HW #7 Solutions

1. Show that $p_n = 10^{-2^n}$ converges quadratically to zero.

Compute $\lim_{n\to\infty} \frac{p_{n+1}}{p_n^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1 = C.$

Since $C \neq 0, \infty$, by definition the sequence converges quadratically to zero.

2. Show that $p_n = 10^{-n^k}$ does not converge quadratically to zero, for any $k > 1$.

Compute $\lim_{n\to\infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n\to\infty} \frac{10^{-[n^k + k n^{k-1} + \dots + kn + 1]}}{10^{-2n^k}} = +\infty$

If you don't see this right away, notice for $n \gg 0$, $\frac{10^{-(n+k)}}{10^{-2n^k}} \approx \frac{10^{-n^k}}{10^{-2n^k}}$

$\approx 10^{n^k}$ which is huge. Since the limit is infinite, the order of converge is not quadratic. Indeed, the denominator p_n^2 goes to zero to quickly, so the order of convergence < 2. Indeed, this is THE POINT: in 1), the exponent is an exponential, in 2) a power.

3. a) Show $p_n = \frac{1}{n^2}$ converges linearly to zero for $n \geq 1$

Compute $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$ Since $1 \neq 0, \infty$,

the converge is linearly.

b) Show that $p_n = \frac{1}{n^k}$ converges linearly to 0, for $n \geq 1$ and any integer $k \geq 1$.

Compute $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{(n)^k}} = \lim_{n\to\infty} \frac{n^k}{(n+1)^k} = \lim_{n\to\infty} \frac{n^k}{n^k + kn^{k-1} + \dots + n + 1}$

$= 1 = C$. Since $C \neq 0, \infty$, the order of convergence is $p = 1$ or linear.

4. Error

5. "Fixed-Slope" $f(x) = 2 - e^{-x}$

(a) $\alpha = -\ln 2 \approx -.6931$

(b) $\varphi(x) = x - \dfrac{f(x)}{g} = x - \dfrac{(2-e^{-x})}{g} = x - \frac{1}{g}(2 - e^{-x})$

$\varphi'(x) = 1 - \frac{1}{g} e^{-x}$

(c) For $x_0 = 2$, $g = 1$, this method converges, but <u>very</u> slowly.

(d) $e_{k+1} = \varphi'(x_k)\, e_k \Rightarrow e_{k+1} = \Big(\underbrace{1 - \frac{1}{g} e^{-x_k}}_{\varphi'(x_k)}\Big) e_k$

(e) As we saw in class, everything depends on $\varphi'(\alpha)$. If $|\varphi'(\alpha)| < 1$, then $x_n \to \alpha$. Computing

$$\varphi'(\alpha) = \varphi'(-\ln 2) = 1 - \frac{1}{g} e^{-(-\ln 2)} = 1 - \frac{1}{g} e^{\ln 2} = 1 - \frac{2}{g}$$

Thus, $|\varphi'(-\ln 2)| < 1$ if, and only if, $\left|1 - \frac{2}{g}\right| < 1$

or $-1 < 1 - \frac{2}{g} < 1$ or $-2 < \frac{-2}{g} < 0$

or $1 > \frac{1}{g} > 0$ $\qquad$ $-2 < 0$ switches inequality

or $g > 1 > 0$

i.e. $\boxed{g > 1}$