Geometric measure on quasi-Fuchsian groups via singular traces. Part I.

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Kleinian groups: basics

We let $\mathrm{SL}(2,\mathbb{C})$ be the group of all 2×2 complex matrices with determinant 1. We identify the group $\mathrm{PSL}(2,\mathbb{C})=\mathrm{SL}(2,\mathbb{C})/\{\pm 1\}$ and its action on the complex sphere $\bar{\mathbb{C}}$ by fractional linear transformations. The matrix

$$g=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} \text{ is identified with the mapping } z\to\frac{g_{11}z+g_{12}}{g_{21}z+g_{22}}.$$

We use the following definition of a Kleinian group.

Definition

Let $G\subset\mathrm{PSL}(2,\mathbb{C})$ be a discrete subgroup. We say that

(a) G is regular at the point $z\in\bar{\mathbb{C}}$ if there exists a neighborhood $U\ni z$ such that $g(U)\cap U=\varnothing$ for all but finitely many $g\in G$.

(b) G is Kleinian if it is regular at some point $z\in\bar{\mathbb{C}}$.

Limit set of a Kleinian group

Definition

Let G be a Kleinian group.

(a) the set of all points at which G is regular is called a regular set of G.

(b) the complement of the regular set is called a limit set of G.

The limit set is denoted by $\Lambda(G)$. In the theory of Kleinian groups it plays the same role as the Julia set in conformal dynamics.

By definition, $\Lambda(G)$ is a closed $G-$invariant set. One can show that $\Lambda(G)$ is nowhere dense.

Usually, $\Lambda(G)$ is a perfect set (i.e., every point is an accumulation point). Exceptions are so-called elementary Kleinian groups, whose limit sets contain at most 2 points.

Fuchsian and quasi-Fuchsian groups

Definition

A Kleinian group G is called

(a) Fuchsian (of the first kind) if its limit set is a circle.

(b) quasi-Fuchsian if its limit set is a Jordan curve.

In this talk, we are mostly interested in finitely generated quasi-Fuchsian groups.
By analogue with the Julia sets, one would expect that a limit set of a quasi-Fuchsian group is a fractal curve. In particular, one would expect that its Hausdorff dimension is strictly greater than 1.

Hausdorff dimension of the limit set

Definition

We say that the Hausdorff dimension of a set $X \subset \mathbb{C}$ is less than q if, for every $\epsilon > 0$, there exist balls $(B(a_i, r_i))_{i \in I}$ such that

$$X \subset \bigcup_{i \in I} B(a_i, r_i), \quad \sum_{i \in I} r_i^q < \epsilon.$$

The infimum of all such q is called the Hausdorff dimension of a set X.

Theorem

Let G be a finitely generated quasi-Fuchsian (but not Fuchsian) group. Its limit set $\Lambda(G)$ has Hausdorff dimension strictly greater than 1.

Geometric measure

Definition

Let G be a Kleinian group. The measure ν on $\bar{\mathbb{C}}$ is called $p-$dimensional geometric (relative to G) if

$$d(\nu \circ g)(z) = |g'(z)|^p d\nu(z)$$

for every $g \in G$.

We are particularly interested in the case when p is the Hausdorff dimension of $\Lambda(G)$.

Theorem (Sullivan)

Let G be a geometrically finite Kleinian group. If p is the Hausdorff dimension of $\Lambda(G)$, then $p < 2$ and there exists a unique $p-$dimensional geometric measure supported on $\Lambda(G)$.

How to express geometric measure?

Geometric measures are contructed by means of a complicated process. It is strongly desirable to have a simple and accessible construction.

Question

Is there any construction of geometric measure from the first principles?

In this talk, we propose such a construction for the special case of finitely generated quasi-Fuchsian groups. The only components of these construction are

1. Riemann mapping theorem
2. Hilbert transform
3. trace on $\mathcal{L}_{1,\infty}$

General information

Let $\mathcal{L}_\infty$ be the $*-$algebra of all bounded operators on a given (separable, infinite dimensional) Hilbert space H.

An operator is called compact if it can be approximated (in norm topology, with any given precision) by a finite rank operator. Spectrum of a compact operator consists of non-zero eigenvalues of finite multiplicity converging to 0 (which may or may not be an eigenvalue).

If A is compact, then $|A|$ is compact. Eigenvalues of $|A|$ are called singular values of A.

For a compact operator A, we define its singular value sequence $\mu(A) = (\mu(k, A))_{k\geq 0}$ by arranging the eigenvalues of $|A|$ in the decreasing order and taking them with multiplicities.

