Calculus III; solutions to graded homework problems #7

Sec 14.6 #26. Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x, y, z) = \tan(x + 2y + 3z),\ (-5, 1, 1)$$

$\nabla f(x, y, z) = \langle \sec^2(x+2y+3z)\cdot 1, \sec^2(x+2y+3z)\cdot 2, \sec^2(x+2y+3z)\cdot 3\rangle$
$\nabla f(-5, 1, 1) = \langle \sec^2(0)\cdot 1, \sec^2(0)\cdot 2, \sec^2(0)\cdot 3\rangle = \langle 1, 2, 3\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(-5, 1, 1)| = \sqrt{14}$.

Sec 14.6 #40. Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

$$x = y^2 + z^2 - 2,\ (-1, 1, 0)$$

Let $F(x, y, z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 - 2$ is the level surface $F(x, y, z) = 2$.

$$F_x(x, y, z) = -1,\ F_y(x, y, z) = 2y,\ F_z(x, y, z) = 2z;$$

then

$$F_x(-1, 1, 0) = -1,\ F_y(-1, 1, 0) = 2,\ F_z(-1, 1, 0) = 0.$$

(a) An equation of the tangent plane is
$-1(x + 1) + 2(y - 1) + 0(z - 0) = 0$ or $-x + 2y = 3$.
(b) The normal line has symmetric equations $\frac{x+1}{-1} = \frac{y-1}{2},\ z = 0$.

Sec 14.7 #8. Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = e^{4y-x^2-y^2}$.

$$f_x = -2xe^{4y-x^2-y^2},\ f_y = (4 - 2y)e^{4y-x^2-y^2},$$

$f_{xx} = (4x^2-2)e^{4y-x^2-y^2},\ f_{xy} = -2x(4-2y)e^{4y-x^2-y^2},\ f_{yy} = (4y^2-16y+14)e^{4y-x^2-y^2}$
Then $f_x = 0$ and $f_y = 0$ implies $x = 0$ and $y = 2$, so the only critical point is $(0, 2)$. $D(0, 2) = (-2e^4)(-2e^4) = 4e^8 > 0$ and $f_{xx}(0, 2) = -2e^4 < 0$, so $f(0, 2) = e^4$ is a local maximum.

Sec 14.7 #31. Find the absolute maximum and minimum values of $f(x, y) = x^4 + y^4 - 4xy + 2$, on the set $D = \{(x, y)|0 \le x \le 3,\ 0 \le y \le 2\}$.

$f_x(x, y) = 4x^3 - 4y$, $f_y(x, y) = 4y^3 - 4x$. The critical points of f are $(0, 0)$, $(1, 1)$, $(-1, -1)$. Only $(1, 1)$ with $f(1, 1) = 0$ is inside D.

On L_1: $y = 0$, $f(x, 0) = x^4 + 2$, $0 \le x \le 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3, 0) = 83$, and its minimum at $x = 0$, $f(0, 0) = 2$. On L_2: $x = 3$, $f(3, y) = y^4 - 12y + 83$, $0 \le y \le 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$,

$f(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3, 0) = 83$. On L_3: $y = 2$, $f(x, 2) = x^4 - 8x + 18$, $0 \leq x \leq 3$, a polynomial in x which attains its minimum at $\sqrt[3]{2}$, $f(\sqrt[3]{2}, 2) = 18 - 6\sqrt[3]{2} \approx 10.4$ and its maximum at $x = 3$, $f(3, 2) = 75$. On L_4: $x = 0$, $f(0, y) = y^4 + 2$, $0 \leq y \leq 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0, 2) = 18$, and its minimum at $y = 0$, $f(0, 0) = 2$. Thus the absolute maximum of f on D is $f(3, 0) = 83$ and the absolute minimum is $f(1, 1) = 0$.