Introduction to the Hypoelliptic Laplacian on a compact group

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But... where does it come from?
What is it good for?
And, most importantly, how do we prove things about it?

Section 1: Frenkel's formula

Let G be a connected, simply connected, compact group, with Lie algebra $\mathfrak{g}$ having nondegenerate bilinear form B.

Specify a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and corresponding maximal abelian subgroup $T \subset G$. We have the co-root lattice

$$\mathrm{CR} = \{X \in \mathfrak{t} \ : \ \exp(X) = 1_G\}$$

and the weight lattice

$$\mathrm{CR}^* = \{\lambda \in \mathfrak{t}^* \ : \ \lambda(X) \in \mathbb{Z}, \quad X \in \mathrm{CR}\}$$

with specified positive subspace $\mathrm{CR}^*_+ \subset \mathrm{CR}^*$.

There is the Casimir element, $\Delta_G \in \mathcal{U}(\mathfrak{g})$.

What is $\mathrm{Tr}(e^{t\Delta_G})$?

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Some possible answers:

1. Spectral (Peter-Weyl) representation: we have

$$\mathrm{Tr}(e^{t\Delta_G}) = \sum_{\lambda \in CR_+^*} \left(\prod_{\alpha > 0} \frac{B(\alpha, \rho + \lambda)}{B(\alpha, \rho)} \right)^2 e^{-tB(\lambda + 2\rho, \lambda)}$$

   where $\prod_{\alpha > 0}$ is the product over positive roots, ρ is the Weyl element.

2. Minakshisundaram-Pleijel expansion: asymptotically, as $t \to 0$, we have

$$\mathrm{Tr}(e^{t\Delta_G}) \sim (4\pi t)^{-\frac{\dim(G)}{2}} (a_0 + a_1 t + a_2 t^2 + \cdots)$$

   where

$$a_n = \frac{\mathrm{Vol(G)}}{n!} (2\pi B(\rho, \rho))^{2n}.$$

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Is there a "better" representation?

Jacobi identity

If $G = \mathbb{T}$ is the unit circle, the answer is *yes*. We have

$$\mathrm{Tr}(e^{t\Delta_{\mathbb{T}}}) = \frac{\mathrm{Vol}(\mathbb{T})}{(4\pi t)^{\frac{1}{2}}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{4t}n^2}.$$

We can prove this via the Poisson summation formula and the fact that

$$\mathrm{Tr}(e^{t\Delta_{\mathbb{T}}}) = \sum_{n \in \mathbb{Z}} e^{-tn^2}$$

Is there something similar for a general Lie group?

Frenkel's formula

It turns out that we get a cleaner statement by considering a "shifted" version

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G})$$

where $H \in \mathfrak{t}$, and $\lambda(e^H)$ is the left shift operator by $e^H \in G$.

Theorem (Èskin (1964), Frenkel (1984))

If $H \in \mathfrak{t}$ is regular (i.e., $\alpha(H) \neq 0$ for all roots α), then

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G})$$
$$= \frac{\mathrm{Vol}(G)e^{4\pi^2 tB(\rho,\rho)^2}}{(4\pi t)^{\frac{\dim(G)}{2}}\sigma(H)} \sum_{\gamma \in CR} \left(\prod_{\alpha > 0} \langle \alpha, H + \gamma \rangle \right) e^{-\frac{1}{4t}B(H+\gamma, H+\gamma)}$$

where σ is the Weyl denominator.

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We can also take $H \to 0$, but the resulting expression for $\mathrm{Tr}(e^{t\Delta_G})$ is more complicated, see Èskin's paper.

Proof of Frenkel's formula

Frenkel proved his formula using the explicit spectral decomposition of Δ_G and the Weyl character formula. From these facts, we have

$$
\operatorname{Tr}(\lambda(e^H)\rho(e^K)e^{t\Delta_G})
$$
$$
= \sum_{\lambda \in CR_+^*} \frac{1}{\sigma(H)\sigma(K)} \left(\sum_{w \in W} \operatorname{sgn}(w)e^{w(\lambda+\rho)(H)} \right)
$$
$$
\cdot \left(\sum_{w \in W} \operatorname{sgn}(w)e^{w(\lambda+\rho)(K)} \right) e^{-tB(\lambda+2\rho,\lambda)}
$$

(here, ρ is the right-regular representation). Some rearrangement, Poisson's summation formula, and sending $K \to 0$ gives the result.

