The Hausdorff measure of Julia sets from singular traces

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What is this talk?

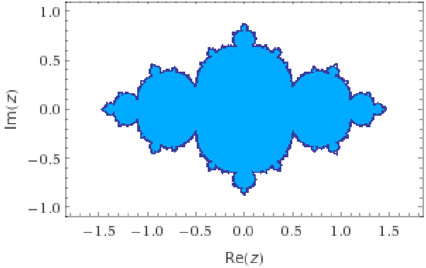
This talk concerns the paper, *The conformal trace theorem for Julia sets of quadratic polynomials* (ETDS, 2017) from myself, A Connes, F Sukochev and D Zanin.

Part I: Conformal Dynamics

Historical background: What is the conformal trace theorem?

Let $c \in \mathbb{C}$ be small, and let $f_c(z) = z^2 + c$. The Julia set J of f_c is the boundary of the set of points z such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

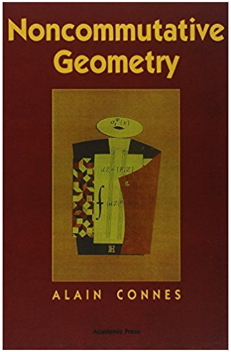
When $c \approx 0$, J is a Jordan curve:



If $c \neq 0$ then this is non-smooth and a kind of fractal.

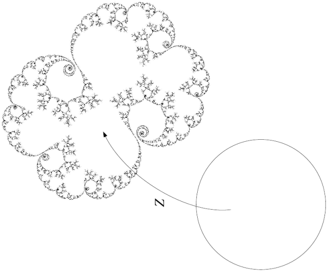
Background (continued)

In 1994 in his book *Noncommutative Geometry*, A. Connes introduced a formula for the Hausdorff measure of a Julia set in terms of his quantised calculus:



Background (continued)

To state the formula properly takes some work, but the key ingredients are as follows: take a Julia set J and let $Z : \mathbb{T} \to J$ be the extension to the boundary of the conformal equivalence between the exterior of the unit disk and the exterior of J:



Z is typically not differentiable, and it is typically not even of bounded variation.

Background (continued)

The conformal trace theorem then states that for all continuous normalised traces φ on the ideal $\mathcal{L}_{1,\infty}$, there is a constant $K(\varphi, p)$ such that:

$$K(\varphi, p) \int_J f \, d\lambda_p = \varphi(M_{f \circ Z} |đZ|^p).$$

where p is the Hausdorff dimension of J and λ_p is the p-dimensional Hausdorff measure. Connes and Sullivan introduced this formula as a way of computing integrals with respect to the Hausdorff measure on Julia sets.

Background (continued)

Despite being announced as early as 1994, Connes and Sullivan's proof of the conformal trace theorem was never published.
In our paper we provided a new proof, using operator integration techniques which did not exist in 1994.

Complex polynomial dynamics

Let f be a polynomial with complex coefficients, and take $z_0 \in \mathbb{C}$. consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ for different choices of $z_0 \in \mathbb{C}$. In particular, since f is a polynomial, exactly one of the following happens:

1. Either $|z_n| \to \infty$.
2. $\{z_n\}_{n\geq 0}$ remains bounded.

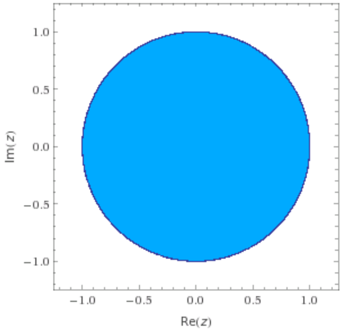
Complex polynomial dynamics

The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = {z_0}^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

1. If $|z_0| < 1$, then $f^k(z_0) \to 0$ as $k \to \infty$.
2. If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
3. If $|z_0| > 1$, then $|f^k(z_0)| \to \infty$ as $k \to \infty$.

Complex polynomial dynamics

Pictorially,



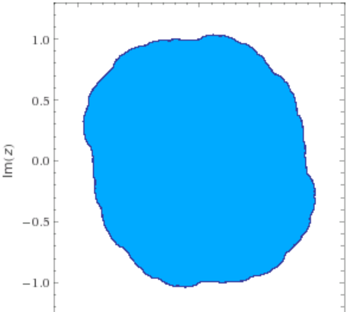
Here, the set of z_0 such that z_n remains bounded is coloured in blue. The set of z_0 such that z_n is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly? Consider $f(z) = z^2 + 0.1 + 0.1i$.

