Geometric measure on quasi-Fuchsian groups via singular traces. Part II.

Dmitriy Zanin
(in collaboration with A. Connes and F. Sukochev)

University of New South Wales

September 18, 2018

What exactly to prove?

We aim to prove $[F, M_Z] \in \mathcal{L}_{p,\infty}$, where p is the Hausdorff dimension of $\Lambda(G)$.
By Peller theorem, it suffices to show that

$$k \in L_{p,\infty}(\mathbb{D}, (1-|z|^2)^{-2} dz d\bar{z}).$$

Here,

$$k(z) = Z'(z)(1-|z|^2), \quad |z| < 1.$$

Fuchsian group conjugate to G

Consider G acting on $\Lambda(G)_{\mathrm{int}}$. Let π be the action of G on the unit disk by the formula

$$g \circ Z = Z \circ \pi(g), \quad g \in G.$$

Every $\pi(g)$ is a conformal automorphism of the unit disk; hence, $\pi(g)$ is fractional linear.

Thus, $\pi(G)$ is a group of fractional linear transformations preserving the unit circle, i.e. a Fuchsian group and its limit set is the unit circle $\mathbb{T}$, thus it is Fuchsian of the first kind. As a group, $\pi(G)$ is isomorphic to G and is, therefore, finitely generated.

No parabolic elements in the conjugate Fuchsian group

We claim that the Fuchsian group $\pi(G)$ does not contain parabolic elements. Assume the contrary: let $g \in G$ be such that $\pi(g)$ is parabolic. Hence, there exists a fixed point $w_0 \in \mathbb{T}$ of $\pi(g)$ such that

$$(\pi(g))^n w \to w_0, \quad n \to \pm\infty$$

for every $w \in \mathbb{D}$. Let $w = Z(z)$, $z \in \Lambda(G)_{\mathrm{int}}$ and let $w_0 = Z(z_0)$, $z_0 \in \Lambda(G)$. Clearly,

$$g^n(z) \to z_0, \quad n \to \pm\infty.$$

Hence, $g \in G$ is parabolic, which is not the case (we made an assumption that G does not contain parabolic elements).

Riemann surface of conjugate Fuchsian group is compact

The assertion below is Theorem 10.4.3 in [Beardon].

Theorem

If Γ is a finitely generated Fuchsian group of the first kind, then Riemann surface $\mathbb{D}/\Gamma$ has finite area.

The assertion below is Corollary 4.2.7 in [Katok].

Theorem

If Γ is a Fuchsian group without parabolic elements such that Riemann surface $\mathbb{D}/\Gamma$ has finite area, then $\mathbb{D}/\Gamma$ is compact.

A combination of these assertions yields that Riemann surface of $\pi(G)$ is compact.

Fundamental domain of conjugate Fuchsian group is compactly supported in $\mathbb{D}$.

The assertion below is a combination of Corollary 4.2.3 and Theorem 3.2.2 in [Katok].

Theorem

If Γ is a Fuchsian group whose Riemann surface $\mathbb{D}/\Gamma$ is compact, then there exists a compact fundamental domain $\mathbb{F}$ of Γ.

In particular, $\pi(G)$ admits a fundamental domain $\mathbb{F}$ which is compactly supported in $\mathbb{D}$.

The usage of compact fundamental domains

Lemma

We have

$$\sup_{z\in\pi(g)\mathbb{F}}(1-|z|^2)|Z'(z)|\leq\frac{\mathrm{const}}{|g_{21}|^2}.$$

Proof.

Let $z=\pi(g)w$ with $w\in\mathbb{F}$. Conformal invariance of hyperbolic metric and the chain rule yield

$$(1-|z|^2)|Z'(z)|=|g'(Z(w))|\cdot(1-|w|^2)|Z'(w)|.$$

Obviously,

$$|g'(Z(w))|=|g_{21}Z(w)+g_{22}|^{-2}=|g_{21}|^{-2}\cdot|Z(w)-g^{-1}(\infty)|^{-2}.$$

The first factor is bounded by $|g_{21}|^{-2}$ and the second one is bounded. □

Critical exponent of the group G

For a Kleinian group G, the series

$$\sum_{g\in G}|g'(z)|^2$$

converges for almost every (with respect to Lebesgue measure) $z\in\bar{\mathbb{C}}$. The critical exponent of G is defined as follows

$$p=\inf\Big\{q:\ \sum_{g\in G}|g'(z)|^q \text{ converges for a.e. } z\in\bar{\mathbb{C}}\Big\}.$$

Proof of the main result, part (a)

G is a quasiconformal deformation of a Fuchsian group of the first kind. In particular, its limit set $\Lambda(G)$ is a quasi-circle. Hence, the Hausdorff dimension of $\Lambda(G)$ is strictly less than 2.