Remarks on Frenkel's formula

If we examine Frenkel's formula

$$\sum_{\lambda \in CR_+^*} \frac{1}{\sigma(H)\sigma(K)} \left(\sum_{w \in W} \text{sgn}(w) e^{w(\lambda+\rho)(H)} \right)$$

$$\cdot \left(\sum_{w \in W} \text{sgn}(w) e^{w(\lambda+\rho)(K)} \right) e^{-tB(\lambda+2\rho,\lambda)}$$

$$= \frac{\text{Vol}(G) e^{4\pi^2 t B(\rho,\rho)^2}}{(4\pi t)^{\frac{\dim(G)}{2}} \sigma(H)} \sum_{\gamma \in CR} \left(\prod_{\alpha>0} \langle \alpha, H+\gamma \rangle \right) e^{-\frac{1}{4t} B(H+\gamma, H+\gamma)}$$

We can see that a sum over the *weight lattice* is converted into a sum over the *coroot lattice*. The exponential $\exp(-tx^2)$ is converted to $\exp(-\frac{1}{t}x^2)$. (as if this were a Fourier transform).

Looking to Bismut

Bismut gave a new proof of Frenkel's formula, which is really a lot more complicated than Frenkel's proof. But it differs in some essential ways:

- Bismut does not use the Peter-Weyl decomposition of G. The proof involves very little information from the representation theory of G.

- The coroot lattice CR of G emerges in a purely geometric way, rather than as the dual of the weight lattice.

Section 2: The Primordial History, or, the Duistermaat–Heckman localisation formula

In the beginning...

In this section, I will attempt to give a "folkloric" story for the origins of the hypoelliptic Laplacian.

I am strongly indebtted to the paper of Choi–Takhtajan (2021) for this explanation.

The Hamiltonian-Lagrangian correspondence

In its most general terms, the path integral method in quantum mechanics relates traces of semigroups to integrals over loop space.

The way it is supposed to work is as follows: X is a manifold, and $\mathcal{L} \in C^\infty(TX)$ is a "Lagrangian". The corresponding "Hamiltonian" $\mathcal{H} \in C^\infty(T^*X)$ is related to $\mathcal{L}$ by the Legendre transform

$$\mathcal{H}(x, p) = \sup_{X \in T_x X} (p(X) - \mathcal{L}(x, X)), \quad (x, p) \in T^*X.$$

We are supposed to have something like the following:

$$\mathrm{Tr}(e^{-t\mathcal{H}(x,-i\partial)}) = \int_{L_t X} e^{-\int_0^t \mathcal{L}(\gamma(s),\dot\gamma(s))\,ds}\mathcal{D}\gamma. \qquad (1.1)$$

Here, $\mathcal{H}(x,-i\partial)$ is an operator on the Hilbert space $L_2(X)$ defined by some kind of quantisation of $\mathcal{H}$, $L_t X$ is the space of loops of "time" t, that is functions $\mathbb{R}/t\mathbb{Z} \to X$, and "$\mathcal{D}\gamma$" is some kind of measure on loop space.

