L_p boundedness of operators in the Androulidakis-Mohsen-Yuncken calculus

Ed McDonald

Penn State University

October 22, 2024

Introduction

This talk is mostly based on my preprint

L_p estimates in the Androulidakis-Mohsen-Yuncken calculus, arXiv:2410.13701.

In turn, this work was based on the paper by Androulidakis, Mohsen and Yuncken that can be found at arXiv:2201.12060.

Part 1: Asteroids

Rules of Asteroids

In the classic game of asteroids, a player controls a spaceship moving on a two dimensional toroidal space, $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. There are two controls available:

1. The spaceship can be rotated,
2. The spaceship can be moved forward.

Configuration space of Asteroids

The configuration space of the game is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, with coordinates (x, y, θ), where (x, y) is the position of the spaceship and θ is its angle.
The controls of the game allow us to move along the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta.$$

Configuration space of Asteroids

The configuration space of the game is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, with coordinates (x, y, θ), where (x, y) is the position of the spaceship and θ is its angle.
The controls of the game allow us to move along the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta.$$

The player moves the spaceship along a path which is parallel to the span of X and Y. That is, the path of the spaceship in configuration space is $\{\gamma(t)\}_{t\geq 0}$, where

$$\dot{\gamma}(t) \in \mathrm{span}\{X_{\gamma(t)}, Y_{\gamma(t)}\}, \quad t \geq 0.$$

Configuration space of Asteroids

The configuration space of the game is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, with coordinates (x, y, θ), where (x, y) is the position of the spaceship and θ is its angle.
The controls of the game allow us to move along the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta.$$

The player moves the spaceship along a path which is parallel to the span of X and Y. That is, the path of the spaceship in configuration space is $\{\gamma(t)\}_{t\geq 0}$, where

$$\dot{\gamma}(t) \in \mathrm{span}\{X_{\gamma(t)}, Y_{\gamma(t)}\}, \quad t \geq 0.$$

Despite there being only two available directions, we can reach any point (x, y, θ) from any other point by traveling on a piecewise-smooth path parallel to X and Y.

Configuration space of Asteroids

The configuration space of the game is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, with coordinates (x, y, θ), where (x, y) is the position of the spaceship and θ is its angle.
The controls of the game allow us to move along the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta.$$

The player moves the spaceship along a path which is parallel to the span of X and Y. That is, the path of the spaceship in configuration space is $\{\gamma(t)\}_{t\geq 0}$, where

$$\dot{\gamma}(t) \in \mathrm{span}\{X_{\gamma(t)}, Y_{\gamma(t)}\}, \quad t \geq 0.$$

Despite there being only two available directions, we can reach any point (x, y, θ) from any other point by traveling on a piecewise-smooth path parallel to X and Y. Physically, this is called a *non-holonomic constraint*.

Asteroids

In general, if we can travel parallel to X and Y then we can approximate paths along $[X, Y]$, by the Lie-Kato-Trotter product formula

$$\exp(t[X,Y]) = \lim_{n\to\infty}(\exp(\frac{t}{n}X)\exp(\frac{t}{n}Y)\exp(-\frac{t}{n}(X+Y))\exp(-\frac{t}{n}Y))^n.$$

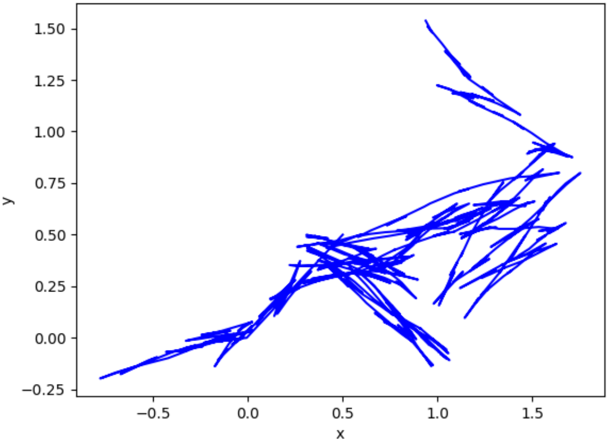
But moving along $[X, Y]$ is harder than moving along X and Y.
In the asteroids example,

$$[X,Y] = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

so $\{X, Y, [X,Y]\}$ form a basis for the tangent space to $\mathbb{T}^3$ at every point.

