly, indicate generators for each subgroup.

6pts.



The group order is shown next to the subgroup.

divisors of 36 are

1, 2, 3, 4, 6, 9, 12, 18, 36

(c) List all generators of $\mathbb{Z}_{36}$, and explain why these elements are generators.

5pts.

generators = k s.t. $\gcd(k, 36) = 1$

$= \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35\}$

There are $\varphi(36) = \varphi(4)\varphi(9)$

$= 2 \cdot 6 = 12$

of them

Prop: $1 \le k < 36$ generates $\mathbb{Z}_{36} \iff \gcd(k, 36) = 1$.

Part II.

6. (10 pts.) Fix $n \in \mathbb{Z}^+$ with $n \geq 2$.

   (a) Prove that for any $\sigma \in S_n$, $\sigma A_n = A_n\sigma$. (Thus, A_n is a normal subgroup of S_n, $A_n \lhd S_n$.)

We will show $\sigma A_n \subseteq A_n \sigma$ and $A_n \sigma \subseteq \sigma A_n$.

Suppose first $x \in \sigma A_n$. Then $x = \sigma \tau$ for τ an even permutation. If σ is even

   (b) Now explicitly list all the cosets of A_n in S_n and give $(S_n : A_n)$.

A_n, $(12)A_n$

7. (10 pts.) Let G be a group and fix an element $a \in G$. Define $C_a = \{x \in G \mid xa = ax\}$. Prove that C_a is a subgroup of G. (This subgroup is called the *centralizer of a in G*.)

Proof: Since $ea = ae$, $e \in C_a$ and $C_a \neq \emptyset$.

Suppose now that $x, y \in C_a$. Then $xa = ax$ and $ya = ay$. Consider xya. Then

$xya = x(ya) = x(ay)$ since $y \in C_a$

$= (xa)y$ by associativity

$= (ax)y$ since $x \in C_a$.

Thus, C_a is closed under products.

8. (10 pts.) Let $\phi : G \to G'$ be a group homomorphism. Prove that if $|G|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G|$.

Now suppose $x \in C_a$ so that $xa = ax$. Then multiplying on the left and right by x^{-1} yields

$xa = ax$

$\Rightarrow x^{-1}(xa)x^{-1} = x^{-1}(ax)x^{-1}$

$\Rightarrow ax^{-1} = x^{-1}a$

$\Rightarrow x^{-1} \in C_a$.

Thus, C_a is closed under inverses. By the 2-step subgroup test, $C_a \leq G$.