MATH253X-UX1 Final Exam Name: Answer Key

Fall 2019

Instructions. You have 120 minutes. Closed book, closed notes, no calculator. **The last page contains some unlabeled theorems from our course.** *Show all your work* to receive full credit.

1. Consider the points $A = (1, 2, -1)$, $B = (-3, 0, 1)$ and $C = (0, 3, 1)$.

(a) Give a parameterization of the straight line segment from A to B. Be sure you state what the parameter may range over.

Solution:

$$\overrightarrow{AB} = \langle -3-1, 0-2, 1+1\rangle = \langle -4, -2, 2\rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle 1, 2, -1\rangle + t\langle -4, -2, 2\rangle$$
$$\Rightarrow \quad \boxed{\mathbf{r}(t) = \langle 1-4t, 2-2t, 2t-1\rangle \ , \ 0 \le t \le 1}$$

(b) Find an equation (*not* a parameterization) for the plane containing A, B, C.

Solution: For the normal vector, we can choose any scalar multiple of $\overrightarrow{AB} \times \overrightarrow{AC}$ for example:

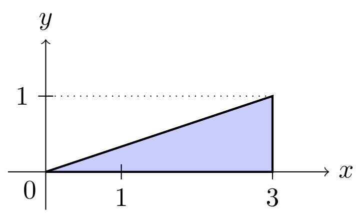
$$\overrightarrow{AC} = \langle 0-1, 3-2, 1+1\rangle = \langle -1, 1, 2\rangle$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = (-2)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = -2\langle 2+1, -(4-1), 2+1\rangle = -2\langle 3, -3, 3\rangle = -6\langle 1, -1, 1\rangle$$
$$\Rightarrow \quad (x-1) - (y-2) + (z+1) = 0 \quad \text{or} \quad \boxed{x - y + z + 2 = 0}$$

2. Sketch the region of integration of

$$\int_0^1 \int_{3y}^3 e^{(x^2)}\, dx\, dy.$$

Then use your sketch to reverse the order of integration and evaluate the integral.

Solution: The region of integration is:



So switching the order of integration we have:

$$\int_0^1 \int_{3y}^3 e^{(x^2)}\, dx\, dy = \int_0^3 \int_0^{\frac{x}{3}} e^{(x^2)}\, dy\, dx$$

And we compute:

$$\int_0^3 \int_0^{\frac{x}{3}} e^{(x^2)}\, dy\, dx = \int_0^3 \left[ye^{(x^2)}\right]_0^{\frac{x}{3}} dx = \int_0^3 \frac{x}{3}e^{(x^2)} - 0\, dx = \left[\frac{e^{(x^2)}}{6}\right]_0^3 = \boxed{\frac{e^9 - 1}{6}}.$$

3. Assume a particle has velocity $\mathbf{v}(t) = (t+1)\mathbf{i} + 2\sqrt{t}\mathbf{j} + (t-1)\mathbf{k}$ for $t \ge 1$ with speed measured in m/s.

(a) Find the time(s) when acceleration is parallel to $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Solution:

$$\mathbf{a}(t) = \left\langle 1, \frac{1}{\sqrt{t}}, 1\right\rangle = c\langle 2, 1, 2\rangle \quad \Rightarrow \quad 1 = 2c \quad \Rightarrow \quad c = \frac{1}{2}$$
$$\Rightarrow \quad \frac{1}{\sqrt{t}} = \frac{1}{2} \quad \Rightarrow \quad \sqrt{t} = 2 \quad \Rightarrow \quad \boxed{t = 4 \text{ s}}$$

(b) Find the distance traveled from $t = 1$ s to $t = 3$ s.

Solution: Let d be the distance traveled.

$$\begin{aligned}
\|\mathbf{v}(t)\| &= \sqrt{(t+1)^2 + (2\sqrt{t})^2 + (t-1)^2} = \sqrt{t^2 + 2t + 1 + 4t + t^2 - 2t + 1} \\
&= \sqrt{2t^2 + 4t + 2} = \sqrt{2(t+1)^2} = (t+1)\sqrt{2}
\end{aligned}$$

$$\Rightarrow \quad d = \int_1^3 \|\mathbf{v}(t)\|\ dt = \int_1^3 (t+1)\sqrt{2}\ dt = \left[\left(\frac{t^2}{2} + t\right)\sqrt{2}\right]_1^3 = \left(\frac{9}{2} + 3 - \frac{1}{2} - 1\right)\sqrt{2} = \boxed{6\sqrt{2}\text{ m}}$$

(c) Find the position vector $\mathbf{r}(t)$ at all times if $\mathbf{r}(1) = 2\mathbf{i} - \dfrac{1}{2}\mathbf{k}$.

