# The Hausdorff measure of Julia sets from singular traces

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## What is this talk?

This talk concerns the paper, *The conformal trace theorem for Julia sets of quadratic polynomials* (ETDS, 2017) from myself, A Connes, F Sukochev and D Zanin.

## What this talk is about

Part I will cover:

1. A brief introduction to complex polynomial dynamics and Julia sets
2. Geometric measure theory (specifically, Hausdorff meausures)

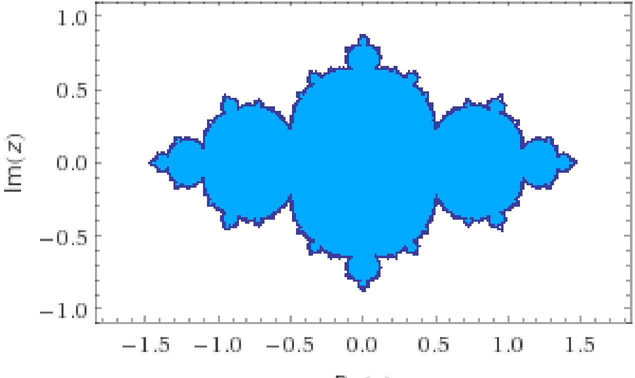
Part II will cover:

1. Statement of the conformal trace theorem, and an outline the proof
2. Prospects for future work.

# Historical background: What is the conformal trace theorem?

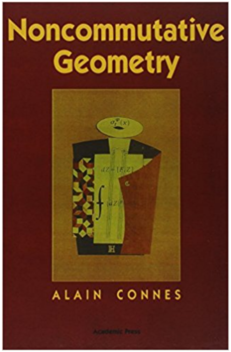
Let $c \in \mathbb{C}$ be small, and let $f_c(z) = z^2 + c$. The Julia set $J$ of $f_c$ is the boundary of the set of points $z$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

When $c \approx 0$, $J$ is a Jordan curve:



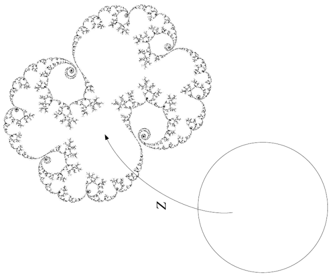
## Background (continued)

In 1994 in his book *Noncommutative Geometry*, A. Connes introduced a formula for the Hausdorff measure of a Julia set in terms of his quantised calculus:



## Background (continued)

To state the formula properly takes some work, but the key ingredients are as follows: take a Julia set $J$ and let $Z : \mathbb{T} \to J$ be the extension to the boundary of the conformal equivalence between the exterior of the unit disk and the exterior of $J$:




$Z$ is typically not differentiable, and it is typically not even of bounded variation.

# Background (continued)

The conformal trace theorem then states that for all continuous normalised traces $\varphi$, there is a constant $C_\varphi$ such that:

$$c_\varphi \int_J f \, d\lambda_p = \varphi(M_{f \circ Z} |\bar{d} Z|^p).$$

where $p$ is the Hausdorff dimension of $J$ and $\lambda_p$ is the $p$-dimensional Hausdorff measure. Connes and Sullivan introduced this formula as a way of computing integrals with respect to the Hausdorff measure on Julia sets.

# Background (continued)

Despite being announced as early as 1994, Connes and Sullivan's proof of the conformal trace theorem was never published.
In our paper we provided a new proof, using operator integration techniques which did not exist in 1994.

## Background (end)

The most technically challenging part of the proof is the so-called "quantised change of variables formula":

$$|[F, M_{f\circ Z}]|^p - |f'(M_Z)|^p|[F, M_Z]|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Where $Z \in C(\mathbb{T})$, and $f$ is a polynomial. It is quite easy to show that:

$$[F, M_{f\circ Z}] - f'(M_Z)[F, M_Z] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$$

but "taking a power $p$" is highly nontrivial.

