The Hausdorff measure of Julia sets from singular traces

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What is this talk?

This talk concerns the paper, *The conformal trace theorem for Julia sets of quadratic polynomials* (ETDS, 2017) from myself, A Connes, F Sukochev and D Zanin.

What this talk is about

Part I will cover:

1. A brief introduction to complex polynomial dynamics and Julia sets
2. Geometric measure theory (specifically, Hausdorff meausures)

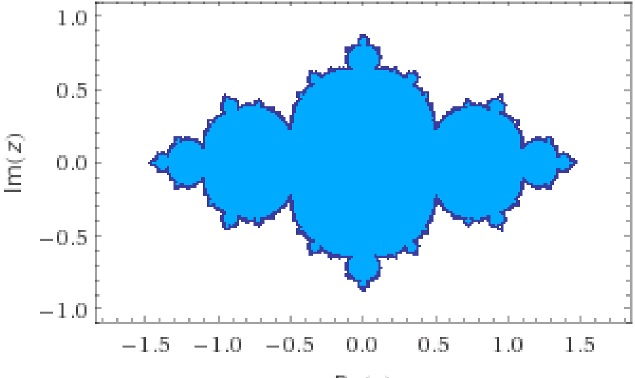
Part II will cover:

1. Statement of the conformal trace theorem, and an outline the proof
2. Prospects for future work.

Historical background: What is the conformal trace theorem?

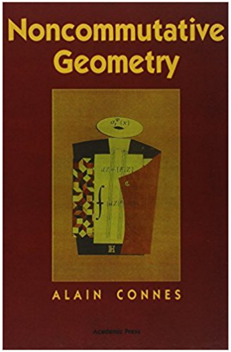
Let $c \in \mathbb{C}$ be small, and let $f_c(z) = z^2 + c$. The Julia set J of f_c is the boundary of the set of points z such that $\{f_c^n(z)\}_{n\geq 0}$ is bounded.

When $c \approx 0$, J is a Jordan curve:



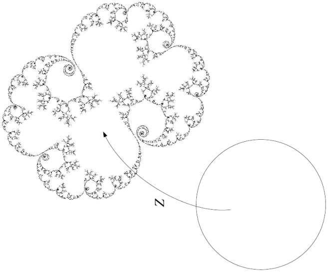
Background (continued)

In 1994 in his book *Noncommutative Geometry*, A. Connes introduced a formula for the Hausdorff measure of a Julia set in terms of his quantised calculus:



Background (continued)

To state the formula properly takes some work, but the key ingredients are as follows: take a Julia set J and let $Z : \mathbb{T} \to J$ be the extension to the boundary of the conformal equivalence between the exterior of the unit disk and the exterior of J:



Z is typically not differentiable, and it is typically not even of bounded variation.

Background (continued)

The conformal trace theorem then states that for all continuous normalised traces φ, there is a constant C_φ such that:

$$c_\varphi \int_J f \, d\lambda_p = \varphi(M_{f\circ Z}|đZ|^p).$$

where p is the Hausdorff dimension of J and λ_p is the p-dimensional Hausdorff measure. Connes and Sullivan introduced this formula as a way of computing integrals with respect to the Hausdorff measure on Julia sets.

Background (continued)

Despite being announced as early as 1994, Connes and Sullivan's proof of the conformal trace theorem was never published.
In our paper we provided a new proof, using operator integration techniques which did not exist in 1994.

Background (end)

The most technically challenging part of the proof is the so-called "quantised change of variables formula":

$$|[F, M_{f\circ Z}]|^p - |f'(M_Z)|^p |[F, M_Z]|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Where $Z \in C(\mathbb{T})$, and f is a polynomial. It is quite easy to show that:

$$[F, M_{f\circ Z}] - f'(M_Z)[F, M_Z] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$$

but "taking a power p" is highly nontrivial.

