Spectral asymptotics and scattering theory in the nilpotent Lie group setting

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(based on joint work with Z. Fan, J. Li, F. Sukochev and D. Zanin)

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March 1, 2023

Introduction

This talk is based on a series of preprints by myself with Zhijie Fan (Wuhan), Ji Li (Macquarie), Fedor Sukochev (UNSW) and Dmitriy Zanin (UNSW).

The first two papers are available:

a. Spectral estimates and asymptotics for stratified Lie groups `arXiv:2201.12349` (with Sukochev and Zanin)
b. Endpoint weak Schatten class estimates and trace formula for commutators of Riesz transforms with multipliers on Heisenberg groups `arXiv:2201.12350` (with Fan, Li, Sukochev and Zanin)

There will also be other papers (currently in preparation).

Plan for this talk

1. Some elementary background on scattering theory (why do we care?)
2. Stratified lie groups and recent developments
3. The future(?)

I will discuss a program (mostly unfinished) to do scattering theory (in the style of Birman-Kato) and related things for maximally hypoelliptic operators (in the style of of Helffer-Nourigat, Androulidakis-Mohsen-Yuncken).

Very elementary scattering theory

If Q is an elliptic and symmetric differential operator

$$Q : C^\infty(X,E) \to C^\infty(X,E)$$

where X is compact and Riemannian, and E is some Hermitian vector bundle, then Q is self-adjoint and has a discrete spectral decomposition

$$Q = \sum_{n=0}^{\infty} \lambda(n,Q) P_n$$

where P_n is a finite rank $L_2(X,E)$-orthogonal projection, and $\{\lambda(n,Q)\}_{n=0}^{\infty}$ enumerates the spectrum in increasing order of absolute value.

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If X is not compact, this is of course not true.

Very elementary scattering theory

Suppose that X is not compact (later, we will simply take $X = \mathbb{R}^d$). If we assume that the geometry of X and E are not so bad and that the coefficients of Q are uniformly bounded in the correct sense, then Q is still self-adjoint but its spectrum is complicated.

Very elementary scattering theory

Suppose that X is not compact (later, we will simply take $X = \mathbb{R}^d$). If we assume that the geometry of X and E are not so bad and that the coefficients of Q are uniformly bounded in the correct sense, then Q is still self-adjoint but its spectrum is complicated.
In this case scattering theory can provide a more useful description.

Very elementary scattering theory

A conventional situation is that we have a differential operator (on $\mathbb{R}^d$),

$$D_1 = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$$

with smooth coefficients $\{a_\alpha\}$ that are constant outside of a compact set, say $a_\alpha(x) = c_\alpha$. If we define

$$D_0 = \sum_{|\alpha|\leq m} c_\alpha\partial^\alpha$$

The spectral theory of D_0 is easy to understand using the Fourier transform. We expect that D_1
Scattering theory is about the solutions to the equation

$$\frac{\partial u}{\partial t} = iD_1 u.$$

Or $u(t) = \exp(itD_1)u(0)$. We want to know when there exists u_+ such that

$$\lim_{t\to\infty} \|\exp(itD_1)u(0) - \exp(itD_0)u_+\| = 0.$$

Stratified Lie groups

Let $\mathfrak{g}$ be a Lie algebra which admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{k=1}^{\infty} \mathfrak{g}_k$$

where $[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$ and $\mathfrak{g}_1$ generates $\mathfrak{g}$. This is called a stratified Lie algebra. Exponentiating $\mathfrak{g}$, we get a simply connected nilpotent Lie group

$$G = \exp(\mathfrak{g}).$$

This is a homeomorphism, and the Lebesgue measure of $\mathfrak{g}$ pushes forward to the Haar measure of G. Suppose that $\mathfrak{g}_1$ has a basis $\{X_1, \ldots, X_m\}$., and G is essentially a Euclidean space $\mathbb{R}^d$ equipped with a family of vector fields

$$X_1, \ldots, X_m$$

with polynomial coefficients satisfying the Hörmander condition at every point.

Ellipticity on stratified Lie groups

The stratification of $\mathfrak{g}$ defines a grading on the algebra of invariant differential operators, $\mathcal{U}(\mathfrak{g})$, on G. Say that an operator $P \in \mathcal{U}(\mathfrak{g})$ has order k if the highest degree term in P is homogeneous of degree k.

Theorem (Helffer-Nourigat, Rockland)

Let $P \in \mathcal{U}(\mathfrak{g})$ have degree k. If for every $\pi \in \widehat{G}_u$ (the unitary dual of G), $\pi(P)$ is injective on H_π^∞ (the smooth vectors), then for every Q of degree less than or equal to k we have

$$\|Qu\|_{L_2(G)} \lesssim \|Pu\|_{L_2(G)} + \|u\|_{L_2(G)}, \quad u \in L_2(G).$$

In particular, P is hypoelliptic.

Ellipticity on stratified Lie groups

Birman's theorem

Suppose that A_1, A_0 are self-adjoint operators on a Hilbert space H. If for any bounded interval $I \subset \mathbb{R}$ we have

$$\chi_I(A_1)(A_1 - A_0)\chi_I(A_0) \in \mathcal{L}_1(H)$$

then the wave operators $W_{\pm}(A_1, A_0)$ exist and are complete.

Birman's theorem for stratified Lie groups

Suppose that

$$D_1 = \sum_{|\alpha|_h \leq m} a_\alpha(x) X^\alpha$$

where each a_α is a smooth function on G equal to a constant (say, c_α) outside a compact set. Then we expect that

$$D_0 = \sum_{|\alpha|_h \leq m} c_\alpha X^\alpha$$

is a good model for D_1 asymptotically, since $D_1 - D_0$ is a differential operator with compactly supported coefficients.

A first result

Theorem

Let $f \in C_c^\infty(G)$, and let $Q \in \mathcal{U}(\mathfrak{g})$ be a Rockland operator. Then for any bounded interval $I \subset \mathbb{R}$, we have

$$M_f \chi_I(Q) \in \mathcal{L}_1(L_2(G)).$$

Thank you for listening!