Lipschitz estimates in Operator ideals

Edward McDonald (Penn State University)

George Mason University, January 2025

January 31, 2025

Introduction

This talk is mostly inspired by the paper

M., Sukochev, Lipschitz estimates in quasi-Banach Schatten ideals. *Math. Ann.* 383 (2022), no.1–2, 571–619.

Plan for this talk

1. Operator Lipschitz functions: some basic concepts and history.
2. Schur multipliers and the Löwner identity
3. Some very light background on Schatten ideals
4. Approximation methods (Besov spaces and wavelets)

Section 1: Operator Lipschitz functions (Some basic concepts and history)

Basic setting

This talk will be about operator theory and functional calculus, but there's no real loss of generality by considering $n \times n$ matrices.

Basic setting

This talk will be about operator theory and functional calculus, but there's no real loss of generality by considering $n \times n$ matrices.

Functional calculus for matrices: Given a self-adjoint matrix $A \in M_n(\mathbb{C})$ and a function $f : \mathbb{R} \to \mathbb{C}$, the $n \times n$ matrix $f(A)$ can be defined by

$$f(A) = Uf(\Lambda)U^*$$

where $A = U\Lambda U^*$ is a diagonalisation, and

$$f\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n) \end{pmatrix}.$$

Operator Lipschitz functions

Recall that the operator norm $\|\cdot\|_\infty$ of a matrix $A \in M_n(\mathbb{C})$ may be defined as

$$\|A\|_\infty := \sup_{|x|_2 \leq 1} |Ax|_2.$$

where $|\cdot|_2$ is the Euclidean norm.

Operator Lipschitz functions

Recall that the operator norm $\|\cdot\|_\infty$ of a matrix $A \in M_n(\mathbb{C})$ may be defined as

$$\|A\|_\infty := \sup_{|x|_2 \leq 1} |Ax|_2.$$

where $|\cdot|_2$ is the Euclidean norm.
A function $f : \mathbb{R} \to \mathbb{C}$ is called *Lipschitz* if there exists a constant c_f such that

$$|f(t) - f(s)| \leq c_f |t - s|, \quad t, s \in \mathbb{R}.$$

Operator Lipschitz functions

Recall that the operator norm $\|\cdot\|_\infty$ of a matrix $A \in M_n(\mathbb{C})$ may be defined as

$$\|A\|_\infty := \sup_{|x|_2 \leq 1} |Ax|_2.$$

where $|\cdot|_2$ is the Euclidean norm.
A function $f : \mathbb{R} \to \mathbb{C}$ is called *Lipschitz* if there exists a constant c_f such that

$$|f(t) - f(s)| \leq c_f |t - s|, \quad t, s \in \mathbb{R}.$$

Similarly, a function $f : \mathbb{R} \to \mathbb{C}$ is said to be *operator Lipschitz* if there exists a constant C_f (independent of n) such that

$$\|f(A) - f(B)\|_\infty \leq C_f \|A - B\|_\infty, \quad A = A^*, B = B^* \in M_n(\mathbb{C}).$$

Operator Lipschitz functions

Recall that the operator norm $\|\cdot\|_\infty$ of a matrix $A \in M_n(\mathbb{C})$ may be defined as

$$\|A\|_\infty := \sup_{|x|_2 \leq 1} |Ax|_2.$$

where $|\cdot|_2$ is the Euclidean norm.

A function $f : \mathbb{R} \to \mathbb{C}$ is called *Lipschitz* if there exists a constant c_f such that

$$|f(t) - f(s)| \leq c_f |t - s|, \quad t, s \in \mathbb{R}.$$

Similarly, a function $f : \mathbb{R} \to \mathbb{C}$ is said to be *operator Lipschitz* if there exists a constant C_f (independent of n) such that

$$\|f(A) - f(B)\|_\infty \leq C_f \|A - B\|_\infty, \quad A = A^*, B = B^* \in M_n(\mathbb{C}).$$

Question (from M. G. Krein, 1950s)

Is every Lipschitz function operator Lipschitz?