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The Hamiltonian-Lagrangian correspondence

One case (and one of the few cases that is well-understood) is for Δ_g, the Laplace operator on a compact Riemannian manifold

$$\frac{1}{2}\Delta_g u = \frac{1}{2}\det(g)^{-\frac{1}{2}}\partial_\alpha(g^{\alpha,\beta}\det(g)^{\frac{1}{2}}\partial_\beta u)$$

This is supposed to be $\mathcal{H}(x, -i\partial)$, where $\mathcal{H}(x, p) = \frac{1}{2}g^{\alpha,\beta}(x)p_\alpha p_\beta$. The corresponding Lagrangian is

$$\mathcal{L}(x, X) = \frac{1}{2}g_{\alpha,\beta}(x)X^\alpha X^\beta$$

and

$$\mathrm{Tr}(e^{\frac{1}{2}t\Delta_g}) = \int_{L_t X} e^{-\frac{1}{2}\int_0^t g_{\alpha,\beta}(\gamma(s))\dot\gamma^\alpha(s)\dot\gamma^\beta(s)\,ds}\,\mathcal{D}\gamma$$

where the integral on the right-hand side can be rigorously defined as a limit of discrete approximations. See Andersson–Driver (1998).

Duistermaat–Heckman formula

Let (X, ω) be a symplectic n-manifold. The symplectic volume form of X is $\frac{1}{(n/2)!} \omega^{\frac{n}{2}} = \exp(\omega)$.

Let $H \in C^\infty(X)$, and suppose that H has isolated and non-degenerate critical points. Laplace's asymptotic formula tells us that as $t \to \infty$, we have

$$\int_X \exp(-tH) \frac{\omega^{n/2}}{(n/2)!} \sim \sum_p \frac{\exp(-tH(p))}{t^{n/2} \det^{\frac{1}{2}}(2\pi \nabla^2 H(p))}$$

Here the sum is over the critical points of H.

Duistermaat–Heckman formula

(X, ω) is a symplectic n-manifold.

Theorem

Suppose that there is a circle action $\mathbb{T} \times X \to X$ with isolated fixed points, and let its derivative be the vector field V. If V is Hamiltonian, with Hamiltonian function H, then

$$\int_X \exp(-tH) \frac{\omega^{n/2}}{(n/2)!} = \sum_p \frac{\exp(-tH(p))}{t^{n/2} \det^{\frac{1}{2}}(2\pi \nabla^2 H(p))}$$

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That is to say, for Hamiltonians coming from a circle action, the Laplace asymptotic expansion is exact.

Proof of Duistermaat–Heckman localisation

Duistermaat-Heckman can be proved by introducing a new parameter b in the following way: It is possible to construct a differential form $\mathcal{V} \in \Omega^{\bullet}(X)$ such that

$$\int_X \exp(-tH + \omega) = \int_Z \exp(-tH + \omega + b\mathcal{V}) \quad b > 0.$$

This is proved by showing that the right hand side is independent of b, and then setting $b = 0$. The limit as $b \to \infty$ is computed using Laplace's asymptotic formula.

Bismut's dream

We would like to compute $\mathrm{Tr}(e^{t\Delta_g})$ by considering

$$\int_{L_t X} e^{-\int_0^t \frac{1}{2} g_{\alpha,\beta}(\gamma(s))\dot{\gamma}^\alpha(s)\dot{\gamma}^\beta(s)\, ds} \mathcal{D}\gamma$$

as an integral of a differential form over the infinite dimensional manifold LX. Note that LX has a $\mathbb{T}$-action given by rotating around the loop. There are reasons to think that this should be a Hamiltonian action, with Hamiltonian function

$$\int_0^1 \frac{1}{2} g_{\alpha,\beta}(\gamma(s))\dot{\gamma}^\alpha(s)\dot{\gamma}^\beta(s)\, ds$$

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This theme was very popular in the 1980s – it led ultimately to Getzler's proof of the local index formula. We could try to mimic the proof of Duistermaat–Heckman by introducing a 1-parameter perturbation of $\mathcal{L}$, something like

$$\int_{LX} e^{-\int_0^t \mathcal{L}(\gamma(s),\dot{\gamma}(s))+b\mathcal{V}\,ds}\mathcal{D}\gamma$$

where $b > 0$, and computing the limit as $b \to \infty$.

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where $b > 0$, and computing the limit as $b \to \infty$. This is **TOO HARD**.

Bismut's dream

Doing Duistermaat–Heckman in loop space is a great idea but it's too difficult to attempt directly.

