Introduction to the Hypoelliptic Laplacian on a compact group

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But... where does it come from?
What is it good for?
And, most importantly, how do we prove things about it?

Section 1: Frenkel's formula

Let G be a connected, simply connected, compact group. There is the Casimir element, $\Delta_G \in \mathcal{U}(\mathfrak{g})$.
What is $\mathrm{Tr}(e^{t\Delta_G})$?
Some possible answers:

1. Spectral (Peter-Weyl) representation: we have

$$\mathrm{Tr}(e^{t\Delta_G}) = \sum_{\lambda \in CR_+^*} \left(\prod_{\alpha>0} \frac{\langle \alpha, \rho+\lambda \rangle}{\langle \alpha, \rho \rangle} \right)^2 e^{-tB(\lambda+2\rho,\lambda)}$$

   where W_+ is the weight lattice, $\prod_{\alpha>0}$ is the product over positive roots, ρ is the Weyl element, and B is the bilinear pairing on $\mathfrak{g}^*$.

2. Minakshisundaram-Pleijel expansion: asymptotically, as $t \to 0$, we have

$$\mathrm{Tr}(e^{t\Delta_G}) \sim (4\pi t)^{-\frac{\dim(G)}{2}} (a_0 + a_1 t + a_2 t^2 + \cdots)$$

   where a_0, a_1, etc. are determined by the geometry of G.

Is there another (more insightful) representation?

Jacobi identity

If $G = \mathbb{T}$ is the unit circle, the answer is *yes*. We have

$$\mathrm{Tr}(e^{t\Delta_{\mathbb{T}}}) = \mathrm{Vol}(\mathbb{T})(4\pi t)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-\frac{1}{4t}n^2}.$$

We can prove this via the Poisson summation formula and the fact that

$$\mathrm{Tr}(e^{t\Delta_{\mathbb{T}}}) = \sum_{n\in\mathbb{Z}} e^{-tn^2}$$

Is there something similar for a general Lie group?

Frenkel's formula

It turns out that $\mathrm{Tr}(e^{t\Delta_G})$ is too difficult to compute. Instead, we can compute a "shifted" version

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G})$$

where $H \in \mathfrak{t}$, and $\lambda(e^H)$ is the left shift operator by $e^H \in G$.

Theorem (Frenkel (1984))

If $H \in \mathfrak{t}$ is regular (i.e., $\alpha(H) \neq 0$ for all roots α), then

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G}) = \frac{\mathrm{Vol}(G)e^{4\pi^2 t|\rho|^2}}{(4\pi t)^{\frac{\dim(G)}{2}}\sigma(H)} \sum_{\gamma \in CR} \left(\prod_{\alpha > 0} \langle 2\pi\alpha, H + \gamma \rangle \right) e^{-\frac{1}{4t}|H+\gamma|^2}$$

where σ is the Weyl denominator.

Proof of Frenkel's formula

Frenkel's formula is proved using the explicit spectral decomposition of Δ_G and the Weyl character formula. From these facts, we have

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G}) = \frac{1}{\sigma(H)} \sum_{\lambda \in CR_+^*} \sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)(H)} e^{-tB(\lambda+2\rho,\lambda)}$$

Some basic facts from representation theory and Poisson's summation formula give the proof.

Section 2: The Primordial History, or, the Berligne-Vergne localisation formula

The Hamiltonian-Lagrangian correspondence

In its most general terms, the path integral method in quantum mechanics relates traces of semigroups to integrals over loop space.
The way it is supposed to work is as follows: X is a manifold, and $\mathcal{L} \in C^\infty(TX)$ is a "Lagrangian". The corresponding "Hamiltonian" $\mathcal{H} \in C^\infty(T^*X)$ is related to $\mathcal{L}$ by the Legendre transform

$$\mathcal{H}(x,p) = \sup_{X \in T_xX} (p(X) - \mathcal{L}(x,X)), \quad (x,p) \in T^*X.$$

The Hamiltonian-Lagrangian correspondence

We are supposed to have something like the following:

$$\mathrm{Tr}(e^{-t\mathcal{H}(x,-i\partial)}) = \int_{LX} e^{-\int_0^t \mathcal{L}(\gamma(s),\gamma'(s))\,ds}\mathcal{D}\gamma. \tag{2.1}$$

Here, $\mathcal{H}(x,-i\partial)$ is an operator on the Hilbert space $L_2(X)$ defined by some kind of quantisation of $\mathcal{H}$, LX is the space of loops, that is functions $S^1 \to X$, and "$\mathcal{D}\gamma$" is some kind of measure on loop space.
Both sides of this formula are problematic, but the left-hand-side is a bit more accessible.

