MATH 253
Spring 2019

Midterm Exam 1

Name: Answer Key

Instructions. You have 120 minutes. Closed book, closed notes, and no calculators allowed. *Show all your work* in order to receive full credit.

1. Consider the point $A(1,-2,0)$ and the line

$$x-2=\frac{y+1}{3}=\frac{z-1}{2}$$

(a) Find the equation of the plane containing A and the line.

Solution: The line direction $\overrightarrow{u}=\langle 1,3,2\rangle$ is in the plane as is $\overrightarrow{AB}$ for any B one the line; take $B(2,-1,1)$. Then $\overrightarrow{AB}=\langle 2-1,-1+2,1-0\rangle=\langle 1,1,1\rangle$. So a normal vector to the plane is:

$$\overrightarrow{u}\times\overrightarrow{AB}=\langle 1,3,2\rangle\times\langle 1,1,1\rangle=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&3&2\\1&1&1\end{vmatrix}=\langle 3(1)-1(2),-(1(1)-1(2)),1(1)-1(3)\rangle=\langle 1,1,-2\rangle$$

and so the equation of the plane is:

$$\boxed{(x-1)+(y+2)-2(z-0)=0}\quad\text{or equivalently}\quad\boxed{x+y-2z+1=0}.$$

(b) Find the distance from A to the line.

Solution:

$$d=\frac{\left\|\overrightarrow{u}\times\overrightarrow{AB}\right\|}{\|\overrightarrow{u}\|}=\frac{\|\langle 1,1,-2\rangle\|}{\|\langle 1,3,2\rangle\|}=\frac{\sqrt{1+1+4}}{\sqrt{1+9+4}}=\sqrt{\frac{6}{14}}=\sqrt{\frac{3}{7}}=\boxed{\frac{\sqrt{21}}{7}}$$

2. Consider the space curve parametrized by:

$$\mathbf{r}(t)=\langle\cos t,\cos t+3\sin t,3\sin t\rangle\,.$$

(a) Show that $\mathbf{r}(t)$ is a parametrization of the intersection of the surfaces $x-y+z=0$ and $9x^2+z^2=9$.

Solution: We need to verify that the components of $\mathbf{r}(t)$ satisfy the equations of the surfaces at all times t:

$$x-y+z=(\cos t)-(\cos t+3\sin t)+(3\sin t)=0\quad\checkmark$$

and

$$9x^2+z^2=9(\cos t)^2+(3\sin t)^2=9\cos^2 t+9\sin^2 t=9\quad\checkmark$$

(b) Show that the tangent line to $\mathbf{r}(t)$ at $t=\dfrac{3\pi}{4}$ is parallel to $\langle 1,4,3\rangle$.

Solution:

$$\mathbf{r}'(t)=\langle -\sin t,-\sin t+3\cos t,3\cos t\rangle$$

and so the tangent line at $t=\dfrac{3\pi}{4}$ has direction:

$$\mathbf{r}'\left(\frac{3\pi}{4}\right)=\left\langle -\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}+3\left(-\frac{\sqrt{2}}{2}\right),-3\frac{\sqrt{2}}{2}\right\rangle$$

$$=\left\langle -\frac{\sqrt{2}}{2},-\frac{4\sqrt{2}}{2},-3\frac{\sqrt{2}}{2}\right\rangle=-\frac{\sqrt{2}}{2}\langle 1,4,3\rangle\,.$$

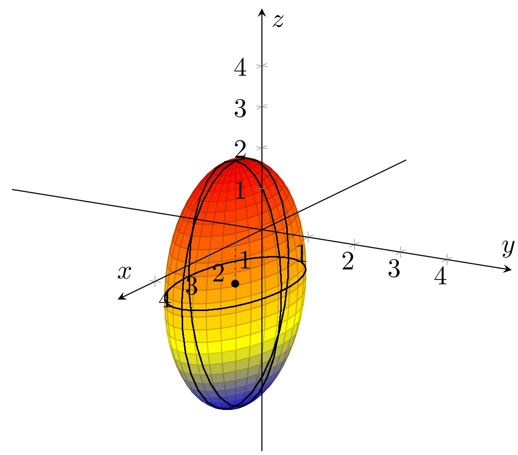
Since the vectors are scalar multiples of each other, then by definition,
$\boxed{\text{the tangent line and } \langle 1,4,3\rangle \text{ are parallel}}$.

