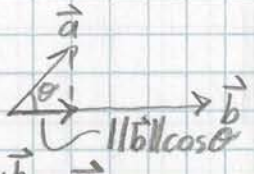
Dot Product: $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$, $0 < \theta \le \pi$

if $\vec{a}\cdot\vec{b} = \begin{cases} 0 & \vec{a}\perp\vec{b} \\ <0 & \theta > \pi/2 \\ >0 & \text{acute angle} \end{cases}$



$\text{proj}_a \vec{b}$

$= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}\frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$

Cross Product: $\vec{a}\times\vec{b}$ in $\mathbb{R}^3$

$\|\vec{a}\times\vec{b}\|$ = area of D

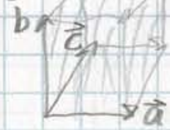
$a\cdot(\vec{a}\times\vec{b}) = 0$ because $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

$\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

$(\vec{a}\cdot\vec{b})\times\vec{c}$ = Never do this.

$\vec{a}\cdot(\vec{b}\times\vec{c})$ = Triple Scalar Product

Volume of parallelopiped

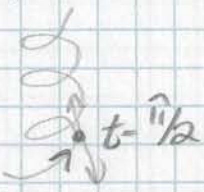


Lines: $l: \vec{P} + t\vec{v}$ — $\vec{v}$ = direction vector

Eg: $P(1,1,1)$ $\vec{v} = \langle \pi, e, e^{\pi}\rangle$

$x(t) = 1 + \pi t$
$y(t) = 1 + et$ or $\langle 1+\pi t, 1+et, 1+e^{\pi}t\rangle$
$z(t) = 1 + e^{\pi}t$

Example: Helix tan line on curve.



$\vec{r}(t) = \langle 2\sin t, 2\cos t, t^2\rangle$

$\vec{r}(\frac{\pi}{2}) = (2, 0, \pi^2/4)$

$P = \vec{r}(\frac{\pi}{2})$

Line $= \vec{P} + t\vec{v}$

$\vec{v} = \vec{r}\,'(t) = \langle 2\cos t, -2\sin t, 2t\rangle$
$\underset{t=\frac{\pi}{2}}{=} \langle 0, -2, \pi\rangle$

Applications of space curves

$\vec{x}(t)$ $\vec{v}(t)$ $\vec{a}(t)$

$\vec{x}\,'(t) = \vec{v}(t)$ $\vec{v}\,'(t) = \vec{a}(t)$

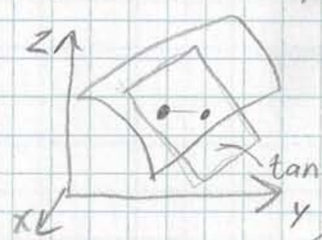
$\vec{x}\,''(t) = \vec{a}(t)$

Planes: $\vec{x} \cdot \vec{n} = \vec{P} \cdot \vec{n}$

$$2x + 3y = 12 \Rightarrow \vec{n} = \langle 2, 3, 0 \rangle$$
$$P(1, \tfrac{10}{3}, 0)$$

Differential. $z = f(x,y)$ $\quad z = 2x^2 + 3y$

$P(1,-1,-1)$



Remember:
$\Delta x = dx$
$\Delta y = dy$
But:
dz = diferential, change in z on tangent plane
Δz = change in function values
new z - old z
$f(x+\Delta x, y+\Delta y) - f(x,y)$

$$dz = \frac{dz}{dx} dx + \frac{dz}{dy} dy$$
$$= f_x(a,b)dx + f_y(a,b)dy \quad z - z_0 = f_x(a,b)dx + f_y(a,b)dy$$

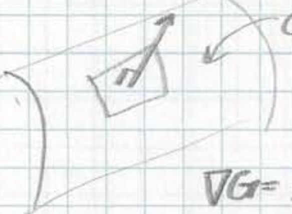
original point
$f(a,b) = z_0$

$z = 2x^2 + 3y \quad f(x,y) = z$
$p(1,-1,-1) \quad f(1,-1) = -1 \qquad z - z_0 = f_x(a,b)(x - x_0) + f_y(a,b)(y - y_0)$

$f_x = 4x \quad f_x(1,-1) = 4$
$f_y = 3 \quad f_y(1,-1) = 3 \qquad z - (-1) = 4(x-1) + 3(y-(-1))$
$\qquad z + 1 = 4(x-1) + 3(y+1)$

Method II - Level Surface way.

