

Quantised Calculus.

Lecture 22: The conformal trace theorem.

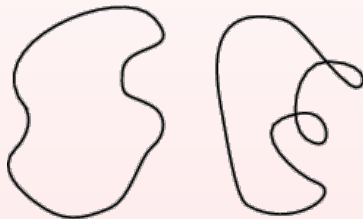
AMSI Summer School, 2019.

Plan for this lecture.

- ① A brief introduction to complex polynomial dynamics and Julia sets
- ② Geometric measure theory (specifically, Hausdorff measures)
- ③ Statement of the conformal trace theorem.

Historical background: Integration on curves

Suppose that you have a simple closed curve \mathcal{C} in the plane.
Something like:



Integration on curves

Let $\gamma : [0, 1) \rightarrow \mathbb{R}^2$ be a parametrisation. Then for a function $f : \mathcal{C} \rightarrow \mathbb{C}$, the arc-length integral is defined as:

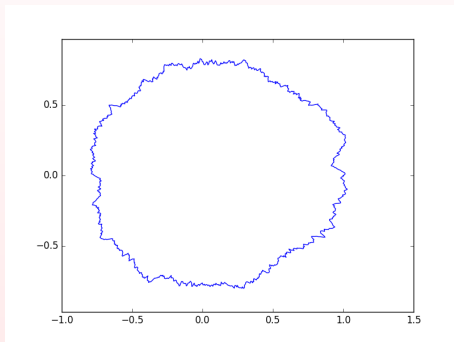
$$\int_{\mathcal{C}} f \, d\ell = \int_0^1 f(\gamma(t)) |\gamma'(t)| \, dt.$$

Of course this only makes sense if \mathcal{C} admits a piecewise-differentiable parametrisation.

But what if it doesn't?

Integration on curves

There are closed curves in the plane which are very “rough” and can look quite “fractal”:



Is there a reasonable way to “integrate” a function on these curves? Quantised calculus provides an answer, at least for a very special class of curves which come from conformal dynamics.

Context: Dynamical systems

Let X be a set and let $f : X \rightarrow X$. Iteration of the function f defines an **iterated function system**. Let $z_0 \in X$, and define

$$z_{n+1} = f(z_n), \quad n \geq 0.$$

Immediate questions which concern us: What is the asymptotic behaviour of z_n ? How does it depend on z_0 ? There is an enormous literature on this theme.

Complex polynomial dynamics

Let f be a polynomial with complex coefficients, and take $z_0 \in \mathbb{C}$. consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n \geq 0}$ for different choices of $z_0 \in \mathbb{C}$. In particular, since f is a polynomial, exactly one of the following happens:

- 1 Either $|z_n| \rightarrow \infty$.
- 2 $\{z_n\}_{n \geq 0}$ remains bounded.

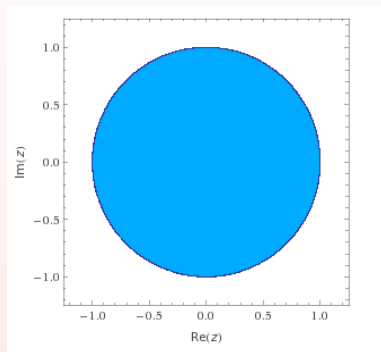
Complex polynomial dynamics

The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = z_0^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

- 1 If $|z_0| < 1$, then $f^k(z_0) \rightarrow 0$ as $k \rightarrow \infty$.
- 2 If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
- 3 If $|z_0| > 1$, then $|f^k(z_0)| \rightarrow \infty$ as $k \rightarrow \infty$.

Complex polynomial dynamics

Pictorially,



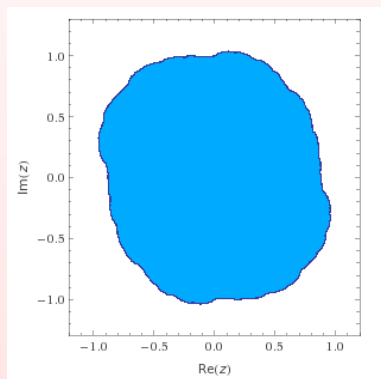
Here, the set of z_0 such that z_n remains bounded is coloured in blue. The set of z_0 such that z_n is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly? Consider $f(z) = z^2 + 0.1 + 0.1i$.

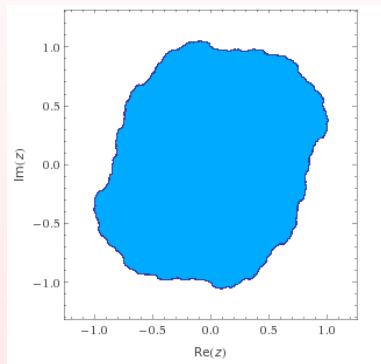
It is not feasible to determine analytically the behaviour of $\{f^n(z)\}_{n \geq 0}$. Instead we use a computer: On a large grid of complex numbers, colour each point z blue if $f^N(z) < 10$ for some suitably large number N .

