Math 215

Practice Problems for the Final Exam

Some of these problems are taken from the review problems in the textbook, at the end of each chapter. The numbers in parentheses are the chapter and problem number.

1. (a) (16#13) Calculate

$$\int_0^1 \int_x^1 e^{x/y}\, dydx$$

by first reversing the order of integration. (b) Find the center of mass of the region D in the first quadrant that lies above the hyperbola $xy = 1$ and the line $y = x$, and below the line $y = 2$, if its density is constant.

2. Find the area of the region enclosed by one loop of the lemniscate given in polar coordinates by $r^2 = \cos(2\theta)$.

3. (a) Find the area of the region enclosed by the cardioid given in polar coordinates by $r = 1 + \cos(\theta)$. (b) Use polar coordinates to evaluate

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy.$$

4. Find the mass and center of mass of the tetrahedron with the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$ whose density is given by $\rho(x,y,z) = x$.

5. Let $f(x,y,z) = x + 2y + z$, and R the solid $x^2 + y^2 + z^2 \leq 4$, $\sqrt{x^2+y^2} \leq z$. Set up an integral to compute $\int\int\int_R f(x,y,z)\, dV$ *(a)* using rectangular coordinates (x,y,z), *(b)* using cylindrical coordinates (r,θ,z), and *(c)* using spherical coordinates (ρ,θ,φ) (do not evaluate the integrals).

6. (a) (16#33) Find the volume of one of the wedges cut from the cylinder $x^2+y^2 = a^2$ by the planes $z = 0$ and $z = mx$. (b) (16#42) Use spherical coordinates to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^2\, dz\, dy\, dx\,.$$

7. Set up a triple integral in cylindrical coordinates to compute the total mass of the solid in the first octant obtained by removing the cylinder $x^2 + y^2 = 1$ from the sphere $x^2 + y^2 + z^2 = 4$, if the density is given by $\rho(x,y,z) = z^2 + \sqrt{x^2+y^2}$ (do not evaluate the integral).

8. Consider the sphere $x^2 + (y-3)^2 + z^2 = 25$ and the cylinder $x^2 + y^2 = 4$. Set up (but do not evaluate) an integral which calculates the volume of the intersection of the inside of the sphere and the inside of the cylinder.

9. (17#10) Find the work done by the force field $\vec{F}(x,y,z) = z\vec{i} + x\vec{j} + y\vec{k}$ in moving a particle from the point $(3,0,0)$ to the point $(0,\pi/3,3)$ (i) along a straight line and (ii) along the helix $x = 3\cos t$, $y = t$, $z = 3\sin t$. Is this force field conservative? Justify your answer.

10. (17#16) Use Green's theorem to evaluate $\int_C (1+\tan x)\,dx + (x^2+e^y)\,dy$ where C is the positively oriented boundary of the region enclosed by the curves $y=\sqrt{x}$, $x=1$, and $y=0$.

11. Consider the vector field

$$\vec{F} \;=\; \frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath}\,.$$

(a) Evaluate directly the line integral of $\vec{F}$ along the unit circle, once around in the counter-clockwise direction. Is $\vec{F}$ conservative?
(b) Compute the curl of $\vec{F}$. Why does your answer not contradict Green's theorem?

12. Let C_1 be the unit circle $x^2+y^2=1$ and C_2 the concentric circle of radius two. Orient both C_1 and C_2 counterclockwise. Suppose that $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ is a vector field on the plane such that

$$\int_{C_1} \vec{F}\cdot d\vec{n} \;=\; 10 \quad \text{and} \quad \int_{C_1} \vec{F}\cdot d\vec{r} \;=\; 17\,.$$

(a) If $\vec{F}$ is smooth on the plane, compute $\int\int_D \operatorname{div}(\vec{F})\,dA$ where D is the domain defined by the inequality $x^2+y^2\le 1$.
(b) If $\vec{F}$ is smooth on the annulus bounded by C_1 and C_2, and $\partial Q/\partial x = \partial P/\partial y$ everywhere on the annulus, compute $\int_{C_2} \vec{F}\cdot d\vec{r}$.

13. (17#19) Is there a vector field $\vec{G}$ such that curl $\vec{G} = 2x\vec{\imath} + 3yz\vec{\jmath} - xz^2\vec{k}$? Justify your answer.

14. (a) Find the center of mass of a thin wire bent into the shape of the quarter-circle $x^2+y^2=a^2$, $x\ge 0$, $y\ge 0$. (b) Find the center of mass of the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant, if the density is constant.

15. (17.7# 40) A fluid has density 1500 and flows with velocity field $\vec{V} = -y\vec{\imath} + x\vec{\jmath} + 2z\vec{k}$. Find the rate of flow outward through the sphere $x^2+y^2+z^2=25$.

16. (14#34) Use the divergence theorem to calculate the surface integral $\int\int_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = x^3\vec{\imath} + y^3\vec{\jmath} + z^3\vec{k}$ and S is the surface of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=2$.

17. (17#33) Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = xy\vec{\imath} + yz\vec{\jmath} + zx\vec{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, oriented counterclockwise as viewed from above.

18. (17#36) Compute the outward flux of $\vec{F}(x,y,z) = (x^2+y^2+z^2)^{-3/2}(x\vec{\imath}+y\vec{\jmath}+z\vec{k})$ through the ellipsoid $4x^2+9y^2+6z^2=36$. Hint: Wouldn't it be easier to compute the flux through a sphere?

19. There will be questions on the exam on the material of the last three lab projects.