Operator Lipschitz functions

Recall that the operator norm $\|\cdot\|_\infty$ of a matrix $A \in M_n(\mathbb{C})$ may be defined as

$$\|A\|_\infty := \sup_{|x|_2 \leq 1} |Ax|_2.$$

where $|\cdot|_2$ is the Euclidean norm.
A function $f : \mathbb{R} \to \mathbb{C}$ is called *Lipschitz* if there exists a constant c_f such that

$$|f(t) - f(s)| \leq c_f |t - s|, \quad t, s \in \mathbb{R}.$$

Similarly, a function $f : \mathbb{R} \to \mathbb{C}$ is said to be *operator Lipschitz* if there exists a constant C_f (independent of n) such that

$$\|f(A) - f(B)\|_\infty \leq C_f \|A - B\|_\infty, \quad A = A^*, B = B^* \in M_n(\mathbb{C}).$$

Question (from M. G. Krein, 1950s)

Is every Lipschitz function operator Lipschitz? That is, does $|f(t) - f(s)| \lesssim |t - s|$ imply that $\|f(A) - f(B)\|_\infty \lesssim \|A - B\|_\infty$?

Operator Lipschitz functions

Answer

No.

Operator Lipschitz functions

Answer

No.

Farforovskaya (1968): There exist Lipschitz functions that are not operator Lipschitz

Operator Lipschitz functions

Answer

No.

Farforovskaya (1968): There exist Lipschitz functions that are not operator Lipschitz

Kato (1973): The absolute value function $f(t) = |t|$ is not operator Lipschitz

Operator Lipschitz functions

Answer

No.

Farforovskaya (1968): There exist Lipschitz functions that are not operator Lipschitz

Kato (1973): The absolute value function $f(t) = |t|$ is not operator Lipschitz

Johnson & Williams (1975): An operator Lipschitz function is differentiable.

Dimension-dependent constant

If f is any Lipschitz function on $\mathbb{R}$, and $A, B \in M_n(\mathbb{C})$ are self-adjoint then

$$\|f(A) - f(B)\|_\infty \leq C_{\mathrm{abs}} \log(1+n) \|f\|_{\mathrm{Lip}} \|A - B\|_\infty$$

where C_{abs} is an absolute constant. This is sharp in the order of growth as $n \to \infty$. I do not know if a sharp estimate for C_{abs} is known.

Operator Lipschitz functions

It is easy to check that sufficiently good functions are operator Lipschitz. Let's check the function $f(x) = e^{i\xi x}$ for $\xi \in \mathbb{R}$. There's a classical integral identity:

$$e^{i\xi A} - e^{i\xi B} = i\xi \int_0^1 e^{i\xi(1-\theta)A}(A-B)e^{i\xi\theta B}\,d\theta.$$

This is called the Duhamel formula, you can check it by differentiating both sides. The triangle inequality implies

$$\|e^{i\xi A} - e^{i\xi B}\|_\infty \leq |\xi|\|A-B\|_\infty.$$

Operator Lipschitz functions

It is easy to check that sufficiently good functions are operator Lipschitz. Let's check the function $f(x) = e^{i\xi x}$ for $\xi \in \mathbb{R}$. There's a classical integral identity:

$$e^{i\xi A} - e^{i\xi B} = i\xi \int_0^1 e^{i\xi(1-\theta)A}(A-B)e^{i\xi\theta B}\, d\theta.$$

This is called the Duhamel formula, you can check it by differentiating both sides. The triangle inequality implies

$$\|e^{i\xi A} - e^{i\xi B}\|_\infty \leq |\xi|\|A-B\|_\infty.$$

By Fourier inversion,

$$\|f(A) - f(B)\|_\infty \leq \|A-B\|_\infty \cdot 2\pi\|\widehat{f'}\|_1.$$

Operator Lipschitz functions

It is easy to check that sufficiently good functions are operator Lipschitz. Let's check the function $f(x) = e^{i\xi x}$ for $\xi \in \mathbb{R}$. There's a classical integral identity:

$$e^{i\xi A} - e^{i\xi B} = i\xi \int_0^1 e^{i\xi(1-\theta)A}(A-B)e^{i\xi\theta B}\,d\theta.$$

