# Spectral asymptotics and scattering theory in the nilpotent Lie group setting

Edward McDonald
(based on joint work with Z. Fan, J. Li, F. Sukochev and D. Zanin)

Penn State University

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# Introduction

This talk is based on a series of preprints by myself with Zhijie Fan (Wuhan), Ji Li (Macquarie), Fedor Sukochev (UNSW) and Dmitriy Zanin (UNSW).

The first two papers are available:

a. Spectral estimates and asymptotics for stratified Lie groups `arXiv:2201.12349` (with Sukochev and Zanin)
b. Endpoint weak Schatten class estimates and trace formula for commutators of Riesz transforms with multipliers on Heisenberg groups `arXiv:2201.12350` (with Fan, Li, Sukochev and Zanin)

There will also be other papers (currently in preparation).

# Plan for this talk

1. Some elementary background on scattering theory
2. Stratified Lie groups and recent developments
3. Singular values, Cwikel's estimates and Birman's theorem.
4. Some new results

# Summary for the minister

In our preprints we have some technical results on the spectra of operators of the form

$$M_f D^{-1} : L_2(G) \to L_2(G)$$

where $G$ is a stratified Lie group, $M_f$ is the operator of pointwise multiplication by a function $f$ on $G$ and $D$ is a positive maximally hypoelliptic differential operator on $G$.
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These results are interesting on their own, but I will discuss a program (mostly unrealized) to do scattering theory (in the style of Birman-Kato) for maximally hypoelliptic operators (in the style of of Helffer-Nourigat, Androulidakis-Mohsen-Yuncken).

# Summary for the minister (continued)

Singular value estimates for operators like $M_f D^{-1}$ have several applications. For example:

- Bound state problems: estimate the number of eigenvalues of $D + M_f$.
- Scattering theory: compare the effect of $M_f$ on the evolution of $\exp(it(D + M_f))$,
- Spectral theory: determine the Weyl asymptotics of general maximally hypoelliptic differential operators.

# Very elementary scattering theory

If $Q$ is an elliptic and symmetric differential operator

$$Q : C^\infty(X,E) \to C^\infty(X,E)$$

where $X$ is compact and Riemannian, and $E$ is some Hermitian vector bundle, then $Q$ is self-adjoint and has a discrete spectral decomposition

$$Q = \sum_{n=0}^{\infty} \lambda(n,Q) P_n$$

where $P_n$ is a finite rank $L_2(X,E)$-orthogonal projection, and $\{\lambda(n,Q)\}_{n=0}^{\infty}$ enumerates the spectrum of $Q$ in increasing order of absolute value.

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where $P_n$ is a finite rank $L_2(X,E)$-orthogonal projection, and $\{\lambda(n,Q)\}_{n=0}^{\infty}$ enumerates the spectrum of $Q$ in increasing order of absolute value.
If $X$ is not compact, this is of course not true.

# Very elementary scattering theory

Suppose that $X$ is not compact (later, we will simply take $X = \mathbb{R}^d$). If we assume that the geometry of $X$ and $E$ are not so bad and that the coefficients of $Q$ are uniformly bounded in the correct sense, then $Q$ is still self-adjoint but its spectrum is complicated.

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Normally, we say that the spectral measure $E^Q$ of $Q$ splits into three mutually singular parts:

$$E^Q = E^Q_{pp} + E^Q_{ac} + E^Q_{sc}$$

the pure point spectrum (the eigenvalues), the absolutely continuous spectrum and the singular continuous spectrum.

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the pure point spectrum (the eigenvalues), the absolutely continuous spectrum and the singular continuous spectrum.
In this case scattering theory can provide a more useful description than decomposing into eigenfunctions.

# Very elementary scattering theory

A very standard situation is that we have a symmetric differential operator (on $\mathbb{R}^d$),

$$D_1 = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$$

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$$D_0 = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

The spectral theory of $D_0$ is easy to understand using the Fourier transform: it is purely absolutely continuous.
We expect that the absolutely continuous spectrum of $D_1$ somehow arises from that of $D_0$.

# Very elementary scattering theory

Scattering theory is about the solutions to the equation

$$\frac{\partial u}{\partial t} = iD_1 u.$$

Or $u(t) = \exp(itD_1)u(0)$. We want to know when there exists $u_+$ such that

$$u(t) \sim e^{itD_0} u_+, \quad t \to \infty$$

or rather

$$\lim_{t\to\infty} \| \exp(itD_1)u(0) - \exp(itD_0)u_+\|_{L_2(X)} = 0.$$

Alternatively, we want to know when there exists a strong operator topology limit

$$W_+(D_1, D_0) := \lim_{t\to\infty} e^{-itD_0} e^{itD_1}.$$

(actually, we are interested in a slight modification of this).

