Dave-Haller's Weyl law and the tangent groupoid

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Introduction

This talk is based on none of my own work, instead I want to advertise the following two papers:

E. Van Erp and R. Yuncken, A groupoid approach to pseudodifferential calculi. *J. Reine Angew. Math.* **756** (2019), 151–182.

and

S. Dave and S. Haller, The heat asymptotics on filtered manifolds *J. Geom. Anal.* **30** (2020), no. 1, 337–389.

Plan for this talk

1. Carnot-Caratheodory geometry
2. The tangent groupoid (of Connes)
3. The H-tangent groupoid (of van Erp and Yuncken)
4. The Volterra calculus
5. Dave-Haller's Weyl law for H-elliptic operators.

Part 1: Asteroids

Rules of Asteroids

In the classic game of asteroids, a player controls a spaceship moving on a two dimensional toroidal space, $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. There are two controls available:

1. (i) The spaceship can be rotated,
2. (ii) The spaceship can be moved forward.

Configuration space of Asteroids

The configuration space of the game is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, with coordinates (x, y, θ), where (x, y) is the position of the spaceship and θ is its angle.
The controls of the game allow us to move along the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta.$$

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The player moves the spaceship along a path which is parallel to the span of X and Y. That is, the path of the spaceship in configuration space is $\{\gamma(t)\}_{t\geq 0}$, where

$$\dot{\gamma}(t) \in \mathrm{span}\{X_{\gamma(t)}, Y_{\gamma(t)}\}, \quad t \geq 0.$$

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Despite there being only two available directions, we can reach any point (x, y, θ) from any other point by travelling on a piecewise-smooth path parallel to X and Y.

Asteroids

In general, if we can travel parallel to X and Y then we can approximate paths along $[X, Y]$, by the Lie-Kato-Trotter product formula

$$\exp(t[X,Y]) = \lim_{n\to\infty}(\exp(\frac{t}{n}X)\exp(\frac{t}{n}Y)\exp(-\frac{t}{n}(X+Y))\exp(-\frac{t}{n}Y))^n.$$

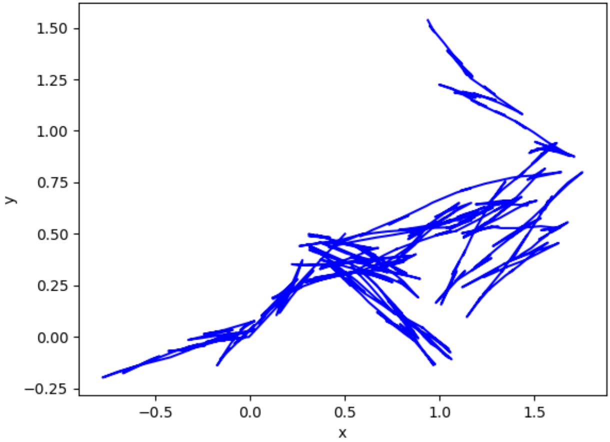
But moving along $[X, Y]$ is harder than moving along X and Y.
In the asteroids example,

$$[X,Y] = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

so $\{X, Y, [X,Y]\}$ form a basis for the tangent space to $\mathbb{T}^3$ at every point.

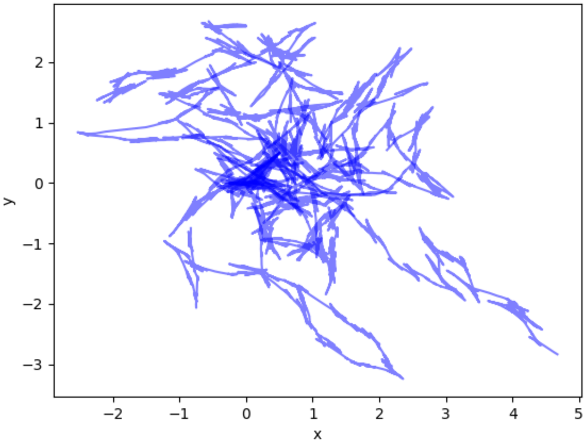
Random walks

A random walk making independent increments in the X and Y directions looks a bit like this:



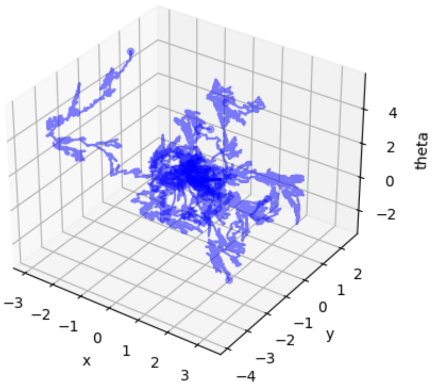
Random walks

Ten realisations of the same random walk look like this:



Random walks

The real path through configuration space is in three dimensions, and looks like this:



Asteroids

Thinking about X and Y as derivations (not just as directions), we should think of X, Y as being order 1 and $[X, Y]$ as being order 2.
The operator

$$\Theta = X^2 + Y^2 = \partial_\theta^2 + (\cos(\theta)\partial_x + \sin(\theta)\partial_y)^2$$

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- Θ is not elliptic, but $(1-\Theta)^{-1}$ does improve regularity somewhat. Why, and by how much?
- The spectrum of Θ is discrete, with a sequence of eigenvalues $0 \leq \lambda(1, -\Theta) \leq \lambda(2, -\Theta) \leq \cdots$ What is their asymptotic behaviour?

Asteroids

Let $\|u\|_s$ denote the standard Hilbert-Sobolev norm of order s of $u \in C^\infty(\mathbb{T}^3)$. That is,

$$\|u\|_s := \|(1 - \partial_x^2 - \partial_y^2 - \partial_z^2)^{\frac{s}{2}} u\|_{L_2(\mathbb{T}^3)}.$$

A highly non-trivial calculation gives us the *sub-elliptic estimates*:

$$\|u\|_{s+\frac{1}{2}} \lesssim_s \|\Theta u\|_s + \|u\|_s.$$

This implies hypoellipticity (i.e., $\Theta u \in C^\infty \Rightarrow u \in C^\infty$) and also the discreteness of the spectrum of Θ.

The Asteroids Weyl law

Theorem (General Weyl law)

Note that an elliptic second order differential operator $P \geq 0$ on a compact d-manifold has discrete spectrum $0 < \lambda(1, P) \leq \lambda(2, P) \leq \lambda(3, P) \leq \cdots$ and there exists a constant c_d such that as $n \to \infty$ we have

$$\lambda(n, P) \sim c_d \mathrm{Vol}(X)^{\frac{2}{d}} n^{\frac{2}{d}}.$$

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Theorem (Weyl law for Asteroids)

The nth smallest eigenvalue of $-\Theta$ *obeys*

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The spectral dimension of Asteroids is 4*, not three!* What is going on here?

Part 2: Carnot manifolds and graded Lie groups

Carnot manifolds

The plane bundle $\mathrm{span}(X, Y) \subset T\mathbb{T}^3$ is an example of a contact structure, and this leads us to what is in general called a Carnot manifold.

Definition

A *Carnot manifold* is a manifold X equipped with a filtration of sub-bundles of TX. That is, there are subbundles $(H^j)_{j=0}^N$ such that

$$0 = H^0 < H^1 < \cdots < H^N = TX$$

and if $E \in \Gamma H^j, F \in \Gamma H^j$, then $[E, F] \in \Gamma H^{j+k}$.

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We should think of the directions in H^j as having "order j". I will concentrate on the example $X = \mathbb{T}^3$, with $N = 2$ and $H^1 = \mathrm{span}\{X, Y\}$.