Solution:

$$\mathbf{r}(t) - \mathbf{r}(1) = \int_1^t \mathbf{v}(u)\ du = \int_1^t \left\langle u+1, 2\sqrt{u}, u-1\right\rangle\ du = \left[\left\langle \frac{u^2}{2} + u, \frac{4u^{\frac{3}{2}}}{3}, \frac{u^2}{2} - u\right\rangle\right]_1^t$$

$$\begin{aligned}
\Rightarrow \quad \mathbf{r}(t) - \left\langle 2, 0, -\frac{1}{2}\right\rangle &= \left\langle \frac{t^2}{2} + t, \frac{4t^{\frac{3}{2}}}{3}, \frac{t^2}{2} - t\right\rangle - \left\langle \frac{1}{2} + 1, \frac{4}{3}, \frac{1}{2} - 1\right\rangle \\
\Rightarrow \quad \mathbf{r}(t) &= \left\langle \frac{t^2}{2} + t, \frac{4t^{\frac{3}{2}}}{3}, \frac{t^2}{2} - t\right\rangle + \left\langle 2 - \frac{3}{2}, 0 - \frac{4}{3}, -\frac{1}{2} + \frac{1}{2}\right\rangle \\
&= \left\langle \frac{t^2}{2} + t, \frac{4t^{\frac{3}{2}}}{3}, \frac{t^2}{2} - t\right\rangle + \left\langle \frac{1}{2}, -\frac{4}{3}, 0\right\rangle \\
\Rightarrow \quad & \boxed{\mathbf{r}(t) = \left\langle \frac{t^2}{2} + t + \frac{1}{2}, \frac{4t^{\frac{3}{2}}}{3} - \frac{4}{3}, \frac{t^2}{2} - t\right\rangle}
\end{aligned}$$

4. Use Lagrange multipliers to find the extreme values of the function $f(x, y) = x^2 - y^2$ along the parabola $x - y^2 = -1$.

Solution: Define $g(x, y) = x - y^2 = -1$ for the constraint. Then extreme values will happen when:

$$\nabla f = \lambda \nabla g\ ,\ g(x, y) = -1 \quad \Rightarrow \quad \langle 2x, -2y\rangle = \lambda\langle 1, -2y\rangle\ ,\ x - y^2 = -1 \quad \Rightarrow \quad \begin{cases} 2x = \lambda \\ -2y = -2\lambda y \\ x - y^2 = -1 \end{cases}$$

From the second equation, we have two cases:

- if $y = 0$ then from the constraint: $x = -1$ so we have the point $(-1, 0)$;
- if $y \neq 0$ then $\lambda = 1$ and plugging into the first equation we have:

$$2x = 1 \quad \Rightarrow \quad x = \frac{1}{2}$$

which when plugged into the constraint gives you $-y^2 = -\dfrac{3}{2}$ so $y = \pm\sqrt{\dfrac{3}{2}}$ and so we have the points $\left(\dfrac{1}{2}, \pm\sqrt{\dfrac{3}{2}}\right)$.

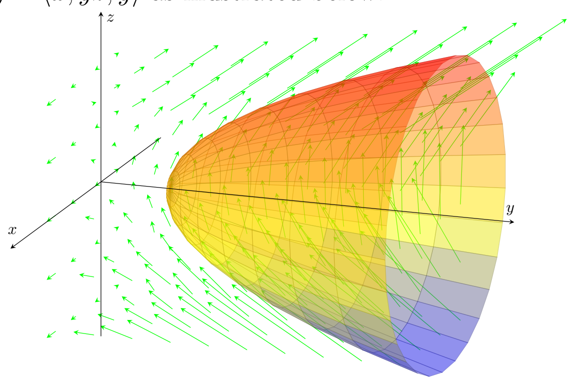
We now compute $f(x, y)$ at all the points found above to find the extreme values:

x	y	$f(x,y)$	
-1	0	1	absolute maximum
$\frac{1}{2}$	$\pm\sqrt{\frac{3}{2}}$	$-\frac{5}{4}$	absolute minimum

5. Consider the surface S parametrized by:

$$\mathbf{r}(u,v) = \left\langle u\cos v, u^2+1, u\sin v \right\rangle \qquad \text{for} \qquad 0 \le u \le 2\,,\ 0 \le v \le 2\pi$$

in the vector field $\mathbf{F}(x,y,z) = \langle x, yz, y\rangle$ as illustrated below:



(a) Use Stokes' theorem to compute the circulation of $\mathbf{F}(x,y,z)$ around the oriented boundary curve C of the surface S NOT directly BUT as a surface integral using the given S.