# Very elementary scattering theory

Let $D_0, D_1$ be self-adjoint operators on some Hilbert space $H$, and let $P_{ac}(D_1)$ be the projection onto the absolutely continuous subspace of $D_1$. Define two operators $W_\pm(D_0, D_1)$ by

$$W_\pm(D_0, D_1) := \mathrm{s-}\lim_{t\to\pm\infty} e^{-itD_0}e^{itD_1}P_{ac}(D_1).$$

These are called the wave operators. We say that the wave operators (if they exist) are *complete* if

$$\mathrm{ran}(W_\pm(D_0, D_1)) = P_{ac}(D_0).$$

# Very elementary scattering theory

Here is the general picture to keep in mind. Suppose for the moment that $D_1$ does not have any singular continuous spectrum. We want to understand the solutions to the Schrödinger equation

$$\frac{du}{dt}(t) = iD_1 u(t), \quad u(0) = u_0.$$

Splitting the initial value $u_0$ into the point and absolutely continuous parts, the solution looks like

$$u(t) = \sum_{\lambda} e^{it\lambda} E^{D_1}(\{\lambda\})u_0 + e^{itD_1} P_{ac}(D_1)u_0.$$

where the sum is over the eigenvalues of $D_1$. If the wave operator $W_+(D_0, D_1)$ exists, then $P_{ac}(D_1)u_0$ looks asymptotically like a function evolving under $D_0$.

$$\lim_{t\to\infty} \|e^{itD_0}u_+ - e^{itD_1}P_{ac}(D_1)u_0\| = 0$$

where

$$u_+ = W_+(D_0, D_1)u_0.$$

# Very elementary scattering theory

With a little more effort, we can compare the solutions of the wave equations

$$\frac{\partial^2 u}{\partial t^2} = D_1 u,\ \frac{\partial^2 u}{\partial t^2} = D_0 u.$$

(this is called acoustical scattering; see Reed-Simon Volume III.)

# Goals of scattering theory

As I see it, the primary goal of the Birman-Kato theory is to understand the absolutely continuous spectrum of an operator $D_1$ by relating it to a simpler operator $D_0$. If the wave operator $W_+(D_0, D_1)$ exists and is complete, then it provides a unitary equivalence between the absolutely continuous subspaces of $D_1$ and $D_0$.
Another important task not directly related to scattering theory is to figure out how many eigenvalues there are in the point spectrum.

# Uses of scattering theory

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# Uses of scattering theory

The Birman-Kato theory has had much application in geometry and topology. Some selected applications:

- Relative index theorems: Suppose that $D_1$ and $D_0$ are odd self-adjoint operators on a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $H$. The relative index of $D_1$ with respect to $D_0$ is

$$\mathrm{ind}(D_1, D_0) = \mathrm{Str}(e^{-tD_1^2} - e^{-tD_0^2})$$

  (provided it exists). The relative index is the differences of the indices of $D_1$ and $D_0$, plus an extra term coming from the continuous spectrum. See Eichhorn *Relative Index Theory* (2008), and also Borisov-Müller-Schrader "Relative Index Theorems and Supersymmetric Scattering Theory" (1988)

# Uses of scattering theory

- Witten index: It is concievable that one could have a non-Fredholm operator $D$ such that

$$\mathrm{wind}(D) := \lim_{t\to\infty} \mathrm{Tr}(e^{-tD^*D} - e^{-tDD^*})$$

exists. This is called the Witten index, and can be expressed in terms of the scattering data of the pair $(|D|, |D^*|)$. See Carey-Gesztesy-Levitina-Sukochev "The spectral shift function and the Witten index" (2016).

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Closely related is the Lax-Phillips scattering theory, with its well-known applications in geometry (see Lax-Phillips *Scattering theory* (1989)) and geometric scattering theory (Melrose, *The Atiyah-Patodi-Singer index theorem* (1992)).

# Birman's theorem

Suppose that $A_1, A_0$ are self-adjoint operators on a Hilbert space $H$. If for any bounded interval $I \subset \mathbb{R}$ we have

$$E^{A_1}(I)(A_1 - A_0)E^{A_0}(I) \in \mathcal{L}_1(H)$$

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It suffices, for example, to have

$$(A_1 - A_0)(1 + A_0^2)^{-N} \in \mathcal{L}_1(H)$$

for sufficiently large $N$.

