Newton

Part a)

a) $\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$, $\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$ given initially (1), (2)

$\mathcal{E} \Rightarrow$ radius $\quad \theta \Rightarrow \rho$ or P

We will try to relate radius and ρ with the given equations.

From (2) $\Rightarrow -\frac{dp(r)}{dr}\frac{r^2}{G\rho(r)} = m(r)$

Integrating both sides $\Rightarrow -\frac{d}{dr}\left[\frac{dp(r)}{dr}\frac{r^2}{G\rho(r)}\right] = \frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$ From (1)

$$\Rightarrow -\frac{d}{dr}\left[\frac{dp(r)}{dr}\frac{r^2}{G\rho(r)}\right] = 4\pi r^2 \rho(r) \quad (\star)$$

From (3) in the document, by integrating both sides: $\frac{dp(r)}{dr} = K\left(1+\frac{1}{n}\right)\rho(r)^{\frac{1}{n}}\frac{d\rho(r)}{dr}$ (Δ)

By combining $(\star)$ and (Δ) $\Rightarrow -\frac{d}{dr}\left[K\left(1+\frac{1}{n}\right)\rho(r)^{1/n}\frac{d\rho(r)}{dr}\frac{r^2}{G\rho(r)}\right] = 4\pi r^2 \rho(r)$

$$\Rightarrow 4\pi r^2 \rho(r) + K\left(1+\frac{1}{n}\right)\frac{d}{dr}\left[\rho(r)^{\frac{1}{n}-1}\frac{d\rho(r)}{dr}r^2\right] = 0$$

Let $r = c_1 \mathcal{E}$ and $dr = c\, d\mathcal{E}$

$$\Rightarrow 4\pi c_1^2 \mathcal{E}^2 \rho(\mathcal{E}) + K\left(1+\frac{1}{n}\right)\frac{1}{c_1}\frac{1}{d\mathcal{E}}\left[\rho(\mathcal{E})^{\frac{1}{n}-1}\frac{d\rho(\mathcal{E})}{d\mathcal{E}}c_1^2\mathcal{E}^2\right] = 0$$

Let $\theta = c_2 \rho$ and $\frac{d\theta}{d\mathcal{E}} = c_2\frac{d\rho}{d\mathcal{E}}$

$$\Rightarrow 4\pi c_1^2 \xi^2 \frac{1}{c_2} \theta + K\left(1+\frac{1}{n}\right) \frac{1}{c_1} \frac{1}{d\xi}\left[c_2^{\cancel{1}-\frac{1}{n}} \theta^{(\frac{1}{n}-1)} \frac{d\theta}{d\xi} \cancel{\frac{1}{c_2}} c_1^2 \xi^2 \right] = 0$$

Using $c_1^2 = \dfrac{K(1+n)}{4\pi G} \rho^{\frac{1}{n}-1}$ and $\theta = \rho\,\rho^{1/2}$

We obtain $$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0$$

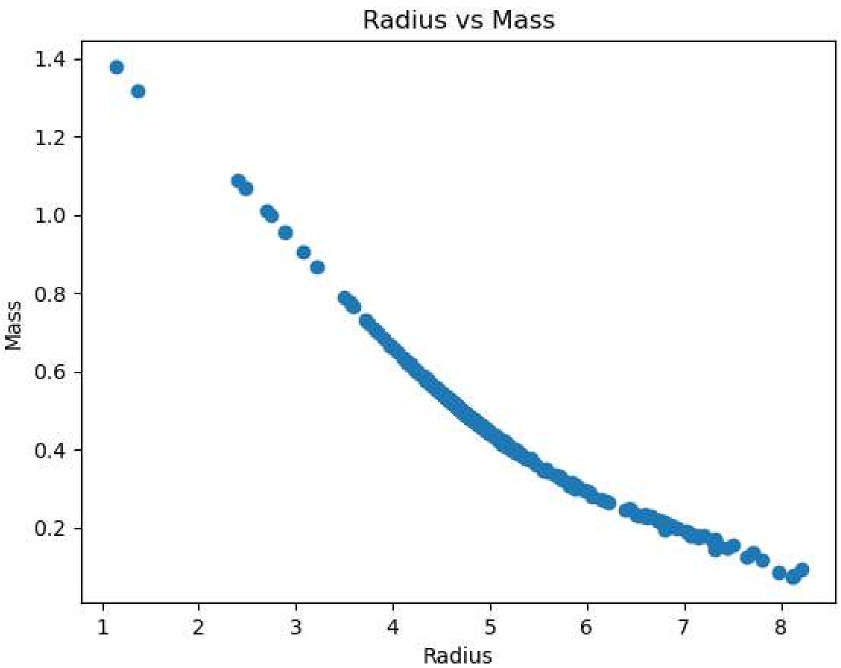
Mathematica:

I defined the function, specified boundary conditions, then used DSolve to get the general solution. Then extracted the solution as a function and expanded it as a power series around ξ = 0:

$$1 - \frac{\xi^2}{6} + \frac{\xi^4}{120}$$

Part b)

