L_p boundedness of operators in the Androulidakis-Mohsen-Yuncken calculus

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Introduction

This talk is mostly based on my preprint

L_p estimates in the Androulidakis-Mohsen-Yuncken calculus, arXiv:2410.13701.

In turn, this work was based on the paper by Androulidakis, Mohsen and Yuncken that can be found at arXiv:2201.12060.

Section 0: Asteroids

Part 1: Asteroids

Rules of Asteroids

In the classic game of asteroids, a player controls a spaceship moving on a two dimensional toroidal space, $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. There are two controls available:

1. (i) The spaceship can be rotated,
2. (ii) The spaceship can be moved forward.

Configuration space of Asteroids

The configuration space of the game is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, with coordinates (x, y, θ), where (x, y) is the position of the spaceship and θ is its angle.
The controls of the game allow us to move along the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta.$$

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$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta.$$

The player moves the spaceship along a path which is parallel to the span of X and Y. That is, the path of the spaceship in configuration space is $\{\gamma(t)\}_{t\geq 0}$, where

$$\dot{\gamma}(t) \in \mathrm{span}\{X_{\gamma(t)}, Y_{\gamma(t)}\}, \quad t \geq 0.$$

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Despite there being only two available directions, we can reach any point (x, y, θ) from any other point by travelling on a piecewise-smooth path parallel to X and Y. Physically, this is called a *non-holonomic constraint*.

Asteroids

In general, if we can travel parallel to X and Y then we can approximate paths along $[X, Y]$, by the Lie-Kato-Trotter product formula

$$\exp(t[X,Y]) = \lim_{n\to\infty}(\exp(\frac{t}{n}X)\exp(\frac{t}{n}Y)\exp(-\frac{t}{n}(X+Y))\exp(-\frac{t}{n}Y))^n.$$

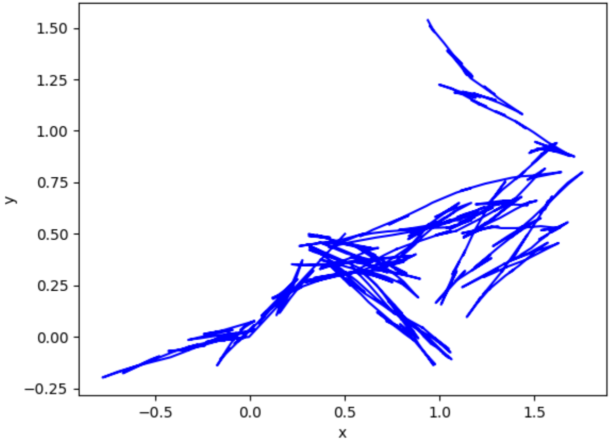
But moving along $[X, Y]$ is harder than moving along X and Y.
In the asteroids example,

$$[X,Y] = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

so $\{X, Y, [X, Y]\}$ form a basis for the tangent space to $\mathbb{T}^3$ at every point.

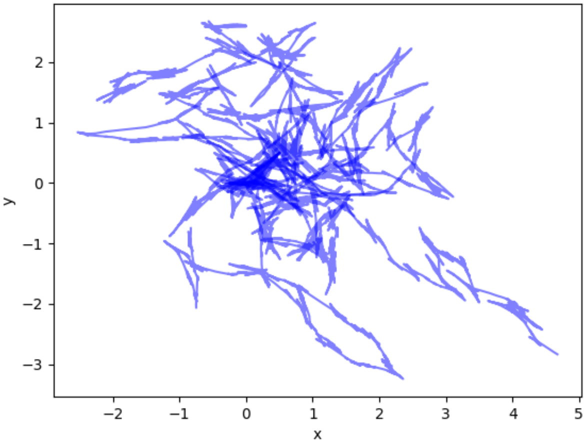
Random walks

A random walk making independent increments in the X and Y directions looks a bit like this:



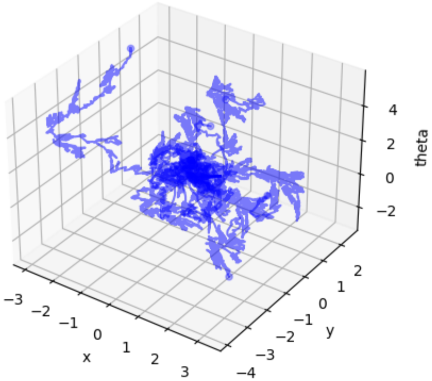
Random walks

Ten realisations of the same random walk look like this:



Random walks

The path through configuration space is in three dimensions, and looks like this:



Asteroids

Thinking about X and Y as derivations (not just as directions), we should think of X, Y as being order 1 and $[X, Y]$ as being order 2.
The operator

$$\Theta = X^2 + Y^2 = \partial_\theta^2 + (\cos(\theta)\partial_x + \sin(\theta)\partial_y)^2$$

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In stochastic analysis, Θ is the generator of the random walks above.

