Numerical Analysis Name : SOLUTIONS
Exam 1 October 15, 2015

Instructions: Show all work for full credit. You may use a calculator for simple 'adding machine'-like computations. There are 110 points on this exam, and you get your grade out of 100.

1. (15 pts.) Consider the function $f(x) = \ln(1+x)$.

   (a) Find the Taylor polynomial $p_4(x)$, a polynomial of degree $n = 4$, that approximates $f(x)$ near the point $x_0 = 0$. Show all work for full credit.

$f(x) = \ln(1+x)$ $\quad f(0) = \ln(1) = 0$

$f'(x) = (1+x)^{-1}$ $\quad f'(0) = 1$

$f^{(2)}(x) = -1(1+x)^{-2}$ $\quad f^{(2)}(0) = -1$

$f^{(3)}(x) = 2(1+x)^{-3}$ $\quad f^{(3)}(0) = 2$

$f^{(4)}(x) = -3!(1+x)^{-4}$ $\quad f^{(4)}(0) = -3!$

$$p_4(x) = f(0) + f'(0)x + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + f^{4}(0)\frac{x^4}{4!}$$

$$= 0 + x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 3!\frac{x^4}{4!}$$

$$\boxed{p_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}$$

   (b) Consider now the nth degree Taylor polynomial $p_n(x)$ approximating $f(x)$ at $x_0 = 0$. Give the value of the error term $R_n(x) = f(x) - p_n(x)$.

$$R_n(x) = f^{(n+1)}(\xi)\frac{x^{n+1}}{(n+1)!} \qquad \therefore\ R_n(x) = \frac{(-1)^{n+1}\, n!}{(1+\xi)^{n+1}}\frac{x^{n+1}}{(n+1)!} = \boxed{\frac{(-1)^{n+1}}{(n+1)}\frac{x^{n+1}}{(1+\xi)^{n+1}}} \text{ for } \xi \in [0,x]$$

$$f^{(n+1)}(x) = (-1)^{n+1}\, n!\,(1+x)^{-(n+1)}$$

   (c) Determine the *smallest* number of terms n such that the absolute error $|R_n(x)| < 10^{-3}$ for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Justify your work by filling in the table below and showing your work.

For $|R_n(x)| < 10^{-3}$, consider $|R_n(x)| = \frac{1}{(n+1)}\frac{|x|^{n+1}}{(1+\xi)^{n+1}}$ ← make big, ← make small, using $x \in [-\frac{1}{2}, \frac{1}{2}]$

$$\frac{1}{(n+1)}\frac{|x|^{n+1}}{(1+\xi)^{n+1}} \le \frac{1}{(n+1)}\frac{(\frac{1}{2})^{n+1}}{(1+-\frac{1}{2})^{n+1}} = \frac{1}{n+1}\frac{(\frac{1}{2})^{n+1}}{(\frac{1}{2})^{n+1}}$$

$= \frac{1}{n+1}$ Thus, make $\frac{1}{n+1} < 10^{-3}$

If $n = 999$, then $1/_{n+1} = 10^{-3}$

If $n = 1000$, then $1/_{n+1} \approx .00099 < 10^{-3}$

k	$R_k(x)$
$n-1 =$ 999	$\lvert R_{n-1}(x)\rvert \le$ 10^{-3}
$n =$ 1000	$\lvert R_n(x)\rvert \le$ .00099...

$\boxed{n = 1000}$

2. (6 pts.) Give the 200th Taylor polynomial $p_{200}(x)$ approximating $g(x) = 2001x^{79} - 1001x^{29} + x - 3$ near $x_0 = 0$ and the error $R_{200}(x)$. Briefly justify your answer.

$p_{200}(x) = g(x)$ $R_{200}(x) = 0$

Justification:

If $n \geq 79$, the Taylor polynomial $p_n(x) = g(x)$ exactly.

3. (15 pts.) Pretend that a computer can only represent the floating point numbers 0, $\pm\infty$, and those which in base 2 have the form

$$\pm 1.a_1 a_2 a_3 \times 2^m,$$

where a_1, $a_2, 1_2, a_3$ are binary digits (i.e., 0 or 1) and m is an integer with $-5 \le m \le 5$. If x is a real number, let $fl(x)$ denote its floating point value and assume that this machine uses *truncation* for finding floating point equivalents. For parts (b)-(e), you must briefly justify your answer for full credit.

(a) Give the **floating point** representation and the **decimal** value of the smallest and largest *positive* floating point numbers on this computer. (Perform scratch work elsewhere.)

