Introduction to the Hypoelliptic Laplacian on a compact group

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But... where does it come from?
What is it good for?
And, most importantly, how do we prove things about it?

Section 1: Frenkel's formula

Let G be a connected, simply connected, compact group, with Lie algebra $\mathfrak{g}$ having nondegenerate bilinear form B.
Specify a Cartan subalgebra $\mathfrak{t} \subset G$ and corresponding maximal abelian subgroup $T \subset G$. We have the co-root lattice

$$\mathrm{CR} = \{X \in \mathfrak{t} \ : \ \exp(X) = 1_G\}$$

and the weight lattice

$$\mathrm{CR}^* = \{\lambda \in \mathfrak{t}^* \ : \ \lambda(X) \in \mathbb{Z}, \quad X \in \mathrm{CR}\}$$

with specified positive subspace $\mathrm{CR}^*_+ \subset \mathrm{CR}^*$.
There is the Casimir element, $\Delta_G \in \mathcal{U}(\mathfrak{g})$.
What is $\mathrm{Tr}(e^{t\Delta_G})$?

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Some possible answers:

1. Spectral (Peter-Weyl) representation: we have

$$\mathrm{Tr}(e^{t\Delta_G}) = \sum_{\lambda \in CR_+^*} \left(\prod_{\alpha > 0} \frac{B(\alpha, \rho + \lambda)}{B(\alpha, \rho)} \right)^2 e^{-tB(\lambda + 2\rho, \lambda)}$$

   where $\prod_{\alpha>0}$ is the product over positive roots, ρ is the Weyl element.

2. Minakshisundaram-Pleijel expansion: asymptotically, as $t \to 0$, we have

$$\mathrm{Tr}(e^{t\Delta_G}) \sim (4\pi t)^{-\frac{\dim(G)}{2}} (a_0 + a_1 t + a_2 t^2 + \cdots)$$

   where

$$a_n = \frac{\mathrm{Vol(G)}}{n!} (2\pi B(\rho, \rho))^{2n}.$$

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Is there a "better" representation?

Jacobi identity

If $G = \mathbb{T}$ is the unit circle, the answer is *yes*. We have

$$\mathrm{Tr}(e^{t\Delta_{\mathbb{T}}}) = \mathrm{Vol}(\mathbb{T})(4\pi t)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-\frac{1}{4t}n^2}.$$

We can prove this via the Poisson summation formula and the fact that

$$\mathrm{Tr}(e^{t\Delta_{\mathbb{T}}}) = \sum_{n\in\mathbb{Z}} e^{-tn^2}$$

Is there something similar for a general Lie group?

Frenkel's formula

It turns out that $\mathrm{Tr}(e^{t\Delta_G})$ is too difficult to compute. Instead, we can compute a "shifted" version

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G})$$

where $H \in \mathfrak{t}$, and $\lambda(e^H)$ is the left shift operator by $e^H \in G$.

Theorem (Èskin (1964), Frenkel (1984))

If $H \in \mathfrak{t}$ is regular (i.e., $\alpha(H) \neq 0$ for all roots α), then

$$\begin{aligned}&\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G})\\ &= \frac{\mathrm{Vol}(G)e^{4\pi^2 tB(\rho,\rho)^2}}{(4\pi t)^{\frac{\dim(G)}{2}}\sigma(H)}\sum_{\gamma\in CR}\left(\prod_{\alpha>0}\langle 2\pi\alpha, H+\gamma\rangle\right)e^{-\frac{1}{4t}B(H+\gamma,H+\gamma)^2}\end{aligned}$$

where σ is the Weyl denominator.

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\end{aligned}$$

where σ is the Weyl denominator.

We can also take $H \to 0$, but the resulting expression for $\mathrm{Tr}(e^{t\Delta_G})$ is more complicated, see Èskin's paper.

