

# The Dixmier trace and the Density of States

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Based on joint work with N. Azamov, E. Hekkelman F. Sukochev and  
D. Zanin.

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This talk is about a connection between the density of states and Dixmier traces, covered in the two papers

- N. Azamov, E. M., F. Sukochev. and D. Zanin *A Dixmier Trace Formula for the Density of States* Comm. Math. Phys. 377 (2020), no. 3, 2597–2628. arXiv:1910.12380.
- N. Azamov, E. Hekkelman, E.M., F. Sukochev and D. Zanin. *An application of singular traces to crystals and percolation*. J. Geom. Phys. 179 (2022), Paper No. 104608, 22 pp. arXiv:2202.03676

and the recent preprint

- E. Hekkelman and E. M. *A general Dixmier trace formula for the density of states on open manifolds* (2023) arXiv:2304.13272

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But their use is not limited to NCG. Today I will talk about one Dixmier trace formula which is meaningful in mathematical physics.

This formula was conjectured and proved on the basis of developments in NCG.

# The basic idea

If  $H$  is a lower-bounded self-adjoint operator (on a Hilbert space) with discrete spectrum (i.e. the spectrum consists only of eigenvalues of finite multiplicity), then we can understand the spectrum through the spectral counting function

$$\begin{aligned} N(t, H) &= \operatorname{Tr}(\chi_{(-\infty, t]}(H)) \\ &= \text{Dimension of the span of the eigenvectors} \\ &\quad \text{corresponding to eigenvalues } \leq t. \end{aligned}$$

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This is less useful when  $H$  has some essential spectrum. What is a good replacement?

Initially we concentrate on Schrödinger operators.

Let  $d > 1$  and  $R > 0$ . Let  $B(0, R)$  be the open ball in  $\mathbb{R}^d$  of radius  $R$ ,

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The operator  $\Delta_R$  is negative semidefinite and has purely discrete spectrum.

# Schrödinger operators on bounded domains

Given  $V \in L_\infty(B(0, R))$ , denote by  $M_V$  the operator of pointwise multiplication by  $V$ . That is,  $(M_V \xi)(t) = V(t)\xi(t)$ .

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For the purposes of this talk a *Schrödinger operator* on  $B(0, R)$  is a linear operator

$$H = -\Delta_R + M_V$$

where  $V$  is real-valued. Since we only consider bounded  $V$ , the operator  $H$  is self-adjoint with domain equal to that of  $\Delta_R$ .

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Since  $H$  is a relatively compact perturbation of  $-\Delta_R$ , the spectrum of  $H$  is discrete and consists only of eigenvalues of finite multiplicity.

So far we have only discussed operators on bounded domains. What about unbounded domains? Let  $\Delta = \sum_{j=1}^d \partial_{x_j}^2$  be the Laplace operator on  $L_2(\mathbb{R}^d)$ . This operator is self-adjoint, has spectrum  $(-\infty, 0]$  and has no eigenvalues.

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What does the spectrum of  $H$  look like? The spectral counting function usually does not make sense for Schrödinger operators on  $\mathbb{R}^d$ , since the spectrum can be continuous.



# The density of states

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**Idea:** (From solid-state physics.) Let  $R > 0$ , and “restrict”  $H$  to  $B(0, R)$ :

$$H_R := -\Delta_R + M_V$$

on  $L_2(B(0, R))$ . The *density of states* of  $H$  is a measure  $\nu_H$  on  $\mathbb{R}$  defined as

$$\nu_H((-\infty, t]) := \lim_{R \rightarrow \infty} \frac{N(t, H_R)}{\text{Vol}(B(0, R))}, \quad t \in \mathbb{R}.$$

Optimistically, this limit exists and indeed defines a measure.

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- If it exists, what is the support of  $\nu_H$ ? Answer: the essential spectrum of  $H$ .
- Does  $\nu_H$  depend on the choice of boundary conditions for  $H_R$ ?  
Answer: No.
- Does it matter that we “restricted”  $H$  to balls  $B(0, R)$ ? What about cubes or other domains? Answer: For many classes of potentials  $V$  (random, almost periodic), the choice of domain is irrelevant. For arbitrary potentials, the domain matters.