This is called the Duhamel formula, you can check it by differentiating both sides. The triangle inequality implies

$$\|e^{i\xi A} - e^{i\xi B}\|_\infty \leq |\xi|\|A-B\|_\infty.$$

By Fourier inversion,

$$\|f(A) - f(B)\|_\infty \leq \|A-B\|_\infty \cdot 2\pi\|\widehat{f'}\|_1.$$

By Cauchy-Schwarz and Plancherel, $\|\widehat{f'}\|_1 \leq \|f'\|_2 + \|f''\|_2$. This is a "good enough" sufficient condition for most purposes.

Peller's theorem

The previous computation was based on Fourier inversion of f and a description of $e^{i\xi A} - e^{i\xi B}$ as an integral (Duhamel's integral).

Peller's theorem

The previous computation was based on Fourier inversion of f and a description of $e^{i\xi A} - e^{i\xi B}$ as an integral (Duhamel's integral). Using a more subtle description of $e^{i\xi A} - e^{i\xi B}$, and handling the Littlewood-Paley components of f individually, V. V. Peller has proved the following:

Theorem (Peller (1990))

If f is Lipschitz and belongs to the homogeneous Besov class $\dot{B}^1_{\infty,1}(\mathbb{R})$ then f is operator Lipschitz.

Peller's theorem

The previous computation was based on Fourier inversion of f and a description of $e^{i\xi A} - e^{i\xi B}$ as an integral (Duhamel's integral). Using a more subtle description of $e^{i\xi A} - e^{i\xi B}$, and handling the Littlewood-Paley components of f individually, V. V. Peller has proved the following:

Theorem (Peller (1990))

If f is Lipschitz and belongs to the homogeneous Besov class $\dot{B}^1_{\infty,1}(\mathbb{R})$ then f is operator Lipschitz.

In other words, if f is Lipschitz and

$$\int_0^\infty \sup_{t\in\mathbb{R}} \frac{|f(t-h) - 2f(t) + f(t+h)|}{h^2}\,dh < \infty$$

then f is operator Lipschitz.

Peller's theorem

The previous computation was based on Fourier inversion of f and a description of $e^{i\xi A} - e^{i\xi B}$ as an integral (Duhamel's integral). Using a more subtle description of $e^{i\xi A} - e^{i\xi B}$, and handling the Littlewood-Paley components of f individually, V. V. Peller has proved the following:

Theorem (Peller (1990))

If f is Lipschitz and belongs to the homogeneous Besov class $\dot{B}^1_{\infty,1}(\mathbb{R})$ then f is operator Lipschitz.

In other words, if f is Lipschitz and

$$\int_0^\infty \sup_{t\in\mathbb{R}} \frac{|f(t-h)-2f(t)+f(t+h)|}{h^2}\,dh < \infty$$

then f is operator Lipschitz. For example, if $f', f'' \in L_\infty(\mathbb{R})$ then f is operator Lipschitz.

Peller's operator Bernstein inequality

The classical Bernstein inequality states that if $f \in L_\infty(\mathbb{R})$ has Fourier transform supported in the interval $[-\sigma, \sigma]$, then

$$\|f\|_{\mathrm{Lip}} \leq C\sigma\|f\|_\infty.$$

Peller's operator Bernstein inequality

The classical Bernstein inequality states that if $f \in L_\infty(\mathbb{R})$ has Fourier transform supported in the interval $[-\sigma, \sigma]$, then

$$\|f\|_{\mathrm{Lip}} \leq C\sigma \|f\|_\infty.$$

Peller's theorem is a consequence of his *operator Bernstein inequality*.

