Recent work and unsolved problems in Quantum Harmonic Analysis

Edward McDonald (Penn State University)

Ghent University

January 13, 2025

Plan for this talk

In this talk I will go over some introductory material in Quantum Harmonic Analysis and propose some future work.

Prologue: Meyer's decomposition

Nemytskij operators

Let $F \in C^{\infty}(\mathbb{R})$. The mapping of functional composition

$$L_{\infty}(\mathbb{R}^d, \mathbb{R}) \to L_{\infty}(\mathbb{R}^d), \quad u \mapsto F(u)$$

is sometimes called a *Nemytskij operator*.

Nemytskij operators

Let $F \in C^\infty(\mathbb{R})$. The mapping of functional composition

$$L_\infty(\mathbb{R}^d, \mathbb{R}) \to L_\infty(\mathbb{R}^d), \quad u \mapsto F(u)$$

is sometimes called a *Nemytskij operator*.

Question: In terms of function spaces, what are the mapping properties of $u \mapsto F(u)$? (This is needed for nonlinear PDE).

Meyer's decomposition

How can we study Nemytskij operators from the point of view of harmonic analysis?

Meyer's decomposition

How can we study Nemytskij operators from the point of view of harmonic analysis?
One method relates Nemytskij operators to pseudodifferential operators via the *Meyer decomposition*.

Meyer's decomposition

How can we study Nemytskij operators from the point of view of harmonic analysis?
One method relates Nemytskij operators to pseudodifferential operators via the *Meyer decomposition*.
Let $\{\Delta_j\}_{j=0}^{\infty}$ be an inhomogeneous Littlewood-Paley decomposition for $\mathbb{R}^d$.
That is, $\Delta_j = \Psi_j(-i\nabla)$ where Ψ_j is supported in the ball of radius 2^j and

$$\sum_{j=0}^{\infty} \Psi_j = 1.$$

Let $S_n = \sum_{j=0}^{n} \Delta_j$.

Meyer's decomposition (cont.)

Assume initially that $u = S_n u$ for some $n \geq 1$. We have

$$F(u) = F(0) + F(S_n u) - F(0) = F(0) + \sum_{j=0}^{\infty} F(S_j u) - F(S_{j-1} u)$$

where $S_{-1} u = 0$. Therefore

$$F(u) = F(0) + \sum_{j=0}^{\infty} \int_0^1 F'(S_{j-1} u + \theta \Delta_j u) \Delta_j u d\theta =: F(0) + m(F, u) u$$

where $m(F, u)$ is the *paradifferential operator*

$$m(F, u) = \sum_{j=0}^{\infty} \int_0^1 F'(S_{j-1} u + \theta \Delta_j u) \Delta_j d\theta.$$

Bourdaud's theorem

Some elementary estimates show for any $u \in L_\infty(\mathbb{R}^d, \mathbb{R})$ and $F \in C^\infty(\mathbb{R})$, $m(F,u)$ is a pseudodifferential operator in the Hörmander class $\Psi^0_{1,1}(\mathbb{R}^d)$ (Stein's "forbidden symbols").

Theorem (Bourdaud (1988))

If $T \in \Psi^0_{1,1}(\mathbb{R}^d)$, then T is bounded on the Sobolev space $W^s_p(\mathbb{R}^d)$ for all $s > 0$ and $1 < p < \infty$.

Bourdaud's theorem

Some elementary estimates show for any $u \in L_\infty(\mathbb{R}^d, \mathbb{R})$ and $F \in C^\infty(\mathbb{R})$, $m(F, u)$ is a pseudodifferential operator in the Hörmander class $\Psi^0_{1,1}(\mathbb{R}^d)$ (Stein's "forbidden symbols").

Theorem (Bourdaud (1988))

If $T \in \Psi^0_{1,1}(\mathbb{R}^d)$, then T is bounded on the Sobolev space $W^s_p(\mathbb{R}^d)$ for all $s > 0$ and $1 < p < \infty$.