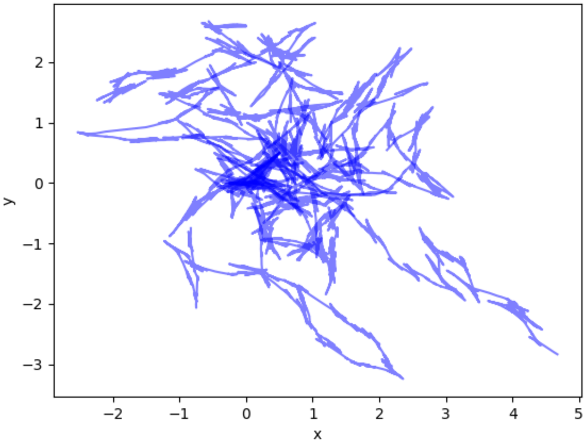
Random walks

A random walk making independent increments in the X and Y directions looks a bit like this:



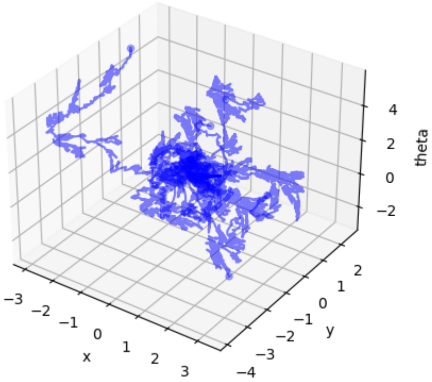
Random walks

Ten realisations of the same random walk look like this:



Random walks

The path through configuration space is in three dimensions, and looks like this:



Asteroids

Thinking about X and Y as derivations (not just as directions), we should think of X, Y as being order 1 and $[X, Y]$ as being order 2.
The operator

$$\Theta = X^2 + Y^2 = \partial_\theta^2 + (\cos(\theta)\partial_x + \sin(\theta)\partial_y)^2$$

is homogeneous of order 2.

Asteroids

Thinking about X and Y as derivations (not just as directions), we should think of X, Y as being order 1 and $[X, Y]$ as being order 2.
The operator

$$\Theta = X^2 + Y^2 = \partial_\theta^2 + (\cos(\theta)\partial_x + \sin(\theta)\partial_y)^2$$

is homogeneous of order 2.

Θ is not elliptic, but $(1-\Theta)^{-1}$ does improve regularity somewhat. Why, and by how much?

Asteroids

Thinking about X and Y as derivations (not just as directions), we should think of X, Y as being order 1 and $[X, Y]$ as being order 2.
The operator

$$\Theta = X^2 + Y^2 = \partial_\theta^2 + (\cos(\theta)\partial_x + \sin(\theta)\partial_y)^2$$

is homogeneous of order 2.

Θ is not elliptic, but $(1-\Theta)^{-1}$ does improve regularity somewhat. Why, and by how much?

In stochastic analysis, Θ is the generator of the random walks above.

Asteroids

Let $\|u\|_{W_2^s(\mathbb{T}^3)}$ denote the standard Hilbert-Sobolev norm of order $s \in \mathbb{R}$ of $u \in C^\infty(\mathbb{T}^3)$. That is,

$$\|u\|_{W_2^s(\mathbb{T}^3)} := \|(1 - \partial_x^2 - \partial_y^2 - \partial_\theta^2)^{\frac{s}{2}} u\|_{L_2(\mathbb{T}^3)}.$$

A highly non-trivial calculation gives us the *sub-elliptic estimates*:

$$\|u\|_{W_2^{s+\frac{1}{2}}(\mathbb{T}^3)} \lesssim_s \|\Theta u\|_{W_2^s(\mathbb{T}^3)} + \|u\|_{W_2^s(\mathbb{T}^3)}.$$

This implies hypoellipticity (i.e., $\Theta u \in C^\infty \Rightarrow u \in C^\infty$.)

More examples

A similar example can be seen on the Lie group $\mathrm{SO}(3)$. The Lie algebra $\mathfrak{so}(3)$ is spanned by three elements, $\{A, B, C\}$, and $[A, B] = C, [C, A] = B$, etc.

More examples

A similar example can be seen on the Lie group $\mathrm{SO}(3)$. The Lie algebra $\mathfrak{so}(3)$ is spanned by three elements, $\{A, B, C\}$, and $[A, B] = C, [C, A] = B$, etc. The operator

$$\Delta = A^2 + B^2 + C^2$$

is the *quadratic Casimir element*. It is elliptic.