# Part I: Basic conformal dynamics

# Complex polynomial dynamics

Let $f$ be a polynomial with complex coefficients, and take $z_0 \in \mathbb{C}$. consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ for different choices of $z_0 \in \mathbb{C}$. In particular, since $f$ is a polynomial, exactly one of the following happens:

1. Either $|z_n| \to \infty$.
2. $\{z_n\}_{n\geq 0}$ remains bounded.

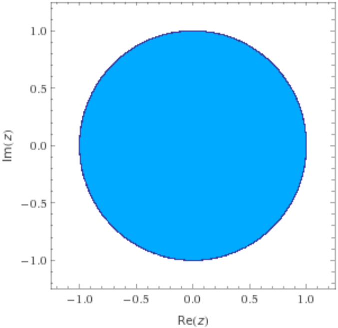
## Complex polynomial dynamics

The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = {z_0}^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

1. If $|z_0| < 1$, then $f^k(z_0) \to 0$ as $k \to \infty$.
2. If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
3. If $|z_0| > 1$, then $|f^k(z_0)| \to \infty$ as $k \to \infty$.

# Complex polynomial dynamics

Pictorially,



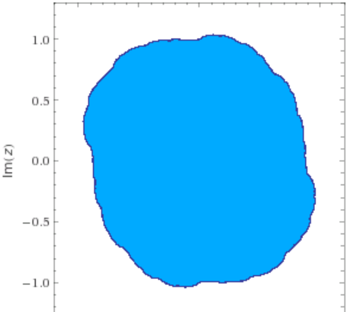

Here, the set of $z_0$ such that $z_n$ remains bounded is coloured in blue. The set of $z_0$ such that $z_n$ is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

# Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly? Consider $f(z) = z^2 + 0.1 + 0.1i$.

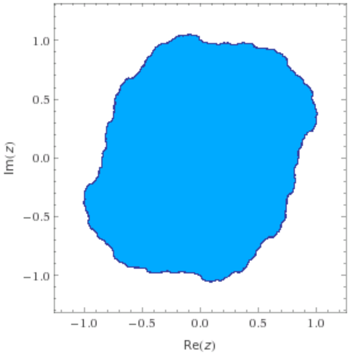
It is not feasible to determine analytically the behaviour of $\{f^n(z)\}_{n\geq 0}$. Instead we use a computer: On a large grid of complex numbers, colour each point $z$ blue if $f^N(z) < 10$ for some suitably large number $N$.

The result looks like this:



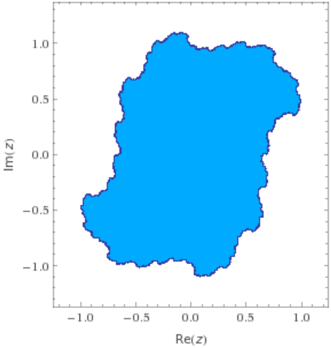
# Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



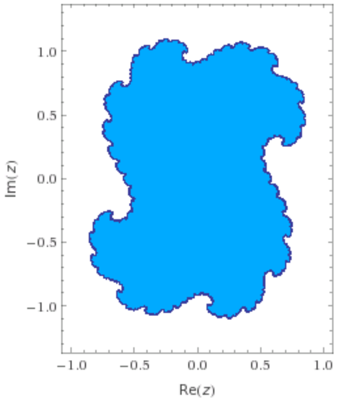
# Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



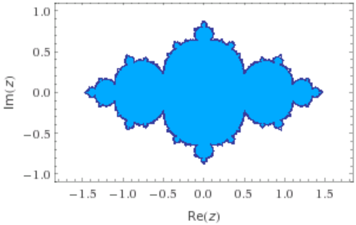
# Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



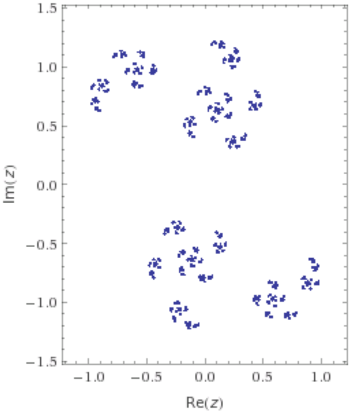
# Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



