MATH 202 Name : SOLUTIONS
Midterm 2 March 23, 2012

Instructions: Show all work for full credit. Poor notation or sloppy work will be penalized.

1. (15 pts.)

   (a) (7 pts.) Prove that $\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ does not exist.

Along the x-axis, as $(x,0)\to(0,0)$, the values $\frac{3xy}{x^2+y^2} = \frac{0}{x^2} = 0$

Along $y=x$, as $(x,x)\to 0$, $\frac{3xy}{x^2+y^2} = \frac{3x^2}{2x^2} = \frac{3}{2}$

Since $0 \neq 3/2$, the limit fails to exist.

   (b) (8 pts.) A particle moves along a curve in 3-space given by $\mathbf{r}(t) = \langle 4\sin(3t), 3t, 4\cos(3t)\rangle$ meters, where t is measured in seconds. Give and evaluate a definite integral that computes the distance traveled by the particle between time $t=0$ seconds and $t=2\pi$ seconds.

Arc length $L = \int_0^{2\pi} \|\vec{r}\,'(t)\|dt = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$= \int_0^{2\pi} \sqrt{(4\cos(3t)\cdot 3)^2 + (3)^2 + (-4\sin(3t)\cdot 3)^2}\, dt$

$= \int_0^{2\pi} \sqrt{144\cos^2(3t) + 9 + 144\sin^2(3t)}\, dt = \int_0^{2\pi} \sqrt{144+9}\, dt$

$= \int_0^{2\pi} \sqrt{153}\, dt = \boxed{2\pi\sqrt{153} \text{ m}}$

2. (8 pts.) Suppose that $w = xy^2 - x^2$ and that at time $t = 0$, $(x, y) = (1, 2)$, $\left.\frac{dx}{dt}\right|_{t=0} = -5$, and $\frac{dy}{dt} = 3$. Use the Multivariable Chain Rule to compute $\frac{dw}{dt}$ when $t = 0$.

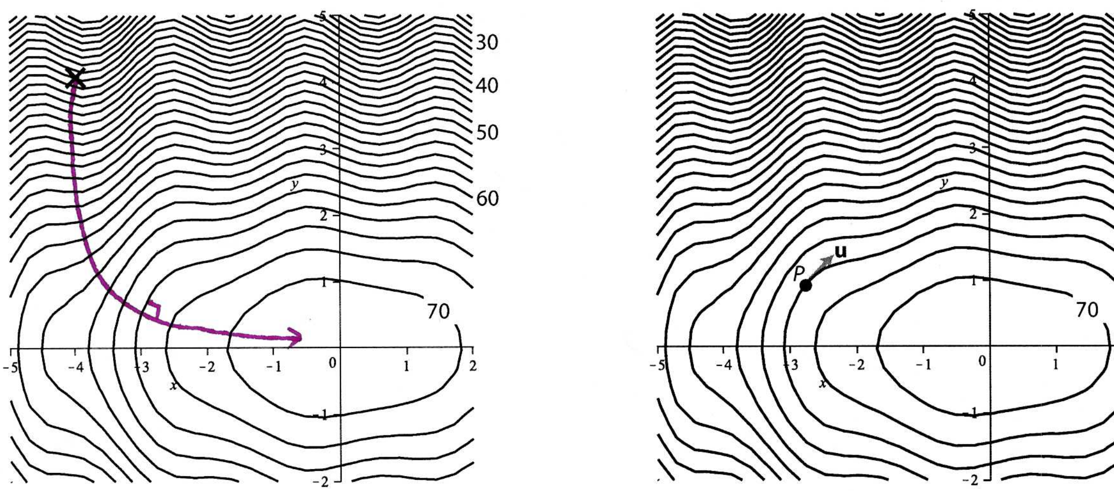
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (y^2 - 2x)(-5) + 2xy\,(3)$$

$$= (2^2 - 2(1))(-5) + 2(1)(2)(3)$$

$$= 2(-5) + 12 = \boxed{2}$$

3. (12 pts. – 6 pts. each) It is well-known that mosquitoes are drawn to CO_2. Below is a contour plot for the concentration $C(x, y)$ of CO_2 at positions (x, y) in feet on the grid shown.



(a) (b)

(a) In Figure (a) on the left, sketch the path of a mosquito placed on the X at $(-4, 4)$ as it continuously moves to maximize its exposure to CO_2. Briefly justify the reasoning for drawing the path you did.

The path follows the gradient which is orthogonal to level curves.

(b) In Figure (b) on the right, give the value of the directional derivative of $C(x, y)$ in the direction indicated by the vector $\mathbf{u}$ at the point P indicated with a dot. Explain your answer briefly.

$D_{\vec{u}} C(P) = 0$ since $\vec{u}$ is tangent to a level curve (i.e. no change)

4. (10 pts.) Find the equation of the tangent plane to the ellipsoid $x^2 + 2y^2 + 9z^2 = 31$ at the point $(2, 3, 1)$.

Let $G(x,y,z) = x^2+2y^2+9z^2-31$, then $\vec{n} = \nabla G(2,3,1)$. (or some non-zero multiple)

$\nabla G = \langle 2x, 4y, 18z \rangle$

$\nabla G(2,3,1) = \langle 2(2), 4(3), 18(1) \rangle = \langle 4, 12, 18 \rangle$

Dividing by 2, $\vec{n} = \langle 2,6,9 \rangle$.

