

# The Hausdorff measure of Julia sets from singular traces

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November 6, 2018

# What is this talk?

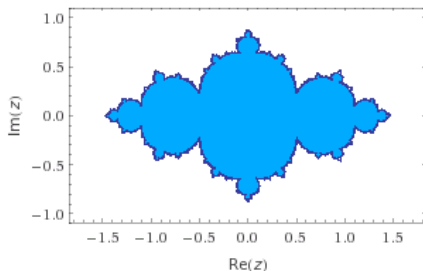
This talk concerns the paper, *The conformal trace theorem for Julia sets of quadratic polynomials* (ETDS, 2017) from myself, A Connes, F Sukochev and D Zanin.

## Part I: Conformal Dynamics

# Historical background: What is the conformal trace theorem?

Let  $c \in \mathbb{C}$  be small, and let  $f_c(z) = z^2 + c$ . The Julia set  $J$  of  $f_c$  is the boundary of the set of points  $z$  such that  $\{f_c^n(z)\}_{n \geq 0}$  is bounded.

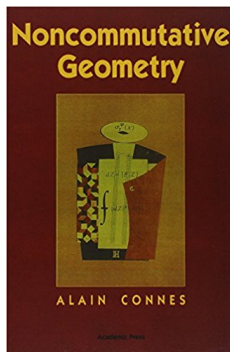
When  $c \approx 0$ ,  $J$  is a Jordan curve:



If  $c \neq 0$  then this is non-smooth and a kind of fractal.

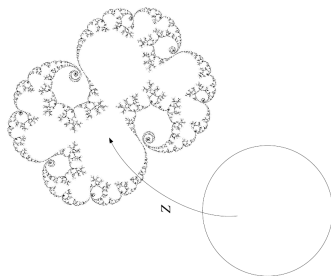
## Background (continued)

In 1994 in his book *Noncommutative Geometry*, A. Connes introduced a formula for the Hausdorff measure of a Julia set in terms of his quantised calculus:



## Background (continued)

To state the formula properly takes some work, but the key ingredients are as follows: take a Julia set  $J$  and let  $Z : \mathbb{T} \rightarrow J$  be the extension to the boundary of the conformal equivalence between the exterior of the unit disk and the exterior of  $J$ :



$Z$  is typically not differentiable, and it is typically not even of bounded variation.

## Background (continued)

The conformal trace theorem then states that for all continuous normalised traces  $\varphi$  on the ideal  $\mathcal{L}_{1,\infty}$ , there is a constant  $K(\varphi, p)$  such that:

$$K(\varphi, p) \int_J f d\lambda_p = \varphi(M_{f \circ Z} |dZ|^p).$$

where  $p$  is the Hausdorff dimension of  $J$  and  $\lambda_p$  is the  $p$ -dimensional Hausdorff measure. Connes and Sullivan introduced this formula as a way of computing integrals with respect to the Hausdorff measure on Julia sets.

## Background (continued)

Despite being announced as early as 1994, Connes and Sullivan's proof of the conformal trace theorem was never published. In our paper we provided a new proof, using operator integration techniques which did not exist in 1994.



# Complex polynomial dynamics

Let  $f$  be a polynomial with complex coefficients, and take  $z_0 \in \mathbb{C}$ . consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of  $\{z_n\}_{n \geq 0}$  for different choices of  $z_0 \in \mathbb{C}$ . In particular, since  $f$  is a polynomial, exactly one of the following happens:

- ① Either  $|z_n| \rightarrow \infty$ .
- ②  $\{z_n\}_{n \geq 0}$  remains bounded.

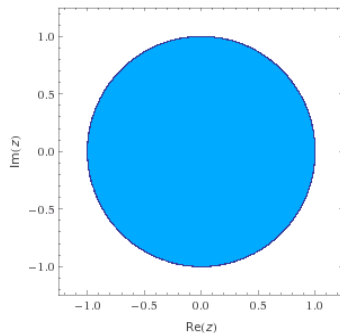
# Complex polynomial dynamics

The simplest nontrivial example is  $f(z) = z^2$ . Then  $z_k = f^k(z_0) = z_0^{2^k}$ , and the behaviour of  $f^k(z_0)$  neatly splits into three separate cases:

- 1 If  $|z_0| < 1$ , then  $f^k(z_0) \rightarrow 0$  as  $k \rightarrow \infty$ .
- 2 If  $|z_0| = 1$ , then  $|f^k(z_0)| = 1$  for all  $k \geq 0$ .
- 3 If  $|z_0| > 1$ , then  $|f^k(z_0)| \rightarrow \infty$  as  $k \rightarrow \infty$ .

