Homework #4
Selected solutions and comments.

4.6 Exercises

5. Show that two splits $Y_0|Y_1$ and $Y_0'|Y_1'$ of a set Y are compatible if and only if $Y_i \subseteq Y_j'$ for some i, j. Further show that $Y_i \subseteq Y_j'$ if and only if $Y_{1-i} \supseteq Y_{1-j}'$.

Solution 5. Suppose that two splits $Y_0|Y_1$ and $Y_0'|Y_1'$ of a set Y are compatible and, without loss of generality, that $Y_1 \cap Y_1' = \emptyset$. Then

$$Y_1 = Y_1 \cap Y = Y_1 \cap (Y_0' \cup Y_1') = (Y_1 \cap Y_0') \cup (Y_1 \cap Y_1') = (Y_1 \cap Y_0') \cup \emptyset = (Y_1 \cap Y_0').$$

Then $Y_1 = Y_1 \cap Y_0'$ and it means $Y_1 \subseteq Y_0'$. The same approach works for 3 other options.

Now suppose that $Y_1 \subseteq Y_0'$. Then $Y_1 \cap Y_1' = \emptyset$ since Y_0' and Y_1' are disjoint. Hence two splits $Y_0|Y_1$ and $Y_0'|Y_1'$ of a set Y are compatible if and only if $Y_i \subseteq Y_j'$ for some i, j.

Also $Y_i \subseteq Y_j'$ if and only if $Y_{1-i} \supseteq Y_{1-j}'$, because taking complements in Y reverses set inclusions.

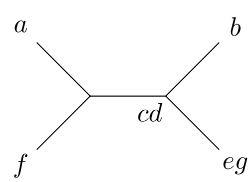
9. For $X = \{a, b, c, d, e, f, g\}$ consider the pairwise compatible splits

$$a|bcdefg, \; eg|abcdf, \; b|acdefg, \; af|bcdeg, \; f|abcdeg.$$

a. By Tree Popping, find an X-tree inducing precisely these splits.

b. Use Tree Popping again, but with the splits in some other order, to again find the same tree.

Solution 9. You should get the tree below regardless of the ordering of the splits. Notice that since there were five splits listed, your final tree should have only five edges. Only three trivial splits were listed.



11. If T_1, T_2 are X-trees with T_2 a refinement of T_1, write $T_1 \le T_2$. Show that " $\le$ " is a partial order on the set of X-trees. Then characterize X-trees that are maximal with respect to " $\le$ ".

Solution 11. For " $\le$ " to be a partial order, you need to show that it is reflexive ($T \le T$), antisymmetric (if $T_1 \le T_2$ and $T_2 \le T_1$, then $T_1 = T_2$), and transitive. These follow easily from definition. A maximal element with respect to $\le$ is a *binary* tree.

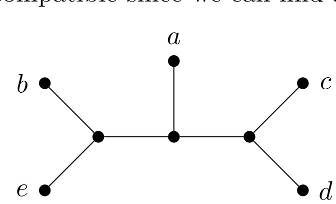
14. A collection of quartets is said to be *compatible* if there is some tree T that displays them all.

a. Show that the quartets $ab|cd$ and $ac|be$ are compatible.

b. Explain why the quartets $ab|cd$, $ac|be$, and $bc|de$ are not compatible.

Solution 14. This problem tripped up a fair number of students. Please review this solution if you were one of them.

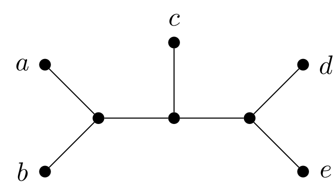
a. The quartets $ab|cd$ and $ac|be$ are compatible since we can find a tree that displays them all.



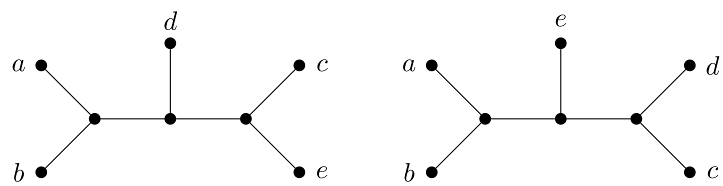
b. The quartets $ab|cd$, $ac|be$, and $bc|de$ are not compatible, because in part (a) we found a tree displaying two of them, but it does not show $bc|de$. The (compatible) quartet involving b, c, d, e displayed by this tree is $be \mid cd$.

15. For a 5-leaf binary phylogenetic X-tree T, $\mathcal{Q}(T)$ has 5 elements. Draw such a 5-leaf tree and give a set of only two of these quartets that still determine T. Give another set of two of these quartets that does not determine T.

Solution 15. Suppose $X = \{a, b, c, d, e\}$. A 5-leaf binary phylogenetic X-tree T can be



A set of only two of these quartets that still determine T is $\{ab|cd, ac|de\}$. A set of two of these quartets that does not determine T is $\{ab|cd, ab|ce\}$ because there are two other options for T except that one showed above. They are



So this set cannot determine T.

16. Generalize the result of the last problem by showing that for any binary phylogenetic X-tree, there are $|X| - 3$ quartets that determine T.

