

Spectral asymptotics and scattering theory in the nilpotent Lie group setting

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(based on joint work with Z. Fan, J. Li, F. Sukochev and D. Zanin)

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This talk is based on a series of preprints by myself with Zhijie Fan (Wuhan), Ji Li (Macquarie), Fedor Sukochev (UNSW) and Dmitriy Zanin (UNSW).

The first two papers are available:

- a Spectral estimates and asymptotics for stratified Lie groups
arXiv:2201.12349 (with Sukochev and Zanin)
- b Endpoint weak Schatten class estimates and trace formula for commutators of Riesz transforms with multipliers on Heisenberg groups
arXiv:2201.12350 (with Fan, Li, Sukochev and Zanin)

There will also be other papers (currently in preparation).

Plan for this talk

- 1 Some elementary background on scattering theory (why do we care?)
- 2 Stratified lie groups and recent developments
- 3 The future(?)

We have some new results on the spectral theory of operators of the form

$$M_f D^{-1} : L_2(G) \rightarrow L_2(G)$$

where G is a stratified Lie group, M_f is the operator of pointwise multiplication by a function f on G and D is a positive maximally hypoelliptic differential operator on G .

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These are interesting on their own, but I will discuss a program (mostly unfinished) to do scattering theory (in the style of Birman-Kato) for maximally hypoelliptic operators (in the style of Helffer-Nourigat, Androulidakis-Mohsen-Yuncken).

Why do we want to know?

Singular value estimates for operators like $M_f D^{-1}$ have several applications. For example:

- Bound state problems: estimate the number of eigenvalues of $D + M_f$.
- Scattering theory: compare the effect of M_f on the evolution of $\exp(it(D + M_f))$,
- Spectral theory: determine the Weyl asymptotics of general maximally hypoelliptic differential operators.

Very elementary scattering theory

If Q is an elliptic and symmetric differential operator

$$Q : C^\infty(X, E) \rightarrow C^\infty(X, E)$$

where X is compact and Riemannian, and E is some Hermitian vector bundle, then Q is self-adjoint and has a discrete spectral decomposition

$$Q = \sum_{n=0}^{\infty} \lambda(n, Q) P_n$$

where P_n is a finite rank $L_2(X, E)$ -orthogonal projection, and $\{\lambda(n, Q)\}_{n=0}^{\infty}$ enumerates the spectrum in increasing order of absolute value.

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If X is not compact, this is of course not true.

Very elementary scattering theory

Suppose that X is not compact (later, we will simply take $X = \mathbb{R}^d$). If we assume that the geometry of X and E are not so bad and that the coefficients of Q are uniformly bounded in the correct sense, then Q is still self-adjoint but its spectrum is complicated.

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In this case scattering theory can provide a more useful description.

Very elementary scattering theory

A conventional situation is that we have a symmetric differential operator (on \mathbb{R}^d),

$$D_1 = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

with smooth coefficients $\{a_\alpha\}$ that are constant outside of a compact set, say $a_\alpha(x) = c_\alpha$. If we define

$$D_0 = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

The spectral theory of D_0 is easy to understand using the Fourier transform. We expect that D_1 “looks similar” to D_0 in some sense. Ideally, D_1 is relatively compact with respect to D_0 .

Another question we might want to ask about D_1 : does it have a discrete spectrum? If so, where is it?

Uses of scattering theory

The Birman-Kato theory is well-known in geometry and topology.

- Relative index theorems: Suppose that D_1 and D_0 are odd self-adjoint operators on a $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert space H . The relative index of D_1 and D_0 is

$$\text{ind}(D_1, D_0) = \text{Str}(e^{-tD_1^2} - e^{-tD_0^2})$$

(provided it exists). The relative index is the differences of the indices of D_1 and D_0 , plus an extra term coming from the continuous spectrum. See Eichhorn *Relative Index Theory* (2008).

- Witten index: It is conceivable that one could have a non-Fredholm operator D such that

$$\text{wind}(D) := \lim_{t \rightarrow \infty} \text{Tr}(e^{-tD^*D} - e^{-tDD^*})$$

exists. This is called the Witten index, and can be expressed in terms of the scattering data of the pair $(|D|, |D^*|)$. See ALCS..(?)

- Richard's topological version of Levinson's theorem.

Related is the Lax-Phillips scattering theory, with its well-known

Very elementary scattering theory

Scattering theory is about the solutions to the equation

$$\frac{\partial u}{\partial t} = iD_1 u.$$

Or $u(t) = \exp(itD_1)u(0)$. We want to know when there exists u_+ such that

$$\lim_{t \rightarrow \infty} \|\exp(itD_1)u(0) - \exp(itD_0)u_+\| = 0.$$

Or, alternatively, when there exists a strong limit

$$W_+(D_1, D_0) := s\text{-}\lim_{t \rightarrow \infty} e^{-itD_0} e^{itD_1}.$$

(actually, we are interested in a slight modification of this).

