A Measure of Fault-Tolerance for Distributed Networks

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Abstract

We consider probabilistic networks having links that are perfectly reliable but nodes that fail randomly and independently with known probabilities. We define a fault-tolerance measure of such network that is directly affected by the choice of its underlying graph and the location of its node-components. A state of the network is tolerant if the currently operating nodes comprise a connected subnetwork. The probability of the network being in a tolerant state is the Fault-Tolerance FT measure of the network. We are concerned with the design of globally-best tolerant networks, that maximize FT for any given set of node-operating probabilities. We study this measure of fault-tolerance by developing combinatorial tools and by determining the optimal networks in both "sparse" and "dense" classes.

Keywords: Fault-tolerance, probabilistic graph, combinatorial optimization

1 Introduction

A major issue for network designers is the capability of a network to operate successfully despite the presence of component failures. Fault-tolerant behavior plays a vital role in the development of highly dependable distributed systems and has become the major obstacle in their realization. The fault-tolerant models that have evolved from probabilistic graphs signify the attention paid to addressing this issue.

In this paper we deal with a very early stage of this fault-tolerant design, by presenting a model of fault-tolerance that is concerned with the state of the network when failures have rendered some nodes inoperable. This model tries to keep the remaining network as connected as possible by selecting topologies or interconnections that can best tolerate failures and enhance performance. This situation typically arises in networks that deploy packet-switching broadcasting techniques since failures of certain nodes may disconnect the network, resulting in loss of communication paths. Similar scenario in LANs of processors running in parallel: failure of processors may interrupt communications between operating processors which would otherwise reallocate the unfinished jobs among themselves. There is also a number of design applications in the defense industry such as in space-borne systems and in communication of missile sites.

A network is represented by an undirected simple graph whose nodes fail with independent probabilities and whose edges are perfectly reliable (usually alternate communication links exist such as redundant cables and microwaves). Such a network is in a *tolerant state* if the graph induced by the surviving nodes is not empty and connected, and the *Fault-Tolerance* $FT(G)$ of a network with underlying graph G, is the probability that the network is in a tolerant state.

Despite the more global performance-measure of a network that this model presents, it has been largely ignored in the literature. The traditional models of fault-tolerance are primarily concerned with the existence of paths between specified operating nodes of the network, whereas this model treats all nodes as equally important for communication.

The design aspect of this fault-tolerance model is concerned with the following problem: Suppose that out of n locations, that is the node set, it is only feasible to establish perfectly reliable communication links between only e pairs of locations. Select these pairs, that is the edges of the graph, so that the fault-tolerance of the network is maximized. Assuming that the operating probabilities of the components are known, the selection of these edges may be done in the following manner: first a topology, i.e. a graph G is chosen; then the locations of the components at the nodes of G are determined based on a *best assignment.*

2 Definitions

More formally, let (n, e) be the class of networks whose underlying graphs are simple undirected with n nodes and e edges. The edges operate perfectly but the nodes fail with independent and known probabilities. Consider a graph $G \in (n, e)$ with a node set $V(G) = \{v_1, v_2, \dots, v_n\}$ and a vector of non-decreasing operating probabilities for the nodes $\vec{P} = (p_1, p_2, \dots, p_n)$. An **assignment function** π is a function from V to $\{1, \dots, n\}$ that assigns to node v the probability $p_{\pi(v)}$.

The **Fault-Tolerance** of a network G, given a vector $\vec{P}$ and assignment π, denoted $FT(G; \vec{P}; \pi)$, is the probability that despite random node failures the remaining network of G is connected. We use the term **tolerant state** of G, to denote any non-empty subset S of nodes that induce a connected subgraph of G and Θ to denote all such states. A tolerant state $S \in \Theta$ contributes $\prod_{v \in S} p_{\pi(v)} \prod_{v \notin S}(1 - p_{\pi(v)})$ to the FT-probability of G. Therefore we can form an algebraic expression for the FT-probability by enumerating all tolerant states of the network G:

$$FT(G; \vec{P}; \pi) = \sum_{S \in \Theta} Pr(S) = \sum_{S \in \Theta} \prod_{v \in S} p_{\pi(v)} \prod_{v \notin S} (1 - p_{\pi(v)})$$

A graph G is said to be **globally-best tolerant** in the class (n, e) if there exists an assignment π such that for any given vector $\vec{P}$ of node-operating probabilities $FT(G; \vec{P}; \pi) \geq FT(G'; \vec{P}; \pi')$, for all networks $G' \in (n, e)$ and all assignments π'.