# Example

An example where the DOS is easily computed is the “free” Hamiltonian, corresponding to  $V = 0$ . We have

$$\frac{d\nu_{-\Delta}}{dt} = \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} \max\{t, 0\}^{\frac{d}{2}-1}.$$

# Example

Another example allowing explicit computation is when  $V$  is radially homogeneous (i.e.,  $V(r\xi) = V(\xi)$  for all  $r > 0$ ). In this case, the DOS exists and

$$\frac{d\nu_{-\Delta+M_V}}{dt} = \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} \cdot \left( \frac{1}{\text{Vol}(S^{d-1})} \int_{S^{d-1}} \max\{t - V(\theta), 0\}^{\frac{d}{2}-1} d\theta \right).$$

See N. Azamov, E. M., F. Sukochev and D. Zanin *The density of states depends on the domain* Oper. Matrices **16**, No. 4 (2022), 1185–1189  
arXiv:2107.09828.

# Insensitivity to localised perturbations

One of the important features of the DOS is that  $\nu_{-\Delta+M_V}$  only cares about the behaviour of  $V$  “at infinity”.

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For example, if  $V_1, V_0 \in L_\infty(\mathbb{R}^d)$  are such that  $V_1 - V_0$  is compactly supported, then

$$\nu_{-\Delta+M_{V_1}} = \nu_{-\Delta+M_{V_0}}.$$

(assuming that both measures exist, of course.)

Let  $\mathcal{H}$  be a Hilbert space. An eigenvalue sequence of a compact operator  $A \in B(\mathcal{H})$  is a sequence  $\{\lambda(k, A)\}_{k \in \mathbb{N}}$  of the eigenvalues of  $A$  listed with multiplicity, such that  $\{|\lambda(k, A)|\}_{k \in \mathbb{N}}$  is non-increasing.

The usual operator trace  $\text{Tr}$  can be characterised for trace class operators  $A \in \mathcal{L}_1 \subset B(\mathcal{H})$  as

$$\text{Tr}(A) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda(k, A).$$

# The weak trace class

A compact operator  $T$  is said to be weak trace-class ( $T \in \mathcal{L}_{1,\infty}$ ) if  $\mu(k, T) = \mathcal{O}(k^{-1})$ . Equivalently,

$$\|T\|_{1,\infty} := \sup_{k \geq 0} (1+k)\lambda(k, |T|) < \infty.$$

The class  $\mathcal{L}_{1,\infty}$  is an an ideal.



# Dixmier traces

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An important class of examples are the famous Dixmier traces.

If  $T \in \mathcal{L}_{1,\infty}$ , then

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An important class of examples are the famous Dixmier traces.

If  $T \in \mathcal{L}_{1,\infty}$ , then

$$\sum_{k=0}^N \lambda(k, T) = \mathcal{O}(\log(N)), \quad N \rightarrow \infty.$$

An extended limit is a linear functional  $\omega \in \ell_\infty(\mathbb{N})^*$  which coincides with the “limit” functional on the subspace of convergent sequences.

If  $T \geq 0$  is a positive element of  $\mathcal{L}_{1,\infty}$ , define  $\text{Tr}_\omega(T)$  as

$$\text{Tr}_\omega(T) = \omega \left( \left\{ \frac{1}{\log(2+N)} \sum_{k=0}^N \mu(k, T) \right\}_{N=0}^\infty \right).$$

# Connes' integral formula

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# Connes' integral formula

A property of Dixmier traces (and all traces on  $\mathcal{L}_{1,\infty}$ ) is that they vanish on finite rank operators. It was probably Connes who realised that this property makes Dixmier traces computable in many situations. (One version of) Connes' integral formula states that if  $f \in C_c^\infty(\mathbb{R}^d)$  then

$$\mathrm{Tr}_\omega(M_f(1 - \Delta)^{-\frac{d}{2}}) = \frac{\mathrm{Vol}(S^{d-1})}{d(2\pi)^d} \int_{\mathbb{R}^d} f(t) dt.$$

This is taking place on the Hilbert space  $\mathcal{H} = L_2(\mathbb{R}^d)$ , and relies on the fact that  $M_f(1 - \Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$ .