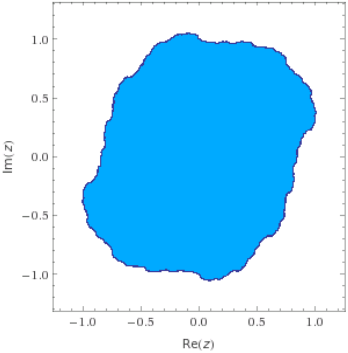
It is not feasible to determine analytically the behaviour of $\{f^n(z)\}_{n \geq 0}$. Instead we use a computer: On a large grid of complex numbers, colour each point z blue if $f^N(z) < 10$ for some suitably large number N.

The result looks like this:



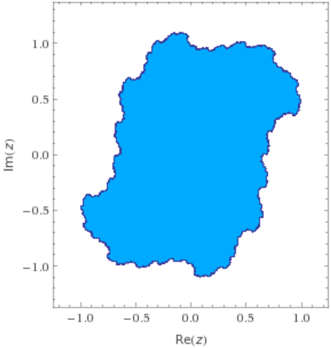
Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



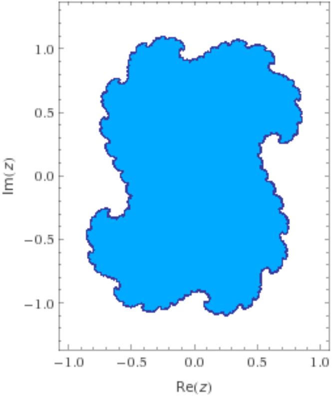
Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



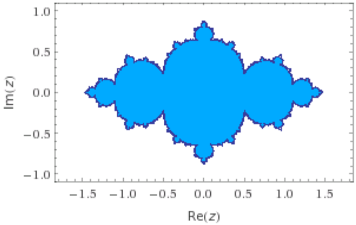
Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



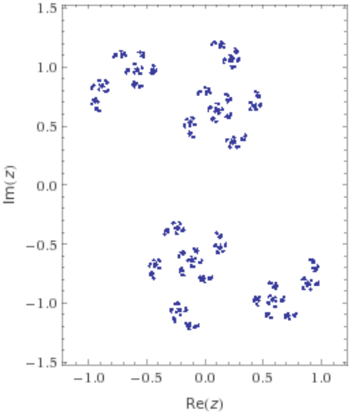
Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



Complex polynomial dynamics

Let try a slightly bigger parameter. Consider
$f(z) = z^2 + 0.5 + 0.5i$,



The Mandelbrot set

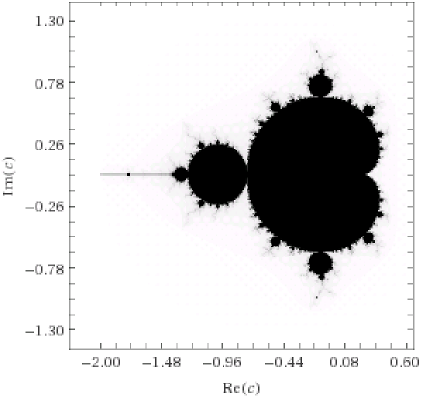
What is going on here? Consider the general polynomial:

$$f_c(z) := z^2 + c$$

with a parameter $c \in \mathbb{C}$. Note: any quadratic polynomial can be transformed into some f_c by an affine change of coordinates.

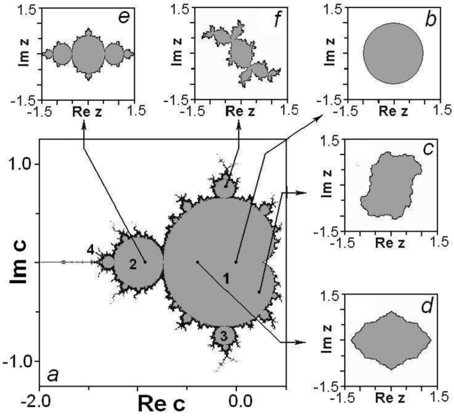
The Mandelbrot set

Consider the case $z_0 = 0$ (for simplicity). For which c is $\{f_c^k(0)\}_{k\geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} \ : \ \{f_c^k(0)\}_{k\geq 0} \text{ is bounded}\}$. M can be approximated by a computer:



The Mandelbrot set

A more informative image is this one:



The Julia set

Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The *Julia set* of f_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

Theorem (Mandelbrot)

The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).

