MATH 253
April 2, 2020

Name: Answer Key

Instructions. (100 points) You have 120 minutes to scan, complete, and upload this exam. In other words, you have up to a maximum of two hours for this exam. Closed book, closed notes, no internet, no calculators, and no help allowed. No cheating of any kind. **Show all your work** in order to receive credit. Incomplete answers with little work shown will be graded harshly.

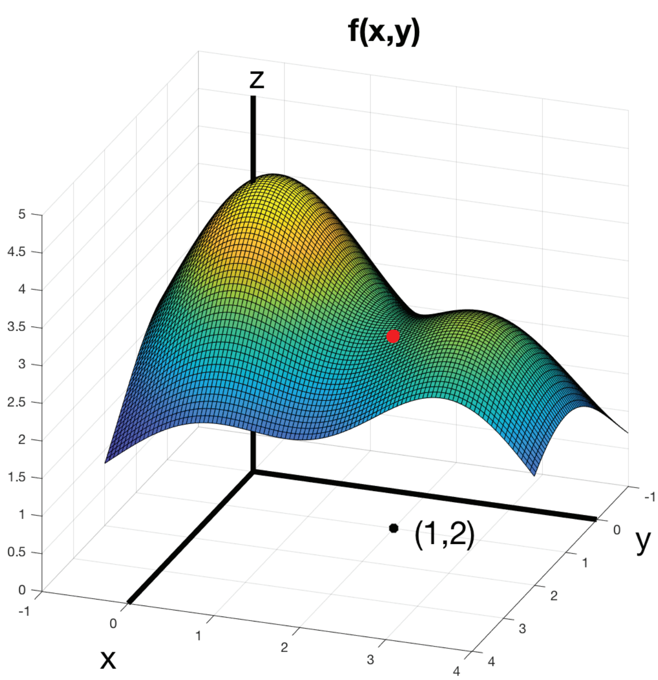
(6^{pts}) **1.** Find the directional derivative $D_{\vec{u}}(1,0)$ of $h(x,y) = x\sin(xy)$ in the direction of $\vec{v} = \langle 3,3\rangle$.

Solution:

- The unit vector $\vec{u}$ in the direction of $\vec{v}$ is $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$.
- The gradient vector is $\big\langle xy\cos(xy) + \sin(xy),\, x^2\cos(xy)\big\rangle$ and evaluating this at $(1,0)$ using that $xy = 0$, we find $\nabla f(1,0) = \big\langle 0 + \sin(0),\, 1^2\cos(0)\big\rangle = \langle 0,\, 1\rangle$.

Since $D_{\vec{u}}(1,0) = \nabla f(1,0)\cdot\vec{u} = \langle 0,\, 1\rangle\cdot\vec{u} = \langle 0,\, 1\rangle \cdot \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right\rangle = \boxed{\dfrac{\sqrt{2}}{2}}$.

(8^{pts}) **2.** The graph of $f(x,y)$ is shown in the figure below with the red point denoting $(1,2,f(1,2))$.



(a) (4 pts) Is $\dfrac{\partial f}{\partial x}(1,2)$ negative, zero, or positive? Explain carefully.

Solution: $\boxed{\dfrac{\partial f}{\partial x}(1,2) < 0}$ since in the positive x-direction, the curve of intersection with the plane $y = 2$ is decreasing at $x = 1$.

(b) (4 pts) Is $\dfrac{\partial f}{\partial y}(1,2)$ negative, zero, or positive? Explain carefully.

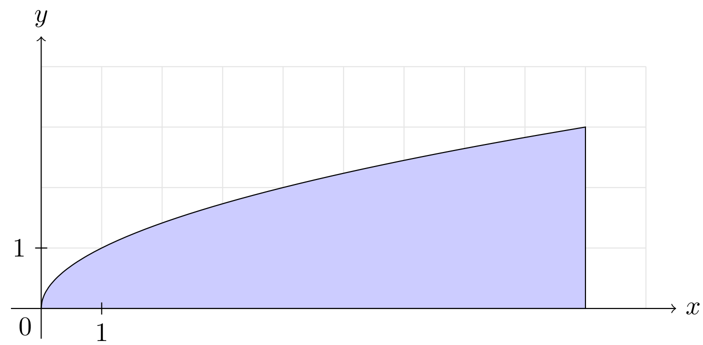
Solution: $\boxed{\dfrac{\partial f}{\partial y}(1,2) > 0}$ since in the positive y-direction, the curve of intersection with the plane $x = 1$ is increasing at $y = 2$.