# Connes' integral formula reconsidered

Restrict attention to a function  $f \in C_c^\infty(\mathbb{R}^d)$  which is radial. There exists  $g \in C_0([0, \infty))$  such that  $f(t) = g(|t|^2)$ . Then Connes' integral formula implies Switch to the Fourier side (recall that  $\text{Tr}_\omega$  is unitarily invariant!) and we get

$$\text{Tr}_\omega(g(-\Delta)(1 + M_{|x|}^2)^{-\frac{d}{2}}) = \frac{\text{Vol}(S^{d-1})}{d} \int_0^\infty g(r) d\mu(r)$$

where  $\mu$  is the measure  $d\mu(r) = \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} r^{\frac{d}{2}-1} dr$ .

This measure  $\mu$  is precisely the density of states measure for the “free” Hamiltonian  $H = -\Delta$ . This is not a coincidence!

# Idea of the formula

Idea: Replace  $-\Delta$  with  $H = -\Delta + M_V$ . An argument using the Riesz representation theorem implies that there exists a measure  $\mu_H$  on  $\mathbb{R}$  such that

$$\mathrm{Tr}_\omega(f(H)(1 + M_{|x|}^2)^{-\frac{d}{2}}) = \frac{\mathrm{Vol}(S^{d-1})}{d} \int_{\mathbb{R}} f \, d\mu_H$$

for all  $f \in C_c(\mathbb{R})$ .

## Theorem

$\nu_H = \mu_H$ .

*That is, if the density of states measure  $\nu_H$  exists, then it is given by the formula*

$$\mathrm{Tr}_\omega(f(H)(1 + M_{|x|}^2)^{-\frac{d}{2}}) = \frac{\mathrm{Vol}(S^{d-1})}{d} \int_{\mathbb{R}} f \, d\nu_H, \quad f \in C_c(\mathbb{R}).$$

# Advantages of the Dixmier trace formula for DOS

1. Simplifies proofs of some properties of DOS, such as insensitivity to localised perturbations described above.
2. Makes some computations very easy. For example, the formula for radially homogeneous potentials stated above was first derived using the Dixmier trace formula.



# Trace formulas

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Since then, we have discovered that those assumptions are not really necessary. We can define a DOS measure (Strichartz has called this a “spectral mass”) quite abstractly.

If  $(X, d_X)$  is a measure space equipped with a Borel measure  $\mu$ , and  $H$  is an operator on  $L_2(X, \mu)$ , we can define a density of states for  $H$  in a similar way to how it was defined for Schrödinger operators on  $\mathbb{R}^d$ .

## General form

Let  $(X, d_X)$  be a metric space equipped with a measure  $\mu$  *satisfying certain assumptions*. Let  $H$  be a self-adjoint operator on  $L_2(X)$  *satisfying certain assumptions* for which the DOS  $\nu_H$  exists *in some particular sense*. Then for a certain (fixed) function  $w \in L_\infty(X)$ , for all Dixmier traces  $\text{Tr}_w$  and for all  $f \in C_c(\mathbb{R})$  we have

We have proved this theorem in two different settings, using totally different methods.

### Azamov–Hekkelman–M.–Sukochev–Zanin (2022)

$X$  a discrete metric space with some geometrical conditions,  $H$  a general self-adjoint operator,  $w(x) = \frac{1}{1+|B(x_0, d_X(x_0, x))|}$ .

### Hekkelman–M. (2023)

$X$  a manifold of bounded geometry plus some geometrical conditions,  $H$  a self-adjoint, lower bounded, uniformly elliptic second-order differential operator,  $w(x) = \frac{1}{1+|B(x_0, d_X(x_0, x))|}$ .

