Quantised Calculus. Lecture 22: The conformal trace theorem.

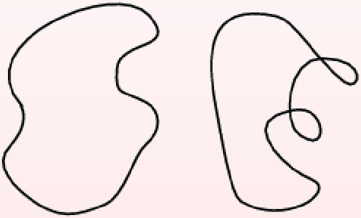
AMSI Summer School, 2019.

Plan for this lecture.

1. A brief introduction to complex polynomial dynamics and Julia sets
2. Geometric measure theory (specifically, Hausdorff meausures)
3. Statement of the conformal trace theorem.

Historical background: Integration on curves

Suppose that you have a simple closed curve $\mathcal{C}$ in the plane. Something like:



Integration on curves

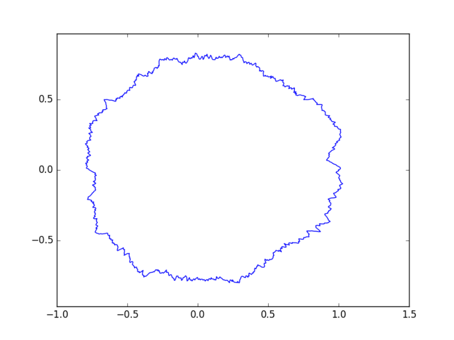
Let $\gamma : [0,1) \to \mathbb{R}^2$ be a parametrisation. Then for a function $f : \mathcal{C} \to \mathbb{C}$, the arc-length integral is defined as:

$$\int_{\mathcal{C}} f \, d\ell = \int_0^1 f(\gamma(t))|\gamma'(t)| \, dt.$$

Of course this only makes sense if $\mathcal{C}$ admits a piecewise-differentiable parametrisation.
But what if it doesn't?

Integration on curves

There are closed curves in the plane which are very "rough" and can look quite "fractal":



Is there a reasonble way to "integrate" a function on these curves? Quantised calculus provides an answer, at least for a very special class of curves which come from conformal dynamics.

Context: Dynamical systems

Let X be a set and let $f : X \to X$. Iteration of the function f defines an iterated function system. Let $z_0 \in X$, and define

$$z_{n+1} = f(z_n), \quad n \geq 0.$$

Immediate questions which concern us: What is the asymptotic behaviour of z_n? How does it depend on z_0? There is an enormous literature on this theme.

Complex polynomial dynamics

Let f be a polynomial with complex coefficients, and take $z_0 \in \mathbb{C}$. consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ for different choices of $z_0 \in \mathbb{C}$. In particular, since f is a polynomial, exactly one of the following happens:

1. Either $|z_n| \to \infty$.
2. $\{z_n\}_{n\geq 0}$ remains bounded.

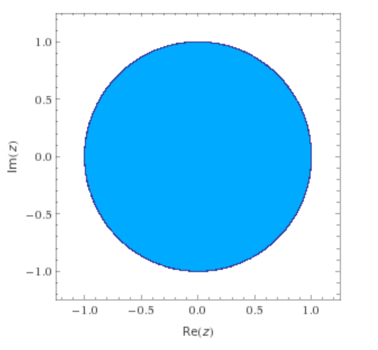
Complex polynomial dynamics

The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = {z_0}^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

1. If $|z_0| < 1$, then $f^k(z_0) \to 0$ as $k \to \infty$.
2. If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
3. If $|z_0| > 1$, then $|f^k(z_0)| \to \infty$ as $k \to \infty$.

Complex polynomial dynamics

Pictorially,



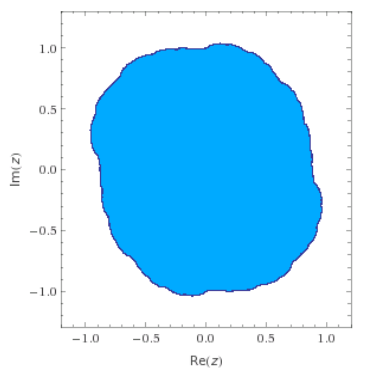
Here, the set of z_0 such that z_n remains bounded is coloured in blue. The set of z_0 such that z_n is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly? Consider $f(z) = z^2 + 0.1 + 0.1i$.

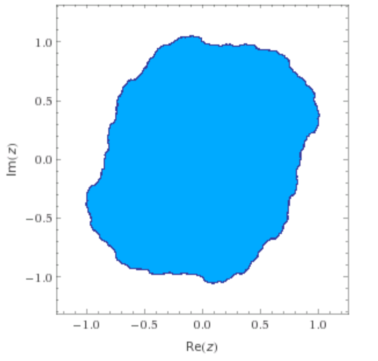
It is not feasible to determine analytically the behaviour of $\{f^n(z)\}_{n\geq 0}$. Instead we use a computer: On a large grid of complex numbers, colour each point z blue if $f^N(z) < 10$ for some suitably large number N.