Proof of Frenkel's formula

Frenkel proved his formula using the explicit spectral decomposition of Δ_G and the Weyl character formula. From these facts, we have

$$\begin{aligned}
&\mathrm{Tr}(\lambda(e^H)\rho(e^K)e^{t\Delta_G}) \\
&= \sum_{\lambda\in CR_+^*} \frac{1}{\sigma(H)\sigma(K)} \left(\sum_{w\in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)(H)}\right) \\
&\quad\cdot \left(\sum_{w\in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)(K)}\right) e^{-tB(\lambda+2\rho,\lambda)}
\end{aligned}$$

(here, ρ is the right-regular representation). Some rearrangement, Poisson's summation formula, and sending $K \to 0$ gives the result.

If we examine Frenkel's formula

$$\sum_{\lambda \in CR_+^*} \frac{1}{\sigma(H)\sigma(K)} \left(\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)(H)} \right)$$

$$\cdot \left(\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)(K)} \right) e$$

$$= \frac{\mathrm{Vol}(G) e^{4\pi^2 t B(\rho,\rho)^2}}{(4\pi t)^{\frac{\dim(G)}{2}} \sigma(H)} \sum_{\gamma \in CR} \left(\prod_{\alpha > 0}\right.$$

We can see that a sum over the *weight lattice* is converted into a sum over the *coroot lattice*. The exponential $\exp(-tx^2)$ is converted to $\exp(-\frac{1}{t}x^2)$ (as if this were a Fourier transform).

Looking to Bismut

Bismut gave a new proof of Frenkel's formula, as the

$$\begin{aligned}&\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G})\\&=\frac{\mathrm{Vol}(G)e^{4\pi^2 tB(\rho,\rho)^2}}{(4\pi t)^{\frac{\dim(G)}{2}}\sigma(H)}\sum_{\gamma\in\mathrm{CR}}\left(\prod_{\alpha>0}\langle 2\pi\alpha, H+\gamma\rangle\right)e^{-\frac{1}{4t}B(H+\gamma,H+\gamma)^2}.\end{aligned}$$

But it differs in some essential ways from Frenkel's proof:

- Bismut does not use the Peter-Weyl decomposition of G. The theorem of the highest weight is not needed! There is no representation theory of G.
- The coroot lattice CR of G emerges in a purely geometric way, rather than as the dual of the weight lattice.

Section 2: The Primordial History, or, the Berline-Vergne localisation formula

Section introduction

In this section, I will attempt to give a "folkloric" story for the origins of the hypoelliptic Laplacian.
I am strongly indebtted to the paper of Choi–Takhtajan (2021) for this explanation.

The Hamiltonian-Lagrangian correspondence

In its most general terms, the path integral method in quantum mechanics relates traces of semigroups to integrals over loop space.
The way it is supposed to work is as follows: X is a manifold, and $\mathcal{L} \in C^\infty(TX)$ is a "Lagrangian". The corresponding "Hamiltonian" $\mathcal{H} \in C^\infty(T^*X)$ is related to $\mathcal{L}$ by the Legendre transform

$$\mathcal{H}(x,p) = \sup_{X \in T_x X} (p(X) - \mathcal{L}(x,X)), \quad (x,p) \in T^*X.$$

The Hamiltonian-Lagrangian correspondence

We are supposed to have something like the following:

$$\mathrm{Tr}(e^{-t\mathcal{H}(x,-i\partial)}) = \int_{L_tX} e^{-\int_0^t \mathcal{L}(\gamma(s),\dot{\gamma}(s))\,ds}\mathcal{D}\gamma. \tag{2.1}$$

Here, $\mathcal{H}(x,-i\partial)$ is an operator on the Hilbert space $L_2(X)$ defined by some kind of quantisation of $\mathcal{H}$, L_tX is the space of loops of "time" t, that is functions $\mathbb{R}/t\mathbb{Z} \to X$, and "$\mathcal{D}\gamma$" is some kind of measure on loop space.