Conclusion: If $F(0) = 0$ and $u \in W^s_p(\mathbb{R}^d, \mathbb{R})$ for some $s > 0$ and $1 < p < \infty$ then $F(u) \in W^s_p(\mathbb{R}^d)$.

Section 2: Quantum Harmonic Analysis

Quantum Euclidean spaces

Let $d \geq 1$ and let θ be a $d \times d$ antisymmetric real matrix. The quantum Euclidean space is an attempt to make rigorous mathematical sense of a space $\mathbb{R}^d_\theta$ with coordinate functions

$$x_1, \ldots, x_d$$

which do not commute, but instead satisfy the relation

$$x_j x_k - x_k x_j = i\theta_{j,k}, \quad 1 \leq j, k \leq d.$$

The object goes by many names (Noncommutative Euclidean space, Quantum Euclidean space, Moyal d-space, Groenwald space, CCR algebras).

Quantum Euclidean spaces

Let $d \geq 1$ and let θ be a $d \times d$ antisymmetric real matrix. The quantum Euclidean space is an attempt to make rigorous mathematical sense of a space $\mathbb{R}^d_\theta$ with coordinate functions

$$x_1, \ldots, x_d$$

which do not commute, but instead satisfy the relation

$$x_j x_k - x_k x_j = i\theta_{j,k}, \quad 1 \leq j, k \leq d.$$

The object goes by many names (Noncommutative Euclidean space, Quantum Euclidean space, Moyal d-space, Groenwald space, CCR algebras).
Much of the literature motivates this from the relation $[q,p] = i\hbar$ from quantum mechanics.

Uncertainty principle

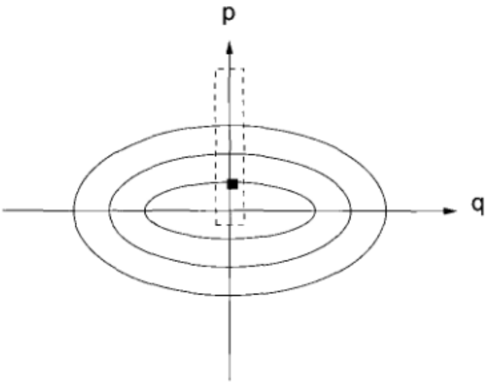

Fig. 1.5 Precise q implying large uncertainty in p.

(Similar pictures can be found in many physics books. This one is from Müller-Kirsten's "Introduction to Quantum mechanics")

Definition of $\mathbb{R}^d_\theta$

$\mathbb{R}^d_\theta$ is a space in the sense of noncommutative geometry: we do not define it directly, but instead we define it in terms of its function spaces $L_\infty(\mathbb{R}^d_\theta)$, $L_2(\mathbb{R}^d_\theta)$, $\mathcal{S}(\mathbb{R}^d_\theta)$, etc.

Definition of $\mathbb{R}^d_\theta$

$\mathbb{R}^d_\theta$ is a space in the sense of noncommutative geometry: we do not define it directly, but instead we define it in terms of its function spaces $L_\infty(\mathbb{R}^d_\theta)$, $L_2(\mathbb{R}^d_\theta)$, $\mathcal{S}(\mathbb{R}^d_\theta)$, etc.The following definition is expedient:

Definition

For $t \in \mathbb{R}^d$, let $\lambda_\theta(t)$ be the unitary operator on $L_2(\mathbb{R}^d)$ given by

$$\lambda_\theta(t)u(s) := e^{i(t,s)}u(s - \frac{1}{2}\theta t), \quad u \in L_2(\mathbb{R}^d).$$

The von Neumann subalgebra of $\mathcal{B}(L_2(\mathbb{R}^d))$ generated by $\{\lambda_\theta(t)\}_{t\in\mathbb{R}^d}$ is denoted $L_\infty(\mathbb{R}^d_\theta)$.