But maybe, we can use the Hamiltonian-Lagrangian correspondence to replace incomprehensible loop space integrals with slightly more comprehensible operator traces.

Can we find an operator $\mathcal{L}_b$ depending on a parameter b such that

$$\mathrm{Tr}(e^{t\Delta_g}) = \mathrm{Tr}(e^{t\mathcal{L}_b})$$

for all $b > 0$, and such that the limit as $b \to \infty$ expresses the trace as a sum over geodesics?

Answer: In a way, yes. Not precisely as stated above, but we can get close enough.

The hypoelliptic Laplacian

Let G be a compact Lie group, we consider Δ_G rather than the general Δ_g. Through some combination of path integral computations, intuition, and guesswork, Bismut found a solution the preceding question.

Theorem (Bismut (2011))

Let $H \in \mathfrak{t}$ be regular. There is a differential operator $\mathcal{L}_b$, depending on a positive real parameter b, acting on the space

$$C^\infty(\mathfrak{g} \times G, \Lambda^\bullet \mathfrak{g})$$

such that for $H \in \mathfrak{t}$ we have

$$\mathrm{Tr}(\lambda(e^H) e^{t(\Delta_G + C)}) = \mathrm{STr}(\lambda(e^H) e^{-t\mathcal{L}_b}), \quad b > 0.$$

where C is a scalar. As $b \to \infty$, it can be proved that $\mathrm{STr}(\lambda(e^H) e^{-t\mathcal{L}_b})$ converges to Frenkel's formula.

Section 3: The form of $\mathcal{L}_b$

Notation

G is a compact Lie group of dimension d, and $\mathfrak{g}$ is its Lie algebra with positive-definite bilinear form B. Let $e_1, \ldots, e_d$ be an orthonormal basis of $\mathfrak{g}$.

We will write the variables of $\mathfrak{g} \times G$ as (y, x), and we have Clifford operators on $\Lambda^\bullet \mathfrak{g}$, denoted

$$c(v) = v \wedge + \iota_v, \quad \widehat{c}(v) = v \wedge - \iota_v, \quad v \in \mathfrak{g}.$$

These satisfy the Clifford relations

$$c(v)c(w) + c(w)c(v) = 2B(v, w), \quad \widehat{c}(v)\widehat{c}(w) + \widehat{c}(w)\widehat{c}(v) = -2B(v, w).$$

Definition of $\mathcal{L}_b$.

It would be nice to have a heuristic derivation of $\mathcal{L}_b$ that follows from the path integral computations, but I am not convinced that such a thing exists. Instead, let's take Bismut at his word and assert the following definition.

Definition

$$\mathcal{L}_b = (D + \frac{1}{b}Q)^2 - D^2$$

where

$$Q = \sum_{j=1}^d c(e_j)y_j - i\widehat{c}(e_j)\partial_{y_j}$$

and

$$D = \sum_{j=1}^d -ic(e_j)\partial_{x_j} + \frac{1}{12}f_{k,l}^j c(e_j)c(e_k)c(e_\ell)$$

is the *Kostant Dirac operator* on G. Note that $D^2 = -\Delta_G + C$.

Structure of $\mathcal{L}_b$.

Expanding out the definitions, we get

$$\mathcal{L}_b = \frac{1}{b^2}Q^2 + \frac{1}{b}(QD + DQ)$$

which turns out to be

$$\mathcal{L}_b = \frac{1}{b^2}\sum_{j=1}^{d} y_j^2 - \partial_{y_j}^2 + \frac{1}{b}\sum_{j=1}^{d} y_j\partial_{x_j} + \text{matrix terms}.$$

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This operator satisfies Hörmander's condition, and is therefore hypoelliptic.

Section 4: The world's most convoluted proof of Frenkel's formula

Ultimately, we want to show the following facts:

- $\lim_{b \to 0} e^{-t\mathcal{L}_b} = e^{-tD^2}$ *in the correct sense*,
- $\lim_{b \to \infty} \mathrm{STr}(\lambda(e^H)e^{-t\mathcal{L}_b}) =$ Frenkel's formula.