The result looks like this:



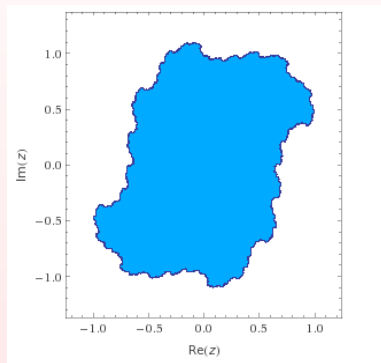
Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



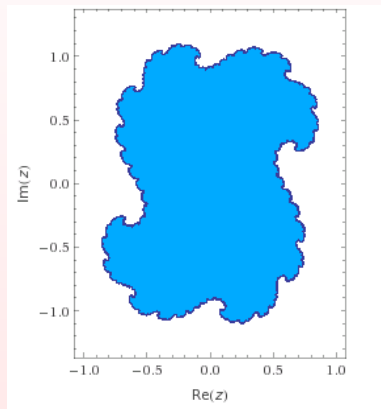
Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



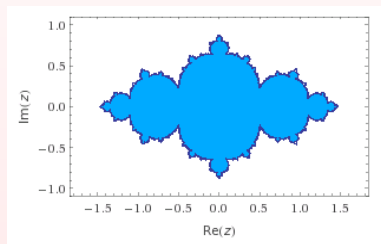
Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



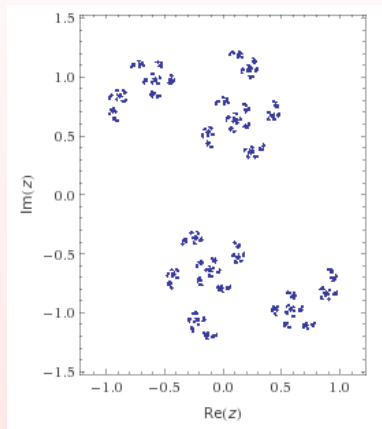
Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



Complex polynomial dynamics

Let try a slightly bigger parameter. Consider
 $f(z) = z^2 + 0.5 + 0.5i$,



The Mandelbrot set

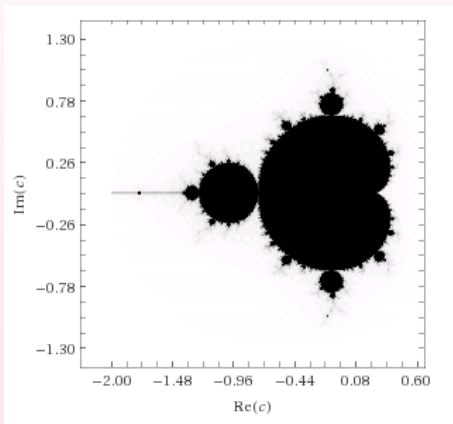
What is going on here? Consider the general polynomial:

$$f_c(z) := z^2 + c$$

with a parameter $c \in \mathbb{C}$. Note: any quadratic polynomial can be transformed into some f_c by an affine change of coordinates.

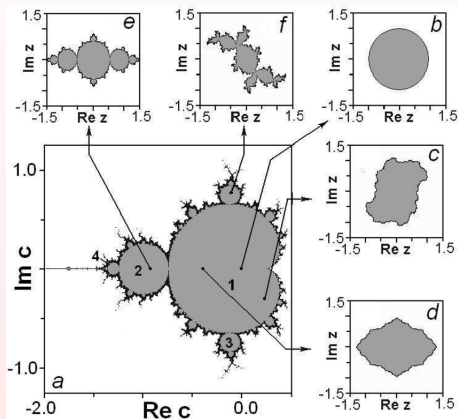
The Mandelbrot set

Consider the case $z_0 = 0$ (for simplicity). For which c is $\{f_c^k(0)\}_{k \geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} : \{f_c^k(0)\}_{k \geq 0} \text{ is bounded}\}$. M can be approximated by a computer:



The Mandelbrot set

A more informative image is this one:



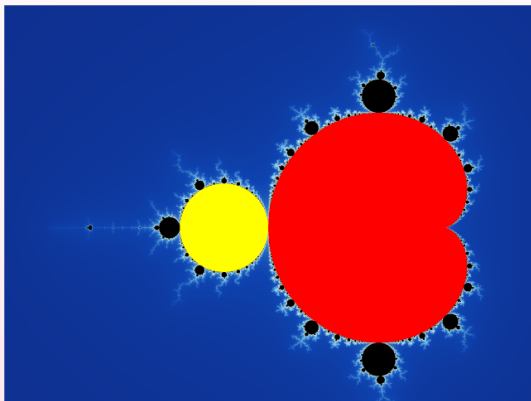
The Julia set

Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The **Julia set** of f_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

Theorem 1 (Mandelbrot)

The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).