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The Hamiltonian-Lagrangian correspondence

One case (and one of the few cases that is well-understood) is for Δ_g, the Laplace operator on a compact Riemannian manifold

$$\frac{1}{2}\Delta_g u = \frac{1}{2}\det(g)^{-\frac{1}{2}}\partial_\alpha(g^{\alpha,\beta}\det(g)^{\frac{1}{2}}\partial_\beta u)$$

This is supposed to be $\mathcal{H}(x, -i\partial)$, where $\mathcal{H}(x,p) = \frac{1}{2}g^{\alpha,\beta}(x)p_\alpha p_\beta$. The corresponding Lagrangian is

$$\mathcal{L}(x, X) = \frac{1}{2}g_{\alpha,\beta}(x)X^\alpha X^\beta$$

and

$$\mathrm{Tr}(e^{\frac{1}{2}t\Delta_g}) = \int_{LX} e^{-\int_0^t \frac{1}{2}g_{\alpha,\beta}(\gamma(s))\dot{\gamma}^\alpha(s)\dot{\gamma}^\beta(s)\,ds}\,\mathcal{D}\gamma$$

where the integral on the right-hand side is rigorously defined as a limit of discrete approximations. See Andersson–Driver (1998).

Berline-Vergne localisation

Suppose that we're trying to compute the integral of a differential form α over a manifold Z, like

$$\int_Z \alpha, \quad \alpha \in \Omega^{\mathrm{top}}(Z).$$

There is a very effective trick for computing this integral if we have some $\mathbb{T}$-action: Assume that $\mathbb{T}$ acts on Z, and α is invariant under this action. We have

Theorem (Berline-Vergne localisation)

$$\int_Z \alpha = \int_{Z^0} \frac{\alpha}{e(N(Z/Z^0))}$$

where Z^0 is the fixed submanifold under the $\mathbb{T}$-action, and $e(N(Z/Z^0))$ is the Euler class of the normal bundle to Z^0 in Z.

What is of greater interest to us is actually the standard proof of the Berline-Vergne formula.

Proof of Berline-Vergne localisation

Berline-Vergne localisation can be proved in the following way. It is possible to construct a differential form $\mathcal{V} \in \Omega^2(X)$ such that

$$\int_Z \alpha = \int_Z \alpha e^{b\mathcal{V}}, \quad b > 0.$$

This is proved by showing that the right hand side is independent of b, and then setting $b = 0$. The limit as $b \to \infty$ is computed using Laplace's asymptotic formula.

Bismut's dream

We would like to compute $\mathrm{Tr}(e^{t\Delta_g})$ by considering

$$\int_{LX} e^{-\int_0^t \mathcal{L}(\gamma(s),\dot{\gamma}(s))\,ds}\mathcal{D}\gamma$$

as an integral of a differential form over the infinite dimensional manifold L_tX. Note that L_tX has a $\mathbb{T}$-action given by rotating around the loop.

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$$\int_{LX} e^{-\int_0^t \mathcal{L}(\gamma(s),\dot{\gamma}(s))+b\mathcal{V}\,ds}\mathcal{D}\gamma$$

where $b > 0$, and computing the limit as $b \to \infty$.

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where $b > 0$, and computing the limit as $b \to \infty$. This is **TOO HARD**.

Bismut's dream

Doing Berline-Vergne in loop space is a great idea but it's too difficult to attempt directly.
But maybe, we can use the Hamiltonian-Lagrangian correspondence to replace incomprehensible loop space integrals with slightly more comprehensible operator traces.

Can we find an operator $\mathcal{L}_b$ depending on a parameter b such that

$$\mathrm{Tr}(e^{t\Delta_g}) = \mathrm{Tr}(e^{t\mathcal{L}_b})$$

for all $b > 0$, and such that the limit as $b \to \infty$ gives an interesting formula for the trace?

Answer: In a way, yes. Not precisely as stated above, but we can get close enough.

The hypoelliptic Laplacian

Let G be a compact Lie group, we consider Δ_G rather than the general Δ_g. Through some combination of path integral computations, intuition, and guesswork, Bismut found a solution the preceding question.

