Lipschitz estimates in quasi-Banach Schatten ideals

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Introduction

This talk is mostly about the paper

M., Sukochev, Lipschitz estimates in quasi-Banach Schatten ideals. *Math. Ann.* 383 (2022), no.1–2, 571–619.

Plan for this talk

1. Operator Lipschitz functions: some basic concepts and history.
2. Some very light background on Schatten ideals: the problem with $0 < p < 1$.
3. Schur multipliers in $0 < p < 1$.
4. Besov spaces and wavelets
5. Future directions

Operator Lipschitz functions

Let H be a (complex and separable) Hilbert space, and denote the operator norm by $\|\cdot\|_\infty$. A function $f:\mathbb{R}\to\mathbb{C}$ is said to be *operator Lipschitz* if there exists a constant C_f such that

$$\|f(A)-f(B)\|_\infty \leq C_f\|A-B\|_\infty, \quad A,B\in\mathcal{B}_{\mathrm{sa}}(H)$$

Question (from Krein)

Is every Lipschitz function operator Lipschitz?

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Is every Lipschitz function operator Lipschitz? That is, does $|f(t)-f(s)|\lesssim|t-s|$ imply that $\|f(A)-f(B)\|_\infty\lesssim\|A-B\|_\infty$?

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Answer

No.

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Kato (1973): The absolute value function $f(t) = |t|$ is not operator Lipschitz

Johnson & Williams (1975): An operator Lipschitz function is differentiable.

Finite-dimensional case

If H is N-dimensional, then

$$\|f(A)-f(B)\|_\infty \leq C_{\mathrm{abs}}\log(1+N)\|f\|_{\mathrm{Lip}}\|A-B\|_\infty$$

where C_{abs} is an absolute constant. This is sharp in the order of growth as $N\to\infty$. I do not know if a sharp estimate for C_{abs} is known.

Operator Lipschitz functions

It is easy to check that sufficiently good functions are operator Lipschitz. Let's check the function $f(x) = e^{i\xi x}$ for $\xi \in \mathbb{R}$. We have

$$e^{i\xi A} - e^{i\xi B} = i\xi \int_0^1 e^{i\xi(1-\theta)A}(A-B)e^{i\xi\theta B}\,d\theta.$$

The integral converges in the Bochner sense. The triangle inequality implies

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$$\|f(A) - f(B)\|_\infty \leq \|A-B\|_\infty \cdot 2\pi \|\widehat{\partial f}\|_1.$$

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By Cauchy-Schwarz, $\|\widehat{\partial f}\|_1 \leq \|f'\|_2 + \|f''\|_2$. This is a "good enough" sufficient condition for most purposes.

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Theorem (Peller (1990))

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In other words, if f is Lipschitz and

$$\int_0^\infty \sup_{t\in\mathbb{R}} \frac{|f(t-h) - 2f(t) + f(t+h)|}{h^2}\, dh < \infty$$

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then f is operator Lipschitz. For example, if $f', f'' \in L_\infty(\mathbb{R})$ then f is operator Lipschitz.

Peller's operator Bernstein inequality

The classical Bernstein inequality states that if $f \in L_\infty(\mathbb{R})$ has Fourier transform supported in the interval $[-\sigma, \sigma]$, then

$$\|f\|_{\mathrm{Lip}} \leq C\sigma\|f\|_\infty.$$

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Peller's theorem is a consequence of his *operator Bernstein inequality*.

Theorem (Peller (1990))

If $f \in L_\infty(\mathbb{R})$ has Fourier transform supported in the interval $[-\sigma, \sigma]$, then

$$\|f\|_{\mathrm{O-Lip}} \leq C\sigma\|f\|_\infty.$$

Here $\|f\|_{\mathrm{O-Lip}}$ is the operator Lipschitz seminorm, i.e.

$$\|f\|_{\mathrm{O-Lip}} := \sup_{A=A^*, B=B^* \in \mathcal{B}(H)} \frac{\|f(A) - f(B)\|_\infty}{\|A - B\|_\infty}.$$

Schatten ideals

If T is a compact operator on H, the singular value sequence of T is defined as

$$\mu(k, T) := \inf\{\|T - R\|_\infty \ : \ \mathrm{rank}(R) \leq k\}, \quad k \geq 0.$$

(Equivalently, $\mu(T) = \{\mu(k, T)\}_{k=0}^\infty$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.)