(some authors take this as the definition of M)

Fixed points

The asymptotic behaviour of $\{f^n(z_0)\}_{n\geq 0}$ is best understood by examining the fixed points ($f(z) = z$) of f.

A fixed point λ is said to be:

1. Attracting if $|f'(\lambda)| < 1$,
2. Repelling if $|f'(\lambda)| > 1$,
3. Neutral if $|f'(\lambda)| = 1$.

An attracting fixed point λ are called *super-attracting* if $f'(\lambda) = 0$.

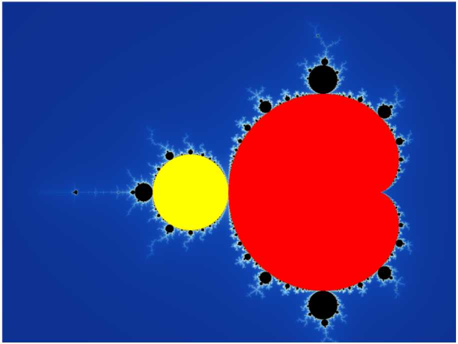
Fixed points (cont.)

The behaviour of $\{f^n(z)\}_{n\geq 0}$ near an attracting fixed point is easily described:
If z is sufficiently close to an attracting fixed point λ, then $f^n(z) \to \lambda$ exponentially fast.
If λ is superattracting, then $f^n(z) \to \lambda$ super-exponentially fast.
The set of all z such that $f^n(z) \to \lambda$ as $n \to \infty$ is called the attracting basin of λ. It is easy to see that an attracting basin is open.

The main cardioid

When does $f(z) = z^2 + c$ have an attracting fixed point? Solve for c: $z^2 - z + c = 0$, $|2z| < 1$.

Let M_0 be the set $\{\frac{z}{2}(1 - \frac{z}{2}) \ : \ |z| < 1\}$. M_0 is an open subset of the Mandelbrot set M called the *main cardioid*, shown below in red:



By definition, if $c \in M_0$ then f_c has an attracting fixed point.

The main cardioid

The significance of the main cardioid is the following theorem:

Theorem

The Julia set $J(f_c)$ of f_c is a Jordan curve (i.e. homeomorphic to a circle) if and only if c is in the main cardioid M_0.

Very rough outline of the proof:

The connected components of $\mathbb{C} \setminus J(f_c)$ correspond to the attracting basin of the fixed points of f_c. For $c \in M_0$, there are exactly two attracting fixed points (one of them at infinity). $\square$

Hausdorff dimension (reminder)

Let (X, d) be a metric space. For $q > 0$, let the q-dimensional Hausdorff content of X be:

$$C_q(X) = \inf\{\sum_{j=0}^{\infty} \mathrm{diam}(U_j)^q \ : \ X \subseteq \bigcup_{j=0}^{\infty} U_j\}.$$

The Hausdorff dimension of X is defined to be the infimum of the set of q such that $C_q(X) = 0$.

Hausdorff dimension of Julia sets

Fact: If c is in the main cardioid M_0 of the Mandelbrot set M, then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.

Conformal dimension

Another closely related notion of dimension is conformal dimension (due to Sullivan).
Let $X \subseteq \mathbb{C}$, with $f : X \to X$, and ν a measure on X. The measure ν is said to be f-conformal with dimension p, if for every open subset $U \subseteq X$ such that $f|U$ is injective, we have:

$$\nu(f(U)) = \int_U |f'|^p \, d\nu.$$

Characterising the Hausdorff measure

Fact: For Julia sets $J(f_c)$, $c \in M_0$, the Hausdorff measure is the unique f_c-conformal measure of dimension equal to p, where p is the Hausdorff dimension.