Solution: First we compute:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & 2u & \sin v \\ -u\sin v & 0 & u\cos v \end{vmatrix} = \left\langle 2u^2\cos v, -u, 2u^2\sin v \right\rangle .$$

Next, we take the curl of the field:

$$\operatorname{curl}\mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & yz & y \end{vmatrix} = \langle 1-y, 0-0, 0-0\rangle = \langle 1-y, 0, 0\rangle\,,$$

and along the surface we have $\operatorname{curl}\mathbf{F}(\mathbf{r}(u,v)) = \left\langle 1-(u^2+1), 0, 0\right\rangle = \left\langle -u^2, 0, 0\right\rangle$. And therefore,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS = \int_0^{2\pi}\int_0^2 \left\langle -u^2,0,0\right\rangle\cdot\left\langle 2u^2\cos v, -u, 2u^2\sin v\right\rangle\,du\,dv = \int_0^{2\pi}\int_0^2 -2u^4\cos v\,du\,dv$$
$$= \left(\int_0^{2\pi}\cos v\,dv\right)\left(\int_0^2 -2u^4\,du\right) = \Big[\sin v\Big]_0^{2\pi}\left[-\frac{2u^5}{5}\right]_0^2 = \boxed{0}$$

(b) Find an equation of the tangent plane to the surface at the point $\left(\dfrac{1}{2}, 2, \dfrac{\sqrt{3}}{2}\right)$.

Solution: We first solve for (u,v) such that:

$$\mathbf{r}(u,v) = \left\langle \frac{1}{2}, 2, \frac{\sqrt{3}}{2}\right\rangle \quad\Rightarrow\quad \begin{cases} u\cos v = \frac{1}{2} \\ u^2+1 = 2 \\ u\sin v = \frac{\sqrt{3}}{2} \end{cases} .$$

From the second equation we get $u^2 = 1$ so with the restriction in domain of u, that means $u = 1$. Then the first equation becomes $\cos v = \dfrac{1}{2}$ so $v = \dfrac{\pi}{3}$ or $v = \dfrac{5\pi}{3}$, but since $\sin v > 0$ for our point (third equation) that means $v = \dfrac{\pi}{3}$. Therefore, the normal vector is:

$$\mathbf{r}\left(1, \frac{\pi}{3}\right) = \left\langle 2u^2\cos v, -u, 2u^2\sin v\right\rangle\Big|_{(1,\frac{\pi}{3})} = \left\langle 1, -1, \sqrt{3}\right\rangle .$$

And so the equation of the tangent plane is:

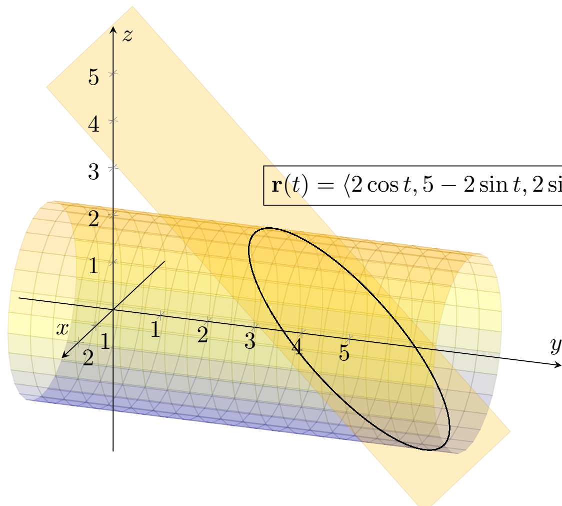
$$\left(x - \frac{1}{2}\right) - (y-2) + \sqrt{3}\left(z - \frac{\sqrt{3}}{2}\right) = 0 \quad \text{or} \quad \boxed{x - y + z\sqrt{3} = 0}\,.$$

6. Sketch the two surfaces

$$x^2 + z^2 = 4, \quad y + z = 5$$

and highlight their curve of intersection. Then give a parameterization of that curve.