My answers:

smallest positive number:

in floating point 1.000×2^{-5} decimal equivalent: $1/32 = .03125$

largest positive number:

in floating point 1.111×2^{5} decimal equivalent: 60

(b) Give an example of a real number x such that $fl(x) \neq x$.

$x = 1/3$ $\quad (1/3)_2 = .01010\overline{1}_2$ so $fl(1/3) = fl(.01010_2) = 1.010 \times 2^{-2}$

$= \frac{1}{4} + \frac{1}{16} = \frac{5}{16} = .3125$

(c) Give an example of two real numbers such that the sum $fl(x+y)$ gives an overflow error.

$x = y = 1.000 \times 2^5 = 32$

Then $x + y = 64 > 60$ (see above)

$64 = 1.000 \times 2^6$ but $6 > m = 5$

(d) Give an example of two non-zero real numbers such that the sum $fl(x+y) = x$.

Take $x = 1 = 1.000 \times 2^0$ then $x+y = 1.00001_2$

$y = 1/32 = 1.000 \times 2^{-5}$, and $fl(x+y) = 1.$

(e) i. Give the definition of machine epsilon ϵ_M. (Give the general definition for any machine.)

ϵ_M is the largest floating point number s.t. $fl(1+\epsilon_M) = 1$

ii. Give the value of machine epsilon ϵ_M for this computer. Justify your answer.

$\epsilon_M = 1.111 \times 2^{-4}$

Why? $1+\epsilon_M = 1.000111_2$ and with truncation, $fl(1+\epsilon_M) = fl(1.000) = 1$

However, $y = 1.000 \times 2^{-3}$, the next largest machine number, satisfies

$fl(1+y) = fl(1.001_2) = 1.001 \times 2^0 \neq 1.$

4. (6 pts.) Use Taylor's Theorem to give an expression for $f(x+h)$ at the point x and then show that the right difference approximation to $f'(x) = \dfrac{f(x+h) - f(x)}{h}$ is $O(h)$.

$f(x+h) = f(x) + f'(x)(x+h-x) + \dfrac{f^{(2)}(\xi)(x+h-x)^2}{2!}$ for some $\xi \in [x, x+h]$

$= f(x) + f'(x)h + f^2(\xi)\dfrac{h^2}{2}$

$\Rightarrow f(x+h) - f(x) = f'(x)h + f^2(\xi)\dfrac{h^2}{2}$

$\Rightarrow \dfrac{f(x+h) - f(x)}{h} = f'(x) + \dfrac{f^2(\xi)}{2}h$ since $\left|\dfrac{f^2(\xi)}{2}\right|$ can be bounded

this says, by definition, $f'(x) = \dfrac{f(x+h) - f(x)}{h} + O(h)$

5. (5 pts.) Consider the system of three linear equations in three unknowns. Set up the augmented matrix and perform **one** step of Gaussian elimination on this system.

$$\begin{aligned} 3x &- 5y &+ 2z &= 1 \\ 6x &+ y & &= 1 \\ 2x &+ 4y &+ 2z &= 3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 3 & -5 & 2 & 1 \\ 6 & 1 & 0 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2}$$

$$\boxed{\left(\begin{array}{ccc|c} 3 & -5 & 2 & 1 \\ 0 & 11 & -4 & -1 \\ 2 & 4 & 2 & 3 \end{array}\right)}$$

Only R_2 should change

6. (5 pts.) Suppose the result of performing Gaussian elimination on a linear system of equations gives the following augmented matrix. Use backward substitution to solve for x, y, and z.

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & 5 \end{array}\right)$$

$\boxed{x = \frac{-3}{2},\ y = 2,\ z = 1}$

$5z = 5 \Rightarrow z = 1$

$y - z = 1 \Rightarrow y - 1 = 1 \Rightarrow y = 2$

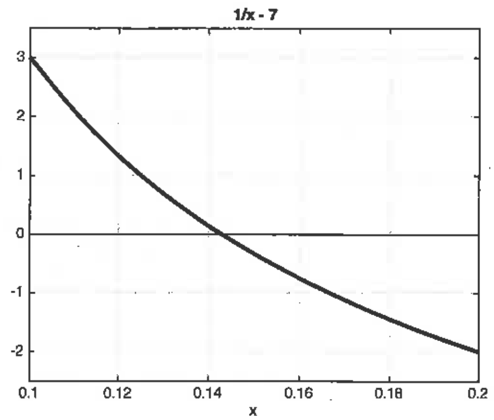
$2x - y + 3z = -2 \Rightarrow 2x - 2 + 3(1) = -2$

$2x + 1 = -2$

$2x = -3$

$x = -3/2$

7. (12 pts.) It is possible to check that $\frac{1}{7} \approx .14286$. Consider the graph of $f(x) = \frac{1}{x} - 7$ for $.1 \leq x \leq .2$ below. In this problem you should round your answer to five significant digits.