Remarks on the definition

1. Note that if $\theta = 0$, this reduces to the description of $L_\infty(\mathbb{R}^d)$ represented as bounded multipliers on $L_2(\mathbb{R}^d)$, and λ_0 is the operator of pointwise multiplication by the exponential function $s \mapsto \exp(i(t,s))$.
2. A simple computation shows that

$$\lambda_\theta(t+s) = \exp(\frac{1}{2}i(t,\theta s))\lambda_\theta(t)\lambda_\theta(s), \quad t,s \in \mathbb{R}^d.$$

   This is the Weyl form of the canonical commutation relations. It is also a twisted unitary representation of the group $\mathbb{R}^d$ by a 2-cocycle.
3. Heuristically, we have

$$\lambda_\theta(t) = \exp(it_1 x_1 + \cdots + it_d x_d)$$

   where

$$[x_j, x_k] = i\theta_{j,k}.$$

Identification of $L_\infty(\mathbb{R}^d_\theta)$

As a von Neumann algebra, $L_\infty(\mathbb{R}^d_\theta)$ is not very complicated: it has type I.

Theorem (Stone-von Neumann)

There is a $*$-algebra isomorphism

$$L_\infty(\mathbb{R}^d_\theta) \approx L_\infty(\mathbb{R}^{\mathrm{ker}(\theta)}) \otimes \mathcal{B}(L_2(\mathbb{R}^{\frac{\mathrm{rank}(\theta)}{2}})).$$

When $\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, this is recognisable as the Schrödinger representation;

$$x_1 = M_x, \quad x_2 = -i\partial_x, \text{ on } L_2(\mathbb{R}_x).$$

Derivatives

There is a group action of $\mathbb{R}^d$ on $L_\infty(\mathbb{R}^d_\theta)$ given on the generators by

$$T_t\lambda_\theta(s) = e^{i(t,s)}\lambda_\theta(s), \quad t, s \in \mathbb{R}^d.$$

This is the action of *translation*. Say that $x \in L_\infty(\mathbb{R}^d_\theta)$ is uniformly smooth if

$$t \mapsto T_t x, \quad \mathbb{R}^d \to L_\infty(\mathbb{R}^d_\theta)$$

is smooth. We can define the derivatives

$$\partial_j x := \frac{d}{dt} T_{te_j}(x)|_{t=0}.$$

Weyl transform

Given $f \in \mathcal{S}(\mathbb{R}^d)$, let

$$\lambda_\theta(f) = \int_{\mathbb{R}^d} f(t)\lambda_\theta(t)\,dt.$$

The space $\mathcal{S}(\mathbb{R}^d_\theta)$ is defined as the image $\lambda_\theta(\mathcal{S}(\mathbb{R}^d))$. This is a $*$-algebra, multiplication is given by twisted convolution

$$\lambda_\theta(f)\lambda_\theta(g) = \lambda_\theta(f *_\theta g).$$

Noncommutative integral

The noncommutative integral τ_θ is defined on $\mathcal{S}(\mathbb{R}^d_\theta)$ by

$$\tau_\theta(\lambda_\theta(f)) := (2\pi)^d f(0).$$

Theorem

τ_θ extends to a semifinite normal trace on $L_\infty(\mathbb{R}^d_\theta)$, giving the pair $(L_\infty(\mathbb{R}^d_\theta), \tau_\theta)$ the structure of a semifinite von Neumann algebra.

We can therefore define L_p spaces, $L_p(L_\infty(\mathbb{R}^d_\theta), \tau_\theta)$, which we abbreviate as $L_p(\mathbb{R}^d_\theta)$.

