b. Show that $C^\vee$ is an affine variety when C is the cubic curve $y = x^3$.

In general, more work is needed to study $C^\vee$. In particular, the method used in the above examples breaks down when there are vertical tangents or singular points. Nevertheless, one can develop a satisfactory theory of what is called the *dual curve* $C^\vee$ of a curve $C \subset k^2$. One can also define the *dual variety* $V^\vee$ of a given irreducible variety $V \subset k^n$.

Chapter 8

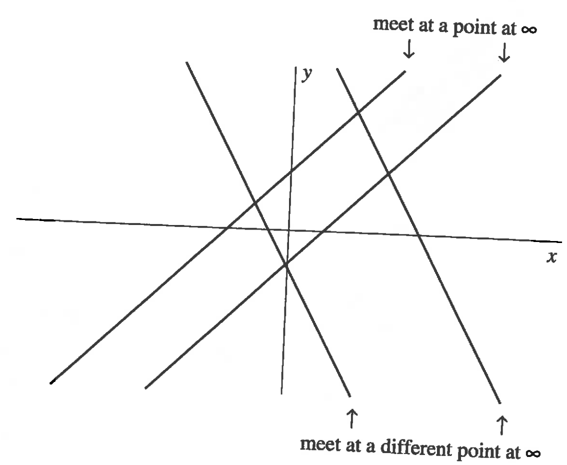
Projective Algebraic Geometry

So far, all of the varieties we have studied have been subsets of affine space k^n. In this chapter, we will enlarge k^n by adding certain "points at ∞" to create n-dimensional projective space $\mathbb{P}^n(k)$. We will then define projective varieties in $\mathbb{P}^n(k)$ and study the projective version of the algebra–geometry correspondence. The relation between affine and projective varieties will be considered in §4; in §5, we will study elimination theory from a projective point of view. By working in projective space, we will get a much better understanding of the Extension Theorem from Chapter 3. The chapter will end with a discussion of the geometry of quadric hypersurfaces.

§1. The Projective Plane

This section will study the projective plane $\mathbb{P}^2(\mathbb{R})$ over the real numbers $\mathbb{R}$. We will see that, in a certain sense, the plane $\mathbb{R}^2$ is missing some "points at ∞," and by adding them to $\mathbb{R}^2$, we will get the projective plane $\mathbb{P}^2(\mathbb{R})$. Then we will introduce *homogeneous coordinates* to give a more systematic treatment of $\mathbb{P}^2(\mathbb{R})$.

Our starting point is the observation that two lines in $\mathbb{R}^2$ intersect in a point, *except* when they are parallel. We can take care of this exception if we view parallel lines as meeting at some sort of point at ∞. As indicated by the following picture, there should be different points at ∞, depending on the direction of the lines:



To approach this more formally, we introduce an equivalence relation on lines in the plane by setting $L_1 \sim L_2$ if L_1 and L_2 are parallel. Then an equivalence class $[L]$ consists of all lines parallel to a given line L. The above discussion suggests that the we should introduce one point at ∞ for each equivalence class $[L]$. We make the following provisional definition.

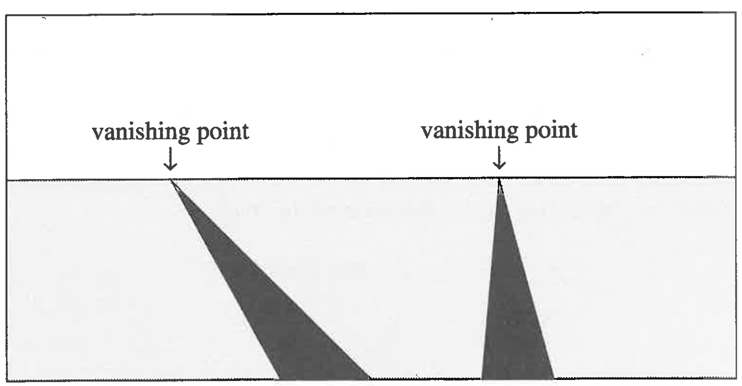
Definition 1. *The* **projective plane over $\mathbb{R}$**, *denoted* $\mathbb{P}^2(\mathbb{R})$, *is the set*

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \{\textit{one point at } \infty \textit{ for each equivalence class of parallel lines}\}.$$

We will let $[L]_\infty$ denote the common point at ∞ of all lines parallel to L. Then we call the set $\overline{L} = L \cup [L]_\infty \subset \mathbb{P}^2(\mathbb{R})$ the *projective line* corresponding to L. Note that two projective lines always meet in exactly one point: If they are not parallel, they meet at a point in $\mathbb{R}^2$; if they are parallel, they meet at their common point at ∞.

At first sight, one might expect that a line in the plane should have two points at ∞, corresponding to the two ways we can travel along the line. However, the reason why we want only one is contained in the previous paragraph: If there were two points at ∞, then parallel lines would have two points of intersection, not one. So, for example, if we parametrize the line $x = y$ via $(x, y) = (t, t)$, then we can approach its point at ∞ using either $t \to \infty$ or $t \to -\infty$.

A common way to visualize points at ∞ is to make a perspective drawing. Pretend that the earth is flat and consider a painting that shows two roads extending infinitely far in different directions:



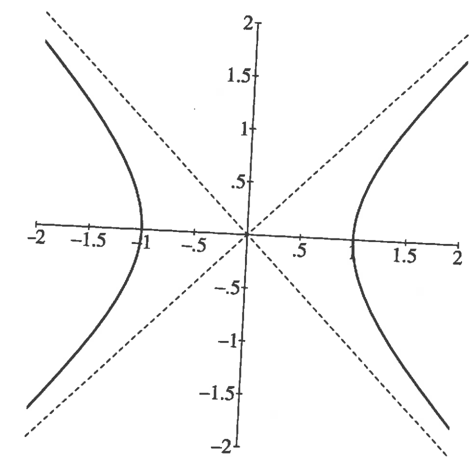
For each road, the two sides (which are parallel, but appear to be converging) meet at the same point on the horizon, which in the theory of perspective is called a *vanishing point*. Furthermore, any line parallel to one of the roads meets at the same vanishing point, which shows that the vanishing point represents the point at ∞ of these lines. The same reasoning applies to any point on the horizon, so that the horizon in the picture represents points at ∞. (Note that the horizon does not contain all of them—it is missing the point at ∞ of lines parallel to the horizon.)

The above picture reveals another interesting property of the projective plane: The points at ∞ form a special projective line, which is called the *line at* ∞. It follows that $\mathbb{P}^2(\mathbb{R})$ has the projective lines $\overline{L} = L \cup [L]_\infty$, where L is a line in $\mathbb{R}^2$, together with the line at ∞. In the exercises, you will prove that two distinct projective lines in $\mathbb{P}^2(\mathbb{R})$ determine a unique point and two distinct points in $\mathbb{P}^2(\mathbb{R})$ determine a unique projective line. Note the symmetry in these statements: When we interchange "point" and "projective line" in one, we get the other. This is an instance of the *principle of duality*, which is one of the fundamental concepts of projective geometry.

For an example of how points at ∞ can occur in other contexts, consider the parametrization of the hyperbola $x^2 - y^2 = 1$ given by the equations

$$x = \frac{1+t^2}{1-t^2},$$
$$y = \frac{2t}{1-t^2}.$$

When $t \neq \pm 1$, it is easy to check that this parametrization covers all of the hyperbola except $(-1, 0)$. But what happens when $t = \pm 1$? Here is a picture of the hyperbola:



If we let $t \to 1^-$, then the corresponding point (x, y) travels along the first quadrant portion of the hyperbola, getting closer and closer to the asymptote $x = y$. Similarly, if $t \to 1^+$, we approach $x = y$ along the fourth quadrant portion of the hyperbola. Hence, it becomes clear that $t = 1$ should correspond to the point at ∞ of the asymptote $x = y$. Similarly, one can check that $t = -1$ corresponds to the point at ∞ of $x = -y$. (In the exercises, we will give a different way to see what happens when $t = \pm 1$.)

Thus far, our discussion of the projective plane has introduced some nice ideas, but it is not entirely satisfactory. For example, it is not really clear why the line at ∞ should be called a projective line. A more serious objection is that we have no unified way of naming points in $\mathbb{P}^2(\mathbb{R})$. Points in $\mathbb{R}^2$ are specified by it coordinates, but points at ∞ are specified by lines. To avoid this asymmetry, we will introduce *homogeneous coordinates* on $\mathbb{P}^2(\mathbb{R})$.

To get homogeneous coordinates, we will need a new definition of projective space. The first step is to define an equivalence relation on nonzero points of $\mathbb{R}^3$ by setting

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$$

if there is a nonzero real number λ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. One can easily check that $\sim$ is an equivalence relation on $\mathbb{R}^3 - \{0\}$. Then we can redefine projective space as follows.

Definition 2. *$\mathbb{P}^2(\mathbb{R})$ is the set of equivalence classes of $\sim$ on $\mathbb{R}^3 - \{0\}$. Thus, we can write*

$$\mathbb{P}^2(\mathbb{R}) = (\mathbb{R}^3 - \{0\})/\sim.$$

If a triple $(x, y, z) \in \mathbb{R}^3 - \{0\}$ corresponds to a point $p \in \mathbb{P}^2(\mathbb{R})$, we say that (x, y, z) are **homogeneous coordinates** *of p.*

At this point, it is not clear that Definitions 1 and 2 give the same object, although we will see shortly that this is the case.

Homogeneous coordinates are different from the usual notion of coordinates in that they are not unique. For example, $(1, 1, 1)$, $(2, 2, 2)$, (π, π, π) and $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ are all homogeneous coordinates of the same point in projective space. But the nonuniqueness of the coordinates is not so bad since they are all multiples of one another.

As an illustration of how we can use homogeneous coordinates, let us define the notion of a projective line.

Definition 3. *Given real numbers A, B, C, not all zero, the set*

$$\{p \in \mathbb{P}^2(\mathbb{R}) : p \text{ has homogeneous coordinates } (x, y, z) \text{ with } Ax + By + Cz = 0\}$$

is called a **projective line** *of $\mathbb{P}^2(\mathbb{R})$.*

An important observation is that if $Ax + By + Cz = 0$ holds for one set (x, y, z) of homogeneous coordinates of $p \in \mathbb{P}^2(\mathbb{R})$, then it holds for *all* homogeneous coordinates of p. This is because the others can be written $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$, so that $A \cdot \lambda x + B \cdot \lambda y + C \cdot \lambda z = \lambda(Ax + By + Cz) = 0$. Later in this chapter, we will use the same idea to define varieties in projective space.

To relate our two definitions of projective plane, we will use the map

$$\mathbb{R}^2 \longrightarrow \mathbb{P}^2(\mathbb{R}) \tag{1}$$

defined by sending $(x, y) \in \mathbb{R}^2$ to the point $p \in \mathbb{P}^2(\mathbb{R})$ whose homogeneous coordinates are $(x, y, 1)$. This map has the following properties.

Proposition 4. *The map (1) is one-to-one and the complement of its image is the projective line H_∞ defined by $z = 0$.*

Proof. First, suppose that (x, y) and (x', y') map to the same point p in $\mathbb{P}^2(\mathbb{R})$. Then $(x, y, 1)$ and $(x', y', 1)$ are homogeneous coordinates of p, so that $(x, y, 1) = \lambda(x', y', 1)$ for some λ. Looking at the third coordinate, we see that $\lambda = 1$ and it follows that $(x, y) = (x', y')$.

Next, let (x, y, z) be homogeneous coordinates of a point $p \in \mathbb{P}^2(\mathbb{R})$. If $z = 0$, then p is on the projective line H_∞. On the other hand, if $z \neq 0$, then we can multiply by $1/z$ to see that $(x/z, y/z, 1)$ gives homogeneous coordinates for p. This shows that p is in the image of the map (1). We leave it as an exercise to show that the image of the map is disjoint from H_∞, and the proposition is proved. □

We will call H_∞ the *line at* ∞. It is customary (though somewhat sloppy) to identify $\mathbb{R}^2$ with its image in $\mathbb{P}^2(\mathbb{R})$, so that we can write projective space as the disjoint union

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup H_\infty.$$

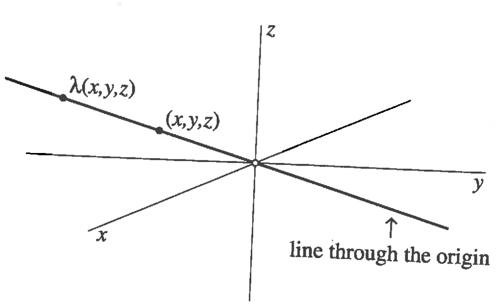
This is beginning to look familiar. It remains to show that H_∞ consists of points at ∞ in our earlier sense. Thus, we need to study how lines in $\mathbb{R}^2$ (which we will call *affine lines*) relate to projective lines. The following table tells the story:

affine line		projective line		point at ∞
$L : y = mx + b$	$\to$	$\overline{L} : y = mx + bz$	$\to$	$(1, m, 0)$
$L : x = c$	$\to$	$\overline{L} : x = cz$	$\to$	$(0, 1, 0)$

To understand this table, first consider a nonvertical affine line L defined by $y = mx + b$. Under the map (1), a point (x, y) on L maps to a point $(x, y, 1)$ of the projective line $\overline{L}$ defined by $y = mx + bz$. Thus, L can be regarded as subset of $\overline{L}$. By Proposition 4, the remaining points of $\overline{L}$ come from where it meets $z = 0$. But the equations $z = 0$ and $y = mx + bz$ clearly imply $y = mx$, so that the solutions are $(x, mx, 0)$. We have $x \neq 0$ since homogeneous coordinates never simultaneously vanish, and dividing by x shows that $(1, m, 0)$ is the unique point of $\overline{L} \cap H_\infty$. The case of vertical lines is left as an exercise.