# Using Birman's theorem

Consider the pair

$$A_1 = c(x)\Delta, A_0 = \Delta = \sum_{j=1}^{d} \partial_{x_j}^2$$

on $\mathbb{R}^d$, where $c$ is a smooth positive function equal to 1 outside a compact set. Then

$$(A_1 - A_0)(1-\Delta)^{-N} = (c(x)-1)\Delta(1-\Delta)^{-N}$$

This belongs to $\mathcal{L}_1$ for sufficiently large $N$, thanks to some old results of Birman-Solomyak.

# Birman-Cwikel-Solomyak estimates

If we want to understand scattering for differential operators (say, on $\mathbb{R}^d$), the analytical problem is to determine when

$$f(x)(1-\Delta)^{-\frac{N}{2}} \in \mathcal{L}_1$$

where $f(x)$ is a pointwise multiplier.
Results such as these are usually attributed to Birman and Solomyak.
M. Cwikel determined the corresponding results for $\mathcal{L}_p$, with $p > 2$.

# Stratified Lie groups (concretely)

Nilpotent Lie groups are one of the best-understood classes of Lie groups, and on a nilpotent Lie group we can further assume a *stratified* structure. In very elementary terms, we can specify a stratified Lie group by the following data: Let $X_1, \ldots, X_m$ (first order differential operators) on $\mathbb{R}^N$. That is,

$$X_j = \sum_{k=1}^{N} p_{j,k}(x_1, \ldots, x_N) \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq m.$$

We assume that:

1. The coefficients $\{p_{j,k}\}_{1 \leq j \leq m,\, 1 \leq k \leq N}$ are polynomial functions on $\mathbb{R}^d$,
2. The differential operators $\{X_1, \ldots, X_m\}$ satisfy the *Hörmander condition*.

# The Hörmander condition

The Hörmander condition is concerned with the Lie algebra generated by $\{X_1, \ldots, X_m\}$. We look at the span of all of the repeated commutators formed by the $\{X_1, \ldots, X_m\}$. Let

$$\begin{aligned} \mathfrak{g}_1 &= \mathrm{span}\{X_1, \ldots, X_m\}, \\ \mathfrak{g}_{k+1} &= \mathrm{span}\{[X_1, \mathfrak{g}_k], \ldots, [X_m, \mathfrak{g}_k]\}, \quad k \geq 1. \end{aligned}$$

Eventually, $\mathfrak{g}_k = \{0\}$ because the coefficients of the operators are polynomials. Let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \cdots +$ . In this special case, the Hörmander condition states that for every $1 \leq j \leq N$, we have $\partial/\partial x_j \in \mathfrak{g}$.
The data $\{X_1, \ldots, X_m\}$ on $\mathbb{R}^N$ is defines *stratified Lie group structure* on $\mathbb{R}^N$.

# Stratified Lie groups (abstractly)

In more abstract terms, let $\mathfrak{g}$ be a Lie algebra which admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$$

where $[\mathfrak{g}_k, \mathfrak{g}_n] \subseteq \mathfrak{g}_{k+n}$ and $\mathfrak{g}_1$ generates $\mathfrak{g}$. This is called a stratified Lie algebra. In terms of our concrete description, $\mathfrak{g}_1 = \mathrm{span}\{X_1, \ldots, X_m\}$. The number

$$Q := \sum_{n=1}^{\infty} n\dim(\mathfrak{g}_n)$$

is called the homogeneous dimension of $\mathfrak{g}$.

# Stratified Lie groups

Exponentiating $\mathfrak{g}$, we get a simply connected nilpotent Lie group

$$G = \exp(\mathfrak{g}).$$

This is a homeomorphism, and the Lebesgue measure of $\mathfrak{g}$ pushes forward to the Haar measure of $G$. We let $X_1, \ldots, X_m$ be a basis for $\mathfrak{g}_1$.

# Ellipticity on stratified Lie groups

The stratification of $\mathfrak{g}$ defines a grading on the algebra of invariant differential operators, $\mathcal{U}(\mathfrak{g})$, on $G$. Say that an operator $P \in \mathcal{U}(\mathfrak{g})$ has order $k$ if the highest homogeneous degree term in $P$ is homogeneous of degree $k$.

