Recent work and unsolved problems in Quantum Harmonic Analysis

Edward McDonald (Penn State University)

Ghent University

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Plan for this talk

In this talk I will go over some introductory material in Quantum Harmonic Analysis and propose some future work.

Prologue: Meyer's decomposition

Nemytskij operators

Let $F \in C^{\infty}(\mathbb{R})$. The mapping of functional composition

$$L_{\infty}(\mathbb{R}^d, \mathbb{R}) \to L_{\infty}(\mathbb{R}^d), \quad u \mapsto F(u)$$

is sometimes called a *Nemytskij operator*.

Nemytskij operators

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Question: In terms of function spaces, what are the mapping properties of $u \mapsto F(u)$? (This is needed for nonlinear PDE).

Meyer's decomposition

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Let $\{\Delta_j\}_{j=0}^{\infty}$ be an inhomogeneous Littlewood-Paley decomposition for $\mathbb{R}^d$.
That is, $\Delta_j = \Psi_j(-i\nabla)$ where Ψ_j is supported in the ball of radius 2^j and

$$\sum_{j=0}^{\infty} \Psi_j = 1.$$

Let $S_n = \sum_{j=0}^{n} \Delta_j$.

Meyer's decomposition (cont.)

Assume initially that $u = S_n u$ for some $n \geq 1$. We have

$$F(u) = F(0) + F(S_n u) - F(0) = F(0) + \sum_{j=0}^{\infty} F(S_j u) - F(S_{j-1} u)$$

where $S_{-1} u = 0$. Therefore

$$F(u) = F(0) + \sum_{j=0}^{\infty} \int_0^1 F'(S_{j-1} u + \theta \Delta_j u) \Delta_j u d\theta =: F(0) + m(F, u) u$$

where $m(F, u)$ is the *paradifferential operator*

$$m(F, u) v = \sum_{j=0}^{\infty} \int_0^1 F'(S_{j-1} u + \theta \Delta_j u) \Delta_j v d\theta.$$

Sometimes this is called *paralinearisation*.

Bourdaud's theorem

Some elementary estimates show for any $u \in L_\infty(\mathbb{R}^d, \mathbb{R})$ and $F \in C^\infty(\mathbb{R})$, $m(F, u)$ is a pseudodifferential operator in the Hörmander class $\Psi^0_{1,1}(\mathbb{R}^d)$ (Stein's "forbidden symbols").

Theorem (Bourdaud (1988))

If $T \in \Psi^0_{1,1}(\mathbb{R}^d)$, then T is bounded on the Sobolev space $W^s_p(\mathbb{R}^d)$ for all $s > 0$ and $1 < p < \infty$.

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Conclusion: If $F(0) = 0$ and $u \in W^s_p(\mathbb{R}^d, \mathbb{R})$ for some $s > 0$ and $1 < p < \infty$ then $F(u) \in W^s_p(\mathbb{R}^d)$.

Section 2: Quantum Harmonic Analysis

Quantum Euclidean spaces

Let $d \geq 1$ and let θ be a $d \times d$ antisymmetric real matrix. The quantum Euclidean space is an attempt to make rigorous mathematical sense of a space $\mathbb{R}^d_\theta$ with coordinate functions

$$x_1, \ldots, x_d$$

which do not commute, but instead satisfy the relation

$$x_j x_k - x_k x_j = i\theta_{j,k}, \quad 1 \leq j, k \leq d.$$

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Much of the literature motivates this from the relation $[q,p] = i\hbar$ from quantum mechanics.

Uncertainty principle

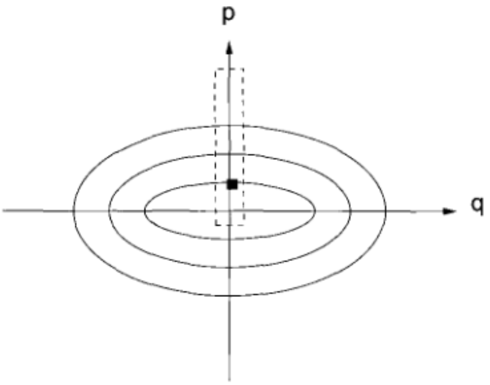

Fig. 1.5 Precise q implying large uncertainty in p.

(Similar pictures can be found in many physics books. This one is from Müller-Kirsten's "Introduction to Quantum mechanics")

Definition of $\mathbb{R}^d_\theta$

$\mathbb{R}^d_\theta$ is a space in the sense of noncommutative geometry: we do not define it directly, but instead we define it in terms of its function spaces $L_\infty(\mathbb{R}^d_\theta)$, $L_2(\mathbb{R}^d_\theta)$, $\mathcal{S}(\mathbb{R}^d_\theta)$, etc.