Used pandas library to obtain data from .cvs file and used the equation: $R = \text{sqrt}(\frac{GM}{10^{\log(g)}}$

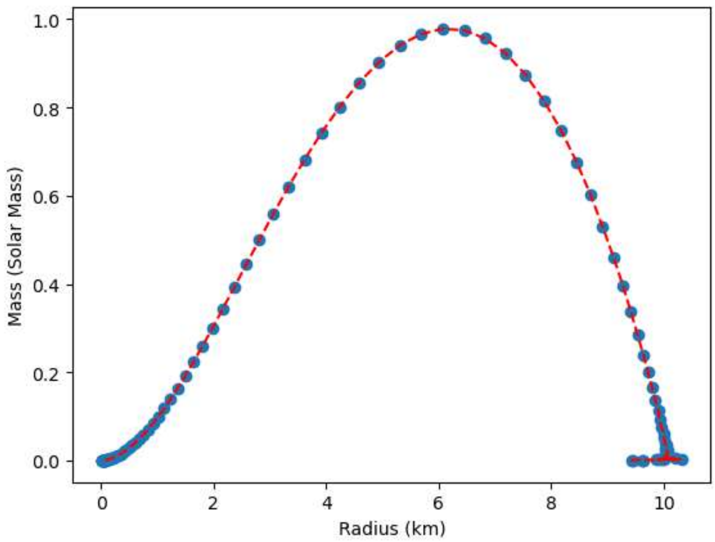


Einstein

Part a)

In this part I gathered all equations under tov_equations function and solved it with scipy.integrate.solve_ivp function with the method “RK45”. I defined a helper function pressure_zero as an “events=” parameter to ensure solve_ivp stops when p = 0.

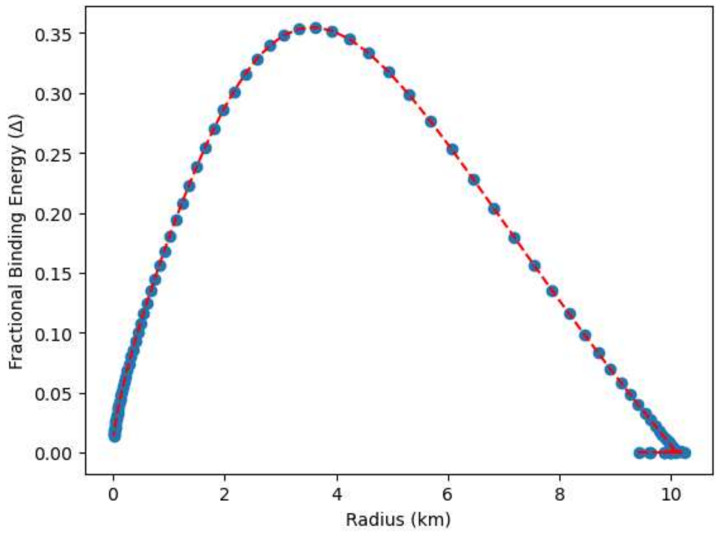
I set K = 100 and ρ values between 1e-6 and 1e3, ensuring a close proximity to the values given in the question prompt:



Part b)

For this part, additionally to the equation system defined in tov_equations function, derivative of baryonic mass term (dmp_dr) was added as a separate equation. Then the same method in part a is used to compute both M and Mp to obtain the binding energy: $\Delta = \frac{M_p - M}{M}$

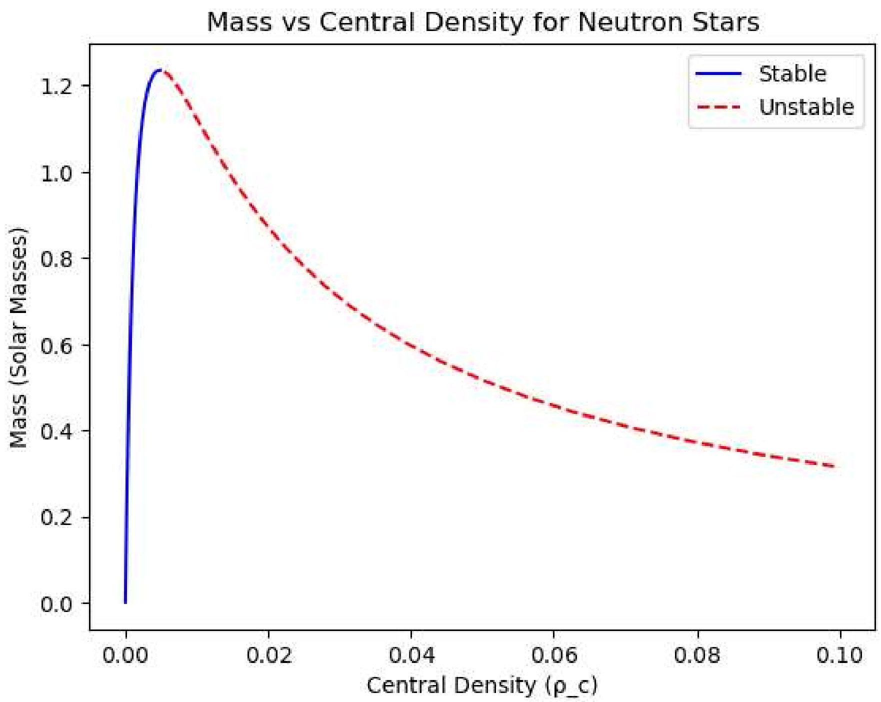
The following graph is plotted with the same values as part a:



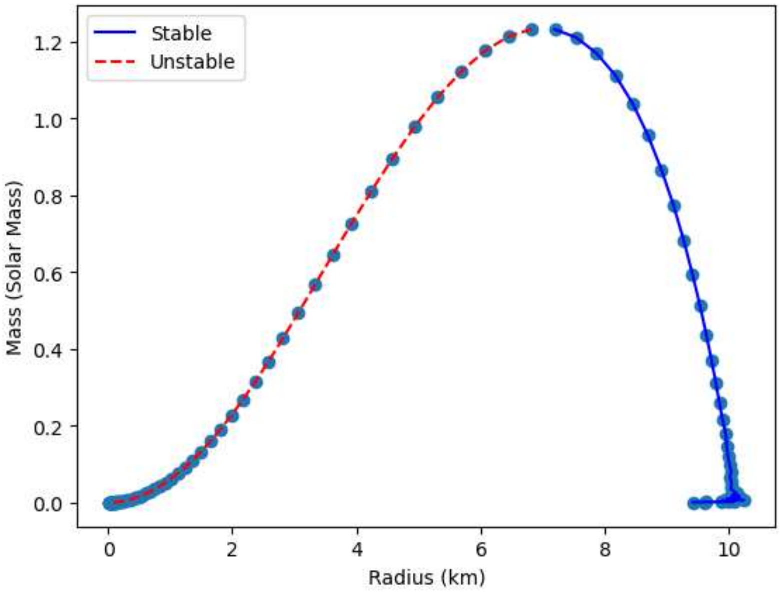
Part c)

In this part $dM_p/d\rho_c$ is computed using numpy.gradient function for different set of parameters for better visualization.

For rho = logspace(-6,-1, 100):



For rho = logspace(-6,3, 100):



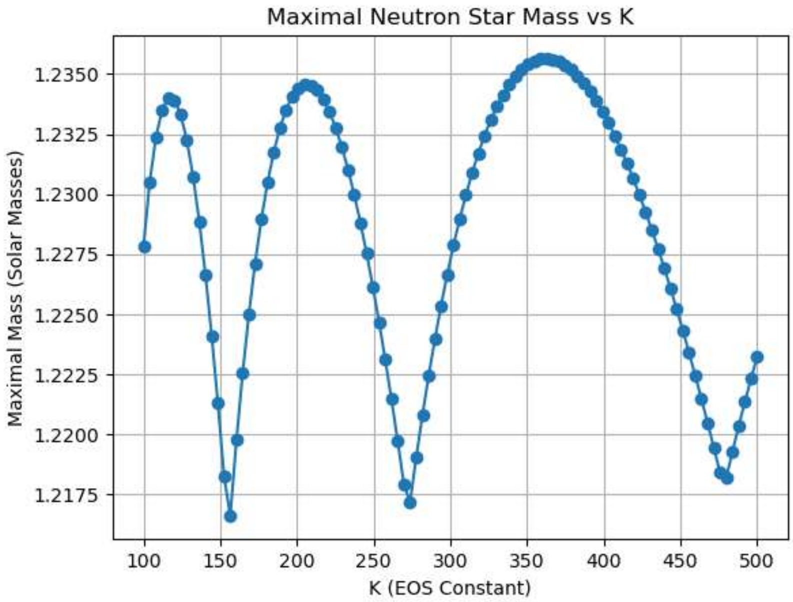
Maximal Neutron Star Mass: 1.23 Solar Masses

Here we see that for the concave function, the density values to the left of what corresponds to 1.23 $M_{\odot}$ are stable and the right side are unstable.

Part d)

In this part, I calculated Mp values with varying K's using solve_ivp with RK45 as before but it kept giving me an inconsistent, wobbly graph. And it never got past the point 1.25 $M_{\odot}$ so according to my calculations every value of K was viable because we can infer that possible K values must result in less mass after solving TOV equations than the maximum observed mass.

I tried solving it using implicit euler because the stability of implicit methods but it didn't converge. I began to implement splicing however I ran out of time. So here is the graph obtained with solve_ivp:



Part e)

I defined the equation: $\nu' = \frac{2M}{r(r-2M)}$ then applied DSolve to obtain general solution. After that I applied boundary conditions to get the final expression:

$$\nu(R) - 2M\left(\frac{\text{Log}[r]}{2M} - \frac{\text{Log}[-2M + r]}{2M}\right) - \text{Log}\left[1 - \frac{2M}{R}\right]$$

which is the same as:

$$\ln\left(1 - \frac{2M}{r}\right) - \ln\left(1 - \frac{2M}{R}\right) + \nu(R)$$