Graded Lie groups

The most fundamental example of a Carnot manifold is a graded nilpotent Lie group. These are important in the general theory as local models.
Let $\mathfrak{g}$ be a Lie algebra which admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$$

where $[\mathfrak{g}_k, \mathfrak{g}_n] \subseteq \mathfrak{g}_{k+n}$. This is called a stratified Lie algebra. Being finite dimensional, it is easy to see that $\mathfrak{g}$ is nilpotent.
The number

$$Q := \sum_{n=1}^{\infty} n\dim(\mathfrak{g}_n)$$

is called the homogeneous dimension of $\mathfrak{g}$.

Graded Lie groups

Exponentiating $\mathfrak{g}$, we get a simply connected Lie group

$$G = \exp(\mathfrak{g}).$$

Nilpotent groups are very special: the exponential mapping is a homeomorphism, and the Lebesgue measure of $\mathfrak{g}$ pushes forward to the Haar measure of G.

Alternative point of view: $\mathfrak{g}$ and G are identical as sets, and G is equipped with the group law determined by the Baker-Campbell-Hausdorff formula. The nilpotency of the group ensures that the BCH series is finite.

The Heisenberg group

The prototypical example is the Heisenberg Lie group. This one has $\mathfrak{g} = \mathrm{span}\{\mathcal{X}, \mathcal{Y}, \mathcal{T}\}$, with $[\mathcal{X}, \mathcal{Y}] = \mathcal{T}$ and all other commutators vanishing. The grading is

$$\mathfrak{g}_1 = \mathrm{span}\{\mathcal{X}, \mathcal{Y}\}, \quad \mathfrak{g}_2 = \mathrm{span}\{\mathcal{T}\}.$$

The homogeneous dimension is $Q = 1 \cdot 2 + 2 \cdot 1 = 4$. The representation theory of the Heisenberg Lie group is well-known since Stone and von Neumann. The unitary irreducible representations are enumerated as

$$\widehat{G} = \{\pi_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{R}} \sqcup \{\pi_s\}_{s\in\mathbb{R}\setminus\{0\}}$$

where $\pi_{\alpha,\beta}$ are the one-dimensional representations $\pi_{\alpha,\beta}(\mathcal{X}) = i\alpha, \pi_{\alpha,\beta}(\mathcal{Y}) = i\beta$ and π_s are the infinite dimensional Schrödinger representations,

$$\pi_s(\mathcal{X}) = \mathrm{sgn}(s)|s|^{\frac{1}{2}}\partial_x, \quad \pi_s(\mathcal{Y}) = -i|s|^{\frac{1}{2}}x.$$

The osculating group

From the data of a Carnot manifold (X, H), we define the *associated graded bundle* $\mathfrak{t}_H X$, which is the graded vector bundle over X formed by

$$\mathfrak{t}_H X = \bigoplus_{n=1}^{N} (\mathfrak{t}_H X)^n, \qquad \text{where} \qquad (\mathfrak{t}_H X)^n = H^n / H^{n-1}.$$

If $X \in \Gamma(\mathfrak{t}_H X)^n$, and $Y \in \Gamma(\mathfrak{t}_H X)^m$, then $[X, Y]$ is a well-defined section of $(\mathfrak{t}_H X)^{n+m}$. Here, ΓE denotes the space of smooth sections of a vector bundle E. In fact more is true, for smooth functions f, g, we then have

$$[fX, gY] - fg[X, Y] \in \Gamma H^{n+m-1}.$$

This implies that the commutator of vector fields descends to a Lie bracket on the the fibres of $\mathfrak{t}_H X$. That is, $\mathfrak{t}_H X$ is a bundle of graded nilpotent Lie groups.
The fibrewise exponential $T_H X := \exp(\mathfrak{t}_H X)$ is called the *osculating group* of the filtration H.

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We should think of the $T_H X$ as being an infinitesimal model for X with its filtration H.

The homogeneous dimension

The number

$$d_H := \sum_{k=1}^{\infty} k \cdot \mathrm{rank}(\mathfrak{t}_H X^k) = \sum_{k=1}^{N} \mathrm{rank}(H^k)$$

is called the homogenous dimension of (X, H). This is the homogeneous dimension of the fibres of $T_H X$ as a graded Lie group.