The table shows that two lines in $\mathbb{R}^2$ meet at the same point at ∞ if and only if they are parallel. For nonvertical lines, the point at ∞ encodes the slope, and for vertical lines, there is a single (but different) point at ∞. Be sure you understand this. In the exercises, you will check that the points listed in the table exhaust all of H_∞. Consequently, H_∞ consists of a unique point at ∞ for every equivalence class of parallel lines. Then $\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup H_\infty$ shows that the projective planes of Definitions 1 and 2 are the same object.

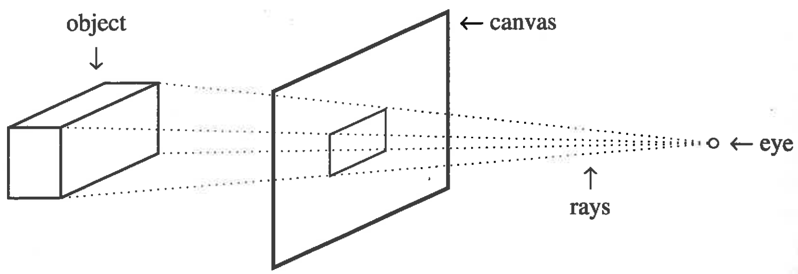
We next introduce a more geometric way of thinking about points in the projective plane. Let (x, y, z) be homogeneous coordinates of a point p in $\mathbb{P}^2(\mathbb{R})$, so that all other homogeneous coordinates for p are given by $\lambda(x, y, z)$ for $\lambda \in \mathbb{R} - \{0\}$. The crucial observation is that these points all lie on the same line through the origin in $\mathbb{R}^3$:



The requirement in Definition 2 that $(x, y, z) \neq (0, 0, 0)$ guarantees that we get a line in $\mathbb{R}^3$. Conversely, given *any* line L through the origin in $\mathbb{R}^3$, a point (x, y, z) on $L - \{0\}$ gives homogeneous coordinates for a uniquely determined point in $\mathbb{P}^2(\mathbb{R})$ [since any other point on $L - \{0\}$ is a nonzero multiple of (x, y, z)]. This shows that we have a one-to-one correspondence

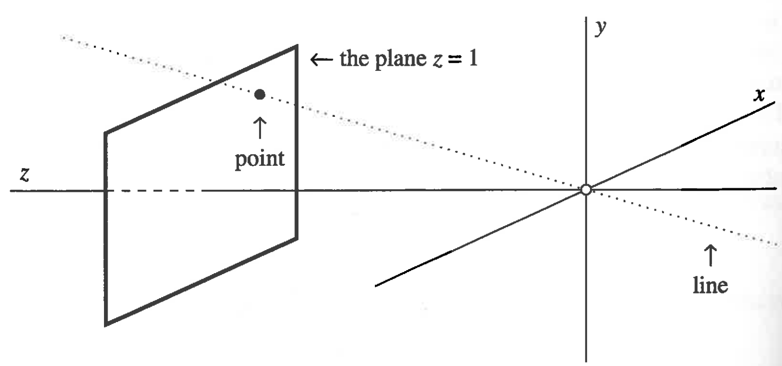
(2) $$\mathbb{P}^2(\mathbb{R}) \cong \{\text{lines through the origin in } \mathbb{R}^3\}.$$

Although it may seem hard to think of a point in $\mathbb{P}^2(\mathbb{R})$ as a line in $\mathbb{R}^3$, there is a strong intuitive basis for this indentification. We can see why by studying how to draw a 3-dimensional object on a 2-dimensional canvas. Imagine lines or rays that link our eye to points on the object. Then we draw according to where the rays intersect the canvas:



Renaissance texts on perspective would speak of the "pyramid of rays" connecting the artist's eye with the object being painted. For us, the crucial observation is that each ray hits the canvas exactly once, giving a one-to-one correspondence between rays and points on the canvas.

To make this more mathematical, we will let the "eye" be the origin and the "canvas" be the plane $z = 1$ in the coordinate system below:



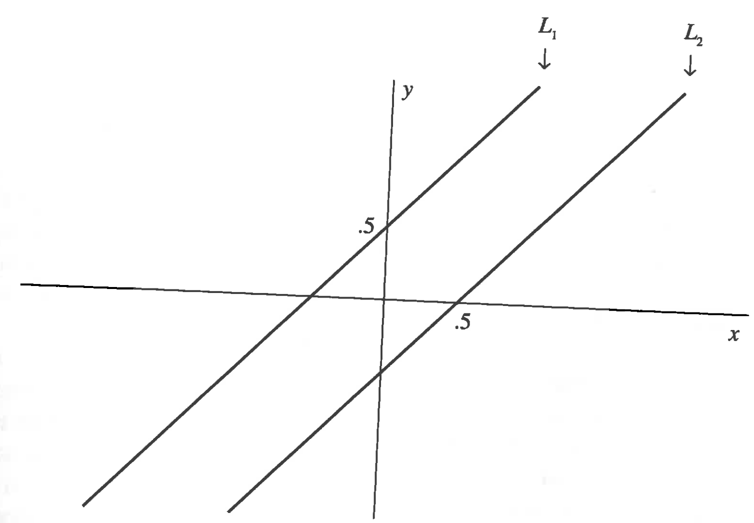
Rather than work with rays (which are half-lines), we will work with lines through the origin. Then, as the picture indicates, every point in the plane $z = 1$ determines a unique line through the origin. This one-to-one correspondence allows us to think of a *point* in the plane as a *line through the origin* in $\mathbb{R}^3$ [which by (2) is a point in $\mathbb{P}^2(\mathbb{R})$]. There are two interesting things to note about this correspondence:

- A point (x, y) in the plane gives the point $(x, y, 1)$ on our "canvas" $z = 1$. The corresponding line through the origin is a point $p \in \mathbb{P}^2(\mathbb{R})$ with homogeneous coordinates $(x, y, 1)$. Hence, the correspondence given above is *exactly* the map $\mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R})$ from Proposition 4.
- The correspondence is not onto since this method will never produce a line in the (x, y)-plane. Do you see how these lines can be thought of as the points at ∞?

In many situations, it is useful to be able to think of $\mathbb{P}^2(\mathbb{R})$ both algebraically (in terms of homogeneous coordinates) and geometrically (in terms of lines through the origin).

As the final topic in this section, we will use homogeneous coordinates to examine the line at ∞ more closely. The basic observation is that although we began with coordinates x and y, once we have homogeneous coordinates, there is nothing special about the extra coordinate z—it is no different from x or y. In particular, if we want, we could regard x and z as the original coordinates and y as the extra one.

To see how this can be useful, consider the parallel lines $L_1 : y = x + 1/2$ and $L_2 : y = x - 1/2$ in the (x, y)-plane:



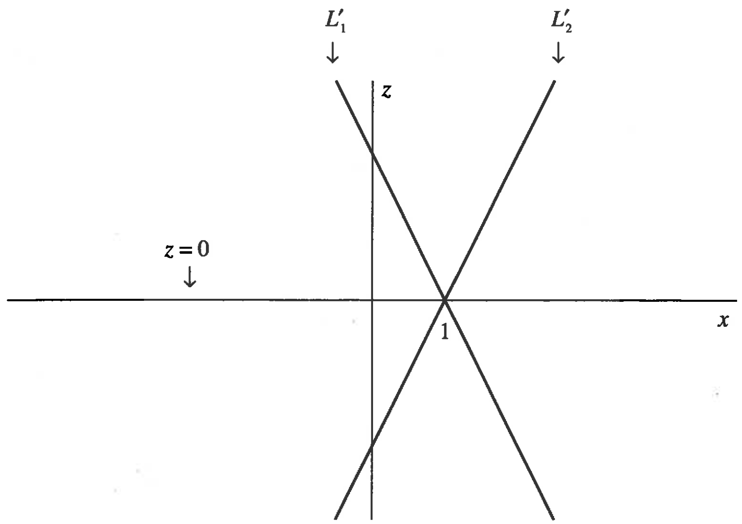
The (x, y)-Plane

We know that these lines intersect at ∞ since they are parallel. But the picture does not show their point of intersection. To view these lines at ∞, consider the projective lines

$$\overline{L}_1 : y = x + (1/2)z,$$
$$\overline{L}_2 : y = x - (1/2)z$$

determined by L_1 and L_2. Now regard x and z as the original variables. Thus, we map the (x, z)-plane $\mathbb{R}^2$ to $\mathbb{P}^2(\mathbb{R})$ via $(x, z) \mapsto (x, 1, z)$. As in Proposition 4, this map is one-to-one, and we can recover the (x, z)-plane inside $\mathbb{P}^2(\mathbb{R})$ by setting $y = 1$. If we do this with the equations of the projective lines $\overline{L}_1$ and $\overline{L}_2$, we get the lines

$L_1' : z = -2x + 2$ and $L_2' : z = 2x - 2$. This gives the following picture in the (x, z)-plane:



The (x, z)-Plane

In this picture, the x-axis is defined by $z = 0$, which is the line at ∞ as we originally set things up in Proposition 4. Note that L_1' and L_2' meet when $z = 0$, which corresponds to the fact that L_1 and L_2 meet at ∞. Thus, the above picture shows how our two lines behave as they approach the line at ∞. In the exercises, we will study what some other common curves look like at ∞.

It is interesting to compare the above picture with the perspective drawing of two roads given earlier in the section. It is no accident that the horizon in the perspective drawing represents the line at ∞. The exercises will explore this idea in more detail.

Another interesting observation is that the Euclidean notion of distance does not play a prominent role in the geometry of projective space. For example, the lines L_1 and L_2 in the (x, y)-plane are a constant distance apart, whereas L_1' and L_2' get closer and closer in the (x, z)-plane. This explains why the geometry of $\mathbb{P}^2(\mathbb{R})$ is quite different from Euclidean geometry.

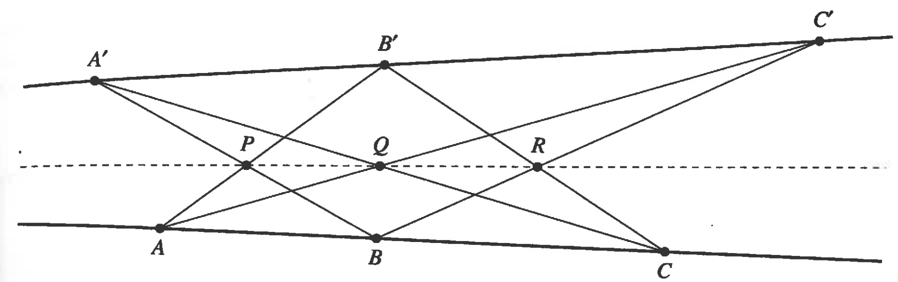
EXERCISES FOR §1

1. Using $\mathbb{P}^2(\mathbb{R})$ as given in Definition 1, we saw that the projective lines in $\mathbb{P}^2(\mathbb{R})$ are $\overline{L} = L \cup [L]_\infty$, and the line at ∞.
   a. Prove that *any* two distinct points in $\mathbb{P}^2(\mathbb{R})$ determine a unique projective line. Hint: There are three cases to consider, depending on how many of the points are points at ∞.
   b. Prove that *any* two distinct projective lines in $\mathbb{P}^2(\mathbb{R})$ meet at a unique point. Hint: Do this case-by-case.
2. There are many theorems that initially look by like theorems in the plane, but which are really theorems in $\mathbb{P}^2(\mathbb{R})$ in disguise. One classic example is Pappus's Theorem, which

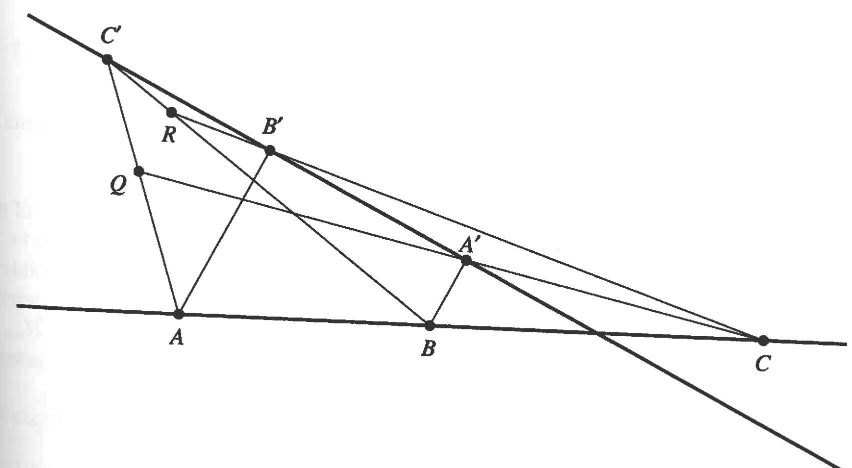
goes as follows. Suppose we have two collinear triples of points A, B, C and A', B', C'. Then let

$$P = \overline{AB'} \cap \overline{A'B},$$
$$Q = \overline{AC'} \cap \overline{A'C},$$
$$R = \overline{BC'} \cap \overline{B'C}.$$

Pappus's Theorem states that P, Q, R are always collinear points. In Exercise 8 of Chapter 6, §4, we drew the following picture to illustrate the theorem:



a. If we let the points on one of the lines go the other way, then we can get the following configuration of points and lines:



Note that P is now a point at ∞. Is Pappus's Theorem still true [in $\mathbb{P}^2(\mathbb{R})$] for the above picture?

b. By moving the point C in the picture for part b, show that you can also make Q a point at ∞. Is Pappus's Theorem still true? What line do P, Q, R lie on? Draw a picture to illustrate what happens.