The result looks like this:



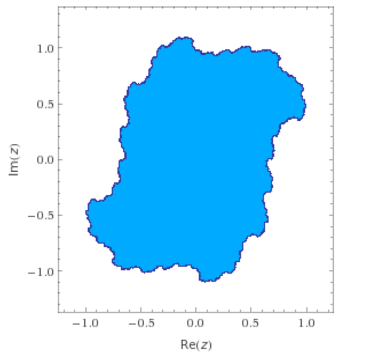
Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



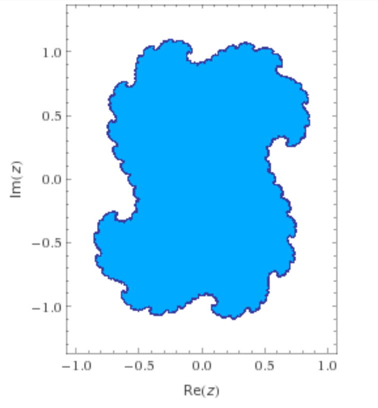
Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



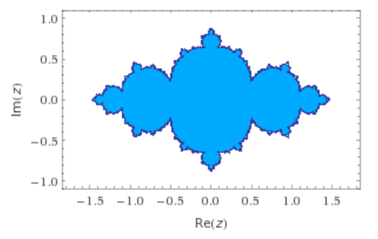
Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



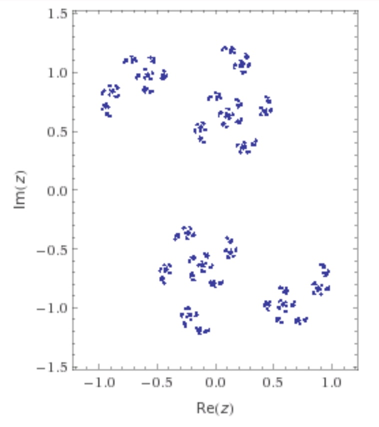
Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



Complex polynomial dynamics

Let try a slightly bigger parameter. Consider
$f(z) = z^2 + 0.5 + 0.5i$,



The Mandelbrot set

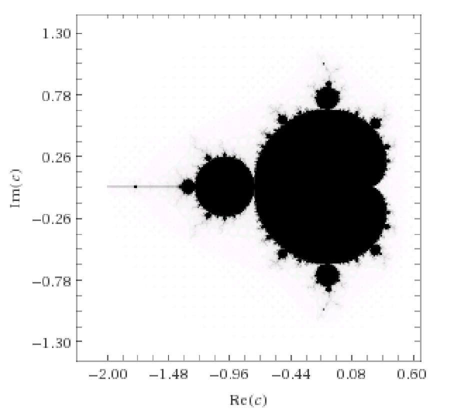
What is going on here? Consider the general polynomial:

$$f_c(z) := z^2 + c$$

with a parameter $c \in \mathbb{C}$. Note: any quadratic polynomial can be transformed into some f_c by an affine change of coordinates.

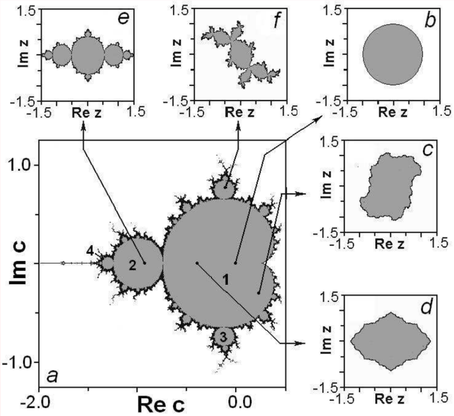
The Mandelbrot set

Consider the case $z_0 = 0$ (for simplicity). For which c is $\{f_c^k(0)\}_{k\geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} \ : \ \{f_c^k(0)\}_{k\geq 0} \text{ is bounded}\}$. M can be approximated by a computer:



The Mandelbrot set

A more informative image is this one:



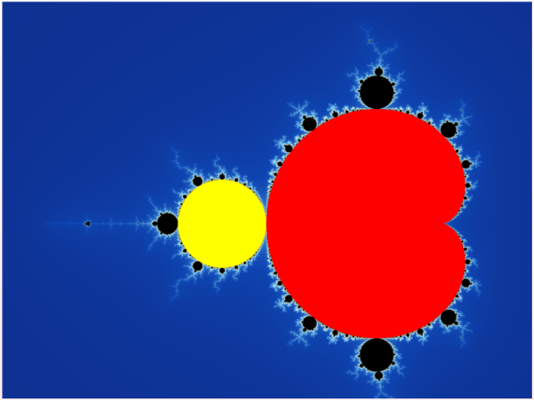
The Julia set

Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The Julia set of f_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

Theorem 1 (Mandelbrot)

The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).

