MATH253X-F01
Fall 2019

Midterm Exam 2

Name: Answer Key

Instructions. You have 60 minutes. Closed book, closed notes, no calculator. *Show all your work* in order to receive full credit.

1. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{3x^2+y^2}.$$

Either show it does not exist, or give strong evidence for suspecting it does.

Solution: Setting $x=0$ and letting $y\to 0$, we have $\lim_{y\to 0}\frac{3y^2}{y^2}=3$. Setting $y=0$ and letting $x\to 0$, we have $\lim_{x\to 0}\frac{x^2}{3x^2}=\frac{1}{3}$. Since these limits are different, $\boxed{\text{the original multivariable limit does not exist.}}$

2. The following table gives some information about a function $f(x,y)$:

(x,y)	f	f_x	f_y
$(-1,3)$	3	2	-1
$(0,1)$	-5	-1	3
$(3,4)$	1	4	-2

(a) Use the chain rule to compute $\frac{dg}{dt}(0)$ where:

$$g(t)=f(t^2-t+3,2e^{-3t}+2).$$

Solution: We have $x(t)=t^2-t+3$ and $y(t)=2e^{-3t}+2$ so $x(0)=3$ and $y(0)=4$. Therefore, $g(0)=f(3,4)$ and

$$\frac{dg}{dt}(0)=f_x(3,4)\frac{dx}{dt}(0)+f_y(3,4)\frac{dy}{dt}(0)=4\Big[2t-1\Big]_{t=0}-2\Big[2(-3)e^{-3t}\Big]_{t=0}=4(-1)-2(-6)=\boxed{8}.$$

(b) Give an equation for the linear (tangent plane) approximation to f at the point $(-1,3)$, and use it to estimate $f(-1.1,3.2)$.

Solution: The linear approximation is:

$$L(x,y)=f(-1,3)+f_x(-1,3)(x+1)+f_y(-1,3)(y-3)\quad\Leftrightarrow\quad\boxed{L(x,y)=3+2(x+1)-(y-3)}.$$

So the approximate value of $f(-1.1,3.2)$ is given by:

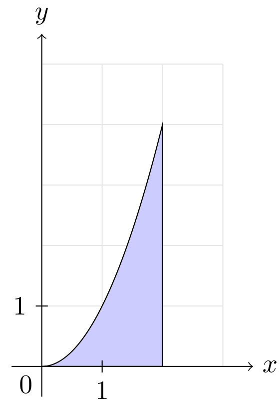
$$L(-1.1,3.2)=3+2(-1.1+1)-(3.2-3)=3-0.2-0.2=\boxed{2.6}.$$

3. Evaluate the integral

$$\int_0^4\int_{\sqrt{y}}^2 e^{\left(x^3+1\right)}\,dx\,dy$$

fully, by first drawing the region of integration, and then reversing the order of integration.

Solution: The bounds indicate that we have $\sqrt{y}\le x\le 2$ and $0\le y\le 4$. The inner bounds being in x, that means that if we drill horizontally left to right, we enter our region on the curve $x=\sqrt{y}$, i.e. $y=x^2$, and exit it on the line $x=2$. Furthermore, the shadow of the region onto the y-axis covers $[0,4]$:



So reversing the order of integration, we have:

$$\int_0^4 \int_{\sqrt{y}}^2 e^{(x^3+1)}\,dx\,dy = \int_0^2 \int_0^{x^2} e^{(x^3+1)}\,dy\,dx = \int_0^2 \Big[y\Big]_{y=0}^{y=x^2} e^{(x^3+1)}\,dx = \int_0^2 x^2 e^{(x^3+1)}\,dx = \begin{vmatrix} u = x^3+1 \\ du = 3x^2\,dx \end{vmatrix}$$

$$= \int_{x=0}^{x=2} \frac{e^u}{3}\,du = \left[\frac{e^u}{3}\right]_{x=0}^{x=2} = \left[\frac{e^{(x^3+1)}}{3}\right]_0^2 = \boxed{\frac{e^9-e}{3}}$$

4. Find and classify (using the Second Derivatives Test) all critical points of

$$f(x,y) = x^2y - 2xy + y^2 - 3y + 1.$$

Solution: The gradient is

$$\nabla f = \langle f_x, f_y \rangle = \big\langle 2xy - 2y, x^2 - 2x + 2y - 3 \big\rangle$$

is defined everywhere and when setting it to the zero vector, we get $f_x = 0 = 2y(x-1)$ for:

- either $y = 0$ then plugging into $f_y = 0$ that means $x^2 - 2x - 3 = 0$ so we get $x = 3$ or $x = -1$;
- or $x = 1$ then plugging into $f_y = 0$ that means $1 - 2 + 2y - 3 = 0$ so $y = 2$.

