Oblig 5 Fys 2140 eoflund / Eirik Lund

2.7 $\Psi(x,0) = \begin{cases} Ax & 0 \le x \le a/2 \\ A(a-x) & a/2 \le x \le a \end{cases}$

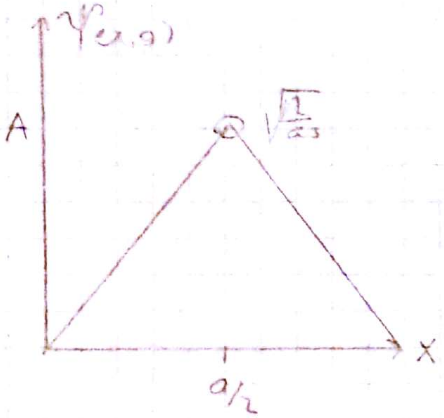
a) finn A ved normering $1 = \int_0^a |\Psi(x,0)|^2 dx$

Siden symmetrisk, så velger $x \in [0, a/2]$ Ax

$|Ax|^2 = A^2x^2$, $A^2\int_0^{a/2} x^2 dx = A\frac{x^3}{3}\Big|_0^{a/2} \Rightarrow \frac{A^2a^3}{24} = 1$

for å få $x \in [0, a]$ ganger vi med 2.

$A = \sqrt{\frac{12}{a^3}}$



Som er maksverdien til bølgefunksjonen

b) finn $\Psi(x,t)$, $\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)e^{-i\frac{\hbar n^2\pi^2}{2ma^2}t}$

for $x \in [0, a/2]$ $c_n = \sqrt{\frac{2}{a}}\sqrt{\frac{12}{a^3}}\int_0^{a/2} \sin\left(\frac{n\pi x}{a}\right) x\, dx$

delvis integrasjon: $v = x$ $v' = 1$

$f = \sin\left(\frac{n\pi x}{a}\right)$ $F = \frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)$

$c_n = \frac{2\sqrt{6}}{a^2}\left(\left[x \cdot \frac{-a}{n\pi}\cos\frac{n\pi x}{a}\right|_0^{a/2} + \int_0^{a/2}\frac{a}{n\pi}\cos\frac{n\pi x}{a}dx\right)$

$c_n = \frac{2\sqrt{6}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2\sqrt{6}}{n\pi}\cos\left(\frac{n\pi}{2}\right)$

$c_n = \begin{cases} 0 & \text{ved } n = 0 \quad \underline{\text{even}} \\ \frac{4\sqrt{6}}{n^2\pi^2} & \text{ved } n = 1, 5, 9, 13 \\ -\frac{4\sqrt{6}}{n^2\pi^2} & \text{ved } n = 3, 7, 11 \end{cases}$ $\Big\}$ odd

c) $|c_1|^2 = E_1$

$$\frac{4^2\sqrt{6}^2}{\pi^4} = \frac{96}{\pi^4} \approx 98.55\%$$ Sannsynlighet for E_1.

d) $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ $\qquad \langle H\rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$

$$|c_n|^2 E_n = \frac{48\hbar^2}{n^2 m a^2}$$

Bruker at $\sum_{n\,odd} \frac{1}{n^2}$ konvergerer mot $\frac{\pi^2}{8}$

$$\langle H\rangle = \frac{48\hbar^2}{\pi^2 n^2 m a^2}\sum_{n\,odd} = \frac{48\hbar^2}{\pi^2 m a^2}\frac{\pi^2}{8} = \frac{6\hbar^2}{ma^2}$$

2.11 $\quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$

$$\psi_1(x) = A\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\, e^{-\frac{m\omega}{2\hbar}x^2}$$

a) $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx$

kaller $\alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$ og $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$

$$\langle x\rangle_{\psi_0} = \alpha\int_{-\infty}^{\infty} x\frac{2}{m\omega x}e^{-u}\,du \qquad u = \frac{m\omega x^2}{2\hbar}$$

$$\langle x\rangle_{\psi_0} = \frac{\alpha}{m\omega}2\cdot 2\left|\frac{1}{e^{\infty}}\right|_0^{\infty} = 0$$ også fordi vi får en odde integrand slik at $\langle x\rangle = 0$

eneste forskjellen mellom ψ_0 og ψ_1 er en skalar på $\sqrt{\frac{2m\omega}{\hbar}}$

så $\langle x\rangle_{\psi_1} = 0$

$$\langle p\rangle_{\psi_0} = m\frac{\partial\langle x\rangle}{\partial t} = 0$$

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$$\langle x^2\rangle_{\psi_0} = \int_{-\infty}^{\infty} x^2|\psi_0(x)|^2 dx = \alpha^2\int_{-\infty}^{\infty} x^2 e^{-\xi^2} dx$$

vi substituerer x^2 med ξ^2

$$\alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2} d\xi$$ vi vet at $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

vi klarer å bli kvitt ξ^2 så integralet blir flg:

$$\alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\frac{\sqrt{\pi}}{2} = \frac{\hbar}{2m\omega} \qquad \langle x^2\rangle_{\psi_0} = \frac{\hbar}{2m\omega}$$

$$\langle x^2\rangle_{\psi_1} = \int_{-\infty}^{\infty} x^2|\psi_1(x)|^2 dx$$ Samme substitusjon for x^2

gir $\langle x^2\rangle_{\psi_1} = 2\alpha^2\frac{\hbar}{m\omega}\left(\frac{\hbar}{m\omega}\right)^{1/2}\int_{-\infty}^{\infty}\xi e^{-\xi^2}d\xi$

$$\langle x^2\rangle_{\psi_1} = 2\sqrt{\frac{m\omega}{\hbar\pi}}\sqrt{\frac{\hbar^2}{m^2\omega^2}}\sqrt{\frac{\hbar}{m\omega}}\frac{3\sqrt{\pi}}{4} = \frac{3}{2}\frac{\hbar}{m\omega}$$

$$\langle p^2\rangle_{\psi_0} = \int_{-\infty}^{\infty}\psi_0\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi_0\, dx$$ pga produkt regel

