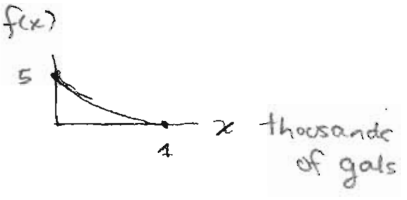
MATH 371 Name : SOLUTIONS
Quiz 4 November 14, 2008

Instructions: Round all answers to two significant digits (i.e. two decimal places, or if your answer is very small, give the first two non-zero digits following the decimal point). Also, you must show your work (integration, etc) to get full credit. There are 20 points on this quiz. Good luck.

1. (6 pts. – 2 pts. each) A Tesoro gas station is supplied with gasoline once a week. Suppose its weekly volume of sales in thousands of gallons is a random variable X with density function

$$f(x) = \begin{cases} 5(1-x)^4, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Sketch the density function for X and give an intuitive reason why its shape might be reasonable.



It is very likely the Tesoro station will sell small amount of gas (in thousands of gals) each week and selling larger amounts is increasingly less likely

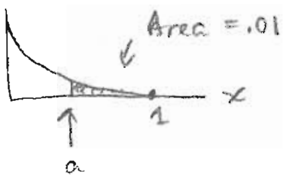
   (b) What is the expected value of X? Give a sentence to explain the meaning of $E(X)$ in terms of sales. (Use units.)

$X \sim \text{Beta}(1, 5)$ ($\alpha = 1$, $\beta = 5$) $\quad \mathbb{E}(X) = \frac{\alpha}{\alpha+\beta} = \frac{1}{6}$ $\Rightarrow$ on average ≈ 166.7 gals are sold per week

   (c) What should the capacity of the tank be so that the probability of the supplies being exhausted in a given week is .01?

$P(X>a) = .01$

find a



The tank is exhausted only if more than a 1000s of gals are sold.

$$P(X>a) = \int_a^1 5(1-x)^4\,dx$$

$$= -(1-x)^5 \Big|_a^1$$

$$= 0 - -(1-a)^5$$

$$= (1-a)^5$$

$(1-a)^5 = .01 \iff a = 1 - \sqrt[5]{.01} \approx .60$

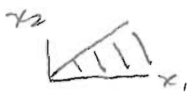
$= 600$ gals of gas

2. (7 pts.) Managers at Fred Meyer are concerned with the promptness of customer service. In particular, let X_1 denote the total time in minutes between a customer's arrival at the store and departure from the checkout counter, and X_2, the time in minutes a customer spends standing in line at the checkout counter and checking out.

The joint density function for X_1 and X_2 can be modeled by

$$f(x_1, x_2) = \begin{cases} e^{-x_1}, & 0 \le x_2 \le x_1 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

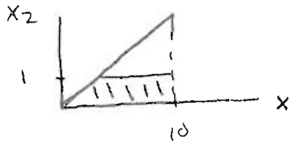
(a) (1 pt.) Sketch the domain of $f(x_1, x_2)$.



(b) (3 pts.) Describe simply in words the meaning of $P(X_1 < 10, X_2 < 1)$. Then give the value of $P(X_1 < 10,\ X_2 < 1)$.

$P(X_1 < 10, X_2 < 1)$ = probability that a customer spends fewer than 10 mins at FM and of the this time, less than one minute is spent waiting on line + checking out

Domain:



$$P(X_1 < 10, X_2 < 1) = \int_0^1\int_{x_2}^{10} e^{-x_1}\,dx_1\,dx_2$$

much easier

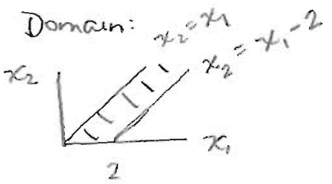
$$= \int_0^1 -e^{-x_2}\Big|_{x_2}^{10} dx_2 = \int_0^1 -e^{-10} + e^{-x_2}\,dx_2 = -e^{-10} - e^{-x_2}\Big|_0^1$$

$$= -e^{-10} - (e^{-1} - 1) = 1 - e^{-10} - e^{-1} \approx .63$$

(c) (3 pts.) Describe simply in words the meaning of $P(X_1 - X_2 < 2)$, and then compute the value of this probability.

