(e) an infinite ring R that is not an integral domain

Bobby

(f) a ring R and a prime ideal I such that I is not maximal

Bobby

(g) a Euclidean domain that is not field

$\mathbb{Z}$, $\mathbb{F}[x]$

(3) (6 points) List all group homomorphisms from $\mathbb{Z}/24\mathbb{Z}$ to $\mathbb{Z}/60\mathbb{Z}$

For $\varphi : \mathbb{Z}/24\mathbb{Z} \to \mathbb{Z}/60\mathbb{Z}$ to be a group homomorphism, we must find conditions on $a \in \mathbb{Z}/60\mathbb{Z}$ such that if $\varphi(1) = a$, then φ is a group homomorphism.

In particular, we need $0 = \varphi(24 \cdot 1) = \varphi(\underbrace{1 + \cdots + 1}_{24 \text{ times}}) = 24 \cdot a$ or

(*) $24a \equiv 0 \mod 60$. Let $d = 12 = \gcd(24, 60)$, then we note that if $a = \frac{60}{d} \cdot k = 5k$ $k = 1, 2, \ldots, 12$, then we have solutions to (*). That is, $a = 5, 10, \ldots, 55, 0 \mod 60$ are all five values of a.

For more detail, note that

$24a \equiv 0 \mod 60 \iff 2 \cdot da \equiv 0 \mod 5 \cdot d \iff 2 \cdot 12 \cdot a \equiv 0 \mod 5 \cdot 12$

$\iff 2a \equiv 0 \mod 5 \iff a \equiv 0 \mod 5 \iff 5 | a$. Then find all a with $0 \leq a < 60$ and $5 | a$.

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(4) (8 points) Determine whether the polynomial $f(x) = x^3 + x^2 + x + 2$ is reducible in each of the following rings. Briefly justify your answer.

(a) $\mathbb{Z}/2\mathbb{Z}[x]$

reducible, $x=1$ is a root

(b) $\mathbb{Z}/3\mathbb{Z}[x]$ $f(1) = 2 \neq 0$ $f(0) \neq 0$.

$f(2) = 8+4+2+2 = 12+4 \not\equiv 0 \mod 3$

irreducible by Dedekind Criterion

(c) $\mathbb{Q}[x]$ Check $\pm 1, \pm 2$

irreducible.

$f(1) = 5$ $f(2) > 0$

$f(-1) = 1$ $f(-2) = -8+4-2+2 \neq 0$

(5) (6 points) Observe that the polynomial $3x^2 + 4x + 3 \in \mathbb{Z}/5\mathbb{Z}[x]$ factors both as $(3x+2)(x+4)$ and as $(4x+1)(2x+3)$. Explain whether or not this illustrates that $\mathbb{Z}/5\mathbb{Z}[x]$ is not a UFD.

Does NOT. Note $\mathbb{Z}/5\mathbb{Z}$ is a field $\Rightarrow$ $\mathbb{Z}/5\mathbb{Z}[x]$ is a UFD.

The point is that $3, 1, 4, 2$ are all units in $\mathbb{Z}/5\mathbb{Z}[x]$. In particular, $3 \cdot 2 = 1$ in $\mathbb{Z}/5\mathbb{Z}$.

$(3x+2)(x+4)$

$= [3(3x+2)][2(x+4)]$ since $3 \cdot 2 = 1$

$= (4x+1)(2x+3)$

i.e. $(3x+2)$ and $(4x+1)$ are associates

and $(x+4)$ and $(2x+3)$ are associates.

From Answer: Dakota did a great job! Look up a proof to Lagrange's Thm in book. See you Monday.