# Complex polynomial dynamics

Let try a slightly bigger parameter. Consider
$f(z) = z^2 + 0.5 + 0.5i$,



# The Mandelbrot set

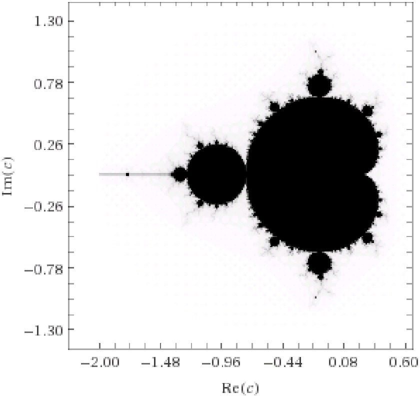
What is going on here? Consider the general polynomial:

$$f_c(z) := z^2 + c$$

with a parameter $c \in \mathbb{C}$. Note: any quadratic polynomial can be transformed into some $f_c$ by an affine change of coordinates.

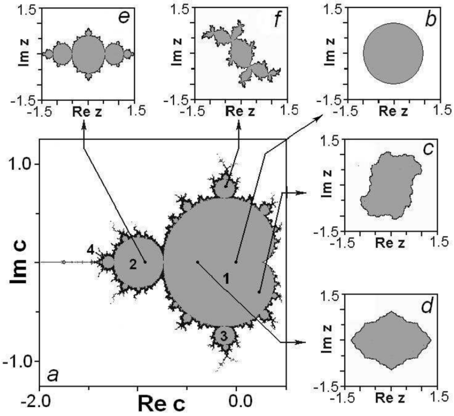
# The Mandelbrot set

Consider the case $z_0 = 0$ (for simplicity). For which $c$ is $\{f_c^k(0)\}_{k\geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} \ : \ \{f_c^k(0)\}_{k\geq 0} \text{ is bounded}\}$. $M$ can be approximated by a computer:



# The Mandelbrot set

A more informative image is this one:



# The Julia set

Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The *Julia set* of $f_c$ is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

**Theorem (Mandelbrot)**

*The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).*

# Fixed points

The asymptotic behaviour of $\{f^n(z_0)\}_{n\geq 0}$ is best understood by examining the fixed points ($f(z) = z$) of $f$.

A fixed point $\lambda$ is said to be:

1. Attracting if $|f'(\lambda)| < 1$,
2. Repelling if $|f'(\lambda)| > 1$,
3. Neutral if $|f'(\lambda)| = 1$.

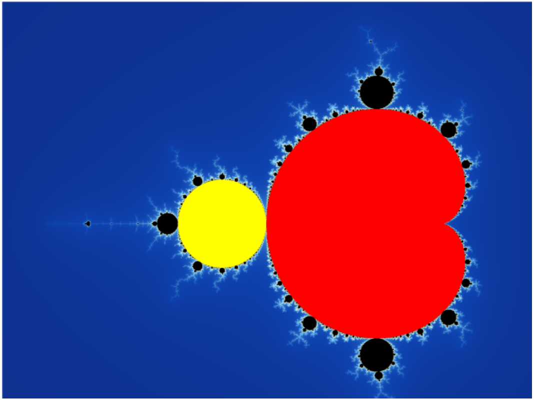
An attracting fixed point $\lambda$ can be called *super-attracting* if $f'(\lambda) = 0$.

# Fixed points (cont.)

The behaviour of $\{f^n(z)\}_{n\geq 0}$ near an attracting fixed point is easily described:
If $z$ is sufficiently close to an attracting fixed point $\lambda$, then $f^n(z) \to \lambda$ exponentially fast.
If $\lambda$ is superattracting, then $f^n(z) \to \lambda$ super-exponentially fast.
The set of all $z$ such that $f^n(z) \to \lambda$ as $n \to \infty$ is called the attracting basin of $\lambda$. It is easy to see that an attracting basin is open.