Harmonic analysis questions

Many authors (such as G. Hong) and most recently M. Ruzhansky and S. Shaimardan and K. Tulenov have addressed the problem of Fourier multipliers on $\mathbb{R}^d_\theta$. I.e., the boundedness of operators

$$m(-i\partial_1, -i\partial_2, \ldots, -i\partial_d) : L_p(\mathbb{R}^d_\theta) \to L_q(\mathbb{R}^d_\theta).$$

In terms of λ_θ, we have

$$m(-i\nabla)\lambda_\theta(f) = \lambda_\theta(mf) = \int_{\mathbb{R}^d} m(\xi)\lambda_\theta(\xi)$$

We can also define Sobolev spaces $W^s_p(\mathbb{R}^d_\theta)$ (González-Pérez, Junge, Parcet) and Besov spaces $B^s_{p,q}(\mathbb{R}^d_\theta)$ (Lafleche). Triebel spaces $F^s_{p,q}(\mathbb{R}^d_\theta)$ are more challenging.

Section 3: PDE on quantum Euclidean spaces

Linear PDE, constant coefficients

Linear PDE with constant coefficients

$$P(-i\nabla)u = f$$

are best studied with Fourier multipliers. Here, things are mostly similar to the commutative case.

Linear PDE, variable coefficients

There is a theory (González-Pérez, Junge, Parcet [GJP]) of elliptic PDE defined in terms of *left* multipliers:

$$\sum_{|\alpha|\leq m} a_\alpha \cdot \partial^\alpha u = f$$

where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.

Linear PDE, variable coefficients

There is a theory (González-Pérez, Junge, Parcet [GJP]) of elliptic PDE defined in terms of *left* multipliers:

$$\sum_{|\alpha|\leq m} a_\alpha \cdot \partial^\alpha u = f$$

where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.
To analyse these, they defined *left* pseudodifferential operators on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u = (2\pi)^{-d} \int_{\mathbb{R}^d} \sigma(\xi)\lambda_\theta(\xi)\tau_\theta(\lambda_\theta(\xi)^* u)\, d\xi.$$

Here, $\sigma : \mathbb{R}^d \to L_\infty(\mathbb{R}^d_\theta)$ is a smooth function satisfying

$$\|\partial_\xi^\alpha \partial^\beta \sigma(\xi)\|_\infty \lesssim (1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

Linear PDE, variable coefficients

There is a theory (González-Pérez, Junge, Parcet [GJP]) of elliptic PDE defined in terms of *left* multipliers:

$$\sum_{|\alpha|\leq m} a_\alpha \cdot \partial^\alpha u = f$$

where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.
To analyse these, they defined *left* pseudodifferential operators on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u = (2\pi)^{-d} \int_{\mathbb{R}^d} \sigma(\xi)\lambda_\theta(\xi)\tau_\theta(\lambda_\theta(\xi)^* u)\, d\xi.$$

Here, $\sigma : \mathbb{R}^d \to L_\infty(\mathbb{R}^d_\theta)$ is a smooth function satisfying

$$\|\partial_\xi^\alpha \partial^\beta \sigma(\xi)\|_\infty \lesssim (1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

Using these operators, [GJP] analysed the L_p-theory of linear elliptic equations.

Unaddressed question: two-sided linear PDE

A trivial modification of [GJP] allows to study *right* PDE

$$\sum_{|\alpha|\leq m} (\partial^\alpha u) a_\alpha = f$$

where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.
Something that no-one has addressed (to my knowledge) is the question of two-sided linear PDE, something like

$$\sum_{|\alpha|\leq m} a_\alpha (\partial^\alpha u) b_\alpha = f.$$

This is a new phenomenon in the noncommutative case.

Unaddressed question: two-sided linear PDE

A trivial modification of [GJP] allows to study *right* PDE

$$\sum_{|\alpha|\leq m} (\partial^\alpha u)a_\alpha = f$$

where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.
Something that no-one has addressed (to my knowledge) is the question of two-sided linear PDE, something like

$$\sum_{|\alpha|\leq m} a_\alpha(\partial^\alpha u)b_\alpha = f.$$

This is a new phenomenon in the noncommutative case.To analyse it, we would need some kind of theory of *bilateral pseudodifferential operators*.

Section 4: Towards a theory of bilateral pseudodifferential operators

Noncommutative Meyer decomposition?