# Complex polynomial dynamics

Pictorially,



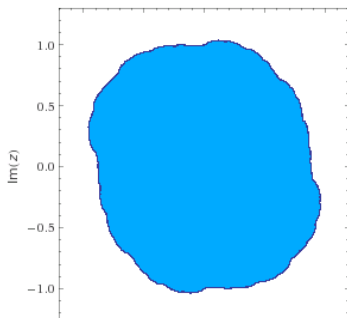
Here, the set of  $z_0$  such that  $z_n$  remains bounded is coloured in blue. The set of  $z_0$  such that  $z_n$  is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

# Complex polynomial dynamics

What if we perturb the polynomial  $f(z) = z^2$  slightly? Consider  $f(z) = z^2 + 0.1 + 0.1i$ .

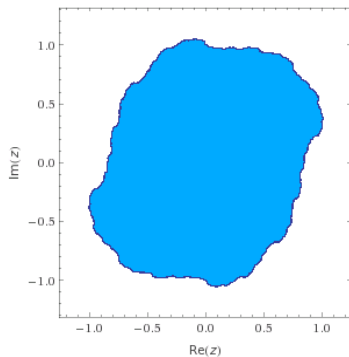
It is not feasible to determine analytically the behaviour of  $\{f^n(z)\}_{n \geq 0}$ . Instead we use a computer: On a large grid of complex numbers, colour each point  $z$  blue if  $f^N(z) < 10$  for some suitably large number  $N$ .

The result looks like this:



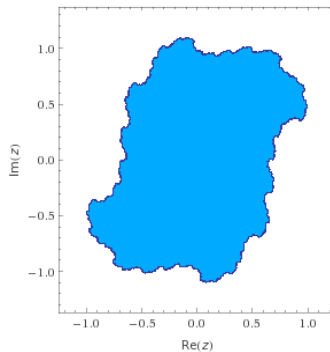
# Complex polynomial dynamics

Try  $f(z) = z^2 + 0.1 - 0.2i$ ,



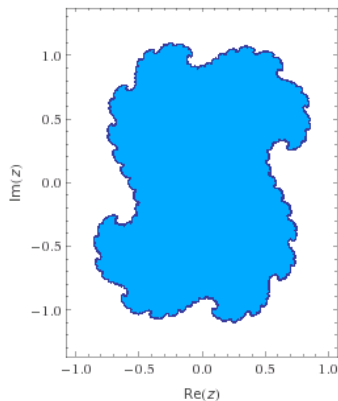
# Complex polynomial dynamics

Try  $f(z) = z^2 + 0.2 - 0.3i$ ,



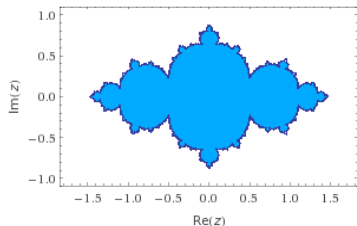
# Complex polynomial dynamics

Try  $f(z) = z^2 + 0.3 - 0.1i$ ,



# Complex polynomial dynamics

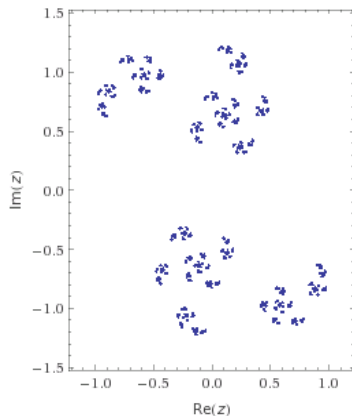
Try  $f(z) = z^2 - 0.7 + 0.001i$ ,





# Complex polynomial dynamics

Let try a slightly bigger parameter. Consider  
 $f(z) = z^2 + 0.5 + 0.5i$ ,



# The Mandelbrot set

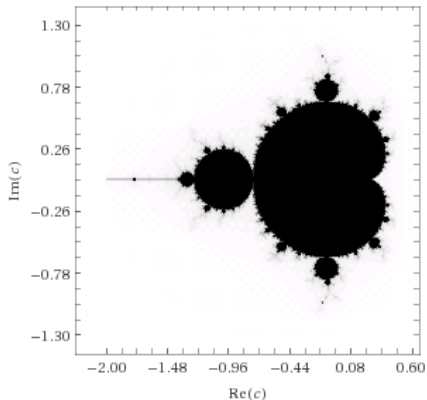
What is going on here? Consider the general polynomial:

$$f_c(z) := z^2 + c$$

with a parameter  $c \in \mathbb{C}$ . Note: any quadratic polynomial can be transformed into some  $f_c$  by an affine change of coordinates.

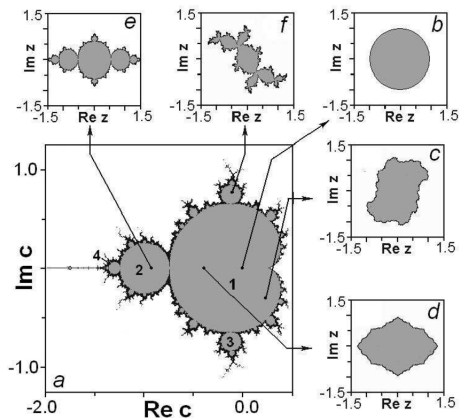
# The Mandelbrot set

Consider the case  $z_0 = 0$  (for simplicity). For which  $c$  is  $\{f_c^k(0)\}_{k \geq 0}$  bounded? Define the Mandelbrot set  $M := \{c \in \mathbb{C} : \{f_c^k(0)\}_{k \geq 0} \text{ is bounded}\}$ .  $M$  can be approximated by a computer:



# The Mandelbrot set

A more informative image is this one:



# The Julia set

Let  $c \in \mathbb{C}$ , and consider  $f_c(z) = z^2 + c$ . The *Julia set* of  $f_c$  is the boundary of the set of points  $z \in \mathbb{C}$  such that  $\{f_c^n(z)\}_{n \geq 0}$  is bounded.

## Theorem (Mandelbrot)

*The Julia set  $J(f_c)$  is connected if and only if  $c \in M$  (the Mandelbrot set).*

(some authors take this as the definition of  $M$ )

# Fixed points

The asymptotic behaviour of  $\{f^n(z_0)\}_{n \geq 0}$  is best understood by examining the fixed points ( $f(z) = z$ ) of  $f$ .

A fixed point  $\lambda$  is said to be:

- ① Attracting if  $|f'(\lambda)| < 1$ ,
- ② Repelling if  $|f'(\lambda)| > 1$ ,
- ③ Neutral if  $|f'(\lambda)| = 1$ .

An attracting fixed point  $\lambda$  are called *super-attracting* if  $f'(\lambda) = 0$ .

## Fixed points (cont.)

The behaviour of  $\{f^n(z)\}_{n \geq 0}$  near an attracting fixed point is easily described:

If  $z$  is sufficiently close to an attracting fixed point  $\lambda$ , then  $f^n(z) \rightarrow \lambda$  exponentially fast.

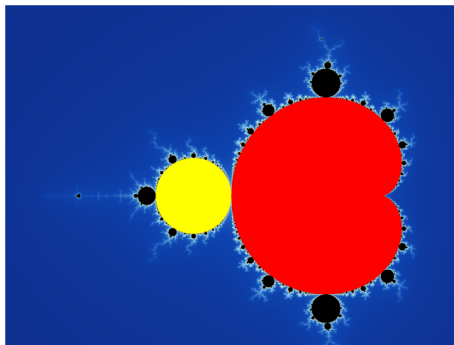
If  $\lambda$  is superattracting, then  $f^n(z) \rightarrow \lambda$  super-exponentially fast.

The set of all  $z$  such that  $f^n(z) \rightarrow \lambda$  as  $n \rightarrow \infty$  is called the attracting basin of  $\lambda$ . It is easy to see that an attracting basin is open.