Hence we found three critical points: $\boxed{(3,0)\ ,\ (-1,0)\ ,\ (1,2)}$.
To classify them, we use the Second Derivatives Test:

$$f_{xx} = 2y \quad , \quad f_{yy} = 2 \quad , \quad f_{xy} = 2x - 2 \quad \Rightarrow \quad d(x,y) = 4y - 4(x-1)^2$$

- $d(3,0) = 4(0) - 4(4) < 0$ so $\boxed{\text{saddle point at } (3,0,1)}$;
- $d(-1,0) = 4(0) - 4(4) < 0$ so $\boxed{\text{saddle point at } (-1,0,1)}$;
- $d(1,2) = 4(2) - 4(0) > 0$ and $f_{xx} = 4 > 0$ so $\boxed{\text{relative minimum at } (1,2)}$.

5. Give an equation for the tangent plane to the surface

$$\frac{xy}{y+z} + e^{-z}\ln(x+2y) = 3$$

at the point $(3,-1,0)$.

Solution: Let $F(x,y,z) = \dfrac{xy}{y+z} + e^{-z}\ln(x+2y)$. Then we find

$$\begin{aligned}
\nabla F(x,y,z) &= \left\langle \frac{y}{y+z} + \frac{e^{-z}}{x+2y}, \frac{x(y+z)-xy(1)}{(y+z)^2} + \frac{2e^{-z}}{x+2y}, \frac{-xy}{(y+z)^2} - e^{-z}\ln(x+2y) \right\rangle \\
&= \left\langle \frac{y}{y+z} + \frac{e^{-z}}{x+2y}, \frac{xz}{(y+z)^2} + \frac{2e^{-z}}{x+2y}, \frac{-xy}{(y+z)^2} - e^{-z}\ln(x+2y) \right\rangle \\
\Rightarrow \quad F(3,-1,0) &= \left\langle \frac{-1}{-1+0} + \frac{1}{3-2}, \frac{3(0)}{(-1+0)^2} + \frac{2}{3-2}, \frac{-3(-1)}{(-1+0)^2} - \ln(3-2) \right\rangle = \langle 2,2,3\rangle
\end{aligned}$$

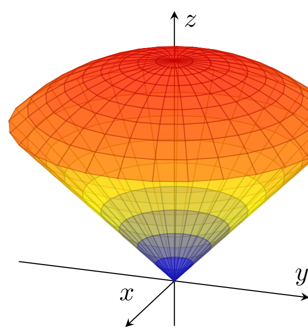
The tangent plane is thus given by

$$2(x-3) + 2(y+1) + 3(z-0) = 0,$$

or

$$\boxed{2x + 2y + 3z = 4}.$$

6. Use polar coordinates to find the volume of the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the top half of the sphere $x^2 + y^2 + z^2 = 6$.



Solution: If we solve for z in the top half of the sphere, we have $z = \sqrt{6 - x^2 - y^2}$ or using polar $z = \sqrt{6 - r^2}$ and that is our top surface whereas the cone $z = \sqrt{x^2+y^2}$ i.e. using polar $z = r$ (for $r \geq 0$) is on the bottom. The base or shadow R in the xy-plane is a disk with radius satisfying

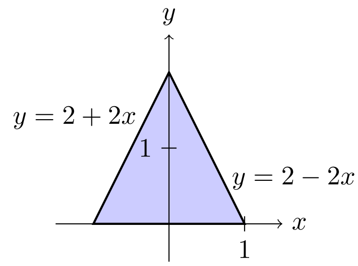
$$\sqrt{6-r^2} = r \quad \Longrightarrow \quad 6 - r^2 = r^2 \quad \Longrightarrow \quad r^2 = 3$$

So here $r = \sqrt{3}$ and the volume is:

$$\begin{aligned}
V &= \iint_R \sqrt{6 - x^2 - y^2} - \sqrt{x^2+y^2}\, dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}} \left[\sqrt{6-r^2} - r\right] r\, dr\, d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\sqrt{3}} r\sqrt{6-r^2} - r^2\, dr\right) \\
&= \Big[\theta\Big]_0^{2\pi} \left[-\frac{1}{2}\left(\frac{2}{3}\right)(6-r^2)^{\frac{3}{2}} - \frac{r^3}{3}\right]_0^{\sqrt{3}} \\
&= 2\pi\left[-\frac{1}{3}(3\sqrt{3}) - \frac{3\sqrt{3}}{3} + \frac{1}{3}(6\sqrt{6}) + 0\right] \\
&= \boxed{4\pi(\sqrt{6} - \sqrt{3})}.
\end{aligned}$$

7. A flat triangular plate is bounded by the lines $y = 2 - 2x$, $y = 2 + 2x$ and the x-axis, where x, y are in m. The mass density is given by

$$\rho(x, y) = y^2 \ \ \text{kg/m}^2.$$



From the symmetry of the plate and the density, you can see that the center of mass of the plate must be on the y-axis, so $\bar{x} = 0$.

(a) Give an expression involving integrals for $\bar{y}$, including appropriate limits of integration.