We can therefore characterise the Hausdorff measure as follows:

Theorem

The Hausdorff measure λ_p of $J(f_c)$ is the unique positive Borel measure such that for all open sets $U \subseteq J(f_c)$ with $f_c|U$ injective, and continuous functions g supported in U, we have:

$$\int_{f_c(U)} g \, d\lambda_p = \int_U (g \circ f_c)|f_c'|^p \, d\lambda_p.$$

The main task of the paper

By the Riesz representation theorem, we have the following:

Theorem

Let ℓ be a bounded linear functional on the space $C(J)$. If:

$$\ell(g) = \frac{1}{2}\ell((g \circ f_c)|f_c'|^p), \quad g \in C(J).$$

then there is a constant c such that:

$$\ell(g) = c \int_J g \, d\lambda_p.$$

Part II: Quantised calculus

Quantised Calculus: A very rapid introduction

Let H be a (complex, separable) Hilbert space. Let $\mathcal{B}(H)$ denote the algebra of bounded operators on H, and let $\|\cdot\|$ denote the operator norm. Given $T \in \mathcal{B}(H)$ and $s \geq 0$, define:

$$\mu(s, T) := \inf\{\|T - R\| \ : \ \mathrm{rank}(R) \leq s\}.$$

The function $s \mapsto \mu(s, T)$ is called the singular value function of T.

Operator $\mathcal{L}_p$ spaces

For $p \in (0, \infty)$, the space $\mathcal{L}_p(H)$ is defined to be the set of operators T such that $\{\mu(n, T)\}_{n \geq 0}$ is in ℓ_p. Similarly, the space $\mathcal{L}_{p,\infty}(H)$ is the set of operators T such that

$$\{n^{\frac{1}{p}}\mu(n, T)\}_{n \geq 0}$$

is bounded. Equivalently, $\{\mu(n, T)\}_{n \geq 0} \in \ell_{p,\infty}$.
The spaces $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ are ideals of $\mathcal{B}(H)$ (in the ring-theoretic sense).

Traces

Let $\mathcal{E}$ be an ideal of $\mathcal{B}(H)$. A functional $\varphi : \mathcal{E} \to \mathbb{C}$ is called a trace if it is invariant under unitary conjugation. That is, for all unitary operators U and $T \in \mathcal{E}$ we have

$$\varphi(UTU^*) = \varphi(T).$$

The most well-known example is the classical trace tr, which can be defined for positive operators $T \geq 0$ by

$$\mathrm{tr}(T) = \sum_{n \geq 0} \mu(n, T) = \int_0^\infty \mu(s, T)\, ds.$$

Other Traces

There are many more traces on ideals of $\mathcal{B}(H)$.
For the ideal $\mathcal{L}_{1,\infty}$:

1. There exists an uncountable infinity of linearly independent non-trivial traces φ.
2. There exist traces φ which are continuous in the sense that $|\varphi(T)| \leq C \sup_{n\geq 0} n\mu(n, T)$.
3. There exist discontinuous traces.
4. Any continuous trace can be written as a linear combination of positive traces.
5. All traces vanish on finite rank operators (They are *singular*).

Generalised limits

Consider a sequence $\{a_n\}_{n\geq 0}$. If a_n does not necessarily converge, there are many methods to assign a "limit" to a_n. For example,

1. The "Cèsaro limit" is $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} a_k$,
2. The "Abel limit" is $\lim_{r\to 1}\sum_{k=0}^{\infty} r^k(a_{k+1}-a_k)$,
3. The "zeta-function regularised limit" is $F(-1)$ where F is the analytic continuation of the function $F(s) = \sum_{k\geq 0}(a_{k+1}-a_k)^{-s}$.

There is another (entirely distinct) method, which is to use an "extended limit" ω.

Extended limits

Let $\ell_\infty(\mathbb{N})$ denote the vector space of bounded sequences $\{x_n\}_{n\geq 0}$. An extended limit ω is a linear functional on ℓ_∞ such that:

- If $x_n \geq 0$ for all $n \geq 0$, then $\omega(\{x_n\}_{n\geq 0}) \geq 0$.
- $\omega(\{x_n\}_{n\geq 0}) = c$ if $\lim_{n\to\infty} x_n = c$ exists.

An uncountable infinity of extended limits exist.

