Quantised Calculus. Lecture 20: Applications of matrix analysis.

AMSI Summer School, 2019.

What you have seen so far

(a) Hilbert and Banach space theory
(b) Spectral theory of compact operators
(c) Classification of ideals of operators by singular values
(d) Traces on ideals, inc. Dixmier traces.

But what good is any of this?

A puzzle

Suppose that you have two coins:

(If you toss one of them, it returns either heads or tails.)
Toss both at once and count the number of heads.
What are the possible outcomes??

A puzzle

Answer:
There are two possible outcomes:

$$\left\{ \frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2} \right\}.$$

Obviously these are no ordinary coins!
(This is not just a word game, there is legitimate mathematics and careful science here.)

A very rapid introduction to quantum mechanics

Quantum mechanics is a physical theory: it is a theoretical framework for predicting the positions, velocities, energies etc. of atomic-scale objects.

As a physical theory, it is extraordinarily successful: physicists can use it to predict the results of experiments to exquisite levels of precision.

In physics courses you will learn about the long and complicated history of how people arrived at this theory through experiment, guesswork and intuition. But for a mathematics course, we take the basic principles as given.

Mathematical quantum mechanics

A quantum mechanical system is given by the following:

(i) A Hilbert space, H (called the state space, unit vectors in H are called pure states)

(ii) An algebra of linear operators on H (called the algebra of observables, and the self-adjoint elements are called observables).

For this course, we will deal exclusively with observables that are compact.

Examples

Usually the way that this works looks like the following: suppose that you have a particle (or some object) which can classically be in any of the positions $\{1, 2, 3, \ldots, n\}$, then the quantum state space is:

$$\ell_2(\{1, 2, 3, \ldots, n\}) = \mathbb{C}^n.$$

If you have a particle which lives in the interval $[0, 1]$, take $H = L_2([0, 1])$.

Examples

Whereas classically a particle has a definite position:

$$x \in [0, 1]$$

a quantum mechanical particle is "fuzzy", it is described by a function:

$$\psi \in L_2([0, 1]).$$

Physicists call this the wavefunction.

Observables and the Born rule

Suppose that we have an observable A (recall: this means a self-adjoint linear operator on H).
The eigenvalues of A are the "possible observed values". Say that A is compact, then by the spectral theorem we have:

$$A = \sum_{n=1}^{\infty} \lambda_n(A)\langle \cdot, e_n\rangle e_n$$

where e_n is an orthonormal basis of eigenvectors of A.
For a given eigenvalue $\lambda \in \{\lambda_n(A)\}_{n=1}^{\infty} = \sigma(A) \setminus \{0\}$, let p_λ denote the projection onto the eigenspace determined by λ.
That is,

$$p_\lambda = \sum_{n\,:\lambda=\lambda_n(A)} \langle \cdot, e_n\rangle e_n.$$

Observables and the Born rule

One of the most basic principles of quantum mechanics is the Born rule:

The Born rule: If the system is in a pure state $\psi \in H$, and we make an observation of A, then the outcome will be $\lambda \in \sigma(A) = \{0\} \cup \{\lambda_n(A)\}_{n=1}^{\infty}$ with probability equal to:

$$P(\lambda|\psi) = |\langle p_\lambda \psi, \psi \rangle|^2 = \|p_\lambda \psi\|_H^2.$$

After the observation is made, **the state of the system will change** to $\frac{p_\lambda \psi}{\|p_\lambda \psi\|_H}$.

Where does this rule come from? Why is the outcome of an observation random? Why does an observation affect the system anyway? No one has ever come up with a convincing physical explanation, but as mathematicians we just have to accept it.

