Numerical Analysis
Exam 2

Name : SOLUTIONS

November 19, 2015

Instructions: Show all work for full credit. You may use a calculator for simple 'adding machine'-like computations.

1. (16 pts.) Consider the function $f(x) = \dfrac{1}{x}$ for $x > 0$ and equally spaced nodes $x_0 = \frac{1}{2}$, $x_1 = 1$, $x_2 = \frac{3}{2}$.

   (a) (2 pts.) Give the degree n of the interpolating polynomial $p_n(x)$ that passes through the points $(x_i, f(x_i))$ for each i?

   $n=2$

   (b) (6 pts.) Find the Lagrange form of the interpolant $p(x)$ that passes through these points. You may leave your answer as the sum of rational functions.

$$p(x) = \frac{2(x-1)(x-3/2)}{(\frac{1}{2}-1)(\frac{1}{2}-3/2)} + 1\,\frac{(x-\frac{1}{2})(x-\frac{3}{2})}{(1-1/2)(1-3/2)} + \frac{2}{3}\,\frac{(x-1/2)(x-1)}{(3/2-1/2)(3/2-1)}$$

$$= \frac{2(x-1)(x-3/2)}{-1/2(-1)} + \frac{(x-1/2)(x-3/2)}{\frac{1}{2}(-\frac{1}{2})} + \frac{2}{3}\,\frac{(x-1/2)(x-1)}{1(1/2)} = \boxed{4(x-1)(x-3/2) - 4(x-1/2)(x-3/2) + \frac{4}{3}(x-1/2)(x-1)}$$

   (c) (4 pts.) Compute the value of the interpolating polynomial you found in (b) at the argument $x = \dfrac{3}{4}$. Then compute the absolute error for this approximation to $f(x)$ at $x = \dfrac{3}{4}$.

Note: $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$, $\frac{3}{4} - 1 = \frac{-1}{4}$, $\frac{3}{4} - \frac{3}{2} = \frac{-3}{4}$ so $p(\frac{3}{4}) = 4(\frac{-1}{4})(\frac{-3}{4}) - 4(\frac{1}{4})(\frac{-3}{4}) + \frac{4}{3}(\frac{1}{4})(\frac{-1}{4})$

$= \frac{3}{4} + \frac{3}{4} - \frac{1}{12} = \frac{17}{12} \approx 1.4167$ $f(\frac{3}{4}) = \frac{4}{3}$ $|f(3/4) - p(3/4)| = |\frac{4}{3} - \frac{17}{12}| = |\frac{-1}{12}| = \frac{1}{12}$

Answer: $p(\frac{3}{4}) = \boxed{17/12}$ and the absolute error is $\boxed{1/12}$

   (d) (4 pts.) Suppose now that $x \in [.5,\ 1.5]$; that is, x is any point in the interval. Give an upper bound for the absolute error between $f(x)$ and $p(x)$ that works for every $x \in [.5,\ 1.5]$. (Round your answer to four decimal places, if necessary.)

Using the cheat sheet, $|f(x) - p(x)| \leq \frac{1}{9\sqrt{3}} h^3 \|f^3\|_{\infty, I}$ for $I = [1/2, 3/2]$

and $h = 1/2$. Thus, $|f(x) - p(x)| \leq \frac{1}{9\sqrt{3}} (\frac{1}{2})^3 \|f^3\|_{\infty, I}$

$\Rightarrow |f(x) - p(x)| \leq \frac{1}{9\sqrt{3}} \frac{1}{8} \cdot 96 = \boxed{\frac{4}{3\sqrt{3}} \approx .7698}$

If $f(x) = x^{-1}$, then

$f'(x) = -x^{-2}$

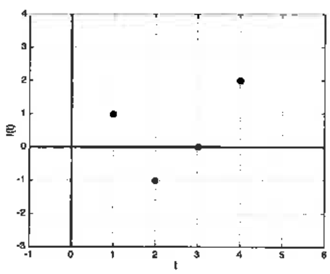
$f^{(2)}(x) = 2x^{-3}$

$f^3(x) = -6x^{-4}$

and $\|f^3\|_{\infty, I} = \frac{6}{(\frac{1}{2})^4} = 96$

2. (18 pts.) Consider the data tabulated in the following table and depicted in the plot below:

t	1	2	3	4
$f(t)$	1	-1	0	2



(a) (3 pts.) On the axes provided above, sketch a rough graph of the interpolating polynomial $p_3(x)$.