Part I: Basic conformal dynamics

Complex polynomial dynamics

Let f be a polynomial with complex coefficients, and take $z_0 \in \mathbb{C}$. consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ for different choices of $z_0 \in \mathbb{C}$. In particular, since f is a polynomial, exactly one of the following happens:

1. Either $|z_n| \to \infty$.
2. $\{z_n\}_{n\geq 0}$ remains bounded.

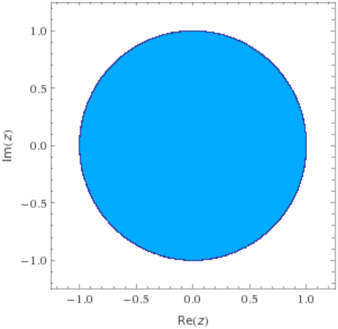
Complex polynomial dynamics

The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = {z_0}^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

1. If $|z_0| < 1$, then $f^k(z_0) \to 0$ as $k \to \infty$.
2. If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
3. If $|z_0| > 1$, then $|f^k(z_0)| \to \infty$ as $k \to \infty$.

Complex polynomial dynamics

Pictorially,



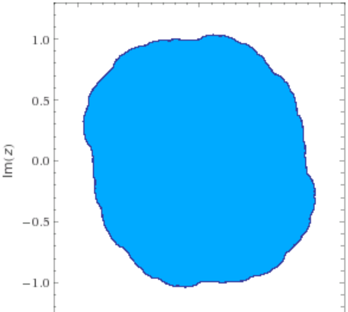
Here, the set of z_0 such that z_n remains bounded is coloured in blue. The set of z_0 such that z_n is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly? Consider $f(z) = z^2 + 0.1 + 0.1i$.

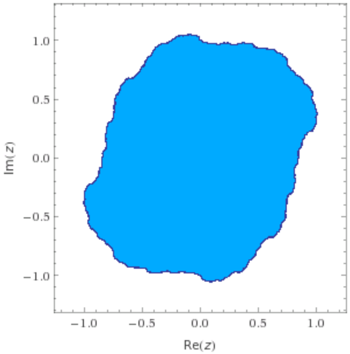
It is not feasible to determine analytically the behaviour of $\{f^n(z)\}_{n\geq 0}$. Instead we use a computer: On a large grid of complex numbers, colour each point z blue if $f^N(z) < 10$ for some suitably large number N.

The result looks like this:



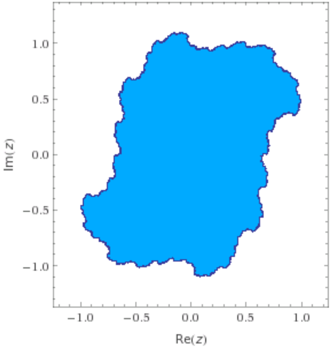
Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



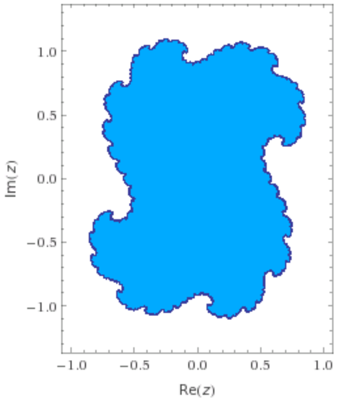
Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



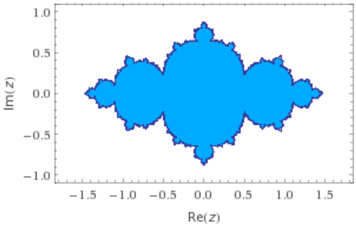
Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



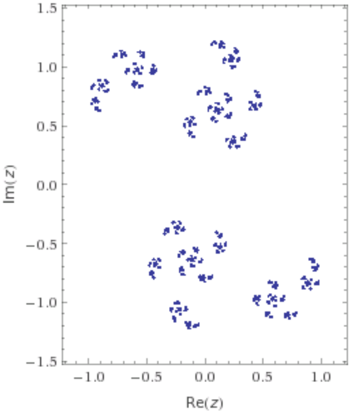
Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



