MATH253X-UX1
Summer 2016

Midterm Exam 2

Name: Answer Key

Instructions. You have 90 minutes. Closed book, closed notes, no calculator. *Show all your work* in order to receive full credit.

1. Show that $\lim_{(x,y)\to(-1,1)} \dfrac{xy+1}{2x^2-y^2-1}$ does not exist.

Solution:

- Setting $x=-1$ and letting $y\to 1$ to approach $(-1,1)$ along the line $(-1,y)$, we see
$$\lim_{y\to 1}\frac{1-y}{1-y^2}=\frac{1}{2}.$$
- Setting $y=1$ and letting $x\to -1$ to approach $(-1,1)$ along the line $(x,1)$, we see
$$\lim_{x\to -1}\frac{x+1}{2x^2-2}=-\frac{1}{4}.$$

Since these limits are different, $\boxed{\text{the original multivariable limit does not exist.}}$

2. Use Lagrange multipliers to find the point(s) on the curve $x^2-2y^2=1$ closest from the point $P(0,2)$.

Solution: We want to minimize the distance from a point on the hyperbolic curve to $P(0,2)$. For simplicity, let $f(x,y)$ be the square of that distance:
$$f(x,y)=(x-0)^2+(y-2)^2=x^2+(y-2)^2.$$
Then our constraint is $g(x,y)=x^2-2y^2=1$ and we need also to satisfy:
$$\nabla f=\lambda\nabla g \quad\Rightarrow\quad \langle 2x,2(y-2)\rangle=\lambda\langle 2x,-4y\rangle \quad\Rightarrow\quad \begin{cases}2x=2\lambda x\\ 2(y-2)=-4\lambda y\end{cases}$$
The first equation has two solutions:

- either $x=0$, then from the constraint, $0-2y^2=1$ which has no real solution for y;
- or $\lambda=1$, then from the second equation:
$$2y-4=-4y \quad\Rightarrow\quad y=\frac{2}{3}$$
and so plugging into the constraint $x^2=1+2\left(\frac{4}{9}\right)=\frac{17}{9}$ so we have the points $\left(\pm\frac{\sqrt{17}}{3},\frac{2}{3}\right)$.

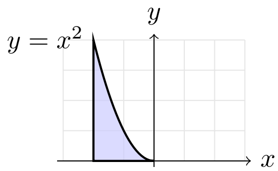
Both have the same $f(x,y)$ value so they are both points we're looking for: $\boxed{\left(\pm\dfrac{\sqrt{17}}{3},\dfrac{2}{3}\right)}$.

3. Find an equation of the tangent plane to the following surface at the point $(x_0,y_0,z_0)=(2,1,-1)$:
$$x\ln y-3yz^2+1=xz.$$
Solution: Let $F(x,y,z)=x\ln y-3yz^2-xz=-1$. Then,
$$\nabla F(2,1,-1)=\left\langle \ln y-z,\frac{x}{y}-3z^2,-6yz-x\right\rangle\Big|_{(2,1,-1)}=\langle 0+1,2-3,6-2\rangle=\langle 1,-1,4\rangle$$
and so the equation of the tangent plane is:
$$(x-2)-(y-1)+4(z+1)=0 \quad\Rightarrow\quad \boxed{x-y+4z+3=0}.$$

4. For each of the iterated integrals below, sketch the region of integration then convert as indicated. DO NOT evaluate.

(a) Rewrite $\int_{-2}^{0}\int_{0}^{x^2} 3xy\,dy\,dx$ in the order $dx\,dy$.

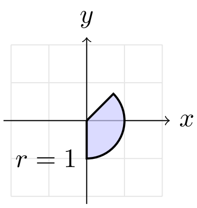
Solution:



$$\boxed{\int_0^4 \int_{-2}^{-\sqrt{y}} 3xy\,dx\,dy}$$

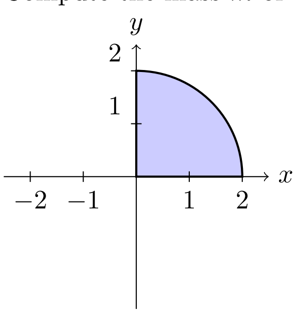
(b) Rewrite $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}}\int_0^1 r^2\,dr\,d\theta$ in rectangular coordinates.