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Definition

For $t \in \mathbb{R}^d$, let $\lambda_\theta(t)$ be the unitary operator on $L_2(\mathbb{R}^d)$ given by

$$\lambda_\theta(t)u(s) := e^{i(t,s)}u(s - \frac{1}{2}\theta t), \quad u \in L_2(\mathbb{R}^d).$$

The von Neumann subalgebra of $\mathcal{B}(L_2(\mathbb{R}^d))$ generated by $\{\lambda_\theta(t)\}_{t\in\mathbb{R}^d}$ is denoted $L_\infty(\mathbb{R}^d_\theta)$.

Remarks on the definition

1. Note that if $\theta = 0$, this reduces to the description of $L_\infty(\mathbb{R}^d)$ represented as bounded multipliers on $L_2(\mathbb{R}^d)$, and λ_0 is the operator of pointwise multiplication by the exponential function $s \mapsto \exp(i(t,s))$.
2. A simple computation shows that

$$\lambda_\theta(t+s) = \exp(\frac{1}{2}i(t,\theta s))\lambda_\theta(t)\lambda_\theta(s), \quad t,s \in \mathbb{R}^d.$$

   This is the Weyl form of the canonical commutation relations. It is also a twisted unitary representation of the group $\mathbb{R}^d$ by a 2-cocycle.
3. Heuristically, we have

$$\lambda_\theta(t) = \exp(it_1x_1 + \cdots + it_dx_d)$$

   where

$$[x_j, x_k] = i\theta_{j,k}.$$

Identification of $L_\infty(\mathbb{R}^d_\theta)$

As a von Neumann algebra, $L_\infty(\mathbb{R}^d_\theta)$ is not very complicated: it has type I.

Theorem (Stone-von Neumann)

There is a $*$-algebra isomorphism

$$L_\infty(\mathbb{R}^d_\theta) \approx L_\infty(\mathbb{R}^{\mathrm{ker}(\theta)}) \otimes \mathcal{B}(L_2(\mathbb{R}^{\frac{\mathrm{rank}(\theta)}{2}})).$$

When $\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, this is recognisable as the Schrödinger representation;

$$x_1 = M_x, \quad x_2 = -i\partial_x, \text{ on } L_2(\mathbb{R}_x).$$

Derivatives

There is a group action of $\mathbb{R}^d$ on $L_\infty(\mathbb{R}^d_\theta)$ given on the generators by

$$T_t\lambda_\theta(s) = e^{i(t,s)}\lambda_\theta(s), \quad t, s \in \mathbb{R}^d.$$

This is the action of *translation*. Say that $x \in L_\infty(\mathbb{R}^d_\theta)$ is uniformly smooth if

$$t \mapsto T_t x, \quad \mathbb{R}^d \to L_\infty(\mathbb{R}^d_\theta)$$

is smooth. We can define the derivatives

$$\partial_j x := \frac{d}{dt} T_{te_j}(x)|_{t=0}.$$

Weyl transform

Given $f \in \mathcal{S}(\mathbb{R}^d)$, let

$$\lambda_\theta(f) = \int_{\mathbb{R}^d} f(t)\lambda_\theta(t)\,dt.$$

The space $\mathcal{S}(\mathbb{R}^d_\theta)$ is defined as the image $\lambda_\theta(\mathcal{S}(\mathbb{R}^d))$. This is a $*$-algebra, multiplication is given by twisted convolution

$$\lambda_\theta(f)\lambda_\theta(g) = \lambda_\theta(f *_\theta g).$$

Noncommutative integral

The noncommutative integral τ_θ is defined on $\mathcal{S}(\mathbb{R}^d_\theta)$ by

$$\tau_\theta(\lambda_\theta(f)) := (2\pi)^d f(0).$$

Theorem

τ_θ extends to a semifinite normal trace on $L_\infty(\mathbb{R}^d_\theta)$, giving the pair $(L_\infty(\mathbb{R}^d_\theta), \tau_\theta)$ the structure of a semifinite von Neumann algebra.

We can therefore define L_p spaces, $L_p(L_\infty(\mathbb{R}^d_\theta), \tau_\theta)$, which we abbreviate as $L_p(\mathbb{R}^d_\theta)$.