(12pts) **3.** Compute the integral

$$I = \int_0^3 \int_{y^2}^9 \frac{1}{x\sqrt{x}+1}\, dx\, dy$$

by drawing the region of integration and then reversing the order of integration.

Solution: The bounds indicate that we have $y^2 \leq x \leq 9$ and $0 \leq y \leq 3$. The inner bounds being in x, that means that if we drill horizontally left to right, we enter our region on the curve $x = y^2$, i.e. $y = \sqrt{x}$ (because $y \geq 0$ here), and exit it on the line $x = 9$. Furthermore, the shadow of the region onto the y-axis covers $[0, 3]$:



So reversing the order of integration, we have:

$$\int_0^3 \int_{y^2}^9 \frac{1}{x\sqrt{x}+1}\, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} \frac{1}{x\sqrt{x}+1}\, dy\, dx = \int_0^9 \Big[y\Big]_{y=0}^{y=\sqrt{x}} \frac{1}{x\sqrt{x}+1}\, dx = \int_0^9 \frac{\sqrt{x}}{x\sqrt{x}+1}\, dx$$

$$= \left| \begin{array}{l} u = x\sqrt{x}+1 \\ du = \dfrac{3\sqrt{x}}{2}\, dx \end{array} \right| = \int_{x=0}^{x=9} \frac{2}{3u}\, du = \left[\frac{2}{3}\ln|u|\right]_{x=0}^{x=9}$$

$$= \left[\frac{2}{3}\ln\left|x\sqrt{x}+1\right|\right]_0^9 = \boxed{\frac{2}{3}\ln 28}$$

(12pts) **4.** Consider the function $f(x, y) = x^2y + y^2 - 4xy + 3y$.

(a) (5 pts) Show that the point $(2, 1/2)$ is a critical point for $f(x, y)$.

Solution: The gradient is

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle 2xy - 4y, x^2 + 2y - 4x + 3 \right\rangle$$

and we verify that $f_x(2, 1/2) = 0$ and $f_y(2, 1/2) = 0$. All at once:

$$\nabla f(2, 1/2) = \left\langle 2(2)(1/2) - 4(1/2), 2^2 + 2(1/2) - 4(2) + 3 \right\rangle = \langle 2 - 2, 4 + 1 - 8 + 3 \rangle = \langle 0, 0 \rangle$$

so $(2, 1/2)$ is a critical point of f.

(b) (7 pts) Use the second derivative test to classify $(2, 1/2)$ as a local minimum, local maximum or saddle point of $f(x, y)$.

Solution: We have:

$$f_{xx} = 2y \quad , \quad f_{yy} = 2 \quad , \quad f_{xy} = 2x - 4 \quad \Rightarrow \quad d(x, y) = 4y - 4(x - 2)^2.$$

Since $d(2, 1/2) = 2 > 0$ and $f_{yy} = 2 > 0$ then $\boxed{(2, 1/2) \text{ is a relative minimum}}$.

(8[pts]) **5.** Find an equation of the tangent plane to the surface

$$x^2 \sin z + yz - \ln y - 2x = 4$$

at the point $(-2, 1, 0)$.

Solution: Let $F(x, y, z) = x^2 \sin z + yz - \ln y - 2x$. Then we find

$$\nabla F(x, y, z) = \left\langle 2x \sin z - 2, z - \frac{1}{y}, x^2 \cos z + y \right\rangle$$

$$\Rightarrow \quad \nabla F(-2, 1, 0) = \left\langle 2(-2) \sin 0 - 2, 0 - \frac{1}{1}, (-2)^2 \cos 0 + 1 \right\rangle = \langle -2, -1, 5 \rangle$$

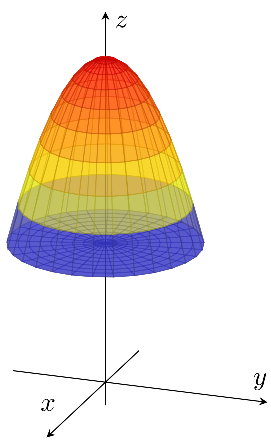
The tangent plane is thus given by using $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ to get

$$\boxed{-2x - y + 5z = 3}.$$

(16[pts]) **6.** Set up, but **DO NOT INTEGRATE**, double integrals for the computations below. A complete answer has limits of integration and the integrand is simplified completely.