Så kan vi omforme til $\langle p^2\rangle_{\psi_0} = \alpha^2\sqrt{m\omega}\,\hbar^{3/2}\int_{-\infty}^{\infty}(1-\xi^2)e^{-\xi^2}d\xi$

$$\langle p^2\rangle_{\psi_0} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{m\omega}\,\hbar^{3/2}$$ komponenten er $\frac{m\omega}{\sqrt{\pi}}\hbar^{\frac{3}{2}-\frac{1}{2}}$

$$\langle p^2\rangle_{\psi_0} = \frac{m\omega}{\sqrt{\pi}}\hbar\underbrace{\int_{-\infty}^{\infty}(1-\xi^2)e^{-\xi^2}d\xi}$$ Som vi vet

er $\frac{\sqrt{\pi}}{2}$ gir endelig $\langle p^2\rangle_{\psi_0} = \frac{1}{2}m\omega\hbar$

$$\langle p^2 \rangle_{\psi_1} = \int_{-\infty}^{\infty} \psi_1 \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \psi_1 \, dx$$

Observerer at ψ_0 var grunntilstand for kvante oscillator fra ligning 2.6 $\psi_n(x) = A_n (a_+)^n \psi_0(x)$. $E_n = (n+\frac{1}{2})\hbar\omega$

vi har E_1 i dette tilfellet $E_1 = \left(\frac{2+1}{2}\right) = \frac{3}{2}$

Så da vet jeg at $\langle p^2 \rangle_{\psi_1} = \frac{3}{2} m\omega\hbar$

Samme observasjon kunne vi også gjort for $\langle x^2 \rangle_{\psi_n}$

b) $\sigma_{x\,\psi_0} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$ $\quad \sigma_{x\,\psi_1} = \sqrt{\frac{3\hbar}{2m\omega}}$

$\sigma_{p\,\psi_0} = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{m\omega\hbar}{2}}$ $\quad \sigma_{p\,\psi_1} = \sqrt{\frac{3m\omega\hbar}{2}}$

usikkerhetsrelasjonen gir $(\sigma_x \sigma_p)_{\psi_0} = \left(\frac{\hbar}{2m\omega} \frac{m\omega\hbar}{2}\right)^{1/2} = \frac{\hbar}{2}$

for ψ_1: $(\sigma_x \sigma_p)_{\psi_1} = \frac{3\hbar}{2m\omega} \frac{3m\omega\hbar}{2} = \frac{3}{2}\hbar$

$\psi_1 > \psi_2$ for $\sigma_x \sigma_p$

c) $\langle T \rangle = \frac{\langle p^2 \rangle}{2m}$ $\quad \langle V \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle$

$\langle T \rangle_{\psi_0} = \frac{\frac{1}{2} m\omega\hbar}{2m} = \frac{\hbar\omega}{4}$, $\langle V \rangle = \frac{\hbar\omega}{4}$

$\langle T \rangle_{\psi_1} = \langle V \rangle_{\psi_1} = \frac{3}{4}\hbar\omega$

$\sum \langle T \rangle_{\psi_0} + \langle V \rangle_{\psi_0} = \frac{1}{2}\hbar\omega = E_0$

for E_1 så får vi $\frac{3}{2}\hbar\omega$

2.12

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad p = i\sqrt{\frac{\hbar}{2m\omega}}(a_+ - a_-)$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi_n|^2 dx = \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^2 (a_+ + a_-)\, dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} (\psi_n^2 a_+ + \psi_n^2 a_-)\, dx \quad \text{pga ortogonalitet}$$

så står disse normalt og $\cos\theta$, $\theta = \frac{\pi}{2}$ gir $\langle x \rangle = 0$

tilsvarende får vi for $\langle p \rangle$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} \psi_n^2 (a_+ + a_-)^2\, dx$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} \psi_n^2 (a_+^2 + a_+a_- + a_-a_+ + a_-^2)\, dx$$

relasjoner $a_+a_- = \frac{1}{\hbar\omega}H - \frac{1}{2}$, $a_-a_+ = \frac{1}{\hbar\omega}H + \frac{1}{2}$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} n\psi_n^2 + n\psi_n^2 + \psi_n^2\, dx$$

vi vet at $\int_{-\infty}^{\infty} |\psi_n|^2 = 1$ så da får vi at

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}(n+n+1) = \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle p^2 \rangle = -\frac{\hbar m\omega}{2}(-2n-1) = \frac{\hbar m\omega (2n+1)}{2}$$

$$\langle T \rangle = \frac{\hbar m\omega}{2}\frac{(2n+1)}{2m} = \frac{\hbar\omega}{4}(2n+1)$$

$$\sigma_x \sigma_p = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \left(\frac{\hbar}{2m\omega}(2n+1)\,\frac{\hbar m\omega}{2}(2n+1)\right)^{1/2}$$

$$= \frac{\hbar}{2}(2n+1) \quad \text{for } n \in [0, N]$$