Quantised Calculus.
Lecture 21: Quantised calculus on the circle.

AMSI Summer School, 2019.

Plan for this lecture

- Introduction to Fourier analysis on the circle.
- Introduction to quantised differentials.

The circle

$\mathbb{T}$ denotes the unit circle in the complex plane:

$$\mathbb{T} := \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}.$$

The circle $\mathbb{T}$ comes equipped with a metric (the arc-length metric). It makes sense therefore to define:

$$C(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C} \ : \ f \text{ is continuous.}\}.$$

Fourier series of continuous functions

Suppose that f is a continuous function on $\mathbb{T}$ ($f \in C(\mathbb{T})$).
Define the uniform norm, $\|f\|_\infty$, of f as:

$$\|f\|_\infty := \sup_{\zeta \in \mathbb{T}} |f(\zeta)| = \max_{\zeta \in \mathbb{T}} |f(\zeta)|.$$

Lemma 1.1

$C(\mathbb{T})$ is a Banach space with the norm $\|\cdot\|_\infty$.

Integration on the circle

Given a function $f \in C(\mathbb{T})$, define the integral of f over $\mathbb{T}$ as being:

$$\int_{\mathbb{T}} f(\zeta)\, d\mathbf{m}(\zeta) := \int_0^1 f(e^{2\pi i t})\, dt.$$

In other words, we are simply identifying $[0, 1)$ with $\mathbb{T}$ via the map $t \mapsto e^{2\pi i t}$, and the space of functions on $\mathbb{T}$ is nothing but the space of functions on $[0, 1)$ (but wrapped into a circle) and continuous functions on $\mathbb{T}$ correspond to periodic functions on $[0, 1]$.

Integration on the circle

For $n \in \mathbb{Z}$, let e_n denote the monomial function:

$$e_n(z) = z^n, \quad z \in \mathbb{T}.$$

With the identification $[0, 1) \to \mathbb{T}$, e_n corresponds to $t \mapsto \exp(2\pi int)$.

Integration on the circle

A function $f : \mathbb{T} \to \mathbb{C}$ is said to be (Lebesgue) measurable if $t \mapsto f(e^{2\pi i t})$ is Lebesgue measurable on $[0,1)$.
Now we can define:

$$L_2(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C} \;:\; f \text{ is measurable and } \int_{\mathbb{T}} |f|^2 \, d\mathbf{m} < \infty\}.$$

Since by definition we have:

$$\int_{\mathbb{T}} |f(\zeta)|^2 \, d\mathbf{m}(\zeta) = \int_0^1 |f(e^{2\pi i t})|^2 \, dt$$

$L_2(\mathbb{T})$ is an inner product space, with the inner product:

$$\langle f, g \rangle = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)} \, d\mathbf{m}(\zeta).$$

An orthonormal basis for $L_2(\mathbb{T})$

This means that the sequence $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L_2(\mathbb{T})$.
As discussed in earlier lectures, it is a basis.

Fourier series on the circle

Given $f \in L_2(\mathbb{T})$ and $n \in \mathbb{Z}$, we can consider the nth Fourier coefficient of f, which is given by:

$$\widehat{f}(n) = \langle f, e_n \rangle = \int_{\mathbb{T}} f(z) z^{-n} \, d\mathbf{m}(z) = \int_0^1 f(e^{2\pi i t}) e^{-2\pi i n t} \, dt.$$

Therefore (using basic Hilbert space theory) we get that for all $f \in L_2(\mathbb{T})$ we have:

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e_n$$

where the sum converges in the L_2-sense, and:

$$\int_{\mathbb{T}} |f|^2 \, d\mathbf{m} = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2.$$

Differential calculus from Fourier series

Note that $e_0(z) = z^0 = 1$. That is, z^0 is just the constant function and hence:

$$\widehat{f}(0) = \langle f, e_0 \rangle = \langle f, 1 \rangle = \int_{\mathbb{T}} f \, d\mathbf{m}.$$

That is: the 0th Fourier coefficient of f is just the integral.

Differential calculus from Fourier series

Consider a function f on $\mathbb{T}$. The angular derivative of f, denoted ∂f is defined by:

$$(\partial f)(e^{2\pi it}) = \frac{1}{2\pi}\frac{d}{dt}f(e^{2\pi it}).$$

Therefore:

$$\partial(e_n) = ine_n, \quad n \in \mathbb{Z}.$$

Differential calculus and Fourier series

So at a purely informal level, we get:

$$\partial f = \sum_{n\in\mathbb{Z}} in\widehat{f}(n)e_n.$$

So: to differentiate a function, just multiply its nth Fourier coefficient by in.

Other Fourier multipliers

Operators like ∂ are just the beginning of a general notion called a Fourier multiplier: Let g be a function on $\mathbb{Z}$. Define $g(\partial)$ as the operator:

$$g(\partial)f = \sum_{n\in\mathbb{Z}} g(n)\widehat{f}(n)e_n.$$

Lemma 1.2

If g is bounded, then $g(\partial)$ is a bounded linear operator on $L_2(\mathbb{T})$.