The Dixmier trace

By definition, if $T \in \mathcal{L}_{1,\infty}$ then $\mu(n, T) = O(n^{-1})$ and hence:

$$\sup_{N \geq 0} \frac{1}{\log(2+N)} \sum_{j=0}^{N} \mu(j, T) < \infty.$$

The most-well known example of a trace on $\mathcal{L}_{1,\infty}$ is obtained by applying a generalised limiting procedure to the above sequence. Let ω be a generalised limit, and for $T \geq 0$ in $\mathcal{L}_{1,\infty}$ define:

$$\mathrm{tr}_\omega(T) = \omega\left(\left\{\frac{\sum_{j=0}^{N} \mu(j, T)}{\log(2+N)}\right\}\right).$$

Then (the linear extension of) tr_ω is a nontrivial trace on the ideal $\mathcal{L}_{1,\infty}$.

The Hilbert transform

Let $\mathbb{T}$ be the unit circle (in the complex plane). The Hilbert space $L_2(\mathbb{T})$ is defined with respect to the arc-length measure (the Haar measure).
There is the trigonometric orthonormal basis for $L_2(\mathbb{T})$,

$$e_n(z) = z^n, \quad n \in \mathbb{Z}, z \in \mathbb{T}.$$

The Hilbert trasform F is defined on the basis e_n by $Fe_n = \mathrm{sgn}(n)e_n$.
For a bounded function f on $\mathbb{T}$, let M_f be the operator $M_f\xi(z) = f(z)\xi(z)$ (pointwise multiplication).

Quantised differentials

If f is a bounded function on $\mathbb{T}$, then pointwise multiplication by f defines a bounded linear operator M_f on $L_2(\mathbb{T})$. Connes calls the commutator $i[F, M_f]$ the "quantised differential" of f.
The name is intended to imply that this is something like a differential df. So we use the symbol $đf$,

$$đf := i[F, M_f].$$

Part III: Description of the formula

Böttcher coordinates

The following is due to L. Böttcher:

Theorem

Let f be a polynomial of degree $d \geq 2$. There exists a conformal map Z:

$$Z : \{|z| > 1\} \to \textit{Attracting basin of } \infty.$$

such that $f(Z(z)) = Z(z^d)$, for all $|z| > 1$.

If $J(f_c)$ is a Jordan curve, then Carathéodory's theorem implies that Z has continuous extension:

$$Z : \mathbb{T} \to J(f_c)$$

such that $Z(z^2) = f_c(Z(z))$ for all $z \in \mathbb{T}$.

Description of the Conformal Trace Formula

Let f be a continuous function on the Julia set $J(f_c)$. Then the operator

$$M_{f\circ Z}|\bar{d}Z|^p = M_{f\circ Z}|[F, M_Z]|^p$$

is some kind of "p-dimensional density" on the Julia set $J(f_c)$ (in the language of quantised calculus).
Motivated by noncommutative geometry, one might guess that the correct way of "integrating" this density is to take a trace.

Description of the Conformal Trace Formula

Lemma

The quantised differential $đZ$ (i.e., the commutator $[F, M_Z]$) is in the ideal $\mathcal{L}_{p,\infty}$.

Hence, the operator $M_{f \circ Z}|đZ|^p$ is in $\mathcal{L}_{1,\infty}$.

Outline of the proof

Let φ be a continuous normalised trace on $\mathcal{L}_{1,\infty}$. Define the linear functional:

$$\ell(g) = \varphi(g(M_Z)|\bar{\partial} Z|^p).$$

Then all that we need to show is:

- $\ell(1) \neq 0$ for at least some trace φ,
- $\ell(g) = \frac{1}{2}\ell((g \circ f_c)|f_c'|^p)$.

Further directions

For all continuous normalised traces φ on the ideal $\mathcal{L}_{1,\infty}$, there is a constant $K(\varphi, p)$ such that:

$$K(\varphi, p) \int_J f \, d\lambda_p = \varphi(M_{f \circ Z} |\bar{d} Z|^p).$$

- What is the dependence of $K(\varphi, c)$ on φ?
- Can a similar result be stated for Julia set of more general polynomials (or even non-polynomials)?
- Can we provide a similar result for Hausdorff measures on other Jordan curves in the plane? (such as a Koch snowflake)

At present, very little is known.

Thank you for listening! Further reading:

Good references for Julia sets and conformal dynamics are,

Carleson and Gamelin, *Complex Dynamics*, 1993

Milnor, *Dynamics in One Complex variable*, 2006.

More information on the quantised calculus may be found in:

Connes, *Noncommutative geometry*, 1994.