MATH 631
Midterm

Name :________________________

October 9, 2018

**Instructions:** This exam is closed book and closed notes, and two hours in length. You may use **only** your brain and blank scratch paper in writing solutions.

The problems in Part I are computational in nature, and full marks are awarded for correct answers. You need only justify your answers if you are explicitly asked to do so. Part II involves writing proofs and is theoretical in nature. You should prove results from first principles and not simply quote statements from the book. Your proofs will be graded not only on correctness, but points will be awarded/taken away for poor writing and exposition. Blank paper is supplied for scratch work, but final responses should be written in the space provided.

**Part I.**

1. (10 pts.) Consider the cylic group $C_{4900} = \langle x \rangle$ of order $4900 = 2^2 \cdot 5^2 \cdot 7^2$.

   (a) (4 pts.) Give the number of generators of $C_{4900}$.

   $\varphi(4900) = \varphi(2^2)\cdot\varphi(5^2)\cdot\varphi(7^2) = 2\cdot 20\cdot 42 = 1{,}680$

   (b) (6 pts.) List explicitly the elements $x^a$, with $0 \le a \le 4899$, of order 10.

   $|x^a| = 10 \iff \dfrac{4900}{(4900,a)} = 10$ i.e. $\gcd(4900,a) = 2\cdot 5\cdot 7^2 = 490$

   Since $\varphi(10) = 1\cdot 4 = 4$, there should be only 4 such values of $a$. The candidates are $a = k\cdot 490$ for $k = 1,2,\dots,10$ and $(k,2)=1$, $(k,5)=1$. i.e. $k = 1, \cancel{2}, 3, \cancel{4}, \cancel{5}, \cancel{6}, 7, \cancel{8}, 9, \cancel{10}$

   So that $k = 1,3,7,9$

   *Answer:* $|x^a| = 10$ if $a =$ $490,\ 3\cdot 490,\ 7\cdot 490,\ 9\cdot 490$ or $a = 490, 1470, 3430, 4410$.

   (If it helps, you can simply give the prime factorizations of $a$. I am not interested in your ability to multiply integers.)

2. (10 pts.) Consider the cyclic groups $\mathbb{Z}/30\mathbb{Z}$ and $C_{18} = \langle x \rangle$ of orders 30 and 18 respectively, and suppose that

$$\begin{aligned}\varphi_a : \mathbb{Z}/30\mathbb{Z} &\to C_{18}\\ 1 &\mapsto x^a\end{aligned}$$

   extends to a well-defined group homomorphism from $\mathbb{Z}/30\mathbb{Z}$ to $C_{18}$.

   (a) (6 pts.) List the values of $a$ with $0 \le a \le 17$ for which this is true. (I.e. The map defines a well-defined group homomorphism.)

   If $\varphi: \mathbb{Z}/30\mathbb{Z} \to C_{18}$ is <u>well-defined</u>, then in particular, $\varphi(30\cdot\bar{1}) = \varphi(\bar{0}) = x^0 = x^{30a}$

   Necessarily, $30a \equiv 0 \mod 18$. Since $(30,18) = 6$, there will be six such values of $a$ obtained via: $5a \equiv 0 \mod 3$ or $2a \equiv 0 \mod 3$ or simply $a = 3$ for starters. Thus, $a = 3,6,9,12,15,0$ are the right values.

   (b) (4 pts.) Give a brief explanation why such a well-defined group homomorphism can not be surjective.

   If $\varphi_a$ is surjective, then $|\mathrm{Im}(\varphi_a)| = 18$. By the First Isomorphism Theorem, $|\mathbb{Z}/30\mathbb{Z}/\ker\varphi| = 18$, but by Lagrange's Theorem $|\ker\varphi|\,\big|\,30$ and therefore $|\mathbb{Z}/30\mathbb{Z}/\ker\varphi_a| = 18$. <u>Nonsense</u>. and $|\mathbb{Z}/30\mathbb{Z}/\ker\varphi_a|\,\big|\,30$.

3. (20 pts. – 4 pts. each) Consider the symmetric group $G = S_7$ and let $\sigma = (1\ 2\ 3\ 6\ 5\ 4\ 7)$ be an 7-cycle.

(a) Express $\sigma$ as the product of (not necessarily disjoint) transpositions.

$(17)(14)(15)(16)(13)(12)$

(b) Compute the number of conjugates of $\sigma$ in $S_7$.

$6! = 720$

(c) Let $\tau$ be the 7-cycle $(3\ 7\ 1\ 4\ 5\ 6\ 2)$. Give an element $\alpha$ that conjugates $\sigma$ to $\tau$, i.e. give $\alpha$ such that $\alpha\sigma\alpha^{-1} = \tau$.

$\sigma = (1\ 2\ 3\ 6\ 5\ 4\ 7)$

$\tau = (3\ 7\ 1\ 4\ 5\ 6\ 2)$

$\alpha = (13)(27)(46)$

(d) Noting that $S_7$ acts on itself by conjugation, explicitly use the Orbit-Stabilizer theorem to find the size of the stabilizer of $\sigma$ under this action and the elements of the Stabilizer subgroup of $S_7$.

Let $\Theta$ be the orbit of $\sigma$ under conjugation by $S_7$, and $G_\sigma$ the stabilizer.

$|\Theta| = [S_7 : G_\sigma] \Rightarrow 6! = 7!/|G_\sigma| \Rightarrow |G_\sigma| = 7$ Since $\langle\sigma\rangle \leq G_\sigma$ and their orders are equal, $\langle\sigma\rangle = G_\sigma$.

The stabilizer of $\sigma$ in this context is better known as $C_{S_7}(\sigma)$. (Using appropriate notation in place of words here is fine.)

(e) Noting that $\sigma \in A_7$, what is the size of the conjugacy class of $\sigma$ in $A_7$? Stated otherwise, how many conjugates in $A_7$ does $\sigma$ have? Briefly, state a result that justifies your answer.

The options are $6!$ or $\frac{6!}{2}$. You need to know that $\Theta_\sigma^{S_7} = \Theta_\sigma^{A_7}$ if, and only if $\sigma$'s disjoint cycle decomposition is NOT comprised of distinct odd integers.

*Answer:* The number of conjugates of $\sigma$ in $A_7$ is $\frac{1}{2}6! = 360$

because .... $\sigma$ does not commute with any odd permutation. Since $\sigma$ is a 7-cycle.

4. (12 pts. – 6 pts. each) In the Table below, list a representative of each isomorphism class for groups $G$ with $|G| = 6$ or $p^2$.

| Group order $\lvert G\rvert$ | Number of isomorphism types | Representatives |
|---|---|---|
| 2 | 1 | $C_2$ |
| 6 | 2 | Abelian $C_6$, non-Abelian $D_3$ (or $S_3$) |
| $p^2$, for $p$ prime | 2 | Abelian: $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$ |