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(Equivalently, $\mu(T) = \{\mu(k, T)\}_{k=0}^\infty$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.) Note that $\|T\|_\infty = \mu(0, T) = \|\mu(T)\|_{\ell_\infty}$. For $1 \leq p < \infty$, the Schatten $\mathcal{L}_p$-norm of a compact operator T is

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \left(\sum_{k=0}^\infty \mu(k, T)^p\right)^{\frac{1}{p}}.$$

Equivalently, $\|T\|_p = \mathrm{Tr}(|T|^p)^{1/p}$. It is not obvious, but this is a norm (i.e. $\|T + S\|_p \leq \|T\|_p + \|S\|_p$.)

$\mathcal{L}_p$-operator Lipschitz functions

A function f on $\mathbb{R}$ is said to be $\mathcal{L}_p$-operator Lipschitz if there exists a constant $C_f > 0$ such that

$$\|f(A) - f(B)\|_p \leq C_f\|A - B\|_p, \quad A, B \in \mathcal{B}_{\mathrm{sa}}(H).$$

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For $p = 2$ this is almost trivial and has been known for approx. 110 years. For $p \neq 2$, this requires some very deep harmonic analysis. There are now essentially four proofs of Potapov-Sukochev. The most recent is due to Conde-Alonso, González-Pérez, Parcet and Tablate and uses operator-valued harmonic analysis.

What about $0 < p < 1$?

For $0 < p < 1$, we can still define

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \mathrm{Tr}(|T|^p)^{\frac{1}{p}}.$$

This is not a norm, merely a quasi-norm. There is no triangle inequality, merely a quasi-triangle inequality

$$\|T + S\|_p \leq 2^{\frac{1}{p}-1}(\|T\|_p + \|S\|_p).$$

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$$\|T + S\|_p \leq 2^{\frac{1}{p}-1}(\|T\|_p + \|S\|_p).$$

Nonetheless, we have

$$\|T + S\|_p^p \leq \|T\|_p^p + \|S\|_p^p.$$

Geometry in $\mathcal{L}_p$.

The unit ball $B = \{T \ : \ \|T\|_p \leq 1\}$ in $\mathcal{L}_p$ is not convex.
I.e., if $\xi_1, \ldots, \xi_n \in B$ then it might happen that

$$\theta_1\xi_1 + \cdots + \theta_n\xi_n \notin B, \quad |\theta_1| + \cdots + |\theta_n| \leq 1.$$

For this reason the theory of integration $\mathcal{L}_p$-valued functions is not straightforward. We could have continuous functions $f \in C([0,1], \mathcal{L}_p)$ whose integral is not in $\mathcal{L}_p$.

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For this reason the theory of integration $\mathcal{L}_p$-valued functions is not straightforward. We could have continuous functions $f \in C([0,1], \mathcal{L}_p)$ whose integral is not in $\mathcal{L}_p$.
Instead, B is only closed under p-convex combinations, i.e.

$$\theta_1\xi_1 + \cdots + \theta_n\xi_n \in B, \quad |\theta_1|^p + \cdots + |\theta_n|^p \leq 1.$$

$\mathcal{L}_p$-Lipschitz functions for $0 < p < 1$.

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At least some functions are, for example $f(t) = (t+\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

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Which functions are Lipschitz in $\mathcal{L}_p$ when $0 < p < 1$?
At least some functions are, for example $f(t) = (t+\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
What about $f(t) = \exp(it\xi)$ for $\xi \in \mathbb{R}$?

Schur multipliers

The usual method to prove $\mathcal{L}_p$-estimates in $0 < p < 1$ is the same as for $p \geq 1$, to use Schur multipliers.
If $A = \{A_{j,k}\}_{j,k}$ and $B = \{B_{j,k}\}_{j,k}$ are matrices of the same size, then $A \circ B := \{A_{j,k}B_{j,k}\}_{j,k}$.

Definition

Let $m = \{m_{j,k}\}_{j,k=1}^n$ be an $n \times n$ matrix. Define

$$\|m\|_{\mathfrak{m}_p} := \sup_{\|B\|_p \leq 1} \|m \circ B\|_p.$$

In general, let $m : \mathbb{R}^2 \to \mathbb{C}$ be bounded and define

$$\|m\|_{\mathfrak{m}_p} := \sup_{x_1,\ldots,x_n,y_1,\ldots,y_n \in \mathbb{R}} \|\{m(x_j, y_k)\}_{j,k=1}^n\|_{\mathfrak{m}_p}.$$

We could also consider the "cb" version. This does not make much difference.