(a) Perform three iterations of the bisection method to approximate $\frac{1}{7}$. Give your answer in the table. (Make sure at each step you list the left endpoint a, the right endpoint b, and the approximation to the root x_n.)

n	a	b	estimate $x_n = c$
1	.10000	.20000	.15000
2	.10000	.15000	.12500
3	.12500	.15000	.13750

(b) Find the value of the absolute error and the relative error for the estimate x_3 from your table above. As part of your answers, give the formulas you use to compute these quantities.

Absolute Error $|\sqrt{7} - .13750| \approx .005371$

Relative Error $\dfrac{\sqrt{7} - .13750}{\sqrt{7}} = .0345$

Answers might vary somewhat depending on when you rounded.

(c) Give a formula involving a, b, and n for an upper bound on the absolute value of the error $|e_n|$ after the nth iteration. Then use this formula to determine how many iterations N you should perform if you want the estimate to satisfy $|e_N| \leq 10^{-8}$.

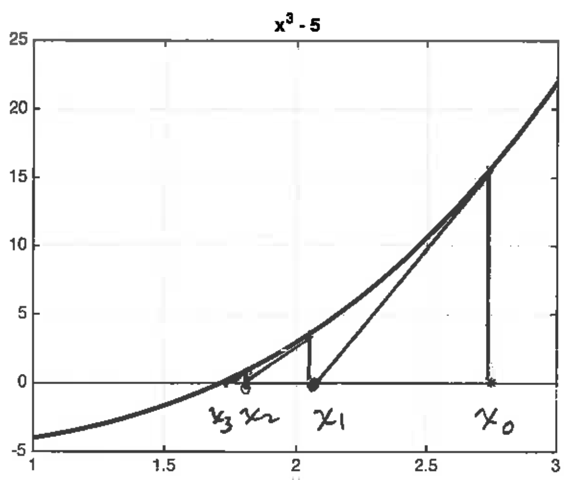
$|e_n| \leq (\frac{1}{2})^n (b-a) = (.5)^n (.2-.1) = .1\,(.5)^n$

If $N = 24$, $e_{24} \approx 5.96 \times 10^{-9}$, but if $N=23$, $e_{23} \approx 1.19 \times 10^{-8}$

Thus, $\boxed{N = 24}$

8. (7 pts.) Below is the graph of $g(x) = x^3 - 5$. This graph and Newton's method can be used to find an approximation to $\sqrt[3]{5}$.

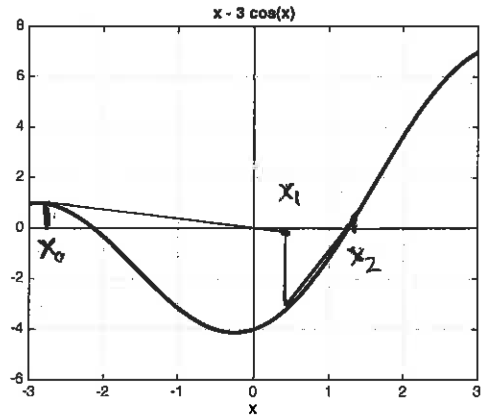
(a) With an initial value of $x_0 = 2.75$, sketch three iterations of Newton's method. (Your picture should highlight the geometric interpretation of Newton's method. Clearly mark x_1, x_2, and x_3 on the graph.)



(b) Will Newton's method converge?

Yes, and FAST!

9. (15 pts.) In this problem you will perform Newton's Method to estimate a solution to the equation $x = 3\cos(x)$. For your convenience a graph of $f(x) = x - 3\cos(x)$ is shown.



(a) Give the formula for computing the $(n+1)$st estimate x_{n+1} from x_n:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n - 3\cos(x_n))}{(1+3\sin(x_n))}$$

4pts.