Solution: We have a cylinder and a slanted plane, so their intersection is elliptic in shape.



7. Find all critical points of the function

$$f(x,y) = x^3 - 6xy + 8y^3$$

and, to the extent possible, determine whether they are local maxima, local minima, or saddle points.

Solution: We have $\nabla f(x,y) = \left\langle 3x^2 - 6y, -6x + 24y^2 \right\rangle$ so partials are continuous everywhere and therefore critical points will be where the gradient is null:

$$\nabla f(x,y) = \mathbf{0} \iff \left\langle 3x^2 - 6y, -6x + 24y^2 \right\rangle = \langle 0, 0\rangle \iff \begin{cases} 3x^2 - 6y = 0 \\ -6x + 24y^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y = \frac{x^2}{2} \\ -x + 4y^2 = 0 \end{cases} \qquad \Rightarrow \begin{cases} y = \frac{x^2}{2} \\ -x + x^4 = 0 \end{cases} \qquad \Rightarrow \begin{cases} y = \frac{x^2}{2} \\ x(-1 + x^3) = 0 \end{cases}$$

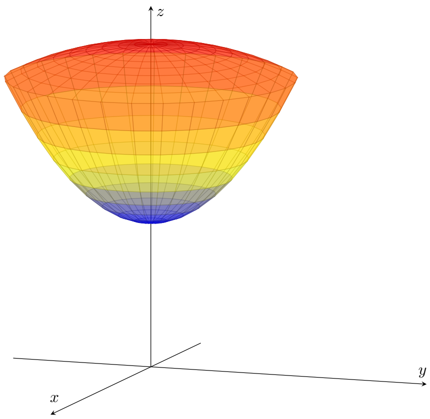
From the second equation we get $x = 0$ or $x = 1$ which when plugged into the first equation gives us respectively $y = 0$ or $y = \dfrac{1}{2}$ so we have two critical points $(0,0)$ and $\left(1, \dfrac{1}{2}\right)$. To classify them, we use the Second Derivatives Test:

$$f_{xx} = 6x \quad , \quad f_{yy} = 48y \quad , \quad f_{xy} = -6$$

$$\Rightarrow \quad d(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 288xy - 36$$

- $d(0,0) = 0 - 36 = -36 < 0$, so $\boxed{(0,0,0) \text{ is a saddle point}}$;
- $d(1,1/2) = 144 - 36 = 108 > 0$, $f_{xx}(1,1/2) = 6 > 0$ so $\boxed{(1,1/2) \text{ is a local minimum}}$.

8. Use **cylindrical coordinates** to find the mass of the solid enclosed below by the paraboloid $z = x^2+y^2+1$ and above by the sphere $x^2+y^2+z^2=5$ if the density function is given by $\rho(x,y,z) = \dfrac{1}{z^2}$.



Solution: In cylindrical coordinates, the paraboloid is $z = r^2+1$, the sphere is $r^2+z^2=5$ and since we have part of the top half, we can solve for $z=\sqrt{5-r^2}$ and the density function remains $\dfrac{1}{z^2}$. For bounds in r. note that the shadow of the solid onto the xy-plane is a disk whose radius corresponds to that of the circle of intersection between the paraboloid and the sphere:

$$r^2+1=\sqrt{5-r^2} \Rightarrow (r^2+1)^2 = 5-r^2 \Rightarrow r^4+2r^2+1=5-r^2 \Rightarrow r^4+3r^2-4=0 \Rightarrow (r^2+4)(r^2-1)=0.$$

So we find $r=1$ and therefore the mass of the solid is:

$$\begin{aligned}
m &= \int_0^{2\pi}\int_0^1\int_{r^2+1}^{\sqrt{5-r^2}} \frac{1}{z^2}\, r\, dz\, dr\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 \left[-\frac{1}{z}\right]_{r^2+1}^{\sqrt{5-r^2}} r\, dr\right) \\
&= \Big[\theta\Big]_0^{2\pi} \int_0^1 -\frac{r}{\sqrt{5-r^2}} + \frac{r}{r^2+1}\, dr = 2\pi\left[\sqrt{5-r^2}+\frac{1}{2}\ln(r^2+1)\right]_0^1 \\
&= 2\pi\left(2+\frac{1}{2}\ln 2 - \sqrt{5} - 0\right) = \boxed{\pi\left(4+\ln 2 - 2\sqrt{5}\right)}.
\end{aligned}$$

