Hi,

The homeworks were good, but here is a good proof of

53. Let G be a group, $h, k \in G$, and let $\varphi : \mathbb{Z} \times \mathbb{Z} \to G$ be defined by $\varphi((m,n)) = h^m k^n$. Give necessary and sufficient conditions, involving h and k, for φ to be a homomorphism.

φ is a homomorphism $\Leftrightarrow$ h and k commute; $hk = kh$

Proof:

$$\begin{aligned}
\varphi \text{ is a homomorphism} &\Leftrightarrow \varphi((2,2)) = \varphi((1,1)) \cdot \varphi((1,1)) \\
&\Leftrightarrow h^2 k^2 = (hk)(hk) \\
&\Leftrightarrow h^{-1}[h^2 k^2]k^{-1} = h^{-1}[hkhk]k^{-1} \\
&\Leftrightarrow hk = kh
\end{aligned}$$

$\square$

54. is similar. G must be Abelian.

50. Let $\varphi : G \to H$ be a group homomorphism. Show that $\varphi[G]$ is Abelian if, and only if, $\forall x, y \in G$, we have $xyx^{-1}y^{-1} \in \ker \varphi$.

Proof: For short, let $K =$ kernel φ. Then

$$\begin{aligned}
\varphi[G] \text{ is Abelian} &\Leftrightarrow \forall x,y \in G,\ \varphi(x)\varphi(y) = \varphi(y)\varphi(x) \\
&\Leftrightarrow \forall x,y \in G,\ \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = e_H \\
&\Leftrightarrow \forall x,y \in G,\ \varphi(xyx^{-1}y^{-1}) = e_H \quad \text{since } \varphi \text{ is a homomorphism} \\
&\Leftrightarrow xyx^{-1}y^{-1} \in K.
\end{aligned}$$

$\square$