Let M_0 be the set $\{\frac{z}{2}(1-\frac{z}{2}) \ : \ |z|<1\}$. M_0 is an open subset of the Mandelbrot set M called the main cardioid, shown below in red:



The main cardioid

The significance of the main cardioid is the following theorem:

Theorem 2

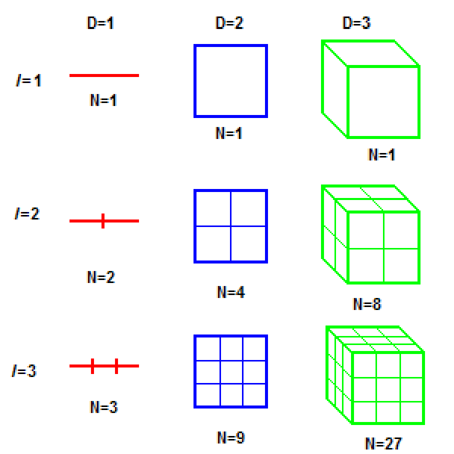
The Julia set $J(f_c)$ of f_c is a Jordan curve (i.e. homeomorphic to a circle) if and only if c is in the main cardioid M_0.

Very rough outline of the proof:

The connected components of $\mathbb{C} \setminus J(f_c)$ correspond to the attracting basin of the fixed points of f_c. For $c \in M_0$, there are exactly two attracting fixed points (one of them at infinity). $\square$

Hausdorff dimension

In geometric measure theory, there are a few different notions of "dimension" of a set $X \subset \mathbb{R}^d$. Generally they are based on the idea of "scaling dimension":



Hausdorff dimension

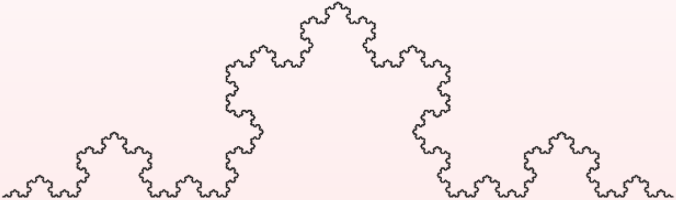
Let $X \subseteq \mathbb{R}^d$. For $q > 0$, let the q-dimensional Hausdorff content of X be:

$$C_q(X) = \inf\{\sum_{j=0}^{\infty} r_j^q \ : \ X \subseteq \bigcup_{j=0}^{\infty} B(x_j, r_j)\}.$$

The Hausdorff dimension of X is defined to be the infimum of the set of q such that $C_q(X) = 0$.

Hausdorff dimension

Very roughly speaking, a set $X \subset \mathbb{R}^d$ has Hausdorff dimension equal to p if it takes 2^p copies of X to cover $2X$.



The von Koch curve has Hausdorff dimension $\frac{\log(4)}{\log(3)}$.

Hausdorff measure

Corresponding to Hausdorff dimension, there is something called a Hausdorff measure.
Roughly speaking, the p-dimensional Hausdorff measure of a set $X \subset \mathbb{R}^d$ is determined by the relation:

$$\lambda_p(B(x, \varepsilon)) \sim \varepsilon^p.$$

Hausdorff dimension of Julia sets

Fact: If c is in the main cardioid M_0 of the Mandelbrot set M, then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.

Let $g : J(f_c) \to \mathbb{C}$ be a continuous function. In his 1994 book Noncommutative geometry, Alain Connes announced a formula for $\int_J g \, d\lambda_p$ given in terms of quantised calculus.

Canonical coordinates

Theorem 3

Let $c \in M_0$ (the main cardioid). There exists a continuous function:

$$Z : \mathbb{T} \to J(f_c)$$

which satisfies $Z(\zeta^2) = f_c(Z(\zeta))$ for all $\zeta \in \mathbb{T}$.

The function Z is a canonical parametrisation of $J(f_c)$.

Description of the Conformal Trace Formula

Let f be a continuous function on the Julia set $J(f_c)$. Then the operator (on $L_2(\mathbb{T})$).

$$M_{f\circ Z}|đZ|^p = M_{f\circ Z}|[F, M_Z]|^p$$

is some kind of "p-dimensional density" on the Julia set $J(f_c)$ (in the language of quantised calculus).

Description of the Conformal Trace Formula

Lemma 4
The quantised differential $\bar{d}Z$ (i.e., the commutator $i[F, M_Z]$) is in the ideal $\mathcal{L}_{p,\infty}(L_2(\mathbb{T}))$.

Hence, the operator $M_{f\circ Z}|\bar{d}Z|^p$ is in $\mathcal{L}_{1,\infty}(L_2(\mathbb{T}))$.

Description of the Conformal Trace Formula

Theorem 5

Let tr_ω be a Dixmier trace. Then there is a constant $K(\omega, c)$ such that for all $f \in C(J(f_c))$,

$$\mathrm{tr}_\omega(M_{f \circ Z}|\bar{\partial} Z|^p) = K(\omega, c) \int_{J(f_c)} f \, d\lambda_p$$

where λ_p is the p-dimensional Hausdorff measure on $J(f_c)$. Also, there exist extended limits ω such that $K(\omega, c) > 0$.

Thank you for listening! Further reading:
Good references for Julia sets and conformal dynamics are,
Carleson and Gamelin, Complex Dynamics, 1993
Milnor, Dynamics in One Complex variable, 2006.
More information on the quantised calculus may be found in:
Connes, Noncommutative geometry, 1994.