Lipschitz estimates in quasi-Banach Schatten ideals

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Introduction

This talk is mostly about the paper

M., Sukochev, Lipschitz estimates in quasi-Banach Schatten ideals. *Math. Ann.* 383 (2022), no.1–2, 571–619.

Plan for this talk

1. Operator Lipschitz functions: some basic concepts and history.
2. Some very light background on Schatten ideals: the problem with $p < 1$.
3. Besov spaces and wavelets

Operator Lipschitz functions

Let H be a (complex and separable) Hilbert space, and denote the operator norm by $\|\cdot\|_\infty$. A function $f:\mathbb{R}\to\mathbb{C}$ is said to be *operator Lipschitz* if there exists a constant C_f such that

$$\|f(A)-f(B)\|_\infty \leq C_f\|A-B\|_\infty,\quad A,B\in\mathcal{B}_{\mathrm{sa}}(H)$$

Question (from Krein)

Is every Lipschitz function operator Lipschitz?

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Question (from Krein)

Is every Lipschitz function operator Lipschitz? That is, does $|f(t)-f(s)|\lesssim|t-s|$ imply that $\|f(A)-f(B)\|_\infty\lesssim\|A-B\|_\infty$?

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Johnson & Williams (1975): An operator Lipschitz function is differentiable.

Finite-dimensional case

If H is N-dimensional, then

$$\|f(A) - f(B)\|_\infty \leq C_{\mathrm{abs}} \log(1+N) \|f\|_{\mathrm{Lip}} \|A - B\|_\infty$$

where C_{abs} is an absolute constant. This is sharp in the order of growth as $N \to \infty$. I do not know if a sharp estimate for C_{abs} is known.

Operator Lipschitz functions

It is easy to check that sufficiently good functions are operator Lipschitz. Let's check the function $f(x) = e^{i\xi x}$ for $\xi \in \mathbb{R}$. We have

$$e^{i\xi A} - e^{i\xi B} = i\xi \int_0^1 e^{i\xi(1-\theta)A}(A-B)e^{i\xi\theta B}\,d\theta.$$

The integral converges in the Bochner sense. The triangle inequality implies

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$$\|f(A) - f(B)\|_\infty \leq \|A-B\|_\infty \cdot 2\pi \|\widehat{\partial f}\|_1.$$

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By Cauchy-Schwarz, $\|\widehat{\partial f}\|_1 \leq \|f'\|_2 + \|f''\|_2$. This is a "good enough" sufficient condition for most purposes.

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Theorem (Peller (1990))

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In other words, if

$$\int_0^\infty \sup_{t\in\mathbb{R}} \frac{|f(t-h) - 2f(t) + f(t+h)|}{h^2}\,dh + \sup_{t\in\mathbb{R},h>0} \frac{|f(t+h)-f(t)|}{h} < \infty$$

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then f is operator Lipschitz. For example, if $f' \in W^1_\infty(\mathbb{R})$ then f is operator Lipschitz.

Peller's operator Bernstein inequality

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$$\|f\|_{\mathrm{Lip}} \leq C\sigma\|f\|_\infty.$$

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Peller's theorem is a consequence of his *operator Bernstein inequality*.

Theorem (Peller (1990))

If $f \in L_\infty(\mathbb{R})$ has Fourier transform supported in the interval $[-\sigma, \sigma]$, then

$$\|f\|_{\mathrm{O-Lip}} \leq C\sigma\|f\|_\infty.$$

Here $\|f\|_{\mathrm{O-Lip}}$ is the operator Lipschitz seminorm, i.e.

$$\|f\|_{\mathrm{O-Lip}} := \sup_{A=A^*, B=B^* \in \mathcal{B}(H)} \frac{\|f(A)-f(B)\|_\infty}{\|A-B\|_\infty}.$$

Schatten ideals

If T is a compact operator on H, the singular value sequence of T is defined as

$$\mu(k, T) := \inf\{\|T - R\|_\infty \ : \ \mathrm{rank}(R) \leq k\}, \quad k \geq 0.$$

(Equivalently, $\mu(T) = \{\mu(k, T)\}_{k=0}^\infty$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.)

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(Equivalently, $\mu(T) = \{\mu(k, T)\}_{k=0}^\infty$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.) Note that $\|T\|_\infty = \mu(0, T) = \|\mu(T)\|_{\ell_\infty}$. For $1 \leq p < \infty$, the Schatten $\mathcal{L}_p$-norm of a compact operator T is

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \left(\sum_{k=0}^\infty \mu(k, T)^p\right)^{\frac{1}{p}}.$$

Equivalently, $\|T\|_p = \mathrm{Tr}(|T|^p)^{1/p}$. It is not obvious, but this is a norm (i.e. $\|T + S\|_p \leq \|T\|_p + \|S\|_p$.)