**Theorem (Helffer-Nourigat, Rockland)**

*Let $P \in \mathcal{U}(\mathfrak{g})$ have degree $k$. If for every $\pi \in \widehat{G} \setminus \{1\}$ (the unitary dual of $G$), $\pi(P)$ is injective on $H_\pi^\infty$ (the smooth vectors), then for every $A$ of degree less than or equal to $k$ we have*

$$\|Au\|_{L_2(G)} \lesssim \|Pu\|_{L_2(G)} + \|u\|_{L_2(G)}, \quad u \in L_2(G).$$

*In particular, $P$ is hypoelliptic.*

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# Ellipticity on stratified Lie groups

Recently some substantial advances have been made in the study of ellipticity on Heisenberg manifolds and more general foliated manifolds. This opens up the possibility to study the scattering theory for maximally hypoelliptic operators. Stratified Lie groups are a good setting for scattering theory, because the spectral theory of invariant differential operators can be fully described using representation theory.

# Some results

Recall that $\{X_1, \ldots, X_m\}$ denotes a basis for $\mathfrak{g}_1$, the first layer of our stratified Lie algebra. By assumption $X_1, X_2, \ldots, X_m$ generate $\mathfrak{g}$. Let

$$\Delta := \sum_{j=1}^{m} X_j^2.$$

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Given a function $f$ on $G$, denote by $M_f$ the (possibly unbounded) operator of pointwise multiplication by $f$. We want to understand the operators

$$M_f(1-\Delta)^{-N}, \quad (1-\Delta)^{-N} M_f (1-\Delta)^{-N}$$

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Why is this?
Among other things, so we can use Birman's theorem to study the scattering of differential operators on $G$.

# A first result

One not-entirely-trivial results we obtained is the following.

**Theorem (M.-Sukochev-Zanin)**

*Let $r > Q$ (recall that $Q$ is the homogeneous dimension) and let $q > 2$. Given $f \in \ell_1(L_q)(G)$ (a function space on $G$), the operator*

$$M_f(1-\Delta)^{-\frac{r}{2}} : L_2(G) \to L_2(G)$$

*is trace class.*

# Reminder on singular values

Given a compact operator $T$ on some Hilbert space, the $(n+1)$-st singular value of $T$ is defined as

$$\mu(n, T) := \inf\{\|T - R\| \ :\ \mathrm{rank}(R) \leq n\}.$$

One say that $T \in \mathcal{L}_{p,\infty}(H)$ if $\mu(n.T) = O(n^{-\frac{1}{p}})$, with

$$\|T\|_{p,\infty} := \sup_{n\geq 0}(n+1)^{\frac{1}{p}}\mu(n, T).$$

# Some results

## Theorem

*Let $G$ be a stratified Lie group with stratification $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$, homogeneous dimension $Q = \sum_{n=1}^{\infty} n \cdot \dim(\mathfrak{g}_n)$ and a fixed sub-Laplacian $\Delta = \sum_{j=1}^{m} X_j^2$, where $\{X_j\}_{j=1}^{m}$ is a basis for $\mathfrak{g}_1$.*

1. *if $p > 2$, then*
$$\|M_f(-\Delta)^{-\frac{Q}{2p}}\|_{p,\infty} \leq c_p\|f\|_{L_p(G)}$$
2. *if $p < 2$ and $q > 2$, then*
$$\|M_f(1-\Delta)^{-\frac{Q}{2p}}\|_{p,\infty} \leq c_{p,q}\|f\|_{\ell_p(L_q)(G)}.$$
3. *if $p = 2$ and $q > 2$, then*
$$\|M_f(1-\Delta)^{-\frac{Q}{2p}}\|_{p,\infty} \leq c_q\|f\|_{\ell_{2,\log}(L_q)(G)}.$$

# Some results

Of course, a similar result holds for Schatten ideals.

**Theorem**

(i) *if* $p > 2$ *and* $r > \frac{Q}{p}$, *then*

$$\|M_f(-\Delta)^{-\frac{r}{2}}\|_p \leq c_{p,r}\|f\|_{L_p(G)}.$$

(ii) *if* $p = 2$ *and* $r > \frac{Q}{p}$, *then*

$$\|M_f(1-\Delta)^{-\frac{r}{2}}\|_p = c_{p,r}\|f\|_{L_p(G)}.$$

(iii) *if* $p < 2$, $r > \frac{Q}{p}$ *and* $q > 2$, *then*

$$\|M_f(1-\Delta)^{-\frac{r}{2}}\|_p \leq c_{p,q,r}\|f\|_{\ell_p(L_q)(G)}.$$

# Birman's theorem for stratified Lie groups

Suppose that

$$D_1 = \sum_{\alpha} a_\alpha(x) X^\alpha$$

where each $a_\alpha$ is a smooth function on $G$ equal to a constant (say, $c_\alpha$) outside a compact set. Then we expect that

$$D_0 = \sum_{\alpha} c_\alpha X^\alpha$$

is a good model for $D_1$ asymptotically, since $D_1 - D_0$ is a differential operator with compactly supported coefficients.
The preceding theorems verify the existence and completeness of the wave operators for $D_1, D_0$.