If we want to study nonlinear PDE, we need to consider Nemytskij operators

$$u \mapsto F(u)$$

where now $F(u)$ is defined for an operator u using functional calculus. Can we do something similar to Meyer's decomposition?

Noncommutative Meyer decomposition?

If we want to study nonlinear PDE, we need to consider Nemytskij operators

$$u \mapsto F(u)$$

where now $F(u)$ is defined for an operator u using functional calculus. Can we do something similar to Meyer's decomposition? Answer: Yes, but we need pseudodifferential operators formed out of both left and right multipliers.

Simple case of a noncommutative Meyer decomposition

Consider the case $F(u) = u^2$. Then

$$F(u) = \sum_{j=0}^{\infty} F(S_j u) - F(S_{j-1} u) = \sum_{j=0}^{\infty} S_{j-1} u \Delta_j u + \Delta_j u S_j u.$$

So the Meyer operator should be the following:

$$m(F, u)v = \sum_{j=0}^{\infty} S_{j-1} u \Delta_j v + \Delta_j v S_j u.$$

Note that here we need both *left* and *right* multiplication. The situation for more general functions F is similar.

Bilateral multiplication

The algebra $L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}$ is the opposite algebra. Given $A \in L_\infty(\mathbb{R}^d_\theta) \otimes L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}$ and $u \in L_\infty(\mathbb{R}^d_\theta)$, denote $A\sharp u$ for the linear extension of the mapping $(a \otimes b)\sharp u = aub$. The Haagerup tensor product $L_\infty(\mathbb{R}^d_\theta) \otimes_h L_\infty(\mathbb{R}^d_\theta)$ has the property that $\sharp$ has a continuous extension

$$\sharp : (L_\infty(\mathbb{R}^d_\theta) \otimes_h L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}) \times L_\infty(\mathbb{R}^d_\theta) \to L_\infty(\mathbb{R}^d_\theta).$$

Proposed double symbol definition

Definition

Let $\rho, \delta_1, \delta_2 \in [0,1]$ and $m \in \mathbb{R}$. A *bisymbol* in the class $S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d \times \mathbb{R}^d_\theta)$ is a function:

$$\sigma : \mathbb{R}^d \to L_\infty(\mathbb{R}^d_\theta) \otimes_h L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}.$$

Such that for all multi-indices $\alpha, \beta_1, \beta_2 \in \mathbb{N}^d$ we have:

$$\|\partial_\xi^\alpha(\partial^{\beta_1} \otimes \partial^{\beta_2})\sigma(\xi)\|_{L_\infty(\mathbb{R}^d_\theta)\otimes_h L_\infty(\mathbb{R}^d_\theta)} \leq C_{\alpha,\beta_1,\beta_2}(1+|\xi|)^{m-\rho|\alpha|+\delta_1|\beta_1|+\delta_2|\beta_2|}$$

Bilateral pseudodifferential operators

Let $\sigma \in S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d \times \mathbb{R}^d_\theta)$. Define an operator T_σ on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u := (2\pi)^{-d} \int_{\mathbb{R}^d} (\sigma(\xi) \sharp \lambda_\theta(\xi)) \tau_\theta(\lambda_\theta(\xi)^* u) \, d\xi$$

Denote the space of such operators by $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$.

Bilateral pseudodifferential operators

Let $\sigma \in S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d \times \mathbb{R}^d_\theta)$. Define an operator T_σ on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u := (2\pi)^{-d} \int_{\mathbb{R}^d} (\sigma(\xi) \sharp \lambda_\theta(\xi)) \tau_\theta(\lambda_\theta(\xi)^* u)\, d\xi$$

Denote the space of such operators by $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$.

Conjecture

- For $\delta_1 + \delta_2 < \rho$, $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$ has similar properties of adjointability, composition and asymptotic convergence to traditional pseudodifferential operators.