More examples

A similar example can be seen on the Lie group $\mathrm{SO}(3)$. The Lie algebra $\mathfrak{so}(3)$ is spanned by three elements, $\{A, B, C\}$, and $[A, B] = C, [C, A] = B$, etc. The operator

$$\Delta = A^2 + B^2 + C^2$$

is the *quadratic Casimir element*. It is elliptic. The operator

$$\Theta = A^2 + B^2$$

is not elliptic, but it is hypoelliptic.

Section 2: Pseudodifferential calculus

What is a pseudodifferential calculus?

One of the best ways of studying differential operators is to build a pseudodifferential calculus.

What is a pseudodifferential calculus?

One of the best ways of studying differential operators is to build a pseudodifferential calculus. Let M be a closed manifold, and let $\mathrm{DO}^k(M)$ be the set of differential operators on M of order $k \geq 0$. That is, locally $P \in \mathrm{DO}^k(M)$ looks like

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha.$$

Let $\mathrm{DO}(M) = \bigcup_{k \geq 0} \mathrm{DO}^m(M)$.

What is a pseudodifferential calculus?

One of the best ways of studying differential operators is to build a pseudodifferential calculus. Let M be a closed manifold, and let $\mathrm{DO}^k(M)$ be the set of differential operators on M of order $k \geq 0$. That is, locally $P \in \mathrm{DO}^k(M)$ looks like

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha.$$

Let $\mathrm{DO}(M) = \bigcup_{k \geq 0} \mathrm{DO}^m(M)$. We could also count "orders" in a non-traditional way as in the Asteroids example, where X and Y have order 1 and $[X, Y]$ has order 2.

What is a pseudodifferential calculus?

$\mathrm{DO}(M)$ is a filtered algebra. That is, $\mathrm{DO}^k(M) \subset \mathrm{DO}^{k+1}(M)$ and

$$\mathrm{DO}^k(M) \cdot \mathrm{DO}^\ell(M) \subseteq \mathrm{DO}^{k+\ell}(M).$$

What we would like to do is embed $\mathrm{DO}(X)$ into a larger filtered algebra $\Psi(M) = \bigcup_{k\in\mathbb{Z}} \Psi^k(M)$, with

$$\mathrm{DO}^k(M) \subseteq \Psi^k(M).$$

Desirable properties include:

- $\Psi^k(M)$ should be closed under the adjoint
- $\Psi^0(M)$ should act as bounded operators on $L_2(M)$.
- $\Psi^{-\infty}(M)$ consists of smoothing operators.
- Negative order operators should be compact.

Desirable properties include:

- $\Psi^k(M)$ should be closed under the adjoint
- $\Psi^0(M)$ should act as bounded operators on $L_2(M)$.
- $\Psi^{-\infty}(M)$ consists of smoothing operators.
- Negative order operators should be compact.

The final three conditions can be included in:

- $\Psi^k(M)$ acts boundedly from $W_2^{s+k}(M)$ to $W_2^s(M)$ for all $s \in \mathbb{R}$.

What is a pseudodifferential calculus?

The most important thing that we want is

- Elliptic operators $D \in \mathrm{DO}^k(M)$ should have a parametrix $Q \in \Psi^{-k}(M)$.

A parametrix is an inverse modulo smoothing, i.e.

$$DQ - 1, QD - 1 \in \Psi^{-\infty}(M).$$

What is a pseudodifferential calculus?

The most important thing that we want is

- Elliptic operators $D \in \mathrm{DO}^k(M)$ should have a parametrix $Q \in \Psi^{-k}(M)$.

A parametrix is an inverse modulo smoothing, i.e.

$$DQ - 1, QD - 1 \in \Psi^{-\infty}(M).$$

In the standard calculus, $P = \sum_{|\alpha| \leq k} a_\alpha(x)\partial^\alpha$ is elliptic if its principal symbol is invertible, i.e.

$$\xi \neq 0 \Rightarrow \sum_{|\alpha|=k} a_\alpha \xi^\alpha \neq 0.$$

Why a pseudodifferential calculus?

There are a lot of things to be done with a pseudodifferential calculus. One of the most important is in deriving *elliptic estimates*.