Analysis of the Born rule

Let us check that the Born rule makes sense. First, the probabilities are in the right range since pure states are unit vectors:

$$0 \leq P(\lambda|\psi) = |\langle p_\lambda \psi, \psi\rangle|^2 \leq \|\psi\|_H^2 \|p_\lambda\| \leq 1.$$

And since $\{e_n\}$ is an orthonormal basis:

$$\begin{aligned}
\sum_{\lambda \in \sigma(A)} P(\lambda|\psi) &= \sum_{\lambda \in \sigma(A)} \langle p_\lambda \psi, \psi\rangle \\
&= \sum_{\lambda \in \sigma(A)} \sum_{\{n \,:\, \lambda_n(A) = \lambda\}} |\langle e_n, \psi\rangle|^2 \\
&= \sum_{n=1}^{\infty} |\langle e_n, \psi\rangle|^2 = \|\psi\|_H^2 \\
&= 1.
\end{aligned}$$

So that we can consider the observed value of A as being a genuine random variable, with range of values equal to $\sigma(A)$, and the probability of observing $\lambda \in \{\lambda_n(A)\}_{n=1}^{\infty}$ is equal to $\|p_\lambda \psi\|_H^2$.

Analysis of the Born rule

The Born rule is also compatible with functional calculus:

- The possible observed values of A^2 are $\lambda_n(A)^2$.
- The possible observed values of $f(A)$ are $f(\lambda_n(A))$,
- Etc...

Note: The possible observed values of $A + B$ are not $\lambda_n(A) + \lambda_n(B)$, since this is different to $\lambda_n(A + B)$.

Expectation values

Suppose that have a state space H and a (compact) observable A.
What is the expected value of A?
If we are in the pure state ψ, this is:

$$\begin{aligned}
\mathbb{E}(A|\psi) &= \sum_{\lambda \in \sigma(A)} \lambda P(\lambda|\psi) \\
&= \sum_{n=1}^{\infty} \lambda_n(A) |\langle e_n, \psi \rangle|^2 \\
&= \sum_{n=1}^{\infty} \lambda_n(A) \langle e_n, \psi \rangle \langle \psi, e_n \rangle \\
&= \sum_{n=1}^{\infty} \langle A e_n, \psi \rangle \langle \psi, e_n \rangle \\
&= \langle A\psi, \psi \rangle.
\end{aligned}$$

Variance and standard deviation

Recall that the variance of a classical real-valued random variable is $\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.
For a quantum observation, we have:

$$\mathbb{E}(A^2|\psi) = \langle A^2\psi, \psi\rangle = \|A\psi\|^2$$

(recall that observables are self-adjoint!) So that:

$$\mathbb{E}((A - \mathbb{E}(A|\psi))^2) = \|(A - \mathbb{E}(A|\psi)1)\psi\|^2.$$

Accordingly the standard deviation is defined to be the square root of the variance:

$$\sigma_{A,\psi} := \|(A - \mathbb{E}(A|\psi)1)\psi\|.$$

Heisenberg's uncertainty principle

The fact that operator multiplication does not commute has some counterintuitive consequences.
Let $\sigma_{A,\psi}$ denote the standard deviation of the random outcome of observing A, and let $\sigma_{B,\psi}$ denote the standard deviation of the random outcome of observing B.

Theorem 1.1

Let A and B be observables, and let ψ be a pure state. Then:

$$\sigma_{A,\psi}\sigma_{B,\psi} \geq |\langle [A, B]\psi, \psi\rangle|.$$

($[A, B]$ denotes the commutator: $[A, B] = AB - BA$.)

The meaning of Heisenberg's uncertainty principle

If A and B are two **noncommuting** observables, then there is a lower bound to the quantity:

$$\sigma_{A,\psi}\sigma_{B,\psi}$$

So (in a given pure state) one cannot simultaneously have high precision on A and B!

Proof of Heisenberg's uncertainty principle

Proof. We can initially centre A and B to have mean zero:

$$A' := A - \mathbb{E}(A|\psi)1, \quad B' := B - \mathbb{E}(B|\psi)1.$$

Note that $[A', B'] = [A, B]$. Then:

$$\begin{aligned}
|\langle [A,B]\psi, \psi\rangle| &= |\langle [A', B']\psi, \psi\rangle| \\
&= |\langle A'B'\psi, \psi\rangle - \langle B'A'\psi, \psi\rangle| \\
&\leq |\langle A'B'\psi, \psi\rangle| + |\langle B'A'\psi, \psi\rangle| \\
&= |\langle B'\psi, A'\psi\rangle| + |\langle A'\psi, B'\psi\rangle| \\
&= 2|\langle A'\psi, B'\psi\rangle|.
\end{aligned}$$

