Section 2.5

6. Use the given lattices to help find the centralizers of every element in the following groups:

   a. D_8
   b. Q_8
   c. S_3
   d. D_{16}

(Schamel)

a.

g	e	r	r^2	r^3	s	rs	r^2s	r^3s
$C_{D_8}(g)$	D_8	$\langle r\rangle$	D_8	$\langle r\rangle$	$\langle s, r^2\rangle$	$\langle rs, r^2\rangle$	$\langle s, r^2\rangle$	$\langle rs, r^2\rangle$

b.

g	1	-1	i	$-i$	j	$-j$	k	$-k$
$C_{Q_8}(g)$	Q_8	Q_8	$\langle i\rangle$	$\langle i\rangle$	$\langle j\rangle$	$\langle j\rangle$	$\langle k\rangle$	$\langle k\rangle$

c.

g	e	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$C_{S_3}(g)$	S_3	$\langle(1\ 2)\rangle$	$\langle(1\ 3)\rangle$	$\langle(2\ 3)\rangle$	$\langle(1\ 2\ 3)\rangle$	$\langle(1\ 3\ 2)\rangle$

d.

g	e	r	r^2	r^3	r^4	r^5	r^6	r^7
$C_{D_{16}}(g)$	D_{16}	$\langle r\rangle$	$\langle r\rangle$	$\langle r\rangle$	D_{16}	$\langle r\rangle$	$\langle r\rangle$	$\langle r\rangle$

g	s	sr	sr^2	sr^3	sr^4	sr^5	sr^6	sr^7
$C_{D_{16}}(g)$	$\langle s, r^4\rangle$	$\langle sr^5, r^4\rangle$	$\langle sr^2, r^4\rangle$	$\langle sr^3, r^4\rangle$	$\langle s, r^4\rangle$	$\langle sr^5, r^4\rangle$	$\langle sr^2, r^4\rangle$	$\langle sr^3, r^4\rangle$

8. In each of the following groups find the normalizer of each subgroup:

   a. S_3
   b. Q_8

(Schamel)

a.

A	$\{e\}$	$\langle(1\ 2)\rangle$	$\langle(1\ 3)\rangle$	$\langle(2\ 3)\rangle$	$\langle(1\ 2\ 3)\rangle$	S_3
$N_{S_3}(A)$	S_3	$\langle(1\ 2)\rangle$	$\langle(1\ 3)\rangle$	$\langle(2\ 3)\rangle$	S_3	S_3

b.

A	$\langle 1\rangle$	$\langle -1\rangle$	$\langle i\rangle$	$\langle j\rangle$	$\langle k\rangle$	Q_8
$N_{Q_8}(A)$	Q_8	Q_8	Q_8	Q_8	Q_8	Q_8

9. Draw the lattices of subgroups of the following groups:
(Baggett)

a. $\mathbb{Z}/16\mathbb{Z}$

$$\begin{tikzcd}[arrows=dash]
\mathbb{Z}/16\mathbb{Z} \arrow[d] \\
\langle \bar{2} \rangle \arrow[d] \\
\langle \bar{4} \rangle \arrow[d] \\
\langle \bar{8} \rangle \arrow[d] \\
\{\bar{0}\}
\end{tikzcd}$$

b. $\mathbb{Z}/24\mathbb{Z}$

$$\begin{tikzcd}[arrows=dash]
 & & & \mathbb{Z}/24\mathbb{Z} \arrow[dl] \arrow[dr] & \\
 & & \langle \bar{2} \rangle \arrow[dl] \arrow[dr] & & \langle \bar{3} \rangle \arrow[dl] \\
 & \langle \bar{4} \rangle \arrow[dl] \arrow[dr] & & \langle \bar{6} \rangle \arrow[dl] & \\
\langle \bar{8} \rangle \arrow[dr] & & \langle \overline{12} \rangle \arrow[dl] & & \\
 & \{\bar{0}\} & & &
\end{tikzcd}$$

c. $\mathbb{Z}/48\mathbb{Z}$

$$
\begin{tikzcd}[column sep=small]
 & & & & \mathbb{Z}/48\mathbb{Z} \arrow[dl, no head] \arrow[dr, no head] & \\
 & & & \langle\bar{2}\rangle \arrow[dl, no head] \arrow[dr, no head] & & \langle\bar{3}\rangle \arrow[dl, no head] \\
 & & \langle\bar{4}\rangle \arrow[dl, no head] \arrow[dr, no head] & & \langle\bar{6}\rangle \arrow[dl, no head] & \\
 & \langle\bar{8}\rangle \arrow[dl, no head] \arrow[dr, no head] & & \langle\bar{12}\rangle \arrow[dl, no head] & & \\
\langle\bar{16}\rangle \arrow[dr, no head] & & \langle\bar{24}\rangle \arrow[dl, no head] & & & \\
 & \{\bar{0}\} & & & &
\end{tikzcd}
$$

Section 3.1

3. Let A be an Abelian group and let B be a subgroup of A. Prove that A/B is Abelian. Give an example of a non-Abelian group G containing a proper normal subgroup N such that G/N is Abelian.

