# HW #10 Solutions

§23 #14. Show that $f(x) = x^2 + 8x - 2$ is irreducible over $\mathbb{Q}$. Is $f(x)$ irreducible over $\mathbb{R}$? Over $\mathbb{C}$?

Let $p = 2$. Then $p \mid 8$ and $p \mid -2$. However, $p \nmid 1$ and $p^2 \nmid -2$. By Eisenstein's Criterion, $f(x)$ is irreducible over $\mathbb{Q}$.

However, $f(x)$ is reducible over $\mathbb{R}$. The quadratic formula gives 2 real roots of $f(x)$. Also, $f(0) = -2$, $f(1) = 7$, and $f(x)$ is continuous, so by the Intermediate Value Theorem from Calculus, $f$ must have a real root in the interval $(0, 1)$.

Since $f(x)$ is reducible over $\mathbb{R}$, it is also reducible over $\mathbb{C}$.

16. Demonstrate that $x^3 + 3x^2 - 8$ is irreducible over $\mathbb{Q}$.

Let $f(x) = x^3 + 3x^2 - 8$ and let $g(x) = f(x+2)$. Then

$$g(x) = (x+2)^3 + 3(x+2)^2 - 8 = (x^3 + 6x^2 + 12x + 8) + (3x^2 + 12x + 12) - 8$$
$$= x^3 + 9x^2 + 24x + 12$$

Let $p = 3$. Then $p \mid 9$, $p \mid 24$, and $p \mid 12$, but $p \nmid 1$ and $p^2 \nmid 12$. By Eisenstein's Criterion, $g(x)$ is irreducible. Therefore, $f(x)$ is irreducible as well.

20. Determine whether the polynomial in $\mathbb{Z}[x]$ satisfies an Eisenstein criterion for irreducibility over $\mathbb{Q}$.

$4x^{10} - 9x^3 + 24x - 18$

The only prime that divides $-9$, $24$, and $-18$ is $p = 3$. However, $p^2 \mid -18$. Thus, the polynomial does not satisfy any Eisenstein criterion.

28. Find all irreducible polynomials of the indicated degree in the given ring.

Degree 3 in $\mathbb{Z}_2[x]$

A polynomial of degree 3 is irreducible if and only if it has no roots. The only polynomials of degree 3 in $\mathbb{Z}_2[x]$ with no roots are $x^3 + x + 1$ and $x^3 + x^2 + 1$.