Exam 2 Name: Solutions

Math 202 F01 04/03/06

1. (15 pts.) A solid hemisphere of radius 5 and uniform density lies so that its flat circular base is on the xy-plane, with center at the origin. By symmetry, it is easy to see that the center of mass $(\bar{x}, \bar{y}, \bar{z})$ has $\bar{x} = 0$ and $\bar{y} = 0$. Using *spherical coordinates*, give an expression for $\bar{z}$ involving integrals. Do *not* evaluate any integrals. Leave your answer in a form where only the evaluation of integrals remains to obtain a numerical answer. (Partial credit will be given for answers in other coordinate systems).

$$\bar{z} = \frac{M_{xy}}{\text{mass}} = \frac{\iiint_R z\,dV}{\iiint_R dV} = \frac{\int_0^{2\pi}\int_0^{\pi/2}\int_0^5 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{\int_0^{2\pi}\int_0^{\pi/2}\int_0^5 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

2. (15 pts.) Find all points satisfying the constraint $x^2 + y^2 = 1$ that maximize the function $f(x, y) = x^3 y$.

Using Lagrange multipliers: Let $g(x,y) = x^2+y^2$

$\nabla f = \lambda \nabla g \Rightarrow (3x^2y, x^3) = \lambda(2x, 2y)$, so

we have 3 equations: (1) $3x^2y = \lambda 2x$

(2) $x^3 = \lambda 2y$

(3) $x^2+y^2=1$

From (1), $\lambda = \frac{3x^2y}{2x} = \frac{3}{2}xy$ (using that $x \neq 0$ at a max, since $f(0,y)=0$ is clearly not the largest value)

Substituting this into (2), yields $x^3 = \frac{3}{2}xy^2 \cdot 2$, so $x^2 = 3y^2$ (again using $x \neq 0$ at a max)

Substituting for x^2 in (3) yields $3y^2+y^2=1 \Rightarrow 4y^2=1 \Rightarrow y^2=\frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

Substituting $y = \pm\frac{1}{2}$ in (3) yields $x^2+\frac{1}{4}=1 \Rightarrow x^2 = \frac{3}{4} \Rightarrow x = \pm\frac{\sqrt{3}}{2}$

We now have 4 candidate points $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$, and substituting them into f shows maxs are at $\pm\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

3. (15 pts.) Consider the function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2$.

(a) Find all critical points of f.

$\nabla f = \vec{0}$

$(6xy - 6x,\ 3x^2 + 3y^2 - 6y) = (0, 0)$

so $6xy - 6x = 0$ and $3x^2 + 3y^2 - 6y = 0$

$0 = 6xy - 6x = 6x(y-1)$ so $x = 0$ or $y = 1$

If $x = 0$, $3x^2 + 3y^2 - 6y = 0$ becomes $3y^2 - 6y = 0$

$3y(y-2) = 0 \Rightarrow y = 0, 2$

If $y = 1$, $3x^2 + 3y^2 - 6y = 0$ becomes $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

so critical points are $(0,0), (0,2), (-1,1), (1,1)$

(b) Use the second derivative test to determine as much as possible about the critical points you found in part (a).

$f_{xx} = 6y - 6$

$f_{xy} = f_{yx} = 6x$

$f_{yy} = 6y - 6$

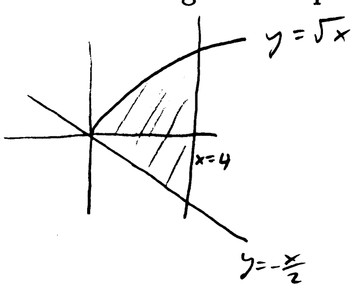
$D = \begin{vmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{vmatrix}$

point	D	f_{xx}	
$(0,0)$	$36 > 0$	$-6 < 0$	$\Rightarrow$ local min
$(0,2)$	$36 > 0$	$6 > 0$	$\Rightarrow$ local max
$(-1,1)$	$-36 < 0$		$\Rightarrow$ saddle
$(1,1)$	$-36 < 0$		$\Rightarrow$ saddle

4. (12 pts.) A metal plate is shaped like the region in the xy-plane bounded by

$$y = -x/2, \quad y = \sqrt{x}, \quad \text{and } x = 4,$$

with x, y, z measured in cm. Electric charge is distributed over the plate, with charge density $\rho(x, y) = x^2y$ coulombs/cm^2. What is the total charge on the plate? Specify units.