G is finitely generated and, by the Ahlfors Finiteness Theorem, G is analytically finite. By Bishop-Jones theorem, G is geometrically finite. In particular, its critical exponent p equals to the Hausdorff dimension.

A few hours of meditation over [Sullivan] deliver that $\{\|g\|_{\infty}^{-2}\}_{g\in G} \in l_{p,\infty}$. Hence, also $\{g_{21}^{-2}\}_{1\neq g\in G} \in l_{p,\infty}$. By the above lemma, we have that

$$k \in L_{p,\infty}(\mathbb{D}, (1-|z|^2)^{-2} dz d\bar{z}).$$

Restatement

Let ν be a finite measure such that

$$\varphi(M_{f\circ Z}|[F,M_Z]|^p)=\int_{\Lambda(G)} f(z)d\nu(z).$$

We aim to show that

$$d(\nu\circ g)(z)=|g'(z)|^p d\nu(z).$$

Equivalently, we want

$$\varphi(M_{f\circ g^{-1}\circ Z}|[F,M_Z]|^p)=\varphi(M_{(f|g'|^p)\circ Z}|[F,M_Z]|^p).$$

Represenation of $\mathrm{SU}(1,1)$ commutes with F

Let

$$(U_h\xi)(z) = \frac{1}{\bar{\beta}z + \bar{\alpha}}\xi(\frac{\alpha z + \beta}{\bar{\beta}z + \bar{\alpha}})$$

for every $\xi \in L_2(\mathbb{T})$ and for every $z \in \mathbb{T}$. Here,

$$h = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1.$$

Lemma

The mapping $h \to U_h$ is a unitary representation of the group $\mathrm{SU}(1,1)$ which commutes with F.

Idea of the proof

We have

$$U_{\pi(g)} \cdot M_{f\circ g^{-1}\circ Z}|[F, M_Z]|^p \cdot U_{\pi(g)}^{-1} = M_{f\circ Z}|[F, M_{g\circ Z}]|^p.$$

Hence,

$$\varphi(M_{f\circ g^{-1}\circ Z}|[F, M_Z]|^p) = \varphi(M_{f\circ Z}|[F, M_{g\circ Z}]|^p).$$

IF WE HAD

$$\big|[F, M_{g\circ Z}]|^p - [F, M_{g\circ Z}]|^p|g'(Z)|^p\big| \in (\mathcal{L}_{1,\infty})_0,$$

then

$$\varphi(M_{f\circ g^{-1}\circ Z}|[F, M_Z]|^p) = \varphi(M_{(f|g'|^p)\circ Z}|[F, M_Z]|^p).$$

Core lemma

Lemma

Let $0 \leq A \in \mathcal{L}_\infty$ *and* $0 \leq B \in \mathcal{L}_{p,\infty}$ *be such that* $[A^{\frac{1}{2}}, B] \in (\mathcal{L}_{p,\infty})_0$, *then*

$$B^p A^p - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^p \in (\mathcal{L}_{1,\infty})_0.$$

Set $A = M_{|g'|\circ Z}$ and $B = [F, M_{f\circ Z}]$. Long but elementary computation shows that

$$|[F, M_{g\circ Z}]|^p - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^p \in (\mathcal{L}_{1,\infty})_0.$$

Applying the lemma, we obtain

$$|[F, M_{g\circ Z}]|^p - B^p A^p \in (\mathcal{L}_{1,\infty})_0.$$

Lemma

Let $X, Y \geq 0$. There exists a Schwartz function g_p such that

$$X^p - Y^p = V - \int_{\mathbb{R}} X^{is} V Y^{-is} g_p(s) ds,$$

where

$$V = X^{p-1}(X - Y) + (X - Y)Y^{p-1}.$$

Proof.

