

Spectral asymptotics and scattering theory in the nilpotent Lie group setting

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(based on joint work with Z. Fan, J. Li, F. Sukochev and D. Zanin)

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March 1, 2023

This talk is based on a series of preprints by myself with Zhijie Fan (Wuhan), Ji Li (Macquarie), Fedor Sukochev (UNSW) and Dmitriy Zanin (UNSW).

The first two papers are available:

- a Spectral estimates and asymptotics for stratified Lie groups
arXiv:2201.12349 (with Sukochev and Zanin)
- b Endpoint weak Schatten class estimates and trace formula for commutators of Riesz transforms with multipliers on Heisenberg groups
arXiv:2201.12350 (with Fan, Li, Sukochev and Zanin)

There will also be other papers (currently in preparation).

Plan for this talk

- ① Some elementary background on scattering theory (why do we care?)
- ② Stratified lie groups and recent developments
- ③ The future(?)

I will discuss a program (mostly unfinished) to do scattering theory (in the style of Birman-Kato) for maximally hypoelliptic operators (in the style of Helffer-Nourigat, Androulidakis-Mohsen-Yuncken).

Very elementary scattering theory

If Q is an elliptic and symmetric differential operator

$$Q : C^\infty(X, E) \rightarrow C^\infty(X, E)$$

where X is compact and Riemannian, and E is some Hermitian vector bundle, then Q is self-adjoint and has a discrete spectral decomposition

$$Q = \sum_{n=0}^{\infty} \lambda(n, Q) P_n$$

where P_n is a finite rank $L_2(X, E)$ -orthogonal projection, and $\{\lambda(n, Q)\}_{n=0}^{\infty}$ enumerates the spectrum in increasing order of absolute value.

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If X is not compact, this is of course not true.

Very elementary scattering theory

Suppose that X is not compact (later, we will simply take $X = \mathbb{R}^d$). If we assume that the geometry of X and E are not so bad and that the coefficients of Q are uniformly bounded in the correct sense, then Q is still self-adjoint but its spectrum is complicated.

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In this case scattering theory can provide a more useful description.

Very elementary scattering theory

A conventional situation is that we have a differential operator (on \mathbb{R}^d),

$$D_1 = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

with smooth coefficients $\{a_\alpha\}$ that are constant outside of a compact set, say $a_\alpha(x) = c_\alpha$. If we define

$$D_0 = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

The spectral theory of D_0 is easy to understand using the Fourier transform. We expect that D_1

Scattering theory is about the solutions to the equation

$$\frac{\partial u}{\partial t} = iD_1 u.$$

Or $u(t) = \exp(itD_1)u(0)$. We want to know when there exists u_+ such that

$$\lim_{t \rightarrow \infty} \|\exp(itD_1)u(0) - \exp(itD_0)u_+\| = 0.$$

Stratified Lie groups

Let \mathfrak{g} be a Lie algebra which admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{k=1}^{\infty} \mathfrak{g}_k$$

where $[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$ and \mathfrak{g}_1 generates \mathfrak{g} . This is called a stratified Lie algebra. Exponentiating \mathfrak{g} , we get a simply connected nilpotent Lie group

$$G = \exp(\mathfrak{g}).$$

This is a homeomorphism, and the Lebesgue measure of \mathfrak{g} pushes forward to the Haar measure of G . Suppose that \mathfrak{g}_1 has a basis $\{X_1, \dots, X_m\}$, and G is essentially a Euclidean space \mathbb{R}^d equipped with a family of vector fields

$$X_1, \dots, X_m$$

with polynomial coefficients satisfying the Hörmander condition at every point.

Ellipticity on stratified Lie groups

The stratification of \mathfrak{g} defines a grading on the algebra of invariant differential operators, $\mathcal{U}(\mathfrak{g})$, on G . Say that an operator $P \in \mathcal{U}(\mathfrak{g})$ has order k if the highest degree term in P is homogeneous of degree k .

Theorem (Helffer-Nourigat, Rockland)

Let $P \in \mathcal{U}(\mathfrak{g})$ have degree k . If for every $\pi \in \widehat{G}_u$ (the unitary dual of G), $\pi(P)$ is injective on H_π^∞ (the smooth vectors), then for every Q of degree less than or equal to k we have

$$\|Qu\|_{L_2(G)} \lesssim \|Pu\|_{L_2(G)} + \|u\|_{L_2(G)}, \quad u \in L_2(G).$$

In particular, P is hypoelliptic.

Ellipticity on stratified Lie groups

Suppose that A_1, A_0 are self-adjoint operators on a Hilbert space H . If for any bounded interval $I \subset \mathbb{R}$ we have

$$\chi_I(A_1)(A_1 - A_0)\chi_I(A_0) \in \mathcal{L}_1(H)$$

then the wave operators $W_{\pm}(A_1, A_0)$ exist and are complete.

Birman's theorem for stratified Lie groups

Suppose that

$$D_1 = \sum_{|\alpha|_h \leq m} a_\alpha(x) X^\alpha$$

where each a_α is a smooth function on G equal to a constant (say, c_α) outside a compact set. Then we expect that

$$D_0 = \sum_{|\alpha|_h \leq m} c_\alpha X^\alpha$$

is a good model for D_1 asymptotically, since $D_1 - D_0$ is a differential operator with compactly supported coefficients.

Theorem

Let $f \in C_c^\infty(G)$, and let $Q \in \mathcal{U}(\mathfrak{g})$ be a Rockland operator. Then for any bounded interval $I \subset \mathbb{R}$, we have

$$M_f \chi_I(Q) \in \mathcal{L}_1(L_2(G)).$$

Thank you for listening!