# The main cardioid

When does  $f(z) = z^2 + c$  have an attracting fixed point? Solve for  $c$ :  $z^2 - z + c = 0$ ,  $|2z| < 1$ .

Let  $M_0$  be the set  $\{\frac{z}{2}(1 - \frac{z}{2}) : |z| < 1\}$ .  $M_0$  is an open subset of the Mandelbrot set  $M$  called the *main cardioid*, shown below in red:



By definition, if  $c \in M_0$  then  $f_c$  has an attracting fixed point.



# The main cardioid

The significance of the main cardioid is the following theorem:

## Theorem

*The Julia set  $J(f_c)$  of  $f_c$  is a Jordan curve (i.e. homeomorphic to a circle) if and only if  $c$  is in the main cardioid  $M_0$ .*

## Very rough outline of the proof:

The connected components of  $\mathbb{C} \setminus J(f_c)$  correspond to the attracting basin of the fixed points of  $f_c$ . For  $c \in M_0$ , there are exactly two attracting fixed points (one of them at infinity). □

# Hausdorff dimension (reminder)

Let  $(X, d)$  be a metric space. For  $q > 0$ , let the  $q$ -dimensional Hausdorff content of  $X$  be:

$$C_q(X) = \inf \left\{ \sum_{j=0}^{\infty} \text{diam}(U_j)^q : X \subseteq \bigcup_{j=0}^{\infty} U_j \right\}.$$

The Hausdorff dimension of  $X$  is defined to be the infimum of the set of  $q$  such that  $C_q(X) = 0$ .

# Hausdorff dimension of Julia sets

Fact: If  $c$  is in the main cardioid  $M_0$  of the Mandelbrot set  $M$ , then the Julia set  $J(f_c)$  is a Jordan curve with Hausdorff dimension  $p \in [1, 2)$ . In fact  $p = 1$  if and only if  $c = 0$ .

# Conformal dimension

Another closely related notion of dimension is conformal dimension (due to Sullivan).

Let  $X \subseteq \mathbb{C}$ , with  $f : X \rightarrow X$ , and  $\nu$  a measure on  $X$ . The measure  $\nu$  is said to be  $f$ -conformal with dimension  $p$ , if for every open subset  $U \subseteq X$  such that  $f|_U$  is injective, we have:

$$\nu(f(U)) = \int_U |f'|^p d\nu.$$

# Characterising the Hausdorff measure

Fact: For Julia sets  $J(f_c)$ ,  $c \in M_0$ , the Hausdorff measure is the unique  $f_c$ -conformal measure of dimension equal to  $p$ , where  $p$  is the Hausdorff dimension.

We can therefore characterise the Hausdorff measure as follows:

## Theorem

*The Hausdorff measure  $\lambda_p$  of  $J(f_c)$  is the unique positive Borel measure such that for all open sets  $U \subseteq J(f_c)$  with  $f_c|_U$  injective, and continuous functions  $g$  supported in  $U$ , we have:*

$$\int_{f_c(U)} g \, d\lambda_p = \int_U (g \circ f_c) |f'_c|^p \, d\lambda_p.$$

# The main task of the paper

By the Riesz representation theorem, we have the following:

## Theorem

*Let  $\ell$  be a bounded linear functional on the space  $C(J)$ . If:*

$$\ell(g) = \ell((g \circ f_c)|f'_c|^p), \quad g \in C(J).$$

*then there is a constant  $c$  such that:*

$$\ell(g) = c \int_J g \, d\lambda_p.$$

## Part II: Quantised calculus

# Quantised Calculus: A very rapid introduction

Let  $H$  be a (complex, separable) Hilbert space. Let  $\mathcal{B}(H)$  denote the algebra of bounded operators on  $H$ , and let  $\|\cdot\|$  denote the operator norm. Given  $T \in \mathcal{B}(H)$  and  $s \geq 0$ , define:

$$\mu(s, T) := \inf\{\|T - R\| : \text{rank}(R) \leq s\}.$$

The function  $s \mapsto \mu(s, T)$  is called the singular value function of  $T$ .