In this paper we characterize networks that optimize the FT-probability over certain classes and we present classes in which no globally-best tolerant network can exist. We expect the reader to be familiar with graph-theoretic terms, such as class of graphs, induced subgraph, connectivity, matching of edges.

We often make use of the following **pivoting formula** which expresses the FT-probability of a network in terms of the FT-probabilities of smaller networks. A node v serves as the "pivot" by considering its operating status. For a graph G(V,E) and a set $S \subseteq V$ of vertices, $G - S$ is the subgraph of G induced on the vertices of $V - S$. Also if $v \in V$, the graph $G - v$ is obtained by removing v from G, and G/v is the graph obtained if we remove v from G and replace the subgraph induced by its neighborhood $N(v)$ in G, by a clique. Assuming a fixed assignment π and a vector $\vec{P}$ we refer to $FT(G;\vec{P};\pi)$ as $FT(G)$ for ease of notation.

Theorem 2.1 (Pivoting formula) *Let p_v be the operating probability of node v of a network G in the class (n,e). Then*

$$FT(G) = (1-p_v)FT(G-v) + p_v[FT(G/v) - \prod_{u \in N(v)} (1-p_u)FT(G-v-N(v))] + p_v \prod_{u \neq v}(1-p_u)$$

Proof. Consider any tolerant state S of G: If S does not contain v (v has failed), then S is a tolerant state of $G - v$ and is counted for in the first term. If S contains v (v operates), then $S - \{v\}$ is still an tolerant state in G/v, since all paths that used v in S are preserved. But G/v may have tolerant states S' that do not contain a neighbor of v, so that $S' \cup \{v\}$ is not an tolerant state of G, thus the second term is required. Finally, the last term accounts for the tolerant state $S = \{v\}$. □

The recursive structure of the pivoting formula suggests an immediate exponential algorithm for the computation of $FT(G)$. As we show, the pivoting formula may be used to establish the global fault-tolerant optimality of certain networks. Since any globally-best tolerant network must remain so when the nodes operate with equal probabilities, we examine this case first.

Consider the simple case where nodes operate with the same probability p. Since any two assignments yield the same result, we ignore π and denote the FT-probability as $FT(G;p)$. The probability of a tolerant S is now $p^i(1-p)^{n-i}$, if $|S| = i$ and G has n nodes. Therefore, the expression for $FT(G;p)$ is a polynomial in p, referred to as the **FT-polynomial**:

$$FT(G;p) = \sum_{i=1}^{n} s_i(G)p^i(1-p)^{n-i} \qquad (1)$$

where the coefficient $s_i(G)$ is the number of connected induced subgraphs of G (tolerant states) having exactly i nodes.

In this special case we call a network $G \in (n,e)$ **locally-best tolerant**, if it maximizes the FT-probability in its class over any "all-equal-to-p" probability vector, or, if $FT(G;p) \geq FT(G';p)$ for all $G' \in (n,e)$ and for all $0 < p < 1$. It is clear that such a property should be studied first since global optimality implies local.

For this, we establish some necessary conditions for a network to be locally-best tolerant: clearly the first two coefficients in (1) are always n and e respectively and if $\kappa(G)$ is the connectivity of G (the least number of nodes whose removal disconnects the graph) then $s_i(G) = \binom{n}{i}$ for $i > n - \kappa(G)$. Thus (1) becomes:

$$FT(G;p) = np(1-p)^{n-1} + ep^2(1-p)^{n-2} + \sum_{i=3}^{n-\kappa} s_i(G)p^i(1-p)^{n-i} + \sum_{i=n-\kappa+1}^{n} \binom{n}{i} p^i(1-p)^{n-i}$$

It is immediate then that for any two networks G, G' in (n,e) with $\kappa(G) = \kappa(G') = \kappa$, the coefficients $s_3, s_4, \ldots, s_{n-\kappa}$ determine the more "tolerant" of the two networks.

Frank [3,4] proved that when p is sufficiently close to one, $FT(G;p)$ is maximized only if G has the maximum possible connectivity κ in its class and the maximum value of $s_{n-\kappa}$ among all networks in its class with maximum κ. We refer to such a network as **κ-tolerant** network. Thus, a necessary condition for a network to be locally-best tolerant is that it be κ-tolerant.