3. Rewrite the following equation in standard form then sketch the surface.

$$9x^2 + 36y^2 + 4z^2 - 18x + 8z = 23$$

Solution:

$$\begin{aligned} 9(x^2 - 2x) + 36y^2 + 4(z^2 + 2z) &= 23 \\ \iff 9\left[(x-1)^2 - 1\right] + 36y^2 + 4\left[(z+1)^2 - 1\right] &= 23 \\ \iff 9(x-1)^2 - 9 + 36y^2 + 4(z+1)^2 - 4 &= 23 \\ \iff 9(x-1)^2 + 36y^2 + 4(z+1)^2 &= 36 \\ \iff \boxed{\frac{(x-1)^2}{4} + y^2 + \frac{(z+1)^2}{9} = 1} \end{aligned}$$



The surface is an ellipsoid.

4. Consider the following planes.

plane 1: $x - y + 4z = 5$
plane 2: $3x - y - z = 2$

(a) Show that the planes are orthogonal.

Solution: We verify that the dot product of the normal vectors is zero:

$$\langle 1, -1, 4\rangle \cdot \langle 3, -1, -1\rangle = 1(3) - (-1) + 4(-1) = 0 \qquad \checkmark$$

(b) Find parametric equations for the line of intersection of the two planes.

Solution: The cross product of the norm vectors is (parallel to) the direction of the line of intersection:

$$\langle 1, -1, 4\rangle \times \langle 3, -1, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 4 \\ 3 & -1 & -1 \end{vmatrix} = \langle -(-1) + 4, -(1(-1) - 3(4)), 1(-1) - 3(-1)\rangle = \langle 5, 13, 2\rangle$$

Now to find a point on that line, set $x = 0$ for example and we are left with solving the system:

$$\begin{cases} -y + 4z = 5 \\ -y - z = 2 \end{cases} \iff \begin{cases} -y + 4z = 5 \\ 5z = 3 \end{cases} \iff \begin{cases} y = 4\left(\frac{3}{5}\right) - 5 \\ z = \frac{3}{5} \end{cases}$$

so we have the point $\left(0, -\frac{13}{5}, \frac{3}{5}\right)$ and hence parametric equations are:

$$\boxed{\begin{cases} x = 5t \\ y = -\frac{13}{5} + 13t \\ z = \frac{3}{5} + 2t \end{cases}}$$

5. Consider the following space curves:

$$\mathbf{r_1}(t) = \left\langle 2t-3, t^2-5t+3, t^3-2 \right\rangle \quad , \quad \mathbf{r_2}(t) = \left\langle -t+2, t-4, 3t^2+2t+1 \right\rangle$$

(a) Find any intersection point(s) of the space curves.

Solution: Switch the parameter to s in the second curve and equate the components:

$$\begin{cases} 2t-3 = -s+2 \\ t^2-5t+3 = s-4 \\ t^3-2 = 3s^2+2s+1 \end{cases} \iff \begin{cases} s = 5-2t \\ t^2-5t+3 = (5-2t)-4 \\ t^3-2 = 3s^2+2s+1 \end{cases} \iff \begin{cases} s = 5-2t \\ t^2-3t+2 = 0 \\ t^3-2 = 3s^2+2s+1 \end{cases}$$

From the second equation, we get two possible values of t and thus from the first equation corresponding values of s:

- if $t = 1$ then $s = 3$ and the third equation becomes:

$$1-2 = 3(9)+2(3)+1 \iff -1 = 34$$

This is not true so no intersection point from this pair of values.

- if $t = 2$ then $s = 1$ and the third equation becomes:

$$8-2 = 3+2+1 \iff 6 = 6$$

This is true so we have one point of intersection:

$$\mathbf{r_1}(2) = \mathbf{r_2}(1) = \langle 1, -3, 6 \rangle$$

that is the point $\boxed{(1,-3,6)}$.

(b) Find the unit tangent vector $\mathbf{T_1}(t)$ for the space curve $\mathbf{r_1}(t)$ at time t.