Harmonic analysis questions

Many authors (such as G. Hong) and most recently M. Ruzhansky and S. Shaimardan and K. Tulenov have addressed the problem of Fourier multipliers on $\mathbb{R}^d_\theta$. I.e., the boundedness of operators

$$m(-i\partial_1, -i\partial_2, \ldots, -i\partial_d) : L_p(\mathbb{R}^d_\theta) \to L_q(\mathbb{R}^d_\theta).$$

In terms of λ_θ, we have

$$m(-i\nabla)\lambda_\theta(f) = \lambda_\theta(mf) = (2\pi)^{-d} \int_{\mathbb{R}^d} m(\xi)\lambda_\theta(\xi)f(\xi)\, d\xi.$$

We can also define Sobolev spaces $W^s_p(\mathbb{R}^d_\theta)$ (González-Pérez, Junge, Parcet) and Besov spaces $B^s_{p,q}(\mathbb{R}^d_\theta)$ (Lafleche). Triebel spaces $F^s_{p,q}(\mathbb{R}^d_\theta)$ are more challenging.

Section 3: PDE on quantum Euclidean spaces

Linear PDE, constant coefficients

Linear PDE with constant coefficients

$$P(-i\nabla)u = f$$

are best studied with Fourier multipliers. Here, things are mostly similar to the commutative case.

Linear PDE, variable coefficients

There is a theory (González-Pérez, Junge, Parcet [GJP]) of elliptic PDE defined in terms of *left* multipliers:

$$\sum_{|\alpha|\leq m} a_\alpha \cdot \partial^\alpha u = f$$

where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.

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where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.
To analyse these, they defined *left* pseudodifferential operators on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u = (2\pi)^{-d} \int_{\mathbb{R}^d} \sigma(\xi)\lambda_\theta(\xi)\tau_\theta(\lambda_\theta(\xi)^* u)\, d\xi.$$

Here, $\sigma : \mathbb{R}^d \to L_\infty(\mathbb{R}^d_\theta)$ is a smooth function satisfying

$$\|\partial_\xi^\alpha \partial^\beta \sigma(\xi)\|_\infty \lesssim (1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

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Using these operators, [GJP] analysed the L_p-theory of linear elliptic equations.

Unaddressed question: two-sided linear PDE

A trivial modification of [GJP] allows to study *right* PDE

$$\sum_{|\alpha|\leq m} (\partial^\alpha u) a_\alpha = f$$

where $a_\alpha \in C^\infty(\mathbb{R}^d_\theta)$.
Something that no-one has addressed (to my knowledge) is the question of two-sided linear PDE, something like

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$$\sum_{|\alpha|\leq m} a_\alpha(\partial^\alpha u)b_\alpha = f.$$

This is a new phenomenon in the noncommutative case.To analyse it, we would need some kind of theory of *bilateral pseudodifferential operators*.

Section 4: Towards a theory of bilateral pseudodifferential operators

Noncommutative Meyer decomposition?

If we want to study nonlinear PDE, we need to consider Nemytskij operators

$$u \mapsto F(u)$$

where now $F(u)$ is defined for an operator u using functional calculus. Can we do something similar to Meyer's decomposition?

Noncommutative Meyer decomposition?

If we want to study nonlinear PDE, we need to consider Nemytskij operators

$$u \mapsto F(u)$$

where now $F(u)$ is defined for an operator u using functional calculus. Can we do something similar to Meyer's decomposition? Answer: Yes, but we need pseudodifferential operators formed out of both left and right multipliers.

Simple case of a noncommutative Meyer decomposition

Consider the case $F(u) = u^2$. Then

$$F(u) = \sum_{j=0}^{\infty} F(S_j u) - F(S_{j-1} u) = \sum_{j=0}^{\infty} S_{j-1} u \Delta_j u + \Delta_j u S_j u.$$

So the Meyer operator should be the following:

$$m(F, u)v = \sum_{j=0}^{\infty} S_{j-1} u \Delta_j v + \Delta_j v S_j u.$$

Note that here we need both *left* and *right* multiplication. The situation for more general functions F is similar.

Bilateral multiplication

The algebra $L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}$ is the opposite algebra. Given $A \in L_\infty(\mathbb{R}^d_\theta) \otimes L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}$ and $u \in L_\infty(\mathbb{R}^d_\theta)$, denote $A\sharp u$ for the linear extension of the mapping $(a \otimes b)\sharp u = aub$. The Haagerup tensor product $L_\infty(\mathbb{R}^d_\theta) \otimes_h L_\infty(\mathbb{R}^d_\theta)$ has the property that $\sharp$ has a continuous extension

$$\sharp : (L_\infty(\mathbb{R}^d_\theta) \otimes_h L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}) \times L_\infty(\mathbb{R}^d_\theta) \to L_\infty(\mathbb{R}^d_\theta).$$