# The main cardioid

When does $f(z) = z^2 + c$ have an attracting fixed point? Solve for $c$: $z^2 - z + c = 0$, $|2z| < 1$.
Let $M_0$ be the set $\{\frac{z}{2}(1 - \frac{z}{2}) \ : \ |z| < 1\}$. $M_0$ is an open subset of the Mandelbrot set $M$ called the *main cardioid*, shown below in red:



# The main cardioid

The significance of the main cardioid is the following theorem:

**Theorem**

*The Julia set $J(f_c)$ of $f_c$ is a Jordan curve (i.e. homeomorphic to a circle) if and only if $c$ is in the main cardioid $M_0$.*

**Very rough outline of the proof:**

The connected components of $\mathbb{C} \setminus J(f_c)$ correspond to the attracting basin of the fixed points of $f_c$. For $c \in M_0$, there are exactly two attracting fixed points (one of them at infinity). $\square$

# Hausdorff dimension (reminder)

Let $(X, d)$ be a metric space. For $q > 0$, let the $q$-dimensional Hausdorff content of $X$ be:

$$C_q(X) = \inf\{\sum_{j=0}^{\infty} r_j^q \ : \ X \subseteq \bigcup_{j=0}^{\infty} B(x_j, r_j)\}.$$

The Hausdorff dimension of $X$ is defined to be the infimum of the set of $q$ such that $C_q(X) = 0$.

# Hausdorff dimension of Julia sets

Fact: If $c$ is in the main cardioid $M_0$ of the Mandelbrot set $M$, then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.

# Minkowski content

A close relative of Hausdorff dimension is Minkowski dimension. Let $X \subseteq \mathbb{R}^d$. If $\delta > 0$, let $S_\delta(X) = \bigcup_{x \in X} B(x, \delta)$, or equivalently:

$$S_\delta(X) = \{z \in \mathbb{R}^d \ : \ \mathrm{dist}(z, X) < \delta\}.$$

If there is some $p > 0$ such that:

$$c\delta^{d-p} \leq |S_\delta(X)| \leq C\delta^{d-p}$$

for all $\delta > 0$, then $X$ is said to have Minkowski dimension $p$. If $0 < c < C < \infty$, then $X$ is said to have finite upper and positive lower $p$-dimensional Minkowski content.

## Conformal dimension

Another closely related notion of dimension is conformal dimension (due to Sullivan).
Let $X \subseteq \mathbb{C}$, with $f : X \to X$, and $\nu$ a measure on $X$. The measure $\nu$ is said to be $f$-conformal with dimension $p$, if for every open subset $U \subseteq X$ such that $f|U$ is injective, we have:

$$\nu(f(U)) = \int_U |f'|^p \, d\nu.$$

# Characterising the Hausdorff measure

Fact: For Julia sets $J(f_c)$, $c \in M_0$, we have:
Hausdorff dimension $=$ Minkowski dimension
and the Hausdorff measure is the unique $f_c$-conformal measure of dimension equal to $p$, where $p$ is the Hausdorff dimension.
Using the Riesz theorem, we can characterise the Hausdorff measure as follows:

## Theorem

*The Hausdorff measure $\lambda_p$ of $J(f_c)$ is the unique positive Borel measure such that for all open sets $U \subseteq J(f_c)$ with $f_c|U$ injective, and continuous functions $g$ supported in $U$, we have:*

$$\int_{f_c(U)} g \, d\lambda_p = \int_U (g \circ f_c)|f_c'|^p \, d\lambda_p.$$

# Characterising the Hausdorff measure (cont.)

Since $f_c(z) = z^2 + c$ we have:

$$f_c^{-1}(f_c(U)) = U \cup -U.$$

We can moreover say the following:

**Theorem**

*The Hausdorff measure $\lambda_p$ is the unique (up to a constant scalar factor) positive Borel measure such that for all continuous functions $g$ on $J(f_c)$,*

$$\int_{J(f_c)} g \, d\lambda_p = \frac{1}{2} \int_{J(f_c)} (g \circ f_c)|f_c'|^p \, d\lambda_p.$$

When we prove the conformal trace theorem, we use the above characterisation of the Hausdorff measure.