Solution:

$$\mathbf{r_1'}(t) = \left\langle 2, 2t-5, 3t^2 \right\rangle \implies \|\mathbf{r_1'}(t)\| = \sqrt{4+(2t-5)^2+9t^4} = \sqrt{9t^4+4t^2-20t+29}$$

$$\implies \boxed{\mathbf{T_1}(t) = \frac{\left\langle 2, 2t-5, 3t^2 \right\rangle}{\sqrt{9t^4+4t^2-20t+29}}}$$

(c) Find the curvature of the space curve $\mathbf{r_2}(t)$ at $t = -1$.

Solution:

$$\mathbf{r_2'}(t) = \langle -1, 1, 6t+2 \rangle \quad \Rightarrow \quad \mathbf{r_2'}(-1) = \langle -1, 1, -4 \rangle$$

$$\mathbf{r_2''}(t) = \langle 0, 0, 6 \rangle \quad \Rightarrow \quad \mathbf{r_2''}(-1) = \langle 0, 0, 6 \rangle$$

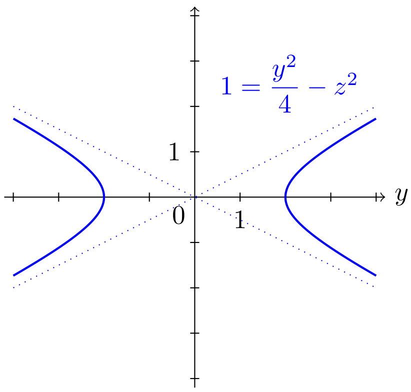
$$\mathbf{r_2'} \times \mathbf{r_2''}\Big|_{t=-1} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -4 \\ 0 & 0 & 6 \end{vmatrix} = \langle 1(6)-0, -(-1(6)-0), -1(0)-0 \rangle = \langle 6, 6, 0 \rangle = 6\,\langle 1, 1, 0 \rangle$$

$$\kappa(-1) = \frac{\|\mathbf{r_2'} \times \mathbf{r_2''}\|}{\|\mathbf{r_2'}\|^3}\Bigg|_{t=-1} = \frac{\|6\,\langle 1,1,0 \rangle\|}{\|\langle -1,1,-4 \rangle\|^3} = \frac{6\sqrt{1+1}}{\left[\sqrt{1+1+16}\right]^3} = \frac{6\sqrt{2}}{18\sqrt{18}} = \boxed{\frac{1}{9}}$$

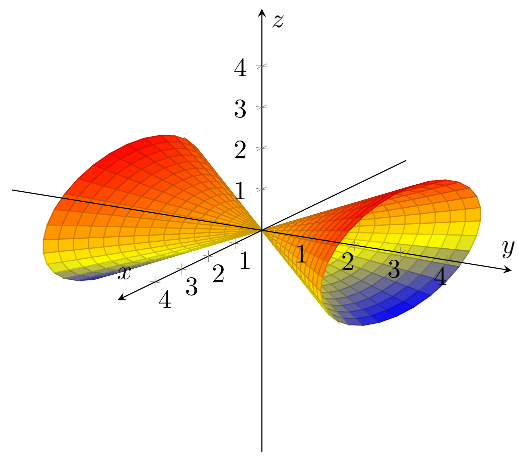
6. For each equation, name the type of surface, sketch the given trace in 2D then the surface in 3D.

(a) $x^2 - y^2 + 4z^2 = 0$ Type of surface: **elliptic cone**

Solution:

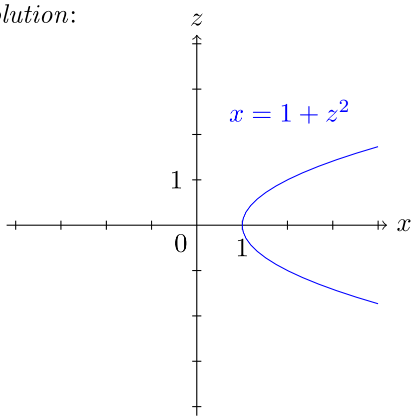


trace: $x = -2$

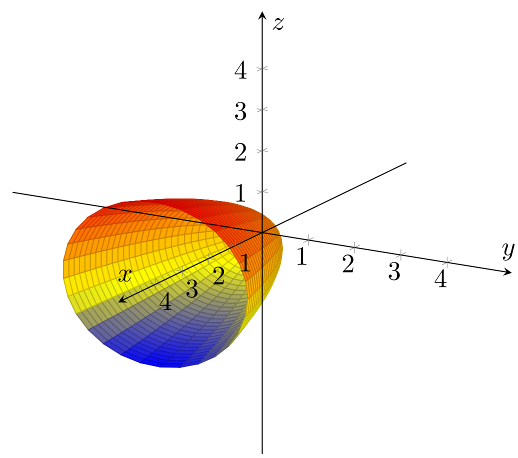


(b) $x = y^2 + z^2$ Type of surface: **circular paraboloid**

Solution:

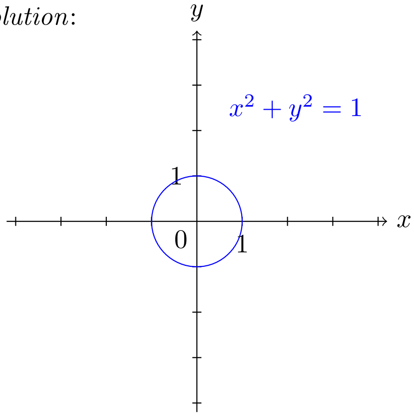


trace: $y = 1$

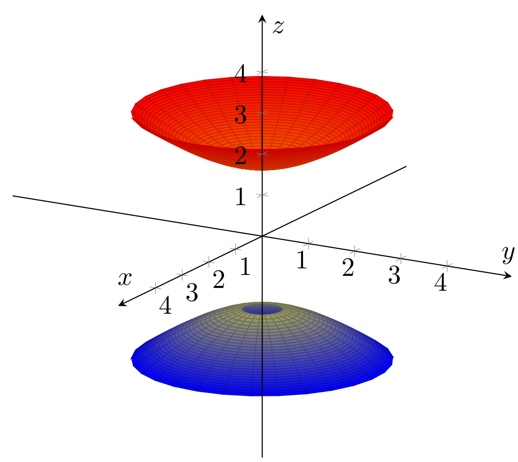


(c) $x^2 + y^2 = z^2 - 3$ Type of surface: **hyperboloid of two sheets**

Solution:



trace: $z = 2$



7. Let $\mathbf{a} = \langle -1, 3, c\rangle$ and $\mathbf{b} = \langle 2, 1, 4\rangle$.

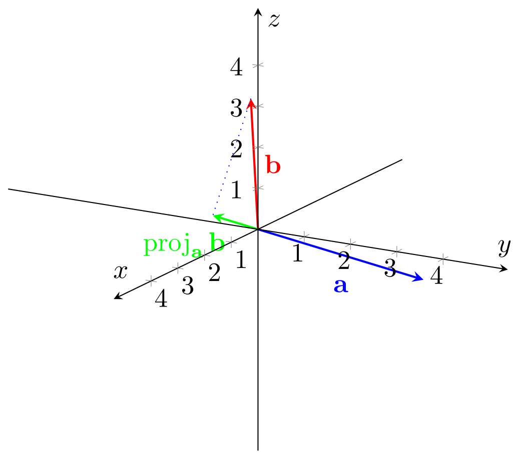
(a) For what value(s) of c will the angle between $\mathbf{a}$ and $\mathbf{b}$ be obtuse (i.e. greater than 90°)?

Solution: The angle is obtuse if the dot product is negative:

$$\langle -1, 3, c\rangle \cdot \langle 2, 1, 4\rangle < 0 \quad \Longleftrightarrow \quad -1(2) + 3(1) + 4c < 0 \quad \Longleftrightarrow \quad \boxed{c < -\frac{1}{4}}$$

(b) Sketch $\mathbf{a}$ and $\mathbf{b}$ in standard position for $c = -1$.

Solution:



(c) Find the vector projection of $\mathbf{b}$ along $\mathbf{a}$ for $c = -1$ and sketch it on the above set of axes (make sure to label it).