# What about the point spectrum?

Recall that if $D_1$ is a self-adjoint operator with no singular continuous spectrum, then the differential equation

$$\frac{du}{dt}(t) = iD_1 u(t), \quad u(0) = u_0.$$

has solution

$$u(t) = \sum_{\lambda} e^{it\lambda} E^{D_1}(\{\lambda\}) u_0 + e^{itD_1} P_{ac}(D_1) u_0.$$

If we know that the wave operator $W_+(D_0, D_1)$ exists, then we have a good understanding of the absolutely continuous part, but what about the discrete part?

# What about the point spectrum?

Cwikel estimates also give us upper bounds for the number of eigenvalues.

**Theorem (Cwikel–Lieb–Rozenblum estimate)**

*Assume that $Q > 2$. Let $V \in L_{\frac{Q}{2}}(G)$ be real-valued. The quadratic form sum $-\Delta \dot{+} M_V$ defines an unbounded self-adjoint operator on $L_2(G)$ with essential spectrum $[0, \infty)$, and*

$$\mathrm{Tr}(\chi_{(-\infty,0)}(-\Delta \dot{+} M_V)) \leq C_G \int_G V_-^{\frac{Q}{2}}.$$

*Here, $V_- = \frac{1}{2}(|V| - V)$ is the negative part of $V$.*

It follows from this that if $f \in L_{\frac{Q}{2}}(G)$, then the operator $-\Delta + M_f$ has absolutely continuous spectrum $[0, \infty)$, which scatters like $-\Delta$. There may be some eigenvalues in $(-\infty, 0)$, but only $C_G \int_G V_-^{\frac{Q}{2}}$ of them.

# Spectral asymptotics

Related to these estimates we have spectral asymptotics. In the following theorem, $\mu$ denotes the singular value function. In particular, the sequence $\{\mu(n,T)\}_{n=0}^{\infty}$ is the sequence of singular values of a compact operator $T$. We give a precise definition of $\mu$ in the next section.

## Theorem

*Let $G$ be a non-abelian stratified Lie group with stratification $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$, homogeneous dimension $Q = \sum_{n=1}^{\infty} n \cdot \dim(\mathfrak{g}_n)$ and a fixed sub-Laplacian $\Delta = \sum_{j=1}^{m} X_j^2$, where $\{X_j\}_{j=1}^{m}$ is a basis for $\mathfrak{g}_1$. Let $k \in \mathbb{N}$ and let $p = \frac{Q}{k}$. then Under some technical assumptions on $f$ (depending on $p$), then there exists the limit*

$$\lim_{t\to\infty} t\mu(t,(1-\Delta)^{-\frac{k}{4}} M_f (1-\Delta)^{-\frac{k}{4}})^p = c_G \int_G f^p.$$

*Here, the constant $c_G > 0$ depends on the stratification and also on the particular choice of the basis in $\mathfrak{g}_1$.*

# Semiclassical corollary

## Corollary

*Assume that* $V \in L_{\frac{Q}{2}}(G)$ *is real-valued. For* $h > 0$*, the operator* $-h^2\Delta \dot{+} M_V$ *can be defined in the sense of quadratic forms. There exists a constant* $c_G > 0$ *such that*

$$\lim_{h\to 0} h^Q \mathrm{Tr}(\chi_{(-\infty,0)}(-h^2\Delta \dot{+} M_V)) = c_G \int_G V_-^{\frac{Q}{2}}.$$

# Brief description of the proofs

How do we go about these results?
The key is to study the "zeta function"

$$z \mapsto \mathrm{Tr}(M_f^z(1-\Delta)^{-\frac{z}{2}}), \quad \Re(z) > Q.$$

This can be computed using the trace on the group von Neumann algebra $VN(G)$, and in turn is computable using the representation of $G$.
The important point is that this zeta function is similar to

$$z \mapsto \mathrm{Tr}((M_f^{\frac{1}{2}}(1-\Delta)^{-\frac{1}{2}}M_f^{\frac{1}{2}})^z)$$

which determines the spectrum of $M_f^{\frac{1}{2}}(1-\Delta)^{-\frac{1}{2}}M_f^{\frac{1}{2}}$.

# The future

These estimates are suboptimal for a number of reasons, one of them being that we state the results for functions on $G$ rather than a general Heisenberg manifold (or an even more general filtered manifold). This is probably not a significant restriction.

Thank you for listening!