# Part II: The Conformal trace theorem

# The main task of the paper

Let $g : J(f_c) \to \mathbb{C}$ be a continuous function. In his 1994 book *Noncommutative geometry*, Alain Connes announced a formula for $\int_J g \, d\lambda_p$ given in terms of his "quantised calculus". We have now completed the proof of this formula.

# The Hilbert transform

Let $\mathbb{T}$ be the unit circle (in the complex plane). The Hilbert space $L_2(\mathbb{T})$ is defined with respect to the arc-length measure (the Haar measure).
There is the trigonometric orthonormal basis for $L_2(\mathbb{T})$,

$$e_n(z) = z^n, \quad n \in \mathbb{Z}, z \in \mathbb{T}.$$

The Hilbert trasform $F$ is defined on the basis $e_n$ by $Fe_n = \mathrm{sgn}(n)e_n$.

# Quantised differentials

If $f$ is a bounded function on $\mathbb{T}$, then pointwise multiplication by $f$ defines a bounded linear operator $M_f$ on $L_2(\mathbb{T})$. Connes calls the commutator $i[F, M_f]$ the “quantised differential” of $f$.
The name is intended to imply that this is something like a differential $df$. So we use the symbol $\text{đ}f$,

$$\text{đ}f := i[F, M_f].$$

## Böttcher coordinates

The following is due to L. Böttcher:

**Theorem**

*Let $f$ be a polynomial of degree $d \geq 2$. There exists a conformal map $Z$:*

$$Z : \{|z| > 1\} \to \textit{Attracting basin of } \infty.$$

*such that $f(Z(z)) = Z(z^d)$, for all $|z| > 1$.*

If $J(f_c)$ is a Jordan curve, then Carathéodory's theorem implies that $Z$ has continuous extension:

$$Z : \mathbb{T} \to J(f_c)$$

such that $Z(z^2) = f_c(Z(z))$ for all $z \in \mathbb{T}$.

# Description of the Conformal Trace Formula

Let $f$ be a continuous function on the Julia set $J(f_c)$. Then the operator

$$M_{f\circ Z}|đZ|^p = M_{f\circ Z}|[F, M_Z]|^p$$

is some kind of "$p$-dimensional density" on the Julia set $J(f_c)$ (in the language of quantised calculus).

Motivated by noncommutative geometry, one might guess that the correct way of "integrating" this density is to take a trace.

# Description of the Conformal Trace Formula

**Lemma**

*The quantised differential $đZ$ (i.e., the commutator $[F, M_Z]$) is in the ideal $\mathcal{L}_{p,\infty}$.*

Hence, the operator $M_{f\circ Z}|đZ|^p$ is in $\mathcal{L}_{1,\infty}$.

# Description of the Conformal Trace Formula

## Theorem

*Let $\varphi$ be a continous trace on $\mathcal{L}_{1,\infty}$. Then there is a constant $K(\varphi, c)$ such that for all $f \in C(J(f_c))$,*

$$\varphi(M_{f\circ Z}|\bar{d}Z|^p) = K(\varphi, c) \int_{J(f_c)} f \, d\lambda_p$$

*where $\lambda_p$ is the p-dimensional Hausdorff measure on $J(f_c)$. Also, there exist traces $\varphi$ such that $K(\varphi, c) > 0$.*

# Equivalent statement of the Conformal trace theorem

Recall that the Hausdorff measure can be characterised as the unique $f_c$-conformal measure of dimension $p$.
Let $\ell_\varphi$ be the linear functional:

$$\ell_\varphi(g) = \varphi(M_{g\circ Z}|\bar{d}Z|^p).$$

By the Riesz theorem, there is some measure $\nu$ such that $\ell_\varphi$ is integration against $\nu$. It therefore suffices to show the following:

1. For all continuous $g : J(f_c) \to \mathbb{C}$, we have:

$$\ell_\varphi(g) = \frac{1}{2}\ell_\varphi((g \circ f_c)|f_c'|^p).$$

2. $\ell_\varphi > 0$, for at least some $\varphi$.

# Non-triviality (cont.)

First, we can show the "non-triviality" component.

