Calculus III; solutions to graded homework problems #8

Sec 14.8 #27. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

$$f(x,y,x) = x^2 + y^2 + z^2, \;\; g(x,y,z) = z^2 - xy - 1$$

$$\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle -\lambda y, -\lambda x, 2\lambda z \rangle.$$

Then $2z = 2\lambda z$ implies $z = 0$ or $\lambda = 1$. If $z = 0$ then $g(x,y,z) = 0$ implies $xy = -1$ or $x = -1/y$. Thus $2x = -\lambda y$ and $2y = -\lambda x$ imply $\lambda = 2/y^2 = 2y^2$ or $y = \pm 1$, $x = \pm 1$. If $\lambda = 1$, then $2x = -y$ and $2y = -x$ imply $x = y = 0$, so $z = \pm 1$. Hence the possible points are $(\pm 1, \mp 1, 0)$, $(0,0,\pm 1)$ and the minimum value of f is $f(0,0,\pm 1) = 1$, so the points closest to the origin are $(0,0,\pm 1)$.

Sec 15.1 #3.

(a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R \sin(x+y)\, dA$, where $R = [0,\pi] \times [0,\pi]$. Take the sample points to be lower left corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

Since $\triangle A = \pi^2/4$, we estimate

(a)

$$\iint_R \sin(x+y) dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i^*, y_j^*) \triangle A$$
$$= f(0,0)\triangle A + f(0,\pi/2)\triangle A + f(\pi/2,0)\triangle A + f(\pi/2,\pi/2)\triangle A$$
$$= 0(\pi^2/4) + 1(\pi^2/4) + 1(\pi^2/4) + 0(\pi^2/4) = \pi^2/2 \approx 4.935$$

(b)

$$\iint_R \sin(x+y) dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i^*, y_j^*) \triangle A$$
$$= f(\pi/4,\pi/4)\triangle A + f(\pi/4,3\pi/4)\triangle A + f(3\pi/4,\pi/4)\triangle A + f(3\pi/4,3\pi/4)\triangle A$$
$$= 1(\pi^2/4) + 0(\pi^2/4) + 0(\pi^2/4) + -1(\pi^2/4) = 0$$

Sec 15.2 #29. Find the volume of the solid in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$.

In the first octant, $z \geq 0$, so $y \leq 3$.

$$V = \int_0^3 \int_0^2 (9 - y^2)\, dx\, dy = \int_0^3 [9x - y^2 x]_{x=0}^{x=2}\, dy$$

$$= \int_0^3 18 - 2y^2\, dy = [18y - 2/3y^3]_0^3 = 36$$

Sec 15.3 #31. The solid enclosed by the parabolic cylinders $y = 1-x^2$ and $y = x^2-1$ and the planes $x+y+z = 2$, $2x+2y-z+10 = 0$.

The curves $y = 1 - x^2$ and $y = x^2 - 1$ intersect at $(\pm 1, 0)$ with $1-x^2 \geq x^2-1$ on $[-1, 1]$. Within this region, the plane $z = 2x+2y+10$ is above the plane $z = 2 - x - y$, so

$$V = \int_{-1}^1 \int_{x^2-1}^{1-x^2} (2x + 2y + 10)\, dy\, dx - \int_{-1}^1 \int_{x^2-1}^{1-x^2} (2 - x - y)\, dy\, dx$$

$$= \int_{-1}^1 \int_{x^2-1}^{1-x^2} (3x + 3y + 8)\, dy\, dx = \int_{-1}^1 [3xy + 3/2\ y^2 + 8y]_{x^2-1}^{1-x^2}\, dx$$

$$= \int_{-1}^1 (-6x^3 - 16x^2 + 6x + 16)\, dx = [-3/2\ x^4 - 16/3\ x^3 + 3x^2 + 16x]_{-1}^1 = 64/3$$