Proposed double symbol definition

Definition

Let $\rho, \delta_1, \delta_2 \in [0,1]$ and $m \in \mathbb{R}$. A *bisymbol* in the class $S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d \times \mathbb{R}^d_\theta)$ is a function:

$$\sigma : \mathbb{R}^d \to L_\infty(\mathbb{R}^d_\theta) \otimes_h L_\infty(\mathbb{R}^d_\theta)^{\mathrm{op}}.$$

Such that for all multi-indices $\alpha, \beta_1, \beta_2 \in \mathbb{N}^d$ we have:

$$\|\partial_\xi^\alpha(\partial^{\beta_1} \otimes \partial^{\beta_2})\sigma(\xi)\|_{L_\infty(\mathbb{R}^d_\theta)\otimes_h L_\infty(\mathbb{R}^d_\theta)} \leq C_{\alpha,\beta_1,\beta_2}(1+|\xi|)^{m-\rho|\alpha|+\delta_1|\beta_1|+\delta_2|\beta_2|}$$

Bilateral pseudodifferential operators

Let $\sigma \in S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d \times \mathbb{R}^d_\theta)$. Define an operator T_σ on $u \in \mathcal{S}(\mathbb{R}^d_\theta)$ by

$$T_\sigma u := (2\pi)^{-d} \int_{\mathbb{R}^d} (\sigma(\xi)\sharp\lambda_\theta(\xi))\tau_\theta(\lambda_\theta(\xi)^* u)\, d\xi$$

Denote the space of such operators by $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$.

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Conjecture

- For $\delta_1 + \delta_2 < \rho$, $\Psi^m_{\rho,\delta_1,\delta_2}(\mathbb{R}^d_\theta)$ has similar properties of adjointability, composition and asymptotic convergence to traditional pseudodifferential operators.

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- For $\rho = 1 > \delta_1 + \delta_2$, and $m = 0$, we have boundedness on $L_p(\mathbb{R}^d_\theta)$ for $1 < p < \infty$.
- In general for $m = 0$, we have boundedness on $W^s_p(\mathbb{R}^d_\theta)$ for $s > 0$.

Why is it interesting?

- If the conjecture is correct, we would have a *genuinely noncommutative* theory of pseudodifferential operators which would allow us to study bilateral PDE and also nonlinear PDE over $\mathbb{R}^d_\theta$.

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- If the conjecture is correct, we would have a *genuinely noncommutative* theory of pseudodifferential operators which would allow us to study bilateral PDE and also nonlinear PDE over $\mathbb{R}^d_\theta$.
- Following the standard commutative proofs does not seem to work, it seems that new ideas are needed.

Thank you for listening!

Further reading

For more about the Meyer decomposition:

- M. Taylor. Partial differential equations III. Nonlinear equations. Second Edition. *Applied Mathematical Sciences* **117**, Springer, New York, 2011.
- H. Bahouri, J. Chemin, R. Danchin. Fourier analysis and nonlinear partial differential equations. *Grundlehren der Mathematischen Wissenschaften* **343**, Springer, Heidelberg, 2011.
- T. Runst, W. Sickel. Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, *De Gruyter Series in Nonlinear Analysis and Applications*, **3**, Walter de Gruyter & Co., Berlin, 1996.

For more about quantum Euclidean spaces, see

- S. Lord, E. M., F. Sukochev, D. Zanin. Singular traces. Vol. 2. Trace formulas. *De Gruyter Studies in Mathematics* **46/2**, De Gruyter, Berlin, 2023.
- L. Lafleche. *On Quantum Sobolev Inequalities,* arXiv:2210.03013.
- A. González-Pérez, M. Junge, J. Parcet. *Singular integrals in quantum Euclidean spaces*, Mem. Amer. Math. Soc., **272**, 2021.
- E. M., F. Sukochev, X. Xiong. *Quantum differentiability—the analytical perspective*, Proc. Sympos. Pure Math., **105**, 2023.

For more about PDE on quantum Euclidean spaces, see

- E. M. *Nonlinear partial differential equations on noncommutative Euclidean spaces*, J. Evol. Equ., **24**, 2024,
- M. Ruzhansky, S. Shaimardan, K. Tulenov. *Sobolev inequality and its applications to nonlinear PDE on noncommutative Euclidean spaces*, arXiv:2408.09100.
- C. Arhancet, L. Hagedorn, C. Kriegler, P. Portal. *The harmonic oscillator on the Moyal-Groenwald plane: an approach via Lie groups and twisted Weyl tuples*, arXiv:2312.06143.