HW #2 Solution to Even Problems

§5 #4 Determine whether the given subset of the complex numbers is a subgroup of the group $\mathbb{C}$ of complex numbers under addition. The set $i\mathbb{R}$ of pure imaginary numbers including 0.

Yes, $i\mathbb{R}$ is a subgroup of $\mathbb{C}$. Firstly, if $a, b \in \mathbb{R}$, then $ia + ib = i(a+b) \in i\mathbb{R}$. Thus, $i\mathbb{R}$ is closed under addition. Secondly, the additive identity $0 = 0i$ is an element of $i\mathbb{R}$. Thirdly, if $ia \in i\mathbb{R}$, then $-ia = i(-a) \in i\mathbb{R}$. Hence, $i\mathbb{R} \leq \mathbb{C}$.

Determine whether the given set of invertible $n \times n$ matrices with real number entries is a subgroup of $GL(n, \mathbb{R})$.

8. The $n \times n$ matrices with determinant 2.

No, this is not a subgroup of $GL(n, \mathbb{R})$. Denote the given set as H. Let $A, B \in H$. Then $\det(A) = 2$ and $\det(B) = 2$. However, $\det(AB) = \det(A)\det(B) = 4$, so $AB \notin H$. Thus, H is not closed under multiplication, and hence H is not a subgroup of $GL(n, \mathbb{R})$.

10. The upper-triangular $n \times n$ matrices with no zeros on the diagonal.

Yes, this is a subgroup of $GL(n, \mathbb{R})$. Denote the given set as K. We have that the product of upper-triangular matrices with no zeros on the diagonal is also an upper-triangular matrix with no zeros on the diagonal. Thus, K is closed under multiplication. Secondly, the identity matrix I_n is an upper-triangular matrix with no zeros on the diagonal. Hence, $I_n \in K$. Thirdly, for every matrix $A \in K$, A^{-1} is also an upper-triangular matrix with no zeros on the diagonal. Hence, $A^{-1} \in K$. Therefore, $K \leq GL(n, \mathbb{R})$.

12. The $n \times n$ matrices with determinant -1 or 1.

Yes, this is a subgroup of $GL(n, \mathbb{R})$. Denote the given set as G. Firstly, we have that if $A, B \in GL(n, \mathbb{R})$, then $\det(A) = \pm 1$ and $\det(B) = \pm 1$. Thus, $\det(AB) = \det(A)\det(B) = \pm 1$. Thus, G is closed under multiplication. Secondly, the identity matrix $I_n \in G$ since $\det(I_n) = 1$. Thirdly, if $A \in G$, then $\det(A) = \pm 1$. Thus,

$\det(A^{-1}) = \frac{1}{\det(A)} = \pm 1$. Hence, $A^{-1} \in G$ as well. Therefore, $G \leq GL(n, \mathbb{R})$.

22. Describe all the elements in the cyclic subgroup of $GL(2, \mathbb{R})$ generated by the given 2×2 matrix.

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus, $\left\langle \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$

26. Which of the following groups are cyclic? For each cyclic group, list all the generators of the group.

$G_1 = \langle \mathbb{Z}, + \rangle$ is cyclic with generators 1 and -1

$G_2 = \langle \mathbb{Q}, + \rangle$ is not cyclic

$G_3 = \langle \mathbb{Q}^+, \cdot \rangle$ is not cyclic

$G_4 = \langle 6\mathbb{Z}, + \rangle$ is cyclic with generators, 6 and -6

$G_5 = \{6^n \mid n \in \mathbb{Z}\}$ under multiplication is cyclic with generators 6 and $1/6$

$G_6 = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ under addition is not cyclic

36. a. Complete Table 5.25 to give the group $\mathbb{Z}_6$ of 6 elements

$\mathbb{Z}_6$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

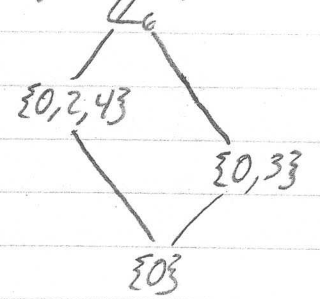
b. Compute the subgroups $\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle$, and $\langle 5 \rangle$ of the group $\mathbb{Z}_6$ given in part (a).

$\langle 0 \rangle = \{0\}$

$\langle 1 \rangle = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$
$\langle 2 \rangle = \{0, 2, 4\}$
$\langle 3 \rangle = \{0, 3\}$
$\langle 4 \rangle = \{0, 4, 2\}$
$\langle 5 \rangle = \{0, 5, 4, 3, 2, 1\} = \mathbb{Z}_6$

c. Which elements are generators for the group $\mathbb{Z}_6$ of part (a)?
1 and 5 are the generators of $\mathbb{Z}_6$ since $\langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_6$.

d. Give the subgroup diagram for the part (b) subgroups of $\mathbb{Z}_6$.



54. For sets H and K, we define the intersection $H \cap K$ by $H \cap K = \{x \mid x \in H \text{ and } x \in K\}$. Show that if $H \leq G$ and $K \leq G$, then $H \cap K \leq G$.

<u>Proof</u>: Firstly, let $a, b \in H \cap K$. Thus, $a, b \in H$ and $a, b \in K$. Since H is closed under the group operation, $ab \in H$. Similarly, $ab \in K$. Thus, $ab \in H \cap K$ and $H \cap K$ is closed under the group operation. Secondly, we have that the identity element e is an element of H and is an element of K since H and K are subgroups of G. Therefore, $e \in H \cap K$. Thirdly, let $a \in H \cap K$. Then $a \in H$ and $a \in K$. Since H is a subgroup of G, $a^{-1} \in H$. Similarly, $a^{-1} \in K$. Hence, $a^{-1} \in H \cap K$. Since $H \cap K$ is closed under the group operation, contains the identity element e of G, and contains inverses, $H \cap K \leq G$. ∎