Asteroids

Let $\|u\|_{W_2^s(\mathbb{T}^3)}$ denote the standard Hilbert-Sobolev norm of order $s \in \mathbb{R}$ of $u \in C^\infty(\mathbb{T}^3)$. That is,

$$\|u\|_{W_2^s(\mathbb{T}^3)} := \|(1-\partial_x^2-\partial_y^2-\partial_\theta^2)^{\frac{s}{2}}u\|_{L_2(\mathbb{T}^3)}.$$

A highly non-trivial calculation gives us the *sub-elliptic estimates*:

$$\|u\|_{W_2^{s+\frac{1}{2}}(\mathbb{T}^3)} \lesssim_s \|\Theta u\|_{W_2^s(\mathbb{T}^3)} + \|u\|_{W_2^s(\mathbb{T}^3)}.$$

This implies hypoellipticity (i.e., $\Theta u \in C^\infty \Rightarrow u \in C^\infty$.)

More examples

A similar example can be performed on the Lie group $\mathrm{SO}(3)$. The Lie algebra $\mathfrak{so}(3)$ is spanned by three elements, $\{A, B, C\}$, and $[A, B] = C, [C, A] = B$, etc.

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$$\Theta = A^2 + B^2$$

is not elliptic, but it is hypoelliptic.

Section 2: Pseudodifferential calculus

What is a pseudodifferential calculus?

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One of the best ways of studying differential operators is to build a pseudodifferential calculus. Let M be a closed manifold, and let $\mathrm{DO}^k(M)$ be the set of differential operators on M of order $m \geq 0$. That is, locally $P \in \mathrm{DO}^k(M)$ looks like

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha.$$

Let $\mathrm{DO}(M) = \bigcup_{k \geq 0} \mathrm{DO}^m(M)$.

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Let $\mathrm{DO}(M) = \bigcup_{k \geq 0} \mathrm{DO}^m(M)$. We could also count "orders" in a non-traditional way as in the asteroids example, where X and Y have order 1 and $[X, Y]$ has order 2.

What is a pseudodifferential calculus?

$\mathrm{DO}(M)$ is a filtered algebra. That is, $\mathrm{DO}^k(M) \subset \mathrm{DO}^{k+1}(M)$ and

$$\mathrm{DO}^k(M) \cdot \mathrm{DO}^\ell(M) \subseteq \mathrm{DO}^{k+\ell}(M).$$

What we would like to do is embed $\mathrm{DO}(X)$ into a larger filtered algebra $\Psi(M) = \bigcup_{k\in\mathbb{Z}} \Psi^k(M)$, with

$$\mathrm{DO}^k(M) \subseteq \Psi^k(M).$$

Desirable properties include:

- $\Psi^k(M)$ should be closed under the adjoint
- $\Psi^0(M)$ should act as bounded operators on $L_2(M)$.
- $\Psi^{-\infty}(M)$ consists of smoothing operators.
- Negative order operators should be compact.

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The final three conditions can be included in:

- $\Psi^k(M)$ acts boundedly from $W_2^{s+k}(M)$ to $W_2^s(M)$ for all $s \in \mathbb{R}$.

What is a pseudodifferential calculus?

The most important thing that we want is

- Elliptic operators $D \in \mathrm{DO}^k(M)$ should have a parametrix $Q \in \Psi^{-k}(M)$.

A parametrix is an inverse modulo smoothing, i.e.

$$DQ - 1, QD - 1 \in \Psi^{-\infty}(M).$$

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A parametrix is an inverse modulo smoothing, i.e.

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In the standard calculus, $P = \sum_{|\alpha| \leq k} a_\alpha(x)\partial^\alpha$ is elliptic if its principal symbol is invertible, i.e.

$$\xi \neq 0 \Rightarrow \sum_{|\alpha|=k} a_\alpha \xi^\alpha \neq 0.$$

Why a pseudodifferential calculus?

There are a lot of things to be done with a pseudodifferential calculus. One of the most important is in deriving *elliptic estimates*.

Lemma

Suppose that $\Psi(M) = \bigcup_{k\in\mathbb{Z}} \Psi^k(M)$ is a pseudodifferential calculus satisfying all of the desirable conditions above. If $D \in \mathrm{DO}^k(M)$ is elliptic, then for every $s \in \mathbb{R}$ there exists a constant $C_{s,D}$ such that

$$\|u\|_{W_2^{s+k}(M)} \leq C_{s,D}\big(\|Du\|_{W_2^s(M)} + \|u\|_{W_2^s(M)}\big).$$

Why a pseudodifferential calculus?