If you made a purely affine version of Pappus's Theorem that took cases b and c into account, the resulting statement would be rather cumbersome. By working in $\mathbb{P}^2(\mathbb{R})$, we cover these cases simultaneously.

3. We will continue the study of the parametrization $(x, y) = ((1+t^2)/(1-t^2), 2t/(1-t^2))$ of $x^2 - y^2 = 1$ begun in the text.

a. Given t, show that (x, y) is the point where the hyperbola intersects the line of slope t going through the point $(-1, 0)$. Illustrate your answer with a picture. Hint: Use the parametrization to show that $t = y/(x + 1)$.
b. Use the answer to part a to explain why $t = \pm 1$ maps to the asymptotes of the hyperbola. Illustrate your answer with a drawing.
c. Using homogeneous coordinates, show that we can write the parametrization as

$$((1 + t^2)/(1 - t^2), 2t/(1 - t^2), 1) = (1 + t^2, 2t, 1 - t^2),$$

and use this to explain what happens when $t = \pm 1$. Does this give the same answer as part b?
d. We can also use the technique of part c to understand what happens when $t \to \infty$. Namely, in the parametrization $(x, y, z) = (1 + t^2, 2t, 1 - t^2)$, substitute $t = 1/u$. Then clear denominators (this is legal since we are using homogeneous coordinates) and let $u \to 0$. What point do you get on the hyperbola?

4. This exercise will study what the hyperbola $x^2 - y^2 = 1$ looks like at ∞.
a. Explain why the equation $x^2 - y^2 = z^2$ gives a well-defined curve C in $\mathbb{P}^2(\mathbb{R})$. Hint: See the discussion following Definition 3.
b. What are the points at ∞ on C? How does your answer relate to Exercise 3?
c. In the (x, z) coordinate system obtained by setting $y = 1$, show that C is still a hyperbola.
d. In the (y, z) coordinate system obtained by setting $x = 1$, show that C is a circle.
e. Use the parametrization of Exercise 3 to obtain a parametrization of the circle from part d.

5. Consider the parabola $y = x^2$.
a. What equation should we use to make the parabola into a curve in $\mathbb{P}^2(\mathbb{R})$?
b. How many points at ∞ does the parabola have?
c. By choosing appropriate coordinates (as in Exercise 4), explain why the parabola is tangent to the line at ∞.
d. Show that the parabola looks like a hyperbola in the (y, z) coordinate system.

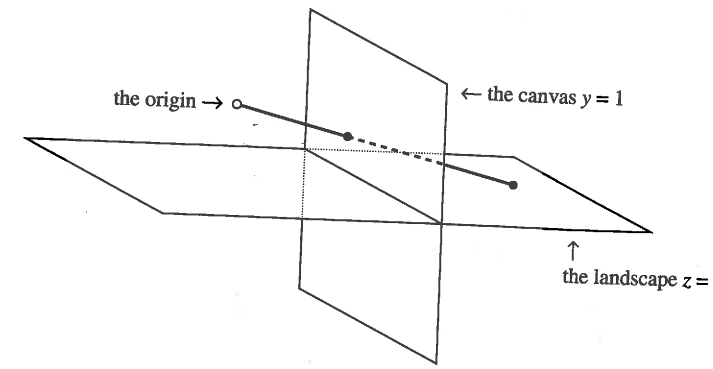
6. When we use the (x, y) coordinate system inside $\mathbb{P}^2(\mathbb{R})$, we only view a piece of the projective plane. In particular, we miss the line at ∞. As in the text, we can use the (x, z) to view the line at ∞. Show that there is exactly one point in $\mathbb{P}^2(\mathbb{R})$ that is visible in neither (x, y) nor (x, z) coordinates. How can we view what is happening at this point?

7. In the proof of Proposition 4, show that the image of the map (2) is disjoint from H_∞.

8. As in the text, the line H_∞ is defined by $z = 0$. Thus, points on H_∞ have homogeneous coordinates $(a, b, 0)$, where $(a, b) \neq (0, 0)$.
a. A vertical affine line $x = c$ gives the projective line $x = cz$. Show that this meets H_∞ at the point $(0, 1, 0)$.
b. Show that a point on H_∞ different from $(0, 1, 0)$ can be written uniquely as $(1, m, 0)$ for some real number m.

9. In the text, we viewed parts of $\mathbb{P}^2(\mathbb{R})$ in the (x, y) and (x, z) coordinate systems. In the (x, z) picture, it is natural to ask what happened to y. To see this, we will study how (x, y) coordinates look when viewed in the (x, z)-plane.
a. Show that (a, b) in the (x, y)-plane gives the point $(a/b, 1/b)$ in the (x, z)-plane.
b. Use the formula of part a to study what the parabolas $(x, y) = (t, t^2)$ and $(x, y) = (t^2, t)$ look like in the (x, z)-plane. Draw pictures of what happens in both (x, y) and (x, z) coordinates.

10. In this exercise, we will discuss the mathematics behind the perspective drawing given in the text. Suppose we want to draw a picture of a landscape, which we will assume is a horizontal plane. We will make our drawing on a canvas, which will be a vertical plane. Our eye will be a certain distance above the landscape, and to draw, we connect a point on the landscape to our eye with a line, and we put a dot where the line hits the canvas:



To give formulas for what happens, we will pick coordinates (x, y, z) so that our eye is the origin, the canvas is the plane $y = 1$, and the landscape is the plane $z = 1$ (thus, the positive z-axis points down).
a. Starting with the point $(a, b, 1)$ on the landscape, what point do we get in the canvas $y = 1$?
b. Explain how the answer to part a relates to Exercise 9. Write a brief paragraph discussing the relation between perspective drawings and the projective plane.

11. As in Definition 3, a projective line in $\mathbb{P}^2(\mathbb{R})$ is defined by an equation of the form $Ax + By + Cz = 0$, where $(A, B, C) \neq (0, 0, 0)$.
a. Why do we need to make the restriction $(A, B, C) \neq (0, 0, 0)$?
b. Show that (A, B, C) and (A', B', C') define the same projective line if and only if $(A, B, C) = \lambda(A', B', C')$ for some nonzero real number λ. Hint: One direction is easy. For the other direction, take two distinct points (a, b, c) and (a', b', c') on the line $Ax + By + Cz = 0$. Show that (a, b, c) and (a', b', c') are linearly independent and conclude that the equations $Xa + Yb + Zc = Xa' + Yb' + Zc' = 0$ have a 1-dimensional solution space for the variables X, Y, Z.
c. Conclude that the set of projective lines in $\mathbb{P}^2(\mathbb{R})$ can be identified with the set $\{(A, B, C) \in \mathbb{R}^3 : (A, B, C) \neq (0, 0, 0)\}/\sim$. This set is called the *dual projective plane* and is denoted $\mathbb{P}^2(\mathbb{R})^\vee$.
d. Describe the subset of $\mathbb{P}^2(\mathbb{R})^\vee$ corresponding to affine lines.
e. Given a point $p \in \mathbb{P}^2(\mathbb{R})$, consider the set $\tilde{p}$ of all projective lines L containing p. We can regard $\tilde{p}$ as a subset of $\mathbb{P}^2(\mathbb{R})^\vee$. Show that $\tilde{p}$ is a projective line in $\mathbb{P}^2(\mathbb{R})^\vee$. We call $\tilde{p}$ the *pencil of lines* through p.
f. The cartesian product $\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})^\vee$ has the natural subset

$$I = \{(p, L) \in \mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})^\vee : p \in L\}.$$

Show that I is described by the equation $Ax + By + Cz = 0$, where (x, y, z) are homogeneous coordinates on $\mathbb{P}^2(\mathbb{R})$ and (A, B, C) are homogeneous coordinates on the dual. We will study varieties of this type in §5.

Parts d, e, and f of this exercise illustrate how collections of naturally defined geometric objects can be given an algebraic structure.

§2. Projective Space and Projective Varieties

The construction of the real projective plane given in Definition 2 of §1 can be generalized to yield projective spaces of any dimension n over any field k. We define an equivalence relation $\sim$ on the nonzero points of k^{n+1} by setting

$$(x_0', \ldots, x_n') \sim (x_0, \ldots, x_n)$$

if there is a nonzero element $\lambda \in k$ such that $(x_0', \ldots, x_n') = \lambda(x_0, \ldots, x_n)$. Then we define projective space as follows.

Definition 1. *n-**dimensional projective space over the field** k, denoted $\mathbb{P}^n(k)$, is the set of equivalence classes of $\sim$ on $k^{n+1} - \{0\}$. Thus,*

$$\mathbb{P}^n(k) = (k^{n+1} - \{0\})/\sim.$$

*Each nonzero $(n+1)$-tuple $(x_0, \ldots, x_n) \in k^{n+1}$ defines a point p in $\mathbb{P}^n(k)$, and we say that $(x_0, \ldots, x_n)$ are **homogeneous coordinates** of p.*

Like $\mathbb{P}^2(\mathbb{R})$, each point $p \in \mathbb{P}^n(k)$ has many sets of homogeneous coordinates. For example, in $\mathbb{P}^3(\mathbb{C})$, the homogeneous coordinates $(0, \sqrt{2}, 0, i)$ and $(0, 2i, 0, -\sqrt{2})$ describe the same point since $(0, 2i, 0, -\sqrt{2}) = \sqrt{2}i(0, \sqrt{2}, 0, i)$. In general, we will write $p = (x_0, \ldots, x_n)$ to denote that $(x_0, \ldots, x_n)$ are homogeneous coordinates of $p \in \mathbb{P}^n(k)$.

As in §1, we can think of $\mathbb{P}^n(k)$ more geometrically as the set of lines through the origin in k^{n+1}. More precisely, you will show in Exercise 1 that there is a one-to-one correspondence

(1) $$\mathbb{P}^n(k) \cong \{\text{lines through the origin in } k^{n+1}\}.$$

Just as the real projective plane contains the affine plane $\mathbb{R}^2$ as a subset, $\mathbb{P}^n(k)$ contains the affine space k^n.

Proposition 2. *Let*

$$U_0 = \{(x_0, \ldots, x_n) \in \mathbb{P}^n(k) : x_0 \neq 0\}.$$

Then the map ϕ taking $(a_1, \ldots, a_n)$ in k^n to the point with homogeneous coordinates $(1, a_1, \ldots, a_n)$ in $\mathbb{P}^n(k)$ is a one-to-one correspondence between k^n and $U_0 \subset \mathbb{P}^n(k)$.