Very elementary scattering theory

With a little more effort, we can compare the solutions of the wave equations

$$\frac{\partial^2 u}{\partial t^2} = D_1 u, \quad \frac{\partial^2 u}{\partial t^2} = D_0 u.$$

(acoustical scattering; see Reed-Simon III)

Very elementary scattering theory

Let D_0, D_1 be self-adjoint operators on some Hilbert space H , and let $P_{ac}(D_1)$ be the projection onto the absolutely continuous subspace of D_1 . Define

$$W_{\pm}(D_0, D_1) := s\text{-}\lim_{t \rightarrow \pm\infty} e^{-itD_0} e^{itD_1} P_{ac}(D_1).$$

These are called the wave operators. We say that the wave operators (if they exist) are *complete* if

$$\text{ran}(W_{\pm}(D_0, D_1)) = P_{ac}(D_0).$$

Birman's theorem

Suppose that A_1, A_0 are self-adjoint operators on a Hilbert space H . If for any bounded interval $I \subset \mathbb{R}$ we have

$$\chi_I(A_1)(A_1 - A_0)\chi_I(A_0) \in \mathcal{L}_1(H)$$

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It suffices, for example, to have

$$(A_1 - A_0)(1 + A_0^2)^{-N} \in \mathcal{L}_1(H)$$

for sufficiently large N .

Using Birman's theorem

Consider the pair

$$A_1 = c(x)\Delta, A_0 = \Delta = \sum_{j=1}^d \partial_{x_j}^2$$

on \mathbb{R}^d , where c is a smooth positive function equal to 1 outside a compact set. Then

$$(A_1 - A_0)(1 - \Delta)^{-N} = (c(x) - 1)\Delta(1 - \Delta)^{-N}$$

This belongs to \mathcal{L}_1 for sufficiently large N , thanks to some old results of Birman-Solomyak.

Stratified Lie groups

Let \mathfrak{g} be a Lie algebra which admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$$

where $[\mathfrak{g}_k, \mathfrak{g}_n] \subseteq \mathfrak{g}_{k+n}$ and \mathfrak{g}_1 generates \mathfrak{g} . This is called a stratified Lie algebra.

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Exponentiating \mathfrak{g} , we get a simply connected nilpotent Lie group

$$G = \exp(\mathfrak{g}).$$

This is a homeomorphism, and the Lebesgue measure of \mathfrak{g} pushes forward to the Haar measure of G . Suppose that \mathfrak{g}_1 has a basis $\{X_1, \dots, X_m\}$, and G is essentially a Euclidean space \mathbb{R}^d equipped with a family of vector fields

$$X_1, \dots, X_m$$

with polynomial coefficients satisfying the Hörmander condition at every point.

Ellipticity on stratified Lie groups

The stratification of \mathfrak{g} defines a grading on the algebra of invariant differential operators, $\mathcal{U}(\mathfrak{g})$, on G . Say that an operator $P \in \mathcal{U}(\mathfrak{g})$ has order k if the highest degree term in P is homogeneous of degree k .

Theorem (Helffer-Nourigat, Rockland)

Let $P \in \mathcal{U}(\mathfrak{g})$ have degree k . If for every $\pi \in \widehat{G}_u$ (the unitary dual of G), $\pi(P)$ is injective on H_π^∞ (the smooth vectors), then for every Q of degree less than or equal to k we have

$$\|Qu\|_{L_2(G)} \lesssim \|Pu\|_{L_2(G)} + \|u\|_{L_2(G)}, \quad u \in L_2(G).$$

In particular, P is hypoelliptic.

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Some results

Recall that $\{X_1, \dots, X_m\}$ denotes a basis for \mathfrak{g}_1 , the first layer of our stratified Lie algebra. By assumption X_1, X_2, \dots, X_m generate \mathfrak{g} . Let

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Given a function f on G , denote by M_f the (possibly unbounded) operator of pointwise multiplication by f . We want to understand the operators

$$M_f(1 - \Delta)^{-N}, \quad (1 - \Delta)^{-N} M_f(1 - \Delta)^{-N}$$

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Why is this?

Among other things, so we can use Birman's theorem to study the scattering of differential operators on G .

A first result

One not-entirely-trivial results we obtained is the following.

Theorem (M.-Sukochev-Zanin)

Let $r > Q$ (recall that Q is the homogeneous dimension) and let $q > 2$. Given $f \in \ell_1(L_q)(G)$ (a function space on G), the operator

$$M_f(1 - \Delta)^{-\frac{r}{2}} : L_2(G) \rightarrow L_2(G)$$

is trace class.

This is a stratified version of an old result of Birman and Solomyak.

