Comments on HW 9.

Ex 45 (1): Show that $\mathbb{Q}_p$ and $\mathbb{R}$ are not isomorphic.

The easiest way to do this is by contradiction. If there were a field isomorphism

$$\varphi : \mathbb{R} \to \mathbb{Q}_p,$$

then $\varphi(p) = p$. (This is true for any isomorphism of fields of characteristic zero.)
Then, $p = \varphi(p) = \varphi(\sqrt{p}^2) = \varphi^2(\sqrt{p})$ and taking norms we find

$$\frac{1}{p} = |\varphi^2(\sqrt{p})|_p = |\varphi(\sqrt{p})|_p^2.$$

Since norms are non-negative and multiplicative, taking square roots yields

$$\frac{1}{\sqrt{p}} = |\varphi(\sqrt{p})|_p$$

which contradicts that the p-adic norm takes values in the set $\{p^n \mid n \in \mathbb{Z}\}$.

Ex 46: Prove that the equation

$$(x^2 - 2)(x^2 - 17)(x^2 - 34) = 0 \tag{1}$$

has a root in $\mathbb{Q}_p$ for all p, but not in $\mathbb{Q}$.

Proof. First, it is clear that $\sqrt{2}, \sqrt{17}, \sqrt{34} \notin \mathbb{Q}$ so (1) has no roots in the rational numbers. However, these roots are real numbers, so this equation has roots for $p = \infty$.

Assume now that $p = 2$. Since $17 \equiv 1 \pmod{8}$, 17 is a square in $\mathbb{Q}_2$ by Exercise 42. If $p = 17$, then $6^2 = 36 \equiv 2 \pmod{17}$ so 2 is a square mod 17. By Proposition 1.43, $x^2 - 2$ has a root in $\mathbb{Q}_{17}$.

For the last case, assume that p is odd but not equal to 17. If either of 2 or 17 are quadratic residues mod p, then (1) has a root in $\mathbb{Q}_p$. If both 2 and 17 are quadratic non-residues mod p, then their product 34 is a quadratic residue mod p and again (1) has a root in $\mathbb{Q}_p$ by Proposition 1.43. □

Ex 44: Alas.