Theorem (Bismut (2011))

Let $H \in \mathfrak{t}$ be regular. There is a differential operator $\mathcal{L}_{b,H}$, depending on a positive real parameter b, acting on the space

$$C^\infty(\mathfrak{g} \times G, \overset{\bullet}{\bigwedge} \mathfrak{g})$$

such that

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G}) = \mathrm{STr}(e^{-t\mathcal{L}_{b,H}}), \quad b > 0.$$

As $b \to \infty$, it can be proved that $\mathrm{STr}(e^{-t\mathcal{L}_{b,H}})$ converges to Frenkel's formula.

The Form of $\mathcal{L}_b$

Notation

G is a compact Lie group of dimension d, and $\mathfrak{g}$ is its Lie algebra. Let $e_1, \ldots, e_d$ be an orthonormal basis of $\mathfrak{g}$.
We will write the variables of $\mathfrak{g} \times G$ as (y, x), and we have some clifford operators on $\bigwedge^\bullet \mathfrak{g}$, denoted

$$c(v) = v \wedge + \iota_v, \quad \widehat{c}(v) = v \wedge - \iota_v, \quad v \in \mathfrak{g}.$$

Definition of $\mathcal{L}_b$.

It would be nice to have a heuristic derivation of $\mathcal{L}_b$ that follows from the path integral computations, but I am not convinced that such a thing exists. Instead, let's take Bismut at his word and assert the following definition.

Definition

$$\mathcal{L}_b = (D + \frac{1}{b}Q)^2 - D^2$$

where

$$Q = \sum_{j=1}^{d} c(e_j)y_j - i\widehat{c}(e_j)\partial_{y_j}$$

and

$$D = \sum_{j=1}^{d} -ic(e_j)\partial_{x_j} + \frac{1}{12}f^j_{k,l}c(e_j)c(e_k)c(e_\ell)$$

is the *Kostant Dirac operator* on G.

Structure of $\mathcal{L}_b$.

Expanding out the definitions, we get

$$\mathcal{L}_b = \frac{1}{b^2} Q^2 + \frac{1}{b}(QD + DQ)$$

which turns out to be

$$\mathcal{L}_b = \frac{1}{b^2} \sum_{j=1}^{d} y_j^2 - \partial_{y_j}^2 + \frac{1}{b} \sum_{j=1}^{d} y_j \partial_{x_j} + \text{matrix terms.}$$

Analytic problems and their solutions

Analysing $\mathcal{L}_b$, we can see that

$$\mathcal{L}_b = \frac{1}{b^2} \underbrace{\sum_{j=1}^{d} y_j^2 - \partial_{y_j}^2}_{\text{Harmonic oscillator on } \mathfrak{g}}$$

$$+ \frac{1}{b} \underbrace{\sum_{j=1}^{d} y_j \partial_{x_j}}_{\text{Generator of geodesic flow, identifying } TG = \mathfrak{g} \times G}$$

$$+ \text{ matrix terms.}$$

This operator is *not elliptic* on $\mathfrak{g} \times G$, but it is hypoelliptic. We are also interested in the heat operator

$$\widehat{\mathcal{L}}_b = \partial_t - \mathcal{L}_b.$$

Scalar part of $\mathcal{L}_b$.

Let's concentrate on the scalar part of $\mathcal{L}_b$,

$$L_b = \frac{1}{b^2}\sum_{j=1}^{d} y_j^2 - \partial_{y_j}^2 + \frac{1}{b}\sum_{j=1}^{d} y_j\partial_{x_j}, \quad (x,y) \in \mathfrak{g}_y \times G_x.$$

Here ∂_{y_j} are the (commuting) derivatives on $\mathfrak{g}$ and ∂_{x_j} are the noncommuting derivatives on G.

A toy model

Life would be a lot easier if the ∂_{x_j} terms were commuting, consider the operator

$$K_b := \frac{1}{b^2}\sum_{j=1}^{d} y_j^2 - \partial_{y_j}^2 + \frac{1}{b}\sum_{j=1}^{d} y_j \partial_{x_j}, \quad (x, y) \in \mathfrak{g}_y \times \mathfrak{g}_x.$$

where now the ∂'_{x_j} commute.

