MATH 202
Midterm 1

Name : SOLUTIONS
February 25, 2008

Instructions: Show all work for full credit.

1. (30 pts. — 6 pts. each) Three points, with coordinates

$$A = (1, 1, 0), \quad B = (0, 2, 1), \quad C = (2, 3, 0),$$

are the vertices of a triangle in 3-dimensional space.

(a) What is the length of the side joining A and B?

$\vec{AB} = (-1, 1, 1)$

$|\vec{AB}| = \sqrt{(-1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

(b) Give a *unit* normal vector to the plane containing the triangle.

$\vec{AB} = (-1,1,1)$

$\vec{AC} = (1,2,0)$

A normal vector is $\vec{AB} \times \vec{AC}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2\hat{i} + \hat{j} - 3\hat{k}$$

Thus, a unit normal is

$$\vec{n} = \frac{(-2, 1, -3)}{|(-2, 1, -3)|} = \boxed{\left(\frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)}$$

(c) Give an equation of the plane containing the triangle.

$\vec{n} = (2, -1, 3)$ $\quad P_0 = A$ $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{a}$

$$2x - y + 3z = (2)(1) + (-1)(1) + (3)(0)$$

$$\boxed{2x - y + 3z = 1}$$

(d) What is the angle formed by the sides meeting at A? (You may leave your answer in a form involving inverse trigonometric functions, and you do not need to rationalize denominators.)

$\vec{AB} \cdot \vec{AC} = |\vec{AB}|\,|\vec{AC}| \cos\theta$

$(-1,1,1) \cdot (1,2,0) = \sqrt{3} \cdot \sqrt{5} \cos\theta$

$1 = \sqrt{15} \cos\theta$

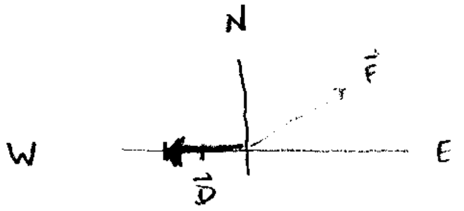
$\cos\theta = \frac{1}{\sqrt{15}}$

$\theta = \arccos\left(\frac{1}{\sqrt{15}}\right)$

(e) What is the area of the triangle?

$$\frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\left|(2, -1, 3)\right| = \frac{1}{2}\sqrt{4+1+9} = \boxed{\frac{\sqrt{14}}{2}} \text{ units}^2$$

2. (10 pts.) In the plane, a constant force $\mathbf{F} = 2\mathbf{i} - \mathbf{j}$ N acts on a particle that is moved due west a total of 2 m. Find the work done.



$\vec{F} = (2,-1) \qquad \vec{D} = (-2,0)$

$W = \vec{F} \cdot \vec{D} = -4$ Newton meters

↑ yikes

3. (10 pts.) In the plane, a particle moves so that it has constant acceleration $\mathbf{a}(t) = 2\mathbf{j}\ m/s^2$.
At $t = 0$, it has velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}\ m/s$.
At time $t = 1$, its position is $\mathbf{r}(1) = 2\mathbf{j}\ m$.
Give a formula for its position, $\mathbf{r}(t)$, at all times t.

$\vec{a}(t) = (0,2)\ m/s^2 \qquad \vec{v}(0) = (1,-1)\ m/s \qquad \vec{r}(1) = (0,2)\ m$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \left(\int 0\,dt\right)\hat{i} + \left(\int 2\,dt\right)\hat{j}$$

$$= c_1\hat{i} + (2t + c_2)\hat{j}$$

$$\vec{v}(0) = (1,-1) \Rightarrow (1,-1) = c_1\hat{i} + (2(0) + c_2)\hat{j}$$

$$\Rightarrow c_1 = 1,\ c_2 = -1$$

$$\vec{v}(t) = \hat{i} + (2t-1)\hat{j} = (1, 2t-1)$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\int dt\right)\hat{i} + \left(\int 2t-1\,dt\right)\hat{j}$$