4pts. (b) Perform two iterations of Newton's Method to estimate a solution to the equation $x = 3\cos(x)$. Use $x_0 = 1$ for your initial guess. (Show all work in computing x_1 and x_2 and round your answers to four decimal places.)

$$x_1 = 1 - \frac{(1 - 3\cos(1))}{1 + 3\sin(1)} \approx 1.1762$$

$$x_2 = 1.1762 - \frac{(1.1762 - 3\cos(1.1762))}{1 + 3\sin(1.1762)} \approx 1.1701$$

Answer:

n	x_n
0	$x_0 = 1.0000$
1	$x_1 = 1.1762$
2	$x_2 = 1.1701$

4pts (c) Let α be the root of the equation $x = 3\cos(x)$ that is near 1, and let e_k denote the error in the kth step $\alpha - x_k$ as usual. Note that $e_0 = \alpha - 1$.

It can be shown that the error in Newton's method satisfies $e_{n+1} = -\frac{1}{2}e_n^2 \frac{f''(\xi_n)}{f'(x_n)}$ for some ξ_n between α and x_n. Use this formula to give an explicit formula for e_1 in terms of e_0. (I.e. compute the derivatives and put them in the right places.)

Then give an upper bound for $|e_1|$ using this formula. (Your answer should be expressed as $|e_1| \leq Ce_0^2$ for some C. Estimate this C.)

$$\boxed{\begin{aligned} f'(x) &= 1+3\sin x \\ f''(x) &= +3\cos x \end{aligned}}$$

$$e_1 = \frac{-1}{2} e_0^2 \frac{f^{(2)}(\xi_1)}{f'(1)} = \frac{-1}{2} e_0^2 \frac{3\cos(\xi_1)}{(1+3\sin(1))} \qquad \xi_1 \in [1, x_1]$$

$$|e_1| \leq \left| \frac{1}{2} e_0^2 \frac{3\cos(\xi_1)}{1+3\sin(1)} \right| \leq \frac{3}{2} \frac{1}{|1+3\sin(1)|} e_0^2 \approx \boxed{.4256\, e_0^2}$$

(d) There is a second root β to the equation $x = 3\cos(x)$ that is roughly $\beta \approx -2.1$. Is it possible to give a starting value x_0 in Newton's method that is close to β, yet the algorithm converges to α? If so, show this graphically in the plot above. If not, explain why this is impossible.

3pts. ↳ Yes. See graph.

10. (8 pts.) In analyzing an (unspecified) algorithm, you discover that the error terms are related by $e_{k+1} = .79e_k$.

Prove or disprove: This algorithm converges.

If the algorithm converges, give the order of convergence for the estimates to the true value.

Answer: (Circle one.) The algorithm DOES / DOES NOT converge.

If $e_{k+1} = .79\, e_k$, then $\lim_{k \to \infty} \frac{e_{k+1}}{e_k} = .79 < 1$ (and non-zero)

Thus, $e_k = (.79)^k e_0$ and the algorithm converges LINEARLY.

So $e_k \to 0$

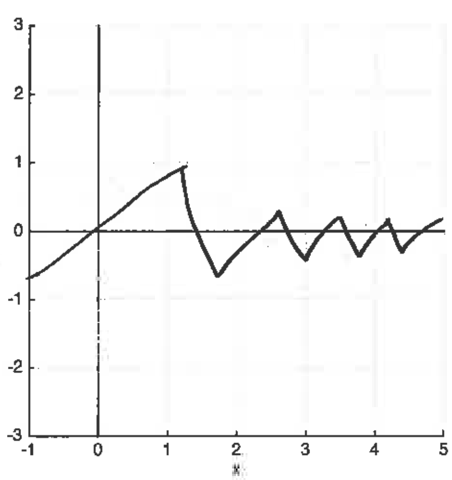
11. (8 pts.) Suppose $f(x)$ is a differentiable function, and that one of two methods discussed in class (right and/or central difference approximations) was used to approximate $f'(a)$ at some value $x = a$. For this algorithm, the values of h were halved with each successive iteration, and the values of the error $e_n = f'(a) - x_n$ are displayed in the table below. Determine whether the right difference approximation or the central difference approximation was used to estimate $f'(a)$. Justify your answer.

n	e_n
1	0.06445292
2	0.01552106
3	0.00384407
4	0.00095877
5	0.00023955
6	0.00005988
7	0.00001497
8	0.00000374
9	0.00000094
10	0.00000023

$e_{n+1}/e_n \approx \frac{1}{4}$ and $\frac{h/2}{h} = \frac{1}{2}$ suggesting the error is $\Theta(h^2)$.

Thus, this must be the CENTRAL DIFFERENCE APPROXIMATION.

12. (8 pts.) Sketch a function $f(x)$ on the axes below for which the bisection method for finding a root of $f(x)$ is **better** than Newton's method. Briefly explain your answer.



Not everywhere differentiable

OR

Really flat near root

OR

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