Dave-Haller's Weyl law and the tangent groupoid

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Introduction

This talk is based on none of my own work, instead I want to advertise the following two papers:

E. Van Erp and R. Yuncken, A groupoid approach to pseudodifferential calculi. *J. Reine Angew. Math.* **756** (2019), 151–182.

and

S. Dave and S. Haller, The heat asymptotics on filtered manifolds *J. Geom. Anal.* **30** (2020), no. 1, 337–389.

Plan for this talk

1. Carnot-Caratheodory geometry
2. The tangent groupoid (of Connes)
3. The H-tangent groupoid (of van Erp and Yuncken)
4. The Volterra calculus
5. Dave-Haller's Weyl law for H-elliptic operators.

Asteroids

In the classic game of asteroids, a player controls a spaceship moving on a two dimensional toroidal space, $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. There are two controls available:

1. (i) The spaceship can be rotated
2. (ii) The spaceship can be moved forward along the direction it is facing.

Asteroids

The configuration space of the game is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, with coordinates (x, y, θ), where (x, y) is the position of the spaceship and θ is its angle.
The controls of the game allow us to move along the vector fields

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The player moves the spaceship along a path which is parallel to the span of X and Y. That is, the path of the spaceship in configuration space is $\{\gamma(t)\}_{t\geq 0}$, where

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Despite there being only two available directions, we can reach any point (x, y, θ) from any other point by travelling parallel to X and Y.

Asteroids

In general, if we can travel parallel to X and Y then we can approximate paths along $[X, Y]$, by the Lie-Kato-Trotter product formula

$$\exp(t[X, Y]) = \lim_{n\to\infty}(\exp(\frac{t}{n}X)\exp(\frac{t}{n}Y)\exp(-\frac{t}{n}(X+Y))\exp(-\frac{t}{n}Y))^n.$$

But moving along $[X, Y]$ is harder than moving along X and Y.
In the asteroids example,

$$[X, Y] = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

so $\{X, Y, [X, Y]\}$ form a basis for the tangent space to $\mathbb{T}^3$ at every point.

Asteroids

Thinking about X and Y as derivations (not just as directions), we should think of X, Y as being order 1 and $[X, Y]$ as being order 2.
The operator

$$\Xi = X^2 + Y^2 = \partial_\theta^2 + (\cos(\theta)\partial_x + \sin(\theta)\partial_y)^2$$

is homogeneous of order 2.

Carnot manifolds

The plane bundle $\mathrm{span}(X, Y) \subset T\mathbb{T}^3$ is an example of a contact structure, and this leads us to what is in general called a Carnot manifold.

Definition

A *Carnot manifold* is a manifold X equipped with a filtration of sub-bundles of TX. That is, there are subbundles $(H^j)_{j=0}^N$ such that

$$0 = H^0 < H^1 < \cdots < H^N = TX$$

and if $E \in \Gamma H^j, F \in \Gamma H^j$, then $[E, F] \in \Gamma H^{j+k}$.

We should think of the directions in H^j as having "order j".

Differential operators on Carnot manifolds

We would like to understand operators coming from Carnot manifolds, such as

$$\Xi = X^2 + Y^2 = \partial_\theta^2 + (\cos(\theta)\partial_x + \sin(\theta)\partial_y)^2$$

from the asteroids example. What do we want to understand?

- Ξ is not elliptic, but $(1-\Xi)^{-1}$ does improve regularity somewhat. Why, and by how much?
- The spectrum of Ξ is discrete, with a sequence of eigenvalues $0 \leq \lambda(1,\Xi) \leq \lambda(2,\Xi) \leq \cdots$. What is their asymptotic behaviour?

The usual way to analyse differential operators is to build a pseudodifferential calculus.

The Connes tangent groupoid

The usual recipe for defining pseudodifferential operators on a manifold X is the following procedure:

1. Identify a class of symbols σ on $\mathbb{R}^d$.
2. Define pseudodifferential operators $\mathrm{Op}(\sigma)$ by a quantisation formula such as

$$\mathrm{Op}(\sigma)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x,\xi)}\sigma(x,\xi)\widehat{f}(\xi)\, d\xi$$

3. Show that the class of pseudodifferential operators just defined is invariant under change of variables
4. A pseudodifferential operator on a manifold X is a linear operator $T : C_c^\infty(X) \to C^\infty(X)$ which has smooth kernel away from the diagonal and which is pseudodifferential in every chart.

This is a little inelegant, is there a better way?

Semiclassical quantisation

Often it is better to quantise a symbol σ into a whole family of operators $\mathrm{Op}_\hbar(\sigma)$ depending on a parameter $\hbar$, by a formula such as

$$\mathrm{Op}_\hbar(\sigma)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x,\xi)}\sigma(x,\hbar\xi)\widehat{f}(\xi)\,d\xi.$$

As $\hbar\to 0$, the noncommutative algebra of $\hbar$-pseudodifferential operators under operator composition is supposed to reduce to the commutative algebra of symbols under pointwise multiplication.

Kernels of pseudodifferential operators

The Schwartz kernel of a pseudodifferential operators with symbol σ is given by the oscillatory integral

$$K(x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}\sigma(x,\xi)\,d\xi$$

(in the distribution sense). If we consider the kernel $K(\cdot,\cdot,\hbar)$ of the $\hbar$-quantisation, we should have

$$K(x,y,\hbar) = (2\pi\hbar)^{-d}\int_{\mathbb{R}^d} e^{i\frac{x-y}{\hbar}\cdot\xi}\sigma(x,\xi)\,d\xi.$$

In the limit as $\hbar\to 0$, what should happen is that this looks more and more like the kernel of a convolution operator. Really, the kernel K of a pseudodifferential operator on a manifold X should be thought of as a function (distribution) on the space

$$\mathbb{T}X = (TX\times\{0\})\sqcup(X\times X\times(0,\infty)).$$

The tangent groupoid

Let X be a manifold, and define the set

$$\mathbb{T}X = (TX \times \{0\}) \sqcup (X \times X \times \mathbb{R}^{\times}).$$

Connes invented a good topology for $\mathbb{T}X$, making it a manifold of dimension $2\dim(X) + 1$. Better yet, this is a *Lie groupoid*, with range and source maps

$$r(x, y, \hbar) = (x, \hbar), \quad s(x, y, \hbar) = (y, \hbar), \quad r((x, z), 0) = s((x, z), 0) = (x, 0)$$

and composition law

$$(x, y, \hbar) \circ (y, w, \hbar) = (x, w, \hbar), \quad ((x, z), 0) \circ ((x, z'), 0) = (x, z + z', 0).$$

The tangent groupoid

Elements of the convolution algebra of the groupoid $\mathbb{T}X$ are distributions $f, g \in \mathcal{D}'(\mathbb{T}X)$, with convolution product

$$(f * g)(x, y, h) = \int_X f(x, w, h) g(w, y, h)\, dw, \quad (f * g)((x, z), 0) = \int_{T_x X} f((x, z$$

That is: the convolution of distributions on $\mathbb{T}X$ looks like composition of kernels of pseudodifferential operators.

Thank you for listening!