Solution:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\langle -1,3,-1\rangle \cdot \langle 2,1,4\rangle}{\|\langle -1,3,-1\rangle\|^2}\mathbf{a} = \frac{-1(2)+3(1)-1(4)}{1+9+1}\mathbf{a} = \boxed{-\frac{3}{11}\mathbf{a} = \left\langle \frac{3}{11}, -\frac{9}{11}, \frac{3}{11}\right\rangle}$$

8. Consider a particle moving in space with ***velocity*** (measured in m/s):

$$\vec{v}(t) = (t^2-4)\vec{\imath} + 3\vec{\jmath} + 3t\sqrt{2}\vec{k}.$$

(a) Find the position vector $\vec{r}(t)$ of the particle at time t if $\vec{r}(1) = 2\vec{\imath} - \vec{\jmath}$.

Solution:

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\frac{t^3}{3} - 4t\right)\vec{\imath} + 3t\vec{\jmath} + \frac{3t^2\sqrt{2}}{2}\vec{k} + \vec{c}$$

$$2\vec{\imath} - \vec{\jmath} = \vec{r}(1) = -\frac{11}{3}\vec{\imath} + 3\vec{\jmath} + \frac{3\sqrt{2}}{2}\vec{k} + \vec{c}$$

$$\Longrightarrow \quad \vec{c} = \left(2 + \frac{11}{3}\right)\vec{\imath} + (-1-3)\vec{\jmath} - \frac{3\sqrt{2}}{2}\vec{k} = \frac{17}{3}\vec{\imath} - 4\vec{\jmath} - \frac{3\sqrt{2}}{2}\vec{k}$$

$$\Longrightarrow \quad \boxed{\vec{r}(t) = \left(\frac{t^3}{3} - 4t + \frac{17}{3}\right)\vec{\imath} + (3t-4)\vec{\jmath} + \frac{3(t^2-1)\sqrt{2}}{2}\vec{k}}$$

Recall the velocity (in m/s):

$$\vec{v}(t) = (t^2 - 4)\vec{\imath} + 3\vec{\jmath} + 3t\sqrt{2}\vec{k}.$$

(b) Find the distance traveled by the particle (i.e. the arc length) between $t = 0$ s and $t = 3$ s.

Solution:

$$\begin{aligned} s(3) = \int_0^3 \|\vec{v}(t)\| \; dt &= \int_0^3 \sqrt{(t^2-4)^2 + 9 + 18t^2} \; dt \\ &= \int_0^3 \sqrt{t^4 - 8t^2 + 16 + 9 + 18t^2} \; dt \\ &= \int_0^3 \sqrt{t^4 + 10t^2 + 25} \; dt \\ &= \int_0^3 \sqrt{(t^2+5)^2} \; dt = \int_0^3 t^2 + 5 \; dt \\ &= \left[\frac{t^3}{3} + 5t\right]_0^3 = 9 + 15 - 0 = \boxed{24 \text{ m}} \end{aligned}$$

(c) Find the tangential component of the acceleration at time t.

Solution: The acceleration is:

$$\vec{a}(t) = 2t\vec{\imath} + 3\sqrt{2}\vec{k}$$

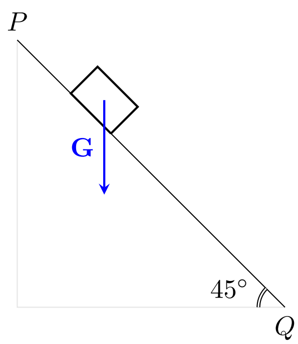
and so the tangential component of acceleration is:

$$\begin{aligned} a_{\vec{T}} &= \frac{\vec{a} \cdot \vec{v}}{\|\vec{v}\|} = \frac{\langle 2t, 0, 3\sqrt{2}\rangle \cdot \langle t^2 - 4, 3, 3t\sqrt{2}\rangle}{t^2+5} \\ &= \frac{2t(t^2-4) + 0(3) + 3\sqrt{2}(3t\sqrt{2})}{t^2+5} = \frac{2t^3 - 8t + 18t}{t^2+5} \\ &= \frac{2t^3 + 10t}{t^2+5} = \boxed{2t} \end{aligned}$$

9. Throughout this problem assume no friction, use 10 m/s^2 as an approximation for the acceleration due to gravity, and don't forget units in your answers. We will consider an ice block of mass 30 kg.

(a) The ice block is brought down along a ramp between P and Q which is at a 45° angle with the horizontal. Find the work done by gravity to move the block down the incline if $\left\|\overrightarrow{PQ}\right\| = 20$ m.