Solution: From the picture below, we need to split the integral. The order $dx\,dy$ is a bit easier as the split is at $y = 0$ but we still need to solve for y when $\theta = \frac{\pi}{4}$ and $r = 1$, i.e. $y = 1\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Inner bounds are from the y-axis $x = 0$, the circle $x^2 + y^2 = 1$, and $y = x$:



$$\boxed{\int_{-1}^{0}\int_0^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\,dx\,dy + \int_0^{\frac{\sqrt{2}}{2}}\int_y^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\,dx\,dy}$$

5. Compute the mass m of the planar lamina with density $\rho(x, y) = y^2$ shown below.



Solution:

$$m = \iint_R y^2\,dA = \int_0^{\frac{\pi}{2}}\int_0^2 r^2\sin^2\theta\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\sin^2\theta\right]_0^2 d\theta = \int_0^{\frac{\pi}{2}} 4\sin^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2(1-\cos(2\theta))\,d\theta = \Big[2\theta - \sin(2\theta)\Big]_0^{\frac{\pi}{2}} = \boxed{\pi}.$$

6. Consider the function:

$$f(x, y) = x^3 - 12xy + 8y^3.$$

(a) Find and classify all critical points of $f(x, y)$.

Solution:

- Find the critical points from solving $\nabla f = \overrightarrow{0}$:

$$\nabla f = \overrightarrow{0} \quad\Rightarrow\quad \left\langle 3x^2 - 12y, -12x + 24y^2\right\rangle = \langle 0, 0\rangle \quad\Rightarrow\quad \begin{cases} x^2 = 4y \\ x = 2y^2 \end{cases} \quad\Rightarrow\quad \begin{cases} 4y^4 = 4y \\ x = 2y^2 \end{cases}$$

The first equation simplifies to $y(y^3-1)=0$ so either $y=0$ or $y=1$. Substituting back into the second equation gives us the two critical points $(0,0)$ and $(2,1)$.

- Apply the Second Partials Test to classify them:

$$f_{xx}=6x\ ,\ f_{yy}=48y\ ,\ f_{xy}=-12 \quad \Rightarrow \quad d=f_{xx}f_{yy}-f_{xy}^2=288xy-144=144(2xy-1)$$

$d(0,0)=-144<0$ so we have a $\boxed{\text{saddle point at } (0,0,0)}$;
$d(2,1)=144(3)>0$ and $f_{xx}(2,1)=12>0$ so $\boxed{f \text{ has a local minimum at } (2,1)}$.

(b) Find the absolute minimum and maximum values of $f(x,y)$ in the rectangular region R defined by $0\le x\le \dfrac{1}{2}$ and $0\le y\le 1$.

Solution: The absolute min/max can happen only at either the critical points within R or on the boundary of R:

- out of the critical points, only $(0,0)$ is part of R;
- we will need to check the vertices $(0,0)$, $(0,1)$, $(1/2,0)$, and $(1/2,1)$;
- along $x=0$ for $0\le y\le 1$:

$$g(y)=f(0,y)=8y^3 \quad \Rightarrow \quad g'(y)=24y^2$$

and $g'(y)=0$ for $y=0$ and we find again $(0,0)$;
- along $x=1/2$ for $0\le y\le 1$:

$$g(y)=f(1/2,y)=\frac{1}{8}-6y+8y^3 \quad \Rightarrow \quad g'(y)=-6+24y^2$$

and $g'(y)=0$ for $y=\pm\frac{1}{2}$; only $(1/2,1/2)$ is in R;
- along $y=0$ for $0\le x\le \frac{1}{2}$:

$$g(x)=f(x,0)=x^3 \quad \Rightarrow \quad g'(x)=3x^2$$

and $g'(x)=0$ for $x=0$ and we find again $(0,0)$;
- along $y=1$ for $0\le x\le \frac{1}{2}$:

$$g(x)=f(x,1)=x^3-12x+8 \quad \Rightarrow \quad g'(x)=3x^2-12$$

and $g'(x)=0$ for $x=\pm 2$; neither points are in R.