Proof. (Bastille) First we show that B is normal. Let $b \in B \leq A$ and let $a \in A$. Then $aba^{-1} \in aBa^{-1}$ and

$$
\begin{aligned}
aba^{-1} &= aa^{-1}b \qquad \text{since } A \text{ is Abelian (and } a^{-1}, b \in A) \\
&= b \in B.
\end{aligned}
$$

Since a, b were chosen arbitrarily, it follows that $aBa^{-1} \subseteq B$ for all $a \in A$; hence B is normal in A. Now consider $A/B = \{aB | a \in A\}$. Let $a_1B, a_2B \in A/B$. Then we have

$$
\begin{aligned}
(a_1B)(a_2B) &= (a_1a_2)B \qquad \text{since } B \text{ is normal so the operation is well-defined} \\
&= (a_2a_1)B \qquad \text{since } a_1, a_2 \in A,\ A \text{ Abelian} \\
&= (a_2B)(a_1B) \qquad \text{since } B \text{ is normal.}
\end{aligned}
$$

Therefore, A/B is Abelian. ☐

<u>Remark</u>: There exist non-Abelian groups G containing a proper normal subgroup N such that G/N is Abelian. For example, take $G = Q_8$ and $N = \{1, -1\}$. Then G is non-Abelian since for example $ij = k \neq -k = ji$, and $N \lhd G$ since for all $a \in Q_8$:

$$
a1a^{-1} = 1 \in N \qquad \text{and} \qquad a(-1)a^{-1} = -1 \in N.
$$

Now we have: $G/N = \{N, iN, jN, kN\}$. Note that if $a, b \in Q_8$, then

- if WLOG $b = \pm 1$ then $ab = ba$ so

$$
(aN)(\pm 1N) = (a(\pm 1))N = ((\pm 1)a)N = (\pm 1N)(aN).
$$

- if $a, b \neq \pm 1$, note that if $ab = c$ then $ba = -c$ but for any coset cN we have $cN = \{c, -c\}$ so

$$
(aN)(bN) = (ab)N = cN = (-c)N = (ba)N = (bN)(aN).
$$

Therefore G/N is Abelian.

27. Let N be a finite subgroup of a group G. Show that $gNg^{-1} \subseteq N$ if and only if $gNg^{-1} = N$. Deduce that $N_G(N) = \{g \in G \mid gNg^{-1} \subseteq N\}$.

 Proof. (Lawless) Clearly, if $gNg^{-1} = N$, then $gNg^{-1} \subseteq N$. Assume $gNg^{-1} \subseteq N$. Consider the map $\varphi : N \to gNg^{-1}$ defined by $\varphi(n) = gng^{-1}$. We can see this map is injective, since if $\varphi(n) = \varphi(m)$, then $gng^{-1} = gmg^{-1}$, and thus $n = m$. Since $|N| < \infty$, and the map is injective, then we know this is a bijection from $N \to gHg^{-1}$. This, combined with our assumption that $gNg^{-1} \subseteq N$ gives us that $gNg^{-1} = N$.

 □

28. Let N be a *finite* subgroup of a group G and assume $N = \langle S \rangle$ for some subset S of G. Prove that an element $g \in G$ normalizes N if and only if $gSg^{-1} \subseteq N$.

 Proof. (Bastille) If g normalizes N then for all $n \in N$, $gng^{-1} \in N$. In particular, for all $s \in S \subseteq N$, $gsg^{-1} \in N$. Hence $gSg^{-1} \subseteq N$.