Schur multipliers and operator-Lipschitz functions

A folk result:

Theorem (Hadamard(?), Schur(?), Löwner(?), Daletskii-Krein(?))

Let $1 \leq p \leq \infty$, and let f be a measurable function on $\mathbb{R}$. Define

$$f^{[1]}(\lambda,\mu) = \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu}, & \lambda \neq \mu \\ f'(\lambda), & \lambda = \mu. \end{cases}$$

Then f is $\mathcal{L}_p$-Lipschitz if and only if $f^{[1]}$ is a bounded Schur multiplier.

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Then f is $\mathcal{L}_p$-Lipschitz if and only if $f^{[1]}$ is a bounded Schur multiplier.

If f is not differentiable then $f'(\lambda)$ does not make sense everywhere, but this is no big deal. Bounded diagonal matrices are Schur multipliers in $\mathcal{L}_p$ for any $1 \leq p \leq \infty$ so we can make $f^{[1]}(\lambda,\lambda)$ any bounded function of λ without changing the result.

Schur multipliers in $\mathcal{L}_p$

One noteworthy difference between $p = 1$ and $p < 1$ is the following example:

Example

Let $I_n = \{\delta_{j,k}\}_{j,k=0}^{n-1}$ be the $n \times n$ identity matrix. Then

$$\|I_n\|_{\mathfrak{m}_p} = n^{\frac{1}{p}-1}.$$

To see this, compute $I_n \circ (\xi \otimes \xi)$ where $\xi = (1, \ldots, 1)$.

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What this means is that restriction to the diagonal

$$\{A_{j,k}\}_{j,k \geq 0} \mapsto \{A_{j,j}\delta_{j,k}\}_{j,k=0}^{\infty}$$

is not bounded in $\mathcal{L}_p$ for any $0 < p < 1$!

Schur multipliers and operator-Lipschitz functions in $\mathcal{L}_p$ for $0 < p < 1$.

Theorem

Let $0 < p \leq \infty$. and let f be a measurable function on $\mathbb{R}$. Then f is $\mathcal{L}_p$-Lipschitz if and only if

$$\sup_{\{\lambda_j\},\{\mu_j\}} \|\{f^{[1]}(\lambda_j, \mu_k)\}_{j,k}\|_{\mathfrak{m}_p} < \infty$$

where the supremum is over all disjoint sequences $\{\lambda_j\}_j$ and $\{\mu_k\}_k$.

Since we only consider disjoint sequences, the diagonal does not enter the picture.

An example

Consider the function

$$f(x) = \sin(x)$$

and $\mu_j = \lambda_j = 2\pi j$. Then

$$f^{[1]}(\lambda_j, \mu_k) = \delta_{j,k}.$$

But this is not a Schur multiplier of $\mathcal{L}_p$! The same reasoning applies to any periodic function f with $f'(\lambda) \neq 0$ for some λ.

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But this is not a Schur multiplier of $\mathcal{L}_p$! The same reasoning applies to any periodic function f with $f'(\lambda) \neq 0$ for some λ. This argument is obviously flawed, we are supposed to consider *disjoint* sequences.

A corrected example

We can fix the previous (wrong) argument by shifting one of the sequences by $\varepsilon > 0$. Let f be a 1-periodic function. Consider the sequences

$$\lambda_j = j + \varepsilon, \mu_k = k, \quad j, k \geq 0.$$

Then

$$f^{[1]}(\lambda_j, \mu_k) = \frac{f(j+\varepsilon) - f(k)}{j - k + \varepsilon} = (f(\varepsilon) - f(0))\frac{1}{j-k+\varepsilon}, \quad j, k \geq 0.$$

If $f(\varepsilon) \neq f(0)$, then we need to consider the matrix

$$\{(j-k+\varepsilon)^{-1}\}_{j,k\geq 0}.$$

This matrix is not diagonal, but a straightforward modification of the argument for $\{\delta_{j,k}\}_{j,k\geq 0}$ shows that it is not a Schur multiplier of $\mathcal{L}_p$ for any $0 < p < 1$.

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This matrix is not diagonal, but a straightforward modification of the argument for $\{\delta_{j,k}\}_{j,k\geq 0}$ shows that it is not a Schur multiplier of $\mathcal{L}_p$ for any $0 < p < 1$. Aleksandrov and Peller have characterised Schur multipliers of $\mathcal{L}_p$ of the Herz-Toeplitz form $m(j-k)$, so we could also use their result.