Let M_0 be the set $\{\frac{z}{2}(1 - \frac{z}{2}) : |z| < 1\}$. M_0 is an open subset of the Mandelbrot set M called the **main cardioid**, shown below in red:



The main cardioid

The significance of the main cardioid is the following theorem:

Theorem 2

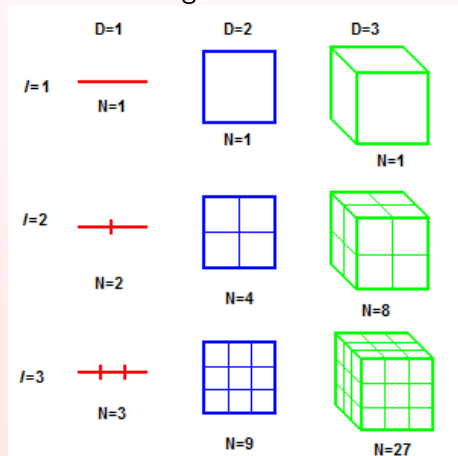
The Julia set $J(f_c)$ of f_c is a Jordan curve (i.e. homeomorphic to a circle) if and only if c is in the main cardioid M_0 .

Very rough outline of the proof:

The connected components of $\mathbb{C} \setminus J(f_c)$ correspond to the attracting basin of the fixed points of f_c . For $c \in M_0$, there are exactly two attracting fixed points (one of them at infinity). \square

Hausdorff dimension

In geometric measure theory, there are a few different notions of “dimension” of a set $X \subset \mathbb{R}^d$. Generally they are based on the idea of “scaling dimension”:



Hausdorff dimension

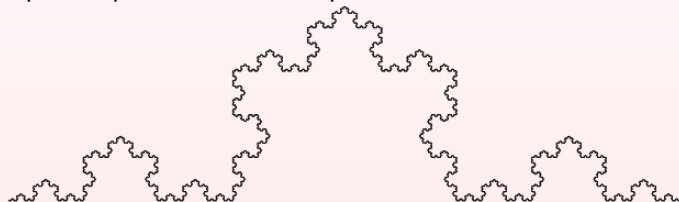
Let $X \subseteq \mathbb{R}^d$. For $q > 0$, let the q -dimensional Hausdorff content of X be:

$$C_q(X) = \inf \left\{ \sum_{j=0}^{\infty} r_j^q : X \subseteq \bigcup_{j=0}^{\infty} B(x_j, r_j) \right\}.$$

The Hausdorff dimension of X is defined to be the infimum of the set of q such that $C_q(X) = 0$.

Hausdorff dimension

Very roughly speaking, a set $X \subset \mathbb{R}^d$ has Hausdorff dimension equal to p if it takes 2^p copies of X to cover $2X$.



The von Koch curve has Hausdorff dimension $\frac{\log(4)}{\log(3)}$.

Hausdorff measure

Corresponding to Hausdorff dimension, there is something called a Hausdorff measure.

Roughly speaking, the p -dimensional Hausdorff measure of a set $X \subset \mathbb{R}^d$ is determined by the relation:

$$\lambda_p(B(x, \varepsilon)) \sim \varepsilon^p.$$

Hausdorff dimension of Julia sets

Fact: If c is in the main cardioid M_0 of the Mandelbrot set M , then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.

Let $g : J(f_c) \rightarrow \mathbb{C}$ be a continuous function. In his 1994 book **Noncommutative geometry**, Alain Connes announced a formula for $\int_J g d\lambda_\rho$ given in terms of quantised calculus.

Canonical coordinates

Theorem 3

Let $c \in M_0$ (the main cardioid). There exists a continuous function:

$$Z : \mathbb{T} \rightarrow J(f_c)$$

which satisfies $Z(\zeta^2) = f_c(Z(\zeta))$ for all $\zeta \in \mathbb{T}$.

The function Z is a **canonical parametrisation** of $J(f_c)$.

Description of the Conformal Trace Formula

Let f be a continuous function on the Julia set $J(f_c)$. Then the operator (on $L_2(\mathbb{T})$).

$$M_{f \circ Z} |\bar{d}Z|^p = M_{f \circ Z} |[F, M_Z]|^p$$

is some kind of “ p -dimensional density” on the Julia set $J(f_c)$ (in the language of quantised calculus).

Description of the Conformal Trace Formula

Lemma 4

The quantised differential $\bar{\partial}Z$ (i.e., the commutator $i[F, M_Z]$) is in the ideal $\mathcal{L}_{p,\infty}(L_2(\mathbb{T}))$.

Hence, the operator $M_{f \circ Z} |\bar{\partial}Z|^p$ is in $\mathcal{L}_{1,\infty}(L_2(\mathbb{T}))$.

Description of the Conformal Trace Formula

Theorem 5

Let tr_ω be a Dixmier trace. Then there is a constant $K(\omega, c)$ such that for all $f \in C(J(f_c))$,

$$\mathrm{tr}_\omega(M_{f \circ Z} |\vec{d}Z|^p) = K(\omega, c) \int_{J(f_c)} f \, d\lambda_p$$

where λ_p is the p -dimensional Hausdorff measure on $J(f_c)$. Also, there exist extended limits ω such that $K(\omega, c) > 0$.

Thank you for listening! Further reading:

Good references for Julia sets and conformal dynamics are,

Carleson and Gamelin, **Complex Dynamics**, 1993

Milnor, **Dynamics in One Complex variable**, 2006.

More information on the quantised calculus may be found in:

Connes, **Noncommutative geometry**, 1994.