Solution:



Set up $\mathbf{G} = \langle 0, -30(10)\rangle = \langle 0, -300\rangle$

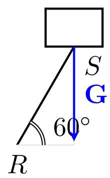
and $\overrightarrow{PQ} = \langle 20\cos 45°, -20\sin 45°\rangle = \langle 10\sqrt{2}, -10\sqrt{2}\rangle$.

Then the work is:

$$W = \mathbf{G} \cdot \overrightarrow{PQ} = \langle 0, -300\rangle \cdot \langle 10\sqrt{2}, -10\sqrt{2}\rangle = \boxed{3000\sqrt{2} \text{ J}}$$

(b) Find the direction ($\odot$ or $\otimes$) and the magnitude of the torque when the weight of the ice block is used at S to rotate an axis placed at R if $\left\|\overrightarrow{RS}\right\| = 6$ m and $\overrightarrow{RS}$ is at a $60°$ angle with the horizontal.

Solution:



Since $\overrightarrow{\tau} = \overrightarrow{RS} \times \mathbf{G}$, by the right hand rule, $\boxed{\text{the direction of the torque is } \otimes}$

And we have $\mathbf{G} = -300\mathbf{j} = -300\langle 0, 1, 0\rangle$ and $\overrightarrow{RS} = \langle 6\cos 60°, 6\sin 60°, 0\rangle = \left\langle 3, 3\sqrt{3}, 0\right\rangle = 3\left\langle 1, \sqrt{3}, 0\right\rangle$.
Therefore,

$$\overrightarrow{\tau} = 3(-300)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \sqrt{3} & 0 \\ 0 & 1 & 0 \end{vmatrix} = -900\langle 0, 0, 1\rangle = -900\mathbf{k}$$

and so its magnitude is $\boxed{900 \text{ Nm}}$.

10. A golf ball takes off from the ground in "Calculus III conditions"[1] with an initial speed of 200 ft/s and at an angle of $50°$ with the horizontal on a flat terrain. Show that the total horizontal distance traveled by the golf ball is

$$x_{\max} = 1250\sin 100° \text{ ft}.$$

Solution: The initial velocity is

$$\mathbf{v}(0) = \langle 200\cos 50°, 200\sin 50°\rangle$$

and since the initial position is $\mathbf{r}(0) = \langle 0, 0\rangle$, we have:

$$\begin{aligned}
\mathbf{a}(t) = \langle 0, -32\rangle \quad &\Longrightarrow \quad \mathbf{v}(t) - \mathbf{v}(0) = \int_0^t \langle 0, -32\rangle\ du = \langle 0, -32u\rangle\Big|_{u=0}^{u=t} = \langle 0, -32t\rangle \\
\Longleftrightarrow \quad & \mathbf{v}(t) = \langle 200\cos 50°, 200\sin 50°\rangle + \langle 0, -32t\rangle = \langle 200\cos 50°, 200\sin 50° - 32t\rangle \\
\Longrightarrow \quad \mathbf{r}(t) - \mathbf{r}(0) &= \int_0^t \langle 200\cos 50°, 200\sin 50° - 32u\rangle\ du = \left\langle 200u\cos 50°, 200u\sin 50° - 16u^2\right\rangle\Big|_{u=0}^{u=t} \\
\Longrightarrow \quad & \mathbf{r}(t) = \left\langle 200t\cos 50°, 200t\sin 50° - 16t^2\right\rangle
\end{aligned}$$

Now we reach $x_{\max}$ when the y-component is back to zero (for some $t_1 > 0$): $\mathbf{r}(t_1) = \langle x_{\max}, 0\rangle$. We solve for t_1 and $x_{\max}$. Starting with the y-component:

$$200t\sin 50° = 16t^2 \quad \Longleftrightarrow \quad t = 0 \quad \text{or} \quad t = 12.5\sin 50°$$

and since $t = 0$ just gives $\mathbf{r}(0) = \langle 0, 0\rangle$, here we have $t_1 = 12.5\sin 50°$ and now we solve from the x-component:

$$x_{\max} = 200(12.5\sin 50°)\cos 50° = 100(12.5)\sin 100° = 1250\sin 100° \text{ ft}. \qquad \checkmark$$

[1] I.e. the acceleration is constant and only due to gravity at 32 ft/s^2. That is we ignore ball spin, air resistance, etc.