MATH 631
ALGEBRA

# FINAL EXAM

FALL 2016
FAUDREE

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**Instructions:**

- This test is closed note and closed book.
- All proofs should be formal.
- Raise your hand if you have a question.

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PART I (Computational): For problems in this section, simple answers are sufficient.

(1) (6 points) List, up to isomorphism, the five groups of order 8. Indicate if they are Abelian or non-Abelian.

(2) (24 points total) Give examples of each of the following or briefly explain why none exist. Make sure you clearly identify the group or ring in question.

(a) a simple group of order 60.

(b) a cyclic nonabelian group.

(c) a ring R in which the group of units is a proper subset of the nonzero elements of $R$

(d) a ring $R$ and a left ideal $I$ of $R$ in which $I$ is not a two-sided ideal

(e) an infinite ring $R$ that is not an integral domain

(f) a ring $R$ and a prime ideal $I$ such that $I$ is not maximal

(g) a Euclidean domain that is not field

(3) (6 points) List all group homomorphisms from $\mathbb{Z}/24\mathbb{Z}$ to $\mathbb{Z}/60\mathbb{Z}$

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(4) (8 points) Determine whether the polynomial $f(x) = x^3 + x^2 + x + 2$ is reducible in each of the following rings. Briefly justify your answer.

(a) $\mathbb{Z}/2\mathbb{Z}[x]$

(b) $\mathbb{Z}/3\mathbb{Z}[x]$

(c) $\mathbb{Q}[x]$

(5) (6 points) Observe that the polynomial $3x^2 + 4x + 3 \in \mathbb{Z}/5\mathbb{Z}[x]$ factors both as $(3x+2)(x+4)$ and as $(4x+1)(2x+3)$. Explain whether or not this illustrates that $\mathbb{Z}/5\mathbb{Z}[x]$ is not a UFD.

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Part II (Long Answer): For problems in this section, formal proofs are required. Each problem is worth 10 points

(1) Prove Lagrange's Theorem: Let $G$ be a finite group and let $H \leq G$. Using first principals, prove that the order of $H$ divides the order of $G$.

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(2) Suppose that $G$ is a group and the center of $G$ has index $n$. Prove that every conjugacy class of $G$ has at most $n$ elements.

(3) Suppose that $R$ is a Euclidean domain. Prove that if $\gcd(a, b) = 1$ and $a$ divides $bc$, then $a$ divides $c$.

(4) Prove that in an integral domain, every prime element is an irreducible element.

(5) Let $R$ be a commutative ring with 1 and let $M$ be an $R$-module. Prove that $\mathrm{Hom}_R(R, M)$ and $M$ are isomorphic as left $R$-modules.