 Now if $gSg^{-1} \subseteq N$, note that since N is finite, so must be S and all its elements have finite order. So we can write $S = \{a_1, a_2, \cdots, a_k\}$ for some fixed k and any $n \in N$ can be expressed in the form:

$$n = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} \qquad \text{where } \alpha_i \geq 0 \text{ since } |a_i| \text{ is finite.}$$

 We also note that for any $a, b \in G$,

$$gabg^{-1} = ga1bg^{-1} = ga(g^{-1}g)bg^{-1} = (gag^{-1})(gbg^{-1}).$$

 So inductively we find that $ga^{\ell}g^{-1} = (gag^{-1})^{\ell}$ for any $\ell \geq 0$. Therefore,

$$\begin{aligned} gng^{-1} &= ga_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} g^{-1} \\ &= \left(ga_1^{\alpha_1} g^{-1}\right)\left(ga_2^{\alpha_2} g^{-1}\right) \cdots \left(ga_k^{\alpha_k} g^{-1}\right) \\ &= \left(ga_1 g^{-1}\right)^{\alpha_1}\left(ga_2 g^{-1}\right)^{\alpha_2} \cdots \left(ga_k g^{-1}\right)^{\alpha_k}. \end{aligned}$$

 But for all $1 \leq i \leq k$, ga_ig^{-1} is an element of gSg^{-1} so $ga_ig^{-1} \in N$ since we assume gSg^{-1} is contained in N, and hence by closure under the operation in N, $\left(ga_ig^{-1}\right)^{\alpha_i} \in N$. Therefore,

$$gng^{-1} = \prod_{i=1}^{k} \left(ga_ig^{-1}\right)^{\alpha_i} \in N,$$

 and hence since n was chosen arbitrarily, we have $gNg^{-1} \subseteq N$, and so g normalizes N. □

31. Prove that if $H \leq G$ and N is a normal subgroup of H then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of G in which N is normal (i.e., is the join of all subgroups H for which $N \trianglelefteq H$).

 Proof. (Mobley) Since H and $N_G(N)$ are both groups, it is sufficient to show that $H \subseteq N_G(N)$. To this end, pick $h \in H$. Since $N \trianglelefteq H$, for all $h \in H$, $hNh^{-1} = N$. Thus, $h \in N_H(N)$. Since H is a subgroup of G, it must be the case that $H \leq N_G(N)$.

 □

 Suppose that $K \leq G$ and $N \trianglelefteq K$. Using arguments similar to those above, we can show that $K \leq N_G(N)$. Thus, any arbitrary normal subgoup of G is contained in $N_G(N)$ and $N_G(N)$ is the largest subgroup of G in which N is normal.

33. Find all normal subgroups of D_8 and for each of these find the isomorphism types of its corresponding quotient.

Proof. (Buchholz) The normal subgroups of D_8 are $< s, r^2 >, < r >, < rs, r^2 >, < r^2 >$, 1 and D_8. The first three of these have index 2 in D_8 and therefore are normal. For the subgroup $< r^2 >$, we note that $rr^2r^{-1} = r^2$ and $sr^2r^{-1} = r^{-2} = r^2$. Since r and s generate D_8, it follows that $g < r^2 > g^{-1} =< r^2 >$ for any $g \in G$. Thus, $< r^2 > \lhd D_8$.
Now we must find the isomorphism types of each corresponding quotients. First note that $|D_8/ < s, r^2 > | = |D_8/ < rs, r^2 > | = |D_8/ < r > | = 2$ so all are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Now $|D_8/ < r^2 > | = 4$ and $D_8/ < r^2 >$ is not cyclic so $D_8/ < r^2 > \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

□

35. Prove that $SL_n(F) \trianglelefteq GL_n(F)$ and describe the isomorphism type of the quotient group.

Proof. (Hazlett) Let $\varphi : GL_n(F) \to F$ such that $\varphi(A) = \det(A)$. Note, $\varphi(A)\varphi(B) = \det(A)\det(B) = \det(AB) = \varphi(AB)$. Then φ is a homomorphism. Note, the kernel of φ is $SL_n(F)$. Consequently $SL_n(F)$ is normal in $GL_n(F)$.