(b) (5 pts.) With the nodes in increasing order, $t_0 = 1 < \cdots < t_3 = 4$, create a divided differences table to find the coefficients c_i in the Newton form of the interpolant. Show all work for credit, and give your answers as rational numbers, not decimals, in the table provided.

k	x_k	$f_0(x_k)$	$f_1(x_k)$	$f_2(x_k)$	$f_3(x_k)$
0	1	1			
			-2		
1	2	-1		$+3/2$	
			1		$-1/3$
2	3	0		$1/2$	
			2		
3	4	2			

(c) (3 pts.) Using your divided differences table from part (a), give the Newton form of the interpolating polynomial $p_3(t)$ for these data.

$$p_3(t) = 1 - 2(t-1) + 3/2\,(t-1)(t-2) - 1/3\,(t-1)(t-2)(t-3)$$

(d) (3 pts.) Now give the Newton form of the interpolating polynomial $p_3^{reverse}(x)$ for the nodes in the *reverse order*: $x_0 = 4$, $x_1 = 3$, $x_2 = 2$, $x_3 = 1$.

$$p_3^{reverse}(t) = 2 + 2(t-4) + \tfrac{1}{2}(t-4)(t-3) - \tfrac{1}{3}(t-4)(t-3)(t-2)$$

(e) (4 pts.) Suppose you used $p_3(t)$ and then $p_3^{reverse}(t)$ to interpolate the value of $f(t)$ at the value $t = \pi$. Which gives the better approximation to $f(\pi)$. Explain.

The values $p_3(\pi)$ and $p_3^{reverse}(\pi)$ are the <u>same</u>

The interpolating polynomial of degree 3 is <u>unique</u> provided the nodes are distinct.

3. (12 pts.) Give two reasons/situations why using the Newton form of the interpolating polyonomial $p_n(x)$ might be preferable to the Lagrange form. (You will be graded on your explanation as well as your answer.)

i. The Newton form of the interpolating polynomial is preferable when ....

you might add another data point since you can just add on a new term $p_{n+1}(x) = p_n(x) + \underbrace{\text{new term}}$ and do not lose your previous computations

ii. The Newton form of the interpolating polynomial is better when ....

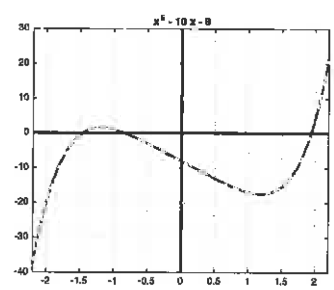
you want to evaluate the interpolant since you can use Horner's method for nested multiplication / evaluating a polynomial.

4. (7 pts.) Suppose you have $n+1$ data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ and the polynomial interpolant $p_n(x)$ has degree strictly less than n. What must be true about these data points? Explain.

The points must lie exactly on a polynomial of degree strictly less than n.

5. (10 pts.) Consider the graph of the quintic function $f(x) = x^5 - 10x - 8$ shown below. This function has only two local extrema, both of which are shown.

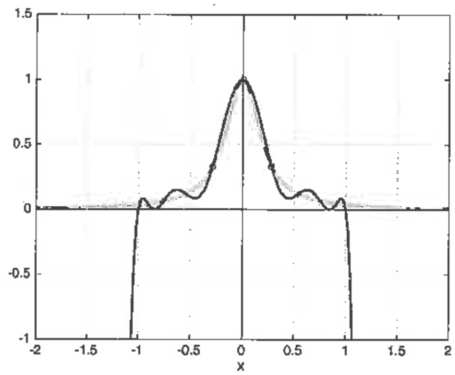
Give a possible initial 'guess' that you might use if you were to use the Durand-Kerner method to find the roots of $f(x)$. Justify why you chose such a guess for starting values as a way to ensure convergence to the roots. (You will be graded both on correctness and completeness of your explanation.)

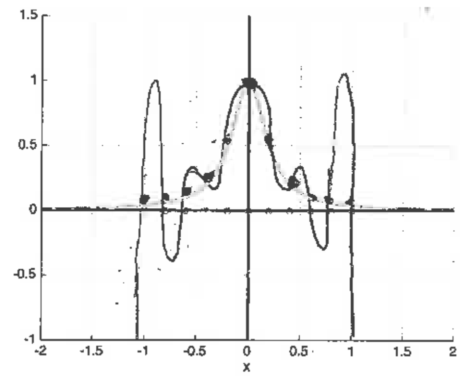


$r = [\underbrace{-1.5 \;\; -.75 \;\; 1.75}_{\text{3 real, estimated from graph}} \;\; \underbrace{i \;\; -i}_{\text{2 complex conjugate roots}}]$

Durand-Kerner requires complex, non-real guesses to approximate complex, non-real roots.