The Laplace operator and the Bessel potential

The Laplace operator Δ is defined as ∂^2. That is,

$$\Delta(e_n) = (in)^2 e_n = -n^2 e_n.$$

This defines an unbounded linear operator on $L_2(\mathbb{T})$, since $\|e_n\|_{L_2(\mathbb{T})} = 1$ but $\|\Delta(e_n)\|_{L_2(\mathbb{T})} = |n|^2 \uparrow \infty$.
A much friendlier operator is the Bessel potential $(1-\Delta)^{-1/2}$.
This is given by:

$$(1-\Delta)^{-1/2} f = \sum_{n\in\mathbb{Z}} (1+n^2)^{-1/2} \widehat{f}(n) e_n.$$

The Bessel potential

Lemma 1.3

The operator $(1-\Delta)^{-1/2}$ *is in the ideal* $\mathcal{L}_{1,\infty}(L_2(\mathbb{T}))$.

Proof.

By the definition of a Fourier multiplier, we have:

$$(1-\Delta)^{-1/2}e_n = (1+n^2)^{-1/2}e_n.$$

So $(1-\Delta)^{-1/2}$ is self-adjoint compact operator with eigenvalues $\{(1+n^2)^{-1/2}\}_{n\in\mathbb{Z}}$. These are positive, and so are also the singular values. Since the sequence $\{(1+n^2)^{-1/2}\}_{n\in\mathbb{Z}}$ is $O(n^{-1})$, it follows that $(1-\Delta)^{-1/2} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{T}))$. $\square$

Multiplication operators

Given $g \in C(\mathbb{T})$, define the pointwise multiplication operator:

$$(M_g f)(\zeta) := g(\zeta)f(\zeta), \quad \zeta \in \mathbb{T}.$$

You have already seen pointwise multiplication operators on $L_2([0,1])$ and these are basically the same.

Lemma 1.4

Each M_g, $g \in C(\mathbb{T})$ is a bounded linear operator on $L_2(\mathbb{T})$ and:

$$\|M_g\| \leq \sup_{\zeta \in \mathbb{T}} |g(\zeta)| = \|g\|_\infty.$$

Connes' integration formula

Now we can finally obtain the first application of singular traces:

Theorem 1.5 (Connes' integral formula for $\mathbb{T}$)

Suppose that $f \in C(\mathbb{T})$, and that Tr_ω is a Dixmier trace. Then:

$$\mathrm{Tr}_\omega(M_f(1-\Delta)^{-1/2}) = \int_{\mathbb{T}} f \, d\mathbf{m}.$$

This is remarkable: such a highly singular object as a Dixmier trace recovers the classical Lebesgue integral.

Fourier series of continuous functions

Before we prove Connes' integration formula, we need to know some facts concerning Fourier series of continuous functions.
If $f \in C(\mathbb{T})$, since in particular $f \in L_2(\mathbb{T})$, we will definitely have:

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e_n$$

where the sum converges in the L_2-sense. But does it converge in the uniform norm?
In fact the answer is no! There exist continuous functions f whose Fourier series does not converge to f in the uniform norm.

Fourier series of continuous functions

Nonetheless it is possible to determine a continuous function as a "sum" of its Fourier series in the uniform norm, if one is willing to relax the meaning of "sum".

Theorem 1.6 (Fejér theorem)

If $f \in C(\mathbb{T})$, define:

$$\sigma_N(f) = \frac{1}{N} \sum_{k=1}^{N} \sum_{j=-k}^{k} \widehat{f}(n) z^n$$

Then $\sigma_N(f)$ converges to f uniformly.

Rotation operators

Suppose that $w = e^{i\theta} \in \mathbb{T}$. The operation of "rotating by an angle θ" is given by:

$$(R_w f)(\zeta) = f(\zeta w^{-1}), \quad \zeta \in \mathbb{T}.$$

We can rotate a basis function e_n as follows:

$$(R_w e_n)(\zeta) = (\zeta w^{-1})^n = w^{-n}\zeta^n = w^{-n} e_n.$$

It is easy to see that:

$$(R_w)^* = R_{\overline{w}}$$

and also that R_w is unitary. Moreover, R_w commutes with Fourier multipliers.

Proof of Connes' integral formula for $\mathbb{T}$

Consider initially the case when $f = e_n$ and let $w \in \mathbb{T}$. Then by the unitary invariance of Tr_ω:

$$\mathrm{Tr}_\omega(M_{e_n}(1-\Delta)^{-1/2}) = \mathrm{Tr}_\omega(R_w M_{e_n}(1-\Delta)^{-1/2} R_w^*).$$

Since rotations commute with Fourier multipliers, and $R_w z^n = w^{-n} z^n$, we have:

$$\mathrm{Tr}_\omega(M_{e_n}(1-\Delta)^{-1/2}) = w^{-n}\mathrm{Tr}_\omega(M_{e_n}(1-\Delta)^{-1/2}).$$

But w was arbitrary! So if $n \neq 0$ we can choose w such that $w^{-n} \neq 1$, and then:

$$\mathrm{Tr}_\omega(M_{e_n}(1-\Delta)^{-1/2}) = 0, \quad n \neq 0.$$

On other other hand, it is easy to compute that $\mathrm{Tr}_\omega((1-\Delta)^{-1/2}) = 1$.