Complex polynomial dynamics

Let try a slightly bigger parameter. Consider
$f(z) = z^2 + 0.5 + 0.5i$,



The Mandelbrot set

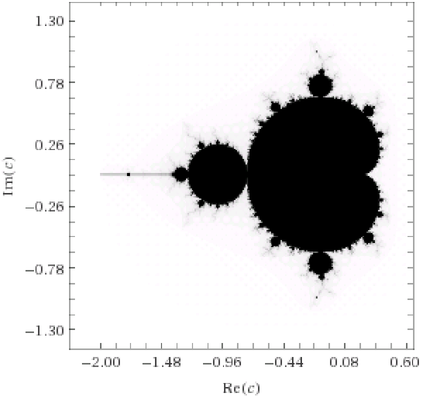
What is going on here? Consider the general polynomial:

$$f_c(z) := z^2 + c$$

with a parameter $c \in \mathbb{C}$. Note: any quadratic polynomial can be transformed into some f_c by an affine change of coordinates.

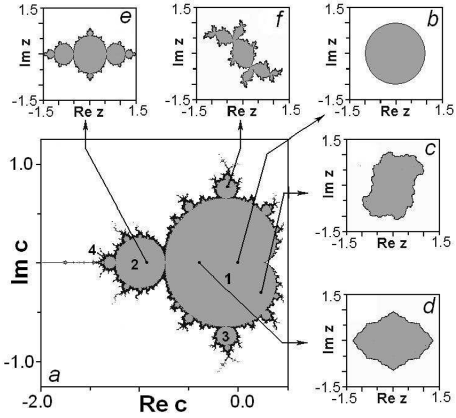
The Mandelbrot set

Consider the case $z_0 = 0$ (for simplicity). For which c is $\{f_c^k(0)\}_{k\geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} \ : \ \{f_c^k(0)\}_{k\geq 0}$ is bounded$\}$. M can be approximated by a computer:



The Mandelbrot set

A more informative image is this one:



The Julia set

Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The *Julia set* of f_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

Theorem (Mandelbrot)

The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).

Fixed points

The asymptotic behaviour of $\{f^n(z_0)\}_{n\geq 0}$ is best understood by examining the fixed points ($f(z) = z$) of f.
A fixed point λ is said to be:

1. Attracting if $|f'(\lambda)| < 1$,
2. Repelling if $|f'(\lambda)| > 1$,
3. Neutral if $|f'(\lambda)| = 1$.

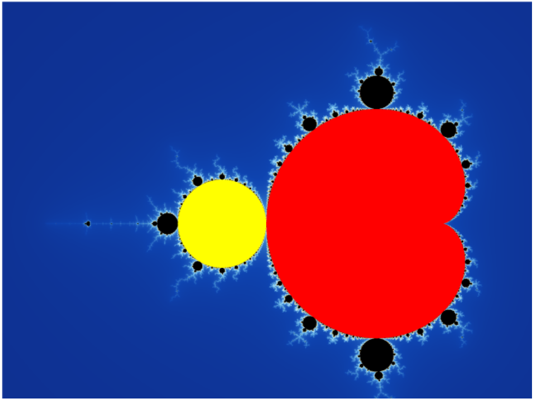
An attracting fixed point λ can be called *super-attracting* if $f'(\lambda) = 0$.

Fixed points (cont.)

The behaviour of $\{f^n(z)\}_{n\geq 0}$ near an attracting fixed point is easily described:
If z is sufficiently close to an attracting fixed point λ, then $f^n(z) \to \lambda$ exponentially fast.
If λ is superattracting, then $f^n(z) \to \lambda$ super-exponentially fast.
The set of all z such that $f^n(z) \to \lambda$ as $n \to \infty$ is called the attracting basin of λ. It is easy to see that an attracting basin is open.