The Hamiltonian-Lagrangian correspondence

One case (and one of the few cases that is well-understood) is for Δ_g, the Laplace operator on a compact Riemannian manifold

$$\frac{1}{2}\Delta_g u = \frac{1}{2}\det(g)^{-\frac{1}{2}}\partial_\alpha(g^{\alpha,\beta}\det(g)^{\frac{1}{2}}\partial_\beta u)$$

This is supposed to be $\mathcal{H}(x, -i\partial)$, where $\mathcal{H}(x,p) = \frac{1}{2}g^{\alpha,\beta}(x)p_\alpha p_\beta$. The corresponding Lagrangian is

$$\mathcal{L}(x, X) = \frac{1}{2}g_{\alpha,\beta}(x)X^\alpha X^\beta$$

and

$$\mathrm{Tr}(e^{\frac{1}{2}t\Delta_g}) = \int_{LX} e^{-\int_0^t \frac{1}{2}g_{\alpha,\beta}(\gamma(s))\dot{\gamma}^\alpha(s)\dot{\gamma}^\beta(s)\,ds}\,\mathcal{D}\gamma$$

where the integral on the right-hand side is rigorously defined as a limit of natural discrete approximations. See Anderson-Driver.

Berligne-Verne localisation

Suppose that we're trying to compute the integral of a differential form α over a manifold Z, like

$$\int_Z \alpha, \quad \alpha \in \Omega^{\mathrm{top}}(Z).$$

There is a very effective trick for computing this integral if we have some $\mathbb{T}$-action: Assume that $\mathbb{T}$ acts on Z, and α is invariant under this action. We have

Theorem (Berligne-Verne localisation)

$$\int_Z \alpha = \int_{Z^0} \frac{\alpha}{e(N(Z/Z^0))}$$

where Z^0 is the fixed submanifold under the $\mathbb{T}$-action, and $e(N(Z/Z^0))$ is the Euler class of the normal bundle to Z^0 in Z.

What is of greater interest to us is actually the standard proof of the Berligne-Verne formula.

Proof of Berligne-Verne localisation

Berligne-Verne localisation can be proved in the following way. It is possible to find a $\mathbb{T}$-invariant differential form $\mathcal{V} \in \Omega^2(X)$ such that

$$\int_Z \alpha = \int_Z \alpha e^{ib\mathcal{V}}, \quad b > 0.$$

This is proved by showing that the right hand side is independent of b, and then setting $b = 0$. The limit as $b \to \infty$ is computed using the method of stationary phase.

Bismut's dream

We would like to compute $\mathrm{Tr}(e^{t\Delta_g})$ by considering

$$\int_{LX} e^{-\int_0^t \mathcal{L}(\gamma(s),\dot{\gamma}(s))\,ds}\mathcal{D}\gamma$$

as an integral of a differential form over the infinite dimensional manifold LX.

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$$\int_{LX} e^{-\int_0^t \mathcal{L}(\gamma(s),\dot{\gamma}(s))+b\mathcal{V}\,ds} \mathcal{D}\gamma$$

where $b > 0$, and computing the limit as $b \to \infty$.

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where $b > 0$, and computing the limit as $b \to \infty$. This is **TOO HARD**.

Bismut's dream

Doing Berligne-Verne in loop space is too difficult, the technicalities in making sense of the path integral are too great.
But maybe, we can use the Hamiltonian-Lagrangian correspondence to replace incomprehensible loop space integrals with slightly more comprehensible operator traces.

Can we find an operator $\mathcal{L}_b$ depending on a parameter b such that

$$\mathrm{Tr}(e^{t\Delta_g}) = \mathrm{Tr}(e^{t\mathcal{L}_b})$$

for all $b > 0$, and such that the limit as $b \to \infty$ gives an interesting formula for the trace?

Answer: In a way, yes. Not precisely as stated above, but we can get close enough.

The hypoelliptic Laplacian

Let G be a compact Lie group, and let $H \in \mathfrak{t}$ be regular.

Theorem

There is a differential operator $\mathcal{L}_{b,H}$, depending on a positive real parameter b, acting on the space

$$C^\infty(\mathfrak{g} \times G, \bigwedge^{\bullet} \mathfrak{g})$$

such that

$$\mathrm{Tr}(\lambda(e^H)e^{t\Delta_G}) = \mathrm{STr}(e^{-t\mathcal{L}_{b,H}}), \quad b > 0.$$

As $b \to \infty$, it can be proved that $\mathrm{STr}(e^{-t\mathcal{L}_{b,H}})$ converges to Frenkel's formula.

Thank you for listening!

Happy Birthday Nigel!