Periodic functions are not $\mathcal{L}_p$-Lipschitz for $0 < p < 1$.

Summarising the preceding reasoning:

Lemma (M. and Sukochev (2022))

Let $0 < p < 1$, and let f be a periodic function on $\mathbb{R}$. Then f is $\mathcal{L}_p$-Lipschitz if and only if it is constant.

What does this imply?

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Let $0 < p < 1$, and let f be a periodic function on $\mathbb{R}$. Then f is $\mathcal{L}_p$-Lipschitz if and only if it is constant.

What does this imply?

- Even C^∞ functions with all derivatives bounded may not be $\mathcal{L}_p$-Lipschitz;
- In particular $f(t) = \exp(it\xi)$, $\xi \neq 0$ is not $\mathcal{L}_p$-Lipschitz for any $0 < p < 1$. This means that methods based on a Fourier decomposition are unlikely to work.

An idea

Consider the matrix

$$\left\{ \frac{c_j - c_k}{j - k + \varepsilon} \right\}_{j,k \geq 0} \tag{0.1}$$

where c_j is a scalar sequence. This is approximately a model for $f^{[1]}(j+\varepsilon, k)$ where f is a function of the form

$$f(x) = \sum_{k \in \mathbb{Z}} c_k \psi(x-k)$$

where ψ is some bump function.

Lemma

If $\sum_j |c_j|^{\frac{p}{1-p}} < \infty$ then (0.1) is a Schur multiplier of $\mathcal{L}_p$.

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If $\sum_j |c_j|^{\frac{p}{1-p}} < \infty$ then (0.1) is a Schur multiplier of $\mathcal{L}_p$.

Why is it $\frac{p}{1-p}$? This is because of the Hölder inequality

$$\left(\sum_j |a_j b_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_j |a_j|^{\frac{p}{1-p}}\right)^{\frac{1-p}{p}} \left(\sum_j |b_j|\right)$$

Sums of shifted bump functions

With considerably more effort, it is possible to prove the following:

Theorem

Let $\psi \in C_c^k(\mathbb{R})$, *where* $k > \frac{2}{p} - 1$. *If* $\{c_k\}_{k\in\mathbb{Z}}$ *is some scalar sequence, and*

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \psi(x-k)$$

then

$$\|f^{[1]}\|_{\mathfrak{m}_p} \lesssim \|\{c_k\}_{k\in\mathbb{Z}}\|_{\ell_{\frac{p}{1-p}}}.$$

What kind of functions can we build out of functions like this?

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What kind of functions can we build out of functions like this?
In fact, basically all of them.

Wavelet methods

What is a good way of approximating a general function from compactly supported C^k-functions?

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Theorem (Daubechies (1988))

For all $k > 0$, there exists a compactly supported C^k function ψ such that the system of translations and dilations

$$\psi_{j,k}(t) := 2^{\frac{j}{2}}\psi(2^j t - k), \quad j,k \in \mathbb{Z}$$

forms an orthonormal basis of $L_2(\mathbb{R})$.

Wavelet methods

Wavelets are analogous to Fourier series, in the sense that if

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

then the coefficients $c_{j,k}$ for $j > N$ represent oscillations of f on the scale $\sim 2^{-N}$. A function of the form

$$f(t) = \sum_{j<N} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

does not oscillate greatly on scales smaller than 2^{-N}. This is similar to functions with Fourier transform supported in $[-2^N, 2^N]$.

An $\mathcal{L}_p$-Lipschitz Bernstein inequality

Theorem (M.-Sukochev (2022))

Let $f \in L_{\frac{p}{1-p}}(\mathbb{R})$ *have Wavelet expansion*

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

where $c_{j,k} = 0$ *for* $k > N$. *Then*

$$\|f^{[1]}\|_{\mathfrak{m}_p} \leq C2^{\frac{N}{p}}\|f\|_{\frac{p}{1-p}}.$$

With $p = 1$, this is the wavelet analogy of Peller's operator Bernstein inequality. For $p < 1$ it is new.

Wavelets and Besov spaces

It follows from the Wavelet Bernstein inequality that Besov spaces have a very simple characterisation in terms of wavelet coefficients.