Bilateral pseudodifferential operators

Let $\sigma \in S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d \times \mathbb{R}^d_\theta)$. Define an operator T_σ on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u := (2\pi)^{-d} \int_{\mathbb{R}^d} (\sigma(\xi) \sharp \lambda_\theta(\xi)) \tau_\theta(\lambda_\theta(\xi)^* u)\, d\xi$$

Denote the space of such operators by $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$.

Conjecture

- For $\delta_1 + \delta_2 < \rho$, $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$ has similar properties of adjointability, composition and asymptotic convergence to traditional pseudodifferential operators.
- For $\rho = 1 > \delta_1 + \delta_2$, and $m = 0$, we have boundedness on $L_p(\mathbb{R}^d_\theta)$ for $1 < p < \infty$.

Bilateral pseudodifferential operators

Let $\sigma \in S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d \times \mathbb{R}^d_\theta)$. Define an operator T_σ on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u := (2\pi)^{-d} \int_{\mathbb{R}^d} (\sigma(\xi) \sharp \lambda_\theta(\xi)) \tau_\theta(\lambda_\theta(\xi)^* u)\, d\xi$$

Denote the space of such operators by $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$.

Conjecture

- For $\delta_1 + \delta_2 < \rho$, $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$ has similar properties of adjointability, composition and asymptotic convergence to traditional pseudodifferential operators.
- For $\rho = 1 > \delta_1 + \delta_2$, and $m = 0$, we have boundedness on $L_p(\mathbb{R}^d_\theta)$ for $1 < p < \infty$.
- In general for $m = 0$, we have boundedness on $W^s_p(\mathbb{R}^d_\theta)$ for $s > 0$.

Why is it interesting?

- If the conjecture is correct, we would have a *genuinely noncommutative* theory of pseudodifferential operators which would allow us to study bilateral PDE and also nonlinear PDE over $\mathbb{R}^d_\theta$.

Why is it interesting?

- If the conjecture is correct, we would have a *genuinely noncommutative* theory of pseudodifferential operators which would allow us to study bilateral PDE and also nonlinear PDE over $\mathbb{R}^d_\theta$.
- Following the standard commutative proofs does not seem to work, it seems that new ideas are needed.

Thank you for listening!

Further reading

For more about the Meyer decomposition:

- M. Taylor. Partial differential equations III. Nonlinear equations. Second Edition. *Applied Mathematical Sciences* **117**, Springer, New York, 2011.
- H. Bahouri, J. Chemin, R. Danchin. Fourier analysis and nonlinear partial differential equations. *Grundlehren der Mathematischen Wissenschaften* **343**, Springer, Heidelberg, 2011.
- T. Runst, W. Sickel. Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, *De Gruyter Series in Nonlinear Analysis and Applications*, **3**, Walter de Gruyter & Co., Berlin, 1996.

For more about quantum Euclidean spaces, see

- S. Lord, E. M., F. Sukochev, D. Zanin. Singular traces. Vol. 2. Trace formulas. *De Gruyter Studies in Mathematics* **46/2**, De Gruyter, Berlin, 2023.
- L. Lafleche. *On Quantum Sobolev Inequalities,* arXiv:2210.03013.
- A. González-Pérez, M. Junge, J. Parcet. *Singular integrals in quantum Euclidean spaces*, Mem. Amer. Math. Soc., **272**, 2021.
- E. M., F. Sukochev, X. Xiong. *Quantum differentiability—the analytical perspective*, Proc. Sympos. Pure Math., **105**, 2023.

For more about PDE on quantum Euclidean spaces, see

- E. M. *Nonlinear partial differential equations on noncommutative Euclidean spaces*, J. Evol. Equ., **24**, 2024,
- M. Ruzhansky, S. Shaimardan, K. Tulenov. *Sobolev inequality and its applications to nonlinear PDE on noncommutative Euclidean spaces*, arXiv:2408.09100.
- C. Arhancet, L. Hagedorn, C. Kriegler, P. Portal. *The harmonic oscillator on the Moyal-Groenwald plane: an approach via Lie groups and twisted Weyl tuples*, arXiv:2312.06143.