The problems we face

The main analytic difficulties we need to deal with are proving *Schauder estimates*

$$\|u\|_{s+2} \lesssim \|\mathcal{L}_b u\|_s + \|u\|_s, \quad \|v\|_{s+2} \lesssim \|(\partial_t - \mathcal{L}_b)v\|_s + \|v\|_s$$

and *heat kernel estimates*

$$\exp(-t\mathcal{L}_b)(p,q) \leq (???)$$

Section 4: The world's most convoluted proof of Frenkel's formula

The limit as $b \to 0$.

In some sense, we are supposed to have

$$\exp(-t\mathcal{L}_b) \to \exp(t\Delta_G).$$

But how does this actually work? After all, they act on different spaces.

The projection onto $\ker(Q)$.

In order to understand how an operator on $L_2(\mathfrak{g} \times G, \bigwedge^\bullet \mathfrak{g})$ converges to an operator on $L_2(G)$, we need to understand how $L_2(G)$ is a subspace of $L_2(\mathfrak{g} \times G, \bigwedge^\bullet \mathfrak{g})$.
Solution: we use a Gaussian function on the $\mathfrak{g}$ fibre. This is like embedding $L_2(\mathbb{T})$ isometrically into $L_2(\mathbb{T} \times \mathbb{R})$ by mapping $f(x)$ to $f(x)e^{-\pi y^2}$.

Definition

Recall that Q is the Witten Dirac,

$$Q = \sum_{j=1}^{d} c(e_j)y_j - \widehat{c}(e_j)\partial_{y_j}.$$

Let P be the kernel projection of Q.

We will embed $L_2(G)$ into $L_2(\mathfrak{g} \times G, \bigwedge^\bullet \mathfrak{g})$ by

$$L_2(G) \cong PL_2(\mathfrak{g} \times G, \bigwedge^\bullet \mathfrak{g})P$$

The 2×2 matrix trick

We want to prove that $\exp(-t\mathcal{L}_b)$ converges to $P\exp(t\Delta_G)P$. Actually it is simpler to work with resolvents, and prove that

$$(\lambda + \mathcal{L}_b)^{-1} \to P(\lambda - \Delta_G)^{-1}P$$

for a suitably large set of $\lambda \in \mathbb{C}$. By some functional calculus, this is equivalent.

Idea: write the resolvent as a 2×2 matrix.

As $b \to \infty$

The much more challenging problem we face is to understand what happens to $\mathrm{STr}(e^{-t\mathcal{L}_b})$ as $b \to \infty$. Note that:

1. Unlike with $b \to 0$, we have to take the supertrace first in order that the limit exists. This is analogous to the local index theorem.
2. This argument will be complicated, in part, because we have to do a simultaneous rescaling of the spatial and clifford variables.
3. It is not possible to compute the limit of $\mathrm{STr}(e^{-t\mathcal{L}_b})$ directly, instead we need to introduce a shift.

Techniques

Going into all the details with the rescaling of variables is too much to attempt here.... Instead I will briefly indicate what we're aiming for. What we expect is that

$$\lim_{b\to\infty} \mathrm{STr}(e^{-t\mathcal{L}_{b,H}}) = \text{A sum over coweights of } G.$$

The left hand side is an integral over the kernel of the operator. How does an integral converge to a sum?

What next?

We have succeeded in proving Frenkel's formula by replacing some elementary representation theory and Poisson's summation formula by the most convoluted arguments imaginable.
We can all agree that having new proofs of old results is worthwhile in itself, but... **What is the point of all this?**

1. Bismut's proof is *totally different* to Frenkel's proof. Strikingly, *no representation theory is needed*. We didn't even need the theorem of the highest weight. The coweight lattice turned up in the sum for completely geometric reasons.
2. Bismut eventually extended his arguments to give totally new results, including his formulas for the heat kernel on symmetric spaces.

Thank you for listening!

Happy Birthday Nigel!