√ #8 Solutions

Let $\phi : \mathbb{Z}_{18} \to \mathbb{Z}_{12}$ be the homomorphism where $\phi(1) = 10$.

a. Find the Kernel K of ϕ

$\phi(0) = 0$

$\phi(6) = 10(6) \bmod 12 = 60 \bmod 12 = 0$

$\phi(12) = 10(12) \bmod 12 = 0$

$K = \{0, 6, 12\}$

b. List the cosets in $\mathbb{Z}_{18}/K$, showing the elements in each coset.

$0+K = \{0, 6, 12\}$ $\quad$ $3+K = \{3, 9, 15\}$

$1+K = \{1, 7, 13\}$ $\quad$ $4+K = \{4, 10, 16\}$

$2+K = \{2, 8, 14\}$ $\quad$ $5+K = \{5, 11, 17\}$

c. Find the group $\phi(\mathbb{Z}_{18})$.

$\phi(\mathbb{Z}_{18}) = \{0, 2, 4, 6, 8, 10\}$

d. Give the correspondence between $\mathbb{Z}_{18}/K$ and $\phi(\mathbb{Z}_{18})$ given by the map μ described in Theorem 34.2.

$0+K \longmapsto 0$ $\quad$ $3+K \longmapsto 6$

$1+K \longmapsto 10$ $\quad$ $4+K \longmapsto 4$

$2+K \longmapsto 8$ $\quad$ $5+K \longmapsto 2$

Another way to say this is $\mu : \mathbb{Z}_{18}/K \to \phi(\mathbb{Z}_{18})$ with

$\mu(a+K) = 10a \bmod 12$

§18 Describe all units in the given ring.

14. $\mathbb{Z}$

The units are $\{1, -1\}$

16. $\mathbb{Z}_5$

The units are $\mathbb{Z}_5^* = \{1, 2, 3, 4\}$

18. $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$

The units are $\{1, -1\} \times \mathbb{Q}^* \times \{1, -1\}$

24. Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z} \times \mathbb{Z}$

We first note that the only ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}$ are $\phi_1(n) = 0$ and $\phi_2(n) = n$. All ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z} \times \mathbb{Z}$ must be component-wise homomorphic. Hence, all of the ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}\times\mathbb{Z}$ are

$\varphi_1(n) = (0,0)$ $\qquad$ $\varphi_3(n) = (0,n)$

$\varphi_2(n) = (n,0)$ $\qquad$ $\varphi_4(n) = (n,n)$

40. Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic. Show that the fields $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic.

Proof: Suppose to the contrary that there is a ring isomorphism $\phi: 2\mathbb{Z}\to 3\mathbb{Z}$. In particular, ϕ would have to be a group isomorphism. Since $2\mathbb{Z}$ and $3\mathbb{Z}$ are both cyclic, it follows that ϕ must map a generator of $2\mathbb{Z}$ to a generator of $3\mathbb{Z}$. Thus, either $\phi(2)=3$ or $\phi(2)=-3$. Since ϕ is a ring isomorphism, $\phi(2)+\phi(2)=\phi(2+2)=\phi(4)=\phi(2\cdot 2)=\phi(2)\cdot\phi(2)$. Thus, $\phi(2)+\phi(2)=\phi(2)\cdot\phi(2)$. If $\phi(2)=3$, then $\phi(2)+\phi(2)=6$ and $\phi(2)\cdot\phi(2)=9$. If $\phi(2)=-3$, then $\phi(2)+\phi(2)=-6$ and $\phi(2)\cdot\phi(2)=9$. Hence, $\phi(2)\neq 3$ and $\phi(2)\neq -3$, which is a contradiction. Thus, there is no ring isomorphism between $2\mathbb{Z}$ and $3\mathbb{Z}$.

A simple proof that $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic as fields is that the equation $x^2+1=0$ has a solution in $\mathbb{C}$, but has no solution in $\mathbb{R}$. ■

44. An element a of a ring R is idempotent if $a^2=a$.

a. Show that the set of all idempotent elements of a commutative ring is closed under multiplication.

Proof: Let I denote the set of idempotent elements of the commutative ring R. Let $a,b\in I$. Then $ab=a^2b^2=abab=(ab)^2$ since R is commutative. Hence, $ab\in I$ and I is closed under multiplication. ■

b. Find all idempotents in the ring $\mathbb{Z}_6\times\mathbb{Z}_{12}$.

For $\mathbb{Z}_6$, $I_{\mathbb{Z}_6}=\{0,1,3,4\}$

For $\mathbb{Z}_{12}$, $I_{\mathbb{Z}_{12}}=\{0,1,4,9\}$

For $\mathbb{Z}_6\times\mathbb{Z}_{12}$, $I_{\mathbb{Z}_6\times\mathbb{Z}_{12}} = I_{\mathbb{Z}_6}\times I_{\mathbb{Z}_{12}}=\{0,1,3,4\}\times\{0,1,4,9\}$

c. Give an example of an idempotent in $M(2,\mathbb{R})$ that is not the zero matrix or the identity.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is idempotent

$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A$

§19 #4 Find all solutions of $x^2+2x+4=0$ in $\mathbb{Z}_6$

$0^2+2(0)+4 \equiv 4 \mod 6$

$1^2+2(1)+4 \equiv 1 \mod 6$

$2^2+2(2)+4 \equiv 0 \mod 6$

$3^2+2(3)+4 \equiv 1 \mod 6$

$4^2+2(4)+4 \equiv 4 \mod 6$

$5^2+2(5)+4 \equiv 3 \mod 6$

$x=2$ is the only solution in $\mathbb{Z}_6$