$$= (t + d_1)\hat{i} + (t^2 - t + d_2)\hat{j}$$

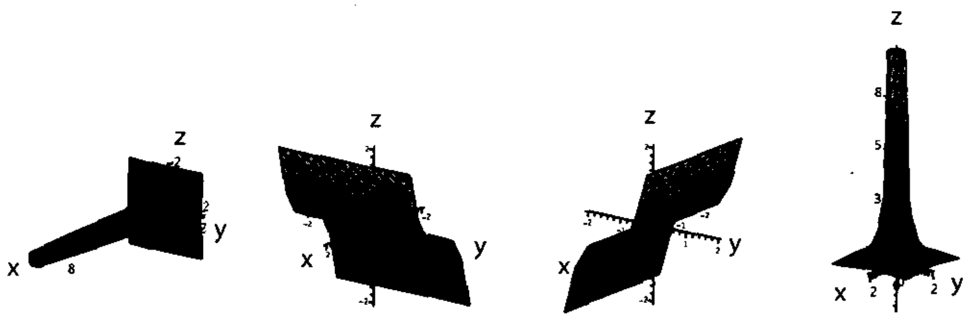
$$\vec{r}(1) = (0,2) \Rightarrow (0,2) = (1 + d_1)\hat{i} + (1^2 - 1 + d_2)\hat{j}$$

$$\text{or } 0 = 1 + d_1 \Rightarrow d_1 = -1$$

$$2 = d_2$$

$$\therefore \boxed{\vec{r}(t) = (t-1,\ t^2 - t + 2)\ m}$$

4. (10 pts. — 5 pts. each: 2 for answer, 3 for explanation) Match the equations with the appropriate graph. (Notice that there are more graphs than equations.) Explain your answer.



A. B. C. D.

(a) $z = y^3$ $\boxed{C.}$

Cubic cylinder along the x-axis

cross-sections parallel

to yz-plane are $z = y^3$

(b) $f(x, y) = \dfrac{1}{x^2 + y^2}$ $\boxed{D}$

radial symmetry $f(x,y) = \frac{1}{r^2}$

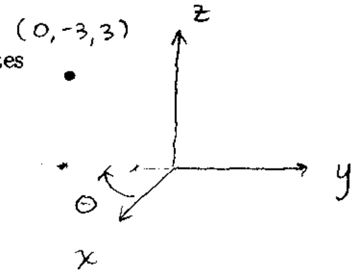
blows up at the origin

as $(x,y) \to (0,0)$, $f(x,y) \to \infty$

5. (10 pts. — 5 pts. each) Consider a point **p** with rectangular coordinates $(0, -3, 3)$.

Express **p** in:

(a) cylindrical coordinates



$x = 0 \quad y = -3 \quad z = 3$

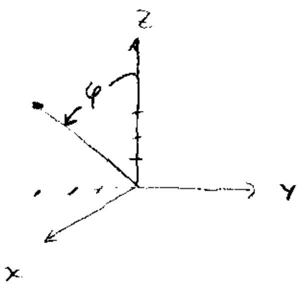
$r = \sqrt{x^2+y^2} = 3$

$\theta = -\frac{\pi}{2}$

$z = 3$

$(r, \theta, z) = \left(3, -\frac{\pi}{2}, 3\right)$

(b) spherical coordinates



$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0^2 + (-3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2}$

$\theta = -\frac{\pi}{2}$

$\varphi = \frac{\pi}{4}$ from inspection

$(\rho, \theta, \varphi) = \left(3\sqrt{2}, -\frac{\pi}{2}, \frac{\pi}{4}\right)$

6. (20 pts.) An object moves along a trajectory so that its position, as a function of time, is given by

$$\mathbf{r}(t) = (t^2,\, 2t,\, \ln(t)).$$

(a) (6 pts.) At what speed is it traveling at time $t = 2$? Compute $|\vec{v}(2)|$

$\vec{v}(t) = \vec{r}'(t) = (2t, 2, \frac{1}{t}) \Rightarrow \vec{v}(2) = (4, 2, \frac{1}{2}) \Rightarrow$ speed is $\sqrt{4^2 + 2^2 + (\frac{1}{2})^2} = \sqrt{\frac{81}{4}} = \boxed{\frac{9}{2}}$

(b) (8 pts.) What is the length of its trajectory between times $t = 1$ and $t = 2$?