$$\iint_R \rho(x,y)\,dA = \int_0^4 \int_{-x/2}^{\sqrt{x}} x^2y\,dy\,dx$$

$$= \int_0^4 \left. \frac{x^2y^2}{2} \right|_{y=-\frac{x}{2}}^{\sqrt{x}} dx$$

$$= \int_0^4 \frac{x^3}{2} - \frac{x^4}{8}\,dx = \left. \frac{x^4}{8} - \frac{x^5}{40} \right|_0^4 = \frac{4^4}{8} - \frac{4^5}{40} = \frac{32}{5}$$

$= 6.4$ coulombs

5. (15 pts.) The density of food available in a fish tank is given by

$$\rho(x,y,z) = xy^2e^{10-z} \text{ calories/m}^3,$$

where x, y, z are measured in m. A particular fish is located at the point $(4,1,9)$, and is interested in swimming in whatever direction will most rapidly increase the density of food in its surroundings.

(a) In what direction should the fish swim? $\nabla\rho(4,1,9) = \left(y^2e^{10-z}, 2xye^{10-z}, -xy^2e^{10-z}\right)\Big|_{(4,1,9)}$

$= (e, 8e, -4e)$ (or $(1,8,-4)$ or $\frac{(1,8,-4)}{\sqrt{1+64+16}} = \left(\frac{1}{9}, \frac{8}{9}, \frac{-4}{9}\right)$)

(b) If the fish swims in the direction you specify in part (a), at what rate will the food density change? Specify units.

$\|\nabla\rho(4,1,9)\| = \|(e, 8e, -4e)\| = \sqrt{e^2+64e^2+16e^2} = 9e \ \frac{\text{calories}/m^3}{m}$

(c) If, due to barriers in its path, the fish is instead forced to swim in the direction given by the vector $(1,-1,0)$, at what rate will the food density it experiences change?

$\nabla\rho(4,1,9) \cdot \frac{(1,-1,0)}{\sqrt{2}} = (e, 8e, -4e)\cdot\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) = \frac{e-8e}{\sqrt{2}} = \frac{-7e}{\sqrt{2}}$

6. (10 pts.) Give the linear approximation of $f(x,y) = \sqrt{x + e^{-2y}}$ valid near the point (8,0). $L(x,y) = f(8,0) + f_x(8,0)(x-8) + f_y(8,0)(y-0)$

$f(8,0) = \sqrt{8+1} = 3$

$f_x(8,0) = \frac{1}{2}\left(x+e^{-2y}\right)^{-\frac{1}{2}}\Big|_{(8,0)} = \frac{1}{6}$

$f_y(8,0) = \frac{1}{2}\left(x+e^{-2y}\right)^{-\frac{1}{2}}\left(-2e^{-2y}\right)\Big|_{(8,0)} = -\frac{1}{3}$

$L(x,y) = 3 + \frac{1}{6}(x-8) - \frac{1}{3}(y-0)$

7. (8 pts.) Explain why $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^3 - x^2}{x^2 + y^2}$ does not exist.

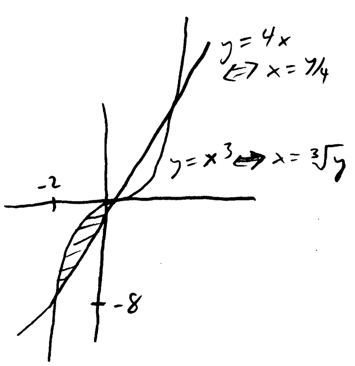
Along x-axis, $y=0$, and $\dfrac{y^3-x^2}{x^2+y^2} = \dfrac{-x^2}{x^2} = -1 \to -1$ as $x \to 0$

Along y-axis, $x=0$, and $\dfrac{y^3-x^2}{x^2+y^2} = \dfrac{y^3}{y^2} = y \to 0$ as $y \to 0$

Since $\dfrac{y^3-x^2}{x^2+y^2}$ approaches two <u>different</u> values as $(x,y)\to(0,0)$ from different directions, the limit cannot exist.

8. (10 pts.) Reverse the order of integration in the following:

$$\int_{-2}^{0}\int_{4x}^{x^3} f(x,y)\,dy\,dx$$



$$= \int_{-8}^{0}\int_{\sqrt[3]{y}}^{y/4} f(x,y)\,dx\,dy$$