$\mathcal{L}_p$-operator Lipschitz functions

A function f on $\mathbb{R}$ is said to be $\mathcal{L}_p$-operator Lipschitz if there exists a constant $C_f > 0$ such that

$$\|f(A) - f(B)\|_p \leq C_f \|A - B\|_p, \quad A, B \in \mathcal{B}_{\mathrm{sa}}(H).$$

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For $p = 2$ this is almost trivial and has been known for approx. 110 years. For $p \neq 2$, this requires some very deep harmonic analysis. Last year, Conde-Alonso, González-Pérez, Parcet and Tablate have a new proof using operator-valued harmonic analysis.

What about $0 < p < 1$?

For $0 < p < 1$, we can still define

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \mathrm{Tr}(|T|^p)^{\frac{1}{p}}.$$

This is not a norm, merely a quasi-norm. There is no triangle inequality, merely a quasi-triangle inequality

$$\|T + S\|_p \leq 2^{\frac{1}{1}-p}(\|T\|_p + \|S\|_p).$$

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$$\|T+S\|_p \leq 2^{\frac{1}{1}-p}(\|T\|_p + \|S\|_p).$$

Nonetheless, we have

$$\|T+S\|_p^p \leq \|T\|_p^p + \|S\|_p^p.$$

Geometry in $\mathcal{L}_p$.

The unit ball $B = \{T \ : \ \|T\|_p \leq 1\}$ in $\mathcal{L}_p$ is not convex.
I.e., if $\xi_1, \ldots, \xi_n \in B$ then it might happen that

$$\theta_1\xi_1 + \cdots + \theta_n\xi_n \notin B, \quad |\theta_1| + \cdots + |\theta_n| \leq 1.$$

For this reason the theory of integration $\mathcal{L}_p$-valued functions is not straightforward. We could have continuous functions $f \in C([0,1], \mathcal{L}_p)$ whose integral is not in $\mathcal{L}_p$.

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For this reason the theory of integration $\mathcal{L}_p$-valued functions is not straightforward. We could have continuous functions $f \in C([0,1], \mathcal{L}_p)$ whose integral is not in $\mathcal{L}_p$.
Instead, B is only closed under p-convex combinations, i.e.

$$\theta_1\xi_1 + \cdots + \theta_n\xi_n \in B, \quad |\theta_1|^p + \cdots + |\theta_n|^p \leq 1.$$

$\mathcal{L}_p$-Lipschitz functions for $0 < p < 1$.

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At least some functions are, for example $f(t) = (t+\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

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Which functions are Lipschitz in $\mathcal{L}_p$ when $0 < p < 1$?
At least some functions are, for example $f(t) = (t+\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
What about $f(t) = \exp(it\xi)$ for $\xi \in \mathbb{R}$?

Periodic functions are not $\mathcal{L}_p$-Lipschitz for $0 < p < 1$.

A first hint that the $0 < p < 1$ case is interesting comes from the following:

Lemma (M. and Sukochev (2022))

Let $0 < p < 1$, and let f be a periodic function on $\mathbb{R}$. Then f is $\mathcal{L}_p$-Lipschitz if and only if it is constant.

What does this imply?

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What does this imply?

- Even C^∞ functions with all derivatives bounded may not be $\mathcal{L}_p$-Lipschitz;
- In particular $f(t) = \exp(it\xi)$, $\xi \neq 0$ is not $\mathcal{L}_p$-Lipschitz for any $0 < p < 1$. This means that methods based on a Fourier decomposition are unlikely to work.

Fourier multipliers

An analogous issue is Fourier multipliers in $L_p(\mathbb{T})$ for $0 < p < 1$.

Theorem

Let $m \in \ell_\infty(\mathbb{Z})$ and $0 < p < 1$. The Fourier multiplier

$$T_m : L_2(\mathbb{T}) \to L_2(\mathbb{T}), \quad T_m(\exp(i\theta n)) = m(n)\exp(i\theta n), \quad n \in \mathbb{Z}$$

is bounded on $L_p(\mathbb{T})$ if and only if m has the form

$$m(n) = \sum_{j=0}^{\infty} c_j \exp(in\zeta_j), \quad n \in \mathbb{Z}$$

where $\sum_j |c_j|^p < \infty$.

In other words, the only $L_p(\mathbb{T})$ multipliers for $0 < p < 1$ are shift operators and p-convex combinations of shifts.

Strategies to get $\mathcal{L}_p$-operator Lipschitz estimates

In the $\mathcal{L}_\infty$ case, we started with a class of functions $\{\exp(i\xi x)\}_{\xi\in\mathbb{R}}$ for which Lipschitz estimates are easy, and derived a more general class by taking convex combinations.

If we could find some set $\{\psi_j\}$ of functions which we know are $\mathcal{L}_p$-Lipschitz, then we could conclude that functions of the form

$$\sum_j c_j\psi_j$$

are also $\mathcal{L}_p$-operator Lipschitz.