The main cardioid

When does $f(z) = z^2 + c$ have an attracting fixed point? Solve for c: $z^2 - z + c = 0$, $|2z| < 1$.
Let M_0 be the set $\{\frac{z}{2}(1 - \frac{z}{2}) \ : \ |z| < 1\}$. M_0 is an open subset of the Mandelbrot set M called the *main cardioid*, shown below in red:



The main cardioid

The significance of the main cardioid is the following theorem:

Theorem

The Julia set $J(f_c)$ of f_c is a Jordan curve (i.e. homeomorphic to a circle) if and only if c is in the main cardioid M_0.

Very rough outline of the proof:

The connected components of $\mathbb{C} \setminus J(f_c)$ correspond to the attracting basin of the fixed points of f_c. For $c \in M_0$, there are exactly two attracting fixed points (one of them at infinity). $\square$

Hausdorff dimension (reminder)

Let (X,d) be a metric space. For $q>0$, let the q-dimensional Hausdorff content of X be:

$$C_q(X)=\inf\{\sum_{j=0}^{\infty} r_j^q \ : \ X\subseteq \bigcup_{j=0}^{\infty} B(x_j,r_j)\}.$$

The Hausdorff dimension of X is defined to be the infimum of the set of q such that $C_q(X)=0$.

Hausdorff dimension of Julia sets

Fact: If c is in the main cardioid M_0 of the Mandelbrot set M, then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.

Minkowski content

A close relative of Hausdorff dimension is Minkowski dimension. Let $X \subseteq \mathbb{R}^d$. If $\delta > 0$, let $S_\delta(X) = \bigcup_{x \in X} B(x, \delta)$, or equivalently:

$$S_\delta(X) = \{z \in \mathbb{R}^d \ : \ \mathrm{dist}(z, X) < \delta\}.$$

If there is some $p > 0$ such that:

$$c\delta^{d-p} \leq |S_\delta(X)| \leq C\delta^{d-p}$$

for all $\delta > 0$, then X is said to have Minkowski dimension p. If $0 < c < C < \infty$, then X is said to have finite upper and positive lower p-dimensional Minkowski content.

Conformal dimension

Another closely related notion of dimension is conformal dimension (due to Sullivan).
Let $X \subseteq \mathbb{C}$, with $f : X \to X$, and ν a measure on X. The measure ν is said to be f-conformal with dimension p, if for every open subset $U \subseteq X$ such that $f|U$ is injective, we have:

$$\nu(f(U)) = \int_U |f'|^p \, d\nu.$$

Characterising the Hausdorff measure

Fact: For Julia sets $J(f_c)$, $c \in M_0$, we have:
Hausdorff dimension $=$ Minkowski dimension
and the Hausdorff measure is the unique f_c-conformal measure of dimension equal to p, where p is the Hausdorff dimension.
Using the Riesz theorem, we can characterise the Hausdorff measure as follows:

Theorem

The Hausdorff measure λ_p of $J(f_c)$ is the unique positive Borel measure such that for all open sets $U \subseteq J(f_c)$ with $f_c|U$ injective, and continuous functions g supported in U, we have:

$$\int_{f_c(U)} g \, d\lambda_p = \int_U (g \circ f_c)|f_c'|^p \, d\lambda_p.$$

Characterising the Hausdorff measure (cont.)

Since $f_c(z) = z^2 + c$ we have:

$$f_c^{-1}(f_c(U)) = U \cup -U.$$

We can moreover say the following:

Theorem

The Hausdorff measure λ_p is the unique (up to a constant scalar factor) positive Borel measure such that for all continuous functions g on $J(f_c)$,

$$\int_{J(f_c)} g \, d\lambda_p = \frac{1}{2} \int_{J(f_c)} (g \circ f_c) |f_c'|^p \, d\lambda_p.$$

When we prove the conformal trace theorem, we use the above characterisation of the Hausdorff measure.