Proof. Since the first component of $\phi(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$ is nonzero, we get a map $\phi : k^n \to U_0$. We can also define an inverse map $\psi : U_0 \to k^n$ as follows. Given $p = (x_0, \ldots, x_n) \in U_0$ since $x_0 \neq 0$ we can multiply the homogeneous coordinates by the nonzero scalar $\lambda = \frac{1}{x_0}$ to obtain $p = (1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$. Then set $\psi(p) = (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in k^n$. We leave it as an exercise for the reader to show that ψ is well-defined and that ϕ and ψ are inverse mappings. This establishes the desired one-to-one correspondence. □

By the definition of U_0, we see that $\mathbb{P}^n(k) = U_0 \cup H$, where

(2) $$H = \{p \in \mathbb{P}^n(k) : p = (0, x_1, \ldots, x_n)\}.$$

If we identify U_0 with the affine space k^n, then we can think of H as the *hyperplane at infinity*. It follows from (2) that the points in H are in one-to-one correspondence with n-tuples $(x_1, \ldots, x_n)$, where two n-tuples represent the same point of H if one is a nonzero scalar multiple of the other (just ignore the first component of points in H). In other words, H is a "copy" of $\mathbb{P}^{n-1}(k)$, the projective space of one smaller dimension. Identifying U_0 with k^n and H with $\mathbb{P}^{n-1}(k)$, we can write

(3) $$\mathbb{P}^n(k) = k^n \cup \mathbb{P}^{n-1}(k).$$

To see what $H = \mathbb{P}^{n-1}(k)$ means geometrically, note that, by (1), a point $p \in \mathbb{P}^{n-1}(k)$ gives a line $L \subset k^n$ going through the origin. Consequently, in the decomposition (3), one should think of p as representing the asymptotic direction of *all* lines in k^n parallel to L. This allows us to regard p as a point at ∞ in the sense of §1, and we recover the intuitive definition of the projective space given there. In the exercises, we will give a more algebraic way of seeing how this works.

A special case worth mentioning is the projective line $\mathbb{P}^1(k)$. Since $\mathbb{P}^0(k)$ consists of a single point (this follows easily from Definition 1), letting $n = 1$ in (3) gives us

$$\mathbb{P}^1(k) = k^1 \cup \mathbb{P}^0(k) = k \cup \{\infty\},$$

where we let ∞ represent the single point of $\mathbb{P}^0(k)$. If we use (1) to think of points in $\mathbb{P}^1(k)$ as lines through the origin in k^2, then the above decomposition reflects the fact these lines are characterized by their slope (where the vertical line has slope ∞). When $k = \mathbb{C}$, it is customary to call

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$$

the *Riemann sphere*. The reason for this name will be explored in the exercises.

For completeness, we mention that there are many other copies of k^n in $\mathbb{P}^n(k)$ besides U_0. Indeed the proof of Proposition 2 may be adapted to yield the following results.

Corollary 3. *For each $i = 0, \ldots n$, let*

$$U_i = \{(x_0, \ldots, x_n) \in \mathbb{P}^n(k) : x_i \neq 0\}.$$

(i) *The points of each U_i are in one-to-one correspondence with the points of k^n.*
(ii) *The complement $\mathbb{P}^n(k) - U_i$ may be identified with $\mathbb{P}^{n-1}(k)$.*
(iii) *We have $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$.*

Proof. See Exercise 5. □

Our next goal is to extend the definition of varieties in affine space to projective space. For instance, we can ask whether it makes sense to consider $\mathbf{V}(f)$ for a polynomial $f \in k[x_0, \ldots, x_n]$. A simple example shows that some care must be taken here. For instance, in $\mathbb{P}^2(\mathbb{R})$, we can try to construct $\mathbf{V}(x_1 - x_2^2)$. The point $p = (x_0, x_1, x_2) = (1, 4, 2)$ appears to be in this set since the components of p satisfy the equation $x_1 - x_2^2 = 0$. However, a problem arises when we note that the same point p can be represented by the homogeneous coordinates $p = 2(1, 4, 2) = (2, 8, 4)$. If we substitute these components into our polynomial, we obtain $8 - 2^2 = 4 \neq 0$. We get different results depending on which homogeneous coordinates we choose.

To avoid problems of this type, we use *homogeneous* polynomials when working in $\mathbb{P}^n(k)$. From Definition 6 of Chapter 7, §1, recall that a polynomial is *homogeneous* of total degree d if every term appearing in f has total degree exactly d. The polynomial $f = x_1 - x_2^2$ in the example is not homogeneous, and this is what caused the inconsistency in the values of f on different homogeneous coordinates representing the same point. For a homogeneous polynomial, this does not happen.

Proposition 4. *Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial. If f vanishes on any one set of homogeneous coordinates for a point $p \in \mathbb{P}^n$, then f vanishes for all homogeneous coordinates of p. In particular $\mathbf{V}(f) = \{p \in \mathbb{P}^n(k) : f(p) = 0\}$ is a well-defined subset of $\mathbb{P}^n(k)$.*

Proof. Let $(a_0, \ldots, a_n)$ and $(\lambda a_0, \ldots, \lambda a_n)$ be homogeneous coordinates for $p \in \mathbb{P}^n(k)$ and assume that $f(a_0, \ldots, a_n) = 0$. If f is homogeneous of total degree k, then every term in f has the form

$$c x_0^{\alpha_0} \cdots x_n^{\alpha_n},$$

where $\alpha_0 + \cdots + \alpha_n = k$. When we substitute $x_i = \lambda a_i$, this term becomes

$$c(\lambda a_0)^{\alpha_0} \cdots (\lambda a_n)^{\alpha_n} = \lambda^k c a_0^{\alpha_0} \cdots a_n^{\alpha_n}.$$

Summing over the terms in f, we find a common factor of λ^k and, hence,

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^k f(a_0, \ldots, a_n) = 0.$$

This proves the proposition. $\square$

Notice that even if f is homogeneous, the equation $f = a$ does not make sense in $\mathbb{P}^n(k)$ when $0 \neq a \in k$. The equation $f = 0$ is special because it gives a well-defined subset of $\mathbb{P}^n(k)$. We can also consider subsets of $\mathbb{P}^n(k)$ defined by the vanishing of a *system* of homogeneous polynomials (possibly of different total degrees). The correct generalization of the affine varieties introduced in Chapter 1, §2 is as follows.

Definition 5. *Let k be a field and let $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]$ be homogeneous polynomials. We set*

$$\mathbf{V}(f_1, \ldots, f_s) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n(k) : f_i(a_0, \ldots, a_n) = 0 \textit{ for all } 1 \leq i \leq s\}.$$

We call $\mathbf{V}(f_1, \ldots, f_s)$ the **projective variety** *defined by $f_1, \ldots, f_s$.*

For example, in $\mathbb{P}^n(k)$, any homogeneous polynomial of degree 1,

$$\ell(x_0, \ldots, x_n) = c_0 x_0 + \cdots + c_n x_n,$$

defines a projective variety $\mathbf{V}(\ell)$ called a *hyperplane*. One example we have seen is the hyperplane at infinity, which was defined as $H = \mathbf{V}(x_0)$. When $n = 2$, we call $\mathbf{V}(\ell)$ a *projective line*, or more simply a *line* in $\mathbb{P}^2(k)$. Similarly, when $n = 3$, we call a hyperplane a *plane* in $\mathbb{P}^3(k)$. Varieties defined by one or more linear polynomials (homogeneous polynomials of degree 1) are called *linear varieties* in $\mathbb{P}^n(k)$. For instance, $\mathbf{V}(x_1, x_2) \subset \mathbb{P}^3(k)$ is a linear variety which is a projective line in $\mathbb{P}^3(k)$.

The projective varieties $\mathbf{V}(f)$ defined by a single homogeneous equation are known collectively as *hypersurfaces*. However, individual hypersurfaces are usually classified according to the total degree of the defining equation. Thus, if f has total degree 2 in $k[x_0, \ldots, x_n]$, we usually call $\mathbf{V}(f)$ a *quadric hypersurface*, or *quadric* for short. For instance, $\mathbf{V}(-x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^3(\mathbb{R})$ is a quadric. Similarly, hypersurfaces defined by equations of total degree 3, 4, and 5 are known as *cubics*, *quartics*, and *quintics*, respectively.

To get a better understanding of projective varieties, we need to discover what the corresponding algebraic objects are. This leads to the notion of *homogeneous ideal*, which will be discussed in §3. We will see that the entire algebra–geometry correspondence of Chapter 4 can be carried over to projective space.

The final topic we will consider in this section is the relation between affine and projective varieties. As we saw in Corollary 3, the subsets $U_i \subset \mathbb{P}^n(k)$ are copies of k^n. Thus, we can ask how affine varieties in $U_i \cong k^n$ relate to projective varieties in $\mathbb{P}^n(k)$. First, if we take a projective variety V and intersect it with one of the U_i, it makes sense to ask whether we obtain an affine variety. The answer to this question is always *yes*, and the defining equations of the variety $V \cap U_i$ may be obtained by a process called *dehomogenization*. We illustrate this by considering $V \cap U_0$. From the proof of Proposition 2, we know that if $p \in U_0$, then p has homogeneous coordinates of the form $(1, x_1, \ldots, x_n)$. If $f \in k[x_0, \ldots, x_n]$ is one of the defining equations of V, then the polynomial $g(x_1, \ldots, x_n) = f(1, x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ vanishes at every point of $V \cap U_0$. Setting $x_0 = 1$ in f produces a "dehomogenized" polynomial g which is usually nonhomogeneous. We claim that $V \cap U_0$ is precisely the affine variety obtained by dehomogenizing the equations of V.

Proposition 6. *Let $V = \mathbf{V}(f_1, \ldots, f_s)$ be a projective variety. Then $W = V \cap U_0$ can be identified with the affine variety $\mathbf{V}(g_1, \ldots, g_s) \subset k^n$, where $g_i(y_1, \ldots, y_n) = f_i(1, y_1, \ldots, y_n)$ for each $1 \leq i \leq s$.*

Proof. The comments before the statement of the proposition show that using the mapping $\psi : U_0 \to k^n$ from Proposition 2, $\psi(W) \subset \mathbf{V}(g_1, \ldots, g_s)$. On the other hand, if $(a_1, \ldots, a_n) \in \mathbf{V}(g_1, \ldots, g_s)$, then the point with homogeneous coordinates $(1, a_1, \ldots, a_n)$ is in U_0 and it satisties the equations

$$f_i(1, a_1, \ldots, a_n) = g(a_1, \ldots, a_n) = 0.$$

Thus, $\phi(\mathbf{V}(g_1, \ldots, g_s)) \subset W$. Since the mappings ϕ and ψ are inverses, the points of W are in one-to-one correspondence with the points of $\mathbf{V}(g_1, \ldots, g_s)$. $\square$

For instance, consider the projective variety

$$V = \mathbf{V}(x_1^2 - x_2x_0, x_1^3 - x_3x_0^2) \subset \mathbb{P}^3(\mathbb{R}). \tag{4}$$

To intersect V with U_0, we dehomogenize the defining equations, which gives us the affine variety

$$\mathbf{V}(x_1^2 - x_2, x_1^3 - x_3) \subset \mathbb{R}^3.$$

We recognize this as the familiar twisted cubic in $\mathbb{R}^3$.

We can also dehomogenize with respect to other variables. For example, the above proof shows that, for any projective variety $V \subset \mathbb{P}^3(\mathbb{R})$, $V \cap U_1$ can be identified with the affine variety in $\mathbb{R}^3$ defined by the equations obtained by setting $g_i(x_0, x_2, x_3) = f_i(x_0, 1, x_2, x_3)$. When we do this with the projective variety V defined in (4), we see that $V \cap U_1$ is the affine variety $\mathbf{V}(1 - x_2x_0, 1 - x_3x_0^2)$. See Exercise 9 for a general statement.

Going in the opposite direction, we can ask whether an affine variety in U_i can be written as $V \cap U_i$ in some projective variety V. The answer is again *yes*, but there is more than one way to do it, and the results can be somewhat unexpected.

One natural idea is to reverse the dehomogenization process described earlier and "homogenize" the defining equations of the affine variety. For example, consider the affine variety $W = \mathbf{V}(x_2 - x_1^3 + x_1^2)$ in $U_0 = \mathbb{R}^2$. The defining equation is not homogeneous, so we do not get a projective variety in $\mathbb{P}^2(\mathbb{R})$ directly from this equation. But we can use the extra variable x_0 to make $f = x_2 - x_1^3 + x_1^2$ homogeneous. Since f has total degree 3, we modify f so that every term has total degree 3. This leads to the homogeneous polynomial

$$f^h = x_2x_0^2 - x_1^3 + x_1^2x_0.$$

Moreover, note that dehomogenizing f^h gives back the original polynomial f in x_1, x_2. The general pattern is the same.

Proposition 7. *Let $g(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ be a polynomial of total degree d.*
(i) *Let $g = \sum_{i=0}^d g_i$ be the expansion of g as the sum of its homogeneous components, where g_i has total degree i. Then*

$$\begin{aligned} g^h(x_0, \ldots, x_n) &= \sum_{i=0}^d g_i(x_1, \ldots, x_n)x_0^{d-i} \\ &= g_d(x_1, \ldots, x_n) + g_{d-1}(x_1, \ldots, x_n)x_0 + \cdots + g_0(x_1, \ldots, x_n)x_0^d \end{aligned}$$

is a homogeneous polynomial of total degree d in $k[x_0, \ldots, x_n]$. We will call g^h the **homogenization** *of g.*
(ii) *The homogenization g can be computed using the formula*

$$g^h = x_0^d \cdot g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

(iii) *Dehomogenizing g^h yields g. That is, $g^h(1, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$.*

(iv) *Let $F(x_0, \ldots, x_n)$ be a homogeneous polynomial and let x_0^e be the highest power of x_0 dividing F. If $f = F(1, x_1, \ldots, x_n)$ is the dehomogenization of F, then $F = x_0^e \cdot f^h$.*

Proof. We leave the proof to the reader as Exercise 10. □

As a result of Proposition 7, given any affine variety $W = \mathbf{V}(g_1, \ldots, g_s) \subset k^n$, we can homogenize the defining equations of W to obtain a projective variety $V = \mathbf{V}(g_1^h, \ldots, g_s^h) \subset \mathbb{P}^n(k)$. Moreover, by part (iii) and Proposition 6, we see that $V \cap U_0 = W$. Thus, our original affine variety W is the *affine portion* of the projective variety V.