Theorem (Peller (1990))

If $f \in L_\infty(\mathbb{R})$ has Fourier transform supported in the interval $[-\sigma, \sigma]$, then

$$\|f\|_{\mathrm{O-Lip}} \leq C\sigma \|f\|_\infty.$$

Here $\|f\|_{\mathrm{O-Lip}}$ is the operator Lipschitz seminorm, i.e.

$$\|f\|_{\mathrm{O-Lip}} := \sup_{A=A^*, B=B^* \in \mathcal{B}(H)} \frac{\|f(A) - f(B)\|_\infty}{\|A - B\|_\infty}.$$

Section 2

Another perspective: Schur multipliers

Let $A, B \in M_n(\mathbb{C})$. The Schur product (a.k.a. Hadamard product, entrywise product...) of A and B is defined by

$$(A_{j,k})_{j,k} \circ (B_{j,k})_{j,k} = (A_{j,k}B_{j,k})_{j,k}.$$

Another perspective: Schur multipliers

Let $A, B \in M_n(\mathbb{C})$. The Schur product (a.k.a. Hadamard product, entrywise product...) of A and B is defined by

$$(A_{j,k})_{j,k} \circ (B_{j,k})_{j,k} = (A_{j,k}B_{j,k})_{j,k}.$$

That is, $A \circ B$ is the matrix formed by multiplying the entries of A and B componentwise.
What is the point of such a thing?

Löwner's formula

The Schur product is very important in linear algebra. One reason for this is that it can be used to *linearise* nonlinear operators.

Löwner's formula

The Schur product is very important in linear algebra. One reason for this is that it can be used to *linearise* nonlinear operators.
If A and B are self-adjoint matrices with eigendecompositions

$$A\xi_j = \lambda_j(A)\xi_j, \quad B\eta_k = \lambda_k(B)\eta_k, \quad 0 \leq j,k < n$$

then

$$f(A)\xi_j = f(\lambda_j(A))\xi_j, \quad f(B)\eta_k = f(\lambda_k(B))\eta_k, \quad 0 \leq j,k < n.$$

K. Löwner noticed that therefore

$$\langle \xi_j, (f(A)-f(B))\eta_k\rangle = \frac{f(\lambda_j(A))-f(\lambda_k(B))}{\lambda_j(A)-\lambda_k(B)}\langle \xi_j, (A-B)\eta_k\rangle$$

for all $0 \leq j,k < n$.

Löwner's formula

In other words, let $\Psi_{f,A,B}$ denote the "Löwner matrix"

$$\Psi_{f,A,B} = \left(\frac{f(\lambda_j(A)) - f(\lambda_k(B))}{\lambda_j(A) - \lambda_k(B)}\right)_{j,k=0}^{n-1}$$

then

$$f(A) - f(B) = \Psi_{f,A,B} \circ (A - B)$$

where the Schur product $\circ$ is computed with respect to the matrix basis $\{\xi_j \otimes \eta_k\}_{j,k=0}^{n-1}$.

Löwner's formula

In other words, let $\Psi_{f,A,B}$ denote the "Löwner matrix"

$$\Psi_{f,A,B} = \left(\frac{f(\lambda_j(A)) - f(\lambda_k(B))}{\lambda_j(A) - \lambda_k(B)}\right)_{j,k=0}^{n-1}$$

then

$$f(A) - f(B) = \Psi_{f,A,B} \circ (A - B)$$

where the Schur product $\circ$ is computed with respect to the matrix basis $\{\xi_j \otimes \eta_k\}_{j,k=0}^{n-1}$.
This means that studying the highly nonlinear relationship

$$A - B \mapsto f(A) - f(B)$$

is reduced to the study of the linear map

$$X \mapsto \Psi_{f,A,B} \circ X.$$

How to study operator Lipschitz functions

Therefore if we want to find the operator Lipschitz functions, we need to characterise those functions f such that for all A and B, we have

$$\|\Psi_{f,A,B} \circ X\|_\infty \leq C_f \|X\|_\infty.$$

Because this is a linear (rather than nonlinear) problem, it turns out to be much easier.