**Theorem**

*Let $\mathcal{C}$ be a Jordan curve with finite upper and positive lower p-Minkowski content, and let $\zeta : \mathbb{T} \to \mathcal{C}$ be the continuous extension of a conformal equivalence of the open unit disk and the interior of $\mathcal{C}$. Then $\bar{d}\zeta \in \mathcal{L}_{p,\infty}$, and for all dilation invariant extended limits $\omega$ such that $\omega \circ \log$ is still dilation invariant, we have:*

$$\mathrm{tr}_\omega(|\bar{d}\zeta|^p) > 0.$$

# Non-triviality (cont.)

Idea of the proof: If $\omega$ satisfies the stated conditions, then:

$$\mathrm{tr}_\omega(|\bar{d}\zeta|^p) = (\omega \circ \log)\left(t \mapsto \frac{1}{t}\mathrm{tr}(|\bar{d}\zeta|^{p(1+1/t)})\right)$$

So it suffices to show that:

$$\liminf_{s\to 0} s \cdot \mathrm{tr}(|\bar{d}\zeta|^{p+s}) > 0.$$

# Non-triviality (cont.)

A theorem of Peller gives us the equivalence:

$$\mathrm{tr}(|\bar{\partial}\zeta|^{p+s}) \approx \int_{\mathbb{D}} |\zeta'(z)|^{p+s}(1-|z|^2)^{p+s-2}\,dzd\overline{z}.$$

According to the Koebe 1/4-theorem,

$$\mathrm{dist}(\zeta(0),\mathcal{C}) \leq |\zeta'(0)| \leq 4\mathrm{dist}(\zeta(0),\mathcal{C})$$

A change of coordinates yields:

$$\frac{1}{4}(1-|z|^2)|\zeta'(z)| \leq \mathrm{dist}(\zeta(z),\mathcal{C}) \leq (1-|z|^2)|\zeta'(z)|$$

for all $z \in \mathbb{D}$.

# Non-triviality (cont.)

In summary:

$$\mathrm{tr}(|\bar{d}\xi|^{p+s}) \approx \int_{\mathrm{int}\mathcal{C}} \mathrm{dist}(z,\mathcal{C})^{p+s-2}\, dz d\bar{z}.$$

The question is now reduced to a purely geometric problem concerning $\mathcal{C}$.

# Non-triviality (cont.)

Split up $\mathrm{int}\mathcal{C}$ into regions:

$$A_k = \{z \in \mathrm{int}\mathcal{C} \ : \ \mathrm{dist}(z,\mathcal{C}) \in [\lambda^{-k-1}, \lambda^{-k})\}.$$

Then,

$$\int_{\mathrm{int}(\mathcal{C}} \mathrm{dist}(z,\mathcal{C})^{p+s-2}\, dzd\overline{z} \geq \sum_{k\geq 0} \lambda^{-k(p+s-2)}|A_k|.$$

with $|A_k| = |S_{\lambda^{1-k}}(\mathcal{C}) \cap \mathrm{int}(\mathcal{C})| - |S_{\lambda^{-k}}(\mathcal{C}) \cap \mathrm{int}(\mathcal{C})| \geq B\lambda^{-k(2-p)}$
for sufficiently big $\lambda$.
This gives,

$$\liminf_{s\to 0} s \cdot \mathrm{tr}(|\bar{\partial}\zeta|^{p+s}) > 0.$$

# Proving conformality

The remaining task is to show that:

$$\ell_\varphi(g) = \frac{1}{2}\ell_\varphi((g\circ f_c)|f_c'|^p).$$

Recall that we have the Böttcher equation:

$$Z(z^2) = f_c(Z(z)).$$

Let $U$ be the partial isometry on $L_2(\mathbb{T})$ given by $U(z^n) = z^{2n}$.
Then,

$$UM_Z U^* = M_{f_c\circ Z}.$$

we also have $U^*U = 1$ and $UU^* = P$ is the projection $P(z^n) = z^n$ if $n$ is even and $0$ if $n$ is odd.
It is important that $U$ commutes with the Hilbert transform $F$.