(a) (8 pts) Compute the volume of the solid that lies below the paraboloid $z = 7 - x^2 - y^2$ and above the plane $z = 3$. **Use polar coordinates and DO NOT EVALUATE**.

Solution:



The shadow of the solid is a disk and its boundary circle corresponds to:

$$7 - x^2 - y^2 = 3 \quad \Longleftrightarrow \quad x^2 + y^2 = 4.$$

So the region of integration R is described by $0 \le \theta \le 2\pi$ and $0 \le r \le 2$ and since the paraboloid is above the plane, we have that the volume is:

$$V = \iint_R 7 - x^2 - y^2 - 3 \, dA = \iint_R 4 - (x^2 + y^2) \, dA$$

$$\Rightarrow \quad \boxed{V = \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) \, r \, dr \, d\theta}$$

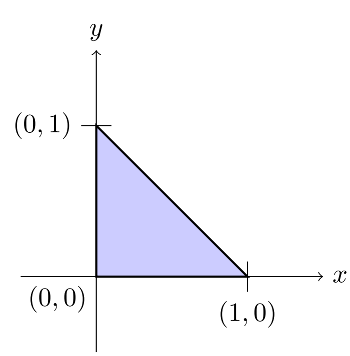
(b) (8 pts) Compute the surface area of the part of the plane $2x + y + z = 4$ that lies above the triangular region in the xy-plane bounded by vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. **Use rectangular coordinates, and DO NOT EVALUATE**.

Solution:

The boundary curves of the region of integration R are $x = 0$, $y = 0$ and $x + y = 1$. So the region R can be written as: $0 \le x \le 1$, $0 \le y \le 1 - x$. Then if we rewrite the plane as $z = 4 - 2x - y$, we have $z_x = -2$ and $z_y = -1$. Therefore the surface area is given by:

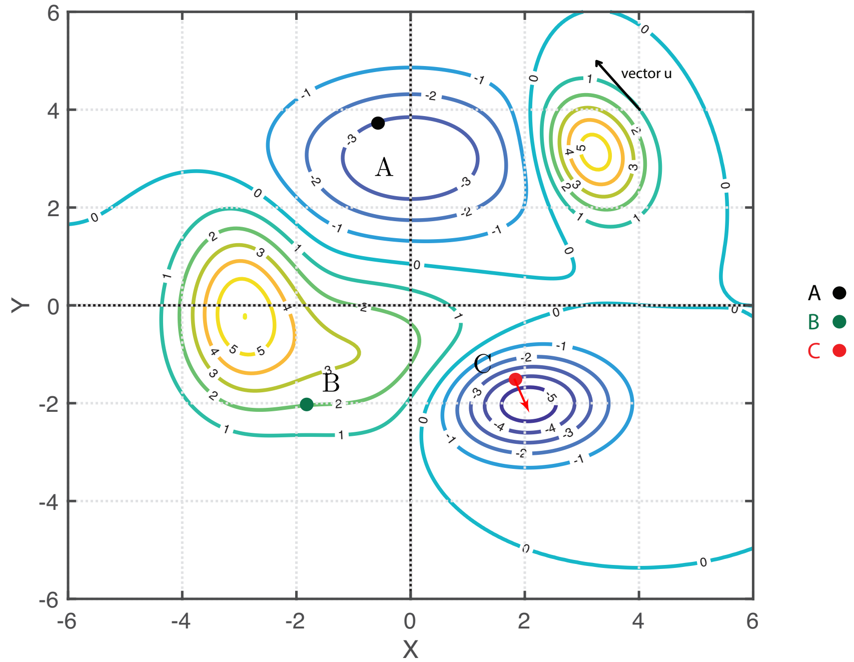
$$SA = \iint_R \sqrt{1 + z_x^2 + z_y^2} \, dA = \iint_R \sqrt{1 + (-2)^2 + (-1)^2} \, dA$$

$$\Rightarrow \quad \boxed{SA = \int_0^1 \int_0^{1-x} \sqrt{6} \, dy \, dx}$$



(14pts) **7.** Consider the contour plot of a function $f(x, y)$ below where $f(x, y)$ gives the temperature in degrees Celsius. Points A, B and C are shown in the figure, and a vector $\mathbf{u}$ too.

Solution:



A is black
B is green
C is red

$\mathbf{u}$ is the vector shown

(a) (4 pts) The magnitude of the gradient vector is largest at which of the three points (A, B, or C)? Why?