Using $\vec{x}\cdot\vec{n} = \vec{p}\cdot\vec{n}$, we get $2x+6y+9z = \langle 2,3,1\rangle \cdot \langle 2,6,9\rangle$

$2x+6y+9z = 4+18+9$

$\boxed{2x+6y+9z = 31}$

5. (10 pts.) Electrical power P in watts is given by

$$P = \frac{V^2}{R},$$

where V is voltage and R is resistance in ohms.

(a) Give a formula for the total differential dP for power.

$dP = \frac{\partial P}{\partial V}\,dV + \frac{\partial P}{\partial R}\,dR$

$= \frac{2V}{R}\,dV + \frac{-V^2}{R^2}\,dR$

$\boxed{dP = \frac{2V}{R}dV - \frac{V^2}{R^2}\,dR}$

(b) If $V = 120$ volts is applied to a 2000-ohm resistor, compute the total differential dP for power.

$dP = \frac{2V}{R}\,dV - \frac{V^2}{R^2}\,dR$

$= \frac{2(120)}{2000}\,dV - \frac{(120)^2}{(2000)^2}\,dR$

$= \frac{240}{2000}\,dV - \frac{120^2}{2000^2}\,dR$

$= \frac{6}{50}\,dV - \frac{9}{2500}\,dR$ watts

$dP = \boxed{\frac{6}{50}\,dV - \frac{9}{2500}\,dR \text{ watts}}$

6. (17 pts.) Suppose the elevation above sea level in tens of meters is given by the function

$$h(x, y) = \frac{y^2}{5 - x} \text{ tens of } m,$$

and a hiker is located at the position $(x, y) = (4, 1)$.

(a) (6 pts.) In what direction from $(4, 1)$ should the hiker move to increase his/her elevation the most?

$\nabla h(4,1)$. This is given by: $\nabla h = \langle \frac{+y^2}{(5-x)^2}, \frac{2y}{(5-x)} \rangle$

So $\nabla h(4,1) = \langle \frac{1^2}{1^2}, \frac{2}{1} \rangle$

$= \boxed{\langle 1,2 \rangle}$

(b) (6 pts.) If the hiker moves in the direction indicated by the vector $\mathbf{v} = \langle 1, -1 \rangle$, what is the rate of change of the hiker's elevation?

Let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}} \langle 1,-1 \rangle$. The rate of change is

$D_{\vec{u}} h(4,1) = \nabla h(4,1) \cdot \vec{u} = \langle 1,2 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1,-1 \rangle$

$= \frac{1}{\sqrt{2}} (1-2) = \boxed{\frac{-1}{\sqrt{2}}}$ or $-\frac{\sqrt{2}}{2}$

(c) (5 pts.) Using your answer to part (b), do you expect the hiker's elevation to rise or fall as the hiker moves in the direction given by $\mathbf{v}$?

To fall or decrease.

The directional derivative is negative.

7. (16 pts.) Consider the function $f(x,y) = -x^3 + 6xy - 3y^2 + 1$.

(a) (8 pts.) Find all critical points of $f(x,y)$.

$\left.\begin{array}{l} f_x(x,y) = -3x^2 + 6y \\ f_y(x,y) = 6x - 6y \end{array}\right\}$ Set both equal to zero and solve.

$\begin{cases} -3x^2 + 6y = 0 \\ 6x - 6y = 0 \rightarrow x = y \end{cases}$

Substituting $x=y$ into top equation gives $-3x^2 + 6x = 0$

$-3x(x-2) = 0$

$x = 0 \quad x = 2$

Since $x=y$, the two critical points are $\boxed{(0,0) \text{ and } (2,2)}$

(b) (8 pts.) Use the second derivative test to determine if the critical points are local maxima, local minima, saddle points or if there is not enough information to tell.

First compute, the second partial derivatives.

$f_{xx}(x,y) = -6x \quad f_{xy}(x,y) = 6 \quad f_{yy}(x,y) = -6$

Form $d = \begin{vmatrix} -6x & 6 \\ 6 & -6 \end{vmatrix} = -36x - 36 = +36(x+1)$

At the critical point $(0,0)$, $d = -36 \Rightarrow \boxed{(0,0) \text{ is a saddle point}}$

At the critical point $(2,2)$, $d = 36$ and $f_{xx}(2,2) = -12 < 0$

Therefore, $\boxed{(2,2) \text{ is a local maximum.}}$

8. (12 pts.) **Use Lagrange multipliers** to find the maximum value of $f(x,y) = xy$ where $x > 0$ and $y > 0$, subject to the constraint $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

The constraint is $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} = 1$.

Require $\nabla f = \lambda \nabla g$ yields $\langle y, x \rangle = \lambda \langle \frac{x}{4}, y \rangle$ and two equations:

E1: $4y = \lambda x$ E2: $x = \lambda y$

The third equation is the constraint: E3: $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Using E2, $\lambda = \frac{x}{y}$ (note $y \neq 0$!) and substituting this into E1 gives

$4y = \left(\frac{x}{y}\right) x$ or $\underline{4y^2 = x^2}$. Now using the underlined,

we substitute $x^2 = 4y^2$ into E3: $\frac{(4y^2)}{8} + \frac{y^2}{2} = 1$ or $y^2 = 1$. Thus,

$y = \pm 1$, but we only consider $y = 1$. Again using the underlined,

$4(1)^2 = x^2 \Rightarrow x = \pm 2$. Since $x > 0$, the $\underline{\text{only}}$ critical point is $(2,1)$

The maximum value is $f(2,1) = 2(1) = 2$

The maximum value is $\underline{\quad 2 \quad}$, and occurs at $(x,y) = \underline{\quad (2,1) \quad}$.