A general solution to the problem of constructing κ-tolerant networks has yet to be discovered, only partial results are known. Using Frank's approach, we can show that when p is sufficiently small, $FT(G;p)$ is maximized only when s_3 is maximized over the class (n,e). We call a network $G \in (n,e)$ **3-tolerant** if $s_3(G) \geq s_3(G')$ for all $G' \in (n,e)$. Thus a second necessary condition is obtained, namely a locally-best tolerant network must be 3-tolerant in its class.

As with κ-tolerant networks, a general solution for the construction of 3-tolerant networks doesn't exist yet. The authors of [1,2], present 3-tolerant networks in sparse and dense classes.

Hence, any attempt to characterize locally-best tolerant networks should first address the issue of characterizing the networks that are both κ-tolerant and 3-tolerant. It is the case though, that such networks do not always exist:

Theorem 2.2 *Locally-best tolerant networks do not exist for all n and e.*

Proof. Consider the class $e = n$. The unique κ-tolerant network is the cycle C_n on n nodes having $\kappa = 2$. However as noted earlier, the authors of [1,2] show that the unique 3-tolerant is the star network plus an edge, having $\kappa = 1$. □

3 Sparse (star) networks

We consider a very common network structure, the most economical in terms of interconnections, the so called "star" network whose FT-probability we examine in this section. Note here that we consider the operating probabilities of the n nodes of a network given in a vector $\vec{P}$, of n ordered probabilities. Also we frequently simplify the notation $FT(G;\vec{P};\pi)$ to $FT(G;\pi)$ or $FT(G;\vec{P})$, or just $FT(G)$, if the vector $\vec{P}$ and/or the assignment π is implied by the context.

With the next claim we establish the local optimality of star-type networks. We call a network S "star" if its underlying graph is $K_{1,n-1}$.

Claim 3.1 *The star network S, is the unique locally-best tolerant network in the class $e = n - 1$.*

Proof: If $t_i(G)$ is the number of subgraphs of G that are trees with i nodes, then any induced connected subgraph with i nodes has at least one tree that spans these nodes:

$$s_i(G) \leq t_i(G) \leq \binom{e}{i-1} \text{ ,for all } i = 3, \ldots, n$$

Now consider the star S with n nodes and e edges, where $e = n - 1$. Since all edges of S are incident upon a node, every coefficient s_i in the FT-polynomial of S realizes the above upper-bound. Therefore since each $s_i(S)$ is maximized, from the FT-polynomial (1) we conclude that S is locally-best tolerant in its class.

For uniqueness, it is enough to observe that any other network in the same class must have at least two independent edges, which implies that t_3 and s_3 of its FT polynomial is not maximum. □

Since any globally-best tolerant network has to be also locally-best tolerant, if there exists a globally-best tolerant network in the class of trees that has to be the star. Before we attempt such a proof, an issue arises that has to be resolved first: having the candidate topology G for global optimality we must determine the placement of the given component-probabilities at the nodes of G, or the **best assignment** of the probabilities in $\vec{P}$ to the nodes of $V(G)$.

For a given network G and vector $\vec{P}$, a **best assignment** π is an assignment with the property

$$FT(G; \vec{P}; \pi) \geq FT(G; \vec{P}; \pi'), \text{ for any other } \pi' \neq \pi$$

Since there is a finite number of possible assignments of the n probabilities to the n nodes of the graph, such a best assignment always exists. This assignment-function has an interesting behavior for a given network. For more see [6].

In order to prove next that the star network is the globally-best tolerant network among all trees, we compute first its best assignment: consider a star network $K_{1,n-1}$ on n points, and an arbitrary vector $\vec{P}$ with ordered non-decreasing probabilities. We claim that a best assignment π assigns probability p_n to the center node. For suppose assignment π assigns probability p_v to its center node v, and probability p_u to a leaf node u. Applying the FT-polynomial on v and u we can write $FT(S; \pi)$ in terms of p_v and p_u:

$$\begin{aligned} FT(S;\pi) \quad &= \quad (1-p_v)p_u \prod_{S-v-u} (1-p_i) + \\ &\quad (1-p_v)(1-p_u)FT(S-v-u) + p_v \end{aligned}$$

Obtain an assignment π' from π by swapping the probabilities p_v and p_u and compute the difference:

$$FT(S;\pi) - FT(S;\pi') = (p_v - p_u)(1 - \prod_{S-v-u} (1-p_i))$$

and it is clear that $FT(S; \pi) > FT(S; \pi') \Longleftrightarrow p_v > p_u$.

