3. Consider the jointly distributed random variables (X, Y) with joint density function

$$f(x,y) = \begin{cases} ce^{-y}, & \text{for } 0 \le x \le e^2 - 1,\ 0 \le y \le \ln(x+1) \\ 0, & \text{otherwise.} \end{cases}$$

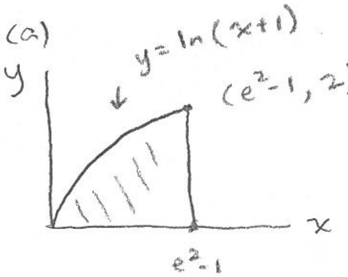
(a) Draw the *support* of the joint density function $f(x, y)$; that is, the region S where $f(x, y) > 0$. Then find the value of c so that $f(x, y)$ is a valid density function on S.

(b) Set up an integral to find the marginal density $f_Y(y)$ and include the domain of this function.

(c) Verify that your marginal density $f_Y(y)$ is correct by integrating it on the support of Y.

(d) Find the value of the conditional probability $P(X \ge 4 \mid Y = \ln(3))$. *Answer:* $\dfrac{e^2-5}{(e^2-3)} \approx .54$.

(a)



$$\int_0^{e^2-1}\int_0^{\ln(x+1)} e^{-y}\,dy\,dx = \int_0^{e^2-1} -e^{-y}\Big|_0^{\ln(x+1)} dx$$

$$= \int_0^{e^2-1} -e^{-\ln(x+1)} - -e^0\,dx = \int_0^{e^2-1} 1 - \frac{1}{x+1}\,dx$$

$$= x - \ln(x+1)\Big|_0^{e^2-1} = (e^2-1) - \ln(e^2-1+1) = e^2 - 3. \text{ Thus, } c = \frac{1}{e^2-3}.$$

(b) With $c = \frac{1}{e^2-3}$, $\quad f_Y(y) = \int_{\text{all } x} ce^{-y}dx = \int_{e^y-1}^{e^2-1} ce^{-y}dx \qquad y = \ln(x+1) \Rightarrow x = e^y - 1$

$$= ce^{-y}\left([e^2-1] - [e^y-1]\right) = ce^{-y}(e^2 - e^y)$$

$$f_Y(y) = \begin{cases} \dfrac{e^{-y}(e^2-e^y)}{e^2-3} = \dfrac{1}{e^2-3}(e^{2-y}-1) & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) $1 \overset{?}{=} \int_0^2 \frac{e^2}{e^2-3}e^{-y} - \frac{1}{e^2-3}\,dy$

$$= -\frac{e^2}{e^2-3}e^{-y}\Big|_0^2 - \frac{1}{e^2-3}\,y\Big|_0^2 = \frac{1}{e^2-3}\left[\left[-e^2e^{-2} + e^2\right] - 2\right] = \frac{1}{e^2-3}\left[-1 + e^2 - 2\right]$$

$$= \frac{e^2-3}{e^2-3} = 1 \checkmark$$

Review Problem # 3

(d) $P(X \geq 4 \mid Y = \ln(3))$

The "correct" density is $f(x \mid y = \ln 3) = \dfrac{f(x, y = \ln 3)}{f_Y(\ln 3)} = \dfrac{\dfrac{e^{-\ln 3}}{e^2 - 3}}{\dfrac{e^{-\ln 3}(e^2 - e^{\ln 3})}{e^2 - 3}}$

$$= \frac{e^{-\ln 3}}{e^{-\ln 3}(e^2 - e^{\ln 3})} = \frac{1}{e^2 - 3}$$

which is easy to check is correct for $2 \leq x \leq e^2 - 1$

Since $f(x,y) = f(y)$ (i.e. no dependency on x) implies $f(x \mid y = \ln 3)$ should be constant constant on an interval of length $(e^2 - 1) - (2) = e^2 - 3$.

Thus, $P(X \geq 4 \mid Y = \ln 3) =$

$$\int_4^{e^2 - 1} f(x \mid y = \ln 3)\, dx$$

$$= \int_4^{e^2 - 1} \frac{1}{e^2 - 3}\, dx$$

$$= \frac{e^2 - 5}{e^2 - 3} \approx .54$$