The asteroids example

When $X = \mathbb{T}^3$ is equipped with the filtration $H^1 = \mathrm{span}\{X, Y\}$, then the fibres of the osculating Lie group are all the Heisenberg group, and the homogeneous dimension is 4.

Part 3: The tangent groupoid

The order of a differential operator

A general differential operator P on a manifold X is a polynomial in vector fields. A filtration H of TX induces a filtration of the algebra $\mathrm{DO}(X)$ of differential operators on X,

$$\mathrm{DO}(X) = \bigcup_{m\geq 0} \mathrm{DO}_H^m(X)$$

where $\mathrm{DO}_H^m(X)$ is the set of differential operators of H-order m. We would like to understand these differential operators better. The standard method is to build a *pseudodifferential calculus*.

The Connes tangent groupoid

The usual recipe for defining pseudodifferential operators on a manifold X is the following procedure:

1. Identify a class of symbols σ on $\mathbb{R}^d$.
2. Define pseudodifferential operators $\mathrm{Op}(\sigma)$ by a quantisation formula such as
$$\mathrm{Op}(\sigma)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x,\xi)}\sigma(x,\xi)\widehat{f}(\xi)\,d\xi$$
3. Show that the class of pseudodifferential operators just defined is invariant under change of variables
4. A pseudodifferential operator on a manifold X is a linear operator $T : C_c^\infty(X) \to C^\infty(X)$ which has smooth kernel away from the diagonal and which is pseudodifferential in every chart.

This is a little inelegant, is there a better way?

Semiclassical quantisation

Often it is better to quantise a symbol σ into a whole family of operators $\mathrm{Op}_\hbar(\sigma)$ depending on a parameter $\hbar$, by a formula such as

$$\mathrm{Op}_\hbar(\sigma)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x,\xi)}\sigma(x,\hbar\xi)\widehat{f}(\xi)\,d\xi.$$

As $\hbar \to 0$, the noncommutative algebra of $\hbar$-pseudodifferential operators under operator composition is supposed to reduce to the commutative algebra of symbols under pointwise multiplication.

Kernels of pseudodifferential operators

The Schwartz kernel of a pseudodifferential operators with symbol σ is given by the oscillatory integral

$$K(x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}\sigma(x,\xi)\,d\xi$$

(in the distribution sense). If we consider the kernel $K(\cdot,\cdot,\hbar)$ of the $\hbar$-quantisation, we should have

$$K(x,y,\hbar) = (2\pi\hbar)^{-d}\int_{\mathbb{R}^d} e^{i\frac{x-y}{\hbar}\cdot\xi}\sigma(x,\xi)\,d\xi.$$

In the limit as $\hbar\to 0$, what should happen is that this looks more and more like the kernel of a convolution operator. Really, the kernel K of a pseudodifferential operator on a manifold X should be thought of as a function (distribution) on the space

$$\mathbb{T}X = (TX\times\{0\})\sqcup(X\times X\times(0,\infty)).$$

The tangent groupoid

Let X be a manifold, and define the set

$$\mathbb{T}X = (TX \times \{0\}) \sqcup (X \times X \times \mathbb{R}^\times).$$

Connes invented a good topology for $\mathbb{T}X$, making it a manifold of dimension $2\dim(X) + 1$.
Roughly speaking, the topology is set up so that

$$\lim_{\varepsilon \to 0} (x, x + \varepsilon z, \varepsilon) = ((x, z), 0).$$

The tangent groupoid

Better yet, this is a *Lie groupoid*, with range and source maps

$$r(x,y,\hbar) = (x,\hbar), \quad s(x,y,\hbar) = (y,\hbar), \quad r((x,z),0) = s((x,z),0) = (x,0)$$

and composition law

$$(x,y,\hbar)\circ(y,w,\hbar) = (x,w,\hbar), \quad ((x,z),0)\circ((x,z'),0) = (x,z+z',0).$$

The tangent groupoid

Elements of the convolution algebra of the groupoid $\mathbb{T}X$ are distributions $f, g \in \mathcal{D}'(\mathbb{T}X)$, with convolution product

$$(f * g)(x, y, h) = \int_X f(x, w, h) g(w, y, h)\, dw,$$
$$(f * g)((x, z), 0) = \int_{T_x X} f((x, z - z'), 0) g((x, z'), 0)\, dz'.$$

That is: the convolution of distributions on $\mathbb{T}X$ looks like composition of kernels of pseudodifferential operators.
Can we use this as a basis for a definition of an algebra of pseudodifferential operators?