Section 3: Some very light background on Schatten ideals

Schatten ideals

If T is a compact operator on H, the singular value sequence of T is defined as

$$\mu(k,T) := \inf\{\|T-R\|_\infty \ : \ \mathrm{rank}(R) \leq k\}, \quad k \geq 0.$$

(Equivalently, $\mu(T) = \{\mu(k,T)\}_{k=0}^{\infty}$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.)

Schatten ideals

If T is a compact operator on H, the singular value sequence of T is defined as

$$\mu(k, T) := \inf\{\|T - R\|_\infty \ : \ \mathrm{rank}(R) \leq k\}, \quad k \geq 0.$$

(Equivalently, $\mu(T) = \{\mu(k, T)\}_{k=0}^\infty$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.) Note that $\|T\|_\infty = \mu(0, T) = \|\mu(T)\|_{\ell_\infty}$.

Schatten ideals

If T is a compact operator on H, the singular value sequence of T is defined as

$$\mu(k, T) := \inf\{\|T - R\|_\infty \ : \ \mathrm{rank}(R) \leq k\}, \quad k \geq 0.$$

(Equivalently, $\mu(T) = \{\mu(k, T)\}_{k=0}^\infty$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.) Note that $\|T\|_\infty = \mu(0, T) = \|\mu(T)\|_{\ell_\infty}$. For $1 \leq p < \infty$, the Schatten $\mathcal{L}_p$-norm of a compact operator T is

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \left(\sum_{k=0}^\infty \mu(k, T)^p\right)^{\frac{1}{p}}.$$

Equivalently, $\|T\|_p = \mathrm{Tr}(|T|^p)^{1/p}$. It is not obvious, but this is a norm (i.e. $\|T + S\|_p \leq \|T\|_p + \|S\|_p$.)

$\mathcal{L}_p$-operator Lipschitz functions

A function f on $\mathbb{R}$ is said to be $\mathcal{L}_p$-operator Lipschitz if there exists a constant $C_f > 0$ such that

$$\|f(A) - f(B)\|_p \leq C_f\|A - B\|_p, \quad A = A^*, B = B^* \in M_n(\mathbb{C})$$

By Loewner's identity, this follows from (actually, is equivalent to)

$$\|\Psi_{f,A,B} \circ X\|_p \leq C_f\|X\|_p, \quad X \in M_n(\mathbb{C}).$$

$\mathcal{L}_2$-operator Lipschitz function

By far the easiest case is $p = 2$, because the $\mathcal{L}_2$ norm of a matrix is the same as the ℓ_2 norm of its entries

$$\|\{A_{j,k}\}_{j,k}\|_2 = \left(\sum_{j,k} |A_{j,k}|^2\right)^{1/2}.$$

Therefore

$$\|M \circ A\|_2 \leq \max\{|M_{j,k}|\}_{j,k}\|A\|_2.$$

It follows that

$$\begin{aligned}\|f(A) - f(B)\|_2 &= \|\Psi_{f,A,B} \circ (A - B)\|_2 \\ &\leq \max_{j,k}\left\{\left|\frac{f(\lambda_j(A)) - f(\lambda_k(B))}{\lambda_j(A) - \lambda_k(B)}\right|\right\}\|A - B\|_2 \\ &\leq \|f\|_{\mathrm{Lip}}\|A - B\|_2.\end{aligned}$$

What about $p \neq 2$?

So any Lipschitz function f is Lipschitz in $\mathcal{L}_2$. What about other Schatten ideals?

Theorem (Potapov and Sukochev (2010))

For $1 < p < \infty$, all Lipschitz functions are $\mathcal{L}_p$-operator Lipschitz.

What about $p \neq 2$?

So any Lipschitz function f is Lipschitz in $\mathcal{L}_2$. What about other Schatten ideals?