Solution: The magnitude of the gradient vector is $\boxed{\text{largest at } C}$. This is because the function $f(x, y)$ is increasing the fastest at C as indicated by the tightness of the contour lines there. (Bonus: The direction of maximal increase is roughly NNW from C.)

(b) (4 pts) A cold-seeking particle is located at C (red dot). Which direction (roughly) should it move to decrease its temperature the most. Draw an arrow on the contour plot to indicate this, or if you do not have a printer, simply make a cartoon drawing that shows where your arrow would be. Explain your answer briefly.

Solution: $\boxed{\text{The direction of maximal } \textbf{decrease} \text{ is in the direction of } -\nabla f(C).}$ Your arrow should point in the direction of the minimum near C (about $(2, -2)$) and, most importantly, your arrow should be orthogonal to the level curve on which C lies.

(c) (3 pts) Consider the point $(3, 3)$.
Is the value $f_{xx}(3, 3)$ negative, positive, or zero? (Circle one.) Why?

Solution: The point $(3, 3)$ is a local maximum, so $\boxed{f_{xx}(3, 3) < 0}$ indicating the $f(x, y)$ is concave down in the x direction as measured from $(3, 3)$.

(d) (3 pts) What is the value of the directional derivative $D_{\vec{u}}(4, 4)$ where $\vec{u}$ is the vector shown in the figure?

Solution: $\boxed{\text{Zero}}$. The vector $\mathbf{u}$ is tangent to a level curve of $f(x, y)$ so the rate of change in that direction is 0.

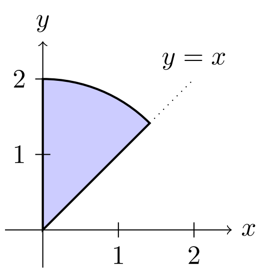
(6pts) **8.** Show that $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{xy+2}{x^2-y-5}$ does not exist.

Solution: We will use two different paths:

- along $x=2$, then $\displaystyle\lim_{(2,y)\to(2,-1)} \frac{xy+2}{x^2-y-5} = \lim_{y\to-1} \frac{2y+2}{4-y-5} = \lim_{y\to-1} \frac{2(y+1)}{-(y+1)} = -2$
- along $y=-1$ then $\displaystyle\lim_{(x,-1)\to(2,-1)} \frac{xy+2}{x^2-y-5} = \lim_{x\to 2} \frac{-x+2}{x^2+1-5} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)(x+2)} = -\frac{1}{4}$

Since these limits are different ($-2 \neq \frac{1}{4}$), $\boxed{\text{the limit does not exist.}}$

(8pts) **9.** Compute the total charge on the lamina pictured below, if the charge density is given by $\sigma(x,y) = 3y$ coulombs/ in^2. Include units in your final answer.



Use polar coordinates because of shape of lamina.

Solution:

$$Q = \iint_R \sigma(x,y)\, dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 (3r\sin\theta)\, r\, dr\, d\theta = \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta\, d\theta\right)\left(\int_0^2 3r^2\, dr\right)$$

$$= \Big[-\cos\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \Big[r^3\Big]_0^2 = \left[0+\frac{\sqrt{2}}{2}\right][8-0] = \boxed{4\sqrt{2}\text{ coulombs}}$$

(10^{pts}) **10.** Use the method of Lagrange multipliers to find the absolute maximum and absolute minimum of the function $f(x,y) = y^2 - x^2$ subject to the constraint $g(x,y) = 4x^2 + y^2 - 36 = 0$.

Solution: Maximize the objective function $f(x,y) = y^2 - x^2$ subject to the constraint is $g(x,y) = 4x^2 + y^2 - 36 = 0$. Therefore,

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad \langle -2x, 2y \rangle = \lambda \langle 8x, 2y \rangle \quad \Longrightarrow \quad \begin{cases} -2x = 8\lambda x \\ 2y = 2\lambda y \end{cases}$$

From the second equation

$$2y - 2\lambda y = 0 \quad \Longrightarrow \quad y(1-\lambda) = 0,$$

there are two solutions:

- either $y = 0$ then from the constraint $4x^2 = 36$ so $x = \pm 3$;
- or $\lambda = 1$ which from the first equation gives us $-2x = 8x$ so $x = 0$; in turns once you plug that into the constraint, you get $y^2 = 36$ so $y = \pm 6$.

Now plugging in these points into f, we get:

x	y	$f(x,y)$	
± 3	0	-9	absolute minimum
0	± 6	36	absolute maximum