2019 Exam 2 #6 Find dist for fnc of cont. random variables

6. (13 pts.) In thousands of dollars, the profit M made by an apartment owner per month is $M = 2 - Y$ where $Y \sim \text{Unif}(0, 3)$ and Y is also measured in thousands of dollars. That is, if a renter causes \$500 of damage to the apartment in a month, then $Y = .5$ and $M = 2 - .5 = 1.5$, or the profit M is \$1,500 that month. Too easy: Better problem $M = 2 - Y^2$ and $Y \sim \text{Unif}(0, 2)$. Middling difficulty: $M = 3 - 2Y$, $Y \sim \text{Unif}(0, 1)$.

(a) (3 pts.) Give the support of the function $M = h(Y)$ for the apartment owner's monthly profit. That is, find the values of m such that the density $f_m(m)$ is non-zero.

$M = 2 - Y^2$

$0 \le Y \le 2$

Strictly decreasing

$\boxed{-2 \le m \le 2}$

(b) (10 pts.) Either using the Method of Distributions functions or the Method of Tranformations, find the density $f_m(m)$ for M.

$h(Y) = m = 2 - Y^2$ strictly decreasing

$h^{-1}(m) = \sqrt{2-m}$ $\qquad \frac{d\,h^{-1}(m)}{dm} = \frac{1}{2}(2-m)^{-1/2} \cdot -1$

$= \frac{-1}{2}(2-m)^{-1/2}$

$f_Y(y) = \frac{1}{2} \quad 0 \le y \le 2$

$f_m(y) = f_y(h^{-1}(m)) \left| \frac{d\,h^{-1}(m)}{dm} \right|$

$= \frac{1}{2} \left| \frac{-1}{2\sqrt{2-m}} \right|$

$= \frac{1}{4\sqrt{2-m}} \qquad -2 \le m < 2$

0 o.w.

$F_M(m) = P(M \le m) = P(2 - Y^2 \le m)$

$= P(Y^2 \ge 2 - m) = P(|Y| \ge \sqrt{2-m})$

$= P(Y \ge \sqrt{2-m}) = 1 - F_Y(\sqrt{2-m})$

$F_Y(y) = \begin{cases} 0 & y < 0 \\ \int_0^y \frac{1}{2}\,d\alpha = \frac{1}{2}y, & 0 \le y \le 2 \\ 1, & y > 2 \end{cases}$

$F_M(m) = \begin{cases} 1 & y < 0 & \text{i.e. } m > 2 \\ 1 - \frac{1}{2}\sqrt{2-m} & 0 \le y < 2 & \text{i.e. } -2 \le m \le 2 \\ 0 & y \ge 2 & \text{i.e. } m < -2 \end{cases}$

7. (10 pts.) Suppose X_1 and X_2 are independent Poisson-distributed random variables where $X_1 \sim \text{Pois}(\lambda_1)$ and $X_2 \sim \text{Pois}(\lambda_2)$. Use the method of moment-generating functions to find the distribution of $U = X_1 + X_2$.

$\therefore$ the density $f_m(m) = \frac{d}{dm} F_m(m) = \begin{cases} 0 & \text{if } m < -2 \text{ or } m \ge 2 \\ \frac{d}{dm}\left(1 - \frac{1}{2}\sqrt{2-m}\right) = \frac{-1}{2}\left(\frac{1}{2}(2-m)^{-1/2}(-1)\right) \end{cases}$

$= \frac{1}{4\sqrt{2-m}} \qquad -2 \le m < 2$

which agrees with the first solution

Another version:

$Y \sim \text{Beta}(1, 3) \qquad M = 2 - Y = h(Y)$

$h(Y)$ strictly decreasing.

$1 \le m \le 2$

$h^{-1}(m) = 2 - m$

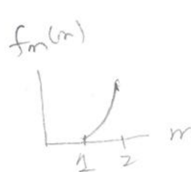
$\left|\frac{d\,h^{-1}(m)}{dm}\right| = |-1| = 1$

$f_Y(y) = \frac{\Gamma(4)}{\Gamma(1)\Gamma(3)}(1-y)^2 = \frac{3!}{2!}(1-y)^2 = 3(1-y)^2$

$0 \le y \le 1$

$\therefore f_m(m) = f_y(h^{-1}(m)) \left| \frac{d\,h^{-1}(m)}{dm} \right|$

$= 3(1-(2-m))^2 |-1| = 3(m-1)^2 \qquad 1 \le m \le 2$



$f_m(m) = \begin{cases} 3(m-1)^2 & 1 \le m \le 2 \\ 0 & \text{o.w.} \end{cases}$

$U \sim$ ____________________

Check: $\int_1^2 3(m-1)^2\,dm = (m-1)^3 \Big|_1^2 = 1 - 0 = 1$ ✓

Note: $f_m(m) \leftrightarrow \text{Beta}(1,3)$ translated!