Theorem (Potapov and Sukochev (2010))

For $1 < p < \infty$, all Lipschitz functions are $\mathcal{L}_p$-operator Lipschitz.

For $p \neq 2$, this requires some very deep harmonic analysis.

What about $p \neq 2$?

So any Lipschitz function f is Lipschitz in $\mathcal{L}_2$. What about other Schatten ideals?

Theorem (Potapov and Sukochev (2010))

For $1 < p < \infty$, all Lipschitz functions are $\mathcal{L}_p$-operator Lipschitz.

For $p \neq 2$, this requires some very deep harmonic analysis. There are now essentially four proofs of Potapov-Sukochev. The most recent is relatively simple and due to Conde-Alonso, González-Pérez, Parcet and Tablate, but still uses advanced operator-valued harmonic analysis.

What about $0 < p < 1$?

For $0 < p < 1$, we can still define

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \mathrm{Tr}(|T|^p)^{\frac{1}{p}}.$$

This is not a norm, merely a quasi-norm. There is no triangle inequality, merely a quasi-triangle inequality

$$\|T + S\|_p \leq 2^{\frac{1}{p}-1}(\|T\|_p + \|S\|_p).$$

What about $0 < p < 1$?

For $0 < p < 1$, we can still define

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \mathrm{Tr}(|T|^p)^{\frac{1}{p}}.$$

This is not a norm, merely a quasi-norm. There is no triangle inequality, merely a quasi-triangle inequality

$$\|T+S\|_p \leq 2^{\frac{1}{p}-1}(\|T\|_p + \|S\|_p).$$

Nonetheless, we have

$$\|T+S\|_p^p \leq \|T\|_p^p + \|S\|_p^p.$$

$\mathcal{L}_p$-Lipschitz functions for $0 < p < 1$.

Which functions are Lipschitz in $\mathcal{L}_p$ when $0 < p < 1$?

$\mathcal{L}_p$-Lipschitz functions for $0 < p < 1$.

Which functions are Lipschitz in $\mathcal{L}_p$ when $0 < p < 1$?
At least some functions are, for example $f(t) = (t+\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

$\mathcal{L}_p$-Lipschitz functions for $0 < p < 1$.

Which functions are Lipschitz in $\mathcal{L}_p$ when $0 < p < 1$?
At least some functions are, for example $f(t) = (t+\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
What about $f(t) = \exp(it\xi)$ for $\xi \in \mathbb{R}$?

Periodic functions are not $\mathcal{L}_p$-Lipschitz for $0 < p < 1$.

Lemma (M. and Sukochev (2022))

Let $0 < p < 1$, and let f be a periodic function on $\mathbb{R}$. Then f is $\mathcal{L}_p$-Lipschitz if and only if it is constant.

What does this imply?

Periodic functions are not $\mathcal{L}_p$-Lipschitz for $0 < p < 1$.

Lemma (M. and Sukochev (2022))

Let $0 < p < 1$, and let f be a periodic function on $\mathbb{R}$. Then f is $\mathcal{L}_p$-Lipschitz if and only if it is constant.

What does this imply?

- Even C^∞ functions with all derivatives bounded may not be $\mathcal{L}_p$-Lipschitz;
- In particular $f(t) = \exp(it\xi)$, $\xi \neq 0$ is not $\mathcal{L}_p$-Lipschitz for any $0 < p < 1$. This means that methods based on a Fourier decomposition are unlikely to work.

Section 4: Approximation methods (Besov spaces and wavelets)

Wavelet methods

What is a good way of approximating a general function by nicer functions?

Theorem (Daubechies (1988))

For all $k > 0$, there exists a compactly supported C^k function ψ such that the system of translations and dilations

$$\psi_{j,k}(t) := 2^{\frac{j}{2}}\psi(2^j t - k), \quad j,k \in \mathbb{Z}$$

forms an orthonormal basis of $L_2(\mathbb{R})$.