Proof of Connes' integral formula for $\mathbb{T}$

Now suppose that f is a function whose Fourier series is finitely supported, $f = \sum_{n=-N}^{N} \widehat{f}(n) e_n$. Then:

$$\begin{aligned}\mathrm{Tr}_\omega(M_f(1-\Delta)^{-1/2}) &= \sum_{n=-N}^{N} \widehat{f}(n)\mathrm{Tr}_\omega(M_{e_n}(1-\Delta)^{-1/2}) \\ &= \widehat{f}(0) = \int_{\mathbb{T}} f\, d\mathbf{m}.\end{aligned}$$

Finally, one can remove the assumption that f has finitely supported Fourier transform by using the continuity of Tr_ω. We have that:

$$\begin{aligned}|\mathrm{Tr}_\omega(M_f(1-\Delta)^{-1/2})| &\leq \|M_f(1-\Delta)^{-1/2}\|_{1,\infty} \\ &\leq \|M_f\|_\infty \|(1-\Delta)^{-1/2}\|_{1,\infty} \\ &\leq \|f\|_\infty \|(1-\Delta)^{-1/2}\|_{1,\infty}.\end{aligned}$$

Proof of Connes' integral formula for $\mathbb{T}$

Let f be continuous. Since $\sigma_N f$ (the Fejèr mean) has finitely supported Fourier transform, we have:

$$\mathrm{Tr}_\omega(M_{\sigma_N f}(1-\Delta)^{-1/2}) = \widehat{\sigma_N f}(0) = \widehat{f}(0).$$

From the Fejér theorem:

$$\lim_{N\to\infty} \|\sigma_N(f) - f\|_\infty = 0,$$

therefore:

$$\begin{aligned}\mathrm{Tr}_\omega(M_f(1-\Delta)^{-1/2}) &= \lim_{N\to\infty} \mathrm{Tr}_\omega(M_{\sigma_N f}(1-\Delta)^{-1/2}) \\ &= \widehat{f}(0) = \int_{\mathbb{T}} f\, d\mathbf{m}. \quad \square\end{aligned}$$

Quantised differentials

The beauty of Connes' integral formula is that ideas from classical calculus can be recovered from the spectral, operator theoretic information of $M_f(1-\Delta)^{-1/2}$.
But there is more to come: Alain Connes also proposed a "spectral" definition of a classical differential df. This is called the quantised differential $đf$, and it has no classical counterpart.

Quantised differentials (cont.)

Define the Hilbert transform F as the Fourier multiplier:

$$F(e_n) = \begin{cases} e_n, & n \geq 0 \\ -e_n, & n < 0. \end{cases}$$

More concisely, $F(e_n) = \mathrm{sgn}(n)e_n$ where sgn is the "sign" function. Then the quantised differential is defined as the operator (on $L_2(\mathbb{T})$) given by:

$$đf := i[F, M_f]$$

where $f \in C(\mathbb{T})$.

Quantised differentials (cont.)

For $f \in C(\mathbb{T})$, we can compute the matrix elements of $đf$ in the Fourier basis:

$$\begin{aligned}
\langle đf e_n, e_m \rangle &= \langle iFM_f e_n, e_m \rangle - \langle iM_f F e_n, e_m \rangle \\
&= i\langle M_f e_n, F e_m \rangle - i\langle M_f F e_n, e_m \rangle \\
&= i\mathrm{sgn}(m)\langle M_f e_n, e_m \rangle - i\mathrm{sgn}(n)\langle M_f e_n, e_m \rangle \\
&= i(\mathrm{sgn}(m) - \mathrm{sgn}(n))\langle M_f e_n, e_m \rangle \\
&= i(\mathrm{sgn}(m) - \mathrm{sgn}(n))\widehat{f}(m-n).
\end{aligned}$$

Quantised differentials (cont.)

Theorem 1.7
For all $f \in C(\mathbb{T})$, $đf$ is a compact operator on $L_2(\mathbb{T})$.

(Proof outline).
First if f has finitely supported Fourier transform, one can see from the matrix representation that $đf$ has finite rank. Then since:

$$\|đf\|_\infty \leq \|FM_f\|_\infty + \|M_f F\|_\infty \leq 2\|f\|_\infty$$

one can use the Fejér theorem:

$$\lim_{N\to\infty} \|đf - đ\sigma_N f\|_\infty \leq 2 \lim_{N\to\infty} \|f - \sigma_N f\|_\infty = 0$$

and therefore $đf$ is a limit in the uniform norm of finite rank operators. □

So $đf$ is an infinitesimal!

The importance of quantised differentials.

Why is $đf$ interesting?

Because $đf$ makes sense when f is merely continuous. No assumption of differentiability is needed! This gives us a possibility to do "calculus" with non-smooth objects like fractals.