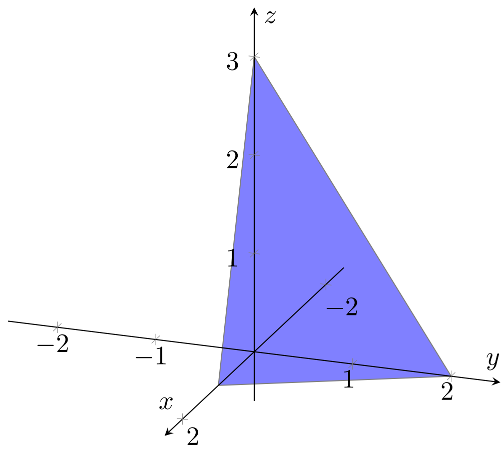
We now plug in all values of those points into f to find the absolute min/max:

x	y	$f(x,y)$	
0	0	0	
0	1	8	absolute max
$\frac{1}{2}$	0	$\frac{1}{8}$	
$\frac{1}{2}$	1	$\frac{17}{8}$	
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{15}{8}$	absolute min

7. Evaluate the following.

(a) the volume below the plane $6x + 3y + 2z = 6$ in the first octant:

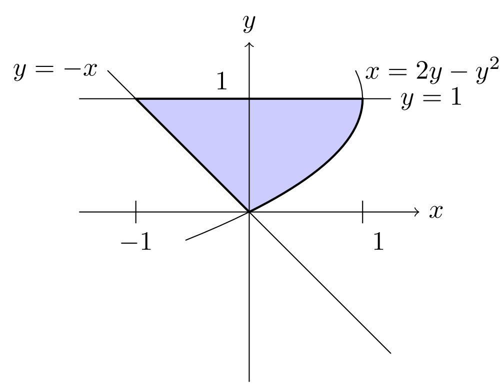
Solution:



Rewrite $2z = 6-6x-3y$ so $z = 3-3x-\frac{3}{2}y$ and the base is bounded (from setting $z = 0$) by the line: $6x + 3y = 6$, i.e. $2x + y = 2$ for $x, y \geq 0$. So we can write $0 \leq y \leq 2 - 2x$ and in x solve for the upper bound by setting $y = 0$ in the line. Then the volume is:

$$\begin{aligned} V &= \int_0^1 \int_0^{2-2x} 3 - 3x - \frac{3}{2}y \, dy \, dx = \int_0^1 \left[(3-3x)y - \frac{3}{4}y^2\right]_{y=0}^{y=2-2x} dx \\ &= \int_0^1 3(1-x)(2-2x) - \frac{3}{4}(2-2x)^2 - 0 \, dx = \int_0^1 6(1-x^2) - 3(1-x)^2 \, dx \\ &= \int_0^1 3(1-x)^2 \, dx = \Big[-(1-x)^3\Big]_0^1 = 0 + 1 = \boxed{1}. \end{aligned}$$

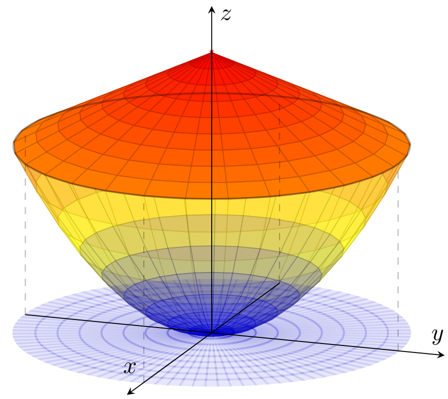
(b) the surface area of the cone $z = \sqrt{x^2 + y^2}$ above the region R bounded by the graphs of $y = -x$, $x = 2y - y^2$, $y = 0$ and $y = 1$ as sketched below:



Solution: The gradient is $\nabla z = \langle z_x, z_y \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ so noting that R is horizontally simple, we have that the surface area of the cone above R is:

$$\begin{aligned} SA &= \iint_R \sqrt{1 + z_x^2 + z_y^2} \, dA = \int_0^1 \int_{x=-y}^{x=2y-y^2} \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} \, dx \, dy = \sqrt{2} \int_0^1 \int_{-y}^{2y-y^2} dx \, dy \\ &= \sqrt{2} \int_0^1 \Big[x\Big]_{-y}^{2y-y^2} dy = \sqrt{2} \int_0^1 2y - y^2 + y \, dy = \sqrt{2} \int_0^1 3y - y^2 \, dy \\ &= \sqrt{2} \left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^1 = \sqrt{2}\left(\frac{3}{2} - \frac{1}{3} - 0\right) = \boxed{\frac{7\sqrt{2}}{6}}. \end{aligned}$$

(c) the volume of the solid bounded by the paraboloid $z = x^2+y^2$ and the inverted cone $z = 6-\sqrt{x^2+y^2}$ using polar coordinates.



Solution: The cone is above the paraboloid and for the base, we have a disk where the radius can be found using the intersection of the surfaces, i.e. set $x^2+y^2 = 6-\sqrt{x^2+y^2}$ or in polar $r^2 = 6-r$ for $r = \sqrt{x^2+y^2} \geq 0$. So $r^2+r-6 = 0$ which has for solutions $r = -3, 2$ and we keep $r = 2$. And so the volume is:

$$V = \int_0^{2\pi}\int_0^2 (6-r-r^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 6r - r^2 - r^3\, dr\, d\theta$$
$$= \int_0^{2\pi}\left[3r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 12 - \frac{8}{3} - 4 - 0\, d\theta = \frac{16}{3}\Big[\theta\Big]_0^{2\pi} = \boxed{\frac{32\pi}{3}}.$$

8. The bee population in a boxed beehive is given at each point (x, y, z) by

$$f(x, y, z) = x^2 + y^2 + xyz.$$

(a) At the point $(3, 1, 2)$, what is the unit direction of greatest decrease in population?

Solution:
$\nabla f(3, 1, 2) = \langle 2x + yz, 2y + xz, xy\rangle|_{(3,1,2)} = \langle 8, 8, 3\rangle$, so the unit direction of greatest decrease is

$$\boxed{-\frac{\nabla f(3,1,2)}{\|\nabla f(3,1,2)\|} = \langle -\frac{8}{\sqrt{137}}, -\frac{8}{\sqrt{137}}, -\frac{3}{\sqrt{137}}\rangle}.$$

(b) Find the directional derivative of f at $(3, 1, 2)$ in the direction of $\mathbf{v} = \langle 1, 2, 2\rangle$?

Solution:
The direction we consider is $\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$, so $\mathbf{u} = \langle 1/3, 2/3, 2/3\rangle$. Then

$$D_{\mathbf{u}}f(3, 1, 2) = \nabla f(3, 1, 2)\cdot \mathbf{u} = \langle 8, 8, 3\rangle \cdot \langle 1/3, 2/3, 2/3\rangle = \frac{8}{3} + \frac{16}{3} + \frac{6}{3} = \boxed{10}.$$

(c) Use the chain rule (no direct substitution) to find $\frac{df}{dt}$ in terms of t if $x(t) = 4 - t^2$, $y(t) = 3t - 2$ and $z(t) = 3t^3 - 1$.

Solution:

$$\begin{aligned}
\frac{df}{dt} &= \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle = \langle 2x + yz, 2y + xz, xy\rangle \cdot \langle -2t, 3, 9t^2\rangle \\
&= (2x + yz)(-2t) + (2y + xz)(3) + (xy)(9t^2) \\
&= -2t(2(4 - t^2) + (3t - 2)(3t^3 - 1)) + 3(2(3t - 2) + (4 - t^2)(3t^3 - 1)) + 9t^2(4 - t^2)(3t - 2) \\
&= -2t(9t^4 - 6t^3 - 2t^2 - 3t + 10) + 3(-3t^5 + 12t^3 + t^2 + 6t - 8) + 9t^2(-3t^3 + 2t^2 + 12t - 8) \\
&= -18t^5 + 12t^4 + 4t^3 + 6t^2 - 20t - 9t^5 + 36t^3 + 3t^2 + 18t - 24 - 27t^5 + 18t^4 + 108t^3 - 72t^2 \\
&= \boxed{-54t^5 + 30t^4 + 148t^3 - 63t^2 - 2t - 24}
\end{aligned}$$