Theorem

Let G be a stratified Lie group with stratification $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$, homogeneous dimension $d_{\text{hom}} = \sum_{n=1}^{\infty} n \cdot \dim(\mathfrak{g}_n)$ and a fixed sub-Laplacian $\Delta = \sum_{j=1}^m X_j^2$, where $\{X_j\}_{j=1}^m$ is a basis for \mathfrak{g}_1 .

i) if $p > 2$, then

$$\|M_f(-\Delta)^{-\frac{d_{\text{hom}}}{2p}}\|_{p,\infty} \leq c_p \|f\|_{L_p(G)}$$

ii) if $p < 2$ and $q > 2$, then

$$\|M_f(1 - \Delta)^{-\frac{d_{\text{hom}}}{2p}}\|_{p,\infty} \leq c_{p,q} \|f\|_{\ell_p(L_q)(G)}.$$

iii) if $p = 2$ and $q > 2$, then

$$\|M_f(1 - \Delta)^{-\frac{d_{\text{hom}}}{2p}}\|_{p,\infty} \leq c_q \|f\|_{\ell_2 \log(L_q)(G)}.$$

Birman's theorem for stratified Lie groups

Suppose that

$$D_1 = \sum_{|\alpha|_h \leq m} a_\alpha(x) X^\alpha$$

where each a_α is a smooth function on G equal to a constant (say, c_α) outside a compact set. Then we expect that

$$D_0 = \sum_{|\alpha|_h \leq m} c_\alpha X^\alpha$$

is a good model for D_1 asymptotically, since $D_1 - D_0$ is a differential operator with compactly supported coefficients.

Spectral asymptotics

Related to these estimates we have spectral asymptotics. In the following theorem, μ denotes the singular value function. In particular, the sequence $\{\mu(n, T)\}_{n=0}^{\infty}$ is the sequence of singular values of a compact operator T . We give a precise definition of μ in the next section.

Theorem

Let G be a non-abelian stratified Lie group with stratification $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$, homogeneous dimension $d_{\text{hom}} = \sum_{n=1}^{\infty} n \cdot \dim(\mathfrak{g}_n)$ and a fixed sub-Laplacian $\Delta = \sum_{j=1}^m X_j^2$, where $\{X_j\}_{j=1}^m$ is a basis for \mathfrak{g}_1 . Let $k \in \mathbb{N}$ and let $p = \frac{d_{\text{hom}}}{k}$. If one of the following conditions holds

- ❶ $p > 1$ and $0 \leq f \in L_p(G)$;
- ❷ $p < 1$ and $0 \leq f \in \ell_p(L_q)(G)$ for some $q > 1$;
- ❸ $p = 1$ and $0 \leq f \in \ell_{1, \log}(L_q)(G)$ for some $q > 1$;

then there exists the limit

$$\lim_{t \rightarrow 0} t \mu(t, (1 - \Delta)^{-\frac{k}{4}} M_f (1 - \Delta)^{-\frac{k}{4}})^p = c_G \int f^p.$$

Corollary

Let $G \neq \mathbb{H}^1$ be a stratified Lie group with stratification $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$, homogeneous dimension $d_{\text{hom}} = \sum_{n=1}^{\infty} n \cdot \dim(\mathfrak{g}_n)$ and a fixed sub-Laplacian $\Delta = \sum_{j=1}^m X_j^2$, where $\{X_j\}_{j=1}^m$ is a basis for \mathfrak{g}_1 . Assume that $V \in L_{\frac{d_{\text{hom}}}{2}}(G)$ is real-valued. For $h > 0$, the operator $-h^2\Delta + M_V$ can be defined in the sense of quadratic forms. There exists a constant $c_G > 0$ such that

$$\lim_{h \rightarrow 0} h^{d_{\text{hom}}} \text{Tr}(\chi_{(-\infty, 0)}(-h^2\Delta + M_V)) = c_G \int_G V_-^{\frac{d_{\text{hom}}}{2}}.$$

Here, $V_- = \frac{1}{2}(|V| - V)$ is the negative part of V .

Theorem (Cwikel–Lieb–Rozenblum estimate)

Assume that $d_{\text{hom}} > 2$. Let $V \in L_{\frac{d_{\text{hom}}}{2}}(G)$ be real-valued. The quadratic form sum

$$-\Delta \dot{+} M_V$$

is well-defined on the form domain $W_2^1(G)$, and defines an unbounded self-adjoint operator on $L_2(G)$ with essential spectrum $[0, \infty)$. The operator $-\Delta \dot{+} M_V$ has finitely many negative eigenvalues, and the total number of eigenvalues less than $-t$ for $t \geq 0$ is bounded by

$$\text{Tr}(\chi_{(-\infty, -t)}(-\Delta \dot{+} M_V)) \leq C_G \int_G (V + t)_-^{\frac{d_{\text{hom}}}{2}}.$$

Theorem

Let G be a stratified Lie group with $d_{\text{hom}} > 4$, and let $f \in (L_{\frac{d_{\text{hom}}}{2}} \cap L_1)(G)$ be real valued. The wave operators exist and are complete for the couple $(-\Delta, -\Delta \dot{+} M_f)$, where $-\Delta \dot{+} M_f$ denotes the quadratic form sum.

We can make similar assertions about pairs of the form (D_1, D_0) , where D_1 and D_0 are maximally hypoelliptic differential operators whose coefficients are asymptotically equal.

These estimates are suboptimal for a number of reasons, one of them being that we state the results for functions on G rather than a general Heisenberg manifold (or an even more general filtered manifold). This is probably not a significant restriction.

Thank you for listening!