Galois Theory

Homework problems

due Thursday, February 12

1. Consider the set $V = \mathbb{Q}(\sqrt[3]{2}) = \left\{a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 \mid a, b, c \in \mathbb{Q}\right\}$. Show that V is a vector space over $\mathbb{Q}$ with basis $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$.

   As an alternative, you can prove the more general statement: Suppose that α is a root of an irreducible polynomial $f(x) = x^3 + n$ for n some integer. Prove that $\{1, \alpha, \alpha^2\}$ is a basis for $V = \{a_0 + a_1\alpha + \cdots + a_s\alpha^s \mid a_i \in \mathbb{Q},\ s \text{ a non-negative integer}\}$, the collection of polynomials in α with rational coefficients. Thus, V has dimension 3 as a vector space.

2. Assume R is a commutative ring with 1. Let $\mathfrak{S}$ be a multiplicatively closed set.

   (a) Suppose that $\mathfrak{p}$ is a prime ideal of R and set $\mathfrak{S} = R \setminus \mathfrak{p}$.

   i. Show that $\mathfrak{S}$ is multiplicatively closed with 1.

   ii. Prove the converse to (i): Namely, that if $\mathfrak{A}$ is an ideal of R and $\mathfrak{S} = R \setminus \mathfrak{A}$ is a multiplicatively closed set with 1, then $\mathfrak{A}$ is a prime ideal. (The upshot of this is that localization of rings takes place at *prime* ideals of R, if you require that $\mathfrak{S}$ contain 1.)

   iii. Define the *localization* $R_\mathfrak{p}$ *of* R *at* $\mathfrak{p}$ with elements

   $$R_\mathfrak{p} = \left\{ \left[\frac{r}{s}\right] \mid r \in R,\ s \in \mathfrak{S} \right\}.$$

   A. By consulting a book, or better yet, thinking about the construction of the quotient field of a domain, define the appropriate equivalence relationship for the elements $\frac{r}{s} = (r, s) \in R \times \mathfrak{S}$ in the classes listed above. (*Hint:* Be a tad careful here. In problem, vi (c) below, we will allow $\mathfrak{S}$ to have zero divisors.)

   iv. Convince yourself that $R_\mathfrak{p}$ is a ring. Convince me that you have done this, but showing me the definition of $\cdot$ in $R_\mathfrak{p}$.

   v. What are the units of $R_\mathfrak{p}$?

   vi. Assume further that $R = \mathbb{Z}$ and $\mathfrak{p} = (5)$. What are the elements of $\mathbb{Z}_{(5)}$? (You can describe them explicitly.) What are the units of $\mathbb{Z}_{(5)}$? What are the prime ideals of $\mathbb{Z}_{(5)}$? What are all the ideals of $\mathbb{Z}_{(5)}$? Draw the ideal lattice diagram for $\mathbb{Z}_{(5)}$.

(b) Let $\mathbb{E}$ denote the positive even integers in the ring $R = \mathbb{Z}$. Note that $\mathbb{E}$ is multiplicatively closed.

Show that the *localization of* $\mathbb{Z}$ *at* $\mathbb{E} \doteq \mathbb{E}^{-1}\mathbb{Z}$ is the rational numbers $\mathbb{Q}$.

(c) Let f be any element of R and $\mathfrak{S} = \{f^n \mid n \in \mathbb{Z}^+ \cup \{0\}\}$. Define R_f, the *localization of* R *at* f, to be the set of equivalence classes of the form $\left[\frac{r}{f^n}\right]$ for $r \in R$, $n \geq 0$ in the 'usual' way.

i. Show that f is nilpotent if, and only if, $R_f = 0$.

ii. Show that if f is not nilpotent, then f becomes a unit in R_f.

3. Express $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$ as a function of the elementary symmetric polynomials.