Strategies to get $\mathcal{L}_p$-operator Lipschitz estimates

We know that if $f_\lambda(t) = (t+\lambda)^{-1}$, where $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then

$$\|f_\lambda\|_{\mathcal{L}_p-\mathrm{Lip}} \leq |\Im(\lambda)|^{-2}.$$

Essentially every smooth function on $\mathbb{R}$ belongs to the closed convex hull of $\{f_\lambda\}_{\Im(\lambda)\neq 0}$.

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I tried for a long time to characterise functions f having a decomposition like

$$f(t) = \sum_{j=0}^{\infty} c_j |\Im(\lambda_j)|^2 f_{\lambda_j}(t)$$

where $\sum_{j=0}^{\infty} |c_j|^p < \infty$, but with no success.

Wavelet methods

It is possible to prove that if f is a compactly supported C^k function where $k > \frac{1}{p}$ then f is $\mathcal{L}_p$-Lipschitz.

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Theorem (Daubechies (1988))

For all $k > 0$, there exists a compactly supported C^k function ψ such that the system of translations and dilations

$$\psi_{j,k}(t) := 2^{\frac{j}{2}}\psi(2^j t - k), \quad j,k \in \mathbb{Z}$$

forms an orthonormal basis of $L_2(\mathbb{R})$.

A new result

Using wavelet methods we can get the following:

Theorem (M. and Sukochev (2022))

Let $0 < p < 1$. Let $f \in \dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ be Lipschitz continuous. Then f is $\mathcal{L}_p$-Lipschitz and

$$\|f(A) - f(B)\|_p \leq C_p(\|f'\|_\infty + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})})\|A - B\|_p, \quad A, B \in \mathcal{B}_{\mathrm{sa}}(H).$$

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In other words, we require that f be Lipschitz and for some $n > \frac{1}{p}$ that

$$\int_0^\infty \left(\int_{-\infty}^\infty \left| \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(t + kh) \right|^{\frac{p}{1-p}} dt \right)^{1-p} \frac{dh}{h^2} < \infty.$$

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For example, if $f' \in W^k_{\frac{p}{1-p}}(\mathbb{R})$ where $k > \frac{1}{p} - 1$ then f is $\mathcal{L}_p$-Lipschitz.

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Wavelets are not new, but their application to this theory is. Some other things we can achieve:

- For all $n \geq 0$, the inequality

$$\sum_{k=0}^{n} \mu(k, f(A) - f(B))^p \lesssim (\|f'\|_\infty + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}}) \sum_{k=0}^{n} \mu(k, A - B)^p$$

  (this recovers the previous result with $n = \infty$.)

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  (this recovers the previous result with $n = \infty$.)

- Hölder-type estimates of the form

$$\|f(A) - f(B)\|_p \lesssim_f \||A-B|^\alpha\|_p$$

  for f in some Besov space.

Wavelet methods

Wavelets are analogous to Fourier series, in the sense that if

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

then the coefficients $c_{j,k}$ for $j > N$ represent oscillations of f on the scale $\sim 2^{-N}$. A function of the form

$$f(t) = \sum_{j<N} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

does not oscillate greatly on scales smaller than 2^{-N}. This is similar to functions with Fourier transform supported in $[-2^N, 2^N]$.

Wavelet Bernstein inequality

How do we use wavelet methods? The key is again a Bernstein inequality.

Theorem (Meyer(?) (1980s))

Let $f \in L_\infty(\mathbb{R})$ have Wavelet expansion

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

where $c_{j,k} = 0$ for $k > N$. Then

$$\|f\|_{\mathrm{Lip}} \leq C2^N \|f\|_\infty.$$

An $\mathcal{L}_p$-Lipschitz Bernstein inequality

Theorem (M.-Sukochev (2022))

Let $f \in L_{\frac{p}{1-p}}(\mathbb{R})$ have Wavelet expansion

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

where $c_{j,k} = 0$ for $k > N$. Then

$$\|f\|_{\mathcal{L}_p-\mathrm{Lip}} \leq C2^{\frac{N}{p}}\|f\|_{\frac{p}{1-p}}.$$

With $p = 1$, this is the wavelet analogy of Peller's operator Bernstein inequality. For $p < 1$ it is new.

Wavelets and Besov spaces

It follows from the Wavelet Bernstein inequality that Besov spaces have a very simple characterisation in terms of wavelet coefficients.

Theorem (Meyer (1986))

Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. Let ψ be a compactly supported C^k wavelet where $k > -s$. Then a distribution $f \in \mathcal{D}'(\mathbb{R})$ belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ if and only if

$$\|f\|_{B^s_{p,q}} \approx \sum_{j\in\mathbb{Z}} 2^{jq(s+\frac{1}{2}-\frac{1}{p})} \left(\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^p\right)^{\frac{q}{p}} < \infty.$$

Using the p-triangle inequality and the $\mathcal{L}_p$-Lipschitz Bernstein inequality, we easily conclude that $\|f\|_{\mathcal{L}_p-\mathrm{Lip}} \lesssim \|f\|_{B^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})}$.

Thank you for listening!