Lemma

Suppose that $\Psi(M) = \bigcup_{k\in\mathbb{Z}} \Psi^k(M)$ *is a pseudodifferential calculus satisfying all of the desirable conditions above. If* $D \in \mathrm{DO}^k(M)$ *is elliptic, then for every* $s \in \mathbb{R}$ *there exists a constant* $C_{s,D}$ *such that*

$$\|u\|_{W_2^{s+k}(M)} \leq C_{s,D}(\|Du\|_{W_2^s(M)} + \|u\|_{W_2^s(M)}).$$

Why a pseudodifferential calculus?

There are a lot of things to be done with a pseudodifferential calculus. One of the most important is in deriving *elliptic estimates*.

Lemma

Suppose that $\Psi(M) = \bigcup_{k\in\mathbb{Z}} \Psi^k(M)$ is a pseudodifferential calculus satisfying all of the desirable conditions above. If $D \in \mathrm{DO}^k(M)$ is elliptic, then for every $s \in \mathbb{R}$ there exists a constant $C_{s,D}$ such that

$$\|u\|_{W_2^{s+k}(M)} \leq C_{s,D}(\|Du\|_{W_2^s(M)} + \|u\|_{W_2^s(M)}).$$

In PDE terms, these estimates imply the *a priori regularity* of solutions to the equation $Du = f$. That is, if $u \in \mathcal{D}'(M)$, $f \in W_2^s(M)$ and $Du = f$, then $u \in W_2^{s+k}(M)$.

Why a pseudodifferential calculus?

Proof.

Let Q be a parametrix for D. For $u \in C^\infty(M)$, write

$$u = QDu + (1 - QD)u.$$

Why a pseudodifferential calculus?

Proof.

Let Q be a parametrix for D. For $u \in C^\infty(M)$, write

$$u = QDu + (1 - QD)u.$$

Since Q and $1 - QD$ are bounded from $W_2^{s+k}(M)$ to $W_2^s(M)$, it follows that

$$\begin{aligned}\|u\|_{W_2^{s+k}(M)} \leq \|Q\|_{W_2^{s+k}(M) \to W_2^s(M)} \|Du\|_{W_2^s(M)} \\ + \|1 - QD\|_{W_2^{s+k}(M) \to W_2^s(M)} \|u\|_{W_2^s(M)}.\end{aligned}$$

□

Elliptic estimates in L_p-spaces

L_2 Sobolev spaces are the easiest to deal with since they are Hilbert spaces, but there are reasons to want L_p elliptic estimates, too. For an elliptic operator $D \in \mathrm{DO}^k(M)$, L_p elliptic estimates take the form

$$\|u\|_{W_p^{s+k}(M)} \leq C_{s,D,p}(\|Du\|_{W_p^s(M)} + \|u\|_{W_p^s(M)}).$$

Elliptic estimates in L_p-spaces

L_2 Sobolev spaces are the easiest to deal with since they are Hilbert spaces, but there are reasons to want L_p elliptic estimates, too. For an elliptic operator $D \in \mathrm{DO}^k(M)$, L_p elliptic estimates take the form

$$\|u\|_{W_p^{s+k}(M)} \leq C_{s,D,p}(\|Du\|_{W_p^s(M)} + \|u\|_{W_p^s(M)}).$$

From the above argument, the key to proving these estimates is to show that operators in $\Psi^k(M)$ act boundedly from $W_p^{s+k}(M)$ to $W_p^s(M)$ for every $s \in \mathbb{R}$.

Why elliptic estimates in L_p-spaces?

We need Sobolev spaces $W^s_p(M)$ for $p \neq 2$ for nonlinear PDE.

Why elliptic estimates in L_p-spaces?

We need Sobolev spaces $W^s_p(M)$ for $p \neq 2$ for nonlinear PDE. Consider $P \in \mathrm{DO}^2(M)$, where $P = -\sum_{j=1}^d X_j^2$, and the X_j are vector fields on M. Suppose that P is elliptic.

Why elliptic estimates in L_p-spaces?

We need Sobolev spaces $W_p^s(M)$ for $p \neq 2$ for nonlinear PDE. Consider $P \in \mathrm{DO}^2(M)$, where $P = -\sum_{j=1}^d X_j^2$, and the X_j are vector fields on M. Suppose that P is elliptic. Let $f \in C^\infty(M)$. Using variational methods, it is easy to show that the PDE

$$Pu + u^3 = f$$

has a weak solution u belonging to $L_4(M)$. In order to show that u is a classical solution, we need to show that it has two orders of differentiability.