The main tasks

Ultimately, we want to show the following facts:

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- $\lim_{b \to \infty} \mathrm{STr}(\lambda(e^H)e^{-t\mathcal{L}_b}) =$ Frenkel's formula.

Of course, to state these questions we need to determine if $\exp(-t\mathcal{L}_b)$ even exists and belongs to the trace class.

The limit as $b \to 0$.

In some sense, we are supposed to have

$$\exp(-t\mathcal{L}_b) \to \exp(-tD^2).$$

But how does this actually work? After all, they act on different spaces.

The projection onto ker(Q).

In order to understand how an operator on $L_2(\mathfrak{g} \times G, \Lambda^\bullet \mathfrak{g})$ converges to an operator on $L_2(G)$, we need to understand how $L_2(G)$ is a subspace of $L_2(\mathfrak{g} \times G, \Lambda^\bullet \mathfrak{g})$.

Solution: we use a Gaussian function on the $\mathfrak{g}$ fibre. This is like embedding $L_2(\mathbb{T})$ isometrically into $L_2(\mathbb{T} \times \mathbb{R})$ by mapping $f(x)$ to $f(x)e^{-\pi y^2}$.

Definition

Recall that Q is the Witten Dirac,

$$Q = \sum_{j=1}^{d} c(e_j)y_j - \widehat{c}(e_j)\partial_{y_j}.$$

Let P be the kernel projection of Q.

It happens that P is one-dimensional, with image a Gaussian function on the $\mathfrak{g}$ coordinates of $\mathfrak{g} \times G$. We will embed $L_2(G)$ into $L_2(\mathfrak{g} \times G, \Lambda^\bullet \mathfrak{g})$ by identifying $L_2(G)$ with

The 2×2 matrix trick

We want to prove that $\exp(-t\mathcal{L}_b)$ converges to $P\exp(t\Delta_G)P$. Actually it is simpler to work with resolvents, and prove that

$$(\lambda - \mathcal{L}_b)^{-1} \to P(\lambda - D^2)^{-1}P$$

for a suitably large set of $\lambda \in \mathbb{C}$. By some functional calculus, this is equivalent.

Idea: write the resolvent as a 2×2 matrix, with respect to the projection P.

The 2×2 matrix trick

Writing $(\lambda + \mathcal{L}_b)$ on $L_2(\mathfrak{g} \times G)$ with respect to the decomposition $\mathrm{im}(P) \oplus \ker(P)$, we have

$$\lambda + \mathcal{L}_b = \begin{pmatrix} \lambda & -\frac{1}{b}QD \\ -\frac{1}{b}DQ & \lambda - \frac{1}{b^2}Q^2 - \frac{1}{b}(QD + DQ) \end{pmatrix}.$$

Using the formula for the inverse of a 2×2 block matrix, we get

$$\lim_{b \to \infty} (\lambda - \mathcal{L}_b)^{-1} = \begin{pmatrix} (\lambda - D^2)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

in some operator topology.

The much more challenging problem we face is to understand what happens to $\mathrm{STr}(e^{-t\mathcal{L}_b})$ as $b \to \infty$. Note that:

1. Unlike with $b \to 0$, we have to take the supertrace first in order that the limit exists. This is analogous to the local index theorem.

2. This argument will be complicated, in part, because we have to do a simultaneous rescaling of the spatial and Clifford variables.

3. It is not possible to compute the limit of $\mathrm{STr}(e^{-t\mathcal{L}_b})$ in general, we need to introduce $\lambda(e^H)$.

Going into all the details with the rescaling of variables is too much to attempt here.... Instead I will briefly indicate what we're aiming for. What we expect is that

$$\lim_{b \to \infty} \mathrm{STr}(\lambda(e^H)e^{-t\mathcal{L}_b}) = \text{A sum over coroots of } G.$$

The left hand side is an integral over the kernel of the operator. How does an integral converge to a sum?