As we mentioned before, though, there are some unexpected possibilities.

Example 8. In this example, we will write the homogeneous coordinates of points in $\mathbb{P}^2(k)$ as (x, y, z). Numbering these as 0, 1, 2, we see that U_2 is the set of points with homogeneous coordinates $(x, y, 1)$, and x and y are coordinates on $U_2 \cong k^2$. Now consider the affine variety $W = \mathbf{V}(g) = \mathbf{V}(y - x^3 + x) \subset U_2$. We know that W is the affine portion $V \cap U_2$ of the projective variety $V = \mathbf{V}(g^h) = \mathbf{V}(yz^2 - x^3 + xz^2)$.

The variety V consists of W together with the points at infinity $V \cap \mathbf{V}(z)$. The affine portion W is the graph of a cubic polynomial, which is a smooth plane curve. The points at infinity, which form the complement of W in V, are given by the solutions of the equations

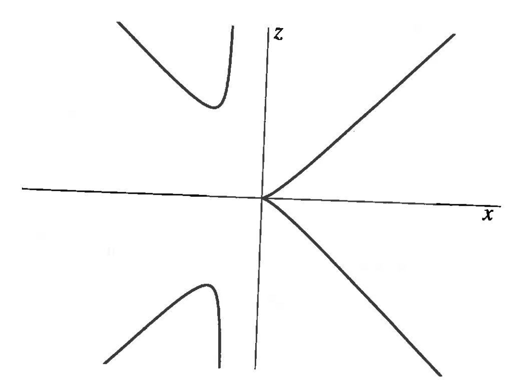
$$\begin{aligned} 0 &= yz^2 - x^3 + xz^2, \\ 0 &= z. \end{aligned}$$

It is an easy to see that the solutions are $z = x = 0$ and since we are working in $\mathbb{P}^2(k)$, we get the unique point $p = (0, 1, 0)$ in $V \cap \mathbf{V}(z)$. Thus, $V = W \cup \{p\}$. An unexpected feature of this example is the nature of the extra point p.

To see what V looks like at p, let us dehomogenize the equation of V with respect to y and study the intersection $V \cap U_1$. We find

$$W' = V \cap U_1 = \mathbf{V}(g^h(x, 1, z)) = \mathbf{V}(z^2 - x^3 + xz^2).$$

From the discussion of singularities in §4 of Chapter 3, one can easily check that p, which becomes the point $(x, z) = (0, 0) \in W'$, is a singular point on W':



Thus, even if we start from a smooth affine variety (that is, one with no singular points), homogenizing the equations and taking the corresponding projective variety may yield a more complicated geometric object. In effect, we were not "seeing the whole picture" in the original affine portion of the variety. In general, given a projective variety $V \subset \mathbb{P}^n(k)$, since $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$, we may need to consider $V \cap U_i$ for each $i = 0, \ldots, n$ to see the whole picture of V.

Our next example shows that simply homogenizing the defining equations can lead to the "wrong" projective variety.

Example 9. Consider the affine twisted cubic $W = \mathbf{V}(x_2 - x_1^2, x_3 - x_1^3)$ in $\mathbb{R}^3$. By Proposition 7, $W = V \cap U_0$ for the projective variety $V = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3) \subset \mathbb{P}^3(\mathbb{R})$. As in Example 8, we can ask what part of V we are "missing" in the affine portion W. The complement of W in V is $V \cap H$, where $H = \mathbf{V}(x_0)$ is the plane at infinity. Thus, $V \cap H = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3, x_0)$, and one easily sees that these equations reduce to

$$\begin{aligned} x_1^2 &= 0, \\ x_1^3 &= 0, \\ x_0 &= 0. \end{aligned}$$

The coordinates x_2 and x_3 are abitrary here, so $V \cap H$ is the projective line $\mathbf{V}(x_0, x_1) \subset \mathbb{P}^3(\mathbb{R})$. Thus, we have $V = W \cup \mathbf{V}(x_0, x_1)$.

Since the twisted cubic W is a curve in $\mathbb{R}^3$, our intuition suggests that it should only have a finite number of points at infinity (in the exercises, you will see that this is indeed the case). This indicates that V is probably too big; there should be a smaller projective variety V' containing W. One way to create such a V' is to homogenize other polynomials that vanish on W. For example, the parametrization (t, t^2, t^3) of W shows that $x_1x_3 - x_2^2 \in \mathbf{I}(W)$. Since $x_1x_3 - x_2^2$ is already homogeneous, we can add it to the defining equations of V to get

$$V' = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3, x_1x_3 - x_2^2) \subset V.$$

Then V' is a projective variety with the property that $V' \cap U_0 = W$, and in the exercises you will show that $V' \cap H$ consists of the single point $p = (0, 0, 0, 1)$. Thus, $V' = W \cup \{p\}$, so that we have a smaller projective variety that restricts to the twisted cubic. The difference between V and V' is that V has an extra component at infinity. In §4, we will show that V' is the smallest projective variety containing W.

In Example 9, the process by which we obtained V was completely straightforward (we homogenized the defining equations of W), yet it gave us a projective variety that was too big. This indicates that something more subtle is going on. The complete answer will come in §4, where we will learn an algorithm for finding the smallest projective variety containing $W \subset k^n \cong U_i$.

EXERCISES FOR §2

1. In this exercise, we will give a more geometric way to describe the construction of $\mathbb{P}^n(k)$. Let $\mathcal{L}$ denote the set of lines through the origin in k^{n+1}.
   a. Show that every element of $\mathcal{L}$ can be represented as the set of scalar multiples of some nonzero vector in k^{n+1}.
   b. Show that two nonzero vectors v' and v in k^{n+1} define the same element of $\mathcal{L}$ if and only if $v' \sim v$ as in Definition 1.
   c. Show that there is a one-to-one correspondence between $\mathbb{P}^n(k)$ and $\mathcal{L}$.
2. Complete the proof of Proposition 2 by showing that the mappings ϕ and ψ defined in that proof are inverses.
3. In this exercise, we will study how a lines in $\mathbb{R}^n$ relate to points at infinity in $\mathbb{P}^n(\mathbb{R})$. We will use the decomposition $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^n \cup \mathbb{P}^{n-1}(\mathbb{R})$ given in (3). Given a line L in $\mathbb{R}^n$, we can parametrize L by the formula $a + bt$, where $a \in L$ and b is a nonzero vector parallel to L. In coordinates, we write this parametrization as $(a_1 + b_1t, \ldots, a_n + b_nt)$.
   a. We can regard L as lying in $\mathbb{P}^n(\mathbb{R})$ using the homogeneous coordinates
   $$(1, a_1 + b_1t, \ldots, a_n + b_nt).$$
   To find out what happens as $t \to \pm\infty$, divide by t to obtain
   $$\left(\frac{1}{t}, \frac{a_1}{t} + b_1, \ldots, \frac{a_n}{t} + b_n\right).$$
   As $t \to \pm\infty$, what point of $H = \mathbb{P}^{n-1}(\mathbb{R})$ do you get?
   b. The line L will have many parametrizations. Show that the point of $\mathbb{P}^{n-1}(\mathbb{R})$ given by part a is the same for all parametrizations of L. Hint: Two nonzero vectors are parallel if and only if one is a scalar multiple of the other.
   c. Parts a and b show that a line L in $\mathbb{R}^n$ has a well-defined point at infinity in $H = \mathbb{P}^{n-1}(\mathbb{R})$. Show that two lines in $\mathbb{R}^n$ are parallel if and only if they have the same point at infinity.
4. When $k = \mathbb{R}$ or $\mathbb{C}$, the projective line $\mathbb{P}^1(k)$ is easy to visualize.
   a. In the text, we called $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. To see why this name is justified, use the parametrization from Exercise 6 of Chapter 1, §3 to show how the plane corresponds to the sphere minus the north pole. Then explain why we can regard $\mathbb{C} \cup \{\infty\}$ as a sphere.
   b. What common geometric object can we use to represent $\mathbb{P}^1(\mathbb{R})$? Illustrate your reasoning with a picture.
5. Prove Corollary 3.
6. This problem studies the subsets $U_i \subset \mathbb{P}^n(k)$.
   a. In $\mathbb{P}^4(k)$, identify the points that are in the subsets U_2, $U_2 \cap U_3$, and $U_1 \cap U_3 \cap U_4$.
   b. Give an identification of $\mathbb{P}^4(k) - U_2$, $\mathbb{P}^4(k) - (U_2 \cup U_3)$, and $\mathbb{P}^4(k) - (U_1 \cup U_3 \cup U_4)$ as a "copy" of another projective space.
   c. In $\mathbb{P}^4(k)$, which points are in $\bigcap_{i=0}^4 U_i$?
   d. In general, describe the subset $U_{i_1} \cap \cdots \cap U_{i_r} \subset \mathbb{P}^n(k)$, where
   $$1 \le i_1 < i_2 < \cdots < i_r \le n.$$
7. In this exercise, we will study when a nonhomogenous polynomial has a well-defined zero set in $\mathbb{P}^n(k)$. Let k be an infinite field. We will show that if $f \in k[x_0, \ldots, x_n]$ is *not* homogeneous, but f vanishes on all homogeneous coordinates of some $p \in \mathbb{P}^n(k)$, then each of the homogeneous components f_i of f (see Definition 6 of Chapter 7, §1) must vanish at p.

a. Write f as a sum of its homogeneous components $f = \sum_i f_i$. If $p = (a_0, \dots, a_1)$, then show that

$$\begin{aligned} f(\lambda a_0, \dots, \lambda a_n) &= \sum f_i(\lambda a_0, \dots, \lambda a_n) \\ &= \sum_i \lambda^i f_i(a_0, \dots, a_n). \end{aligned}$$

b. Deduce that if f vanishes for all $\lambda \neq 0 \in k$, then $f_i(a_0, \dots, a_n) = 0$ for all i.

8. By dehomogenizing the defining equations of the projective variety V, find equations for the indicated affine varieties.
   a. Let $\mathbb{P}^2(k)$ have homogeneous coordinates (x, y, z) and let $V = \mathbf{V}(x^2 + y^2 - z^2) \subset \mathbb{P}^2(\mathbb{R})$. Find equations for $V \cap U_0$, $V \cap U_2$. (Here U_0 is where $x \neq 0$ and U_2 is where $z \neq 0$.) Sketch each of these curves and think about what this says about the projective variety V.
   b. $V = \mathbf{V}(x_0x_2 - x_3x_4, x_0^2x_3 - x_1x_2^2) \subset \mathbb{P}^4$ and find equations for the affine variety $V \cap U_0 \subset k^4$. Do the same for $V \cap U_3$.
9. Let $V = \mathbf{V}(f_1, \dots, f_s)$ be a projective variety defined by homogeneous polynomials $f_i \in k[x_0, \dots, x_n]$. Show that the subset $W = V \cap U_i$ can be identified with the affine variety $\mathbf{V}(g_1, \dots, g_s) \subset k^n$ defined by the dehomogenized polynomials

$$g_j(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = f_j(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n),$$

   where the 1 is substituted for x_i in f_j. Hint: Follow the proof of Proposition 6, using Corollary 3.
10. Prove Proposition 7.
11. Using part (iv) of Proposition 7, show that if $f \in k[x_1, \dots, x_n]$ and $F \in k[x_0, \dots, x_n]$ is any homogeneous polynomial satisfying $F(1, x_1, \dots, x_n) = f(1, x_1, \dots, x_n)$, then $F = x_0^e f^h$ for some $e \geq 0$.
12. What happens if we apply the homogenization process of Proposition 7 to a polynomial g that is itself homogeneous?
13. In Example 8, we were led to consider the variety $W' = \mathbf{V}(z^2 - x^3 + xz^2) \subset k^2$. Show carefully that $(x, z) = (0, 0)$ is a singular point of W'. Hint: Use Definition 3 from Chapter 3, §4.
14. For each of the following affine varieties W, apply the homogenization process given in Proposition 7 to write $W = V \cap U_0$, where V is a projective variety. In each case identify $V - W = V \cap H$, where H is the hyperplane at infinity.
   a. $W = \mathbf{V}(y^2 - x^3 - ax - b) \subset \mathbb{R}^2$, $a, b \in \mathbb{R}$. Is the point $V \cap H$ singular here? Hint: Let the homogeneous coordinates on $\mathbb{P}^2(\mathbb{R})$ be (z, x, y), so that U_0 is where $z \neq 0$.
   ? b. $W = \mathbf{V}(x_1x_3 - x_2^2, x_1^2 - x_2) \subset \mathbb{R}^3$. Is there an extra component at infinity here?
   c. $W = \mathbf{V}(x_3^2 - x_1^2 - x_2^2) \subset \mathbb{R}^3$.
15. From Example 9, consider the twisted cubic $W = \mathbf{V}(x_2 - x_1^2, x_3 - x_1^3) \subset \mathbb{R}^3$.
   a. If we parametrize W by (t, t^2, t^3) in $\mathbb{R}^3$, show that as $t \to \pm\infty$, the point $(1, t, t^2, t^3)$ in $\mathbb{P}^3(\mathbb{R})$ approaches $(0, 0, 0, 1)$. Thus, we expect W to have one point at infinity.
   b. Now consider the projective variety

$$V' = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3, x_1x_3 - x_2^2) \subset \mathbb{P}^3(\mathbb{R}).$$