$f(x,y) = 2x^2 + 3y$ at $(1,2,8)$



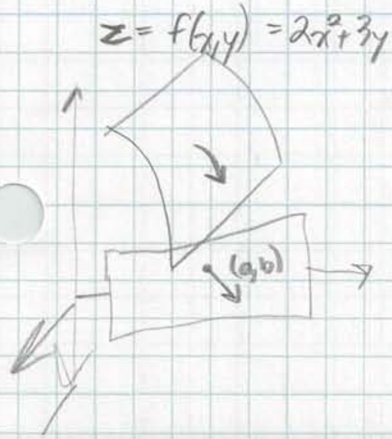
$z = 2x^2 + 3y$
$G(x,y,z) = z - 2x^2 - 3y = 0$

∇G = normal vector for tangent plane

$\vec{n} = \nabla G(x,y,z) = \langle -4x, -3, -1 \rangle$
$(1,2,8) = \langle -4, -3, 1 \rangle = \vec{n}$

Directional Derivatives:

$z = f(x,y) = 2x^2 + 3y$



$D_{\vec{u}} f(a,b) = 0$ $\quad f(x,y)$ is constant
< 0 $\quad f(x,y)$ is decreasing
> 0 $\quad f(x,y)$ is increasing

convert to unit vector $\vec{u}$ if needed

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

Example:
$\vec{v} = \langle 3, 4 \rangle \quad P(1,1)$
$\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$
$= \langle 4, 3 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$
$= \boxed{\frac{24}{5}}$

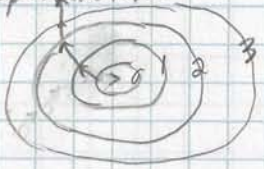
increasing

Gradient:

In $\mathbb{R}^2$ ~ always orthogonal to level curves

~ points in direction of maximal increase of f at (a,b)

Application:



$-\nabla f(a,b)$ points in direction of maximal decrease

$\|\nabla f(a,b)\|$ = rate of maximum increas

2nd Partials Test: $f(x,y)$

Find critical points: $f_x = 0$ and $f_y = 0$

or

one of them is undefined $f(x,y) = (x^2+y^2)^{1/2}$

$f_x = x(x^2+y^2)^{-1/2}$ $f_y = y(x^2+y^2)^{-1/2}$

undefined at $(0,0)$

$f(x,y) = 2x^2 + 3y^2$

$f_x = 4x$ $f_y = 6y$

$(0,0)$

$d = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$

where $f_{xy} = f_{yx}$

$d < 0 \Rightarrow$ saddle point

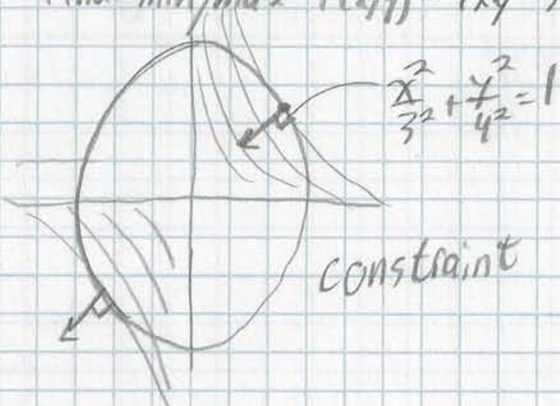
$d > 0 \begin{cases} f_{xx} > 0 \Rightarrow \text{local min} \\ f_{xx} < 0 \Rightarrow \text{local max} \end{cases}$

$d = 0 \Rightarrow$ No information

LeGrange Multipliers

Optimize $f(x,y)$ subject to restraint $g(x,y) = K$

find min/max $f(x,y) = 4xy$ subject to $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$



$\nabla f = \lambda \nabla g$

$\nabla f = \langle 4y, 4x \rangle$ $\nabla g = \langle 2x/9, 2y/16 \rangle$

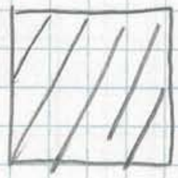
Eq_1: $4y = \lambda \frac{2x}{9}$ Eq_2: $4x = \frac{2y}{16}\lambda$

Eq_3: $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Solve for x and y and then

evaluate $f(x,y)$.