6. (15 pts.) Consider the graph of the 'Runge example' given below, where $f(x) = \dfrac{1}{1+25x^2}$ is plotted on the interval $[-1,\ 1]$. On the left, you can see the graph of the polynomial interpolant when the Chebyshev nodes are used, and on the right you can see a plot of $f(x)$ and equally spaced nodes.





(a) (4 pts.) Explain by looking at the graph on the left how you can tell that the Chebyshev nodes, *and not equally spaced nodes*, were used to construct the polynomial interpolant.

The error near the endpoints $|f(x) - p(x)|$ is smallish

(b) (2 pts.) What is the degree n of the polynomial interpolant pictured on the left?

11 nodes $\Rightarrow$ $n = 10$

(c) (5 pts.) To the best of your ability, sketch on the axes on the right a polynomial interpolant to $f(x)$ at the equally spaced nodes displayed there. In the space here, briefly explain what features you are trying to highlight.

See above. You should show $\lim_{n \to \pm\infty} p_{10}(x) = -\infty$

and wild oscillation near the endpoints. The plot should be symmetric about the y-axis.

(d) (4 pts.) Using the Theorem on error in polynomial interpolation, explain carefully why the Chebyshev nodes might/might not be a better choice than equally spaced nodes for interpolation.

The Chebyshev nodes minimize the quantity

$|(x - x_0) \cdots (x - x_{10})|$ in the error formula

$$|f(x) - p_n(x)| \le \frac{1}{(n+1)!} \|f^{(n+1)}\|_{\infty, I} (x - x_0) \cdots (x - x_n)$$

Thus, they are a better choice

7. (10 pts.)

(a) Give the definition that a sequence p_n converges quadratically to 0.

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n^2} = C \qquad C \neq 0, \infty$$

(b) Prove or disprove: The sequence $p_n = \dfrac{1}{3^{2^n}}$ converges quadratically to 0.

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n^2} = \lim_{n\to\infty} \frac{\frac{1}{3^{2^{n+1}}}}{\left(\frac{1}{3^{2^n}}\right)^2} = \lim_{n\to\infty} \frac{(3^{2^n})^2}{3^{2^{n+1}}} = \lim_{n\to\infty} \frac{3^{2\cdot 2^n}}{3^{2^{n+1}}} = \lim_{n\to\infty} \frac{3^{2^{n+1}}}{3^{2^{n+1}}} = 1 \neq 0, \infty$$

8. (12 pts.) Recall the 'fixed slope' method for finding a root of a polynomial $f(x)$, a variation on Newton's method. This problem will consider convergence of this method to a root of the function $f(x) = \ln(1-x)$.

(a) (1 pt.) Give the value of the single root α of $f(x) = \ln(1-x)$.
Answer. $\alpha =$ ___ 0 ___

(b) (1 pt.) In the 'fixed slope' method, we will use the value of the $g = -2$ in place of the derivative: $x_{n+1} = x_n - \dfrac{f(x_n)}{g}$. Substitute and simplify this expression.

$$x_{n+1} = x_n + \frac{1}{2}\ln(1-x_n)$$

(c) (2 pts.) Give the iterative function $\varphi(x)$ for this algorithm.

$$\varphi(x) = x + \frac{1}{2}\ln(1-x)$$

(d) (3 pts.) Letting $e_k = \alpha - x_k$ denote the error in the kth iteration as usual, give an expression for e_{k+1} as a function e_k that involves iterative function. Simplify.

$e_{k+1} = \varphi'(x_k)e_k \qquad \varphi'(x) = 1 - \dfrac{1}{2(1-x)}$. Therefore,

$$e_{k+1} = \left(1 - \frac{1}{2}\cdot\frac{1}{1-x}\right)e_k$$

(e) (5 pts.) Suppose the intial guess to a root of $f(x)$ is a negative number x_0. Will this fixed slope method converge to the root α? Justify your answer.

You need to test if $|\varphi'(\alpha)| < 1$. $\qquad \varphi'(\alpha) = \varphi'(0) = 1 - \frac{1}{2}\cdot\frac{1}{1-0} = \frac{1}{2}$

Since $|\varphi'(0)| < 1$, yes, the method would converge.