# Operator $\mathcal{L}_p$ spaces

For  $p \in (0, \infty)$ , the space  $\mathcal{L}_p(H)$  is defined to be the set of operators  $T$  such that  $\{\mu(n, T)\}_{n \geq 0}$  is in  $\ell_p$ . Similarly, the space  $\mathcal{L}_{p,\infty}(H)$  is the set of operators  $T$  such that

$$\{n^{\frac{1}{p}} \mu(n, T)\}_{n \geq 0}$$

is bounded. Equivalently,  $\{\mu(n, T)\}_{n \geq 0} \in \ell_{p,\infty}$ .

The spaces  $\mathcal{L}_p$  and  $\mathcal{L}_{p,\infty}$  are ideals of  $\mathcal{B}(H)$  (in the ring-theoretic sense).

# Traces

Let  $\mathcal{E}$  be an ideal of  $\mathcal{B}(H)$ . A functional  $\varphi : \mathcal{E} \rightarrow \mathbb{C}$  is called a trace if it is invariant under unitary conjugation. That is, for all unitary operators  $U$  and  $T \in \mathcal{E}$  we have

$$\varphi(UTU^*) = \varphi(T).$$

The most well-known example is the classical trace  $\mathrm{tr}$ , which can be defined for positive operators  $T \geq 0$  by

$$\mathrm{tr}(T) = \sum_{n \geq 0} \mu(n, T) = \int_0^\infty \mu(s, T) ds.$$

# Other Traces

There are many more traces on ideals of  $\mathcal{B}(H)$ .

For the ideal  $\mathcal{L}_{1,\infty}$ :

- 1 There exists an uncountable infinity of linearly independent non-trivial traces  $\varphi$ .
- 2 There exist traces  $\varphi$  which are continuous in the sense that  $|\varphi(T)| \leq C \sup_{n \geq 0} n\mu(n, T)$ .
- 3 There exist discontinuous traces.
- 4 Any continuous trace can be written as a linear combination of positive traces.
- 5 All traces vanish on finite rank operators (They are *singular*).

# Generalised limits

Consider a sequence  $\{a_n\}_{n \geq 0}$ . If  $a_n$  does not necessarily converge, there are many methods to assign a “limit” to  $a_n$ . For example,

- 1 The “Cèsaro limit” is  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k$ ,
- 2 The “Abel limit” is  $\lim_{r \rightarrow 1} \sum_{k=0}^{\infty} r^k (a_{k+1} - a_k)$ ,
- 3 The “zeta-function regularised limit” is  $F(-1)$  where  $F$  is the analytic continuation of the function

$$F(s) = \sum_{k \geq 0} (a_{k+1} - a_k)^{-s}.$$

There is another (entirely distinct) method, which is to use an “extended limit”  $\omega$ .

# Extended limits

Let  $\ell_\infty(\mathbb{N})$  denote the vector space of bounded sequences  $\{x_n\}_{n \geq 0}$ .  
An extended limit  $\omega$  is a linear functional on  $\ell_\infty$  such that:

- If  $x_n \geq 0$  for all  $n \geq 0$ , then  $\omega(\{x_n\}_{n \geq 0}) \geq 0$ .
- $\omega(\{x_n\}_{n \geq 0}) = c$  if  $\lim_{n \rightarrow \infty} x_n = c$  exists.

An uncountable infinity of extended limits exist.

# The Dixmier trace

By definition, if  $T \in \mathcal{L}_{1,\infty}$  then  $\mu(n, T) = O(n^{-1})$  and hence:

$$\sup_{N \geq 0} \frac{1}{\log(2 + N)} \sum_{j=0}^N \mu(j, T) < \infty.$$

The most-well known example of a trace on  $\mathcal{L}_{1,\infty}$  is obtained by applying a generalised limiting procedure to the above sequence. Let  $\omega$  be a generalised limit, and for  $T \geq 0$  in  $\mathcal{L}_{1,\infty}$  define:

$$\mathrm{tr}_\omega(T) = \omega \left( \left\{ \frac{\sum_{j=0}^N \mu(j, T)}{\log(2 + N)} \right\} \right).$$

Then (the linear extension of)  $\mathrm{tr}_\omega$  is a nontrivial trace on the ideal  $\mathcal{L}_{1,\infty}$ .