Let $\psi : GL_n(F)/SL_n(F) \to F \setminus \{0\}$ such that $\psi(ASL_n(F)) = \det(A)$. We claim that $ASL_n(F)$ is the set of all things with determinant equal to $\det(A)$. Suppose we have a matrix B such that $\det(B) = \det(A)$. Then $\det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{1}{\det(A)}\det(A) = 1$. So $A^{-1}B \in SL_n(F)$ and $ASL_n(F) = BSL_n(F)$. Choose $C \in ASL_n(F)$. Then $C = AS$ where $S \in SL_n(F)$. Hence $\det(C) = \det(AS) = \det(A)\det(S) = \det(A)$. Consequently $ASL_n(F)$ is the set of all matrixes in $GL_n(F)$ with the same determinant as A. We can conclude then that ψ is not only well defined but also an injection. Also, given $f \in F\setminus\{0\}$ the $n \times n$ matrix H with $h_{1,1} = f$, $h_{i,i} = 1$ for $2 \le i \le n$ and $h_{i,j} = 0$ otherwise has the property $\det(H) = f$. Hence $\psi(HSL_n(F)) = f$. So ψ is a surjection. Hence ψ is a bijection. Finally, note that $\psi(ASL_n(F))\psi(BSL_n(F)) = \det(A)\det(B) = \det(AB) = \psi(ABSL_n(F))$. Thus ψ is a homomorphism. This implies that ψ is an isomorphism between $GL_n(F)/SL_n(F)$ and $F \setminus \{0\}$. □

36. Prove that if $G/Z(G)$ is cyclic then G is abelian.

Proof. (Gillispie) Suppose $G/Z(G)$ is cyclic with generator $xZ(G)$.
Proposition 3.2.4 shows us that the sets $aZ(G)$ partition G and if we pick some $g \in G$, we know that $g \in gZ(G)$
. We also know that there exists some $n \in \mathbb{N}$ s.t.$gZ(G) = x^nZ(G)$. Thus there exists some $z \in Z(G)$ so that $g = x^nz$
. Now, pick $g, h \in G$. We showed that there are $m, n \in \mathbb{Z}$ and $z_1, z_2 \in Z(G)$ so that $g = x^nz_1$ and $h = x^mz_2$. Notice that since z_1 and z_2 commute with anything in G we have

$$\begin{aligned}
gh &= x^nz_1x^mz_2 \\
&= x^nx^mz_1z_2 \\
&= x^mx^nz_2z_1 \\
&= x^mz_2x^nz_1 \\
&= hg
\end{aligned}$$

And so G is abelian.□

□

41. Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy \,|\, x, y \in G\rangle$ is a normal subgroup of G and that G/N is Abelian.

Proof (Granade). We claim that for all $g \in G$, $gN = Ng$. To see this, note that it suffices to show that for all $g \in G$ and $n \in N$, there exists n' such that $gn = n'g$. Thus, pick $x, y, g \in G$. By the definition of N, $x^{-1}y^{-1}xy \in N$. Next, let $x' = gxg^{-1}$ and $y' = gyg^{-1}$, so that $x = g^{-1}x'g$ and $y = g^{-1}y'g$. Substituting, we get that:

$$\begin{aligned}
gx^{-1}y^{-1}xy &= g\left(g^{-1}x'^{-1}g\right)\left(g^{-1}y'^{-1}g\right)\left(g^{-1}x'g\right)\left(g^{-1}y'g\right) \\
&= \left(gg^{-1}\right)x'^{-1}\left(gg^{-1}\right)y'^{-1}\left(gg^{-1}\right)x'\left(gg^{-1}\right)y'g \\
&= x'^{-1}y'^{-1}x'y'g
\end{aligned}$$

But then, $x'^{-1}y'^{-1}x'y' \in N$ and so $x'^{-1}y'^{-1}x'y'g \in Ng$. Therefore, $gN \subseteq Ng$. Reversing the argument above gives that $gN = Ng$, as required. $\square$

Corollary. *G/N is Abelian.*

Proof (Granade). Let $aN, bN \in G/N$. Then, we claim that $abN = baN$. It is thus sufficient to show that $ab\,(ba)^{-1} \in N$. But then, $ab\,(ba)^{-1} = aba^{-1}b^{-1} \in N$. $\square$

42. Assume both H and K are normal subgroups of G with $H \cap K = \{1\}$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$.

Proof. (Baggett) Take any elements $x \in H$ and $y \in K$. Since H is normal, we have that $y^{-1}xy \in y^{-1}Hy = H$; since $x^{-1} \in H$ and H is closed under multiplication, $x^{-1}y^{-1}xy \in H$. Similarly, we have that $x^{-1}y^{-1}x \in x^{-1}Kx = K$; since $y \in K$ and K is closed under multiplication, $x^{-1}y^{-1}xy \in K$. Thus, $x^{-1}y^{-1}xy \in H \cap K$. However, $H \cap K = \{1\}$, so $x^{-1}y^{-1}xy = 1$. Equivalently, we have that $xy = yx$. $\square$