$L = \int_1^2 |\vec{r}'(t)|\,dt = \int_1^2 (2t + \frac{1}{t})\,dt$

$|\vec{r}'(t)| = \sqrt{(2t)^2 + 2^2 + (\frac{1}{t})^2} = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{(2t + \frac{1}{t})^2}$

$= 2t + \frac{1}{t}$ since $1 \le t \le 2$

$= t^2 + \ln|t| \Big|_1^2$

$= (4 + \ln 2) - (1 + \ln 1)$

$= \boxed{3 + \ln 2}$

(c) (6 pts.) Give a parameterization of the line tangent to the trajectory at $\mathbf{r}(2)$.

Point $P = \vec{r}(2) = (4, 4, \ln 2)$

Direction vector $\vec{v} = \vec{r}'(2) = (4, 2, \frac{1}{2})$

$\therefore\ (4, 4, \ln 2) + t(4, 2, \frac{1}{2})$ $t \in \mathbb{R}$

OR $x = 4 + 4t$, $y = 4 + 2t$, $z = \ln 2 + \frac{1}{2}t$

7. (10 pts.) The temperature (in $°C$) at each point (x, y), $-2 \le x, y, \le 2$, on a 4×4 metal plate is given by $T(x, y) = 10 - x^2 + y$.

(a) (6 pts.) Draw a contour plot of T that shows the level curves (i.e, isotherms) where $T = 9$, 10, and 11.

$T = 9$: $9 = 10 - x^2 + y$

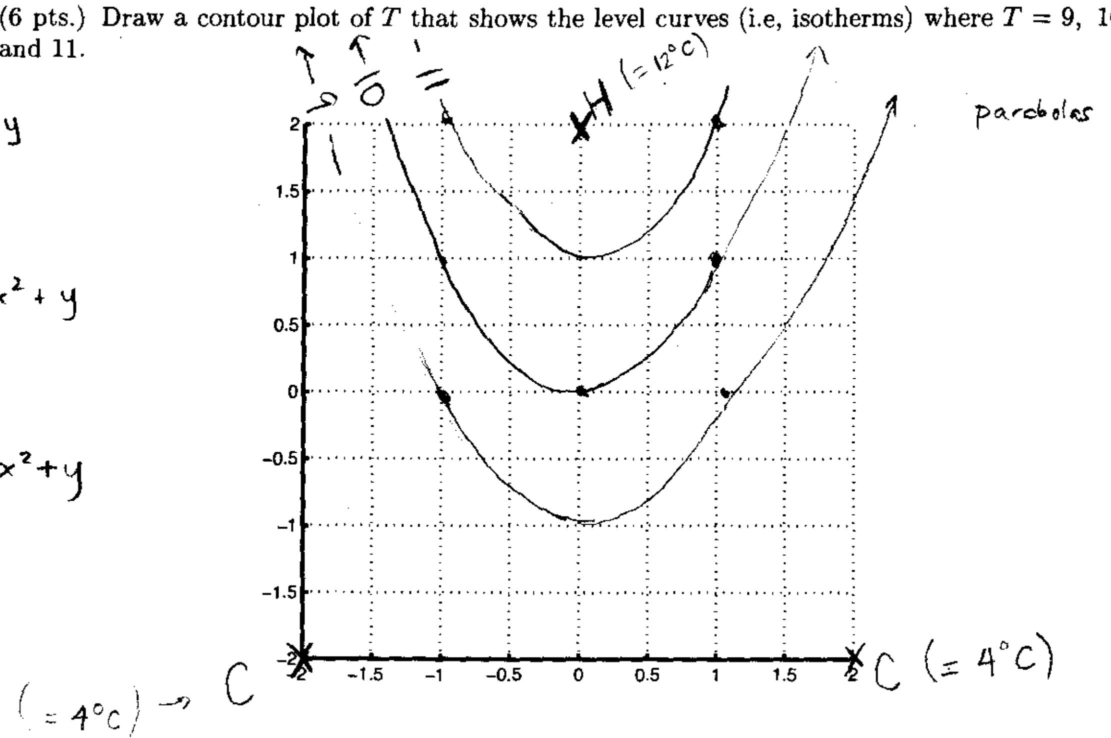
$y = x^2 - 1$

$T = 10$: $10 = 10 - x^2 + y$

$y = x^2$

$T = 11$: $11 = 10 - x^2 + y$

$y = x^2 + 1$



(b) (4 pts.) Using the contour plot above, indicate with an 'H' and 'C' the hottest and coldest points on the metal plate (with coordinates $-2 \le x, y \le 2$).