Part II: The Conformal trace theorem

The main task of the paper

Let $g : J(f_c) \to \mathbb{C}$ be a continuous function. In his 1994 book *Noncommutative geometry*, Alain Connes announced a formula for $\int_J g \, d\lambda_p$ given in terms of his "quantised calculus". We have now completed the proof of this formula.

The Hilbert transform

Let $\mathbb{T}$ be the unit circle (in the complex plane). The Hilbert space $L_2(\mathbb{T})$ is defined with respect to the arc-length measure (the Haar measure).
There is the trigonometric orthonormal basis for $L_2(\mathbb{T})$,

$$e_n(z) = z^n, \quad n \in \mathbb{Z}, z \in \mathbb{T}.$$

The Hilbert trasform F is defined on the basis e_n by
$Fe_n = \mathrm{sgn}(n)e_n$.

Quantised differentials

If f is a bounded function on $\mathbb{T}$, then pointwise multiplication by f defines a bounded linear operator M_f on $L_2(\mathbb{T})$. Connes calls the commutator $i[F, M_f]$ the "quantised differential" of f.
The name is intended to imply that this is something like a differential df. So we use the symbol $\not{d}f$,

$$\not{d}f := i[F, M_f].$$

Böttcher coordinates

The following is due to L. Böttcher:

Theorem

Let f be a polynomial of degree $d \geq 2$. There exists a conformal map Z:

$$Z : \{|z| > 1\} \to \textit{Attracting basin of } \infty.$$

such that $f(Z(z)) = Z(z^d)$, for all $|z| > 1$.

If $J(f_c)$ is a Jordan curve, then Carathéodory's theorem implies that Z has continuous extension:

$$Z : \mathbb{T} \to J(f_c)$$

such that $Z(z^2) = f_c(Z(z))$ for all $z \in \mathbb{T}$.

Description of the Conformal Trace Formula

Let f be a continuous function on the Julia set $J(f_c)$. Then the operator

$$M_{f\circ Z}|đZ|^p = M_{f\circ Z}|[F, M_Z]|^p$$

is some kind of "p-dimensional density" on the Julia set $J(f_c)$ (in the language of quantised calculus).
Motivated by noncommutative geometry, one might guess that the correct way of "integrating" this density is to take a trace.

Description of the Conformal Trace Formula

Lemma

The quantised differential $đZ$ (i.e., the commutator $[F, M_Z]$) is in the ideal $\mathcal{L}_{p,\infty}$.

Hence, the operator $M_{f\circ Z}|đZ|^p$ is in $\mathcal{L}_{1,\infty}$.

Description of the Conformal Trace Formula

Theorem

Let φ be a continous trace on $\mathcal{L}_{1,\infty}$. Then there is a constant $K(\varphi, c)$ such that for all $f \in C(J(f_c))$,

$$\varphi(M_{f\circ Z}|đZ|^p) = K(\varphi, c) \int_{J(f_c)} f \, d\lambda_p$$

where λ_p is the p-dimensional Hausdorff measure on $J(f_c)$. Also, there exist traces φ such that $K(\varphi, c) > 0$.

Equivalent statement of the Conformal trace theorem

Recall that the Hausdorff measure can be characterised as the unique f_c-conformal measure of dimension p.
Let ℓ_φ be the linear functional:

$$\ell_\varphi(g) = \varphi(M_{g\circ Z}|\bar{d}Z|^p).$$

By the Riesz theorem, there is some measure ν such that ℓ_φ is integration against ν. It therefore suffices to show the following:

1. For all continuous $g : J(f_c) \to \mathbb{C}$, we have:

$$\ell_\varphi(g) = \frac{1}{2}\ell_\varphi((g\circ f_c)|f_c'|^p).$$

2. $\ell_\varphi > 0$, for at least some φ.

Non-triviality (cont.)

First, we can show the "non-triviality" component.