The zoom action

The tangent groupoid $\mathbb{T}X$ comes equipped with a natural action of $\mathbb{R}^\times$:

$$\alpha_\lambda(x, y, \hbar) = (x, y, \lambda^{-1}\hbar), \quad \alpha_\lambda((x, z), 0) = (x, \lambda z).$$

This is the *zoom action* (or the Debord-Skandalis action). This turns out to be the key to characterising the kernels of $\hbar$-pseudodifferential operators. Terminology: a distribution $P \in \mathcal{D}'(\mathbb{T}X)$ is called r-fibred if (roughly speaking) $P(x, y, \hbar)$ is smooth in the x and $\hbar$ variables.

Theorem (van Erp-Yuncken (2019))

An r-fibred distribution $P \in \mathcal{D}'(\mathbb{T}X)$ is the family of kernels of a classical pseudodifferential operator of order m if and only if for all $\lambda > 0$, we have

$$\lambda^{-m-\dim(X)} P \circ \alpha_\lambda^{-1} - P \in C^\infty(\mathbb{T}_H X).$$

Note to experts: I am ignoring issues of support and choice of measure

The H-tangent groupoid

An advantage of the groupoid viewpoint is that it can easily be adapted to build pseudodifferential calculi in non-classical situations.

Definition (van Erp-Yuncken (2019))

Let (X, H) be a Carnot manifold. The H-tangent groupoid is defined as a set by

$$\mathbb{T}_H X = T_H X \times \{0\} \sqcup X \times X \times \mathbb{R}^\times.$$

There is an appropriate topology on $\mathbb{T}_H X$ making it a manifold, similar to the tangent groupoid but taking the filtration into account.

Van Erp and Yuncken build algebras of pseudodifferential operators inside the convolution algebra of $\mathbb{T}_H X$ and prove its basic properties.

Part 4: Dave and Haller's Weyl law

H-elliptic differential operators

Let (X, H) be a Carnot manifold. Recall that we built a bundle of nilpotent Lie algebras $\mathfrak{t}_H X$ out of X.
Given a differential operator $P \in \mathrm{DO}^m_H(X)$, we can form its principal cosymbol $\sigma_m(P)$, which is an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_H X)$.
The rule is that if $X \in \Gamma H^j$ is a vector field, then at $x \in X$, $\sigma_j(X)_x = X_x + H^{j-1}_x$ is the image of X in the quotient H^j_x / H^{j-1}_x.

Definition

A differential operator $P \in \mathrm{DO}^m_H(X)$ is called *Rockland* if for all $x \in X$ and all nontrivial unitary irreducible representations π of $T_H X_x$, the image $\pi(\sigma_m(P)_x)$ is injective.

Rockland operators are the Carnot analogy of elliptic operators. In the unfiltered case, the Rockland condition amounts to the invertibility of the principal symbol away from zero.

Theorem (Dave-Haller (2020))

Let $P \geq 0$ be a Rockland operator of order m on a Carnot manifold (X, H). Then P has discrete spectrum, and there is a constant c_P such that the nth smallest eigenvalue obeys

$$\lambda(n, P) \sim c_P n^{\frac{m}{d_H}}.$$

The constant is computable in terms of the principal cosymbol of P and the Plancherel measure of the osculating group.

This explains the operator Θ in the Asteroids example: there the osculating group was the Heisenberg group, we had $m = 2$ and $d_H = 4$. The H-ellipticity is easily checked using the well-known representation theory of the Heisenberg group.

Thank you for listening!