Proof.

Let Q be a parametrix for D. For $u \in C^\infty(M)$, write

$$u = QDu + (1 - QD)u.$$

Since Q is bounded from $W_2^{s+k}(M)$ to $W_2^s(M)$, and $1 - QP$ has order $-\infty$, it follows that

$$\|u\|_{W_2^{s+k}(M)} \leq \|Q\|_{W_2^{s+k}(M)\to W_2^s(M)}\|Du\|_{W_2^s(M)}+\|1-QD\|_{W_2^{s+k}(M)\to W_2^s(M)}\|u\|$$

$\square$

Elliptic estimates in L_p-spaces

L_2 Sobolev spaces are the easiest to deal with since they are Hilbert spaces, but there are reasons to want L_p elliptic estimates, too. For an elliptic operator $D \in \mathrm{DO}^k(M)$, L_p elliptic estimates take the form

$$\|u\|_{W_p^{s+k}(M)} \leq C_{s,D,p}(\|Du\|_{W_p^s(M)} + \|u\|_{W_p^s(M)}).$$

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$$\|u\|_{W_p^{s+k}(M)} \leq C_{s,D,p}(\|Du\|_{W_p^s(M)} + \|u\|_{W_p^s(M)}).$$

From the above argument, the key to proving these estimates is to show that operators in $\Psi^k(M)$ act boundedly from $W_p^{s+k}(M)$ to $W_p^s(M)$.

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$$Pu + u^3 = f$$

has a weak solution u belonging to $L_4(M)$. In order to show that u is a classical solution, we need to show that it has two orders of differentiability.

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has a weak solution u belonging to $L_4(M)$. In order to show that u is a classical solution, we need to show that it has two orders of differentiability. If we know elliptic estimates in $p = \frac{4}{3}$, then

$$\begin{aligned} \|u\|_{W^2_{\frac{4}{3}}(M)} &\lesssim \|Pu\|_{L_{\frac{4}{3}}(M)} + \|u\|_{L_{\frac{4}{3}}(M)} \\ &\lesssim \|f\|_{L_{\frac{4}{3}}} + \|u\|_{L_4(M)}. \end{aligned}$$

How to build a pseudodifferential calculus

Specialise to $M = \mathbb{R}^d$. (* not compact, but close enough.)
It is easy enough to build parametrices for *constant coefficient* differential operators. These can be realised as convolution operators

$$\mathrm{Op}(k)u(x) := \int_{\mathbb{R}^d} k(z)u(x+z)\,dz, \quad u \in C_c^\infty(\mathbb{R}^d).$$

Here, k is a distribution on $\mathbb{R}^d$ and the integral should be interpreted as distributional pairing.

How to build a pseudodifferential calculus

To build parametrices for non-translation-invariant operators, we need to consider operators of the form

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^d} K(x,z)u(x+z)\,dz, \quad u \in C_c^\infty(\mathbb{R}^d)$$

where K is smooth in the x-variable and potentially singular in the z variable.

How to build a pseudodifferential calculus

In general, for operators on a manifold M we need to consider

$$\mathrm{Op}(K)u(x) = \int_{TM_x} K(x,z)u(\exp_x z)\,dz, \quad u \in C_c^\infty(M)$$

where K is a distribution on the tangent bundle TM, ad $\exp_x$ is an exponential mapping.

How to build a pseudodifferential calculus

This leads to the question: what are the right conditions on a kernel K which ensure that all the operators of the form $\mathrm{Op}(K)$ form a good pseudodifferential calculus?

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This leads to the question: what are the right conditions on a kernel K which ensure that all the operators of the form $\mathrm{Op}(K)$ form a good pseudodifferential calculus? Following a suggestion from Androulidakis and Skandalis, it turns out that a good idea is to introduce another parameter.

Theorem (Van Erp-Yuncken (2019))

Consider the class of distributions $(x,z,\hbar) \mapsto K(x,z,\hbar)$ which are smooth in $(x,\hbar)$ and compactly supported in z. If for all $\lambda > 0$ the difference

$$\lambda^{-k-d}K(x,\lambda^{-1}z,\lambda\hbar) - K(x,z,\hbar)$$

is smooth in $(x,z,\hbar)$, then $K(x,\cdot,1)$ is the kernel of a classical pseudodifferential operator of order k.
Conversely, every classical pseudodifferential operator arises in this way.