$P(X_1 - X_2 < 2)$ = Probability that fewer than 2 minutes of the total time a customer spends at FM is spent shopping and not at the check-out counter (including waiting in line

Domain:



$$P(X_1 - X_2 < 2) = \int_0^\infty\int_{x_2}^{x_2+2} e^{-x_1}\,dx_1\,dx_2$$

$$= \int_0^\infty -e^{-x_1}\Big|_{x_2}^{x_2+2} dx_2$$

$$= \int_0^\infty -e^{-x_2-2} + e^{-x_2}\,dx_2$$

$$= \int_0^\infty e^{-x_2}(-e^{-2} + 1)\,dx_2$$

$$= (1 - e^{-2})\int_0^\infty e^{-x_2}\,dx_2$$

$$= (1 - e^{-2}) \lim_{R\to\infty} -e^{-x_2}\Big|_0^R$$

$$= (1 - e^{-2}) \lim_{R\to\infty} (-e^{-R} + 1)$$

$$= 1 - e^{-2} \approx .86$$

3. (7 pts.) Let Y_m and Y_f denote the proportions of time (out of one workday) during which one male and one female employee performed their assigned work tasks. The joint density of Y_m and Y_f can be modeled by

$$f(y_m, y_f) = \begin{cases} y_m + y_f, & 0 \le y_m \le 1,\ 0 \le y_f \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 pts.) Find the marginal density functions for Y_m and Y_f. (Include the domains in your answers.)

Marginal density for M:

$$f_m(y_m) = \int_0^1 y_m + y_f \, dy_f$$

$$= y_m y_f + \tfrac{1}{2} y_f^2 \Big|_0^1$$

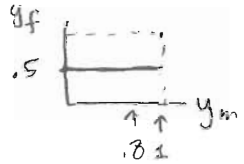
$$= \boxed{y_m + \tfrac{1}{2} \qquad 0 \le y_m \le 1}$$

Since the joint density is symmetric, as is the domain of $f(y_m, y_f)$,

$$\boxed{f_{y_f}(y_f) = y_f + \tfrac{1}{2} \qquad 0 \le y_f \le 1}$$

(b) (3 pts.) If the female employee spends exactly 50% of her time working on assigned duties, find the probability that the male employee works more than 80% of the day on assigned duties.

$$P(Y_m > .8 \mid Y_f = .5) = \int_{.8}^{1} f(y_m \mid y_f = .5)\, dy_m$$



$$f(y_m \mid y_f = .5) = \frac{f(y_m, .5)}{f_{y_f}(.5)} = \frac{y_m + .5}{.5 + .5} = y_m + .5$$

$$\Rightarrow P(Y_m > .8 \mid Y_f = .5) = \int_{.8}^{1} y_m + .5 \, dy_m$$

$$= \tfrac{1}{2} y_m^2 + \tfrac{1}{2} y_m \Big|_{.8}^{1}$$

$$= 1 - \left(\tfrac{1}{2}(.8)^2 + \tfrac{1}{2}(.8)\right) = .28$$

(c) (2 pts.) Set up, but do not evaluate, an expression to compute $P(Y_m \le .5 \mid Y_f \le .5)$. A complete answer includes precise limits of integration.

$$P(Y_m \le .5 \mid Y_f \le .5) = \frac{P(Y_m \le .5,\ Y_f \le .5)}{P(Y_f \le .5)} = \frac{\int_0^{.5}\int_0^{.5} y_m + y_f \, dy_m\, dy_f}{\int_0^{.5} y_f + \tfrac{1}{2}\, dy_f}$$