It suffices to prove the assertion for the case when X and Y have finite spectra. Multiplying equality on the left by $\chi_{\{\lambda\}}(X)$ and on the right by $\chi_{\{\mu\}}(Y)$, it suffices to prove that

$$\lambda^p - \mu^p = (\lambda - \mu)(\lambda^{p-1} + \mu^{p-1}) \cdot \Big(1 - \int_{\mathbb{R}} \lambda^{is} \mu^{-is} g_p(s) ds\Big).$$

This is a commutative assertion which can be verified directly. □

Lemma

Let $A, B \geq 0$. We have

$$B^p A^p - Y^p = T(0) - \int_{\mathbb{R}} T(s) g_p(s) ds,$$

where $Y = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ and

$$T(s) = B^{p-1+is}[B, A^{p-\frac{1}{2}+is}]A^{\frac{1}{2}} Y^{-is} + B^{is}[B, A^{\frac{1}{2}+is}]A^{\frac{1}{2}} Y^{p-1-is}.$$

Again, it suffices to prove the assertion for the case when B has finite spectrum.

Proof.

If $B = \sum_j \lambda_j p_j$, then

$$B^p A^p - Y^p = \sum_j p_j((\lambda_j A)^p - Y^p).$$

Applying the preceding lemma to $X = \lambda_j A$ and Y, we obtain

$$B^p A^p - Y^p = \sum_j p_j \Big(V_j - \int_{\mathbb{R}} (\lambda_j A)^{is} V_j Y^{-is} g_p(s) ds \Big) =$$

$$= \Big(\sum_j p_j V_j \Big) - \int_{\mathbb{R}} \Big(\sum_j p_j (\lambda_j A)^{is} V_j Y^{-is} \Big) g_p(s) ds.$$

Here,

$$V_j = (\lambda_j A)^{p-1}(\lambda_j A - Y) + (\lambda_j A - Y) Y^{p-1}.$$

$\square$

Proof.

Note that

$$\sum_j p_j V_j = \sum_j p_j(\lambda_j^p A^p - \lambda_j^{p-1} A^{p-1} Y + \lambda_j A Y^{p-1} - Y^p) =$$

$$= \Big(\sum_j \lambda_j^p p_j\Big) A^p - \Big(\sum_j \lambda_j^{p-1} p_j\Big) A^{p-1} Y +$$

$$+ \Big(\sum_j \lambda_j p_j\Big) A Y^{p-1} - \Big(\sum_j p_j\Big) Y^p =$$

$$= B^p A^p - B^{p-1} A^{p-1} Y + BAY^{p-1} - Y^p = T(0).$$

Similarly,

$$\sum_j p_j (\lambda_j A)^{is} V_j Y^{-is} = T(s).$$

$\square$

Proof of the core lemma I

We are now ready to prove the core lemma.

Proof.

We write

$$B^p A^p - Y^p = B^{p-1} \cdot I + II \cdot Y^{p-1},$$

where

$$I = [B, A^{p-\frac{1}{2}}]A^{\frac{1}{2}} + \int_{\mathbb{R}} B^{is}[B, A^{p-\frac{1}{2}+is}]A^{\frac{1}{2}} Y^{-is} g_p(s)ds,$$

$$II = [B, A^{\frac{1}{2}}]A^{\frac{1}{2}} + \int_{\mathbb{R}} B^{is}[B, A^{\frac{1}{2}+is}]A^{\frac{1}{2}} Y^{-is} g_p(s)ds.$$

By Hölder inequality, it suffices to show that $I, II \in (\mathcal{L}_{p,\infty})_0$. □

Proof of the core lemma II

Consider I. We have

$$[B, A^{p-\frac{1}{2}+is}] \in (\mathcal{L}_{p,\infty})_0$$

and

$$\|[B, A^{p-\frac{1}{2}+is}]\|_{p,\infty} \leq (1+|s|)\|A\|_\infty^{p-1}\|[B, A^{\frac{1}{2}}]\|_{p,\infty}.$$

The integrand is measurable in weak operator topology. Since $(\mathcal{L}_{p,\infty})_0$ is a separable Banach space, it follows that integrand is Bochner measurable in $(\mathcal{L}_{p,\infty})_0$. Since

$$\int_{\mathbb{R}} (1+|s|)|g_p(s)|ds < \infty,$$

it follows that the integrand is Bochner integrable. Hence, $I \in (\mathcal{L}_{p,\infty})_0$. Similarly, $II \in (\mathcal{L}_{p,\infty})_0$.

THANK YOU FOR YOUR ATTENTION