Filtered pseudodifferential calculus

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Let $0 = H^0 < H^1 < \cdots < H^N = TM$ be a filtration of the tangent bundle of a manifold M. Let $\mathrm{DO}_H(M) = \bigcup_{k\geq 0} \mathrm{DO}_H^k(M)$ be the algebra of differential operators with orders counted so that vector fields in H^j have order j, for all $1 \leq j \leq N$. There is a natural notion of "elliptic" for $\mathrm{DO}_H(M)$ (the Rockland condition). There is also a natural scale of Sobolev spaces adapted to the filtration.

Theorem

There exists a pseudodifferential calculus $\Psi_H(M) = \bigcup_{k\in\mathbb{Z}} \Psi_H^k(M)$ extending $\mathrm{DO}_H(M)$, with all of the desirable properties described above.

Section 3: Operators defined by submersions

Limitations of the exponential map

Recall that pseudodifferential operators on a manifold M were defined by some choice of exponential map $\exp_x$ by

$$\mathrm{Op}(K)u(x) = \int_{TM_x} K(x,z)u(\exp_x(z))\,dz.$$

There are lots of circumstances where we need to integrate over a space of dimension larger than TM_x, so that operators are locally defined by integrals of the form

$$\mathrm{Op}(K)u(x) = \int_V K(x,z)u(\Lambda_x(z))\,dz$$

where V is a linear space and $\dim(V) \geq \dim(M)$, K is a distribution on $M \times V$, and $\Lambda_x : V \to M$ is a submersion, smoothly depending on $x \in X$.

Example: the Grushin operator

The operator on $\mathbb{R}^2$ with coordinates (x, y) given by

$$G = \partial_x^2 + x^2\partial_y^2$$

is called the *Grushin operator*. We can write it as a sum of squares of vector fields,

$$G = X_1^2 + X_2^2, \quad X_1 := \partial_x,\ X_2 := x\partial_y.$$

X_1 and X_2 do not span the tangent space at points where $x = 0$. Instead we need to include a third vector field

$$X_3 := [X_1, X_2] = \partial_y.$$

Example: the Grushin operator

The natural differential calculus associated to the Grushin operator is built in the following way: let K be a distribution on $\mathbb{R}^2 \times \mathbb{R}^3$, and let

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^3} K(x,z)u(\exp(z_1X_1 + z_2X_2 + x_3X_3)x)\,dz, \quad u \in C_c^\infty(\mathbb{R}^2).$$

where exp is the exponential of vector fields.

The AMY calculus for the Grushin operator

Theorem

Let $(x, z, \hbar) \mapsto K(x, z, \hbar)$ *denote a distribution on* $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}$ *which is smooth in* $(x, \hbar)$ *and compactly supported in* z*. Assume that for every* $\lambda > 0$ *the difference*

$$\lambda^{-k-4} K(x, \lambda^{-1} z_1, \lambda^{-1} z_2, \lambda^{-2} z_3, \lambda\hbar) - K(x, z, \hbar)$$

is a smooth function of $(x, z, \hbar)$*. Let* Ψ^k *denote the operators of the form*

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^3} K(x, z) u(\exp(z_1 X_1 + z_2 X_2 + x_3 X_3)x)\, dz, \quad u \in C_c^\infty(\mathbb{R}^2).$$

Then $\Psi = \bigcup_{k \in \mathbb{Z}} \Psi^k$ *has all* of the desirable properties of a pseudodifferential calculus.*

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Then $\Psi = \bigcup_{k\in\mathbb{Z}} \Psi^k$ has all of the desirable properties of a pseudodifferential calculus.*

(* The definition of ellipticity is complicated in this example, and they did not prove that elliptic operators have parametrices in the calculus.)

The general case

In general, let $U \subset \mathbb{R}^d$ be an open subset. Let $X_1, \ldots, X_N$ be smooth vector fields on U which span the tangent space at every point. Let $\{w_1, \ldots, w_N\}$ be positive integers. Say that X_j has weight w_j.

The general case

In general, let $U \subset \mathbb{R}^d$ be an open subset. Let $X_1, \ldots, X_N$ be smooth vector fields on U which span the tangent space at every point. Let $\{w_1, \ldots, w_N\}$ be positive integers. Say that X_j has weight w_j. Similar to the Grushin example, we can build an algebra of operators of the form

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^N} K(x,z)u(\exp(z_1X_1 + \cdots + z_NX_N)x)\,dz.$$

An "approximate homogeneity" condition, taking into account the weights, gives a nice pseudodifferential calculus $\Psi(U)$.

L_p estimates

Finally, I can promote my own work:

Theorem (M. (2024))

Order zero operators in the AMY calculus are bounded on L_p spaces, for $1 < p < \infty$.

This implies (although not obviously) the L_p elliptic regularity of certain differential operators like the Grushin operator, and also some nonlinear PDE.

Thank you for listening!