# Proving conformality (cont.)

Then,

$$\begin{aligned}\varphi(M_{g\circ Z}|\text{đ}Z|^p) &= \varphi(U^*UM_{g\circ Z}|\text{đ}Z|^p)\\ &= \varphi(M_{g\circ f_c\circ Z}U|[F,M_Z]|^pU^*)\\ &= \varphi(M_{g\circ Z}|[F,M_{f_c\circ Z}]|^pUU^*)\\ &= \varphi(M_{g\circ f_c\circ Z}|\text{đ}(f_c\circ Z)|^pP).\end{aligned}$$

A further argument using unitary invariance shows that:

$$\varphi(M_{g\circ f_c\circ Z}|\text{đ}(f_c\circ Z)|^pP) = \frac{1}{2}\varphi(M_{g\circ f_c\circ Z}|\text{đ}(f_c\circ Z)|^p)$$

The following is called by Connes a "quantised chain rule". Let $f$ be a polynomial. Then:

$$|đ(f \circ Z)|^p - |f'(M_Z)|^p |đZ|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Proving the above identity is the most technically challenging part of the proof, requiring the operator integration tricks developed in the preceding talk.

# The quantised chain rule (cont.)

We have,

$$[M_f, [F, M_Z]] = [[M_f, F], M_Z]$$

Since $f$ is a polynomial, the commutator $[M_f, F]$ is finite rank. Thus, we have:

$$M_{z^n}[F, M_Z]M_{z^{-n}} - [F, M_Z] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}.$$

This (highly nontrivally) implies that:

$$|M_{z^n}[F, M_Z]M_{z^{-n}}|^p - |[F, M_Z]|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Since $M_{z^n}$ is unitary, we get:

$$[M_{z^n}, |[F, M_Z]|^p] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

# The quantised chain rule (cont.)

Hence for all polynomials $f$,

$$[M_f, |\bar{d}Z|^p] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

An approximation argument allows us to state the above for all continuous $f$ as well.
Fact: we also have $[M_f, |\bar{d}Z|] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$.
Since $[M_Z^n, [F, M_Z]] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$, we have that:

$$[F, M_Z^n] \in nM_Z^{n-1}[F, M_Z] + \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}.$$

# The quantised chain rule (cont.)

By linearity,

$$[F, f(M_Z)] - f'(M_Z)[F, M_Z] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}.$$

So,

$$|[F, f(M_Z)]|^p - |f'(M_Z)[F, M_Z]|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

Using the identity $|AB| = ||A|B|$, and $[M_{|f'\circ Z|^{1/2}}, |[F, M_Z]|] \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{p,\infty}}$ we have

$$|đf \circ Z|^p - (|f'(M_Z)|^{1/2}|đZ||f'(M_Z)|^{1/2})^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

# The quantised chain rule (cont.)

Using the advanced operator integration techniques from the preceding talk:

$$|f'(M_Z)|^p|đZ|^p-(|f'(M_Z)|^{1/2}|đZ||f'(M_Z)|^{1/2})^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

This finally gives us the quantised chain rule:

$$|đ(f\circ Z)|^p - |f'(M_Z)|^p|đZ|^p \in \overline{\mathrm{FiniteRank}}^{\mathcal{L}_{1,\infty}}.$$

The quantised chain rule gives us:

$$\varphi(M_{g\circ f_c\circ Z}|đ(f_c \circ Z)|^p) = \varphi(M_{g\circ f_c\circ Z}|f_c'(M_Z)|^p|đZ|^p).$$

Which is exactly what we needed.

# Further directions

- What is the dependence of $K(\varphi, c)$ on $\varphi$?
- Can a similar result be stated for Julia set of more general polynomials (or even non-polynomials)?
- Can we provide a similar result for Hausdorff measures on other Jordan curves in the plane? (such as a Koch snowflake)

At present, very little is known.

Thank you for listening! Further reading:

Good references for Julia sets and conformal dynamics are,

Carleson and Gamelin, *Complex Dynamics*, 1993

Milnor, *Dynamics in One Complex variable*, 2006.

More information on the quantised calculus may be found in:

Connes, *Noncommutative geometry*, 1994.