Solution 16. A binary phylogenetic X-tree T has exactly $|X| = n$ trivial splits and $n - 3$ non-trivial splits. To determine T, (that is, to be able to draw T correctly), we need to determine the internal edges of T. For any internal edge of T, removing this edge (including its vertices) partitions the n taxa into four non-empty sets. Choosing one taxon from each set and considering the induced quartet on these taxon labels gives a quartet tree that determines that particular internal edge.

18. Consider the three trees $(a, (b, (c, d)))$, $((a, b), (c, d))$, and $(((a, b), c), d)$ on taxa $X = \{a, b, c, d\}$.

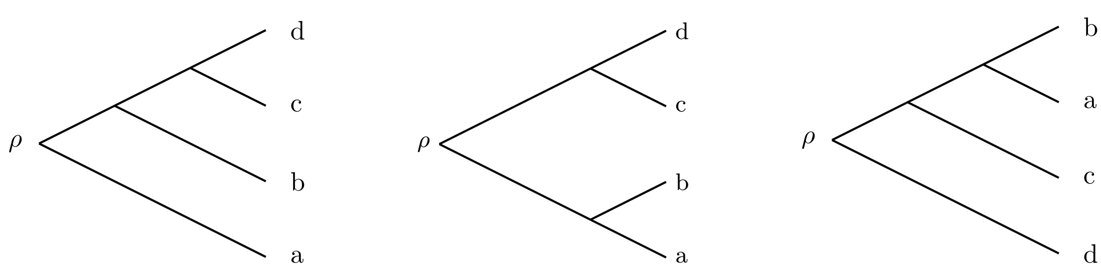
a. Construct the strict consensus tree, treating these as rooted.

b. Construct the strict consensus tree, treating these as unrooted 4-taxon trees. Is your answer the same as the unrooted version of the tree from (a)?

c. Construct the majority-rule consensus tree, treating these as rooted.

Solution 18.

a. These three trees is drawn below.

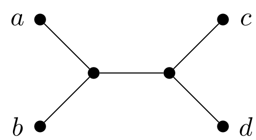


The strict consensus tree is: the rooted star tree.

b. The list of splits displayed on the three unrooted version of these trees is

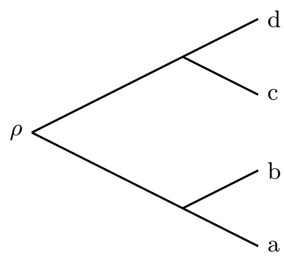
$$a|bcd,\ b|acd,\ c|abd,\ d|abc,\ cd|ab,\ ab|cd,\ ab|cd.$$

Then the strict consensus tree is:



This is not the same as the unrooted version of the tree from (a).

c. For constructing the majority-rule consensus tree, the non-trivial clades that appear in two out of the three (= a majority) of the trees are $\{a, b\}$, and $\{c, d\}$. The majority consensus tree is:



21. Consider the three trees shown below.

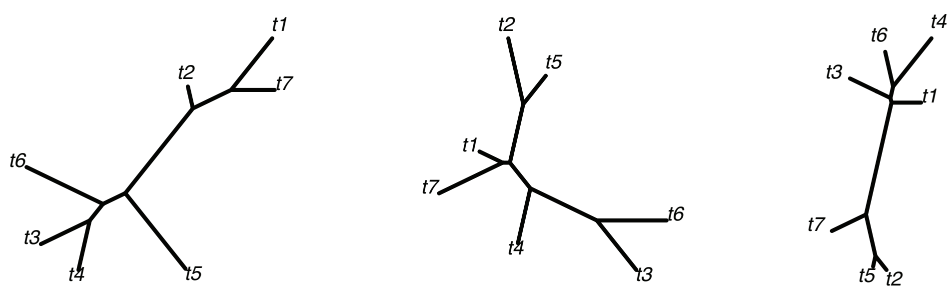


Figure 4.1: Trees T_1, T_2, T_3.

(a) For each of the three trees, list all splits displayed on the tree.

(b) Construct the strict consensus tree.

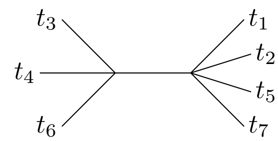
(c) Construct the majority rule consensus tree.

Solution 21.

a. In addition to the trivial splits, the list of splits for each of them is:

$$\begin{aligned}
T_1: &\quad t_3t_4 \mid t_1t_2t_5t_6t_7,\ \boxed{t_3t_4t_6 \mid t_1t_2t_5t_7},\ t_3t_4t_5t_6 \mid t_1t_2t_7,\ \boxed{t_2t_3t_4t_5t_6 \mid t_1t_7},\\
T_2: &\quad \boxed{t_2t_5 \mid t_1t_3t_4t_6t_7},\ \boxed{t_1t_7 \mid t_2t_3t_4t_5t_6},\ \boxed{t_3t_4t_6 \mid t_1t_2t_5t_7},\ t_1t_2t_4t_5t_7 \mid t_3t_6,\\
T_3: &\quad t_4t_6 \mid t_1t_2t_3t_5t_7,\ \boxed{t_3t_4t_6 \mid t_1t_2t_5t_7},\ t_1t_3t_4t_6 \mid t_2t_5t_7,\ \boxed{t_1t_3t_4t_6t_7 \mid t_2t_5},
\end{aligned}$$

b. Using the trivial splits and the red split above, the strict consensus tree is:



c. The majority-rule consensus tree should contain the trivial splits, and the ones in blue, cyan, and red above.