Instructions:

- This test is closed note and closed book.
- All proofs should be formal.
- Raise your hand if you have a question.

PART I (Computational): For problems in this section, simple answers are sufficient.

(1) (10 points) Draw a lattice diagram for the abelian group $\mathbb{Z}/18\mathbb{Z}$.

(2) (10 points) Consider the symmetric group S_4.

(a) List all conjugates of $\sigma = (123)$.

(b) Find $C_{S_4}(\sigma)$, the centralizer of σ in S_4.

(3) (15 points) List up to isomorphism all abelian groups of order 72.

(4) (15 points) Let $Z_{450} = \langle x \rangle$ denote the cyclic group of order $450 = 2 \cdot 3^2 \cdot 5^2$.

(a) Compute the number of generators of Z_{450}.

(b) List all elements of Z_{450} of order 9.

(c) List all group homomorphisms from Z_{450} to $\mathbb{Z}/21\mathbb{Z}$.

Part II (Long Answer): For problems in this section, formal proofs are required. Each problem is worth 10 points

(1) Prove that a group G of order $56 = 2^3 \cdot 7$ is not simple.

(2) Let G be a finite group. Let H be a subgroup of G and let N be a normal subgroup of G. Prove that if $\gcd(|H|, |G : N|) = 1$, then $H \leq N$.

(3) Let G be a nonabelian group of order p^3, where p is a prime number. Prove that the center of G is of order p.

(4) (a) State formally what it means for the subgroup H to be characteristic in G.

(b) State formally the definition of the commutator subgroup G' of the group G.

(c) Prove that if $K \trianglelefteq G$, then $K' \trianglelefteq G$.

(5) Let G be a finite group and A a set. Suppose G acts on A and let $a \in A$. Prove that the size of the orbit of a under this action is equal to the index of the stabilizer G_a in G under this action. (That is, prove $|\mathcal{O}_a| = |G : G_a|$. Note that this is one part of a proposition we proved in class. Referencing that proposition is not acceptable. You should prove this using first principles.)