HW 2 problems

1. Chap 1, #4, second part in slightly modified form. That is, compute the dimension of the vector space of homogeneous polynomials in n variables of degree exactly equal to d.

2. Consider the morphism of affine spaces given by $\phi : \mathbb{A}^2 \to \mathbb{A}^4$ where

$$(u, v) \mapsto (u + v, \quad u + 2v, \quad u - v, \quad 2u - v).$$

   (a) Using ideas we developed in class, solve the implicitization problem for this morphism. That is, find the polynomial equations in y_1, y_2, y_3, y_4 such that any point $p = (y_1, y_2, y_3, y_4) \in \text{Image}(\phi)$ must satisfy.

   (b) Informally but with justification, answer the following question

   "What is the dimension of Image(ϕ)?"

3. Use the multivariate division algorithm developed in class to express $f = x^2y^2 + xy^3$ in the form

$$f = a_1 f_1 + a_2 f_2 + r$$

   for $f_1 = xy^2 - 1$ and $f_2 = y + 1$ under the lexicographic ordering $>_{\text{lex}}$.

   (a) Do this with the divisors ordered by (f_1, f_2).

   (b) Do this with the divisors ordered by (f_2, f_1).

   (c) Indeed, $f \in I = \langle f_1, f_2 \rangle$. Express f as a linear combination of f_1 and f_2, if you can.

4. Order the following monomials with respect to the $>_{\text{lex}}$, $>_{\text{grlex}}$, and $>_{\text{grevlex}}$ monomial orderings. In your answer, justify one of the inequalities (*e.g.* $f >_{\text{grevlex}} g$ since ....) for each monomial order, and just report the others.

$$x^8, \quad xyz, \quad x^2y, \quad xz, \quad xy, \quad z^5, \quad x^5yz^2, \quad x^4y^2z^2$$

5. Sort the following polynomial with respect to the $>_{\text{lex}}$, $>_{\text{grlex}}$, and $>_{\text{grevlex}}$ monomial orderings.

   (a) $f(x, y, z) = 2x + 3y + z + x^2 - z^2 + x^3$

   (b) $g(x, y, z) = 5x^2yx - x^5yz^4 + 3xyz^3 - xy^4$

6. Using $>_{\text{grlex}}$, find an element $g \in \langle f_1, f_2 \rangle = \langle 2xy^2 - x, 3x^2y - y - 1 \rangle \subset \mathbb{R}[x]$ whose remainder on division by (f_1, f_2) is nonzero. *Hint*: you can find such a g where the remainder is g itself.