HW 2

The due date for these problems is Monday, February 8 at the beginning of class.

1. For each n, let
$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$
denote the nth partial sum of the harmonic series. Show that this sequence is not Cauchy under the Euclidean metric. (Hint: Show $|s_{2n} - s_n| \geq \frac{1}{2}$.)

2. (a) Prove, by induction, that if $d : M \times M \to \mathbb{R}$ is a metric, then for any $z_0, z_1, z_2, \ldots, z_n \in M$,
$$d(z_0, z_n) \leq d(z_0, z_1) + d(z_1, s_2) + \cdots + d(z_{n-1}, z_n).$$
   (b) Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Show that if there exist $c, r \in \mathbb{R}$, with $0 \leq r < 1$, such that $|a_{n+1} - a_n| \leq cr^n$, then the sequence is Cauchy under the Euclidean metric. (You will need part (a) and a closed form expression for $1 + r + r^2 + \cdots + r^k$.)

3. Explain how the previous problem shows that if $n.d_1d_2d_3\ldots$, with $n \in \mathbb{Z}$ and $d_i \in \{0, 1, 2, \ldots, 9\}$ is a decimal expansion for a real number, then the sequence of rational numbers
$$n.d_1,\ n.d_1d_2,\ n.d_1d_2d_3,\ \ldots$$
is Cauchy. State and explain the truth of a similar statement for base b expansions of real numbers.