The b → ∞ argument

Let e be the identity element of G. We have

$$\text{STr}(e^{-t\mathcal{L}_b}) = \text{Vol}(G) \int_{\mathfrak{g}} \text{str}(\exp(-t\mathcal{L}_b)((e, Y), (e, Y)))\, dY.$$

(I am ignoring $\lambda(e^H)$ for now.) First, there is a (quite subtle!) change of variables and Getzler rescaling of the Clifford terms. There is an operator P_b, which I will not define, such that

$$\text{str}(e^{-t\mathcal{L}_b}) = \text{str}(e^{-tP_b})$$

The $b \to \infty$ argument

Theorem ((approximate))

Let ϕ_t be the geodesic flow. We have something like

$$|\exp(-tP_b)((e, Y), (e, Y))| \lesssim \exp(-\frac{c}{t}\text{dist}((e, Y), \phi_t(e, Y))^2).$$

So, the kernel will be small away from those points $Y \in \mathfrak{g}$ such that $\phi_t(e, Y) = (e, Y)$.

The $b \to \infty$ argument

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So, the kernel will be small away from those points $Y \in \mathfrak{g}$ such that $\phi_t(e, Y) = (e, Y)$. If $Y \in \mathfrak{t}$, these are precisely the co-roots!
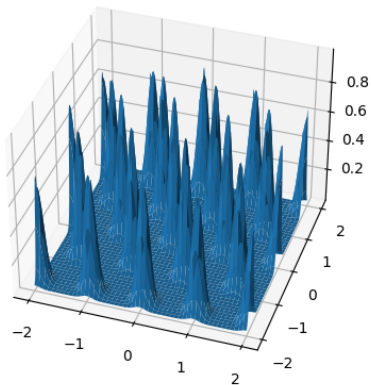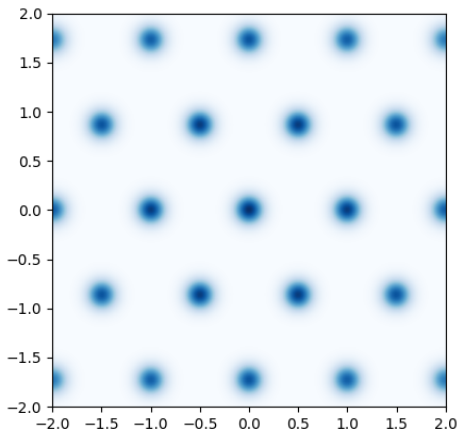
Theorem ((approximate))

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So, the kernel will be small away from those points $Y \in \mathfrak{g}$ such that $\phi_t(e, Y) = (e, Y)$. If $Y \in \mathfrak{t}$, these are precisely the co-roots! It happens that the $\lambda(e^H)$ term that I ignored is essential for localising the integral to $\mathfrak{t}$.

The plan for the argument

Somehow, this is all supposed to come together to deliver a new proof of Frenkel's formula.

Section 5: Analytic problems and their solutions

The problems we face

The main analytic difficulties we need to deal with are proving *subelliptic estimates*

$$\|u\|_{s+\varepsilon} \lesssim \|\mathcal{L}_b u\|_s + \|u\|_s, \quad \|v\|_{s+\varepsilon} \lesssim \|(\partial_t - \mathcal{L}_b)v\|_s + \|v\|_s$$

where $\|\cdot\|_s$ is a scale of Sobolev spaces on $\mathfrak{g} \times G$ or $\mathfrak{g} \times G \times \mathbb{R}$. These are needed to prove that the exponential exists is at least trace-class, and also to establish the regularity of the kernel of $\exp(-t\mathcal{L}_b)$.

The problems we face

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These are needed to prove that the exponential exists is at least trace-class, and also to establish the regularity of the kernel of $\exp(-t\mathcal{L}_b)$.
We also need some way to obtain estimates for

$$|\exp(-tP_b)((e, Y), (e, Y))|$$

which exhibit the desired concentration on the co-roots. I will concentrate on this latter problem.