# The Hilbert transform

Let  $\mathbb{T}$  be the unit circle (in the complex plane). The Hilbert space  $L_2(\mathbb{T})$  is defined with respect to the arc-length measure (the Haar measure).

There is the trigonometric orthonormal basis for  $L_2(\mathbb{T})$ ,

$$e_n(z) = z^n, \quad n \in \mathbb{Z}, z \in \mathbb{T}.$$

The Hilbert transform  $F$  is defined on the basis  $e_n$  by  $Fe_n = \operatorname{sgn}(n)e_n$ .

For a bounded function  $f$  on  $\mathbb{T}$ , let  $M_f$  be the operator  $M_f\xi(z) = f(z)\xi(z)$  (pointwise multiplication).

# Quantised differentials

If  $f$  is a bounded function on  $\mathbb{T}$ , then pointwise multiplication by  $f$  defines a bounded linear operator  $M_f$  on  $L_2(\mathbb{T})$ . Connes calls the commutator  $i[F, M_f]$  the “quantised differential” of  $f$ .

The name is intended to imply that this is something like a differential  $df$ . So we use the symbol  $\bar{d}f$ ,

$$\bar{d}f := i[F, M_f].$$



## Part III: Description of the formula

# Böttcher coordinates

The following is due to L. Böttcher:

## Theorem

*Let  $f$  be a polynomial of degree  $d \geq 2$ . There exists a conformal map  $Z$ :*

$$Z : \{|z| > 1\} \rightarrow \text{Attracting basin of } \infty.$$

*such that  $f(Z(z)) = Z(z^d)$ , for all  $|z| > 1$ .*

If  $J(f_c)$  is a Jordan curve, then Carathéodory's theorem implies that  $Z$  has continuous extension:

$$Z : \mathbb{T} \rightarrow J(f_c)$$

such that  $Z(z^2) = f_c(Z(z))$  for all  $z \in \mathbb{T}$ .

## Description of the Conformal Trace Formula

Let  $f$  be a continuous function on the Julia set  $J(f_c)$ . Then the operator

$$M_{f \circ Z} |\bar{\partial} Z|^p = M_{f \circ Z} |[F, M_Z]|^p$$

is some kind of “ $p$ -dimensional density” on the Julia set  $J(f_c)$  (in the language of quantised calculus).

Motivated by noncommutative geometry, one might guess that the correct way of “integrating” this density is to take a trace.

# Description of the Conformal Trace Formula

## Lemma

*The quantised differential  $\bar{d}Z$  (i.e., the commutator  $[F, M_Z]$ ) is in the ideal  $\mathcal{L}_{p,\infty}$ .*

Hence, the operator  $M_{f \circ Z} |\bar{d}Z|^p$  is in  $\mathcal{L}_{1,\infty}$ .

# Outline of the proof

Let  $\varphi$  be a continuous normalised trace on  $\mathcal{L}_{1,\infty}$ . Define the linear functional:

$$\ell(g) = \varphi(g(M_Z)|dZ|^p).$$

Then if we can show that:

- $\ell(1) \neq 0$  for at least some trace  $\varphi$ ,
- $\ell(g) = \frac{1}{2}\ell((g \circ f_c)|f'_c|^p)$ .

The proof would be complete.

## Further directions

For all continuous normalised traces  $\varphi$  on the ideal  $\mathcal{L}_{1,\infty}$ , there is a constant  $K(\varphi, p)$  such that:

$$K(\varphi, p) \int_J f d\lambda_p = \varphi(M_{f \circ Z} |dZ|^p).$$

- What is the dependence of  $K(\varphi, c)$  on  $\varphi$ ?
- Can a similar result be stated for Julia set of more general polynomials (or even non-polynomials)?
- Can we provide a similar result for Hausdorff measures on other Jordan curves in the plane? (such as a Koch snowflake)

At present, very little is known.

Thank you for listening! Further reading:

Good references for Julia sets and conformal dynamics are,

Carleson and Gamelin, *Complex Dynamics*, 1993

Milnor, *Dynamics in One Complex variable*, 2006.

More information on the quantised calculus may be found in:

Connes, *Noncommutative geometry*, 1994.