Ideals and infinitesimals

An ideal $\mathcal{I}$ in $\mathcal{L}_\infty$ is a linear subspace (usually *not* closed in norm) such that $A \in \mathcal{I}$ and $B \in \mathcal{L}_\infty$ implies that $AB, BA \in \mathcal{I}$. Ideal is called principal if it is generated by a single element. Every non-trivial ideal in $\mathcal{L}_\infty$ consists of compact operators.

Principal ideal generated by the diagonal operator $\mathrm{diag}((\frac{1}{(k+1)^{1/p}})_{k\geq 0})$ is called $\mathcal{L}_{p,\infty}$. For every $p > 0$, it is quasi-Banach (see next page). Equivalently,

$$\mathcal{L}_{p,\infty} = \Big\{A \in \mathcal{L}_\infty : \ \mu(k, A) = O((k+1)^{-\frac{1}{p}})\Big\}.$$

In Connes ideology, these are "infinitesimals of order $\frac{1}{p}$".

Quasi-Banach ideals

Definition

An ideal $\mathcal{I}$ in $\mathcal{L}_\infty$ is called quasi-Banach when equipped with a complete quasi-norm $\|\cdot\|_{\mathcal{I}}$ such that

$$\|AB\|_{\mathcal{I}}, \|BA\|_{\mathcal{I}} \leq \|A\|_{\mathcal{I}}\|B\|_\infty.$$

For example, a natural quasi-norm on the ideal $\mathcal{L}_{p,\infty}$ is given by the formula

$$\|A\|_{p,\infty} = \sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,A).$$

When equipped with this quasi-norm, $\mathcal{L}_{p,\infty}$ becomes a quasi-Banach ideal. In fact, for $p>1$ its natural quasi-norm is equivalent to a norm.
We let $(\mathcal{L}_{p,\infty})_0$ to be the closure of finite rank operators with respect to the quasi-norm $\|\cdot\|_{p,\infty}$.

Traces on ideals

Definition

Let $\mathcal{I}$ be an ideal in $\mathcal{L}_\infty$. Linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ is called trace if

$$\varphi(AB) = \varphi(BA), \quad A \in \mathcal{I}, \quad B \in \mathcal{L}_\infty.$$

Equivalently, for all unitary $U \in \mathcal{L}_\infty$,

$$\varphi(U^{-1}AU) = \varphi(A), \quad A \in \mathcal{I}.$$

Traces on ideals: $\mathcal{L}_{p,\infty}$, $p > 1$

Example

For $p > 1$, ideal $\mathcal{L}_{p,\infty}$ does not carry any trace.

Proof.

If $X \in \mathcal{L}_{p,\infty}$, then there exist $(X_k)_{k=1}^{20} \subset \mathcal{L}_{p,\infty}$ and $(Y_k)_{k=1}^{20} \subset \mathcal{L}_\infty$ such that

$$X = \sum_{k=1}^{20} [X_k, Y_k].$$

Hence, for every trace φ, we have

$$\varphi(X) = \sum_{k=1}^{20} \varphi(X_k Y_k) - \varphi(Y_k X_k) = 0.$$

$\square$

Traces on ideals: $\mathcal{L}_{1,\infty}$

Ideal $\mathcal{L}_{1,\infty}$ carries a plethora of traces. The most famous one is due to Dixmier.

Definition

Let ω be a free ultrafilter on $\mathbb{Z}_+$. The mapping

$$\mathrm{Tr}_\omega : A \to \lim_{n\to\omega} \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu(k,A), \quad 0 \leq A \in \mathcal{L}_{1,\infty}$$

is additive. Its linear extension to $\mathcal{L}_{1,\infty}$ is called Dixmier trace.

Traces on $\mathcal{L}_{1,\infty}$: further properties

1. Every Dixmier trace is positive.
2. Every positive trace on $\mathcal{L}_{1,\infty}$ is continuous.
3. Every continuous trace on $\mathcal{L}_{1,\infty}$ is a linear combination of positive ones.
4. There are continuous traces on $\mathcal{L}_{1,\infty}$ which are not Dixmier traces.
5. There are traces on $\mathcal{L}_{1,\infty}$ which fail to be continuous.
6. There are $2^{2^{\mathbb{N}}}$ continuous traces on $\mathcal{L}_{1,\infty}$.
7. Every trace on $\mathcal{L}_{1,\infty}$ vanishes on $\mathcal{L}_1$.
8. Every continuous trace on $\mathcal{L}_{1,\infty}$ vanishes on $(\mathcal{L}_{1,\infty})_0$.