Wavelet methods

Wavelets are analogous to Fourier series, in the sense that if

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

then the coefficients $c_{j,k}$ for $j > N$ represent oscillations of f on the scale $\sim 2^{-N}$. A function of the form

$$f(t) = \sum_{j<N} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

does not oscillate greatly on scales smaller than 2^{-N}. This is similar to functions with Fourier transform supported in $[-2^N, 2^N]$.

An $\mathcal{L}_p$-Lipschitz Bernstein inequality

Theorem (M.-Sukochev (2022))

Let $f \in L_{\frac{p}{1-p}}(\mathbb{R})$ have Wavelet expansion

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(t)$$

where $c_{j,k} = 0$ for $k > N$. Then

$$\|f^{[1]}\|_{\mathfrak{m}_p} \leq C2^{\frac{N}{p}}\|f\|_{\frac{p}{1-p}}.$$

With $p = 1$, this is the wavelet analogy of Peller's operator Bernstein inequality. For $p < 1$ it is new.

It follows from the Wavelet Bernstein inequality that Besov spaces have a very simple characterisation in terms of wavelet coefficients.

Theorem (Meyer (1986))

Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. Let ψ be a compactly supported C^k wavelet where $k > -s$. Then a distribution $f \in \mathcal{D}'(\mathbb{R})$ belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ if and only if

$$\|f\|_{B^s_{p,q}} \approx \sum_{j\in\mathbb{Z}} 2^{jq(s+\frac{1}{2}-\frac{1}{p})} \left(\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^p\right)^{\frac{q}{p}} < \infty.$$

A new result

Using the p-triangle inequality and the $\mathcal{L}_p$-Lipschitz Bernstein inequality, we get the following:

Theorem (M. and Sukochev (2022))

Let $0 < p < 1$. Let $f \in \dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ be Lipschitz continuous. Then f is $\mathcal{L}_p$-Lipschitz and for all self-adjoint matrices A and B,

$$\|f(A) - f(B)\|_p \leq C_p(\|f\|_{\mathrm{Lip}} + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})})\|A - B\|_p$$

A new result

Using the p-triangle inequality and the $\mathcal{L}_p$-Lipschitz Bernstein inequality, we get the following:

Theorem (M. and Sukochev (2022))

Let $0 < p < 1$. Let $f \in \dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ be Lipschitz continuous. Then f is $\mathcal{L}_p$-Lipschitz and for all self-adjoint matrices A and B,

$$\|f(A)-f(B)\|_p \leq C_p(\|f\|_{\mathrm{Lip}} + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})})\|A-B\|_p$$

In other words, we require that f be Lipschitz and for some $n > \frac{1}{p}$ that

$$\int_0^\infty \left(\int_{-\infty}^\infty \left|\sum_{k=0}^n \binom{n}{k}(-1)^{n-k} f(t+kh)\right|^{\frac{p}{1-p}}\,dt\right)^{1-p} \frac{dh}{h^2} < \infty.$$

A new result

Using the p-triangle inequality and the $\mathcal{L}_p$-Lipschitz Bernstein inequality, we get the following:

Theorem (M. and Sukochev (2022))

Let $0 < p < 1$. Let $f \in \dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ be Lipschitz continuous. Then f is $\mathcal{L}_p$-Lipschitz and for all self-adjoint matrices A and B,

$$\|f(A) - f(B)\|_p \leq C_p(\|f\|_{\mathrm{Lip}} + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})})\|A - B\|_p$$

In other words, we require that f be Lipschitz and for some $n > \frac{1}{p}$ that

$$\int_0^\infty \left(\int_{-\infty}^\infty \left|\sum_{k=0}^n \binom{n}{k}(-1)^{n-k} f(t+kh)\right|^{\frac{p}{1-p}} dt\right)^{1-p} \frac{dh}{h^2} < \infty.$$

For example, $f', \ldots f^{(k)} \in L_{\frac{p}{1-p}}(\mathbb{R})$ where $k > \frac{1}{p} - 1$ is sufficient.

Thank you for listening!