Theorem

Let $\mathcal{C}$ be a Jordan curve with finite upper and positive lower p-Minkowski content, and let $\zeta : \mathbb{T} \to \mathcal{C}$ be the continuous extension of a conformal equivalence of the open unit disk and the interior of $\mathcal{C}$. Then $đ\zeta \in \mathcal{L}_{p,\infty}$, and for all dilation invariant extended limits ω such that $\omega \circ \log$ is still dilation invariant, we have:

$$\mathrm{tr}_\omega(|đ\zeta|^p) > 0.$$

Non-triviality (cont.)

Idea of the proof: If ω satisfies the stated conditions, then:

$$\mathrm{tr}_\omega(|\bar{d}\zeta|^p) = (\omega \circ \log)\left(t \mapsto \frac{1}{t}\mathrm{tr}(|\bar{d}\zeta|^{p(1+1/t)})\right)$$

So it suffices to show that:

$$\liminf_{s\to 0} s \cdot \mathrm{tr}(|\bar{d}\zeta|^{p+s}) > 0.$$

Non-triviality (cont.)

A theorem of Peller gives us the equivalence:

$$\mathrm{tr}(|\bar{d}\zeta|^{p+s}) \approx \int_{\mathbb{D}} |\zeta'(z)|^{p+s}(1-|z|^2)^{p+s-2}\, dz d\bar{z}.$$

According to the Koebe 1/4-theorem,

$$\mathrm{dist}(\zeta(0), \mathcal{C}) \leq |\zeta'(0)| \leq 4\mathrm{dist}(\zeta(0), \mathcal{C})$$

A change of coordinates yields:

$$\frac{1}{4}(1-|z|^2)|\zeta'(z)| \leq \mathrm{dist}(\zeta(z), \mathcal{C}) \leq (1-|z|^2)|\zeta'(z)|$$

for all $z \in \mathbb{D}$.

Non-triviality (cont.)

In summary:

$$\mathrm{tr}(|\bar{d}\xi|^{p+s}) \approx \int_{\mathrm{int}\mathcal{C}} \mathrm{dist}(z,\mathcal{C})^{p+s-2}\, dzd\bar{z}.$$

The question is now reduced to a purely geometric problem concerning $\mathcal{C}$.

Non-triviality (cont.)

Split up $\mathrm{int}\mathcal{C}$ into regions:

$$A_k = \{z \in \mathrm{int}\mathcal{C} \ : \ \mathrm{dist}(z, \mathcal{C}) \in [\lambda^{-k-1}, \lambda^{-k})\}.$$

Then,

$$\int_{\mathrm{int}(\mathcal{C}} \mathrm{dist}(z, \mathcal{C})^{p+s-2}\, dz d\bar{z} \geq \sum_{k \geq 0} \lambda^{-k(p+s-2)} |A_k|.$$

with $|A_k| = |S_{\lambda^{1-k}}(\mathcal{C}) \cap \mathrm{int}(\mathcal{C})| - |S_{\lambda^{-k}}(\mathcal{C}) \cap \mathrm{int}(\mathcal{C})| \geq B\lambda^{-k(2-p)}$
for sufficiently big λ.
This gives,

$$\liminf_{s \to 0} s \cdot \mathrm{tr}(|\bar{d}\zeta|^{p+s}) > 0.$$

Proving conformality

The remaining task is to show that:

$$\ell_\varphi(g) = \frac{1}{2}\ell_\varphi((g\circ f_c)|f_c'|^p).$$

Recall that we have the Böttcher equation:

$$Z(z^2) = f_c(Z(z)).$$

Let U be the partial isometry on $L_2(\mathbb{T})$ given by $U(z^n) = z^{2n}$.
Then,

$$UM_Z U^* = M_{f_c\circ Z}.$$

we also have $U^*U = 1$ and $UU^* = P$ is the projection $P(z^n) = z^n$ if n is even and 0 if n is odd.
It is important that U commutes with the Hilbert transform F.

Proving conformality (cont.)