Theorem 3.1 *Given a vector $\vec{P}$ of n non-decreasing probabilities, let π the assignment that maps p_n to the center node of the star network. Then the star with the assignment π is the unique globally-best tolerant network in the class of tree-networks.*

Proof: We prove the above by induction on n: for $n = 1$ or 2, $K_{1,n-1}$ is the unique tree and the assignment π is the only one possible.

Assume that the theorem is true for trees with $n-1$ nodes, and consider an arbitrary tree with n nodes T_n, under some assignment π over the vector $\vec{P}$. Select a leaf-node x of T_n, and let r be the node adjacent to x. Apply the pivoting formula on x to obtain $FT(T_n)$.

Let now $K_{1,n-1}$ with its best assignment π that assigns p_n to the center node. Select a leaf-node with probability $p_{\pi(x)}$ and for ease of notation call it x again - call c the center node. Apply the pivoting formula on x to obtain $FT(K_{1,n-1})$, and take the difference:

$$\begin{aligned} &FT(K_{1,n-1}) - FT(T_n) = \\ &\quad FT(K_{1,n-1} - x) - FT(T_n - x) + \\ &\quad p_{\pi(x)}[(1 - p_{\pi(r)})FT(T_n - x - r) - \\ &\quad (1 - p_n)FT(K_{1,n-1} - x - c)] \end{aligned} \qquad (2)$$

Observing that $K_{1,n-1} - x - c$ consists of $n-2$ "trivial" tolerant states - while $T_n - x - r$ must have at least one edge (if not, then T_n=star), we can show by enumerating the tolerant states that the expression in the square brackets is positive. Using the hypothesis it follows that $FT(K_{1,n-1}) > FT(T_n)$, which completes the induction. □

Theorem 3.2 *There is no globally-best tolerant network in the class $e = n - 1$.*

Proof: By the previous theorem, the only candidate in this class is the (locally-best tolerant) star network S with the assignment π that assigns the largest probability p_n to its center node c. Obtain the network G from S by deleting an edge $\{c, v\}$ and placing an edge between nodes u and w. Then we get

$$\begin{aligned} &FT(G;\pi) - FT(S;\pi) = \\ &\quad p_{\pi(u)}p_{\pi(w)} \prod_{i \neq u,w} (1 - p_{\pi(i)}) - p_{\pi(v)}p_{\pi(c)} \end{aligned}$$

It is easy to see that the above difference becomes positive for vectors $\vec{P}$ having $p_{\pi(i)} = \frac{1}{2}$ for all $i \neq v$ and $p_{\pi(v)} \to 0$. Thus, the star is not globally-best tolerant when we consider disconnected networks. □

4 Dense networks

We examine next the fault-tolerant behavior of very "dense" networks, i.e. networks whose underlying graphs have number of edges close to the complete graph. The next theorem presents and proves the locally-best tolerant network in the class $e \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$. Define D_n to be the network in the above class with graph $K_n - M$, where M denotes a matching (independent set of edges).

Theorem 4.1 *The network D_n is the locally-best tolerant network in its class.*

Proof: Since $s_1 = n$ and $s_2 = e$ (constants in the (n, e) class), it suffices to show that D_n maximizes the s_i coefficients of the FT-polynomial for each $i = 3, 4, \ldots, n$. An obvious upper bound for each s_i is $\binom{n}{i}$.

Consider the network D_n with its graph $K_n - M$, where $|M| \leq \lfloor \frac{n}{2} \rfloor$. All the nodes of $K_n - M$ have degree $n-1$ or $n-2$. Therefore each induced subgraph on 3 or more nodes is always connected, so that each $s_i(K_n - M)$ realizes the upper bound. Thus, D_n is locally-best tolerant in its class.

For uniqueness, suppose that $G \in (n, e)$ and $G \not\cong K_n - M$. Then G has a node of degree $n-3$ or less, so that there exists a set of three or more nodes which induces a disconnected graph. Hence, at least one s_i of G is not maximized. □

We attempt next to prove the global optimality of D_n in the general case when the survival probabilities of the nodes can vary. We use the following lemma to determine the best assignment π for the graph $K_n - M$ of D_n, over a vector $\vec{P}$.