   Show that $V' \cap U_0 = W$ and that $V' \cap H = \{(0, 0, 0, 1)\}$.
   c. Let $V = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3)$ be as in Example 9. Prove that $V = V' \cup \mathbf{V}(x_0, x_1)$. This shows that V is a union of two proper projective vareities.
16. A homogeneous polynomial $f \in k[x_0, \dots, x_n]$ can also be used to define the *affine* variety $C = \mathbf{V}_a(f)$ in k^{n+1}, where the subscript denotes we are working in affine space. We call C the *affine cone* over the projective variety $V = \mathbf{V}(f) \subset \mathbb{P}^n(k)$. We will see why this is so in this exercise.

   a. Show that if C contains the point $P \neq (0, \dots, 0)$, then C contains the whole line through the origin in k^{n+1} spanned by P.
   b. Now, consider the point p in $\mathbb{P}^n$ with homogeneous coordinates P. Show that p is in the projective variety V if and only if the line through the origin determined by P is contained in C. Hint: See (1) and Exercise 1.
   c. Deduce that C is the union of the collection of lines through the origin in k^{n+1} corresponding to the points in V via (1). This explains the reason for the "cone" terminology since an ordinary cone is also a union of lines through the origin. Such a cone is given by part c of Exercise 14.
17. Homogeneous polynomials satisfy an important relation known as *Euler's Formula*. Let $f \in k[x_0, \dots, x_n]$ be homogeneous of total degree d. Then Euler's formula states that

$$\sum_{i=0}^{n} x_i \cdot \frac{\partial f}{\partial x_i} = d \cdot f.$$

   a. Verify Euler's Formula for the homogeneous polynomial $f = x_0^3 - x_1x_2^2 + 2x_1x_3^2$.
   b. Prove Euler's Formula (in the case $k = \mathbb{R}$) by considering $f(\lambda x_0, \cdots, \lambda x_n)$ as a function of λ and differentiating with respect to λ using the chain rule.
18. In this exercise, we will consider the set of hyperplanes in $\mathbb{P}^n(k)$ in greater detail.
   a. Show that two homogeneous linear polynomials,

$$\begin{aligned} 0 &= a_0x_0 + \cdots + a_nx_n, \\ 0 &= b_0x_0 + \cdots + b_nx_n, \end{aligned}$$

   define the same hyperplane in $\mathbb{P}^n(k)$ if and only if there is some $\lambda \neq 0 \in k$ such that $b_i = \lambda a_i$ for all $i = 0, \dots, n$. Hint: Generalize the argument given for Exercise 11 of §1.
   b. Show that the map sending the hyperplane with equation $a_0x_0 + \cdots + a_nx_n = 0$ to the vector $(a_0, \dots, a_n)$ gives a one-to-one correspondence

$$\phi : \{\text{hyperplanes in } \mathbb{P}^n(k)\} \longrightarrow (k^{n+1} - \{0\})/\sim,$$

   where $\sim$ is the equivalence relation of Definition 1. The set on the right is called the *dual projective space* and is denoted $\mathbb{P}^n(k)^\vee$. Geometrically, the points of $\mathbb{P}^n(k)^\vee$ are hyperplanes in $\mathbb{P}^n(k)$.
   c. Describe the subset of $\mathbb{P}^n(k)^\vee$ corresponding to the hyperplanes containing $p = (1, 0, \dots, 0)$.
19. Let k be an algebraically closed field ($\mathbb{C}$, for example). Show that every homogeneous polynomial $f(x_0, x_1)$ in two variables with coefficients in k can be completely factored into linear homogeneous polynomials in $k[x_0, x_1]$:

$$f(x_0, x_1) = \prod_{i=1}^{d} (a_ix_0 + b_ix_1),$$

   where d is the total degree of f. Hint: First, dehomogenize f.
20. In §4 of Chapter 5, we introduced the *pencil* defined by two hypersurfaces $V = \mathbf{V}(f)$, $W = \mathbf{V}(g)$. The elements of the pencil were the hypersurfaces $\mathbf{V}(f + cg)$ for $c \in k$. Setting $c = 0$, we obtain V as an element of the pencil. However, W is not (usually) an element of the pencil when it is defined in this way. To include W in the pencil, we can proceed as follows.

a. Let (a, b) be homogeneous coordinates in $\mathbb{P}^1(k)$. Show that $\mathbf{V}(af+bg)$ is well-defined in the sense that all homogeneous coordinates (a, b) for a given point in $\mathbb{P}^1(k)$ yield the same variety $\mathbf{V}(af+bg)$. Thus, we obtain a family of varieties parametrized by $\mathbb{P}^1(k)$, which is also called the *pencil* of varieties defined by V and W.
b. Show that both V and W are contained in the pencil $\mathbf{V}(af+bg)$.
c. Let $k = \mathbb{C}$. Show that every affine curve $\mathbf{V}(f) \subset \mathbb{C}^2$ defined by a polynomial f of total degree d is contained in a pencil of curves $\mathbf{V}(aF+bG)$ parametrized by $\mathbb{P}^1(\mathbb{C})$, where $\mathbf{V}(F)$ is a union of lines and G is a polynomial of degree strictly less than d. Hint: Consider the homogeneous components of f. Exercise 19 will be useful.

21. When we have a curve parametrized by $t \in k$, there are many situations where we want to let $t \to \infty$. Since $\mathbb{P}^1(k) = k \cup \{\infty\}$, this suggests that we should let our parameter space be $\mathbb{P}^1(k)$. Here are two examples of how this works.
a. Consider the parametrization $(x, y) = ((1+t^2)/(1-t^2), 2t/(1-t^2))$ of the hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$. To make this projective, we first work in $\mathbb{P}^2(\mathbb{R})$ and write the parametrization as

$$((1+t^2)/(1-t^2), 2t/(1-t^2), 1) = (1+t^2, 2t, 1-t^2)$$

(see Exercise 3 of §1). The next step is to make t projective. Given $(a, b) \in \mathbb{P}^1(\mathbb{R})$, we can write it as $(1, t) = (1, b/a)$ provided $a \neq 0$. Now substitute $t = b/a$ into the right-hand side and clear denominators. Explain why this gives a well-defined map $\mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$.
b. The twisted cubic in $\mathbb{R}^3$ is parametrized by (t, t^2, t^3). Apply the method of part a to obtain a projective parametrization $\mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^3(\mathbb{R})$ and show that the image of this map is precisely the projective variety V' from Example 9.

§3. The Projective Algebra–Geometry Dictionary

In this section, we will study the algebra–geometry dictionary for projective varieties. Our goal is to generalize the theorems from Chapter 4 concerning the $\mathbf{V}$ and $\mathbf{I}$ correspondences to the projective case, and, in particular, we will prove a projective version of the Nullstellensatz.

To begin, we note one difference between the affine and projective cases on the algebraic side of the dictionary. Namely, in Definition 5 of §2, we introduced projective varieties as the common zeros of collections of *homogeneous* polynomials. But being homogeneous is *not* preserved under the sum operation in $k[x_0, \ldots, x_n]$. For example, if we add two homogeneous polynomials of different total degrees, the sum will *never* be homogeneous. Thus, if we form the ideal $I = \langle f_1, \ldots, f_s \rangle \subset k[x_0, \ldots, x_n]$ generated by a collection of homogeneous polynomials, I will contain many *nonhomogeneous* polynomials and these would not be candidates for the defining equations of a projective variety.

Nevertheless, each element of I vanishes on all homogeneous coordinates of every point of $V = \mathbf{V}(f_1, \ldots, f_s)$. This follows because each $g \in I$ has the form

$$g = \sum_{j=1}^{s} A_j f_j \tag{1}$$

for some $A_j \in k[x_0, \ldots, x_n]$. Substituting any homogeneous coordinates of a point in V into g will yield zero since each f_i is zero there.

A more important observation concerns the homogeneous components of g. Suppose we expand each A_j as the sum of its homogeneous components:

$$A_j = \sum_{i=1}^{d} A_{ji}.$$

If we substitute these expressions into (1) and collect terms of the same total degree, it can be shown that the homogeneous components of g also lie in the ideal $I = \langle f_1, \ldots, f_s \rangle$. You will prove this claim in Exercise 2.

Thus, although I contains nonhomogeneous elements g, we see that I also contains the homogeneous components of g. This observation motivates the following definition of a special class of ideals in $k[x_0, \ldots, x_n]$.

Definition 1. *An ideal I in $k[x_0, \ldots, x_n]$ is said to be* **homogeneous** *if for each $f \in I$, the homogeneous components f_i of f are in I as well.*

Most ideals *do not* have this property. For instance, let $I = \langle y - x^2 \rangle \subset k[x, y]$. The homogeneous components of $f = y - x^2$ are $f_1 = y$ and $f_2 = -x^2$. Neither of these polynomials is in I since neither is a multiple of $y - x^2$. Hence, I is not a homogeneous ideal. However, we have the following useful characterization of when an ideal is homogeneous.

Theorem 2. *Let $I \subset k[x_0, \ldots, x_n]$ be an ideal. Then the following are equivalent:*
(i) *I is a homogeneous ideal of $k[x_0, \ldots, x_n]$.*
(ii) *$I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s$ are homogeneous polynomials.*
(iii) *A reduced Groebner basis of I (with respect to any monomial ordering) consists of homogeneous polynomials.*

Proof. The proof of (ii) $\Rightarrow$ (i) was sketched above (see also Exercise 2). To prove (i) $\Rightarrow$ (ii), let I be a homogeneous ideal. By the Hilbert Basis Theorem, we have $I = \langle F_1, \ldots, F_t \rangle$ for some polynomials $F_j \in k[x_0, \ldots, x_n]$ (not necessarily homogeneous). If we write F_j as the sum of its homogeneous components, say $F_j = \sum_i F_{ji}$, then each $F_{ji} \in I$ since I is homogeneous. Let I' be the ideal generated by the homogeneous polynomials F_{ji}. Then $I \subset I'$ since each F_j is a sum of generators of I'. On the other hand, $I' \subset I$ since each of the homogeneous components of the F_j is in I. This proves $I = I'$ and it follows that I has a basis of homogeneous polynomials. Finally, the equivalence (ii) $\Leftrightarrow$ (iii) will be covered in Exercise 3. □

As a result of Theorem 2, for any homogeneous ideal $I \subset k[x_0, \ldots, x_n]$ we may define

$$\mathbf{V}(I) = \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in I\},$$

as in the affine case. We can prove that $\mathbf{V}(I)$ is a projective variety as follows.

Proposition 3. *Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal and suppose that $I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s$ are homogeneous. Then*

$$\mathbf{V}(I) = \mathbf{V}(f_1, \ldots, f_s),$$

so that $\mathbf{V}(I)$ is a projective variety.

Proof. We leave the easy proof as an exercise. □

One way to create a homogeneous ideal is to consider the ideal generated by the defining equations of a projective variety. But there is another way that a projective variety can give us a homogeneous ideal.

Proposition 4. *Let $V \subset \mathbb{P}^n(k)$ be a projective variety and let*

$$\mathbf{I}(V) = \{f \in k[x_0, \ldots, x_n] : f(a_0, \ldots, a_n) = 0 \text{ for all } (a_0, \ldots, a_n) \in V\}.$$

(This means that f must be zero for all homogeneous coordinates of all points in V.) If k is infinite, then $\mathbf{I}(V)$ is a homogeneous ideal in $k[x_0, \ldots, x_n]$.

Proof. $\mathbf{I}(V)$ is closed under sums and closed under products by elements of $k[x_0, \ldots, x_n]$ by an argument exactly parallel to the one for the affine case. Thus, $\mathbf{I}(V)$ is an ideal in $k[x_0, \ldots, x_n]$. Now take $f \in \mathbf{I}(V)$ and fix a point $p \in V$. By assumption, f vanishes at *all* homogeneous coordinates $(a_0, \ldots, a_n)$ of p. Since k is infinite, then Exercise 7 of §2 implies that each homogeneous component f_i of f vanishes at $(a_0, \ldots, a_n)$. This shows that $f_i \in \mathbf{I}(V)$ and, hence, $\mathbf{I}(V)$ is homogeneous. □

Thus, we have all the ingredients for a dictionary relating projective varieties in $\mathbb{P}^n(k)$ and homogeneous ideals in $k[x_0, \ldots, x_n]$. The following theorem is a direct generalization of part (i) of Theorem 7 of Chapter 4, §2 (the affine ideal–variety correspondence).