9. Let S be the closed surface that encloses the eighth of the unit ball centered at the origin for which $x \geq 0$, $y \leq 0$ and $z \leq 0$, oriented outward. Use Gauss' Divergence Theorem and **spherical coordinates** to fully SET UP an integral computing the flux out of S of the vector field $\mathbf{F}(x,y,z) = \langle x^2, -2yx, xz\rangle$. DO NOT EVALUATE.

Solution: By the divergence theorem, we can rewrite the flux:

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_Q \operatorname{div}\mathbf{F}\, dV$$

where Q is the solid with boundary S. The divergence of the field is:

$$\operatorname{div}\mathbf{F}(x,y,z) = P_x + Q_y + R_z = 2x - 2x + x = x.$$

Now if we rewrite our solid Q in spherrical coordinates, we have $0 \leq \rho \leq 1$ (for the unit ball), $-\dfrac{\pi}{2} \leq \theta \leq 0$ (for $x \geq 0$, $y \leq 0$), and $\dfrac{\pi}{2} \leq \phi \leq \pi$ (for $z \leq 0$). And since $x = \rho\cos\theta\sin\phi$ we have:

$$\oiint_S \mathbf{F}\cdot\mathbf{N}\, dS = \iiint_Q \operatorname{div}\mathbf{F}\, dV = \iiint_Q x\, dV = \int_{\frac{\pi}{2}}^{\pi}\int_{-\frac{\pi}{2}}^{0}\int_0^1 (\rho\cos\theta\sin\phi)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$\Rightarrow \boxed{\oiint_S \mathbf{F}\cdot\mathbf{N}\, dS = \int_{\frac{\pi}{2}}^{\pi}\int_{-\frac{\pi}{2}}^{0}\int_0^1 \rho^3\cos\theta\sin^2\phi\, d\rho\, d\theta\, d\phi}.$$

10. Consider the vector field

$$\mathbf{F}(x,y,z) = \langle e^{x-y} - z\sin(xz), z^2 - e^{x-y}, 2yz - x\sin(xz)\rangle.$$

(a) Find a potential function for $\mathbf{F}(x,y,z)$.

Solution: We have that for any potential function f, $\mathbf{F}(x,y,z) = \langle P,Q,R\rangle = \langle f_x, f_y, f_z\rangle$. So,

$$\begin{aligned}
f(x,y,z) &= \int P\,dx = \int e^{x-y} - z\sin(xz)\,dx = e^{x-y} + \cos(xz) + C_1(y,z)\\
f(x,y,z) &= \int Q\,dy = \int z^2 - e^{x-y}\,dy = yz^2 + e^{x-y} + C_2(x,z)\\
f(x,y,z) &= \int R\,dx = \int 2yz - x\sin(xz)\,dz = yz^2 + \cos(xz) + C_3(x,y)\\
\Rightarrow &\quad \boxed{f(x,y,z) = e^{x-y} + \cos(xz) + yz^2\ (+C)}
\end{aligned}$$

(b) Use your answer to part (a) to evaluate the work done by $\mathbf{F}$ if a particle follows a helical path from the point $(2,0,0)$ to the point $(2,0,1)$, spiraling counterclockwise one time around the z-axis.

Solution: By the Fundamental Theorem of Line Integrals,

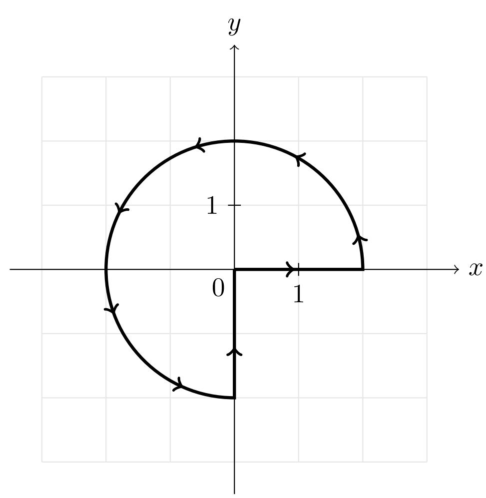
$$\begin{aligned}
W &= \int_C \mathbf{F}\cdot d\mathbf{r} = f(2,0,1) - f(2,0,0)\\
&= e^{2-0} + \cos(2(1)) + 0 - \left(e^{2-0} + \cos(2(0)) + 0\right)\\
&= e^2 + \cos 2 - e^2 - 1 = \boxed{\cos 2 - 1}\,.
\end{aligned}$$