Lemma 4.1 *Suppose that G has a full-degree node v and a node u not of full-degree. If π is an assignment of G under $\vec{P}$, and π' is defined by switching the probabilities on the nodes v and u i.e. $\pi'(v) = \pi(u)$, $\pi'(u) = \pi(v)$ and $\pi'(w) = \pi(w)$ whenever $w \neq u, v$, then*

$$FT(G;\pi) > FT(G;\pi') \Leftrightarrow p_{\pi(v)} > p_{\pi(u)} \square$$

The lemma can be easily verified by applying the pivoting formula on the nodes u and v.

Consider now the graph $K_n - M$. If the matching M is not complete, then there is at least one full degree node. By lemma (4.1), given a vector $\vec{P}$, a best assignment π for $K_n - M$ assigns the highest probabilities to the full degree nodes. Having fixed the, say $n-k$, full degree nodes of $K_n - M$, we next determine the best assignment of the rest of the probabilities $\{p_1, p_2, \ldots, p_k\}$ to the remaining k nodes, each having degree $n-2$.

Claim 4.1 *A best assignment π of the k probabilities $p_1 \leq p_2 \leq \ldots \leq p_k$ to the remaining k nodes of $K_n - M$, assigns probabilities to the end points of a matching edge whose difference is as large as possible, that is, there is an ordering of the matching edges, say $\{u_1, v_1\}, \ldots, \{u_{\frac{k}{2}}, v_{\frac{k}{2}}\}$ such that $\pi(u_i) = i$ and $\pi(v_i) = k - i + 1$, for each $i = 1, 2, \ldots, \frac{k}{2}$.*

Proof: Consider any assignment ψ that violates the condition satisfied by π, i.e. there is a matching edge $\{u, v\}$ and an index i such that $\psi(u) = i$ and $\psi(v) = r \neq k - i + 1$. Pick the smallest such i so that there is another matching edge $\{u', v'\}$ with $\psi(u') = k - i + 1$ and $\psi(v') = s \neq i$. By minimality of i it follows that:

$$p_i \leq p_s \text{ and } p_r \leq p_{k-i+1} \tag{3}$$

Obtain a new assignment ψ' by switching the probabilities p_r and p_{k-i+1}. We prove next that $FT(D_n;\psi') \geq FT(D_n;\psi)$.

Observe that the graph $K_n - M$ fails to be in a tolerant state when only the end-points of a matching edge operate or when all nodes fail. Thus, we are led to:

$$FT(D_n;\psi') - FT(D_n;\psi) =$$
$$= \quad (p_{k-i+1} - p_r)(p_s - p_i) \prod_{j \neq i, k-i+1, r, s} (1 - p_j)$$

Because of the conditions given in (3) we conclude from the above expression that

$$FT(D_n;\psi') \geq FT(D_n;\psi)$$

Continuing in this fashion, we ultimately arrive at an assignment having the property given in the claim. Thus π is a best assignment. To see that π is also unique up to a permutation of matching edges it is enough to retrace the argument in the case when $0 < p_1 < p_2 < \ldots < p_n < 1$ for then the inequalities obtained in the proof are strict. □

Theorem 4.2 *There is no globally-best tolerant network in the class $e \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$.*

Proof: Consider the graph $K_n - M$ of the network D_n, where M is a matching; we know it to be locally-best tolerant. Let π be a best assignment of the probabilities of a vector $\vec{P}$ to the nodes of $K_n - M$. Suppose that $\{v, u\}, \{w, z\} \notin E(K_n - M)$ where u, v, w, z are nodes of $K_n - M$. Obtain the graph G from $K_n - M$ by deleting the edge $\{v, z\}$ and adding the edge $\{v, u\}$, and give G the same assignment π. Then we get

$$FT(G) - FT(D_n) =$$
$$p_{\pi(v)}[(p_{\pi(u)} - p_{\pi(z)}) - p_{\pi(w)}p_{\pi(u)}(1 - p_{\pi(z)})] \cdot \prod_{i \neq v, u, w, z} (1 - p_{\pi(i)})$$

The above expression is seen to be positive for vectors $\vec{P}$ having $p_{\pi(i)} = \frac{1}{2}$ for $i \neq u$ and $p_{\pi(u)} > \frac{5}{8}$ □

5 Conclusions

We have proved the local fault-tolerant optimality of star-type networks and dense networks that are missing only a matching from the complete network. Of the above two types of networks, only the star achieves global optimality in its class, and it is the only one known to us so far. It turns out that global optimality is a very difficult property for a network to satisfy. We conjecture that under this model of fault-tolerance no other class contains a globally-best tolerant network. Another strong indication that supports this conjecture is in the class of complete bipartite networks ([5]). □

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