Why elliptic estimates in L_p-spaces?

We need Sobolev spaces $W^s_p(M)$ for $p \neq 2$ for nonlinear PDE. Consider $P \in \mathrm{DO}^2(M)$, where $P = -\sum_{j=1}^d X_j^2$, and the X_j are vector fields on M. Suppose that P is elliptic. Let $f \in C^\infty(M)$. Using variational methods, it is easy to show that the PDE

$$Pu + u^3 = f$$

has a weak solution u belonging to $L_4(M)$. In order to show that u is a classical solution, we need to show that it has two orders of differentiability. If we know elliptic estimates in $p = \frac{4}{3}$, then

$$\begin{aligned} \|u\|_{W^2_{\frac{4}{3}}(M)} &\lesssim \|Pu\|_{L_{\frac{4}{3}}(M)} + \|u\|_{L_{\frac{4}{3}}(M)} \\ &\lesssim \|f\|_{L_{\frac{4}{3}}} + \|u\|_{L_4(M)}. \end{aligned}$$

How to build a pseudodifferential calculus

Specialise to $M = \mathbb{R}^d$.

How to build a pseudodifferential calculus

Specialise to $M = \mathbb{R}^d$. (* not compact, but close enough.)

How to build a pseudodifferential calculus

Specialise to $M = \mathbb{R}^d$. (* not compact, but close enough.) It is easy enough to build parametrices for *constant coefficient* differential operators. These can be realised as convolution operators

$$\mathrm{Op}(k)u(x) := \int_{\mathbb{R}^d} k(z)u(x+z)\,dz, \quad u \in C_c^\infty(\mathbb{R}^d).$$

Here, k is a distribution on $\mathbb{R}^d$ and the integral should be interpreted as distributional pairing.

How to build a pseudodifferential calculus

To build parametrices for non-translation-invariant operators, we need to consider operators of the form

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^d} K(x,z)u(x+z)\,dz, \quad u \in C_c^\infty(\mathbb{R}^d)$$

where K is smooth in the x-variable and potentially singular in the z variable.

How to build a pseudodifferential calculus

In general, for operators on a manifold M we need to consider

$$\mathrm{Op}(K)u(x) = \int_{TM_x} K(x,z)u(\exp_x z)\,dz, \quad u \in C_c^\infty(M)$$

where K is a distribution on the tangent bundle TM, and $\exp_x$ is an exponential mapping.

How to build a pseudodifferential calculus

This leads to the question: what are the right conditions on a kernel K which ensure that all the operators of the form $\mathrm{Op}(K)$ form a good pseudodifferential calculus?

How to build a pseudodifferential calculus

This leads to the question: what are the right conditions on a kernel K which ensure that all the operators of the form $\mathrm{Op}(K)$ form a good pseudodifferential calculus? Following a suggestion from Androulidakis and Skandalis, it turns out that a good idea is to introduce another parameter.

Theorem (Van Erp-Yuncken (2019))

Consider the class of distributions $(x, z, \hbar) \mapsto K(x, z, \hbar)$ which are smooth in $(x, \hbar)$ and compactly supported in z. If for all $\lambda > 0$ the difference

$$\lambda^{-k-d} K(x, \lambda^{-1} z, \lambda\hbar) - K(x, z, \hbar)$$

is smooth in $(x, z, \hbar)$, then $K(x, \cdot, 1)$ is the kernel of a classical pseudodifferential operator of order k.
Conversely, every classical pseudodifferential operator arises in this way.

Filtered pseudodifferential calculus

One of the advantages of the van Erp-Yuncken point of view is that it generalises neatly to non-traditional weightings.

Filtered pseudodifferential calculus

One of the advantages of the van Erp-Yuncken point of view is that it generalises neatly to non-traditional weightings.
Let $0 = H^0 < H^1 < \cdots < H^N = TM$ be a filtration of the tangent bundle of a manifold M. Let $\mathrm{DO}_H(M) = \bigcup_{k\geq 0} \mathrm{DO}_H^k(M)$ be the algebra of differential operators with orders counted so that vector fields in H^j have order j, for all $1 \leq j \leq N$. There is a natural notion of "elliptic" for $\mathrm{DO}_H(M)$ (the Rockland condition). There is also a natural scale of Sobolev spaces adapted to the filtration.