The method of Gaffney and Gårding

I will illustrate the method we have with a very simple case: consider a diffusion operator

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where the X_j's are antisymmetric vector fields satisfying the Hörmander condition on some open set $U \subset \mathbb{R}^N$.

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where the X_j's are antisymmetric vector fields satisfying the Hörmander condition on some open set $U \subset \mathbb{R}^N$. I will sketch out an argument that for $p, q \in U$ we have

$$\exp(tK)(p,q) \lesssim \exp(-\frac{c}{t}\mathrm{dist}_{cc}(p,q)^2)$$

where dist_{cc} is the Carnot-Caratheodory distance defined by $\{X_j\}$.

Energy methods

Let $u_0 \in L_2(U)$, and $u_t := \exp(tK)u_0$. A simple computation yields the following:

Lemma

Let $\phi \in C^\infty(U)$. We have

$$\frac{d}{dt}\|\phi u_t\|^2 \leq 2\sum_j \|X_j(\phi)u_t\|^2.$$

Gaffney and Gårding's argument is to choose ϕ in an ingenious way.

The ingenious choice of ϕ.

In particular, for $\phi = \exp(s\psi)$ where s is a parameter to be chosen later, we have

$$\frac{d}{dt}\|\exp(s\psi)u_t\|^2 \leq 2s^2\|\exp(s\psi)u_t\|^2 \left(\sum_j \|X_j(\psi)\|_\infty^2\right).$$

By Gronwall's inequality, it follows that

$$\|\exp(s\psi)u_t\| \leq \|\exp(s\psi)u_0\|\exp(s^2 t^2 \sum_j \|X_j(\psi)\|_\infty^2).$$

The ingenious choice of ϕ.

Now select smooth functions ν and μ such that $u_0 = \nu u_0$, $\exp(s\psi)\nu = \nu$ and $\exp(s\psi)\mu = \exp(s)\mu$. It follows that

$$\|\mu u_t\| = \exp(-s)\|\exp(s\psi)\mu u_t\| \le \|\mu\|_\infty \exp(-s + s^2 t k^2)\|\nu u_0\|.$$

here,

$$k^2 = \sum_j \|X_j(\psi)\|_\infty^2.$$

If we select $s = \frac{1}{2tk^2}$, this minimises $s^2 t k^2 - s$, and we get

$$\|\mu u_t\| \le \exp(-\frac{1}{4t}k^{-2})\|u_0\|.$$

The infimum of k is equivalent to one over the Carnot–Caratheodory distance between the supports of ν and μ. Hence,

$$\|\mu \exp(tK)\nu\|_{L_2 \to L_2} \leq \exp(-\frac{d^2}{4t})\|\mu\|\|\nu\|.$$

where d is the Carnot–Caratheodory distance between the supports of ν and μ.

Complex interpolation

From the L_2-estimate and an argument using Sobolev embedding and hypoellipticity, we arrive at

$$\exp(tK)(p, q) \lesssim \exp(-\frac{c}{t}\mathrm{dist}_{cc}(p, q)^2).$$

Something much more interesting happens if we instead analyse an operator like

$$K = \sum_j X_j^2 + Z$$

where Z is a new vector field. We can still do the argument, but we need to choose ϕ which moves along with the flow of Z. This is how we arrive at the "sub-Gaussian bounds with flow" needed for the concentration of the kernel of $\exp(-tP_b)$ on the co-roots.

Section 6: Conclusions

What next?

We have succeeded in proving Frenkel's formula by replacing some elementary representation theory and Poisson's summation formula by the most convoluted arguments imaginable.

We can all agree that having new proofs of old results is worthwhile in itself, but... **What is the point of all this?**

1. Bismut's proof is *totally different* to Frenkel's proof. Strikingly, *no representation theory is needed*. We didn't even need the theorem of the highest weight. The coroot lattice turned up in the sum for completely geometric reasons.

2. Bismut eventually extended his arguments to give totally new results, including his formulas for the heat kernel on symmetric spaces.

Thank you for listening!

Happy Birthday Nigel!