Theorem 5. *Let k be an infinite field. Then the maps*

$$\textit{projective varieties} \xrightarrow{\mathbf{I}} \textit{homogeneous ideals}$$

and

$$\textit{homogeneous ideals} \xrightarrow{\mathbf{V}} \textit{projective varieties}$$

are inclusion-reversing. Furthermore, for any projective variety, we have

$$\mathbf{V}(\mathbf{I}(V)) = V,$$

so that $\mathbf{I}$ is always one-to-one.

Proof. The proof is the same as in the affine case. □

To illustrate the use of this theorem, let us show that every projective variety can be decomposed into irreducible components. As in the affine case, a variety $V \subset \mathbb{P}^n(k)$ is *irreducible* if it cannot be written as a union of two strictly smaller projective varieties.

Theorem 6. *Let k be an arbitrary field.*
(i) *Given a descending chain of projective varieties in $\mathbb{P}^n(k)$,*

$$V_1 \supset V_2 \supset V_3 \supset \cdots,$$

there is an integer N such that $V_N = V_{N+1} = \cdots$.
(ii) *Every projective variety $V \subset \mathbb{P}^n(k)$ can be written uniquely as a finite union of irreducible projective varieties*

$$V = V_1 \cup \cdots \cup V_m,$$

where $V_i \not\subset V_j$ for $i \neq j$.

Proof. Since $\mathbf{I}$ is inclusion-reversing, we get the ascending chain of homogeneous ideals

$$\mathbf{I}(V_1) \subset \mathbf{I}(V_2) \subset \mathbf{I}(V_3) \subset \cdots$$

in $k[x_0, \ldots, x_n]$. Then the Ascending Chain Condition (Theorem 7 of Chapter 2, §5) implies that $\mathbf{I}(V_N) = \mathbf{I}(V_{N+1}) = \cdots$ for some N. By Theorem 5, $\mathbf{I}$ is one-to-one and (i) follows immediately.

As in the affine case, (ii) is an immediate consequence of (i). See Theorems 2 and 4 of Chapter 4, §6. □

The relation between operations such as sums, products, and intersections of homogeneous ideals and the corresponding operations on projective varieties is also the same as in affine space. We will consider these topics in more detail in the exercises below.

We define the radical of a homogeneous ideal as usual:

$$\sqrt{I} = \{f \in k[x_0, \ldots, x_n] : f^n \in I \text{ for some } n \geq 1\}.$$

As we might hope, the radical of a homogeneous ideal is always itself homogeneous.

Proposition 7. *Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal. Then $\sqrt{I}$ is also a homogeneous ideal.*

Proof. If $f \in \sqrt{I}$, then $f^m \in I$ for some $m \geq 1$. Now decompose f into its homogeneous components

$$f = \sum_i f_i = f_{max} + \sum_{i<max} f_i,$$

where f_{max} is the nonzero homogeneous component of maximal total degree in f. Expanding the power f^m, it is easy to show that

$$(f^m)_{max} = (f_{max})^m.$$

Since I is a homogeneous ideal, $(f^m)_{max} \in I$. Hence, $(f_{max})^m \in I$, which shows that $f_{max} \in \sqrt{I}$.

If we let $g = f - f_{max} \in \sqrt{I}$ and repeat the argument, we get $g_{max} \in \sqrt{I}$. But g_{max} is also one of the homogeneous components of f. Applying this reasoning repeatedly shows that all homogeneous components of f are in $\sqrt{I}$. Since this is true for all $f \in \sqrt{I}$, Definition 1 implies that $\sqrt{I}$ is a homogeneous ideal. □

The final part of the algebra–geometry dictionary concerns what happens over an algebraically closed field k. Here, we expect an especially close relation between projective varieties and homogeneous ideals. In the affine case, the link was provided by two theorems proved in Chapter 4, the Weak Nullstellensatz and the Strong Nullstellensatz. Let us recall what these theorems tell us about an ideal $I \subset k[x_1, \ldots, x_n]$:

- (The Weak Nullstellensatz) $\mathbf{V}_a(I) = \emptyset$ in $k^n \iff I = k[x_1, \ldots, x_n]$.
- (The Strong Nullstellensatz) $\sqrt{I} = \mathbf{I}_a(\mathbf{V}_a(I))$ in $k[x_1, \ldots, x_n]$.

(To prevent confusion, we use $\mathbf{I}_a$ and $\mathbf{V}_a$ to denote the affine versions of $\mathbf{I}$ and $\mathbf{V}$.) It is natural to ask if these results extend to projective varieties and homogeneous ideals.

The answer, surprisingly, is *no*. In particular, the Weak Nullstellensatz fails for certain homogeneous ideals. To see how this can happen, consider the ideal $I = \langle x_0, \ldots, x_n \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$. Then $\mathbf{V}(I) \subset \mathbb{P}^n(\mathbb{C})$ is defined by the equations $x_0 = \cdots = x_n = 0$. The only solution is $(0, \ldots, 0)$, but this is impossible since we never allow all homogeneous coordinates to vanish simultaneously. It follows that $\mathbf{V}(I) = \emptyset$, yet $I \neq \mathbb{C}[x_0, \ldots, x_n]$.

Fortunately, $I = \langle x_0, \ldots, x_n \rangle$ is one of the few ideals for which $\mathbf{V}(I) = \emptyset$. The following projective version of the Weak Nullstellensatz describes *all* homogeneous ideals with no projective solutions.

Theorem 8 (The Projective Weak Nullstellensatz). *Let k be algebraically closed and let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. Then the following are equivalent:*

(i) $\mathbf{V}(I) \subset \mathbb{P}^n(k)$ *is empty.*

(ii) *Let G be a reduced Groebner basis for I (with respect to some monomial ordering). Then for each $0 \leq i \leq n$, there is $g \in G$ such that* $\text{LT}(g)$ *is a non-negative power of x_i.*

(iii) *For each $0 \leq i \leq n$, there is an integer $m_i \geq 0$ such that $x_i^{m_i} \in I$.*

(iv) *There is some $r \geq 1$ such that $\langle x_0, \ldots, x_n \rangle^r \subset I$.*

Proof. The ideal I gives us the projective variety $V = \mathbf{V}(I) \subset \mathbb{P}^n(k)$. In this proof, we will also work with the *affine* variety $C_V = \mathbf{V}_a(I) \subset k^{n+1}$. Note that C_V uses the same ideal I, but now we look for solutions in the affine space k^{n+1}. We call C_V the *affine cone* of V. If we interpret points in $\mathbb{P}^n(k)$ as lines through the origin in k^{n+1}, then C_V is the union of the lines determined by the points of V (see Exercise 16 of §2 for the details of how this works). In particular, C_V contains all homogeneous coordinates of the points of V.

To prove (ii) $\Rightarrow$ (i), first suppose that we have a Groebner basis where, for each i, there is $g \in G$ with $\text{LT}(g) = x_i^{m_i}$ for some $m_i \geq 0$. Then Theorem 6 of Chapter 5, §3 implies that C_V is a finite set. But suppose there is a point $p \in V$. Then *all* homogeneous coordinates of p lie in C_V. If we write these in the form $\lambda(a_0, \ldots, a_n)$, we see that there are infinitely many since k is algebraically closed and, hence, infinite. This contradiction shows that $V = \mathbf{V}(I) = \emptyset$.

Turning to (iii) $\Rightarrow$ (ii), let G be a reduced Groebner basis for I. Then $x_i^{m_i} \in I$ implies that the leading term of some $g \in G$ divides $x_i^{m_i}$, so that $\text{LT}(g)$ must be a power of x_i.

The proof of (iv) $\Rightarrow$ (iii) is obvious since $\langle x_0, \ldots, x_n \rangle^r \subset I$ implies $x_i^r \in I$ for all i.

It remains to prove (i) $\Rightarrow$ (iv). We first observe that $V = \emptyset$ implies

$$C_V \subset \{(0, \ldots, 0)\} \text{ in } k^{n+1}.$$

This follows because a nonzero point $(a_0, \ldots, a_n)$ in the affine cone C_V would give homogeneous coordinates of a point in $V \subset \mathbb{P}^n(k)$, which would contradict $V = \emptyset$. Then, applying I_a, we obtain

$$\mathbf{I}_a(\{(0, \ldots, 0)\}) \subset \mathbf{I}_a(C_V).$$

We know $\mathbf{I}_a(\{(0, \ldots, 0)\}) = \langle x_0, \ldots, x_n \rangle$ (see Exercise 7 of Chapter 4, §5) and the affine version of the Strong Nullstellensatz implies $\mathbf{I}_a(C_V) = \mathbf{I}_a(\mathbf{V}_a(I)) = \sqrt{I}$ since k is algebraically closed. Combining these facts, we conclude that

$$\langle x_0, \ldots, x_n \rangle \subset \sqrt{I}.$$

However, in Exercise 12 of Chapter 4, §3 we showed that if some ideal is contained in $\sqrt{I}$, then a power of the ideal lies in I. This completes the proof of the theorem. □

From part (ii) of the theorem, we get an algorithm for determining if a homogeneous ideal has projective solutions over an algebraically closed field. In Exercise 10, we will discuss other conditions which are equivalent to $\mathbf{V}(I) = \emptyset$ in $\mathbb{P}^n(k)$.

Once we exclude the ideals described in Theorem 8, we get the following form of the Nullstellensatz for projective varieties.

Theorem 9 (The Projective Strong Nullstellensatz). *Let k be an algebraically closed field and let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. If $V = \mathbf{V}(I)$ is a nonempty projective variety in $\mathbb{P}^n(k)$, then we have*

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

Proof. As in the proof of Theorem 8, we will work with the projective variety $V = \mathbf{V}(I) \subset \mathbb{P}^n(k)$ and its affine cone $C_V = \mathbf{V}_a(I) \subset k^{n+1}$. We first claim that

$$\mathbf{I}_a(C_V) = \mathbf{I}(V) \tag{2}$$

when $V \neq \emptyset$. To see this, suppose that $f \in \mathbf{I}_a(C_V)$. Given $p \in V$, any homogeneous coordinates of p lie in C_V, so that f vanishes at all homogeneous coordinates of p. By definition, this implies $f \in \mathbf{I}(V)$. Conversely, take $f \in \mathbf{I}(V)$. Since any nonzero point of C_V gives homogeneous coordinates for a point in V, it follows that f vanishes on $C_V - \{0\}$. It remains to show that f vanishes at the origin. Since $\mathbf{I}(V)$ is a homogeneous ideal, we know that the homogeneous components f_i of f also vanish on V. In particular, the constant term of f, which is the homogeneous component f_0 of total degree 0, must vanish on V. Since $V \neq \emptyset$, this forces $f_0 = 0$, which means that f vanishes at the origin. Hence, $f \in \mathbf{I}_a(C_V)$, and (2) is proved.

By the affine form of the Strong Nullstellensatz, we know that $\sqrt{I} = \mathbf{I}_a(\mathbf{V}_a(I))$. Then, using (2), we obtain

$$\sqrt{I} = \mathbf{I}_a(\mathbf{V}_a(I)) = \mathbf{I}_a(C_V) = \mathbf{I}(V) = \mathbf{I}(\mathbf{V}(I)),$$

which completes the proof of the theorem. □

Now that we have the Nullstellensatz, we can complete the projective ideal–variety correspondence begun in Theorem 5. A radical homogeneous ideal in $k[x_0, \dots, x_n]$ is a homogenous ideal satisfying $\sqrt{I} = I$. As in the affine case, we have a one-to-one correspondence between projective varieties and radical homogeneous ideals, provided we exclude the cases $\sqrt{I} = \langle x_0, \dots, x_n \rangle$ and $\sqrt{I} = \langle 1 \rangle$.

Theorem 10. *Let k be an algebraically closed field. If we restrict the correspondences of Theorem 5 to nonempty projective varieties and radical homogeneous ideals properly contained in* $\langle x_0, \dots, x_n \rangle$, *then*

$$\{\textit{nonempty projective varieties}\} \xrightarrow{\mathbf{I}} \left\{ \begin{array}{c} \textit{radical homogeneous ideals} \\ \textit{properly contained in } \langle x_0, \dots, x_n \rangle \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \textit{radical homogeneous ideals} \\ \textit{properly contained in } \langle x_0, \dots, x_n \rangle \end{array} \right\} \xrightarrow{\mathbf{V}} \{\textit{nonempty projective varieties}\}$$

are inclusion-reversing bijections which are inverses of each other.