Filtered pseudodifferential calculus

One of the advantages of the van Erp-Yuncken point of view is that it generalises neatly to non-traditional weightings.

Let $0 = H^0 < H^1 < \cdots < H^N = TM$ be a filtration of the tangent bundle of a manifold M. Let $\mathrm{DO}_H(M) = \bigcup_{k\geq 0} \mathrm{DO}^k_H(M)$ be the algebra of differential operators with orders counted so that vector fields in H^j have order j, for all $1 \leq j \leq N$. There is a natural notion of "elliptic" for $\mathrm{DO}_H(M)$ (the Rockland condition). There is also a natural scale of Sobolev spaces adapted to the filtration.

Theorem

There exists a pseudodifferential calculus $\Psi_H(M) = \bigcup_{k\in\mathbb{Z}} \Psi^k_H(M)$ *extending* $\mathrm{DO}_H(M)$, *with all of the desirable properties described above.*

Filtered pseudodifferential calculus

One of the advantages of the van Erp-Yuncken point of view is that it generalises neatly to non-traditional weightings.
Let $0 = H^0 < H^1 < \cdots < H^N = TM$ be a filtration of the tangent bundle of a manifold M. Let $\mathrm{DO}_H(M) = \bigcup_{k\geq 0}\mathrm{DO}^k_H(M)$ be the algebra of differential operators with orders counted so that vector fields in H^j have order j, for all $1 \leq j \leq N$. There is a natural notion of "elliptic" for $\mathrm{DO}_H(M)$ (the Rockland condition). There is also a natural scale of Sobolev spaces adapted to the filtration.

Theorem

There exists a pseudodifferential calculus $\Psi_H(M) = \bigcup_{k\in\mathbb{Z}}\Psi^k_H(M)$ extending $\mathrm{DO}_H(M)$, with all of the desirable properties described above.

(Pseudodifferential calculi for this and similar situations have been described by Folland, Rothschild, Stein, Melin, Goodman, Beals, Greiner, Stanton, Taylor, Ponge, Street, and others. Van Erp and Yuncken's contribution is in simplifying the definition.)

The asteroids example

In the Asteroids example, we have $M = \mathbb{T}^3$, and the sections of H^1 and H^2 are given by

$$\Gamma H^1 = \mathrm{span}\{X, Y\}, \quad \Gamma H^2 = \mathrm{span}\{\Gamma H^1, [X, Y]\}.$$

Section 3: Operators defined by submersions

Limitations of the exponential map

Recall that pseudodifferential operators on a manifold M were defined by some choice of exponential map $\exp_x$ by

$$\mathrm{Op}(K)u(x) = \int_{TM_x} K(x,z)u(\exp_x(z))\,dz.$$

There are lots of circumstances where we need to integrate over a space of dimension larger than TM_x, so that operators are locally defined by integrals of the form

$$\mathrm{Op}(K)u(x) = \int_V K(x,z)u(\Lambda_x(z))\,dz$$

where V is a linear space and $\dim(V) \geq \dim(M)$, K is a distribution on $M \times V$, and $\Lambda_x : V \to M$ is a submersion, smoothly depending on $x \in X$.

Example: the Grushin operator

The operator on $\mathbb{R}^2$ with coordinates (x, y) given by

$$G = \partial_x^2 + x^2 \partial_y^2$$

is called the *Grushin operator*. We can write it as a sum of squares of vector fields,

$$G = X_1^2 + X_2^2, \quad X_1 := \partial_x, \; X_2 := x\partial_y.$$

X_1 and X_2 do not span the tangent space at points where $x = 0$. Instead we need to include a third vector field

$$X_3 := [X_1, X_2] = \partial_y.$$

Example: the Grushin operator

The natural differential calculus associated to the Grushin operator is built in the following way: let K be a distribution on $\mathbb{R}^2 \times \mathbb{R}^3$, and let

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^3} K(x,z)u(\exp(z_1X_1 + z_2X_2 + z_3X_3)x)\,dz, \quad u \in C_c^\infty(\mathbb{R}^2).$$

where exp is the exponential of vector fields.

