Comments on HW 7

Here are some brief comments on HW 7.

p. 38, Ex 35: Construct a polynomial in $\mathbb{Z}[x]$ with a root mod 2, but no root in $\mathbb{Q}_p$.

Solution: The book suggests $f(x) = x^2 - 3$ and many other examples are possible. We use the one suggested by the book.

It is easy to see that if $x = 1$, then

$$f(1) \equiv 1^2 - 3 \equiv -2 \equiv 0 \pmod 2.$$

Now suppose that $x \in \mathbb{Q}_p$ and that x were a root of $f(x)$. Computing 2-adic norms, we see that $|x|_2^2 = 1$ from which it follows that $|x|_2 = 1$. Thus, x is a 2-adic integer, $x \in \mathbb{Z}_2$. However, we can check that $f(x)$ has no root $\pmod 4$ and by Theorem 1.42, it follows that x has no root in $\mathbb{Z}_p$.

p. 38, Ex 38: Use Hensel's Lemma to justify the existence of a root in $\mathbb{Z}_7$. Then compute that the first three digits in the 7-adic expansion are $\ldots 642_\wedge$ (or -1 times that).