Heisenberg's uncertainty principle

Then by the Cauchy-Schwarz inequality:

$$|\langle [A,B]\psi,\psi\rangle| \leq 2\|A'\psi\|\|B'\psi\|.$$

So:

$$\begin{aligned} |\langle [A,B]\psi,\psi\rangle| &\leq 2\|(A-\mathbb{E}(A|\psi)1)\psi\|\|(B-\mathbb{E}(B|\psi)1)\psi\| \\ &= 2\sigma_{A,\psi}\sigma_{B,\psi}. \quad \square \end{aligned}$$

A computable example

Let us consider $H = \mathbb{C}^2$. Or in other words: $H = \ell_2(\{0,1\})$. This is a quantum system with two classical states (a qubit). Let $e_0 = (1,0)$ and $e_1 = (0,1)$ be the standard basis vectors of $\mathbb{C}^2$. A pure state $\psi \in \mathbb{C}^2$ looks like $\psi = \psi_0 e_0 + \psi_1 e_1$ where $|\psi_0|^2 + |\psi_1|^2 = 1$. If you like ψ is a "smeared out" or "fuzzy" version of a number which is either zero or one (a bit).
Consider the following observable:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \langle \cdot, e_0 \rangle e_0.$$

This trivially has eigenvalues equal to $\{0,1\}$ and corresponding eigenvectors e_0 and e_1.

A computable example (continued)

The probability of observing 0 is:

$$|\langle e_0, \psi\rangle|^2 = |\psi_0|^2$$

and the probability of observing 1 is:

$$|\langle e_1, \psi\rangle|^2 = |\psi_1|^2.$$

If 0 is observed, the state will change to $\frac{\psi_0}{|\psi_0|} e_0$, and if 1 is observed then the state will change to $\frac{\psi_1}{|\psi_1|} e_1$.

About those coins...

Recall the strange coins at the beginning of the lecture? Here is how I made them:
Again consider $H = \mathbb{C}^2 = \ell_2(\{0,1\})$, and consider the following two observables:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and:

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

As we said earlier, the possible observed values of A are 0 and 1.

About those coins... (cont.)

It is easy to compute the eigenvalues of B:

$$\det(B - \lambda 1) = \left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} = 0$$

yields eigenvalues $\{0, 1\}$.
Interpret $\psi \in \ell_2(\{0, 1\})$ as representing the state of a "quantum coin". Think of a measurement of 1 from A as being "heads" and 0 as being tails. Similarly with B.
What happens when we measure $A + B$?

About those coins... (cont.)

The eigenvalues of $A + B$ are the roots of $\det(A + B - \lambda I)$, which means:

$$\left(\frac{3}{2} - \lambda\right)\left(\frac{1}{2} - \lambda\right) - \frac{1}{4} = 0$$

yields:

$$\lambda^2 - 2\lambda + \frac{1}{2} = 0.$$

So:

$$\lambda_0(A + B) = 1 + \frac{\sqrt{2}}{2}, \quad \lambda_1(A + B) = 1 - \frac{\sqrt{2}}{2}.$$

About those coins... (cont.)

What we have here are two random variables which, when individually measured, yield the values $\{0, 1\}$, but when their sum is measured yields $\{\frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\}$.
This strange behaviour happens precisely because they do not commute.
In case you were wondering: there are actual physical objects which behave like this. Look up the Stern-Gerlach experiment.

Quantum infinitesimals

Suppose that T is a self-adjoint compact operator.
We have (from the spectral theorem):

$$T = \sum_{n=1}^{\infty} \lambda_n(T)\langle \cdot, e_n\rangle e_n$$

and T has a "spectrum of possible values" $\{\lambda_n(T)\}_{n=1}^{\infty}$.
Quantum mechanically, there will exist a state ψ such that for all n,

$$P(T \text{ is observed with value } \leq \frac{1}{n}|\psi)$$

is positive.
This is another reason why compact quantum observables are something like infinitesimals.