# The local index theorem

Let  $M$  be a (possibly non-compact) Riemannian  $d$ -manifold with a spinor bundle  $S \rightarrow M$ . Assume that  $S$  has a grading  $\gamma \in \Gamma(\text{End}(S))$ , and associated Dirac operator  $D$ . With respect to the grading, write

$$D = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}.$$

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## Theorem (Patodi, Gilkey)

For all  $x \in M$ , we have

$$\lim_{t \rightarrow 0} \text{Tr}_x(\gamma_x e^{-tD^2}(x, x)) |dx| = \hat{A}(R)(x)^{(d)}.$$

# The integrated local index theorem

Write the local index theorem in a slightly different way. For all  $f \in C_c(M)$ , we have

$$\lim_{t \rightarrow 0} \text{Tr}(\gamma e^{-tD^2} M_f) = \int_M \hat{A}(R) f.$$

The “lim” on the left hand side is necessary.

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The “lim” on the left hand side is necessary.

**McKean–Singer formula:** if  $f = 1$  (and  $M$  is of course compact) then no limit is needed!  $\text{Tr}(\gamma e^{-tD^2})$  is independent of  $t$ , and equal to  $\text{Index}(D_+)$ . The conventional explanation for this is to expand  $\text{Tr}(\gamma e^{-tD^2})$  as

$$\text{Tr}(\gamma e^{-tD^2}) = \sum_{k=0}^{\infty} e^{-t\lambda_k(D_+D_-)} - e^{-t\lambda_k(D_-D_+)} = \text{Index}(D_+).$$



# The integrated local index theorem (cont.)

If  $f$  is not necessarily constant, we get

$$\frac{d}{dt} \text{Tr}(\gamma e^{-tD^2} M_f) = \frac{1}{2} \text{Tr}(\gamma e^{-tD^2} D[D, M_f]).$$

If  $f$  is not constant on connected components of  $M$ , then  $[D, M_f]$  is non-zero, and  $\text{Tr}(\gamma e^{-tD^2} M_f)$  will probably depend on  $t$ .

# The Roe index theorem

Suppose that  $M$  has a *regular exhaustion*. That is,  $M$  is an increasing union of compact subsets  $M = \bigcup_{n=0}^{\infty} M_n$ , satisfying certain technical conditions. Given  $f \in L_{\infty}(M)$ , define

$$\phi_n(x) = \frac{1}{|M_n|} \int_{M_n} f.$$

By the Banach-Alaoglu theorem, there exists a weak\*-limit point  $m$  of the sequence  $\{\phi_n\}_{n=0}^{\infty}$ . That is,  $m$  is a functional such that

$$m\{f\} = \lim_{n \rightarrow \infty} \frac{1}{|M_n|} \int_{M_n} f$$

whenever the limit exists, and extended to all bounded functions  $f$ .

# The Roe index theorem (cont.)

The local index theorem implies that for every  $n$ ,

$$\lim_{t \rightarrow 0} \frac{1}{|M_n|} \int_{M_n} \text{Tr}_x(\gamma_x e^{-tD^2}(x, x)) dx = \frac{1}{|M_n|} \int_{M_n} \widehat{A}(R)^{(d)}.$$

Roe proved that under suitable uniformity assumptions on the metric and the spinor bundle  $S$  you can take the “limit” as  $n \rightarrow \infty$ , and get

$$\lim_{t \rightarrow 0} m\{x \mapsto \text{Tr}_x(\gamma_x e^{-tD^2}(x, x))\} = m\{\widehat{A}(R)\}.$$

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**Roe’s non-compact McKean–Singer identity:** The limit on the left hand side is unnecessary, and we have

$$m\{x \mapsto P_{\ker(D_+)}(x, x) - P_{\ker(D_-)}(x, x)\} = m\{\widehat{A}(R)\}.$$

This is the Roe index theorem.

# A Dixmier trace perspective on Roe's index theorem

Under certain technical

# A “Dixmier–McKean–Singer” formula

## Theorem (Hekkelman-M. (2023))

*Under (admittedly quite restrictive) technical assumptions, the Roe index is given by the Dixmier trace formula*

$$\mathrm{Tr}_\omega(\gamma \exp(-tD^2)M_w) = \mathrm{Ind}_\tau(D).$$

Thank you for listening!