Theorem (Meyer (1986))

Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. Let ψ be a compactly supported C^k wavelet where $k > -s$. Then a distribution $f \in \mathcal{D}'(\mathbb{R})$ belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ if and only if

$$\|f\|_{B^s_{p,q}} \approx \sum_{j\in\mathbb{Z}} 2^{jq(s+\frac{1}{2}-\frac{1}{p})} \left(\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^p\right)^{\frac{q}{p}} < \infty.$$

A new result

Using the p-triangle inequality and the $\mathcal{L}_p$-Lipschitz Bernstein inequality, we get the following:

Theorem (M. and Sukochev (2022))

Let $0 < p < 1$. *Let* $f \in \dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ *be Lipschitz continuous. Then* f *is* $\mathcal{L}_p$*-Lipschitz and*

$$\|f(A) - f(B)\|_p \leq C_p(\|f'\|_\infty + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})})\|A - B\|_p, \quad A, B \in \mathcal{B}_{\mathrm{sa}}(H).$$

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In other words, we require that f be Lipschitz and for some $n > \frac{1}{p}$ that

$$\int_0^\infty \left(\int_{-\infty}^\infty \left|\sum_{k=0}^n \binom{n}{k}(-1)^{n-k} f(t + kh)\right|^{\frac{p}{1-p}} dt\right)^{1-p} \frac{dh}{h^2} < \infty.$$

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For example, $f', \ldots f^{(k)} \in L_{\frac{p}{1-p}}(\mathbb{R})$ where $k > \frac{1}{p} - 1$ is sufficient.

Future directions

If A and B are two bounded operators, then

$$f(A+B) = f(A) + T^{A,A}_{f^{[1]}}(B) + T^{A,A,A}_{f^{[2]}}(B,B) + \cdots + T^{A,A,\cdots,A+B}_{f^{[k+1]}}(B,B,\cdots,B).$$

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Here $T_{f^{[k]}}^{A,\cdots,A}$ are *multiple operator integrals*, essentially multilinear Schur multipliers. If $B \in \mathcal{L}_p$, we would like to understand when the Taylor remainder

$$f(A+B) - f(A) - T_{f^{[1]}}^{A,A}(B) - \cdots - T_{f^{[n]}}^{A,A,\cdots,A}(B,\cdots,B)$$

also belongs to $\mathcal{L}_p$.

Bilinear Schur multipliers

Let $m : \mathbb{R}^3 \to \mathbb{C}$ be a function of three variables. Given finitely supported matrices $X, Y \in \ell_\infty(\mathbb{R}^2)$, let

$$m \circ (X, Y)(\lambda, \nu) = \sum_{\mu} m(\lambda, \mu, \nu) X(\lambda, \mu) Y(\mu, \nu).$$

Definition (Bilinear Schur multipliers)

Given $0 < p_1, p_2, p_3 \leq \infty$, let

$$\|m\|_{\mathfrak{m}(\mathcal{L}_{p_1} \times \mathcal{L}_{p_2}, \mathcal{L}_{p_3})} = \sup_{\|A\|_{p_1} \leq 1, \|B\|_{p_2} \leq 1} \|m \circ (X, Y)\|_{p_3}.$$

Similarly we can define n-linear Schur multipliers for $n > 2$.

Multilinear Schur multipliers

Theorem (Potapov-Sukochev-Skripka, Le Merdy-Skripka)

If

$$1 < p < \infty.$$

and $f^{(n)} \in C_b(\mathbb{R})$, *then*

$$\|f^{[n]}\|_{\mathfrak{m}(\mathcal{L}_{np}^n, \mathcal{L}_p)} \leq C\|f^{(n)}\|_\infty.$$

Theorem (Peller)

If $1 \leq p \leq \infty$, *then*

$$\|f^{[n]}\|_{\mathfrak{m}(\mathcal{L}_{np}^n, \mathcal{L}_p)} \leq C\|f\|_{B^n_{\infty,1}(\mathbb{R})}$$

Question: could this be extended to $0 < p < 1$?

Multilinear Schur multipliers

The best I have so far weakens $\mathcal{L}_{np}$ to $\mathcal{L}_p$.

Theorem

Let $0 < p \leq 1$. If $f \in B^{\frac{n}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ then $f^{[n]} \in \mathfrak{m}(\mathcal{L}_p^n, \mathcal{L}_p)$.

This implies that if $f \in B^{n/p}_{\frac{p}{1-p},p}(\mathbb{R})$, $A \in \mathcal{B}_{\mathrm{sa}}(H)$ and $X = X^* \in \mathcal{L}_p$, then the function

$$t \mapsto f(A + tX)$$

admits an n-term Taylor expansion with remainder term in $\mathcal{L}_p$ with quasi-norm of size $O(t^n)$.

Thank you for listening!