Solution: Setting up the integrals is easier in $dx\ dy$ since it requires a split in $dy\ dx$. Drilling horizontally left to right, we always enter the plate on $y = 2 + 2x$, that is $x = \frac{y}{2} - 1$ and we always exit the plate on $y = 2 - 2x$, that is $x = 1 - \frac{y}{2}$. The projection of the plate onto the y-axis covers $[0, 2]$. So we have:

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\ dA}{\iint_R \rho(x,y)\ dA} \implies \boxed{\bar{y} = \frac{\int_0^2 \int_{\frac{y}{2}-1}^{1-\frac{y}{2}} y^3\ dx\ dy}{\int_0^2 \int_{\frac{y}{2}-1}^{1-\frac{y}{2}} y^2\ dx\ dy}}$$

(b) The total mass of the plate is $m = \dfrac{4}{3}$ kg. Use this to calculate $\bar{y}$.

Solution:

$$\begin{aligned}
M_x &= \int_0^2 \int_{\frac{y}{2}-1}^{1-\frac{y}{2}} y^3\ dx\ dy = \int_0^2 \left[xy^3\right]_{x=\frac{y}{2}-1}^{x=1-\frac{y}{2}} dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \left(\frac{y}{2} - 1\right)\right) y^3\ dy = \int_0^2 (2-y)y^3\ dy \\
&= \begin{vmatrix} u = 2-y & du = -dy \\ dv = y^3\ dy & v = \frac{y^4}{4} \end{vmatrix} = \left[\frac{(2-y)y^4}{4}\right]_0^2 - \int_0^2 -\frac{y^4}{4}\ dy \\
&= 0 - 0 + \left[\frac{y^5}{20}\right]_0^2 = \frac{32}{20} - 0 = \frac{8}{5}
\end{aligned}$$

$$\Rightarrow \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{8}{5}}{\frac{4}{3}} = \frac{8}{5}\left(\frac{3}{4}\right) \quad \Rightarrow \quad \boxed{\bar{y} = \frac{6}{5}\ \text{m}}$$

8. Use Lagrange multipliers to find the maximum product of two positive numbers satisfying $x^2 + y = 6$.

Solution: We have that our objective function is the product so $f(x, y) = xy$ and the constraint is $g(x, y) = x^2 + y = 6$. Therefore,

$$\nabla f = \lambda \nabla g \implies \langle y, x \rangle = \lambda \langle 2x, 1 \rangle \implies \begin{cases} y = 2\lambda x \\ x = \lambda \end{cases}$$

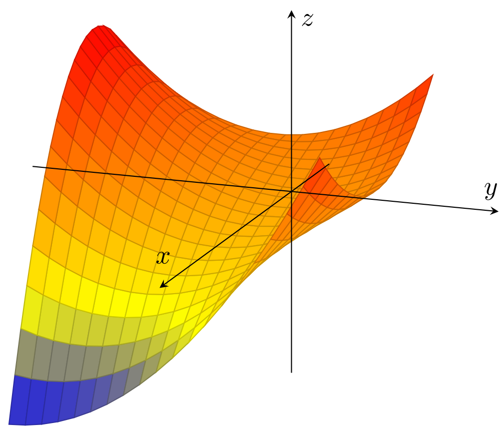
Substituting $\lambda = x$ in the first equation, we get: $y = 2x^2$. Now plugging that into the constraint:

$$x^2 + 2x^2 = 6 \quad \Rightarrow \quad 3x^2 = 6 \quad \Rightarrow \quad x^2 = 2$$

and we have a restriction for positive numbers so $x = \sqrt{2}$ and thus $y = 2x^2 = 4$. This in turns means that the maximum product:

$$f_{\max} = f(\sqrt{2}, 4) = \boxed{4\sqrt{2}}.$$

9. Let $f(x, y) = x^2y - x + y^2$.



(a) Compute the directional derivative of f when moving in the direction of $-\mathbf{j}$ when you are at the point $(1, -1)$. Interpret your result in terms of change in values of f.

Solution: We have that:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle 2xy - 1, x^2 + 2y \right\rangle \quad \Longrightarrow \quad \nabla f(1, -1) = \left\langle 2(1)(-1) - 1, 1^2 + 2(-1) \right\rangle = \langle -3, -1 \rangle .$$

Note that $-\mathbf{j}$ is already a unit vector so the directional derivative is:

$$D_{-\mathbf{j}} f(1, -1) = \nabla f(1, -1) \cdot (-\mathbf{j}) = \langle -3, -1 \rangle \cdot \langle 0, -1 \rangle = \boxed{1}.$$

Since the directional derivative is positive, $\boxed{\text{values of } f \text{ will increase in the direction of } -\mathbf{j} \text{ from } (1, -1)}$.

(b) Give the direction and magnitude of maximum decrease of f when at the point $(1, -1)$.

Solution: Direction of maximum decrease will be opposite the gradient and magnitude will be its norm.

$$\boxed{\text{direction: } \langle 3, 1 \rangle \qquad , \qquad \text{magnitude: } \sqrt{10}}$$

(c) Fully set up bounds and integrand for computing the surface area of f over the region $[-1, 2] \times [-2, 1]$. DO NOT EVALUATE.

Solution:

$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA \quad \Rightarrow \quad \boxed{SA = \int_{-1}^{2} \int_{-2}^{1} \sqrt{1 + (2xy - 1)^2 + (x^2 + 2y)^2}\, dy\, dx}$$