Optimization on bounded domains.



1- find min/max on interior
2- check for min/max on boundary
then choose largest/smallest

Surface Area:

$f(x,y) = 2x^2 + 3y$

Graph formula

$$\iint_R dS = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA = \iint \qquad dx\,dy$$

Parameterized Surface

$\vec{r}(u,v) = (e^u, v, u) \Rightarrow x = e^z$

$$\iint_R dS = \iint_R \|\vec{r}_u \times \vec{r}_v\|\, dA$$

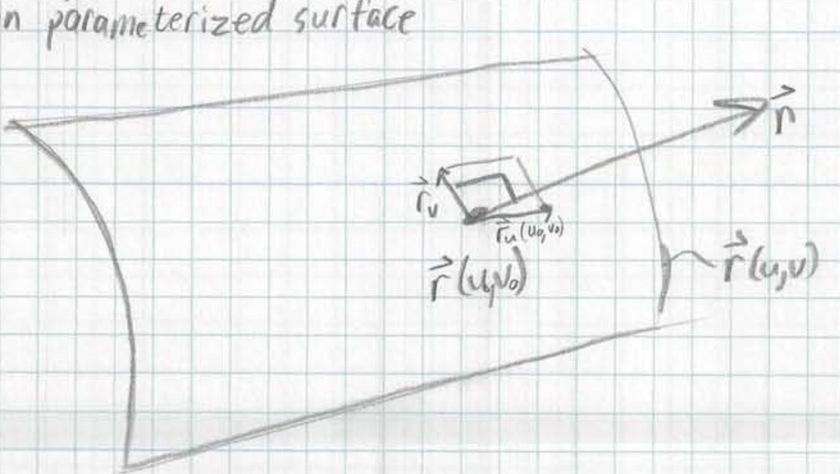
Arc Length:

$$L = \int_C ds = \int_{t_i}^{t_f} \|\vec{r}\,'(t)\|\, dt \quad \text{calc III}$$

$\cancel{\text{if } y=f(x) = \sqrt{1+f_x^2}}$ } Calc II Not on test

$t_i < t \leq t_f$

Finding normal vector on parameterized surface



$\Rightarrow \vec{n} = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$

$\|\vec{n}\| = \|\vec{r}_u \times \vec{r}_v\|$

$=$ area of tile.

Page 1121 Summary of line and Surface integrals

$ds = \|\vec{r}\,'(t)\|\,dt$ Arc Length

$$\int f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,\|\vec{r}\,'(t)\|\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \text{Work}$$
$$= \int_a^b \vec{F}(x(t),y(t),z(t))\cdot\vec{r}\,'(t)\,dt$$

Surface Integrals

$$dS = \sqrt{1+f_x^2+f_y^2}\,dA$$

$$\iint_S f(x,y,z)\,dS$$

Graph

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iint_R \vec{F}\cdot\langle -g_x(x,y), -g_y(x,y), 1\rangle\,dA$$

Flux integrals

Parametric:

$$dS = \|\vec{r}_u \times \vec{r}_v\|\,dA$$

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

review Polar, Cylindrical, and spherical Coordinates

Polar	Cylindrical	Spherical
$r=\sqrt{x^2+y^2}$	$r=\sqrt{x^2+y^2}$	$x=\rho\sin\varphi\cos\theta$
$x=r\cos\theta$	$x=r\cos\theta$	$y=\rho\sin\varphi\sin\theta$
$y=r\sin\theta$	$y=r\sin\theta$	$z=\rho\cos\varphi$
	$z=z$	$\rho=\sqrt{x^2+y^2+z^2}$
		$r=\rho\sin\varphi$