Proof. First, it is an easy consequence of Theorem 8 that the only radical homogeneous ideals I with $\mathbf{V}(I) = \emptyset$ are $\langle x_0, \dots, x_n \rangle$ and $k[x_0, \dots, x_n]$. See Exercise 10 for the details. A second observation is that if I is a homogeneous ideal different from $k[x_0, \dots, x_n]$, then $I \subset \langle x_0, \dots, x_n \rangle$. This will also be covered in Exercise 9.

These observations show that the radical homogeneous ideals with $\mathbf{V}(I) \neq \emptyset$ are precisely those which satisfy $I \subsetneq \langle x_0, \dots, x_n \rangle$. Then the rest of the theorem follows as in the affine case, using Theorem 9. □

We also have a correspondence between *irreducible* projective varieties and homogeneous *prime* ideals, which will be studied in the exercises.

EXERCISES FOR §3

1. In this exercise, you will study the question of determining when a principal ideal $I = \langle f \rangle$ is homogeneous by elementary methods.
   a. Show that $I = \langle x^2y - x^3 \rangle$ is a homogeneous ideal in $k[x, y]$ without appealing to Theorem 2. Hint: Each element of I has the form $g = A \cdot (x^2y - x^3)$. Write A as the sum of its homogeneous components and use this to determine the homogeneous components of g.
   b. Show that $\langle f \rangle \subset k[x_0, \dots, x_n]$ is a homogeneous ideal if and only if f is a homogeneous polynomial without using Theorem 2.
2. This exercise gives some useful properties of the homogeneous components of polynomials.
   a. Show that if $f = \sum_i f_i$ and $g = \sum_i g_i$ are the expansions of two polynomials as the sums of their homogeneous components, then $f = g$ if and only if $f_i = g_i$ for all i.
   b. Show that if $f = \sum_i f_i$ and $g = \sum_j g_j$ are the expansions of two polynomials as the sums of their homogeneous components, then the homogeneous components h_k of the product $h = f \cdot g$ are given by $h_k = \sum_{i+j=k} f_i \cdot g_j$.
   c. Use parts a and b to carry out the proof (sketched in the text) of the implication (ii) ⇒ (i) from Theorem 2.

3. This exercise will study how the algorithms of Chapter 2 interact with homogeneous polynomials.
   a. Suppose we use the division algorithm to divide a homogeneous polynomial f by homogeneous polynomials $f_1, \dots, f_s$. This gives an expression of the form $f = a_1f_1 + \cdots + a_sf_s + r$. Prove that the quotients $a_1, \dots, a_s$ and remainder r are homogeneous polynomials (possibly zero). What is the total degree of r?
   b. If f, g are homogeneous polynomials, prove that the S-polynomial $S(f, g)$ is homogeneous.
   c. By analyzing the Buchberger algorithm, show that a homogeneous ideal has a homogeneous Groebner basis.
   d. Prove the implication (ii) ⇔ (iii) of Theorem 2.
4. Suppose that an ideal $I \subset k[x_0, \dots, x_n]$ has a basis G consisting of homogeneous polynomials.
   a. Prove that G is a Groebner basis for I with respect to lex order if and only if it is a Groebner basis for I with respect to grlex (assuming that the variables are ordered the same way).
   b. Conclude that, for a homogeneous ideal, the reduced Groebner basis for lex and grlex are the same.
5. Prove Proposition 3.
6. In this exercise we study the algebraic operations on ideals introduced in Chapter 4 for homogeneous ideals. Let $I_1, \dots, I_k$ be homogeneous ideals in $k[x_0, \dots, x_n]$.
   a. Show that the ideal sum $I_1 + \cdots + I_k$ is also homogeneous. Hint: Use Theorem 2.
   b. Show that the intersection $I_1 \cap \cdots \cap I_k$ is also a homogeneous ideal.
   c. Show that the ideal product $I_1 \cdots I_k$ is a homogeneous ideal.
7. The interaction between the algebraic operations on ideals in Exercise 6 and the corresponding operations on projective varieties is the same as in the affine case. Let $I_1, \dots, I_k$ be homogeneous ideals in $k[x_0, \dots, x_n]$ and let $V_i = \mathbf{V}(I_i)$ be the corresponding projective variety in $\mathbb{P}^n(k)$.
   a. Show that $\mathbf{V}(I_1 + \cdots + I_k) = \bigcap_{i=1}^k V_i$.
   b. Show that $\mathbf{V}(I_1 \cap \cdots \cap I_k) = \mathbf{V}(I_1 \cdot \cdots \cdot I_k) = \bigcup_{i=1}^k V_i$.
8. Let $f_1, \dots, f_s$ be homogeneous polynomials of total degrees $d_1 < d_2 \leq \cdots \leq d_s$ and let $I = \langle f_1, \dots, f_s \rangle \subset k[x_0, \dots, x_n]$
   a. Show that if g is another homogeneous polynomial of degree d_1 in I, then g must be a constant multiple of f_1. Hint: Use parts a and b of Exercise 2.
   b. More generally, show that if the total degree of g is d, then g must be an element of the ideal $I_d = \langle f_i : \deg(f_i) \leq d \rangle \subset I$.
9. This exercise will study some properties of the ideal $I_0 = \langle x_0, \dots, x_n \rangle \subset k[x_0, \dots, x_n]$.
   a. Show that every proper homogeneous ideal in $k[x_0, \dots, x_n]$ is contained in I_0.
   b. Show that the rth power I_0^r is the ideal generated by the collection of monomials in $k[x_0, \dots, x_n]$ of total degree exactly r and deduce that every homogeneous polynomial of degree $\geq r$ is in I_0^r.
   c. Let $V = \mathbf{V}(I_0) \subset \mathbb{P}^n(k)$ and $C_V = \mathbf{V}_a(I_0) \subset k^{n+1}$. Show that $\mathbf{I}_a(C_V) \neq \mathbf{I}(V)$, and explain why this does not contradict equation (2) in the text.
10. Given a homogeneous ideal $I \subset k[x_0, \dots, x_n]$, where k is algebraically closed, prove that $\mathbf{V}(I) = \emptyset$ in $\mathbb{P}^n(k)$ is equivalent to either of the following two conditions:
    (i) There is some $r \geq 1$ such that every homogeneous polynomial of total degree $\geq r$ is contained in I.
    (ii) The radical of I is either $\langle x_0, \dots, x_n \rangle$ or $k[x_0, \dots, x_n]$.
    Hint: For (i), use Exercise 9, and for (ii), use the proof of Theorem 8 to show that $\langle x_0, \dots, x_n \rangle \subset \sqrt{I}$.

11. A homogeneous ideal is said to be *prime* if it is prime as an ideal in $k[x_0, \dots, x_n]$.
    a. Show that a homogeneous ideal $I \subset k[x_0, \dots, x_n]$ is prime if and only if whenever the product of two *homogeneous* polynomials F, G satisfies $F \cdot G \in I$, then $F \in I$ or $G \in I$.
    b. Let I be a homogeneous ideal. Show that the projective variety $\mathbf{V}(I)$ is irreducible if and only if I is prime. Hint: Consider the proof of the corresponding statement in the affine case (Proposition 3 of Chapter 4, §5).
    c. Let k be algebraically closed. Show that the mappings $\mathbf{V}$ and $\mathbf{I}$ induce a one-to-one correspondence between homogeneous prime ideals in $k[x_0, \dots, x_n]$ properly contained in $\langle x_0, \dots, x_n \rangle$ and nonempty irreducible projective varieties in $\mathbb{P}^n(k)$.
12. Prove that a homogeneous prime ideal is a radical ideal in $k[x_0, \dots, x_n]$

§4. The Projective Closure of an Affine Variety

In §2, we showed that any affine variety could be regarded as the affine portion of a projective variety. Since this can be done in more than one way (see Example 9 of §2), we would like to find the *smallest* projective variety containing a given affine variety. As we will see, there is an algorithmic way to do this.

Given homogeneous coordinates $x_0, \dots, x_n$ on $\mathbb{P}^n(k)$, we have the subset $U_0 \subset \mathbb{P}^n(k)$ defined by $x_0 \neq 0$. If we identify U_0 with k^n using Proposition 2 of §2, then we get coordinates $x_1, \dots, x_n$ on k^n. As in §3, we will use $\mathbf{I}_a$ and $\mathbf{V}_a$ for the affine versions of $\mathbf{I}$ and $\mathbf{V}$.

We first discuss how to homogenize an ideal of $k[x_1, \dots, x_n]$. Given $I \subset k[x_1, \dots, x_n]$, the standard way to produce a homogeneous ideal $I^h \subset k[x_0, \dots, x_n]$ is as follows.

Definition 1. *Let I be an ideal in $k[x_1, \dots, x_n]$. We define the* **homogenization of** I *to be the ideal*

$$I^h = \langle f^h : f \in I \rangle \subset k[x_0, \dots, x_n],$$

where f^h is the homogenization of f as in Proposition 7 of §2.

Naturally enough, we have the following result.

Proposition 2. *For any ideal $I \subset k[x_1, \dots, x_n]$, the homogenization I^h is a homogeneous ideal in $k[x_0, \dots, x_n]$.*

Proof. See Exercise 1. □

Definition 1 is not entirely satisfying as it stands because it does not give us a *finite* generating set for the ideal I^h. There is a subtle point here. Given a particular finite generating set $f_1, \dots, f_s$ for $I \subset k[x_1, \dots, x_n]$, it is always true that $\langle f_1^h, \dots, f_s^h \rangle$ is a homogeneous ideal contained in I^h. However, as the following example shows, I^h can be *strictly larger* than $\langle f_1^h, \dots, f_s^h \rangle$.

Example 3. Consider $I = \langle f_1, f_2 \rangle = \langle x_2 - x_1^2, x_3 - x_1^3 \rangle$, the ideal of the affine twisted cubic in $\mathbb{R}^3$. If we homogenize f_1, f_2, then we get the ideal $J = \langle x_2x_0 - x_1^2, x_3x_0^2 - x_1^3 \rangle$ in $\mathbb{R}[x_0, x_1, x_2, x_3]$. We claim that $J \neq I^h$. To prove this, consider the polynomial

$$f_3 = f_2 - x_1 f_1 = x_3 - x_1^3 - x_1(x_2 - x_1^2) = x_3 - x_1x_2 \in I.$$

Then $f_3^h = x_0x_3 - x_1x_2$ is a homogeneous polynomial of degree 2 in I^h. Since the generators of J are also homogeneous, of degrees 2 and 3, respectively, if we had an equation $f_3^h = A_1 f_1^h + A_2 f_2^h$, then using the expansions of A_1 and A_2 into homogeneous components, we would see that f_3^h was a constant multiple of f_1^h. (See Exercise 7 of §3 for a general statement along these lines.) Since this is clearly false, we have $f_3^h \notin J$, and, thus, $J \neq I^h$.

Hence, we may ask whether there is some reasonable method for computing a finite generating set for the ideal I^h. The answer is given in the following theorem. A *graded* monomial order in $k[x_1, \dots, x_n]$ is one that orders first by total degree:

$$x^\alpha > x^\beta$$

whenever $|\alpha| > |\beta|$. Note that grlex and grevlex are graded orders, whereas lex is not.

Theorem 4. *Let I be an ideal in $k[x_1, \dots, x_n]$ and let $G = \{g_1, \dots, g_s\}$ be a Groebner basis for I with respect to a graded monomial order in $k[x_1, \dots, x_n]$. Then $G^h = \{g_1^h, \dots, g_s^h\}$ is a basis for $I^h \subset k[x_0, \dots, x_n]$.*

Proof. We will prove the theorem by showing the stronger statement that G^h is actually a Groebner basis for I^h with respect to an appropriate monomial order in $k[x_0, \dots, x_n]$.

Every monomial in $k[x_0, \dots, x_n]$ can be written

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_0^d = x^\alpha x_0^d,$$

where x^α contains no x_0 factors. Then we can extend the graded order $>$ on monomials in $k[x_1, \dots, x_n]$ to a monomial order $>_h$ in $k[x_0, \dots, x_n]$ as follows:

$$x^\alpha x_0^d >_h x^\beta x_0^e \iff x^\alpha > x^\beta \quad \text{or} \quad x^\alpha = x^\beta \text{ and } d > e.$$

In Exercise 2, you will show that this defines a monomial order in $k[x_0, \dots, x_n]$. Note that under this ordering, we have $x_i >_h x_0$ for all $i \geq 1$.

For us, the most important property of the order $>_h$ is given in the following lemma.

Lemma 5. *If $f \in k[x_1, \dots, x_n]$ and $>$ is a graded order on $k[x_1, \dots, x_n]$, then*

$$\mathrm{LM}_{>_h}(f^h) = \mathrm{LM}_{>}(f).$$

Proof of Lemma. Since $>$ is a graded order, for any $f \in k[x_1, \dots, x_n]$, $\mathrm{LM}_{>}(f)$ is one of the monomials x^α appearing in the homogeneous component of f of *maximal* total degree. When we homogenize, this term is unchanged. If $x^\beta x_0^e$ is any one of the