The AMY calculus for the Grushin operator

Theorem

Let $(x,z,\hbar) \mapsto K(x,z,\hbar)$ denote a distribution on $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}$ which is smooth in $(x,\hbar)$ and compactly supported in z. Assume that for every $\lambda > 0$ the difference

$$\lambda^{-k-4} K(x, \lambda^{-1} z_1, \lambda^{-1} z_2, \lambda^{-2} z_3, \lambda\hbar) - K(x,z,\hbar)$$

is a smooth function of $(x,z,\hbar)$. Let Ψ^k denote the operators of the form

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^3} K(x,z) u(\exp(z_1X_1 + z_2X_2 + z_3X_3)x)\, dz, \quad u \in C_c^\infty(\mathbb{R}^2).$$

Then $\Psi = \bigcup_{k\in\mathbb{Z}} \Psi^k$ has all of the desirable properties of a pseudodifferential calculus.*

The AMY calculus for the Grushin operator

Theorem

Let $(x,z,\hbar) \mapsto K(x,z,\hbar)$ *denote a distribution on* $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}$ *which is smooth in* $(x,\hbar)$ *and compactly supported in* z*. Assume that for every* $\lambda > 0$ *the difference*

$$\lambda^{-k-4} K(x, \lambda^{-1} z_1, \lambda^{-1} z_2, \lambda^{-2} z_3, \lambda\hbar) - K(x,z,\hbar)$$

is a smooth function of $(x,z,\hbar)$*. Let* Ψ^k *denote the operators of the form*

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^3} K(x,z) u(\exp(z_1 X_1 + z_2 X_2 + z_3 X_3)x)\, dz, \quad u \in C_c^\infty(\mathbb{R}^2).$$

Then $\Psi = \bigcup_{k\in\mathbb{Z}} \Psi^k$ *has all* of the desirable properties of a pseudodifferential calculus.*

(* The definition of ellipticity is complicated in this example, and they did not prove that elliptic operators have parametrices in the calculus.)

The general case

In general, let $U \subset \mathbb{R}^d$ be an open subset. Let $X_1, \ldots, X_N$ be smooth vector fields on U which span the tangent space at every point. Let $\{w_1, \ldots, w_N\}$ be positive integers. Say that X_j has weight w_j.

The general case

In general, let $U \subset \mathbb{R}^d$ be an open subset. Let $X_1, \ldots, X_N$ be smooth vector fields on U which span the tangent space at every point. Let $\{w_1, \ldots, w_N\}$ be positive integers. Say that X_j has weight w_j. Similar to the Grushin example, we can build an algebra of operators of the form

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^N} K(x,z)u(\exp(z_1X_1 + \cdots + z_NX_N)x)\,dz.$$

An "approximate homogeneity" condition, taking into account the weights, gives a nice pseudodifferential calculus $\Psi(U)$.

The general case

In general, let $U \subset \mathbb{R}^d$ be an open subset. Let $X_1, \ldots, X_N$ be smooth vector fields on U which span the tangent space at every point. Let $\{w_1, \ldots, w_N\}$ be positive integers. Say that X_j has weight w_j. Similar to the Grushin example, we can build an algebra of operators of the form

$$\mathrm{Op}(K)u(x) = \int_{\mathbb{R}^N} K(x,z)u(\exp(z_1X_1 + \cdots + z_NX_N)x)\,dz.$$

An "approximate homogeneity" condition, taking into account the weights, gives a nice pseudodifferential calculus $\Psi(U)$.
A similar pseudodifferential calculus for the same situation was described by Street. It is unclear (to me) what the relationship is between the two constructions.

L_p estimates

Finally, I can promote my own work:

Theorem (M. (2024))

Order zero operators in the AMY calculus are bounded on L_p spaces, for $1 < p < \infty$.

This implies (although not obviously) L_p elliptic estimates for certain differential operators like the Grushin operator, and also some nonlinear PDE.

L_p-estimates

Corollary

Order k operators in the AMY calculus are bounded from $W_p^{s+k}(U)$ to $W_p^s(U)$ for all $s \in \mathbb{R}$, and elliptic operators obey L_p elliptic estimates.*

L_p-estimates

Corollary

Order k operators in the AMY calculus are bounded from $W_p^{s+k}(U)$ to $W_p^s(U)$ for all $s \in \mathbb{R}$, and elliptic* operators obey L_p elliptic estimates.

(* ellipticity really means "maximal hypoellipticity")

Thank you for listening!