11. Use Green's theorem to compute

$$\int_C \left(e^{\cos x} - x^2 y\right)\,dx + (\arctan y + xy)\,dy$$

over the closed curve C made up of the line segment from $(0,0)$ to $(2,0)$, then three quarters around the circle $x^2+y^2=4$ until $(0,-2)$ then the line segment back to the origin.

Solution: We verify that C is oriented counterclockwise:



Let $I = \int_C \left(e^{\cos x} - x^2 y\right)\, dx + \left(\arctan y + xy\right)\, dy$. Then by Green's theorem,

$$\begin{aligned}
I &= \iint_R (\arctan y + xy)_x - (e^{\cos x} - x^2 y)_y \, dA = \iint_R y + x^2 \, dA = \int_0^{\frac{3\pi}{2}} \int_0^2 (r\sin\theta + r^2\cos^2\theta)\, r\, dr\, d\theta \\
&= \int_0^{\frac{3\pi}{2}} \int_0^2 r^2 \sin\theta + r^3\cos^2\theta \, dr\, d\theta = \int_0^{\frac{3\pi}{2}} \left[\frac{r^3}{3}\sin\theta + \frac{r^4}{4}\cos^2\theta\right]_0^2 d\theta = \int_0^{\frac{3\pi}{2}} \frac{8}{3}\sin\theta + 4\cos^2\theta \, d\theta \\
&= \int_0^{\frac{3\pi}{2}} \frac{8}{3}\sin\theta + 2(1+\cos 2\theta)\, d\theta = \left[-\frac{8}{3}\cos\theta + 2\theta + \sin 2\theta\right]_0^{\frac{3\pi}{2}} \\
&= 0 + 3\pi + 0 - \left(-\frac{8}{3} + 0 + 0\right) = \boxed{3\pi + \frac{8}{3}}
\end{aligned}$$

12. Let $f(x,y) = \dfrac{x}{y^2} + x^2 y$.

(a) What is the directional derivative of f at $(2,1)$ when moving towards $(0,2)$? What does it mean for function values?

Solution:

$$\nabla f(x,y) = \left\langle \frac{1}{y^2} + 2xy, -\frac{2x}{y^3} + x^2 \right\rangle \quad \Rightarrow \quad \nabla f(2,1) = \langle 1+4, -4+4\rangle = \langle 5, 0\rangle$$

$$\mathbf{v} = \langle 0-2, 2-1\rangle = \langle -2, 1\rangle \quad \Rightarrow \quad \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$D_{\mathbf{u}} f(2,1) = \nabla f(2,1)\cdot \mathbf{u} = \langle 5,0\rangle \cdot \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = -\frac{10}{\sqrt{5}} + 0 = \boxed{-2\sqrt{5}}$$

and so $\boxed{\text{the function values decrease}}$.

(b) Let $x(s,t) = s^2 t$ and $y(s,t) = 2s - t$. Use the appropriate chain rule to find $\dfrac{\partial f}{\partial t}$ (no direct substitution). Your final answer should only contain s and t but DO NOT simplify.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{1}{y^2} + 2xy\right)(s^2) + \left(-\frac{2x}{y^3} + x^2\right)(-1) \\
&= \boxed{\frac{s^2}{(2s-t)^2} + 2s^4 t(2s-t) + \frac{2s^2 t}{(2s-t)^3} - s^4 t^2}
\end{aligned}$$

Some formulas from theorems in the course:

$$f(B) - f(A) = \int_{AB} \nabla f \cdot d\mathbf{r}$$

$$\oint_{C=\partial R} P\, dx + Q\, dy = \iint_R Q_x - P_y \, dA$$

$$\oint_{C=\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

$$\oiint_{S=\partial V} \mathbf{F}\cdot d\mathbf{S} = \iiint_V \operatorname{div}\mathbf{F}\, dV$$