Then,

$$\begin{aligned}\varphi(M_{g\circ Z}|\eth Z|^p) &= \varphi(U^*UM_{g\circ Z}|\eth Z|^p)\\ &= \varphi(M_{g\circ f_c\circ Z}U|[F,M_Z]|^pU^*)\\ &= \varphi(M_{g\circ Z}|[F,M_{f_c\circ Z}]|^pUU^*)\\ &= \varphi(M_{g\circ f_c\circ Z}|\eth(f_c\circ Z)|^pP).\end{aligned}$$

A further argument using unitary invariance shows that:

$$\varphi(M_{g\circ f_c\circ Z}|\eth(f_c\circ Z)|^pP) = \frac{1}{2}\varphi(M_{g\circ f_c\circ Z}|\eth(f_c\circ Z)|^p)$$

The following is called by Connes a “quantised chain rule”. Let f be a polynomial. Then:

$$|\bar{d}(f \circ Z)|^p - |f'(M_Z)|^p |\bar{d}Z|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Proving the above identity is the most technically challenging part of the proof, requiring the operator integration tricks developed in the preceding talk.

The quantised chain rule (cont.)

We have,

$$[M_f, [F, M_Z]] = [[M_f, F], M_Z]$$

Since f is a polynomial, the commutator $[M_f, F]$ is finite rank. Thus, we have:

$$M_{z^n}[F, M_Z]M_{z^{-n}} - [F, M_Z] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}.$$

This (highly nontrivally) implies that:

$$|M_{z^n}[F, M_Z]M_{z^{-n}}|^p - |[F, M_Z]|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Since M_{z^n} is unitary, we get:

$$[M_{z^n}, |[F, M_Z]|^p] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

The quantised chain rule (cont.)

Hence for all polynomials f,

$$[M_f, |đZ|^p] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

An approximation argument allows us to state the above for all continuous f as well.
Fact: we also have $[M_f, |đZ|] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$.
Since $[M_Z^n, [F, M_Z]] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$, we have that:

$$[F, M_Z^n] \in nM_Z^{n-1}[F, M_Z] + \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}.$$

The quantised chain rule (cont.)

By linearity,

$$[F, f(M_Z)] - f'(M_Z)[F, M_Z] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}.$$

So,

$$|[F, f(M_Z)]|^p - |f'(M_Z)[F, M_Z]|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Using the identity $|AB| = ||A|B|$, and $[M_{|f'\circ Z|^{1/2}}, |[F, M_Z]|] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$ we have

$$|\bar{d} f \circ Z|^p - (|f'(M_Z)|^{1/2}|\bar{d} Z||f'(M_Z)|^{1/2})^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

The quantised chain rule (cont.)

Using the advanced operator integration techniques from the preceding talk:

$$|f'(M_Z)|^p|đZ|^p-(|f'(M_Z)|^{1/2}|đZ||f'(M_Z)|^{1/2})^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

This finally gives us the quantised chain rule:

$$|đ(f\circ Z)|^p - |f'(M_Z)|^p|đZ|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

The quantised chain rule gives us:

$$\varphi(M_{g\circ f_c\circ Z}|đ(f_c\circ Z)|^p) = \varphi(M_{g\circ f_c\circ Z}|f_c'(M_Z)|^p|đZ|^p).$$

Which is exactly what we needed.

Further directions

- What is the dependence of $K(\varphi, c)$ on φ?
- Can a similar result be stated for Julia set of more general polynomials (or even non-polynomials)?
- Can we provide a similar result for Hausdorff measures on other Jordan curves in the plane? (such as a Koch snowflake)

At present, very little is known.

Thank you for listening! Further reading:

Good references for Julia sets and conformal dynamics are,

Carleson and Gamelin, *Complex Dynamics*, 1993

Milnor, *Dynamics in One Complex variable*, 2006.

More information on the quantised calculus may be found in:

Connes, *Noncommutative geometry*, 1994.