Morphism from $C(\Lambda(G))$ to $C(\mathbb{T})$

For quasi-Fuchsian group G, the limit set is a Jordan curve. Hence, it divides the complex sphere $\bar{\mathbb{C}}$ into 2 simply connected parts: $\Lambda(G)_{\rm int}$ and $\Lambda(G)_{\rm ext}$.
Riemann mapping theorem says that a simply connected domain $\Lambda(G)_{\rm int}$ is conformally equivalent to the unit disk $\mathbb{D}$. Let $Z : \mathbb{D} \to \Lambda(G)_{\rm int}$ be the conformal equivalence.
By Caratheodory theorem, Z extends to a homeomorphism $Z : \mathbb{T} \to \Lambda(G)$.
We now have a natural morphism

$$C(\Lambda(G)) \to C(\mathbb{T}) \quad f \to f \circ Z.$$

The Hilbert transform

The Hilbert space $L_2(\mathbb{T})$ is defined with respect to the arc-length measure (the Haar measure).
There is the trigonometric orthonormal basis for $L_2(\mathbb{T})$,

$$e_n(z) = z^n, \quad n \in \mathbb{Z}, z \in \mathbb{T}.$$

The Hilbert transform F is defined on the basis e_n by $Fe_n = \mathrm{sgn}(n) e_n$.

Main theorem

The assertion below was proposed (without rigorous proof) in the "Noncommutative Geometry" by Connes. Complete proof appeared in a recent paper by the authors.

Theorem

Let G be a finitely generated quasi-Fuchsian group without parabolic elements ane let p be the Hausdorff dimension of $\Lambda(G)$.

(a) $[F, M_Z] \in \mathcal{L}_{p,\infty}$

(b) for every continuous trace φ on $\mathcal{L}_{1,\infty}$ and for every $f \in C(\Lambda(G))$ we have

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = c_\varphi \int_{\Lambda(G)} f(z)d\nu(z),$$

where ν is the unique $p-$dimensional geometric measure on $\Lambda(G)$.

(c) there is a positive trace φ on $\mathcal{L}_{1,\infty}$ such that $c_\varphi > 0$.

Where $[F, M_h]$ belongs?

It is a classical result by Kronecker that $[F, M_h]$ is finite rank if and only if h is a rational function.
Nehari proved that $[F, M_h]$ is bounded if and only if h has bounded mean oscillation (in short, $h \in \mathrm{BMO}$).
It is immediate that $[F, M_h] \in \mathcal{L}_2$ if and only if h belongs to the Sobolev space $W^{\frac{1}{2},2}$.

Question

When $[F, M_h]$ belongs to $\mathcal{L}_{p,\infty}$?

Peller theorem

The answer to the question above can be derived from the results by Peller.

Theorem

The operator $[F, M_h]$ belongs to $\mathcal{L}_p$ if and only if h belongs to a Besov space $B_p^{\frac{1}{p}}$.

We want to apply this result to the function $h = Z$. For this function, we only know the behavior inside $\mathbb{D}$, while its behavior on the boundary is much harder to investigate. It, therefore, makes sense to restate Peller's result in terms of analytic extension of the function h to the unit disk.

Restatement of Peller's result for $\mathcal{L}_p$, $1 \leq p \leq 2$

Theorem

Suppose h admits an extension to the unit disk. The operator $[F, h]$ belongs to $\mathcal{L}_1$ if and only if $h'' \in L_1(\mathbb{D})$.

It is immediate that $[F, M_h] \in \mathcal{L}_2$ if and only if $h'' \in L_2(\mathbb{D}, (1-|z|^2)dzd\bar{z})$. An intepolation argument yields

Theorem

Suppose h admits an extension to the unit disk. The operator $[F, h]$ belongs to $\mathcal{L}_p$ if and only if $h'' \in L_p(\mathbb{D}, (1-|z|^2)^{2p-2}dzd\bar{z})$.

Peller-type result for $\mathcal{L}_{p,\infty}$, $1 < p < 2$

Preceding theorem can be simplified for $p > 1$ as follows:

Theorem

Suppose h admits an extension to the unit disk. The operator $[F, h]$ belongs to $\mathcal{L}_p$ if and only if $h' \in L_p(\mathbb{D}, (1 - |z|^2)^{p-2} dz d\bar{z})$.

An interpolation argument yields

Theorem

Suppose h admits an extension to the unit disk. The operator $[F, h]$ belongs to $\mathcal{L}_{p,\infty}$ if and only if $k \in L_{p,\infty}(\mathbb{D}, (1 - |z|^2)^{-2} dz d\bar{z})$. Here,

$$k(z) = h'(z)(1 - |z|^2), \quad |z| < 1.$$

TO BE CONTINUED